Common Fixed Point Results for Family of Generalized Multivalued F-Contraction Mappings in Ordered Metric Spaces

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Abstract In this paper, we study the existence of common fixed points of family of multivalued mappings satisfying generalized F-contractive conditions in ordered metric spaces. These results establish some of the general common fixed point theorems for family of multivalued maps.

Keywords Common fixed point · Multivalued mapping · F-contraction · Ordered metric space

1 Introduction and Preliminaries

Markin [16] initiated the study of fixed points for multivalued nonexpansive and contractive maps. Later, a useful and interesting fixed point theory for such maps was developed. Later, a rich and interesting fixed point theory for such multivalued maps was developed; see, for instance [6, 7, 9–11, 14, 15, 18–20, 23]. The theory of multivalued maps has various applications in convex optimization, dynamical systems, commutative algebra, differential equations and economics. Recently, Wardowski [25] introduced a new contraction called F-contraction and proved a fixed point result as a generalization of the Banach contraction principle. Abbas et al. [3] obtained common fixed point results by employed the F-contraction condition. Further in this direction, Abbas et al. [4] introduced a notion of generalized F-contraction mapping and employed there results to obtain a fixed point of a generalized nonexpansive mappings on star shaped subsets of normed linear spaces. Minak et al. [17] proved some fixed point results for Ciric type generalized F-contractions

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on complete metric spaces. Recently, [5] established some fixed point results for multivalued F-contraction maps on complete metrics spaces.

The aim of this paper is to prove common fixed points theorems for family of multivalued generalized F-contraction mappings without using any commutativity condition in partially ordered metric space. These results extend and unify various comparable results in the literature [12, 13, 21, 22].

We begin with some basic known definitions and results which will be used in the sequel. Throughout this article, $\mathbb{N}$, $\mathbb{R}^+$, $\mathbb{R}$ denote the set of natural numbers, the set of positive real numbers and the set of real numbers, respectively.

Let $\mathcal{F}$ be the collection of all mappings $F : \mathbb{R}^+ \to \mathbb{R}$ that satisfy the following conditions:

(F_1) F is strictly increasing, that is, for all $a, b \in \mathbb{R}^+$ such that $a < b$ implies that $F(a) < F(b)$.
(F_2) For every sequence $\{a_n\}$ of positive real numbers, $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} F(a_n) = -\infty$ are equivalent.
(F_3) There exists $\lambda \in (0, 1)$ such that $\lim_{a\to 0^+} a^\lambda F(\lambda) = 0$.

Definition 20.1 ([25]) Let (X, d) be a metric space and $F \in \mathcal{F}$. A mapping $f : X \to X$ is said to be an F-contraction on X if there exists $\tau > 0$ such that $d(fx, fy) > 0$ implies that

$$\tau + F(d(fx, fy)) \leq F(d(x, y))$$

for all $x, y \in X$.

Wardowski [25] gave the following result.

Theorem 20.1 *Let (X, d) be a complete metric space and mapping $f : X \to X$ be and $F-$contraction. Then there exists a unique x in X such that $x = fx$. Moreover, for any $x_0 \in X$, the iterative sequence $x_{n+1} = f(x_n)$ converges to x.*

Kannan [12] has proved a fixed point theorem for a single valued self mapping T of a metric space X satisfying the property

$$d(Tx, Ty) \leq h\{d(x, Tx) + d(y, Ty)\}$$

for all x, y in X and for a fixed where $h \in [0, \frac{1}{2})$.

Ciric [8] considered a mapping $T : X \to X$ satisfying the following contractive condition:

$$d(Tx, Ty) \leq q \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\},$$

where $q \in [0, 1)$. He proved the existence of a fixed point when X is a T-orbitally complete metric space.

Latif and Beg [13] extended mappings considered by Kannan to multivalued mappings and introduced the notion of a K-multivalued mapping. Rus [21] coined the term R-multivalued mapping, which is a generalization of a K-multivalued mapping (see also, [2]). Abbas and Rhoades [1] studied common fixed point problems for multivalued mappings and introduced the notion of generalized R-multivalued mappings which in turn generalizes R-multivalued mappings.

Let (X, d) be a metric space. Denote by $P(X)$ be the family of all nonempty subsets of X, and by $P_{cl}(X)$ the family of all nonempty closed subsets of X.

A point x in X is called fixed point of a multivalued mapping $T : X \to P_{cl}(X)$ provided $x \in Tx$. The collection of all fixed point of T is denoted by $Fix(T)$.

Recall that, a map $T : X \to P_{cl}(X)$ is said to be upper semicontinuous, if for $x_n \in X$ and $y_n \in Tx_n$ with $x_n \to x_0$ and $y_n \to y_0$, implies $y_0 \in Tx_0$ (see [24]).

