Crossed Product Algebras for Piece-Wise Constant Functions

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Abstract In this paper we consider algebras of functions that are constant on the sets of a partition. We describe the crossed product algebras of the mentioned algebras with $\mathbb{Z}$. We show that the function algebra is isomorphic to the algebra of all functions on some set. We also describe the commutant of the function algebra and finish by giving an example of piece-wise constant functions on a real line.

Keywords Piecewise constant · Crossed products · Maximal commutative subalgebra

1 Introduction

An important direction of investigation for any class of non-commutative algebras and rings, is the description of commutative subalgebras and commutative subrings. This is because such a description allows one to relate representation theory, non-commutative properties, graded structures, ideals and subalgebras, homological and other properties of non-commutative algebras to spectral theory, duality, algebraic geometry and topology naturally associated with commutative algebras. In representation theory, for example, semi-direct products or crossed products play a central

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role in the construction and classification of representations using the method of induced representations. When a non-commutative algebra is given, one looks for a subalgebra such that its representations can be studied and classified more easily and such that the whole algebra can be decomposed as a crossed product of this subalgebra by a suitable action.

When one has found a way to present a non-commutative algebra as a crossed product of a commutative subalgebra by some action on it, then it is important to know whether the subalgebra is maximal commutative, or if not, to find a maximal commutative subalgebra containing the given subalgebra. This maximality of a commutative subalgebra and related properties of the action are intimately related to the description and classification of representations of the non-commutative algebra.

Some work has been done in this direction [2, 4, 6] where the interplay between topological dynamics of the action on one had and the algebraic property of the commutative subalgebra in the C^*-crossed product algebra $C(X) \rtimes \mathbb{Z}$ being maximal commutative on the other hand are considered. In [4], an explicit description of the (unique) maximal commutative subalgebra containing a subalgebra $\mathcal{A}$ of $\mathbb{C}^X$ is given. In [3], properties of commutative subrings and ideals in non-commutative algebraic crossed products by arbitrary groups are investigated and a description of the commutant of the base coefficient subring in the crossed product ring is given. More results on commutants in crossed products and dynamical systems can be found in [1, 5] and the references therein.

In this article, we take a slightly different approach. We consider algebras of functions that are constant on the sets of a partition, describe the crossed product algebras of the mentioned algebras with $\mathbb{Z}$ and show that the function algebra is isomorphic to the algebra of all functions on some set. We also describe the commutant of the function algebra and finish by giving an example of piece-wise constant functions on a real line.

2 Definitions and a Preliminary Result

Let $\mathcal{A}$ be any commutative algebra. Using the notation in [4], we let $\psi : \mathcal{A} \to \mathcal{A}$ be any algebra automorphism on $\mathcal{A}$ and define

$$\mathcal{A} \rtimes_{\psi} \mathbb{Z} := \{f : \mathbb{Z} \to \mathcal{A} : f(n) = 0 \text{ except for a finite number of } n\}.$$

It can be proved that $\mathcal{A} \rtimes_{\psi} \mathbb{Z}$ is an associative $\mathbb{C}-$algebra with respect to point-wise addition, scalar multiplication and multiplication defined by *twisted convolution*, $*$ as follows;

$$(f * g)(n) = \sum_{k\in\mathbb{Z}} f(k).\psi^k(g(n-k)),$$

where ψ^k denotes the $k-$fold composition of ψ with itself for positive k and we use the obvious definition for $k \leq 0$.

Definition 1 $\mathcal{A} \rtimes_\psi \mathbb{Z}$ as described above is called the crossed product algebra of $\mathcal{A}$ and $\mathbb{Z}$ under ψ.

A useful and convenient way of working with $\mathcal{A} \rtimes_\psi \mathbb{Z}$, is to write elements $f, g \in \mathcal{A} \rtimes_\psi \mathbb{Z}$ in the form $f = \sum_{n\in\mathbb{Z}} f_n\delta^n$ and $g = \sum_{n\in\mathbb{Z}} g_m\delta^m$ where $f_n = f(n),\ g_m = g(m)$ and

$$\delta^n(k) = \begin{cases} 1, & \text{if } k = n, \\ 0, & \text{if } k \neq n. \end{cases}$$

Then addition and scalar multiplication are canonically defined and multiplication is determined by the relation

$$(f_n\delta^n) * (g_m\delta^m) = f_n\psi^n(g_m)\delta^{n+m}, \tag{1}$$

where $m, n \in \mathbb{Z}$ and $f_n, g_m \in \mathcal{A}$.

Definition 2 By the commutant $\mathcal{A}'$ of $\mathcal{A}$ in $\mathcal{A} \rtimes_\psi \mathbb{Z}$ we mean

$$\mathcal{A}' := \{f \in \mathcal{A} \rtimes_\psi \mathbb{Z} : fg = gf \text{ for every } g \in \mathcal{A}\}.$$

It has been proven [4] that the commutant $\mathcal{A}'$ is commutative and thus, is the unique maximal commutative subalgebra containing $\mathcal{A}$. For any $f, g \in \mathcal{A} \rtimes_\psi \mathbb{Z}$, that is, $f = \sum_{n\in\mathbb{Z}} f_n\delta^n$ and $g = \sum_{m\in\mathbb{Z}} g_m\delta^m$, then $fg = gf$ if and only if

$$\forall r : \sum_{n\in\mathbb{Z}} f_n\phi^n(g_{r-m}) = \sum_{m\in\mathbb{Z}} g_m\phi^m(f_{r-m}).$$

Now let X be any set and $\mathcal{A}$ an algebra of complex valued functions on X. Let $\sigma : X \to X$ be any bijection such that $\mathcal{A}$ is invariant under σ and σ^{-1}, that is for every $h \in \mathcal{A},\ h \circ \sigma \in \mathcal{A}$ and $h \circ \sigma^{-1} \in \mathcal{A}$. Then (X, σ) is a discrete dynamical system and σ induces an automorphism $\tilde{\sigma} : \mathcal{A} \to \mathcal{A}$ defined by,

$$\tilde{\sigma}(f) = f \circ \sigma^{-1}.$$

Our goal is to describe the commutant of $\mathcal{A}$ in the crossed product algebra $\mathcal{A} \rtimes_{\tilde{\sigma}} \mathbb{Z}$ for the case where $\mathcal{A}$ is the algebra of functions that are constant on the sets of a partition. First we have the following results.

Definition 3 For any nonzero $n \in \mathbb{Z}$, we set

$$Sep^n_{\mathcal{A}}(X) = \{x \in X \mid \exists h \in \mathcal{A} \ :\ h(x) \neq \tilde{\sigma}^n(h)(x)\}. \tag{2}$$

The following theorem has been proven in [4].

