A New Two-Step Proximal Algorithm of Solving the Problem of Equilibrium Programming

Sergey I. Lyashko and Vladimir V. Semenov

Dedicated to Boris Polyak on the occasion of his 80th Birthday

Abstract We propose a new iterative two-step proximal algorithm for solving the problem of equilibrium programming in a Hilbert space. This method is a result of extension of L.D. Popov's modification of Arrow-Hurwicz scheme for approximation of saddle points of convex-concave functions. The convergence of the algorithm is proved under the assumption that the solution exists and the bifunction is pseudo-monotone and Lipschitz-type.

Keywords Equilibrium problem • Variational inequality • Two-step proximal algorithm • Bifunction • Pseudomonotonicity • Lipschitz condition • Convergence

1 Introduction

Throughout this chapter, we assume that H is a real Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. The symbol $\rightharpoonup$ denote weak convergence.

Let C be a nonempty closed convex subset of H and $F : C \times C \to \mathbb{R}$ be a bifunction with $F(x,x) = 0$ for all $x \in C$. Consider the following equilibrium problem in the sense of Blum and Oettli [12]:

$$\text{find } x \in C \text{ such that } \quad F(x,y) \geq 0 \quad \forall\, y \in C. \tag{1}$$

The equilibrium problem (1) (problem of equilibrium programming, Ky Fan inequality) is very general in the sense that it includes, as special cases, many applied mathematical models such as: variational inequalities, fixed point problems, optimization problems, saddle point problems, Nash equilibrium point problems in

S.I. Lyashko • V.V. Semenov (✉)
Department of Computational Mathematics, Kiev National Taras Shevchenko University, 64, Vladimirskaya Str., Kiev 01601, Ukraine
e-mail: lyashko.serg@gmail.com; semenov.volodya@gmail.com

B. Goldengorin (ed.), *Optimization and Its Applications in Control and Data Sciences*, Springer Optimization and Its Applications 115,
DOI 10.1007/978-3-319-42056-1_10

non-cooperative games, complementarity problems, see [3, 5, 6, 10, 12, 14, 17, 25] and the references therein. This problem is interesting because it allows to unify all these particular problems in a convenient way. In recent years, many methods have been proposed for solving equilibrium and related problems [2–10, 14, 16, 26, 28, 32, 35–37]. The solution approximation methods for the equilibrium problem are often based on the resolvent of equilibrium bifunction (see, for instance [14]) where at each iterative step a strongly monotone regularization equilibrium problem is solved. It is also called the proximal point method [16, 18, 20, 26, 37].

The variational inequality problem is a special case of the equilibrium problem. For solving the variational inequality in Euclidean space, Korpelevich [21] introduced the extragradient method where two metric projections onto feasible sets must be found at each iterative step. This method was setted in Hilbert spaces by Nadezhkina and Takahashi [27]. Some extragradient-like algorithms proposed for solving variational inequality problems can be found in [19, 33, 34, 38]. In 2011, the authors in [13, 22] have replaced the second projection onto any closed convex set in the extragradient method by one onto a half-space and proposed the subgradient extragradient method for variational inequalities in Hilbert spaces, see also [15, 39].

In recent years, the extragradient method has been extended to equilibrium problems for monotone (more general, pseudomonotone) and Lipschitz-type continuous bifunctions and studied both theoretically and algorithmically [1, 31, 40]. In this methods we must solve two strongly convex minimization problems on a closed convex constrained set at each iterative step. We note that similar methods have been previously proposed and studied by Antipin [2–4].

In 1980, Russian mathematician Popov [30] introduced very interesting modification of Arrow-Hurwicz scheme for approximation of saddle points of convex-concave functions in Euclidean space. Let X and Y are closed convex subset of Euclidean spaces $\mathbb{R}^d$ and $\mathbb{R}^p$, respectively, and $L : X \times Y \to \mathbb{R}$ be a differentiable convex-concave function. Then, the method [30] approximation of saddle points of L on $X \times Y$ can be written as

$$\begin{cases} x_1, \bar{x}_1 \in X, \quad y_1, \bar{y}_1 \in Y, \quad \lambda > 0, \\ x_{n+1} = P_X \left(x_n - \lambda L_1'(\bar{x}_n, \bar{y}_n) \right), \; y_{n+1} = P_Y \left(y_n + \lambda L_2'(\bar{x}_n, \bar{y}_n) \right), \\ \bar{x}_{n+1} = P_X \left(x_{n+1} - \lambda L_1'(\bar{x}_n, \bar{y}_n) \right), \; \bar{y}_{n+1} = P_Y \left(y_{n+1} + \lambda L_2'(\bar{x}_n, \bar{y}_n) \right), \end{cases}$$

where P_X and P_Y are metric projection onto X and Y, respectively, L_1' and L_2' are partial derivatives. Under some suitable assumptions, Popov proved the convergence of this method.

