Multiplication and Composition in Weighted Modulation Spaces

Maximilian Reich and Winfried Sickel

Abstract We study the existence of the product of two weighted modulation spaces. For this purpose, we discuss two different strategies. The more simple one allows transparent proofs in various situations. However, our second method allows a closer look onto associated norm inequalities under restrictions in the Fourier image. This will give us the opportunity to treat the boundedness of composition operators.

Keywords Weighted modulation spaces · Short-time Fourier transform · Frequency-uniform decomposition · Multiplication of distributions · Multiplication algebras · Composition of functions

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1 Introduction

Since modulation spaces have been introduced by Feichtinger [7] they have become canonical for both time-frequency and phase-space analysis. However, in recent time modulation spaces have been found useful also in connection with linear and nonlinear partial differential equations, see e.g., Wang et al. [35–38], Ruzhansky et al. [26], or Bourdaud et al. [5]. Investigations of partial differential equations require partly different tools than used in time-frequency and phase-space analysis. In particular, Fourier multipliers, pointwise multiplication and composition of functions need to be studied. In our contribution, we will concentrate on pointwise multiplication and composition of functions. Already Feichtinger [7] was aware of the importance of pointwise multiplication in modulation spaces. In the meanwhile

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several authors have studied this problem, we refer, e.g., to [6, 13, 29, 30, 32]. In Sect. 3, we will give a survey about the known results. Therefore, we will discuss two different proof strategies. The more simple one, due to Toft [30, 32] and Sugimoto et al. [29], allows transparent proofs in various situations, in particular one can deal with those situations where the modulation spaces form algebras with respect to pointwise multiplication. As a consequence, Sugimoto et al. [29] are able to deal with composition operators on modulation spaces induced by analytic functions. Our second method, much more complicated, allows a closer look onto associated norm inequalities under restrictions in the Fourier image. This will give us the possibility to discuss the boundedness of composition operators on weighted modulation spaces based on a technique which goes back to Bourdaud [3], see also Bourdaud et al. [5] and Reich et al. [23]. Our approach will allow to deal with the boundedness of nonlinear operators $T_f : g \mapsto f \circ g$ without assuming f to be analytic. However, as the case of $M^s_{2,2}$ shows, our sufficient conditions are not very close to the necessary conditions. There is still a certain gap.

The paper is organized as follows. In Sect. 2, we collect what is needed about the weighted modulation spaces we are interested in. The next section is devoted to the study of pointwise multiplication. In particular, we are interested in embeddings of the type

$$M^{s_1}_{p,q} \cdot M^{s_2}_{p,q} \hookrightarrow M^{s_0}_{p,q}\, ,$$

where s_1, s_2, p and q are given and we are asking for an optimal s_0. These results will be applied to problems around the regularity of composition of functions in Sect. 4. For convenience of the reader we also recall what is known in the more general situation

$$M^{s_1}_{p_1,q_1} \cdot M^{s_2}_{p_2,q_2} \hookrightarrow M^{s_0}_{p,q}\, .$$

Special attention will be paid to the algebra property. Here, the known sufficient conditions are supplemented by necessary conditions, see Theorem 3.5. Also only partly new is our main result in Sect. 3 stated in Theorem 3.22. Here we investigate multiplication of distributions (possibly singular) with regular functions (which are not assumed to be C^∞). Partly we have found necessary and sufficient conditions also in this more general situation. Finally, Sect. 4 deals with composition operators. As direct consequences of the obtained results for pointwise multiplication we can deal with the mappings $g \mapsto g^\ell$, $\ell \geq 2$, see Sect. 4.1. In Sect. 4.4, we shall investigate $g \mapsto f \circ g$, where f is not assumed to be analytic. Sufficient conditions, either in terms of a decay for $\mathcal{F} f$ or in terms of regularity of f, are given.

Notation

We introduce some basic notation. As usual, $\mathbb{N}$ denotes the natural numbers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{Z}$ the integers and $\mathbb{R}$ the real numbers, $\mathbb{C}$ refers to the complex numbers. For a real number a, we put $a_+ := \max(a, 0)$. For $x \in \mathbb{R}^n$ we use $\|x\|_\infty := \max_{j=1,\ldots,n} |x_j|$. Many times we shall use the abbreviation $\langle \xi \rangle := (1 + |\xi|^2)^{\frac{1}{2}}$, $\xi \in \mathbb{R}^n$.

The symbols $c, c_1, c_2, \ldots, C, C_1, C_2, \ldots$ denote positive constants which are independent of the main parameters involved but whose values may differ from line to line. The notation $a \lesssim b$ is equivalent to $a \le Cb$ with a positive constant C. Moreover, by writing $a \asymp b$ we mean $a \lesssim b \lesssim a$.

Let X and Y be two Banach spaces. Then the symbol $X \hookrightarrow Y$ indicates that the embedding is continuous. By $\mathcal{L}(X, Y)$ we denote the collection of all linear and continuous operators which map X into Y. By $C_0^\infty(\mathbb{R}^n)$ the set of compactly supported infinitely differentiable functions $f : \mathbb{R}^n \to \mathbb{C}$ is denoted. Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$. The topological dual, the class of tempered distributions, is denoted by $\mathcal{S}'(\mathbb{R}^n)$ (equipped with the weak topology). The Fourier transform on $\mathcal{S}(\mathbb{R}^n)$ is given by

$$\mathcal{F}\varphi(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\, \varphi(x)\, dx\,, \qquad \xi \in \mathbb{R}^n\,.$$

The inverse transformation is denoted by $\mathcal{F}^{-1}$. We use both notations also for the transformations defined on $\mathcal{S}'(\mathbb{R}^n)$.

Convention. If not otherwise stated all functions will be considered on the Euclidean n-space $\mathbb{R}^n$. Therefore $\mathbb{R}^n$ will be omitted in notation.

2 Basics on Modulation Spaces

2.1 Definitions

A general reference for definition and properties of weighted modulation spaces is Gröchenig's monograph [10, Chap. 11].

Definition 2.1 Let $\phi \in \mathcal{S}$ be nontrivial. Then the short-time Fourier transform of a function f with respect to ϕ is defined as

$$V_\phi f(x, \xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(s)\overline{\phi(s-x)} e^{-is\cdot\xi}\, ds \qquad (x, \xi \in \mathbb{R}^n).$$

The function ϕ is usually called the window function. For $f \in \mathcal{S}'$ the short-time Fourier transform $V_\phi f$ is a continuous function of at most polynomial growth on $\mathbb{R}^{2n}$, see [10, Theorem 11.2.3].