Theorem 1 *The unique maximal commutative subalgebra of $\mathcal{A} \rtimes_{\tilde{\sigma}} \mathbb{Z}$ that contains $\mathcal{A}$ is precisely the set of elements*

$$A' = \left\{ \sum_{n\in\mathbb{Z}} f_n \delta^n \mid \textit{for all } n \in \mathbb{Z}: \; f_n|_{Sep^n_{\mathcal{A}}(X)} \equiv 0 \right\}. \tag{3}$$

We observe that since $\tilde{\sigma}(f) = f \circ \sigma^{-1}$, then

$$\tilde{\sigma}^2(f) = \tilde{\sigma}(f \circ \sigma^{-1}) = (f \circ \sigma^{-1}) \circ \sigma^{-1} = f \circ \sigma^{-2},$$

and hence for every $n \in \mathbb{Z}$, $\tilde{\sigma}^n(f) = f \circ \sigma^{-n}$. Therefore, by taking $X = \mathbb{R}$ and $\mathcal{A}$ as the algebra of constant functions on X we have: for every $x \in X$ and every $h \in \mathcal{A}$,

$$\tilde{\sigma}^n(h)(x) := h \circ \sigma^{-n}(x) = h(\sigma^{-n}(x)) = h(x),$$

since h is a constant function. It follows that in this case $Sep^n_{\mathcal{A}}(X) = \emptyset$. Therefore in this case, $\mathcal{A}' = \mathcal{A} \rtimes_{\tilde{\sigma}} \mathbb{Z}$.

3 Algebra of Piece-Wise Constant Functions

Let X be any set, J a countable set and $\mathbb{P} = \{X_j \; : \; j \in J\}$ be a partition of X; that is $X = \cup_{r\in J} X_r$ where $X_r \neq \emptyset$ and $X_r \cap X_{r'} = \emptyset$ if $r \neq r'$.

Let $\mathcal{A}$ be the algebra of piece-wise constant complex-valued functions on X. That is

$$\mathcal{A} = \{h \in \mathbb{C}^X \; : \; \text{for every } j \in J \; : \; h(X_j) = \{c_j\}\}.$$

Let $\sigma : X \to X$ be a bijection on X. The lemma below gives the necessary and sufficient conditions for (X, σ) to be a dynamical system.

Lemma 1 *The following are equivalent.*

1. *The algebra $\mathcal{A}$ is invariant under σ and σ^{-1}.*
2. *For every $i \in J$ there exists $j \in J$ such that $\sigma(X_i) = X_j$.*

Proof We recall that the algebra $\mathcal{A}$ is invariant under σ if and only if for every $h \in \mathcal{A}, \; h \circ \sigma \in \mathcal{A}$.

Obviously, if for every $i \in J$ there exists a unique $j \in J$ such that $\sigma(X_i) = X_j$, then

$$(h \circ \sigma)(X_i) = h(\sigma(X_i)) = h(X_j) = \{c_j\}.$$

Thus $h \circ \sigma \in \mathcal{A}$.

Conversely, suppose $\mathcal{A}$ is invariant under σ but 2. does not hold. Let $x_1, x_2 \in X_j$ and $X_r, X_{r'} \in \mathbb{P}$ such that $\sigma(x_1) \in X_r$ and $\sigma(x_2) \in X_{r'}$. Let $h : X \to \mathbb{C}$ be the function defined by

$$h(x) = \begin{cases} 1 & \text{if } x \in X_r, \\ 0 & \text{otherwise.} \end{cases}$$

Then $h \in \mathcal{A}$. But $h \circ \sigma(x_1) = 1$ and $h \circ \sigma(x_2) = 0$. Thus $h \notin \mathcal{A}$, which contradicts the assumption. □

The following lemma asserts that any bijection $\sigma_2 : X \to X$ that preserves the structure of a partition essentially produces the same algebra of functions.

Lemma 2 *Let* $\mathbb{P}_1 = \{X_j \ : \ j \in J\}$ *and* $\mathbb{P}_2 = \{Y_j \ : \ j \in J\}$ *be partitions of the sets* X *and* Y *respectively, and let*

$$\mathcal{A}_X = \{h \in \mathbb{C}^X : \textit{for every } j \in J : \ h(X_j) = \{c_j\}\},$$

and

$$\mathcal{A}_Y = \{h \in \mathbb{C}^Y : \textit{for every } j \in J : \ h(Y_j) = \{d_j\}\}.$$

Then $\mathcal{A}_X$ *is isomorphic to* $\mathcal{A}_Y$.

Proof Choose points $x_i \in X$ and $y_i \in Y$ such that $x_i \in X_i$ if and only if $y_i \in Y_i \; \forall\, i \in J$ and let $\mu : \mathcal{A}_X \to \mathcal{A}_Y$ be a function defined by

$$\mu(f)(y) = f(x_i) \text{ if } y \in Y_j, \ \forall \ j \in J. \tag{4}$$

It is enough to prove that μ is an algebra isomorphism.

- Let $f, g \in \mathcal{A}_X$ and let $\alpha, \beta \in \mathbb{C}$. Then if $y \in Y$, then $y \in Y_i$ for some $i \in J$, therefore,

$$\begin{aligned}\mu(\alpha f + \beta g)(y) &= (\alpha f + \beta g)(x_i)\\ &= \alpha f(x_i) + \beta g(x_i)\\ &= \alpha\mu(f)(y) + \beta\mu(g)(y)\\ &= [\alpha\mu(f) + \beta\mu(g)](y).\end{aligned}$$

  Therefore μ is linear since y was arbitrary.
- For every $f, g \in \mathcal{A}_X$ and $y \in Y$ $(y \in Y_i)$,

$$\begin{aligned}\mu(fg)(y) &= (fg)(x_i)\\ &= f(x_i)g(x_i)\\ &= \mu(f)(y)\mu(g)(y)\\ &= [\mu(f)\mu(g)](y).\end{aligned}$$

  Thus μ is a multiplicative homomorphism.
- Now, suppose $f, g \in \mathcal{A}_X$ such that $f \neq g$. Then there exists $i \in J$ such that $f(x_i) \neq g(x_i)$, $x_i \in X_i$. Therefore, if $y \in Y_i$,

$$\mu(f)(y) = f(x_i) \neq g(x_i) = \mu(g)y.$$

Therefore μ is injective.

- Finally, suppose $h \in \mathcal{A}_Y$ and let $f \in \mathcal{A}_X$ be defined by $f(x) = h(y_i)$. If $y \in Y$, then $y \in Y_i$ for some $i \in J$, and hence,

$$h(y) = h(y_i) = f(x) = f(x_i) = \mu(f)(y).$$

It follows that μ is onto and hence an algebra isomorphism.

□

Theorem 2 *Let $\mathbb{P}_1 = \{X_j : j \in J\}$ and $\mathbb{P}_2 = \{Y_j : j \in J\}$ be partitions of two sets X and Y and $\mathcal{A}_X$ and $\mathcal{A}_Y$ be algebras of functions that are constant on the sets of the partitions $\mathbb{P}_1$ and $\mathbb{P}_2$ respectively. Let $\sigma_1 : X \to X$ and $\sigma_2 : Y \to Y$ be bijections such that $\mathcal{A}_X$ is invariant under σ_1 (and σ_1^{-1}) and $\mathcal{A}_Y$ is invariant under σ_2 (and σ_2^{-1}) and that $\sigma_1(X_i) = X_j$ whenever $\sigma_2(Y_i) = Y_j$ for all $i, j \in J$. Suppose $\tilde{\sigma_1} : \mathcal{A}_X \to \mathcal{A}_X$ is the automorphism on $\mathcal{A}_X$ induced by σ_1, and $\tilde{\sigma_2} : \mathcal{A}_Y \to \mathcal{A}_Y$ is the automorphism on $\mathcal{A}_Y$ induced by σ_2. Then*