In this chapter, we have been motivated and inspired by the results of the authors in [30, 31], proposed a new two-step proximal algorithm for solving equilibrium problems. This algorithm is the extension of Popov method [30].

The set of solutions of the equilibrium problem (1) is denoted $EP(F, C)$. Further, we assume that the solution set $EP(F, C)$ is nonempty.

Here, for solving equilibrium problem (1), we assume that the bifunction F satisfies the following conditions:

(A1) $F(x,x) = 0$ for all $x \in C$;
(A2) for all x, $y \in C$ from $F(x,y) \geq 0$ it follows that $F(y,x) \leq 0$ (pseudo-monotonicity);
(A3) for all $x \in C$ the function $F(x,\cdot)$ is convex and lower semicontinuous on C;
(A4) for all $y \in C$ the function $F(\cdot, y)$ is weakly upper semicontinuous on C;
(A5) for all x, y, $z \in C$ the next inequality holds

$$F(x,y) \leq F(x,z) + F(z,y) + a\,\|x-z\|^2 + b\,\|z-y\|^2,$$

where a, b are positive constants (Lipschitz-type continuity);
(A6) for all bounded sequences (x_n), (y_n) from C we have

$$\|x_n - y_n\| \to 0 \;\Rightarrow\; F(x_n, y_n) \to 0.$$

It is easy to show that under the assumptions (A1)–(A4), we have

$$x \in EP(F,C) \quad \Leftrightarrow \quad x \in C: \; F(y,x) \leq 0 \;\; \forall\, y \in C.$$

In particular, the set $EP(F,C)$ is convex and closed (see, for instance [31]).

The hypothesis (A5) was introduced by Mastroeni [25]. It is necessary to imply the convergence of the auxiliary principle method for equilibrium problems. For example, the bifunction $F(x,y) = (Ax, y-x)$ with k-Lipschitz operator $A : C \to H$ satisfies (A5). Actually,

$$\begin{aligned} F(x,y) - F(x,z) - F(z,y) &= (Ax, y-x) - (Ax, z-x) - (Az, y-z) = \\ &= (Ax - Az, y-z) \leq \|Ax - Az\|\,\|y-z\| \leq k\,\|x-z\|\,\|y-z\| \leq \\ &\leq \frac{k}{2}\,\|x-z\|^2 + \frac{k}{2}\,\|y-z\|^2. \end{aligned}$$

This implies that F satisfies the condition (A5) with $a = b = k/2$.

The condition (A6) is satisfied by bifunction $F(x,y) = (Ax, y-x)$ with Lipschitz operator $A : C \to H$.

2 The Algorithm

Let $g : H \to \mathbb{R} \cup \{+\infty\}$ be a convex, lower semicontinuous, and proper. The proximity operator of a function g is the operator $\operatorname{prox}_g : H \to \operatorname{dom} g \subseteq H$ ($\operatorname{dom} g$ denotes the effective domain of g) which maps every $x \in H$ to the unique minimizer of the function $g + \|\cdot - x\|^2/2$, i.e.,

$$\forall x \in H \quad \operatorname{prox}_g x = \operatorname{argmin}_{y \in \operatorname{dom} g} \left\{ g(y) + \frac{1}{2}\|y-x\|^2 \right\}.$$

We have

$$z = \operatorname{prox}_g x \quad \Leftrightarrow \quad g(y) - g(z) + (z - x, y - z) \geq 0 \quad \forall y \in \operatorname{dom} g.$$

Proximity operators have attractive properties that make them particularly well suited for iterative minimization algorithms. For instance, prox_g is firmly nonexpansive and its fixed point set is precisely the set of minimizers of g. For detailed accounts of the proximity operators theory, see [11].