Definition 2.2 Let $1 \le p, q \le \infty$. Let $\phi \in \mathcal{S}$ be a fixed window and assume $s \in \mathbb{R}$. Then the weighted modulation space $M^s_{p,q}$ is the collection of all $f \in \mathcal{S}'$ such that

$$\|f\|_{M^s_{p,q}} = \Big(\int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} |V_\phi f(x, \xi)\, \langle\xi\rangle^s|^p dx \Big)^{\frac{q}{p}} d\xi \Big)^{\frac{1}{q}} < \infty$$

(with obvious modifications if $p = \infty$ and/or $q = \infty$).

Formally these spaces $M^s_{p,q}$ depend on the window ϕ. However, for different windows ϕ_1, ϕ_2 the resulting spaces coincide as sets and the norms are equivalent, see [10, Proposition 11.3.2]. For that reason we do not indicate the window in the notation (we do not distinguish spaces which differ only by an equivalent norm).

Remark 2.3 (i) General references with respect to weighted modulation spaces are Feichtinger [7], Gröchenig [10, Chap. 11], Gol'dman [9], Guo et al. [11], Toft [30–32], Triebel [34] and Wang et al. [38] to mention only a few.

(ii) There is an important special case. In case of $p = q = 2$ we obtain $M^s_{2,2} = H^s$ in the sense of equivalent norms, see Feichtinger [7], Gröchenig [10, Proposition 11.3.1]. Here H^s is nothing but the standard Sobolev space built on L_2, at least for $s \in \mathbb{N}$. In general H^s is the collection of all $f \in \mathcal{S}'$ such that

$$\|f\|_{H^s} := \Big(\int_{\mathbb{R}^n} (1+|\xi|^2)^s\, |\mathcal{F}f(\xi)|^2\, d\xi\Big)^{1/2} < \infty .$$

For us of great use will be another alternative approach to the spaces $M^s_{p,q}$. This will be more close to the standard techniques used in connection with Besov spaces. We shall use the so-called frequency-uniform decomposition, see e.g., Wang [37]. Therefore, let $\rho : \mathbb{R}^n \mapsto [0, 1]$ be a Schwartz function which is compactly supported in the cube

$$Q_0 := \{\xi \in \mathbb{R}^n : -1 \le \xi_i \le 1,\ i = 1, \dots, n\} .$$

Moreover, we assume

$$\rho(\xi) = 1 \qquad \text{if} \quad |\xi_i| \le \frac{1}{2}, \qquad i = 1, 2, \dots, n.$$

With $\rho_k(\xi) := \rho(\xi - k)$, $\xi \in \mathbb{R}^n$, $k \in \mathbb{Z}^n$, it follows

$$\sum_{k\in\mathbb{Z}^n} \rho_k(\xi) \ge 1 \qquad \text{for all} \quad \xi \in \mathbb{R}^n .$$

Finally, we define

$$\sigma_k(\xi) := \rho_k(\xi)\Big(\sum_{k\in\mathbb{Z}^n} \rho_k(\xi)\Big)^{-1}, \qquad \xi \in \mathbb{R}^n , \quad k \in \mathbb{Z}^n .$$

The following properties are obvious:

- $0 \le \sigma_k(\xi) \le 1$ for all $\xi \in \mathbb{R}^n$;
- $\operatorname{supp} \sigma_k \subset Q_k := \{\xi \in \mathbb{R}^n : -1 \le \xi_i - k_i \le 1,\ i = 1, \dots, n\}$;
- $\sum_{k\in\mathbb{Z}^n} \sigma_k(\xi) \equiv 1$ for all $\xi \in \mathbb{R}^n$;
- There exists a constant $C > 0$ such that $\sigma_k(\xi) \ge C$ if $\max_{i=1,\dots,n} |\xi_i - k_i| \le \frac{1}{2}$;
- For all $m \in \mathbb{N}_0$ there exist positive constants C_m such that for $|\alpha| \le m$

$$\sup_{k\in\mathbb{Z}^n}\ \sup_{\xi\in\mathbb{R}^n} |D^\alpha \sigma_k(\xi)| \le C_m\, .$$

We shall call the mapping

$$\Box_k f := \mathcal{F}^{-1}\,[\sigma_k(\xi)\,\mathcal{F}f(\xi)]\,(\cdot), \qquad k\in\mathbb{Z}^n, \quad f\in\mathcal{S}'\,,$$

frequency-uniform decomposition operator.

As it is well-known there is an equivalent description of the modulation spaces by means of the frequency-uniform decomposition operators.

Proposition 2.4 *Let $1\le p,q\le\infty$ and assume $s\in\mathbb{R}$. Then the weighted modulation space $M^s_{p,q}$ consists of all tempered distributions $f\in\mathcal{S}'$ such that*

$$\|f\|^*_{M^s_{p,q}} = \Big(\sum_{k\in\mathbb{Z}^n}\langle k\rangle^{sq}\,\|\Box_k f\|^q_{L^p}\Big)^{\frac{1}{q}} < \infty\, .$$

Furthermore, the norms $\|f\|_{M^s_{p,q}}$ and $\|f\|^*_{M^s_{p,q}}$ are equivalent.

We refer to Feichtinger [7] or Wang and Hudzik [37]. In what follows, we shall work with both characterizations. In general, we shall use the same notation $\|\cdot\|_{M^s_{p,q}}$ for both norms.

Lemma 2.5 *(i) The modulation space $M^s_{p,q}$ is a Banach space.*
(ii) $M^s_{p,q}$ is independent of the choice of the window $\rho\in C_0^\infty$ in the sense of equivalent norms.
(iii) $M^s_{p,q}$ is continuously embedded into $\mathcal{S}'$.
(iv) $M^s_{p,q}$ has the Fatou property, i.e., if $(f_m)_{m=1}^\infty\subset M^s_{p,q}$ is a sequence such that $f_m \rightharpoonup f$ (weak convergence in $\mathcal{S}'$) and

$$\sup_{m\in\mathbb{N}} \|\, f_m\,\|_{M^s_{p,q}} < \infty\,,$$

then $f\in M^s_{p,q}$ follows and

$$\|\, f\,\|_{M^s_{p,q}} \le \sup_{m\in\mathbb{N}} \|\, f_m\,\|_{M^s_{p,q}} < \infty\, .$$

Proof For (i), (ii), (iii) we refer to [10].