Definition 20.2 Let X be a nonempty set. Then $(X, d, \preceq)$ is called partially ordered metric space if and only if d is a metric on a partially ordered set $(X, \preceq)$.

We define $\Delta_1, \Delta_2 \subseteq X \times X$ as follows:

$$\Delta_1 = \{(x, y) \in X \times X \mid x \preceq y\},$$
$$\Delta_2 = \{(x, y) \in X \times X \mid x \prec y\}.$$

Definition 20.3 A subset Γ of a partially ordered set X is said to be well-ordered if every two elements of Γ are comparable.

2 Common Fixed Point Theorems

In this section, we obtain common fixed point theorems for family of multivalued mappings. We begin with the following result.

Theorem 20.2 *Let $(X, d, \preceq)$ be an ordered complete metric space and $\{T_i\}_{i=1}^m : X \to P_{cl}(X)$ be family of multivalued mappings. Suppose that for every $(x, y) \in \Delta_1$ and $u_x \in T_i(x)$, there exists $u_y \in T_{i+1}(y)$ for $i \in \{1, 2, \ldots, m\}$ (with $T_{m+1} = T_1$ by convention) such that, $(u_x, u_y) \in \Delta_2$ implies*

$$\tau + F\left(d(u_x, u_y)\right) \leq F(M(x, y; u_x, u_y)), \tag{1}$$

where τ is a positive real number and

$$M(x, y; u_x, u_y) = \max\left\{d(x, y), d(x, u_x), d(y, u_y), \frac{d\left(x, u_y\right) + d\left(y, u_x\right)}{2}\right\}.$$

Then the following statements hold:

(i) $Fix(T_i) \neq \emptyset$ for any $i \in \{1, 2, \ldots, m\}$ if and only if $Fix(T_1) = Fix(T_2) = \cdots = Fix(T_m) \neq \emptyset$.

(ii) $Fix(T_1) = Fix(T_2) = \cdots = Fix(T_m) \neq \emptyset$ *provided that any one* T_i *for* $i \in \{1, 2, \ldots, m\}$ *is upper semicontinuous.*

(iii) $\cap_{i=1}^{m} Fix(T_i)$ *is well-ordered if and only if* $\cap_{i=1}^{m} Fix(T_i)$ *is singleton set.*

Proof To prove (i), let $x^* \in T_k(x^*)$ for any $k \in \{1, 2, \ldots, m\}$. Assume that $x^* \notin T_{k+1}(x^*)$, then there exists an $x \in T_{k+1}(x^*)$ with $(x^*, x) \in \Delta_2$ such that

$$\tau + F\left(d(x^*, x)\right) \leq F(M(x^*, x^*; x^*, x)),$$

where

$$\begin{aligned} M(x^*, x^*; x^*, x) &= \max\left\{d(x^*, x^*), d(x^*, x^*), d(x, x^*), \frac{d(x^*, x) + d(x^*, x^*)}{2}\right\} \\ &= d(x, x^*), \end{aligned}$$

implies that

$$\tau + F\left(d(x^*, x)\right) \leq F(d(x^*, x)),$$

a contradiction as $\tau > 0$. Thus $x^* = x$. Thus $x^* \in T_{k+1}(x^*)$ and so $Fix(T_k) \subseteq Fix(T_{k+1})$. Similarly, we obtain that $Fix(T_{k+1}) \subseteq Fix(T_{k+2})$ and continuing this way, we get $Fix(T_1) = Fix(T_2) = \cdots = Fix(T_k)$. The converse is straightforward.

To prove (ii), suppose that x_0 is an arbitrary point of X. If $x_0 \in T_{k_0}(x_0)$ for any $k_0 \in \{1, 2, \ldots, m\}$, then by using (i), the proof is finished. So we assume that $x_0 \notin T_{k_0}(x_0)$ for any $k_0 \in \{1, 2, \ldots, m\}$. Now for $i \in \{1, 2, \ldots, m\}$, if $x_1 \in T_i(x_0)$, then there exists $x_2 \in T_{i+1}(x_1)$ with $(x_1, x_2) \in \Delta_2$ such that

$$\tau + F\left(d(x_1, x_2)\right) \leq F(M(x_0, x_1; x_1, x_2)),$$

where

$$\begin{aligned} M(x_0, x_1; x_1, x_2) &= \max\left\{d(x_0, x_1), d(x_0, x_1), d(x_1, x_2), \frac{d(x_0, x_2) + d(x_1, x_1)}{2}\right\} \\ &= \max\left\{d(x_0, x_1), d(x_1, x_2), \frac{d(x_0, x_2)}{2}\right\} \\ &= \max\{d(x_0, x_1), d(x_1, x_2)\}. \end{aligned}$$