Now we extend the Popov method [30] to an equilibrium problem (1). In Algorithm 1 we are going to describe, in order to be able to obtain its convergence, the parameter λ must satisfy some condition (see convergence Theorem 1).

Algorithm 1. For $x_1, y_1 \in C$ generate the sequences $x_n, y_n \in C$ with the iterative scheme

$$\begin{cases} x_{n+1} = \operatorname{prox}_{\lambda F(y_n,\cdot)} x_n = \operatorname{argmin}_{y \in C} \left\{ \lambda F(y_n, y) + \frac{1}{2} \|y - x_n\|^2 \right\}, \\ y_{n+1} = \operatorname{prox}_{\lambda F(y_n,\cdot)} x_{n+1} = \operatorname{argmin}_{y \in C} \left\{ \lambda F(y_n, y) + \frac{1}{2} \|y - x_{n+1}\|^2 \right\}, \end{cases}$$

where $\lambda > 0$.

Extragradient method for the equilibrium problem (1) has the form

$$\begin{cases} y_n = \operatorname{prox}_{\lambda F(x_n,\cdot)} x_n, \\ x_{n+1} = \operatorname{prox}_{\lambda F(y_n,\cdot)} x_n, \end{cases}$$

where $\lambda > 0$ [31]. A distinctive and attractive feature of the Algorithm 1 consists in the fact that in the iterative step is used only one function $F(y_n, \cdot)$.

Remark 1. If $F(x, y) = (Ax, y - x)$, then Algorithm 1 takes the form:

$$\begin{cases} x_1 \in C, \ \ y_1 \in C, \\ x_{n+1} = P_C(x_n - \lambda A y_n), \\ y_{n+1} = P_C(x_{n+1} - \lambda A y_n), \end{cases}$$

where P_C is the operator of metric projection onto the set C.

A particular case of the scheme from the Remark 1 was proposed by Popov [30] for search of saddle points of convex-concave functions, which are defined on finite-dimensional Euclidean space. In recent works Malitsky and Semenov [23, 24] proved the convergence of this algorithm for variational inequalities with monotone and Lipschitz operators in infinite-dimensional Hilbert space, and proposed some modifications of this algorithm.

For substantiation of the iterative Algorithm 1 we note first, that if for some number $n \in \mathbb{N}$ next equalities are satisfied

$$x_{n+1} = x_n = y_n \tag{2}$$

than $y_n \in EP(F, C)$ and the following stationarity condition holds

$$y_k = x_k = y_n \quad \forall\, k \geq n.$$

Actually, the equality

$$x_{n+1} = \text{prox}_{\lambda F(y_n, \cdot)} x_n$$

means that

$$F(y_n, y) - F(y_n, x_{n+1}) + \frac{(x_{n+1} - x_n, y - x_{n+1})}{\lambda} \geq 0 \quad \forall y \in C.$$

From (2) it follows that

$$F(y_n, y) \geq 0 \;\; \forall y \in C,$$

i.e. $y_n \in EP(F, C)$.

Taking this into account the practical variant of the Algorithm 1 can be written as

Algorithm 2. Choose $x_1 \in C$, $y_1 \in C$, $\lambda > 0$, and $\varepsilon > 0$.

Step 1. For x_n and y_n compute

$$x_{n+1} = \text{prox}_{\lambda F(y_n, \cdot)} x_n.$$

Step 2. If $\max\{\|x_{n+1} - x_n\|, \|y_n - x_n\|\} \leq \varepsilon$, then STOP, else compute

$$y_{n+1} = \text{prox}_{\lambda F(y_n, \cdot)} x_{n+1}.$$

Step 3. Set $n := n + 1$ and go to Step 1.

Further, we assume that for all numbers $n \in \mathbb{N}$ the condition (2) doesn't hold. In the following section the weak convergence of the sequences (x_n), (y_n) generated by the Algorithm 1 is proved.

3 Convergence Results

To prove the convergence we need next facts.

Lemma 1. *Let non-negative sequences* (a_n), (b_n) *such that*

$$a_{n+1} \leq a_n - b_n.$$

Then exists the limit $\lim_{n\to\infty} a_n \in \mathbb{R}$ *and* $\sum_{n=1}^{\infty} b_n < +\infty$.