We comment on a proof of (iv). Therefore, we follow [8] and work with the norm $\|\cdot\|^*_{M^s_{p,q}}$. From the assumption, we obtain that for all $k\in\mathbb{Z}^n$ and $x\in\mathbb{R}^n$,

$$\mathcal{F}^{-1}\,[\sigma_k\,\mathcal{F}f_m](x) = (2\pi)^{-n/2}\, f_m(x-\cdot)(\sigma_k) \to f(x-\cdot)(\sigma_k) = \mathcal{F}^{-1}\,[\sigma_k\,\mathcal{F}f](x)$$

as $m \to \infty$. Fatou's lemma yields

$$\sum_{|k|\le N} \Big(\int_{\mathbb{R}^n} |\mathcal{F}^{-1}[\sigma_k \mathcal{F} f](x)|^p dx \Big)^{\frac{q}{p}} \le \liminf_{m\to\infty} \sum_{|k|\le N} \Big(\int_{\mathbb{R}^n} |\mathcal{F}^{-1}[\sigma_k \mathcal{F} f_m](x)|^p dx \Big)^{\frac{q}{p}} .$$

An obvious monotonicity argument completes the proof. ∎

2.2 Embeddings

Obviously the spaces $M^s_{p,q}$ are monotone in s and q. But they are also monotone with respect to p. To show this we recall Nikol'skij's inequality, see e.g., Nikol'skij [21, 3.4] or Triebel [33, 1.3.2].

Lemma 2.6 *Let $1 \le p \le q \le \infty$ and f be an integrable function with* $\operatorname{supp} \mathcal{F} f \subset B(y, r)$*, i.e., the support of the Fourier transform of f is contained in a ball with radius $r > 0$ and center in $y \in \mathbb{R}^n$. Then it holds*

$$\| f \|_{L_q} \le C r^{n(\frac{1}{p}-\frac{1}{q})} \| f \|_{L_p}$$

with a constant $C > 0$ independent of r and y.

This implies $\|\Box_k f\|_{L_q} \le c\, \|\Box_k f\|_{L_p}$ if $p \le q$ with c independent of k and f which results in the following corollary (by using the norm $\|\cdot\|^*_{M^s_{p,q}}$).

Corollary 2.7 *Let $s_0 > s$, $p_0 < p$ and $q_0 < q$. Then the following embeddings hold and are continuous:*

$$M^{s_0}_{p,q} \hookrightarrow M^s_{p,q}\,, \qquad M^s_{p_0,q} \hookrightarrow M^s_{p,q}$$

and

$$M^s_{p,q_0} \hookrightarrow M^s_{p,q}\,;$$

i.e., for all p, q, $1 \le p, q \le \infty$, we have

$$M^s_{1,1} \hookrightarrow M^s_{p,q} \hookrightarrow M^s_{\infty,\infty}\,.$$

Of some importance are embeddings with respect to different metrics. To find sufficient conditions is not difficult when working with $\|\cdot\|^*_{M^s_{p,q}}$. A bit more tricky are the necessity parts. We refer to the recent paper by Guo et al. [11].

Proposition 2.8 *Let $s_0, s_1 \in \mathbb{R}$ and $1 \le p_0, p_1 \le \infty$. Then*

$$M^{s_0}_{p_0,q_0} \hookrightarrow M^{s_1}_{p_1,q_1}$$

holds if and only if either

- $p_0 \le p_1$ *and* $s_0 - s_1 > n\Big(\frac{1}{q_1} - \frac{1}{q_0}\Big)$
- *or* $p_0 \le p_1$, $s_0 = s_1$ *and* $q_0 = q_1$.

Remark 2.9 Embeddings of modulation spaces are treated at various places, we refer to Feichtinger [7], Wang and Hudzik [37], Cordero and Nicola [6], Iwabuchi [13] and Guo et al. [11].

The weighted modulation spaces $M^s_{p,q}$ cannot distinguish between boundedness and continuity (as Besov spaces). Let C_{ub} denote the class of all uniformly continuous and bounded functions $f : \mathbb{R}^n \to \mathbb{C}$ equipped with the supremum norm. If $f \in M^s_{p,q}$ is a regular distribution it is determined (as a function) almost everywhere. We shall say that f is a continuous function if there is one continuous function g which equals f almost everywhere.

Corollary 2.10 *Let $s \in \mathbb{R}$ and $1 \le p, q \le \infty$. Then the following assertions are equivalent:*

- $M^s_{p,q} \hookrightarrow L_\infty$;
- $M^s_{p,q} \hookrightarrow C_{ub}$;
- $M^s_{p,q} \hookrightarrow M^0_{\infty,1}$;
- *either* $s \ge 0$ *and* $q = 1$ *or* $s > n/q'$.

Proof We shall work with $\|\cdot\|^*_{M^s_{p,q}}$.
Step 1. Sufficiency. By Proposition 2.8 it will be enough to show $M^0_{\infty,1} \hookrightarrow C_{ub}$. From the definition of $M^0_{\infty,1}$ it follows that

$$\sum_{k\in\mathbb{Z}^n} \Box_k f(x)$$

is pointwise convergent (for all $x \in \mathbb{R}^n$). Furthermore, since $\Box_k f \in C^\infty$, there is a continuous representative in the equivalence class f, given by $\sum_{k\in\mathbb{Z}^n} \Box_k f(x)$. In what follows, we shall work with this representative. Boundedness of $f \in M^0_{\infty,1}$ is obvious, we have

$$|f(x)| = |\sum_{k\in\mathbb{Z}^n} \Box_k f(x)| \le \| f \|_{M^0_{\infty,1}} .$$

It remains to prove uniform continuity. For fixed $\varepsilon > 0$ we choose N such that

$$\sum_{|k|>N} \| \Box_k f \|_{L_\infty} < \varepsilon/2 .$$

In case $|k| \le N$ we observe that

$$|\Box_k f(x) - \Box_k f(y)| \le \| \nabla(\Box_k f) \|_{L_\infty} |x-y| \, .$$

It follows from [33, Theorem 1.3.1] that

$$\| \nabla(\Box_k f) \|_{L_\infty} \le c_1 \, \| (M \Box_k f) \|_{L_\infty}$$

with a constant c_1 independent of f and k. Here M denotes the Hardy–Littlewood maximal function. In the quoted reference, the assumption $\Box_k f \in \mathcal{S}$ is used. A closer look at the proof shows that $\Box_k f \in L_1^{\ell oc}$ satisfying

$$\int_{Q_k} |\Box_k f(x)| \, dx \le c_2 \, (1+|k|)^N \, , \qquad k \in \mathbb{Z}^n \, ,$$

for some $N \in \mathbb{N}$ is sufficient. Since $\Box_k f \in L_\infty$ this is obvious. Consequently we obtain

$$\begin{aligned} |\Box_k f(x) - \Box_k f(y)| &\le c_1 \, \| (M \Box_k f) \|_{L_\infty} \, |x-y| \le c_1 \, \| \Box_k f \|_{L_\infty} \, |x-y| \\ &\le c_2 \, \| f \|_{L_\infty} \, |x-y| \, , \end{aligned}$$

where in the last step we used the standard convolution inequality $\| g * h \|_{L_\infty} \le \| g \|_{L_1} \| f \|_{L_\infty}$. This implies uniform continuity of $\Box_k f$ and therefore of $\sum_{|k| \le N} \Box_k f$. In particular, we find

$$\begin{aligned} |f(x) - f(y)| &= \Big| \sum_{k \in \mathbb{Z}^n} (\Box_k f(x) - \Box_k f(y)) \Big| \\ &\le \sum_{|k|>N} (|\Box_k f(x)| + |\Box_k f(y)|) + c_2 \, \| f \|_{L_\infty} \, |x-y| \sum_{|k| \le N} 1 \\ &\le \varepsilon + c_2 \, \| f \|_{L_\infty} \, |x-y| \, (2N+1)^n \, . \end{aligned}$$