$$\tilde{\sigma_2} \circ \mu = \mu \circ \tilde{\sigma_1}. \tag{5}$$

where μ is given by (4). Moreover, for every $n \in \mathbb{Z}$,

$$\tilde{\sigma_2}^n \circ \mu = \mu \circ \tilde{\sigma_1}^n. \tag{6}$$

Proof Let $y \in X$ such that $y \in Y_i$ for some $i \in J$. Then for every $f \in \mathcal{A}$,

$$\begin{aligned}(\tilde{\sigma_2} \circ \mu)(f)(y) &= (\mu f) \circ \sigma_2^{-1}(y) \\ &= (\mu f)(\sigma_2^{-1}(y)) \\ &= f(\sigma_1^{-1}(x_i)) \\ &= (f \circ \sigma_1^{-1})(x_i) \\ &= \mu(f \circ \sigma_1^{-1})(y) \\ &= \mu[\tilde{\sigma_1}(f)](y) \\ &= [\mu \circ \tilde{\sigma_1}](f)(y).\end{aligned}$$

Since y is arbitrary, we have

$$(\tilde{\sigma_2} \circ)(f)\mu = \mu \circ \tilde{\sigma_1}(f)$$

for every $f \in \mathcal{A}$. And since f is arbitrary,

$$\tilde{\sigma_2} \circ \mu = \mu \circ \tilde{\sigma_1}.$$

Now from (5), we have

$$\tilde{\sigma_2}^2 \circ \mu = \tilde{\sigma_2} \circ (\tilde{\sigma_2} \circ \mu) = \tilde{\sigma_2} \circ (\mu \circ \tilde{\sigma_1}) = (\tilde{\sigma_2} \circ \mu) \circ \tilde{\sigma_1} = (\mu \circ \tilde{\sigma_1}) \circ \tilde{\sigma_1} = \mu \circ \tilde{\sigma_1}^2.$$

Therefore the relation (6) holds for $n = 2$.

Now suppose the relation (6) holds for k. Then:

$$\tilde{\sigma_2}^{k+1} \circ \mu = \tilde{\sigma_2} \circ (\tilde{\sigma_2}^k \circ \mu) = \tilde{\sigma_2} \circ (\mu \circ \tilde{\sigma_1}^k) = (\tilde{\sigma_2} \circ \mu) \circ \tilde{\sigma_1}^k = (\mu \circ \tilde{\sigma_1}) \circ \tilde{\sigma_1}^k = \mu \circ \tilde{\sigma_1}^{k+1}.$$

Therefore, from the induction principle,

$$\tilde{\sigma_2}^n \circ \mu = \mu \circ \tilde{\sigma_1}^n.$$

□

Remark 1 From Theorem 2 above, we get two nice results. The first is that if $\mathbb{P}_1 = \mathbb{P}_2$ are partitions of X and $\sigma_1, \sigma_2 : X \to X$ are bijections on X which preserve the structure of the partition, they will give rise to the same automorphism. That is, suppose $\mathbb{P}_1 = \{X_j \ : \ j \in J\}$ is a partition of X and $\sigma_1, \sigma_2 : X \to X$ are bijections on X such that, if $\sigma_1(X_i) = X_j$, then $\sigma_2(X_i) = X_j$, for all $i, j \in J$. Let $\tilde{\sigma} : \mathcal{A} \to \mathcal{A}$ be the automorphism on induced by σ, that is, for every $h \in \mathcal{A}$,

$$\tilde{\sigma}(h) = h \circ \sigma^{-1}.$$

Then for every $f \in \mathcal{A}$,

$$\tilde{\sigma_1}(f) = \tilde{\sigma_2}(f).$$

This is given by the fact that if $\mathbb{P}_1 = \mathbb{P}_2$, then in (5), we can take $\mu = id$.

The second is the following important theorem.

Theorem 3 *Let $\mathbb{P}_1 = \{X_j : j \in J\}$ and $\mathbb{P}_2 = \{Y_j : j \in J\}$ be partitions of two sets X and Y and $\mathcal{A}_X$ and $\mathcal{A}_Y$ be algebras of functions that are constant on the sets of the partitions $\mathbb{P}_1$ and $\mathbb{P}_2$ respectively. Let $\sigma_1 : X \to X$ and $\sigma_2 : Y \to Y$ be bijections such that $\mathcal{A}_X$ is invariant under σ_1 (and σ_1^{-1}) and $\mathcal{A}_Y$ is invariant under σ_2 (and σ_2^{-1}) and that $\sigma_1(X_i) = X_j$ whenever $\sigma_2(Y_i) = Y_j$ for all $i, j \in J$. Suppose $\tilde{\sigma_1} : \mathcal{A}_X \to \mathcal{A}_X$ is the automorphism on $\mathcal{A}_X$ induced by σ_1, and $\tilde{\sigma_2} : \mathcal{A}_Y \to \mathcal{A}_Y$ is the automorphism on $\mathcal{A}_Y$ induced by σ_2. Then the crossed product algebras $\mathcal{A} \rtimes_{\tilde{\sigma_1}} \mathbb{Z}$ and $\mathcal{A} \rtimes_{\tilde{\sigma_2}} \mathbb{Z}$ are isomorphic.*

Proof We need to construct the an isomorphism between the crossed product algebras $\mathcal{A}_X \rtimes_{\tilde{\sigma_1}} \mathbb{Z}$ and $\mathcal{A}_Y \rtimes_{\tilde{\sigma_1}} \mathbb{Z}$. Using the notation in [4], we let $f := \sum_{n \in \mathbb{Z}} f_n \delta^n$ be an element in $\mathcal{A}_X \rtimes_{\tilde{\sigma_1}} \mathbb{Z}$. Define a function $\mu : \mathcal{A}_X \rtimes_{\tilde{\sigma_1}} \mathbb{Z} \to \mathcal{A}_Y \rtimes_{\tilde{\sigma_2}} \mathbb{Z}$ be defined by

$$\tilde{\mu}\left(\sum_{n \in \mathbb{Z}} f_n \delta_1^n\right) = \sum_{n \in \mathbb{Z}} \mu(f_n) \delta_2^n, \tag{7}$$

where μ is defined in (4). Then, since μ is an algebra isomorphism, it is enough to prove that $\tilde{\mu}$ is multiplicative. To this end, we let $f := \sum_{n \in \mathbb{Z}} f_n \delta_1^n$ and $g := \sum_{m \in \mathbb{Z}} g_m \delta_1^m$ be arbitrary elements in $\mathcal{A}_X \rtimes_{\tilde{\sigma_1}} \mathbb{Z}$, then we prove that $\tilde{\mu}$ is multiplicative on the generators $f_n \delta_1^n$ and $g_m \delta_1^m$ respectively. Using (1) we have

$$\begin{aligned}\tilde{\mu}((f_n\delta_1^n) * (g_m\delta_1^m)) &= \tilde{\mu}(f_n\tilde{\sigma_1}^n(g_m)\delta_1^{n+m}) \\ &= \mu(f_n\tilde{\sigma_1}^n(g_m))\delta_2^{n+m} \\ &= [\mu(f_n)\mu(\tilde{\sigma_1}^n(f_m))]\delta_2^{n+m} \\ &= \mu(f_n)\tilde{\sigma_2}^n(\mu(f_m))\delta_2^{n+m} \quad \text{by (6)} \\ &= \tilde{\mu}(f_n\delta_2^n) * \tilde{\mu}(f_m\delta_2^m).\end{aligned}$$