Lemma 2 (Opial [29]). *Let the sequence* (x_n) *of elements from Hilbert space* H *converges weakly to* $x \in H$. *Then for all* $y \in H \setminus \{x\}$ *we have*

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|.$$

We start the analysis of the convergence with the proof of important inequality for sequences (x_n) and (y_n), generated by the Algorithm 1.

Lemma 3. *Let sequences* (x_n), (y_n) *be generated by the Algorithm 1, and let* $z \in EP(F, C)$. *Then, we have*

$$\|x_{n+1} - z\|^2 \leq \|x_n - z\|^2 - (1 - 2\lambda b)\|x_{n+1} - y_n\|^2 - \\ - (1 - 4\lambda a)\|y_n - x_n\|^2 + 4\lambda a \|x_n - y_{n-1}\|^2. \quad (3)$$

Proof. We have

$$\|x_{n+1} - z\|^2 = \|x_n - z\|^2 - \|x_n - x_{n+1}\|^2 + 2(x_{n+1} - x_n, x_{n+1} - z) = \\ = \|x_n - z\|^2 - \|x_n - y_n\|^2 - \|y_n - x_{n+1}\|^2 - \\ -2(x_n - y_n, y_n - x_{n+1}) + 2(x_{n+1} - x_n, x_{n+1} - z). \quad (4)$$

From the definition of points x_{n+1} and y_n it follows that

$$\lambda F(y_n, z) - \lambda F(y_n, x_{n+1}) \geq (x_{n+1} - x_n, x_{n+1} - z), \quad (5)$$

$$\lambda F(y_{n-1}, x_{n+1}) - \lambda F(y_{n-1}, y_n) \geq -(x_n - y_n, y_n - x_{n+1}). \quad (6)$$

Using inequalities (5), (6) to estimate inner products in (4), we get

$$\|x_{n+1} - z\|^2 \leq \|x_n - z\|^2 - \|x_n - y_n\|^2 - \|y_n - x_{n+1}\|^2 + \\ +2\lambda \{F(y_n, z) - F(y_n, x_{n+1}) + F(y_{n-1}, x_{n+1}) - F(y_{n-1}, y_n)\}. \quad (7)$$

From pseudomonotonicity of the bifunction F and $z \in EP(F, C)$ it follows that

$$F(y_n, z) \leq 0,$$

and Lipschitz-type continuity F guaranties the satisfying of inequality

$$-F(y_n, x_{n+1}) + F(y_{n-1}, x_{n+1}) - F(y_{n-1}, y_n) \leq$$
$$\leq a \|y_{n-1} - y_n\|^2 + b \|y_n - x_{n+1}\|^2 .$$

Using the above estimations (7), we get

$$\|x_{n+1} - z\|^2 \leq \|x_n - z\|^2 - \|x_n - y_n\|^2 - \|y_n - x_{n+1}\|^2 +$$
$$+2\lambda a \|y_{n-1} - y_n\|^2 + 2\lambda b \|y_n - x_{n+1}\|^2 . \tag{8}$$

The term $\|y_{n-1} - y_n\|^2$ we estimate in the next way

$$\|y_{n-1} - y_n\|^2 \leq 2 \|y_{n-1} - x_n\|^2 + 2 \|y_n - x_n\|^2 .$$

Taking this into account (8), we get the inequality

$$\|x_{n+1} - z\|^2 \leq \|x_n - z\|^2 - \|x_n - y_n\|^2 - \|y_n - x_{n+1}\|^2 +$$
$$+4\lambda a \|y_{n-1} - x_n\|^2 + 4\lambda a \|y_n - x_n\|^2 + 2\lambda b \|y_n - x_{n+1}\|^2 ,$$

i.e. the inequality (3). □

Proceed directly to proof of the convergence of the algorithm. Let $z \in EP(F, C)$. Assume

$$a_n = \|x_n - z\|^2 + 4\lambda a \|y_{n-1} - x_n\|^2 ,$$
$$b_n = (1 - 4\lambda a) \|y_n - x_n\|^2 + (1 - 4\lambda a - 2\lambda b) \|y_n - x_{n+1}\|^2 .$$