Choosing $\delta = (c_2 \, \| f \|_{L_\infty} \, (2N+1)^n)^{-1} \, \varepsilon$ we arrive at

$$|f(x) - f(y)| < 2 \, \varepsilon \qquad \text{if} \qquad |x-y| < \delta \, .$$

Step 2. Necessity. Let $\psi \in \mathcal{S}$ be a real-valued function such that $\psi(0) = 1$ and

$$\operatorname{supp} \mathcal{F}\psi \subset \{\xi : \max_{j=1,\ldots,n} |\xi_j| < \varepsilon\} \qquad \text{with} \quad \varepsilon < 1/2.$$

We define f by

$$\mathcal{F} f(\xi) := \sum_{k \in \mathbb{Z}^n} a_k \, \mathcal{F}\psi(\xi - k) \, .$$

Clearly,

$$\Box_k f(x) = a_k\, e^{ikx}\, \psi(x)\,, \qquad k \in \mathbb{Z}^n\,.$$

Substep 2.1. Let $s = 0$ and $1 \le p \le \infty$. The above arguments imply $f \in M^0_{p,q}$ if and only if $(a_k)_k \in \ell_q$. On the other hand,

$$f(x) = \psi(x) \sum_{k \in \mathbb{Z}^n} a_k\, e^{ikx} \tag{2.1}$$

which implies that f is unbounded in 0 if $\sum_{k\in\mathbb{Z}^n} a_k = \infty$. Choosing

$$a_k := \begin{cases} (k_1\, \log(2+k_1))^{-1} & \text{if} \quad k_1 \in \mathbb{N}\,, \quad k = (k_1, 0, \ldots\, 0)\,; \\ 0 & \text{otherwise;} \end{cases}$$

then $f \in M^0_{p,q} \setminus L_\infty$, $q > 1$, follows.

Substep 2.2. Let $1 \le p \le \infty$ and $q = \infty$. Then we choose $a_k := \langle k \rangle^{-n}$. It follows $f \in M^n_{p,\infty}$ but $f(0) = +\infty$.

Substep 2.3. Let $1 \le p \le \infty$, $1 < q < \infty$ and $s = n/q'$. Then, with $\delta > 0$, we choose

$$a_k := \begin{cases} \langle k \rangle^{-n}\ \ (\log\langle k \rangle)^{-(1+\delta)/q} & \text{if} \quad |k| > 0\,; \\ 0 & \text{otherwise.} \end{cases}$$

It follows

$$\begin{aligned} \|\, f\,\|_{M^{n/q'}_{p,q}} &= \|\psi\|_{L_p} \left(\sum_{|k|>0} \langle k \rangle^{-nq+nq/q'}\ (\log\langle k \rangle)^{-(1+\delta)} \right)^{\frac{1}{q}} \\ &= \|\psi\|_{L_p} \left(\sum_{|k|>0} \langle k \rangle^{-n}\ (\log\langle k \rangle)^{-(1+\delta)} \right)^{\frac{1}{q}} < \infty\,. \end{aligned}$$

On the other hand, we have

$$f(0) = \sum_{|k|>0} \langle k \rangle^{-n}\ \ (\log\langle k \rangle)^{-(1+\delta)/q} = \infty$$

if $(1+\delta)/q \le 1$. Hence, for choosing $\delta = q - 1$ the claim follows. ■

Remark 2.11 Sufficient conditions for embeddings of modulation spaces into spaces of continuous functions can be found at several places, in particular in Feichtinger's original paper [7]. We did not find references for the necessity.

3 Pointwise Multiplication in Modulation Spaces

We are interested in embeddings of the type

$$M^{s_1}_{p,q} \cdot M^{s_2}_{p,q} \hookrightarrow M^{s_0}_{p,q} ,$$

where s_1, s_2, p and q are given and we are asking for an optimal s_0. These results will be applied in connection with our investigations on the regularity of compositions of functions in Sect. 4. However, several times we shall deal with the slightly more general problem

$$M^{s_1}_{p_1,q} \cdot M^{s_2}_{p_2,q} \hookrightarrow M^{s_0}_{p,q} , \qquad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} .$$

In view of Corollary 2.7 this always yields

$$M^{s_1}_{p_1,q} \cdot M^{s_2}_{p_2,q} \hookrightarrow M^{s_0}_{p,q} , \qquad \frac{1}{p} \le \frac{1}{p_1} + \frac{1}{p_2} .$$

For convenience of the reader we also recall what is known in the more general situation

$$M^{s_1}_{p_1,q_1} \cdot M^{s_2}_{p_2,q_2} \hookrightarrow M^{s_0}_{p,q} .$$

At first we shall deal with the algebra property. Afterwards we turn to the existence of the product in more general situations.

3.1 On the Algebra Property

The main aim consists in giving necessary and sufficient conditions for the embedding $M^s_{p,q} \cdot M^s_{p,q} \hookrightarrow M^s_{p,q}$. To prepare this we recall a nice identity due to Toft [30], see also Sugimoto et al. [29].

Lemma 3.1 *Let $\varphi_1, \varphi_2 \in \mathcal{S}$ be nontrivial. Let $f, g \in L_2^{\ell oc}$ such that there exist $c > 0$ and $M > 0$ with*

$$\int_{Q_k} |f(x)|^2 + |g(x)|^2 \, dx \le c \, (1 + |k|)^M , \qquad k \in \mathbb{Z}^n .$$

For all $x, \xi \in \mathbb{R}^n$ the following identity takes place

$$V_{\varphi_1 \cdot \varphi_2}(fg)(x,\xi) = (2\pi)^{-n/2} \int V_{\varphi_1}(f)(x, \xi - \eta) \, V_{\varphi_2}(g)(x,\eta) \, d\eta . \tag{3.1}$$

Proof The main tool will be the Plancherel identity. Observe, that for any fixed $x \in \mathbb{R}^n$ the functions $f(t)\,\overline{\varphi_1(t-x)}$, $\overline{g(t)}\,\varphi_2(t-x)$ belong to L_2 and therefore their Fourier transforms as well. For brevity we put

$$I := \int V_{\varphi_1}(f)(x, \xi - \eta)\, V_{\varphi_2}(g)(x, \eta)\, d\eta\,.$$