Now, if $M(x_0, x_1; x_1, x_2) = d(x_1, x_2)$ then

$$\tau + F\left(d(x_1, x_2)\right) \leq F(d(x_1, x_2)),$$

a contradiction as $\tau > 0$. Therefore $M(x_0, x_1; x_1, x_2) = d(x_0, x_1)$ and we have

$$\tau + F\left(d(x_1, x_2)\right) \leq F\left(d(x_0, x_1)\right).$$

Next for this $x_2 \in T_{i+1}(x_1)$, there exists $x_3 \in T_{i+2}(x_2)$ with $(x_2, x_3) \in \Delta_2$ such that

$$\tau + F\left(d(x_2, x_3)\right) \leq F(M(x_1, x_2; x_2, x_3)),$$

where

$$\begin{aligned} M(x_1, x_2; x_2, x_3) &= \max\left\{d(x_1, x_2), d(x_1, x_2), d(x_2, x_3), \frac{d(x_1, x_3) + d(x_2, x_2)}{2}\right\} \\ &= \max\{d(x_1, x_2), d(x_2, x_3)\}. \end{aligned}$$

Now, if $M(x_1, x_2; x_2, x_3) = d(x_2, x_3)$ then

$$\tau + F\left(d(x_2, x_3)\right) \leq F(d(x_2, x_3)),$$

a contradiction as $\tau > 0$. Therefore $M(x_1, x_2; x_2, x_3) = d(x_1, x_2)$ and we have

$$\tau + F\left(d(x_2, x_3)\right) \leq F\left(d(x_1, x_2)\right).$$

Continuing this process, for $x_{2n} \in T_i(x_{2n-1})$, there exist $x_{2n+1} \in T_{i+1}\left(x_{2n}\right)$ with $(x_{2n}, x_{2n+1}) \in \Delta_2$ such that

$$\tau + F\left(d(x_{2n}, x_{2n+1})\right) \leq F\left(M(x_{2n-1}, x_{2n}; x_{2n}, x_{2n+1})\right),$$

where

$$\begin{aligned} M(x_{2n-1}, x_{2n}; x_{2n}, x_{2n+1}) &= \max\Bigg\{d(x_{2n-1}, x_{2n}), d(x_{2n-1}, x_{2n}), d(x_{2n}, x_{2n+1}), \\ &\qquad \frac{d(x_{2n-1}, x_{2n+1}) + d(x_{2n}, x_{2n})}{2}\Bigg\} \\ &= \left\{d(x_{2n-1}, x_{2n}), d(x_{2n}, x_{2n+1}), \frac{d(x_{2n-1}, x_{2n+1})}{2}\right\} \\ &\leq d(x_{2n-1}, x_{2n}), \end{aligned}$$

that is,

$$\tau + F\left(d(x_{2n}, x_{2n+1})\right) \leq F\left(d(x_{2n-1}, x_{2n})\right).$$

Similarly, for $x_{2n+1} \in T_{i+1}(x_{2n})$, there exist $x_{2n+2} \in T_{i+2}\left(x_{2n+1}\right)$ such that for $(x_{2n+1}, x_{2n+2}) \in \Delta_2$ implies

$$\tau + F\left(d(x_{2n+1}, x_{2n+2})\right) \leq F\left(d(x_{2n}, x_{2n+1})\right).$$

Hence, we obtain a sequence $\{x_n\}$ in X such that for $x_n \in T_i(x_{n-1})$, there exist $x_{n+1} \in T_{i+1}\left(x_n\right)$ with $(x_n, x_{n+1}) \in \Delta_2$ such that

$$\tau + F\left(d(x_n, x_{n+1})\right) \leq F\left(d(x_{n-1}, x_n)\right).$$

Therefore

$$F\left(d(x_n,x_{n+1})\right) \le F\left(d(x_{n-1},x_n)\right) - \tau \le F\left(d(x_{n-2},x_{n-1})\right) - 2\tau$$
$$\le \cdots \le F\left(d(x_0,x_1)\right) - n\tau. \quad (2)$$

From (2), we obtain $\lim_{n\to\infty} F\left(d(x_n,x_{n+1})\right) = -\infty$ that together with (F_2) gives

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0.$$

From (F_3), there exists $\lambda \in (0,1)$ such that

$$\lim_{n\to\infty} [d(x_n,x_{n+1})]^\lambda F\left(d(x_n,x_{n+1})\right) = 0.$$

From (2), we have

$$[d(x_n,x_{n+1})]^\lambda F\left(d(x_n,x_{n+1})\right) - [d(x_n,x_{n+1})]^\lambda F\left(d(x_0,x_{n+1})\right)$$
$$\le -n\tau [d(x_n,x_{n+1})]^\lambda \le 0.$$