Then inequality (3) takes form

$$a_{n+1} \leq a_n - b_n.$$

The following condition are required

$$0 < \lambda < \frac{1}{2(2a + b)}.$$

Then from Lemma 1 we can conclude that exists the limit

$$\lim_{n\to\infty} \left(\|x_n - z\|^2 + 4\lambda a \|y_{n-1} - x_n\|^2\right)$$

and

$$\sum_{n=1}^{\infty} \left((1 - 4\lambda a) \|y_n - x_n\|^2 + (1 - 4\lambda a - 2\lambda b) \|y_n - x_{n+1}\|^2\right) < +\infty.$$

Whence we obtain

$$\lim_{n\to\infty}\|y_n - x_n\| = \lim_{n\to\infty}\|y_n - x_{n+1}\| = \lim_{n\to\infty}\|x_n - x_{n+1}\| = 0 \tag{9}$$

and convergence of the sequence $(\|x_n - z\|)$ for all $z \in EP(F, C)$. In particular, sequences (x_n), (y_n) are bounded.

Now we consider the subsequence (x_{n_k}), which converges weakly to the point $\bar{z} \in C$. Then from (9) it follows that $y_{n_k} \rightharpoonup \bar{z}$. Show that $\bar{z} \in EP(F, C)$. We have

$$F(y_n, y) \geq F(y_n, x_{n+1}) + \frac{(x_{n+1} - x_n, x_{n+1} - y)}{\lambda} \quad \forall y \in C. \tag{10}$$

Passing to the limit (10) taking into account (9) and conditions (A4), (A6), we get

$$F(\bar{z}, y) \geq \limsup_{k\to\infty} F(y_{n_k}, y) \geq \lim_{k\to\infty}\{F(y_{n_k}, x_{n_k+1}) +$$
$$+ \frac{(x_{n_k+1} - x_{n_k}, x_{n_k+1} - y)}{\lambda}\Bigg\} = 0 \quad \forall y \in C,$$

i.e. $\bar{z} \in EP(F, C)$.

Now we show that $x_n \rightharpoonup \bar{z}$. Then from (9) it follows that $y_n \rightharpoonup \bar{z}$. Assume the converse. Let exists the subsequence (x_{m_k}) such that $x_{m_k} \rightharpoonup \tilde{z}$ and $\tilde{z} \neq \bar{z}$. It is clear that $\tilde{z} \in EP(F, C)$. Use the Lemma 2 twice. We have

$$\lim_{n\to\infty}\|x_n - \bar{z}\| = \lim_{k\to\infty}\|x_{n_k} - \bar{z}\| < \lim_{k\to\infty}\|x_{n_k} - \tilde{z}\| = \lim_{n\to\infty}\|x_n - \tilde{z}\| =$$
$$= \lim_{k\to\infty}\|x_{m_k} - \tilde{z}\| < \lim_{k\to\infty}\|x_{m_k} - \bar{z}\| = \lim_{n\to\infty}\|x_n - \bar{z}\|,$$

it is impossible. So, sequence (x_n) converges weakly to $\bar{z} \in EP(F, C)$.

Thus, we obtain the following result.

Theorem 1. *Let H be a Hilbert space, $C \subseteq H$ is nonempty convex closed set, for bifunction $F : C \times C \to \mathbb{R}$ conditions (A1)–(A6) are satisfied and $EP(F, C) \neq \emptyset$. Assume that $\lambda \in \left(0, \frac{1}{2(2a+b)}\right)$. Then sequences (x_n), (y_n) generated by the Algorithm 1 converge weakly to the solution $\bar{z} \in EP(F, C)$ of the equilibrium problem (1), and* $\lim_{n\to\infty}\|x_n - y_n\| = 0$.

Remark 2. The asymptotics $\lim_{n\to\infty}\|x_n - y_n\| = 0$ can be specified up to the following:

$$\liminf_{n\to\infty}\sqrt{n}\|x_n - y_n\| = 0. \tag{11}$$

Indeed, if (11) does not hold, then $\|x_n - y_n\| \geq \mu n^{-1/2}$ for some $\mu > 0$ and all sufficiently large n. Hence, the series $\sum\|x_n - y_n\|^2$ diverges. We have obtained an contradiction.

4 Conclusion and Future Work

In this work we have proposed a new iterative two-step proximal algorithm for solving the equilibrium programming problem in the Hilbert space. The method is the extension of Popov's modification [30] for Arrow-Hurwitz scheme for search of saddle points of convex-concave functions. The convergence of the algorithm is proved under the assumption that the solution exists and the bifunction is pseudo-monotone and Lipschitz-type.