Applying the Plancherel identity, we conclude

$$\begin{aligned} I &= \int \mathcal{F}(f(t)\,\overline{\varphi_1(t-x)}\, e^{-i\xi t})(-\eta)\, \overline{\mathcal{F}(\overline{g(t)}\,\varphi_2(t-x))(-\eta)}\, d\eta \\ &= \int f(t)\,\overline{\varphi_1(t-x)}\, e^{-i\xi t}\, \overline{\overline{g(t)}\,\varphi_2(t-x))}\, dt \\ &= \int f(t)\, g(t)\, \overline{\varphi_1(t-x)\,\varphi_2(t-x)}\, e^{-i\xi t}\, dt \\ &= (2\pi)^{n/2}\, V_{\varphi_1 \cdot \varphi_2}(fg)(x, \xi)\,. \end{aligned}$$

The proof is complete. ■

Remark 3.2 It is clear that the assertion does not extend very much. For example, if $f, g \in L_p^{\ell oc}$ for some $p < 2$ then the above claim is not true. We may take

$$f(x) = g(x) = \psi(x)\, |x|^{-n/2}\,, \qquad x \in \mathbb{R}^n\,,$$

where ψ is a smooth and compactly supported cut-off function s.t. $\psi(0) = 1$. Then $f \cdot g$ is not longer a distribution, i.e., the integral

$$V_{\varphi_1 \cdot \varphi_2}(fg)(x, \xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(s)\, g(s)\, \overline{\varphi_1(s-x)\,\varphi_2(s-x)}\, e^{-is\cdot\xi}\, ds$$

does not make sense in general.

In [29, 30], the identity (3.1) is applied either in case $f, g \in \mathcal{S}$ or $f, g \in L_\infty$. Here, we shall apply it in the wider context of Lemma 3.1.

Lemma 3.3 *Let* $1 \le p, q \le \infty$ *and assume* $M^s_{p,q} \hookrightarrow M^0_{\infty,1}$. *Then there exists a constant c such that*

$$\| f \cdot g \|_{M^s_{p,q}} \le c \Big(\| f \|_{M^0_{\infty,1}}\, \| g \|_{M^s_{p,q}} + \| f \|_{M^s_{p,q}}\, \| g \|_{M^0_{\infty,1}} \Big)$$

holds for all $f, g \in M^s_{p,q}$.

Proof The main idea in the proof consists in the fact that the modulation space can be characterized by different window functions. Since $M^0_{\infty,1} \hookrightarrow L_\infty$ we know that f, g satisfy the conditions in Lemma 3.1. Hence

$$\| f \cdot g \|_{M^s_{p,q}} = \Big\{ \int \Big[\int \Big| V_{\varphi^2}(f \cdot g)(x,\xi) \Big|^p dx \Big]^{q/p} \langle \xi \rangle^{sq} d\xi \Big\}^{1/q} \tag{3.2}$$
$$\leq \Big\{ \int \Big[\int \Big| \int V_\varphi f(x,\xi-\eta) V_\varphi g(x,\eta) d\eta \Big|^p dx \Big]^{q/p} \langle \xi \rangle^{sq} d\xi \Big\}^{1/q} .$$

We split the integration with respect to η into two parts

$$\Omega_\xi := \{\eta : \ |\xi-\eta| \geq |\eta|\} \quad \text{and} \quad \Gamma_\xi := \{\eta : \ |\xi-\eta| < |\eta|\} , \quad \xi \in \mathbb{R}^n . \tag{3.3}$$

It follows

$$\| f \cdot g \|_{M^s_{p,q}} \leq 2^s \, (A + B) ,$$

where

$$A := \Big\{ \int \Big[\int \Big| \int_{\Omega_\xi} V_\varphi f(x,\xi-\eta) \, (1+|\xi-\eta|^2)^{s/2} V_\varphi g(x,\eta) \, d\eta \Big|^p dx \Big]^{q/p} d\xi \Big\}^{1/q}$$

and

$$B := \Big\{ \int \Big[\int \Big| \int_{\Gamma_\xi} V_\varphi f(x,\xi-\eta) \, V_\varphi g(x,\eta) \, (1+|\eta|^2)^{s/2} d\eta \Big|^p dx \Big]^{q/p} d\xi \Big\}^{1/q} .$$

We continue by applying the generalized Minkowski inequality, see [18, Theorem 2.4]. This yields

$$A \leq \int \Big\{ \int \Big[\int |V_\varphi f(x,\xi-\eta) \, \langle \xi-\eta \rangle^s \, V_\varphi g(x,\eta) \, |^p \, dx \Big]^{q/p} d\xi \Big\}^{1/q} d\eta$$
$$\leq \int \Big(\sup_{x\in\mathbb{R}^n} |V_\varphi g(x,\eta)| \Big) \Big\{ \int \Big[\int |V_\varphi f(x,\xi-\eta) \, \langle \xi-\eta \rangle^s \, |^p \, dx \Big]^{q/p} d\xi \Big\}^{1/q} d\eta$$
$$= \| g \|_{M^0_{\infty,1}} \, \| f \|_{M^s_{p,q}} .$$

Analogously one can prove

$$B \leq \| f \|_{M^0_{\infty,1}} \, \| g \|_{M^s_{p,q}} .$$

The proof is complete. ■

Remark 3.4 (i) We proved a bit more than stated. In fact, we have shown

$$\| f \cdot g \|_{M^s_{p,q}} \leq 2^s \Big(\| f \|_{M^0_{\infty,1}} \, \| g \|_{M^s_{p,q}} + \| f \|_{M^s_{p,q}} \, \| g \|_{M^0_{\infty,1}} \Big)$$

But here one has to notice that the norm on the left-hand side is generated by the window φ^2, whereas the norms on the right-hand side are generated by the window φ.

(ii) Lemma 3.3 has been proved by Sugimoto et al. [29]. For partial results with a different proof we refer to Feichtinger [7].

Next, we turn to necessary and sufficient conditions for the algebra property.

Theorem 3.5 *Let $1 \le p, q \le \infty$ and $s \in \mathbb{R}$. Then $M^s_{p.q}$ is an algebra with respect to pointwise multiplication if and only if either $s \ge 0$ and $q = 1$ or $s > n/q'$.*

Remark 3.6 (i) By Corollary 2.10 the Theorem 3.5 can be reformulated as

$$M^s_{p.q} \quad \text{is an algebra} \qquad \Longleftrightarrow \qquad M^s_{p.q} \hookrightarrow L_\infty\,.$$

This is in some sense natural because otherwise one could increase local singularities by pointwise multiplication.
(ii) Theorem 3.5 has a partial counterpart for Besov spaces. Here one knows that $B^s_{p,q}$ is an algebra if and only if $B^s_{p,q} \hookrightarrow L_\infty$ and $s > 0$. We refer to Peetre [22, Theorem 11, p. 147], Triebel [33, Theorem 2.8.3] (sufficiency) and to [25, Theorem 4.6.4/1] (necessity).