Therefore $\tilde{\mu}$ is multiplicative on the generators $f_n\delta^n$ and since μ is linear, it is multiplicative on the elements $f = \sum_{n\in\mathbb{Z}} f_n\delta^n \in \mathcal{A}_X \rtimes_{\tilde{\sigma}} \mathbb{Z}$. □

Remark 2 In Lemma 1 we proved the necessary and sufficient condition on a bijection $\sigma : X \to X$ such that the algebra $\mathcal{A}_X$ is invariant under σ, that is, for every $i \in J$ there exists $j \in J$ such that $\sigma(X_i) = X_j$ where the X_i form a partition for X. From this, it can be shown that $\mathcal{A}$ is isomorphic to $\mathbb{C}^J$, where by $\mathbb{C}^J$ we denote the space of complex sequences indexed by J. This can be done by constructing an isomorphism between $\mathcal{A}_X$ and $\mathbb{C}^J$ via σ as follows.

Let $\tau : J \to J$ be a map such that $\tau(i) = j$ is equivalent to $\sigma(X_i) = X_j$ for all $i, j \in J$. Then τ is a bijection that plays the same role as σ_2 in Lemma 2. Therefore, using the same Lemma, we deduce that the algebra $\mathcal{A}$ is isomorphic to $\mathbb{C}^J$. In Theorem 3, we have shown a method of constructing an isomorphism between the crossed product algebras $\mathcal{A}_X \rtimes_{\tilde{\sigma_1}} \mathbb{Z}$ and $\mathcal{A}_Y \rtimes_{\tilde{\sigma_2}} \mathbb{Z}$, when $\mathcal{A}_X$ and $\mathcal{A}_Y$ are isomorphic. It follows that the crossed product algebra $\mathcal{A}_X \rtimes_{\tilde{\sigma_1}} \mathbb{Z}$ is isomorphic to $\mathbb{C}^J \rtimes_{\tilde{\tau}} \mathbb{Z}$, where $\tilde{\tau}$ follows the same definition as $\tilde{\sigma}$.

In the next section we describe the commutant of our algebra $\mathcal{A}_X$ in the crossed product algebra $\mathcal{A}_X \rtimes_{\tilde{\sigma}} \mathbb{Z}$.

3.1 Maximal Commutative Subalgebra

We take the same partition $\mathbb{P} = \cup_{j\in J} X_j$ and a bijection $\sigma : X \to X$ such that for all $i \in J$, there exists $j \in J$ such that $\sigma(X_i) = X_j$. For $k \in \mathbb{Z}_{>0}$, let

$$C_k := \{x \in X \mid k \text{ is the smallest positive integer such that } x, \sigma^k(x) \in X_j \text{ for some } j \in J\}. \tag{8}$$

According to Theorem 1, the unique maximal commutative subalgebra of $\mathcal{A} \rtimes_{\tilde{\sigma}} \mathbb{Z}$ that contains $\mathcal{A}$ is precisely the set of elements

$$A' = \left\{ \sum_{n\in\mathbb{Z}} f_n\delta^n \mid \text{for all } n \in \mathbb{Z} : \ f_n|_{Sep^n_{\mathcal{A}}(X)} \equiv 0 \right\}, \tag{9}$$

where $Sep^n_{\mathcal{A}}(X)$ is given by (2). We have the following theorem which gives the description of $Sep^n_{\mathcal{A}}(X)$ in this case and is crucial in the description if the maximal commutative subalgebra.

Theorem 4 *Let $\sigma : X \to X$ be a bijection on X as given above, $\tilde{\sigma} : \mathcal{A}_X \to \mathcal{A}_X$ be the automorphism on $\mathcal{A}_X$ induced by σ and C_k be given by (8). Then for every $n \in \mathbb{Z}$,*

$$Sep^n_{\mathcal{A}}(X) = \left\{ \bigcup_{k \nmid n} C_k \cup C_\infty \right\}, \tag{10}$$

where

$$C_\infty = \{X_j \in \mathbb{P} \ : \ \sigma^k(X_j) \neq X_j \ \forall k \ \geq 1\}.$$

Proof 1. If $n \equiv 0 \pmod k$ and $x \in X_j \in C_k$, the we can write $n = mk$ for some $m \in \mathbb{Z}$. Then, since $\sigma^k(X_j) = X_j$ it follows that $\sigma^{-k}(X_j) = X_j$ and therefore for every $h \in \mathcal{A}$,

$$\tilde{\sigma}^n(h)(x) = \tilde{\sigma}^{mk}(h)(x) = (h \circ \sigma^{-mk})(x) = h(\sigma^{-mk}(x)) = h(x),$$

since x and $\sigma^{-mk}(x) \in X_j$ for all $m \in \mathbb{Z}$.

2. If $n \not\equiv 0 \pmod k$, we can write $n = mk + j$ where $m, j \in \mathbb{Z}$ with $1 \leq j < k$. It follows that for every $x \in X_j \in C_k$,

$$\begin{aligned} \tilde{\sigma}^n(h)(x) &= \tilde{\sigma}^{mk+j}(h)(x) \\ &= (h \circ \sigma^{-mk+j})(x) \\ &= h(\sigma^{-mk+j}(x)) \\ &= \tilde{\sigma}^j(h)(x). \end{aligned}$$

But k is the smallest integer such that $\sigma^k(X_j) = X_j$. Therefore since $j < k$,

$$\tilde{\sigma}^j(h)(x) \neq h(x).$$

Hence

$$\begin{aligned} Sep^n_{\mathcal{A}}(X) &= \{x \in X \mid \exists\, h \in \mathcal{A} : h(x) \neq \tilde{\sigma}^n(h)(x)\} \\ &= \begin{cases} \left\{\cup_{j\,:\,X_j \notin C_k} X_j\right\} & \text{if } n \equiv 0 \pmod k, \\ \left\{\cup_{j\,:\,X_j \in C_k} X_j\right\} & \text{if } n \not\equiv 0 \pmod k, \end{cases} \end{aligned}$$

and if $x \in C_\infty$, then obviously $x \in Sep^n_{\mathcal{A}}$ for every $n \geq 1$, or simply

$$Sep^n_{\mathcal{A}}(\mathbb{R}) = \bigcup_{k \nmid n} C_k \cup C_\infty.$$

From the above theorem, the description of the maximal commutative subalgebra in $\mathcal{A} \rtimes_{\tilde{\sigma}} \mathbb{Z}$ can be done as follows.