On taking limit as $n \to \infty$ we obtain

$$\lim_{n\to\infty} n[d(x_n,x_{n+1})]^\lambda = 0.$$

Hence $\lim_{n\to\infty} n^{\frac{1}{\lambda}} d(x_n,x_{n+1}) = 0$ and there exists $n_1 \in \mathbb{N}$ such that $n^{\frac{1}{\lambda}} d(x_n,x_{n+1}) \le 1$ for all $n \ge n_1$. So we have

$$d(x_n,x_{n+1}) \le \frac{1}{n^{1/\lambda}}$$

for all $n \ge n_1$. Now consider $m, n \in \mathbb{N}$ such that $m > n \ge n_1$, we have

$$d\left(x_n,x_m\right) \le d\left(x_n,x_{n+1}\right) + d\left(x_{n+1},x_{n+2}\right) + \cdots + d\left(x_{m-1},x_m\right)$$
$$\le \sum_{i=n}^{\infty} \frac{1}{i^{1/\lambda}}.$$

By the convergence of the series $\sum_{i=1}^{\infty} \frac{1}{i^{1/\lambda}}$, we get $d\left(x_n,x_m\right) \to 0$ as $n, m \to \infty$. Therefore $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists an element $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$.

Now, if T_i is upper semicontinuous for any $i \in \{1,2,\ldots,m\}$, then as $x_{2n} \in X$, $x_{2n+1} \in T_i\left(x_{2n}\right)$ with $x_{2n} \to x^*$ and $x_{2n+1} \to x^*$ as $n \to \infty$ implies that $x^* \in T_i\left(x^*\right)$. Thus from (i), we get $x^* \in T_1\left(x^*\right) = T_2\left(x^*\right) = \cdots = T_m\left(x^*\right)$.

Finally to prove (iii), suppose the set $\cap_{i=1}^m Fix\left(T_i\right)$ is a well-ordered. We are to show that $\cap_{i=1}^m Fix\left(T_i\right)$ is singleton. Assume on contrary that there exist u and v such that $u, v \in \cap_{i=1}^{m} Fix\,(T_i)$ but $u \neq v$. As $(u, v) \in \Delta_2$, so for $(u_x, v_y) \in \Delta_2$ implies

$$\begin{aligned}\tau + F\,(d(u, v)) &\leq F(M(u, v; u, v)) \\ &= F\left(\max\left\{d(u, v), d(u, u), d(v, v), \frac{d\,(u, v) + d\,(v, u)}{2}\right\}\right) \\ &= F\,(d\,(u, v))\,,\end{aligned}$$

a contradiction as $\tau > 0$. Hence $u = v$. Conversely, if $\cap_{i=1}^{m} Fix\,(T_i)$ is singleton, then it follows that $\cap_{i=1}^{m} Fix\,(T_i)$ is a well-ordered. □

The following corollary extends and generalizes Theorem 4.1 of [13] and Theorem 3.4 of [21] for two maps in ordered metric spaces.

Corollary 20.1 *Let $(X, d, \preceq)$ be an ordered complete metric space and $T_1, T_2 : X \to P_{cl}(X)$ be two multivalued mappings. Suppose that for every $(x, y) \in \Delta_1$ and $u_x \in T_i(x)$, there exists $u_y \in T_j(y)$ for $i, j \in \{1, 2\}$ with $i \neq j$ such that, $(u_x, u_y) \in \Delta_2$ implies*

$$\tau + F\left(d(u_x, u_y)\right) \leq F(M(x, y; u_x, u_y)), \tag{3}$$

where τ is a positive real number and

$$M(x, y; u_x, u_y) = \max\left\{d(x, y), d(x, u_x), d(y, u_y), \frac{d\left(x, u_y\right) + d\,(y, u_x)}{2}\right\}.$$

Then the following statements hold:

(1) $Fix(T_i) \neq \emptyset$ for any $i \in \{1, 2\}$ if and only if $Fix(T_1) = Fix(T_2) \neq \emptyset$.
(2) $Fix(T_1) = Fix(T_2) \neq \emptyset$ provided that T_1 or T_2 is upper semicontinuous.
(3) $Fix(T_1) \cap Fix(T_2)$ is well-ordered if and only if $Fix(T_1) \cap Fix(T_2)$ is singleton set.