To prepare the proof, we need the following lemma which is of interest for its own.

Lemma 3.7 *Let $1 \le p, q < \infty$ and $s \in \mathbb{R}$. Let $f \in \mathcal{S}'$ and let there exists a constant $c > 0$ such that*

$$\| f \cdot g\|_{M^s_{p,q}} \le c\, \|g\|_{M^s_{p,q}}$$

holds for all $g \in \mathcal{S}$. Then $f \in L_\infty$ follows.

Proof Let $T_f(g) := f \cdot g$, $g \in \mathcal{S}$. Let $\mathring{M}^s_{p,q}$ denote the closure of $\mathcal{S}$ in $M^s_{p,q}$. Hence, there is a unique extension of T_f to a continuous operator belonging to $\mathcal{L}(\mathring{M}^s_{p,q}, M^s_{p,q})$. Next we employ duality. We fix p, q and s ($1 \le p, q \le \infty$, $s \in \mathbb{R}$). Let (g, h) denote the standard dual pairing on $\mathcal{S}' \times \mathcal{S}$. Then

$$\|g\| := \sup \Big\{ |(g, h)| \ : \quad h \in \mathcal{S},\ \|h\|_{M^{-s}_{p',q'}} \le 1 \Big\}$$

is an equivalent norm on $M^s_{p,q}$, see Feichtinger [7] or Toft [30]. In view of this equivalent norm our assumption on T_f implies $\mathcal{L}(M^{-s}_{p',q'}, M^{-s}_{p',q'})$. Next we continue by complex interpolation. Let $0 < \Theta < 1$. It is known that

$$M^s_{p,q} = [M^{s_1}_{p_1,q_1}, M^{s_2}_{p_2,q_2}]_\Theta$$

if $1 \le p_1, q_1 < \infty$, $1 \le p_2, q_2 \le \infty$, $s_1, s_2 \in \mathbb{R}$ and

$$s := (1-\Theta)s_1 + \Theta\, s_2\,, \quad \frac{1}{p} := \frac{1-\Theta}{p_1} + \frac{\Theta}{p_2}\,, \quad \frac{1}{q} := \frac{1-\Theta}{q_1} + \frac{\Theta}{q_2}\,,$$

see Feichtinger [7]. Thanks to the interpolation property of the complex method we conclude

$$T_f \in \mathcal{L}\Big([\mathring{M}^s_{p,q}, M^{-s}_{p',q'}]_{1/2}, [\mathring{M}^s_{p,q}, M^{-s}_{p',q'}]_{1/2}\Big)\,.$$

Because of $\mathring{M}^s_{p,q} = M^s_{p,q}$ if $\max(p,q) < \infty$ we find

$$T_f \in \mathcal{L}\Big(M^0_{2,2}, M^0_{2,2}\Big) = \mathcal{L}\Big(L_2, L_2\Big)\,.$$

But this implies $f \in L_\infty$. ■

Proof of Theorem 3.5.
Step 1. Sufficiency is covered by Lemma 3.3.
Step 2. Necessity in case $1 \le p, q < \infty$ and $s \in \mathbb{R}$. In view of Lemma 3.7 the embedding $M^s_{p,q} \cdot M^s_{p,q} \hookrightarrow M^s_{p,q}$ implies $M^s_{p,q} \subset L_\infty$.
Step 3. To treat the remaining cases $\max(p,q) = \infty$ we argue by using explicit counterexamples.

Substep 3.1. Let $1 \le p \le \infty$, $s = 0$ and $1 < q \le \infty$. We assume that $M^0_{p,q}$ is an algebra. This implies the existence of a constant $c > 0$ such that

$$\|\, f \cdot g \,\|_{M^0_{p,q}} \le c \,\|\, f \,\|_{M^0_{p,q}} \,\|\, g \,\|_{M^0_{p,q}} \qquad (3.4)$$

holds for all $f, g \in M^0_{p,q}$. Let

$$f(x) = \psi(x) \sum_{k=1}^{\infty} a_k\, e^{ikx_1}\,, \qquad x = (x_1, \ldots, x_n) \in \mathbb{R}^n\,,$$

be as in (2.1). Then, as shown above,

$$\|\, f \,\|_{M^0_{p,q}} = \|\psi\|_{L_p}\, \|(a_k)_k\|_{\ell_q}$$

follows. Let

$$f_N(x) := \psi(x) \sum_{k=1}^{N} a_k\, e^{ikx_1}\,, \qquad x = (x_1, \ldots, x_n) \in \mathbb{R}^n\,, \quad N \in \mathbb{N}\,.$$

Obviously $f_N \in \mathcal{S}$. We assume that

$$\operatorname{supp} \mathcal{F}\psi \subset \{\xi : \max_{j=1,\ldots,n} |\xi_j| < \varepsilon\} \qquad \text{with} \quad \varepsilon < 1/4.$$

Then, because of

$$f_N(x) \cdot f_N(x) = \psi^2(x) \sum_{m=2}^{2N} \Big(\sum_{k=1}^{m-1} a_k\, a_{m-k} \Big) e^{imx} ,$$

we conclude

$$\| f_N \cdot f_N \|_{M^0_{p,q}} = \| \psi^2 \|_{L_p} \bigg(\sum_{m=2}^{2N} \Big| \sum_{k=1}^{m-1} a_k\, a_{m-k} \Big|^q \bigg)^{1/q} .$$

Inequality (3.4) implies

$$\bigg(\sum_{m=2}^{2N} \Big| \sum_{k=1}^{m-1} a_k\, a_{m-k} \Big|^q \bigg)^{1/q} \le c\, \frac{\|\psi\|^2_{L_p}}{\|\psi^2\|_{L_p}} \Big(\sum_{k=1}^{N} |a_k|^q \Big)^{2/q} .$$

Clearly, in case $q > 1$ this is impossible in this generality. Explicit counterexamples are given by

$$a_k := k^{-1/q} \qquad \text{if} \quad 1 < q < \infty$$

and

$$a_k = 1 \qquad \text{if} \quad q = \infty\, .$$

In case $1 < q < \infty$ (3.4) yields

$$\| f_N \cdot f_N \|_{M^0_{p,q}} \asymp N^{1-1/q} \qquad \text{and} \qquad \| f_N \|^2_{M^0_{p,q}} \asymp (\log N)^{2/q} .$$

For $q = \infty$ we obtain

$$\| f_N \cdot f_N \|_{M^0_{p,\infty}} \asymp N \qquad \text{and} \qquad \| f_N \|^2_{M^0_{p,\infty}} \asymp 1 .$$

For $N \to \infty$ we find a contradiction in both situations.