In one of a forthcoming work we'll consider the next regularized variant of the algorithm that converges strongly

$$\begin{cases} x_{n+1} = \operatorname{prox}_{\lambda F(y_n,\cdot)} (1-\alpha_n)\, x_n, \\ y_{n+1} = \operatorname{prox}_{\lambda F(y_n,\cdot)} (1-\alpha_{n+1})\, x_{n+1}, \end{cases}$$

where $\lambda > 0$, (α_n) is infinitesimal sequence of positive numbers. Also we plan to study the variant of the method using Bregman's distance instead of Euclidean.

The interesting question is the substantiation of using Algorithm 1 as the element of an iterative method for equilibrium problem with a priori information, described in the form of inclusion to the fixed points set of quasi-nonexpansive operator.

Another promising area is the development of Algorithm 1 variants for solving stochastic equilibrium problems.

Acknowledgements We are grateful to Yura Malitsky, Yana Vedel for discussions. We are very grateful to the referees for their really helpful and constructive comments. Vladimir Semenov thanks the State Fund for Fundamental Researches of Ukraine for support.

References

1. Anh, P.N.: Strong convergence theorems for nonexpansive mappings and Ky Fan inequalities. J. Optim. Theory Appl. **154**, 303–320 (2012)
2. Antipin, A.S.: Equilibrium programming: gradient methods. Autom. Remote Control **58**(8), Part 2, 1337–1347 (1997)
3. Antipin, A.S.: Equilibrium programming: proximal methods. Comput. Math. Math. Phys. **37**, 1285–1296 (1997)
4. Antipin, A.: Equilibrium programming problems: prox-regularization and prox-methods. In: Recent Advances in Optimization. Lecture Notes in Economics and Mathematical Systems, vol. 452, pp. 1–18. Springer, Heidelberg (1997)
5. Antipin, A.S.: Extraproximal approach to calculating equilibriums in pure exchange models. Comput. Math. Math. Phys. **46**, 1687–1998 (2006)
6. Antipin, A.S.: Multicriteria equilibrium programming: extraproximal methods. Comput. Math. Math. Phys. **47**, 1912–1927 (2007)
7. Antipin, A.S., Vasil'ev, F.P., Shpirko, S.V.: A regularized extragradient method for solving equilibrium programming problems. Comput. Math. Math. Phys. **43**(10), 1394–1401 (2003)
8. Antipin, A.S., Artem'eva, L.A., Vasil'ev, F.P.: Multicriteria equilibrium programming: extragradient method. Comput. Math. Math. Phys. **50**(2), 224–230 (2010)