Example 20.1 Let $X = \{x_n = \frac{n(n+1)}{2} : n \in \{1, 2, 3, \ldots\}\}$ endow with usual order $\leq$. Let

$$\begin{aligned}\Delta_1 &= \{(x, y) : x \leq y \;\; where \;\; x, y \in X\} \; and \\ \Delta_2 &= \{(x, y) : x < y \;\; where \;\; x, y \in X\}.\end{aligned}$$

Define $T_1, T_2 : X \to P_{cl}(X)$ as follows:

$$\begin{aligned}T_1\,(x) &= \{x_1\} \;\; for \;\; x \in X, \\ T_2\,(x) &= \begin{cases}\{x_1\} & , \quad x = x_1 \\ \{x_1, x_{n-1}\}\,, & \quad x = x_n, \quad for \;\; n > 1.\end{cases}\end{aligned}$$

Take $F(\alpha) = \ln \alpha + \alpha$, $\alpha > 0$ and $\tau = 1$. For a Euclidean metric d on X, and $(u_x, u_y) \in \Delta_2$, we consider the following cases:

(i) If $x = x_1$, $y = x_m$, for $m > 1$, then for $u_x = x_1 \in T_1(x)$, there exists $u_y = x_{m-1} \in T_2(y)$, such that

$$\begin{aligned} d(u_x, u_y)e^{d(u_x,u_y)-M(x,y;u_x,u_y)} &\leq d(u_x, u_y)e^{d(u_x,u_y)-d(x,y)} \\ &= \frac{m^2 - m - 2}{2}e^{-m} \\ &< \frac{m^2 + m - 2}{2}e^{-1} \\ &= e^{-1}d(x, y) \\ &\leq e^{-1}M(x, y; u_x, u_y). \end{aligned}$$

(ii) If $x = x_n$, $y = x_{n+1}$ with $n > 1$, then for $u_x = x_1 \in T_1(x)$, there exists $u_y = x_{n-1} \in T_2(y)$, such that

$$\begin{aligned} d(u_x, u_y)e^{d(u_x,u_y)-M(x,y;u_x,u_y)} &\leq d(u_x, u_y)e^{d(u_x,u_y)-[\frac{d(x,u_y)+d(y,u_x)}{2}]} \\ &= \frac{n^2 - n - 2}{2}e^{\frac{-3n-2}{2}} \\ &< \frac{n^2 + 4n}{2}e^{-1} \\ &= e^{-1}\left[\frac{d(x, u_y) + d(y, u_x)}{2}\right] \\ &\leq e^{-1}M(x, y; u_x, u_y). \end{aligned}$$

(iii) When $x = x_n$, $y = x_m$ with $m > n > 1$, then for $u_x = x_1 \in T_1(x)$, there exists $u_y = x_{n-1} \in T_2(y)$, such that

$$\begin{aligned} d(u_x, u_y)e^{d(u_x,u_y)-M(x,y;u_x,u_y)} &\leq d(u_x, u_y)e^{d(u_x,u_y)-d(x,u_x)} \\ &= \frac{n^2 - n - 2}{2}e^{-n} \\ &< \frac{n^2 + n - 2}{2}e^{-1} \\ &= e^{-1}d(x, u_x) \\ &\leq e^{-1}M(x, y; u_x, u_y). \end{aligned}$$

Now we show that for $x, y \in X$, $u_x \in T_2(x)$; there exists $u_y \in T_1(y)$ such that $(u_x, u_y) \in \Delta_2$ and (3) of Corollary 20.1 is satisfied. For this, we consider the following cases:

(i) If $x = x_n,\ y = x_1$ with $n > 1$, we have for $u_x = x_{n-1} \in T_2(x)$, there exists $u_y = x_1 \in T_1(y)$, such that

$$\begin{aligned} d(u_x, u_y)e^{d(u_x,u_y)-M(x,y;u_x,u_y)} &\le d(u_x, u_y)e^{d(u_x,u_y)-d(x,y)} \\ &= \frac{n^2-n-2}{2}e^{-n} \\ &< \frac{n^2+n-2}{2}e^{-1} \\ &= e^{-1}d(x, y) \\ &\le e^{-1}M(x, y; u_x, u_y). \end{aligned}$$

(ii) In case $x = x_n,\ y = x_m$ with $m > n > 1$, then for $u_x = x_{n-1} \in T_2(x)$, there exists $u_y = x_1 \in T_2(y)$, such that

$$\begin{aligned} d(u_x, u_y)e^{d(u_x,u_y)-M(x,y;u_x,u_y)} &\le d(u_x, u_y)e^{d(u_x,u_y)-d(y,u_y)} \\ &= \frac{n^2-n-2}{2}e^{n^2-n-m^2-m} \\ &< \frac{m^2+m-2}{2}e^{-1} \\ &= e^{-1}d(y, u_y) \\ &\le e^{-1}M(x, y; u_x, u_y). \end{aligned}$$

Hence all the conditions of Corollary 20.1 are satisfied. Moreover, $x_1 = 1$ is the unique common fixed point of T_1 and T_2 with $Fix(T_1) = Fix(T_2)$.