Substep 3.2. Let $1 \le p \le \infty$, $q = \infty$ and $0 < s \le n$. We argue as in Substep 3.1 and assume $M^s_{p,\infty}$ is an algebra with respect to pointwise multiplication. This leads to the existence of a constant $c > 0$ such that

$$\| f \cdot g \|_{M^s_{p,\infty}} \le c\, \| f \|_{M^s_{p,\infty}} \| g \|_{M^s_{p,\infty}}$$

holds for all $f, g \in M^s_{p,\infty}$. We choose

$$f(x) = g(x) = f_N(x) := \psi(x) \sum_{\|k\|_\infty \le N} a_k\, e^{ikx} , \qquad x \in \mathbb{R}^n ,$$

and obtain

$$\| f_N \|_{M^s_{p,q}} = \|\psi\|_{L_p} \Big(\sum_{\|k\|_\infty \le N} |a_k \, \langle k\rangle^s|^q \Big)^{1/q} ,$$

$$\| f_N \cdot f_N \|_{M^s_{p,q}} = \|\psi^2\|_{L_p} \Bigg(\sum_{\|m\|_\infty \le 2N} \langle m\rangle^{sq} \Big| \sum_{\substack{k:\ \|k\|_\infty \le N \\ \|m-k\|_\infty \le N}} a_k \, a_{m-k} \Big|^q \Bigg)^{1/q} .$$

In case $s < n$ we choose $a_k := 1$ for all k and obtain

$$\| f_N \cdot f_N \|_{M^s_{p,\infty}} \asymp N^{n+s} \qquad \text{and} \qquad \| f_N \|^2_{M^s_{p,\infty}} \asymp N^{2s} .$$

This yields a contradiction if $s < n$. For $s = n$ we consider $a_k := \langle k\rangle^{-n}$ for all k. This yields

$$\log N \lesssim \| f_N \cdot f_N \|_{M^n_{p,\infty}} \qquad \text{and} \qquad \| f_N \|^2_{M^n_{p,\infty}} \asymp 1 ,$$

yielding a contradiction as well.

Substep 3.3. Let $s < 0$ and $1 \le p, q \le \infty$. We choose $a_k := \langle k\rangle^{2|s|}$ for all k and obtain

$$N^{3|s|+n+n/q} \lesssim \| f_N \cdot f_N \|_{M^s_{p,q}} \qquad \text{and} \qquad \| f_N \|^2_{M^s_{p,q}} \asymp N^{2|s|+2n/q} .$$

For $N \to \infty$ this implies $|s| + n \le n/q$. Since $|s| > 0$ this is impossible. The proof is complete. ∎

Corollary 3.8 *Let $1 \le p, q \le \infty$ and $s \ge 0$. Then $M^s_{p,q} \cap M^0_{\infty,1}$ is an algebra with respect to pointwise multiplication and there exist a constant c such that*

$$\| f \cdot g \|_{M^s_{p,q}} \le c \Big(\| f \|_{M^0_{\infty,1}} \| g \|_{M^s_{p,q}} + \| f \|_{M^s_{p,q}} \| g \|_{M^0_{\infty,1}} \Big)$$

holds for all $f, g \in M^s_{p,q} \cap M^0_{\infty,1}$.

Proof The same arguments as in Lemma 3.3 apply. ∎

Remark 3.9 Corollary 3.8 has a counterpart for Besov spaces. Here, one knows that $B^s_{p,q} \cap L_\infty$ is an algebra if $1 \le p, q \le \infty$ and $s > 0$. We refer to Peetre [22, Theorem 11, p. 147] and to [25, Theorem 4.6.4/2].

3.2 More General Products of Functions

Here, we consider the problem

$$M^{s_1}_{p_1,q_1} \cdot M^{s_2}_{p_2,q} \hookrightarrow M^s_{p,q} .$$

As a first result, we mention a generalization of Lemma 3.3.

Lemma 3.10 *Let* $1 \le p_1, p_2, q \le \infty$ *and* $s \ge 0$. *We put* $1/p := (1/p_1) + (1/p_2)$. *If* $p \in [1, \infty]$, *then there exists a constant c such that*

$$\| f \cdot g \|_{M^s_{p,q}} \le c \Big(\| f \|_{M^0_{p_1,1}} \| g \|_{M^s_{p_2,q}} + \| f \|_{M^s_{p_1,q}} \| g \|_{M^0_{p_2,1}} \Big)$$

holds for all $f \in M^s_{p_1,q} \cap M^0_{p_1,1}$ *and all* $g \in M^s_{p_2,q} \cap M^0_{p_2,1}$.

Proof We argue similar as above but using Hölder's inequality with respect to p before applying the generalized Minkowski inequality. ■

Remark 3.11 Observe that $M^0_{p_1,1}, M^0_{p_2,1} \hookrightarrow M^0_{\infty,1} \hookrightarrow L_\infty$.

Lemma 3.12 *Let* $1 \le p_1, p_2, q \le \infty$ *and* $s \le 0$. *We put* $1/p := (1/p_1) + (1/p_2)$. *If* $p \in [1, \infty]$, *then there exists a constant c such that*

$$\| f \cdot g \|_{M^s_{p,q}} \le c \, \| f \|_{M^{|s|}_{p_1,1}} \| g \|_{M^s_{p_2,q}} \tag{3.5}$$

holds for all $f \in M^{|s|}_{p_1,1}$, $g \in M^s_{p_2,q}$ *such that g satisfies* $g \in L^{\ell oc}_2$ *and*

$$\int_{Q_k} |g(x)|^2 \, dx \le C \, (1 + |k|)^M , \tag{3.6}$$

for some $C > 0$ *and* $M > 0$ *independent of* $k \in \mathbb{Z}^n$.