Theorem 5 *Let $\mathcal{A}_X$ be the algebra of piece-wise constant functions $f : X \to \mathbb{C}$, $\sigma : X \to X$ any bijection on X, $\tilde{\sigma} : \mathcal{A}_X \to \mathcal{A}_X$ the automorphism on $\mathcal{A}_X$ induced by σ and C_k be as described above. Then the unique maximal commutative subalgebra of $\mathcal{A}_X \rtimes_{\tilde{\sigma}} \mathbb{Z}$ that contains $\mathcal{A}_X$ is given by*

$$\mathcal{A}' = \left\{ \sum_{n\in\mathbb{Z}\,:\,k|n} \left(\sum_{j_n\in J} a_{j_n} \chi_{X_{j_n}} \right) \delta^n \right\}.$$

Proof From (9) we have that the unique maximal commutative subalgebra of $\mathcal{A}_X \rtimes_{\tilde{\sigma}} \mathbb{Z}$ that contains $\mathcal{A}_X$ is precisely the set of elements

$$\mathcal{A}' = \left\{ \sum_{n\in\mathbb{Z}} f_n \delta^n \mid \text{for all } n \in \mathbb{Z} :\ f_n|_{Sep^n_{\mathcal{A}}(X)} \equiv 0 \right\},$$

and from (2),

$$Sep^n_{\mathcal{A}}(\mathbb{R}) = \bigcup_{k\nmid n} C_k. \tag{11}$$

Combining the two results and using the definition of $h_n \in \mathcal{A}_X$ as

$$h_n = \sum_{j_n\in J} a_{j_n} \chi_{X_{j_n}},$$

we get

$$\mathcal{A}' = \left\{ \sum_{n\in\mathbb{Z}\,:\,k|n} \left(\sum_{j_n\in J} a_{j_n} \chi_{X_{j_n}} \right) \delta^n \right\}.$$

□

It can be observed from the results in Theorem 4 that it is possible to have $Sep^n_{\mathcal{A}}(X) = X$ for all $n \in \mathbb{Z}$. For example, suppose J is infinite and let $\sigma : X \to X$ be a bijection such that $\sigma(X_j) = X_{j+1}$ for every $j \in J$. Then it is easily seen that in in this case $Sep^n_{\mathcal{A}}(X) = X$. However, this is not possible if J is finite since in this case σ acts like a permutation on a finite group. In the following section, we treat one such a case. We let $X = \mathbb{R}$ and $\mathcal{A}_X$ be the algebra of piece-wise constant functions on $\mathbb{R}$ with N fixed jump points, where $N \geq 1$ is an integer. In order to work in the setting described before, we treat jump points as intervals of zero length. Then $\mathbb{R}$ is partitioned into $2N + 1$ sub-intervals.

4 Algebra of Piece-Wise Constant Functions on the Real Line with N Fixed Jump Points

Let $\mathcal{A}$ be the algebra of piece-wise constant functions $f : \mathbb{R} \to \mathbb{R}$ with N fixed jumps at points $t_1, t_2, \ldots, t_N$. Partition $\mathbb{R}$ into $N + 1$ intervals $I_0, I_1, \ldots, I_N$ where $I_\alpha =]t_\alpha, t_{\alpha+1}[$ with $t_0 = -\infty$ and $t_{N+1} = \infty$. By looking at jump points as intervals of zero length, we can write $\mathbb{R} = \cup I_\alpha$ where I_α is as described above for $\alpha = 0, 1, \ldots N$ and $I_\alpha = \{t_\alpha\}$ if $\alpha > N$. Then for every $h \in \mathcal{A}$ we have

$$h(x) = \sum_{\alpha=0}^{2N} a_\alpha \chi_{I_\alpha}(x), \tag{12}$$

where χ_{I_α} is the characteristic function of I_α. As in the preceding section, we let $\sigma : \mathbb{R} \to \mathbb{R}$ be any bijection on $\mathbb{R}$ and let $\tilde{\sigma} : \mathcal{A} \to \mathcal{A}$ be the automorphism on $\mathcal{A}$ induced by σ. Then we have the following lemma which gives the necessary and sufficient conditions for $(\mathbb{R}, \sigma)$ to be a discrete dynamical system.

Lemma 3 *The algebra $\mathcal{A}$ is invariant under both σ and σ^{-1} if and only if the following conditions hold.*

1. *σ (and σ^{-1}) maps the each jump point t_k, $k = 1, \ldots, N$ onto another jump point.*
2. *σ maps every interval I_α, $\alpha = 0, 1, \ldots N$ bijectively onto any of the other intervals $I_0, I_1 \ldots I_N$.*

Proof Obviously, if the two conditions hold, then $\mathcal{A}$ is invariant under σ. So we suppose that $\mathcal{A}$ is invariant under σ and prove that the two conditions must hold.

1. Suppose $\sigma(t_k) = t_0 \notin \{t_1, t_2, \ldots, t_N\}$ for some $k \in \{1, 2, \ldots, N\}$. Then, since σ is onto, there exists $x_0 \in \mathbb{R}$ such that $\sigma(x_0) = t_k$, that is, there exists a non jump point that is mapped onto a jump point. We show that this is not possible.
   Let
   $$h(x) = \begin{cases} 1 & \text{if } \ x = t_k, \\ 0 & \text{otherwise.} \end{cases}$$
   Then $h \in \mathcal{A}$. But
   $$h \circ \sigma(x) = \begin{cases} 1 & \text{if } \ \sigma(x) = t_k, \\ 0 & \text{otherwise,} \end{cases} = \begin{cases} 1 & \text{if } \ x = x_0, \\ 0 & \text{otherwise.} \end{cases}$$
   Therefore $h \circ \sigma \notin \mathcal{A}$ which is a contradiction, implying that σ does not map a non jump point onto a jump point, proving the first condition.

2. Consider the bijection $\sigma : \mathbb{R} \to \mathbb{R}$ defined by

$$\sigma(x) = \begin{cases} x & \text{if } \; x \neq t_k^{'} \text{ or } t_k^{''}, \\ t_k^{''} & \text{if } x = t_k^{'}, \\ t_k^{'} & \text{if } x = t_k^{''}, \end{cases} \tag{13}$$

where $t_k^{'} \in I_k$ and $t_k^{''} \in I_{k+1}$ for some $k \in \{1, 2, \ldots, N\}$. Then σ is a bijection that permutes the jump points. Let $h \in \mathcal{A}$. Then using (12) and for the σ in Eq. (13) above, we have:

$$\sigma(x) = \begin{cases} h(x) & \text{if } \; x \neq t_k^{'} \text{ or } t_k^{''}, \\ a_{k+1} & \text{if } x = t_k^{'}, \\ a_k & \text{if } x = t_k^{''}. \end{cases}$$

Therefore, $h \circ \sigma$ has jumps at points $t_1, \ldots, t_N, t_k^{'}, t_k^{''}$ implying that $h \circ \sigma \notin \mathcal{A}$. □

The following theorem gives the description of $Sep^n_{\mathcal{A}}(\mathbb{R})$ for any $n \in \mathbb{Z}$.