The following result generalizes Theorem 3.4 of [21] and Theorem 3.4 of [22].

Theorem 20.3 *Let $(X, d, \preceq)$ be an ordered complete metric space and $\{T_i\}_{i=1}^m : X \to P_{cl}(X)$ be family of multivalued mappings. Suppose that for every $(x, y) \in \Delta_1$ and $u_x \in T_i(x)$, there exists $u_y \in T_{i+1}(y)$ for $i \in \{1, 2, \ldots, m\}$ (with $T_{m+1} = T_1$ by convention) such that, $(u_x, u_y) \in \Delta_2$ implies*

$$\tau + F\left(d(u_x, u_y)\right) \le F(M_2(x, y; u_x, u_y)), \tag{4}$$

where τ is a positive real number and

$$M_2(x, y; u_x, u_y) = \alpha d(x, y) + \beta d(x, u_x) + \gamma d(y, u_y) + \delta_1 d\left(x, u_y\right) + \delta_2 d\left(y, u_x\right),$$

and $\alpha, \beta, \gamma, \delta_1, \delta_2 \ge 0$, $\delta_1 \le \delta_2$ with $\alpha + \beta + \gamma + \delta_1 + \delta_2 \le 1$. Then the following statements hold:

(I) $Fix(T_i) \ne \emptyset$ for any $i \in \{1, 2, \ldots, m\}$ if and only if $Fix(T_1) = Fix(T_2) = \cdots = Fix(T_m) \ne \emptyset$.

(II) $Fix(T_1) = Fix(T_2) = \cdots = Fix(T_m) \neq \emptyset$ *provided that any one* T_i *for* $i \in \{1, 2, \ldots, m\}$ *is upper semicontinuous.*

(III) $\cap_{i=1}^m Fix(T_i)$ *is well-ordered if and only if* $\cap_{i=1}^m Fix(T_i)$ *is singleton set.*

Proof To prove (I), let $x^* \in T_k(x^*)$ for any $k \in \{1, 2, \ldots, m\}$. Assume that $x^* \notin T_{k+1}(x^*)$, then there exists an $x \in T_{k+1}(x^*)$ with $(x^*, x) \in \Delta_2$ such that

$$\tau + F\left(d(x^*, x)\right) \leq F(M_2(x^*, x^*; x^*, x)),$$

where

$$\begin{aligned} M_2(x^*, x^*; x^*, x) &= \alpha d(x^*, x^*) + \beta d(x^*, x^*) + \gamma d(x, x^*) \\ &\quad + \delta_1 d(x^*, x) + \delta_2 d(x^*, x^*) \\ &= (\gamma + \delta_1) d(x, x^*), \end{aligned}$$

implies that

$$\begin{aligned} \tau + F\left(d(x^*, x)\right) &\leq F((\gamma + \delta_1) d(x^*, x)) \\ &\leq F(d(x^*, x)), \end{aligned}$$

a contradiction as $\tau > 0$. Thus $x^* = x$. Thus $x^* \in T_{k+1}(x^*)$ and so $Fix(T_k) \subseteq Fix(T_{k+1})$. Similarly, we obtain that $Fix(T_{k+1}) \subseteq Fix(T_{k+2})$ and continuing this way, we get $Fix(T_1) = Fix(T_2) = \cdots = Fix(T_k)$. The converse is straightforward.

To prove (II), suppose that x_0 is an arbitrary point of X. If $x_0 \in T_{k_0}(x_0)$ for any $k_0 \in \{1, 2, \ldots, m\}$, then by using (I), the proof is finishes. So we assume that $x_0 \notin T_{k_0}(x_0)$ for any $k_0 \in \{1, 2, \ldots, m\}$. Now for $i \in \{1, 2, \ldots, m\}$, if $x_1 \in T_i(x_0)$, then there exists $x_2 \in T_{i+1}(x_1)$ with $(x_1, x_2) \in \Delta_2$ such that

$$\tau + F\left(d(x_1, x_2)\right) \leq F(M_2(x_0, x_1; x_1, x_2)),$$

where

$$\begin{aligned} M_2(x_0, x_1; x_1, x_2) &= \alpha d(x_0, x_1) + \beta d(x_0, x_1) + \gamma d(x_1, x_2) \\ &\quad + \delta_1 d(x_0, x_2) + \delta_2 d(x_1, x_1) \\ &\leq (\alpha + \beta + \delta_1) d(x_0, x_1) + (\gamma + \delta_1) d(x_1, x_2). \end{aligned}$$