9. Antipin, A.S., Jacimovic, M., Mijailovic, N.: A second-order continuous method for solving quasi-variational inequalities. Comput. Math. Math. Phys. **51**(11), 1856–1863 (2011)
10. Antipin, A.S., Artem'eva, L.A., Vasil'ev, F.P.: Extraproximal method for solving two-person saddle-point games. Comput. Math. Math. Phys. **51**(9), 1472–1482 (2011)
11. Bauschke, H.H., Combettes, P.L.: Convex Analysis and Monotone Operator Theory in Hilbert Spaces. Springer, New York (2011)
12. Blum, E., Oettli, W.: From optimization and variational inequalities to equilibrium problems. Math. Stud. **63**, 123–145 (1994)
13. Censor, Y., Gibali, A., Reich, S.: The subgradient extragradient method for solving variational inequalities in Hilbert space. J. Optim. Theory Appl. **148**(2), 318–335 (2011)
14. Combettes, P.L., Hirstoaga, S.A.: Equilibrium programming in Hilbert spaces. J. Nonlinear Convex Anal. **6**(1), 117–136 (2005)
15. Denisov, S.V., Semenov, V.V., Chabak, L.M.: Convergence of the modified extragradient method for variational inequalities with non-Lipschitz operators. Cybern. Syst. Anal. **51**, 757–765 (2015)
16. Flam, S.D., Antipin, A.S.: Equilibrium programming using proximal-like algorithms. Math. Program. **78**, 29–41 (1997)
17. Giannessi, F., Maugeri, A., Pardalos, P.M.: Equilibrium Problems: Nonsmooth Optimization and Variational Inequality Models. Kluwer Academic, New York (2004)
18. Iusem, A.N., Sosa, W.: On the proximal point method for equilibrium problems in Hilbert spaces. Optimization **59**, 1259–1274 (2010)
19. Khobotov, E.N.: Modification of the extra-gradient method for solving variational inequalities and certain optimization problems. USSR Comput. Math. Math. Phys. **27**(5), 120–127 (1987)
20. Konnov, I.V.: Application of the proximal point method to nonmonotone equilibrium problems. J. Optim. Theory Appl. **119**, 317–333 (2003)
21. Korpelevich, G.M.: The extragradient method for finding saddle points and other problems. Ekonomika i Matematicheskie Metody **12**, 747–756 (1976) (In Russian)
22. Lyashko, S.I., Semenov, V.V., Voitova, T.A.: Low-cost modification of Korpelevich's methods for monotone equilibrium problems. Cybern. Syst. Anal. **47**, 631–639 (2011)
23. Malitsky, Yu.V., Semenov, V.V.: An extragradient algorithm for monotone variational inequalities. Cybern. Syst. Anal. **50**, 271–277 (2014)
24. Malitsky, Yu.V., Semenov, V.V.: A hybrid method without extrapolation step for solving variational inequality problems. J. Glob. Optim. **61**, 193–202 (2015)
25. Mastroeni, G.: On auxiliary principle for equilibrium problems. In: Daniele, P., et al. (eds.) Equilibrium Problems and Variational Models, pp. 289–298. Kluwer Academic, Dordrecht (2003)
26. Moudafi, A.: Proximal point methods extended to equilibrium problems. J. Nat. Geom. **15**, 91–100 (1999)
27. Nadezhkina, N., Takahashi, W.: Weak convergence theorem by an extragradient method for nonexpansive mappings and monotone mappings. J. Optim. Theory Appl. **128**, 191–201 (2006)
28. Nurminski, E.A.: The use of additional diminishing disturbances in Fejer models of iterative algorithms. Comput. Math. Math. Phys. **48**, 2154–2161 (2008)
29. Opial, Z.: Weak convergence of the sequence of successive approximations for nonexpansive mappings. Bull. Am. Math. Soc. **73**, 591–597 (1967)
30. Popov, L.D.: A modification of the Arrow-Hurwicz method for search of saddle points. Math. Notes Acad. Sci. USSR **28**(5), 845–848 (1980)
31. Quoc, T.D., Muu, L.D., Hien, N.V.: Extragradient algorithms extended to equilibrium problems. Optimization **57**, 749–776 (2008)
32. Semenov, V.V.: On the parallel proximal decomposition method for solving the problems of convex optimization. J. Autom. Inf. Sci. **42**(4), 13–18 (2010)
33. Semenov, V.V.: A strongly convergent splitting method for systems of operator inclusions with monotone operators. J. Autom. Inf. Sci. **46**(5), 45–56 (2014)
34. Semenov, V.V.: Hybrid splitting methods for the system of operator inclusions with monotone operators. Cybern. Syst. Anal. **50**, 741–749 (2014)

35. Semenov, V.V.: Strongly convergent algorithms for variational inequality problem over the set of solutions the equilibrium problems. In: Zgurovsky, M.Z., Sadovnichiy, V.A. (eds.) Continuous and Distributed Systems, pp. 131–146. Springer, Heidelberg (2014)
36. Stukalov, A.S.: An extraproximal method for solving equilibrium programming problems in a Hilbert space. Comput. Math. Math. Phys. **46**, 743–761 (2006)
37. Takahashi, S., Takahashi, W.: Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces. J. Math. Anal. Appl. **331**, 506–515 (2007)
38. Tseng, P.: A modified forward-backward splitting method for maximal monotone mappings. SIAM J. Control Optim. **38**(2), 431–446 (2000)
39. Verlan, D.A., Semenov, V.V., Chabak, L.M.: A strongly convergent modified extragradient method for variational inequalities with non-Lipschitz operators. J. Autom. Inf. Sci. **47**(7), 31–46 (2015)
40. Vuong, P.T., Strodiot, J.J, Nguyen, V.H.: Extragradient methods and linesearch algorithms for solving Ky Fan inequalities and fixed point problems. J. Optim. Theory Appl. **155**, 605–627 (2012)