Theorem 6 *Let $\mathcal{A}$ be an algebra of piece-wise constant functions with N fixed jumps at points $t_1, \ldots, t_N$, $\sigma : \mathbb{R} \to \mathbb{R}$ be any bijection on $\mathbb{R}$ such that $\mathcal{A}$ is invariant under σ and let $\tilde{\sigma} : \mathcal{A} \to \mathcal{A}$ be the automorphism on $\mathcal{A}$ induced by σ. Let*

$$C_k := \big\{x \in \mathbb{R} \mid k \text{ is the smallest positive integer such that } x, \sigma^k(x) \in I_\alpha \\ \text{for some } \alpha = 0, \ldots 2N\big\}. \tag{14}$$

Then for every $n \in \mathbb{Z}$,

$$Sep^n_{\mathcal{A}}(\mathbb{R}) = \bigcup_{k \nmid n} C_k. \tag{15}$$

Proof See Theorem 4 and observe that $C_\infty = \emptyset$ in this case. □

Example 1 Let $\mathcal{A}$ be the algebra of piece-wise constant functions with 4−fixed jump points at t_1, t_2, t_3, t_4. Partition $\mathbb{R}$ into five subintervals $I_0, \ldots, I_4$ where $I_\alpha =]t_\alpha, t_{\alpha+1}[$ with $t_0 = -\infty$ and $t_5 = \infty$.

Let $\sigma : \mathbb{R} \to \mathbb{R}$ be a bijection such that $\sigma(I_0) = I_1,\ \sigma(I_1) = I_2,\ \sigma(I_2) = I_0$, $\sigma(I_3) = I_4$ and $\sigma(I_4) = I_3$. It follows that $\sigma^3(I_0) = I_0,\ \sigma^3(I_1) = I_1$ and $\sigma^3(I_2) = I_2$. But $\sigma^j(I_\alpha) \neq I_\alpha$ for $\alpha = 0, 1, 2$ and $1 \leq j < 3$.

Also $\sigma^2(I_3) = I_3,\ \sigma^2(I_4) = I_4$ but $\sigma^j(I_\alpha) \neq I_\alpha$ if $j \not\equiv 0 \pmod 2$ with $\alpha = 3, 4$. Therefore:

$$\begin{aligned} Sep^n_{\mathcal{A}}(\mathbb{R}) &= \{x \in \mathbb{R} \mid \exists\, h \in \mathcal{A} : h(x) \neq \tilde{\sigma}^n(h)(x)\} \\ &= \mathbb{R} \setminus \{\{I_3 \cup I_4\} \cup \{t_k :\ \sigma^2(t_k) = t_k,\ \ k = 1, 2, 3, 4\}\} \text{ if } n \equiv 0 \pmod 2 \\ &= \{I_0 \cup I_1 \cup I_2\} \cup \{t_k :\ \sigma^2(t_k) \neq t_k,\ \ k = 1, 2, 3, 4\} \text{ if } n \equiv 0 \pmod 2, \end{aligned}$$

and

$$
\begin{aligned}
Sep^n_{\mathcal{A}}(\mathbb{R}) &= \{x \in \mathbb{R} \mid \exists\, h \in \mathcal{A} : h(x) \neq \tilde{\sigma}^n(h)(x)\} \\
= \mathbb{R} \setminus \{I_0 \cup \{I_1 \cup I_2\} &\cup \{t_k :\ \sigma^3(t_k) = t_k,\ \ k = 1, 2, 3, 4\}\} \text{ if } n \equiv 0 \pmod 3 \\
= \{I_3 \cup I_4\} &\cup \{t_k :\ \sigma^3(t_k) \neq t_k,\ \ k = 1, 2, 3, 4\} \text{ if } n \equiv 0 \pmod 3.
\end{aligned}
$$

From these results we have the following theorem.

Theorem 7 *Let $\mathcal{A}$ be the algebra of piece-wise constant functions $f : \mathbb{R} \to \mathbb{R}$ with N fixed jumps at points $t_1, t_2, \ldots, t_N$. Partition $\mathbb{R}$ into $N + 1$ intervals $I_0, I_1, \ldots, I_N$ where $I_\alpha =]t_\alpha, t_{\alpha+1}[$ with $t_0 = -\infty$ and $t_{N+1} = \infty$ and $I_M = \{t_\alpha\}$ for $N + 1 \leqslant M \leqslant 2N$. Let $\sigma : \mathbb{R} \to \mathbb{R}$ be any bijection on $\mathbb{R}$ such that $\mathcal{A}$ is invariant under σ and let $\tilde{\sigma} : \mathcal{A} \to \mathcal{A}$ be the automorphism on $\mathcal{A}$ induced by σ. Let*

$$
\begin{aligned}
C_k := \big\{x \in \mathbb{R} \mid k \text{ is the smallest positive integer such that } x, \sigma^k(x) \in I_\alpha \\
\text{for some } \alpha = 0, \ldots 2N\big\}.
\end{aligned} \tag{16}
$$

Then the unique maximal commutative subalgebra of $\mathcal{A} \rtimes_{\tilde{\sigma}} \mathbb{Z}$ that contains $\mathcal{A}$ is given by

$$
\mathcal{A}' = \left\{ \sum_{n \in \mathbb{Z}\,:\, k|n} \left(\sum_{\alpha_n = 0}^{2N} a_{\alpha_n} \chi_{I_{\alpha_n}} \right) \delta^n \right\}.
$$

Proof From (9) we have that the unique maximal commutative subalgebra of $\mathcal{A} \rtimes_{\tilde{\sigma}} \mathbb{Z}$ that contains $\mathcal{A}$ is precisely the set of elements

$$
\mathcal{A}' = \left\{ \sum_{n \in \mathbb{Z}} f_n \delta^n \mid \text{for all } n \in \mathbb{Z} :\ f_n|_{Sep^n_{\mathcal{A}}(X)} \equiv 0 \right\},
$$

and from (11),

$$
Sep^n_{\mathcal{A}}(\mathbb{R}) = \bigcup_{k \nmid n} C_k.
$$

Combining the two results and using the definition of $h_n \in \mathcal{A}$ as

$$
h_n = \sum_{\alpha_n = 0}^{2N} a_{\alpha_n} \chi_{I_{\alpha_n}},
$$

we get

$$
\mathcal{A}' = \left\{ \sum_{n \in \mathbb{Z}\,:\, k|n} \left(\sum_{\alpha_n = 0}^{2N} a_{\alpha_n} \chi_{I_{\alpha_n}} \right) \delta^n \right\}.
$$

□

5 Some Examples

In this section we give some examples of how our results hold for well known simple cases. We treat two cases of piece-wise constant functions on the real line; those with one fixed jump point and those with two fixed jump points.

5.1 *Piece-Wise Constant Functions with One Jump Point*

Let $\mathcal{A}$ be the collection of all piece-wise constant functions on the real line with one fixed jump point t_0. Following the methods in the previous section $\mathbb{R}$ is partitioned into three intervals $I_0 = (-\infty, t_0), \quad I_1 = (t_0, \infty)$ and $I_2 = \{t_0\}$. Then we can write $h \in \mathcal{A}$ as

$$h = \sum_{\alpha=0}^{2} a_\alpha \chi_{I_\alpha} = a_0 \chi_{I_0} + a_1 \chi_{I_1} + a_2 \chi_{I_2}. \tag{17}$$

Let $\sigma : \mathbb{R} \to \mathbb{R}$ be any bijection on $\mathbb{R}$ and let $\tilde{\sigma}$ be the automorphism on $\mathcal{A}$ induced by σ. Note that by the first part of Lemma 3, invariance of the algebra $\mathcal{A}$ implies that $\sigma(t_0) = t_0$. It follows therefore that $\sigma(I_0) = I_0$ or $\sigma(I_0) = I_1$. We treat these two cases below.