Now, if $d(x_0, x_1) \leq d(x_1, x_2)$, then we have

$$\begin{aligned} \tau + F\left(d(x_1, x_2)\right) &\leq F((\alpha + \beta + \gamma + 2\delta_1) d(x_1, x_2)) \\ &\leq F(d(x_1, x_2)), \end{aligned}$$

a contradiction. Therefore

$$\tau + F\left(d(x_1, x_2)\right) \leq F\left(d(x_0, x_1)\right).$$

Next for this $x_2 \in T_{i+1}(x_1)$, there exists $x_3 \in T_{i+2}(x_2)$ with $(x_2, x_3) \in \Delta_2$ such that

$$\tau + F(d(x_2, x_3)) \leq F(M_2(x_1, x_2; x_2, x_3)),$$

where

$$\begin{aligned} M_2(x_1, x_2; x_2, x_3) &= \alpha d(x_1, x_2) + \beta d(x_1, x_2) + \gamma d(x_2, x_3) \\ &\quad + \delta_1 d(x_1, x_3) + \delta_2 d(x_2, x_2) \\ &\leq (\alpha + \beta + \delta_1) d(x_1, x_2) + (\gamma + \delta_1) d(x_2, x_3). \end{aligned}$$

Now, if $d(x_1, x_2) \leq d(x_2, x_3)$ then

$$\begin{aligned} \tau + F(d(x_2, x_3)) &\leq F((\alpha + \beta + \gamma + 2\delta_1) d(x_2, x_3)) \\ &\leq F(d(x_2, x_3)), \end{aligned}$$

a contradiction as $\tau > 0$. Therefore

$$\tau + F(d(x_2, x_3)) \leq F(d(x_1, x_2)).$$

Continuing this process, for $x_{2n} \in T_i(x_{2n-1})$, there exist $x_{2n+1} \in T_{i+1}(x_{2n})$ with $(x_{2n}, x_{2n+1}) \in \Delta_2$ such that

$$\tau + F(d(x_{2n}, x_{2n+1})) \leq F(M_2(x_{2n-1}, x_{2n}; x_{2n}, x_{2n+1})),$$

where

$$\begin{aligned} M_2(x_{2n-1}, x_{2n}; x_{2n}, x_{2n+1}) &= \alpha d(x_{2n-1}, x_{2n}) + \beta d(x_{2n-1}, x_{2n}) + \gamma d(x_{2n}, x_{2n+1}) \\ &\quad + \delta_1 d(x_{2n-1}, x_{2n+1}) + \delta_2 d(x_{2n}, x_{2n}) \\ &\leq (\alpha + \beta + \delta_1) d(x_{2n-1}, x_{2n}) + (\gamma + \delta_1) d(x_{2n}, x_{2n+1}) \\ &\leq d(x_{2n-1}, x_{2n}), \end{aligned}$$

that is,

$$\tau + F(d(x_{2n}, x_{2n+1})) \leq F(d(x_{2n-1}, x_{2n})).$$

Similarly, for $x_{2n+1} \in T_{i+1}(x_{2n})$, there exist $x_{2n+2} \in T_{i+2}(x_{2n+1})$ such that for $(x_{2n+1}, x_{2n+2}) \in \Delta_2$ implies

$$\tau + F(d(x_{2n+1}, x_{2n+2})) \leq F(d(x_{2n}, x_{2n+1})).$$

Hence, we obtain a sequence $\{x_n\}$ in X such that for $x_n \in T_i(x_{n-1})$, there exist $x_{n+1} \in T_{i+1}(x_n)$ with $(x_n, x_{n+1}) \in \Delta_2$ such that

$$\tau + F(d(x_n, x_{n+1})) \leq F(d(x_{n-1}, x_n)).$$

Therefore

$$F\left(d(x_n, x_{n+1})\right) \leq F\left(d(x_{n-1}, x_n)\right) - \tau \leq F\left(d(x_{n-2}, x_{n-1})\right) - 2\tau$$
$$\leq \cdots \leq F\left(d(x_0, x_1)\right) - n\tau. \tag{5}$$

From (4), we obtain $\lim_{n\to\infty} F\left(d(x_n, x_{n+1})\right) = -\infty$ that together with (F_2) gives

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

Follows the arguments those in proof of Theorem 20.2, $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists an element $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$.

Now, if T_i is upper semicontinuous for any $i \in \{1, 2, \ldots, m\}$, then as $x_{2n} \in X$, $x_{2n+1} \in T_i\left(x_{2n}\right)$ with $x_{2n} \to x^*$ and $x_{2n+1} \to x^*$ as $n \to \infty$ implies that $x^* \in T_i\left(x^*\right)$. Thus from (I), we get $x^* \in T_1\left(x^*\right) = T_2\left(x^*\right) = \cdots = T_m\left(x^*\right)$.

Finally to prove (III), suppose the set $\cap_{i=1}^{m} Fix\left(T_i\right)$ is a well-ordered. We are to show that $\cap_{i=1}^{m} Fix\left(T_i\right)$ is singleton. Assume on contrary that there exist u and v such that $u, v \in \cap_{i=1}^{m} Fix\left(T_i\right)$ but $u \neq v$. As $(u, v) \in \Delta_2$, so for $(u_x, v_y) \in \Delta_2$ implies

$$\tau + F\left(d(u, v)\right) \leq F(M_2(u, v; u, v)),$$

where

$$\begin{aligned} M_2(u, v; u, v) &= \alpha d(u, v) + \beta d(u, u) + \gamma d(v, v) \\ &\quad + \delta_1 d\left(u, v\right) + \delta_2 d\left(v, u\right) \\ &= \left(\alpha + \delta_1 + \delta_2\right) d\left(x, y\right), \end{aligned}$$

that is,

$$\begin{aligned} \tau + F\left(d(u, v)\right) &\leq F\left(\left(\alpha + \delta_1 + \delta_2\right) d\left(x, y\right)\right) \\ &\leq F\left(d\left(u, v\right)\right), \end{aligned}$$

a contradiction as $\tau > 0$. Hence $u = v$. Conversely, if $\cap_{i=1}^{m} Fix\left(T_i\right)$ is singleton, then it follows that $\cap_{i=1}^{m} Fix\left(T_i\right)$ is a well-ordered. □

The following corollary extends Theorem 3.1 of [21], in the case of family of mappings in ordered metric space.

Corollary 20.2 *Let* $(X, d, \preceq)$ *be an ordered complete metric space and* $\{T_i\}_{i=1}^{m} : X \to P_{cl}(X)$ *be family of multivalued mappings. Suppose that for every* $(x, y) \in \Delta_1$ *and* $u_x \in T_i(x)$, *there exists* $u_y \in T_{i+1}(y)$ *for* $i \in \{1, 2, \ldots, m\}$ *(with* $T_{m+1} = T_1$ *by convention) such that,* $(u_x, u_y) \in \Delta_2$ *implies*

$$\tau + F\left(d(u_x, u_y)\right) \leq F(\alpha d\left(x, y\right) + \beta d(x, u_x) + \gamma d(y, u_y)]), \tag{6}$$

where τ is a positive real number and $\alpha, \beta, \gamma \geq 0$ with $\alpha, \beta, \gamma \leq 1$. Then the conclusions obtained in Theorem 20.3 remains true.

The following corollary extends Theorem 4.1 of [13].

Corollary 20.3 *Let $(X, d, \preceq)$ be an ordered complete metric space and $\{T_i\}_{i=1}^{m} : X \to P_{cl}(X)$ be family of multivalued mappings. Suppose that for every $(x, y) \in \Delta_1$ and $u_x \in T_i(x)$, there exists $u_y \in T_{i+1}(y)$ for $i \in \{1, 2, \ldots, m\}$ (with $T_{m+1} = T_1$ by convention) such that, $(u_x, u_y) \in \Delta_2$ implies*

$$\tau + F\left(d(u_x, u_y)\right) \leq F(h[d(x, u_x) + d(y, u_y)]), \tag{7}$$

where τ is a positive real number and $h \in [0, \frac{1}{2}]$. Then the conclusions obtained in Theorem 20.3 remain true.

Corollary 20.4 *Let $(X, d, \preceq)$ be an ordered complete metric space and $\{T_i\}_{i=1}^{m} : X \to P_{cl}(X)$ be family of multivalued mappings. Suppose that for every $(x, y) \in \Delta_1$ and $u_x \in T_i(x)$, there exists $u_y \in T_{i+1}(y)$ for $i \in \{1, 2, \ldots, m\}$ (with $T_{m+1} = T_1$ by convention) such that, $(u_x, u_y) \in \Delta_2$ implies*

$$\tau + F\left(d(u_x, u_y)\right) \leq F(d(x, y)), \tag{8}$$

where τ is a positive real number. Then the conclusions obtained in Theorem 20.3 remain true.

The above corollary extends Theorem 4.1 of [13].

3 Conclusion

Recently many results appeared in the literature giving the problems related to the common fixed point for multivalued maps. In this paper we obtained the results for existence of common fixed points of family of maps that satisfying generalized F-contractions in ordered structured metric spaces. We presented some examples to show the validity of established results.

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