5.1.1 $\sigma(I_0) = I_0$

In this case (and by bijectivity of σ), we have that $\sigma(I_1) = I_1$ and since $\sigma(t_0) = t_0$, then for every $x \in \mathbb{R}, \ h \in \mathcal{A}$ and $n \in \mathbb{Z}$

$$\tilde{\sigma}^n h(x) := h \circ \sigma^{-n}(x) = h(x),$$

since x and $\sigma^{-n}(x)$ will lie in the same interval. Therefore, all intervals $I_\alpha, \ \alpha = 0, 1, 2$ belong to C_1 and hence

$$Sep^n_{\mathcal{A}}(\mathbb{R}) = \bigcup_{k \nmid n} C_k = \emptyset.$$

Therefore, the maximal commutative subalgebra will be given by

$$\begin{aligned} \mathcal{A}' = &\left\{ \sum_{n\in\mathbb{Z}} f_n \delta^n \mid \text{for all } n \in \mathbb{Z} : \ f_n|_{Sep^n_{\mathcal{A}}(X)} \equiv 0 \right\} \\ &= \mathcal{A} \rtimes_{\tilde{\sigma}} \mathbb{Z}. \end{aligned}$$

5.1.2 $\sigma(I_0) = I_1$

In this case (and by bijectivity of σ), we have that $\sigma(I_1) = I_0$ and since $\sigma(t_0) = t_0$, then for every $x \in \mathbb{R},\ h \in \mathcal{A}$ and $n \in \mathbb{Z}$ such that $2 \mid n$ we have

$$\tilde{\sigma}^n h(x) := h \circ \sigma^{-n}(x) = h(x),$$

since x and $\sigma^{-n}(x)$ will lie in the same interval. And for odd n, $\tilde{\sigma}^n(h)(x) = h(x)$ if and only if $x = t_0$. Therefore, we have,

$$C_1 = \{I_\alpha \mid \sigma(I_\alpha) = I_\alpha\} = I_2,$$

and

$$C_2 = \{I_\alpha \mid \sigma^2(I_\alpha) = I_\alpha\} = I_0 \cup I_1.$$

Therefore,

$$Sep^n_{\mathcal{A}}(\mathbb{R}) = \bigcup_{k \nmid n} C_k = \begin{cases} C_2 & \text{if } k = 1, \\ \emptyset & \text{if } k = 2. \end{cases}$$

Therefore, the maximal commutative subalgebra will be given by

$$\begin{aligned} \mathcal{A}' &= \left\{ \sum_{n\in\mathbb{Z}\,:\,k|n} \left(\sum_{\alpha_n=0}^{2} a_{\alpha_n} \chi_{I_{\alpha_n}} \right) \delta^n \right\} \\ &= \left\{ \sum_{n\in\mathbb{Z}\,:\,2|n} \left(\sum_{\alpha=0}^{2} a_{\alpha_n} \chi_{I_\alpha} \right) \delta^n \right\} \\ &= \left\{ \sum_{m\in\mathbb{Z}} (a_{0,m}\chi_{I_0} + a_{1,m}\chi_{I_1} + a_{2,m}\chi_{I_2})\delta^{2m} + \sum_{m\in\mathbb{Z}} \left(a_{2,m}\chi_{I_2}\right) \delta^{2m+1} \right\}. \end{aligned}$$

5.2 *Piece-Wise Constant Functions with Two Jump Points*

Let $\mathcal{A}$ be the collection of all piece-wise constant functions on the real line with two fixed jump points at t_0 and t_1. Following the methods in the previous section $\mathbb{R}$ is partitioned into intervals $I_0 =]-\infty, t_0[,\quad I_1 =]t_0, t_1[\quad I_2 =]t_1, \infty[,\quad I_3 = \{t_0\}$ and $I_4 = \{t_1\}$. Then we can write $h \in \mathcal{A}$ as

$$h = \sum_{\alpha=0}^{4} a_\alpha \chi_{I_\alpha}. \tag{18}$$

Let $\sigma : \mathbb{R} \to \mathbb{R}$ be any bijection on $\mathbb{R}$ and let $\tilde{\sigma}$ be the automorphism on $\mathcal{A}$ induced by σ. Note that by the first part of Lemma 3, invariance of the algebra $\mathcal{A}$ implies that $\sigma(t_0) = t_0$ (and $\sigma(t_1) = t_1$) or $\sigma(t_0) = t_1$ (in which case $\sigma(t_1) = t_0$). Below we give a description for the maximal commutative subalgebra of $\mathcal{A} \rtimes_{\tilde{\sigma}} \mathbb{Z}$ for different types of σ.

5.2.1 $\sigma(I_\alpha) = I_\alpha$ for all $\alpha = 0, \ldots, 4$

This case is similar to the one in Sect. 5.1.1 in the sense that, for every $x \in \mathbb{R},\ h \in \mathcal{A}$ and $n \in \mathbb{Z}$

$$\tilde{\sigma}^n h(x) := h \circ \sigma^{-n}(x) = h(x),$$

since x and $\sigma^{-n}(x)$ will lie in the same interval. Therefore, all intervals $I_\alpha,\ \alpha = 0, \ldots, 4$ belong to C_1 and hence

$$Sep^n_{\mathcal{A}}(\mathbb{R}) = \bigcup_{k \nmid n} C_k = \emptyset.$$

Therefore, the maximal commutative subalgebra will be given by

$$\mathcal{A}' = \left\{ \sum_{n\in\mathbb{Z}} f_n \delta^n \mid \text{for all } n \in \mathbb{Z} :\ f_n|_{Sep^n_{\mathcal{A}}(X)} \equiv 0 \right\} = \mathcal{A} \rtimes_{\tilde{\sigma}} \mathbb{Z}.$$

5.2.2 $\sigma(I_0) = I_1,\ \sigma(I_1) = I_0$ and $\sigma(I_\alpha) = I_\alpha,\ \alpha = 2, 3, 4$

In this case (and by bijectivity of σ), we have that $\sigma(I_1) = I_0$ and therefore for every $x \in \mathbb{R},\ h \in \mathcal{A}$ and $n \in \mathbb{Z}$ such that $2 \mid n$ we have

$$\tilde{\sigma}^n h(x) := h \circ \sigma^{-n}(x) = h(x),$$

since x and $\sigma^{-n}(x)$ will lie in the same interval. And for odd n, $\tilde{\sigma}^n(h)(x) = h(x)$ if and only if $x \in I_2 \cup I_3 \cup I_4$. Therefore, we have,

$$C_1 = \{I_\alpha \mid \sigma(I_\alpha) = I_\alpha\} = I_2 \cup I_3 \cup I_4,$$

and

$$C_2 = \{I_\alpha \mid \sigma^2(I_\alpha) = I_\alpha\} = I_0 \cup I_1.$$

Therefore,

$$Sep_{\mathcal{A}}^{n}(\mathbb{R}) = \bigcup_{k \nmid n} C_k = \begin{cases} C_2 & \text{if } k = 1, \\ \emptyset & \text{if } k = 2. \end{cases}$$

It follows that for $n \in \mathbb{Z}$ such that $2 \mid n$, the maximal commutative subalgebra will be given by

$$\begin{aligned} \mathcal{A}_1' &= \left\{ \sum_{n \in \mathbb{Z}\,:\, k|n} \left(\sum_{\alpha_n=0}^{2N} a_{\alpha_n} \chi_{I_{\alpha_n}} \right) \delta^n \right\} \\ &= \left\{ \sum_{n \in \mathbb{Z}\,:\, 2|n} \left(\sum_{\alpha_n=0}^{2N} a_{\alpha_n} \chi_{I_{\alpha_n}} \right) \delta^n \right\} \\ &= \sum_{m \in \mathbb{Z}} \left(\sum_{\alpha=0}^{4} a_{\alpha,m} \chi_{I_\alpha} \right) \delta^{2m}. \end{aligned}$$

And for odd n, we have

$$\mathcal{A}_2' = \sum_n (a_{2,m}\chi_{I_2} + a_{3,m}\chi_{I_3} + a_{4,m}\chi_{I_4})\delta^n.$$

Therefore, the commutant $\mathcal{A}'$ is given by:

$$\mathcal{A} = \left\{ \sum_{m \in \mathbb{Z}} \left(\sum_{\alpha=0}^{4} a_{\alpha,m} \chi_{I_\alpha} \right) \delta^{2m} + \sum_{m \in \mathbb{Z}} (a_{2,m}\chi_{I_2} + a_{3,m}\chi_{I_3} + a_{4,m}\chi_{I_4})\delta^{2m+1} \right\}.$$

Similar results can be obtained for the following cases

1. $\sigma(I_0) = I_1, \quad \sigma(I_1) = I_0, \quad \sigma(I_3) = I_4, \quad \sigma(I_4) = I_3$ and $\sigma(I_2) = I_2$.
2. $\sigma(I_0) = I_2, \quad \sigma(I_2) = I_0$ and $\sigma(I_\alpha) = I_\alpha, \quad \alpha = 1, 3, 4$.
3. $\sigma(I_0) = I_2, \quad \sigma(I_2) = I_0, \quad \sigma(I_3) = I_4 \quad \sigma(I_4) = I_3$ and $\sigma(I_1) = I_1$.
4. $\sigma(I_1) = I_2, \quad \sigma(I_2) = I_1$ and $\sigma(I_\alpha) = I_\alpha \quad \alpha = 0, 3, 4$.
5. $\sigma(I_1) = I_2, \quad \sigma(I_2) = I_1, \quad \sigma(I_3) = I_4, \quad \sigma(I_4) = I_3$ and $\sigma(I_0) = I_0$.

Since in all these cases, $\sigma^2(I_\alpha) = I_\alpha, \ \alpha = 0, \ldots, 4$.

5.2.3 $\sigma(I_0) = I_1, \ \sigma(I_1) = I_2, \ \sigma(I_2) = I_0$ and $\sigma(I_\alpha) = I_\alpha, \ \alpha = 3, 4$

In this case, using similar methods we have,

$$C_1 = \{I_\alpha \mid \sigma(I_\alpha) = I_\alpha\} = \cup I_3 \cup I_4, \quad C_2 = \emptyset,$$

and

$$C_3 = \{I_\alpha \mid \sigma^3(I_\alpha) = I_\alpha\} = I_0 \cup I_1 \cup I_2.$$

Therefore,

$$Sep^n_{\mathcal{A}}(\mathbb{R}) = \bigcup_{k \nmid n} C_k = \begin{cases} C_3 & \text{if } k \neq 3, \\ \emptyset & \text{if } k = 3. \end{cases}$$

It follows that for $n \in \mathbb{Z}$ such that $3 \mid n$, the maximal commutative subalgebra will be given by

$$\begin{aligned} \mathcal{A}'_1 &= \left\{ \sum_{n \in \mathbb{Z}\,:\,k|n} \left(\sum_{\alpha_n=0}^{2N} a_{\alpha_n} \chi_{I_{\alpha_n}} \right) \delta^n \right\} \\ &= \left\{ \sum_{n \in \mathbb{Z}\,:\,3|n} \left(\sum_{\alpha=0}^{4} a_{\alpha_n} \chi_{I_\alpha} \right) \delta^n \right\} \\ &= \sum_{m \in \mathbb{Z}} \left(\sum_{\alpha=0}^{4} a_{\alpha,m} \chi_{I_\alpha} \right) \delta^{3m}. \end{aligned}$$

If $3 \nmid n$, then

$$\mathcal{A}'_2 = \sum_n (a_{3,n}\chi_{I_3} + a_{4,n}\chi_{I_4})\delta^n.$$

Therefore:

$$\mathcal{A}' = \left\{ \sum_{m \in \mathbb{Z}} \left(\sum_{\alpha=0}^{4} a_{\alpha,m} \chi_{I_\alpha} \right) \delta^{3m} + \sum_n (a_{3,n}\chi_{I_3} + a_{4,n}\chi_{I_4})\delta^n \right\}.$$

5.2.4 $\sigma(I_0) = I_1,\ \sigma(I_1) = I_2,\ \sigma(I_2) = I_0$ and $\sigma(I_3) = I_4,\ \sigma(I_4) = I_3$

In this case, using similar methods we have,

$$C_1 = \emptyset, \quad C_2 = I_3 \cup I_4,$$

and

$$C_3 = \{I_\alpha \mid \sigma^3(I_\alpha) = I_\alpha\} = I_0 \cup I_1 \cup I_2.$$

Therefore,

$$Sep^n_{\mathcal{A}}(\mathbb{R}) = \bigcup_{k \nmid n} C_k = \begin{cases} \mathbb{R} \setminus C_3 & \text{if } k = 3, \\ \mathbb{R} \setminus C_2 & \text{if } k = 2, \\ \mathbb{R} & \text{if } k = 1. \end{cases}$$

It follows that for $n \in \mathbb{Z}$ such that $3 \mid n$, the maximal commutative subalgebra will be given by

$$\begin{aligned}\mathcal{A}'_1 &= \left\{ \sum_{n\in\mathbb{Z}\,:\,k|n} \left(\sum_{\alpha_n=0}^{2N} a_{\alpha_n} \chi_{I_{\alpha_n}} \right) \delta^n \right\} \\ &= \left\{ \sum_{n\in\mathbb{Z}\,:\,3|n} \left(\sum_{\alpha_n=0}^{2N} a_{\alpha_n} \chi_{I_{\alpha_n}} \right) \delta^n \right\} \\ &= \sum_{m\in\mathbb{Z}} \left(a_{0,m}\chi_{I_0} + a_{1,m}\chi_{I_1} + a_{2,m}\chi_{I_2} \right) \delta^{3m}.\end{aligned}$$

If $2 \mid n$, then

$$\mathcal{A}'_2 = \sum_{m\in\mathbb{Z}} (a_{3,m}\chi_{I_3} + a_{4,m}\chi_{I_4})\delta^{2m},$$

and for all other values of n, $\mathcal{A}' = \mathcal{A}$. Hence:

$$\mathcal{A}' = \left\{ \sum_{m\in\mathbb{Z}} \left(a_{0,m}\chi_{I_0} + a_{1,m}\chi_{I_1} + a_{2,m}\chi_{I_2} \right) \delta^{3m} + \sum_{m\in\mathbb{Z}} (a_{3,m}\chi_{I_3} + a_{4,m}\chi_{I_4})\delta^{2m} \right\}.$$

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