Proof Point of departure is the formula (3.2). Instead of the splitting in (3.3) we use now the elementary inequality

$$1 + |\eta|^2 \le 2 \, (1 + |\xi|^2) \, (1 + |\xi - \eta|^2)$$

which implies

$$(1 + |\xi|^2)^{s/2} \le 2^{|s|/2} \, (1 + |\xi - \eta|^2)^{|s|/2} \, (1 + |\eta|^2)^{s/2} .$$

This leads to the estimate

$$\begin{aligned} & \| f \cdot g \|_{M^s_{p,q}} \\ & \lesssim \Big\{ \int \Big[\int \Big| \int V_\varphi f(x, \xi - \eta) \, \langle \xi - \eta \rangle^{|s|} \, V_\varphi g(x, \eta) \, \langle \eta \rangle^s \, d\eta \Big|^p dx \Big]^{q/p} d\xi \Big\}^{1/q} \\ & = \Big\{ \int \Big[\int \Big| \int V_\varphi f(x, \tau) \, \langle \tau \rangle^{|s|} \, V_\varphi g(x, \xi - \tau) \, \langle \xi - \tau \rangle^s \, d\tau \Big|^p dx \Big]^{q/p} d\xi \Big\}^{1/q} . \end{aligned}$$

We continue by applying the generalized Minkowski inequality and Hölder's inequality (with respect to p) and obtain

$$\begin{aligned}
&\| f \cdot g \|_{M^s_{p,q}} \\
&\lesssim \int \Big\{ \int \Big[\| V_\varphi f(x,\tau) \, \langle \tau \rangle^{|s|} \|_{L_{p_1}} \; \| V_\varphi g(x, \xi - \tau) \, \langle \xi - \tau \rangle^s \|_{L_{p_2}} \Big]^q \, d\xi \Big\}^{1/q} d\tau \\
&\lesssim \int \| V_\varphi f(x,\tau) \, \langle \tau \rangle^{|s|} \|_{L_{p_1}} d\tau \; \| g \|_{M^s_{p_2,q}} \\
&\lesssim \| f \|_{M^{|s|}_{p_1,1}} \, \| g \|_{M^s_{p_2,q}} \, .
\end{aligned}$$

■

Remark 3.13 Observe that $M^{|s|}_{p_1,1} \hookrightarrow M^{|s|}_{\infty,1} \hookrightarrow L_\infty$. In addition we would like to mention that the constant c in (3.5) does not depend on the constant C in (3.6).

We recall a final result of Cordero and Nicola [6] concentrating on $s = 0$. These authors study $M^0_{p_1,q_1} \cdot M^0_{p_2,q_2} \hookrightarrow M^0_{p,q}$.

Proposition 3.14 *Let* $1 \le p_1, p_2, q_1, q_2 \le \infty$. *Then* $M^0_{p_1,q_1} \cdot M^0_{p_2,q_2} \hookrightarrow M^0_{p,q}$ *holds if and only if*

$$\frac{1}{p} \le \frac{1}{p_1} + \frac{1}{p_2} \qquad \textit{and} \qquad 1 + \frac{1}{q} \le \frac{1}{q_1} + \frac{1}{q_2} \, .$$

Remark 3.15 (i) Proposition 3.14 shows that in case $s = 0$ in Lemma 3.12 we proved an optimal estimate.
(ii) Necessity of the restrictions in Proposition 3.14 is shown by studying products of Gaussian functions. For extensions of Proposition 3.14 to the case of products with more than two factors we refer to Guo et al. [11] and Toft [30].

3.3 *Products of a Distribution with a Function*

Up to now, we considered only products of either L_∞-functions or $L_2^{\ell oc}$-functions with L_∞-functions. But now we turn to the product of a distribution with a function which is not assumed to be C^∞. This requires a definition.

The Definition of the Product in $\mathcal{S}'$

Let $\psi \in \mathcal{S}$ be a function in C_0^∞ such that $\psi(\xi) = 1$ in a neighbourhood of the origin. We define

$$S^j f(x) = \mathcal{F}^{-1}[\psi(2^{-j}\xi) \, \mathcal{F} f(\xi)](x), \quad j = 0, 1, \ldots \, .$$

The Paley-Wiener theorem tells us that $S^j f$ is anentire analytic function of exponential type. Hence, if $f, \; g \in \mathcal{S}'$ the products $S^j f \cdot S^j g$ makes sense for any j. Further,

$$\lim_{j\to\infty} \mathcal{F}^{-1}[\psi(2^{-j}\xi)\, \mathcal{F}f(\xi)](\cdot) = f \qquad \text{(convergence in } \mathcal{S}')$$

for any $f \in \mathcal{S}'$.

Definition 3.16 Let $f,\ g \in \mathcal{S}'$. We define

$$f \cdot g = \lim_{j\to\infty} S^j f \cdot S^j g$$

whenever the limit on the right-hand side exists in $\mathcal{S}'$. We call $f \cdot g$ the product of f and g.

Remark 3.17 In defining the product we followed a usual practice, see e.g., [22], [33, 2.8], [14, 15] and [25, 4.2]. For basic properties of this notion, we refer to [14, 15] and [25, 4.2].

Theorem 3.18 *Let* $1 \le p_1, p_2, q \le \infty$ *and* $s \le 0$. *We put* $1/p := (1/p_1) + (1/p_2)$. *If* $p \in [1, \infty]$, *then there exists a constant c such that*

$$\| f \cdot g \|_{M^s_{p,q}} \le c \, \| f \|_{M^{|s|}_{p_1,1}} \, \| g \|_{M^s_{p_2,q}})$$

holds for all $f \in M^{|s|}_{p_1,1}$ *and* $g \in M^s_{p_2,q}$.

Proof We have to show that the limit of $(S^j f \cdot S^j g)_j$ exists in $\mathcal{S}'$. The remaining assertions, $\lim_{j\to\infty} S^j f \cdot S^j g \in M^s_{p,q}$ and the norm estimates will follow by employing the Fatou property, see Lemmas 2.5 and 3.12.
Step 1. Let $1 \le q < \infty$. We have

$$\lim_{j\to\infty} \|S^j f - f\|_{M^s_{p,q}} = 0 \qquad \text{for all} \quad f \in M^s_{p,q}\,.$$

In addition it is easily seen that

$$\sup_{j\in\mathbb{N}_0} \|S^j f\|_{M^s_{p,q}} \le \|\mathcal{F}^{-1}\psi\|_{L_1} \, \| f \|_{M^s_{p,q}} \tag{3.7}$$

holds for all $f \in M^s_{p,q}$. Hence, we conclude by means of Lemma 3.12

$$\begin{aligned}
&\|S^k f\, S^k g - S^j f\, S^j g\|_{M^s_{p,q}} \\
&\qquad \le \|(S^k f - S^j f)\, S^k g\|_{M^s_{p,q}} + \|S^j f\, (S^k g - S^j g)\|_{M^s_{p,q}} \\
&\qquad \le c\, (\| S^k g \|_{M^s_{p_2,q}} \, \| S^k f - S^j f \|_{M^{|s|}_{p_1,1}} + \| S^k g - S^j g \|_{M^s_{p_2,q}} \, \| S^j f \|_{M^{|s|}_{p_1,1}})
\end{aligned}$$

the convergence of $(S^k f \cdot S^k g)_k$ in $M^s_{p,q}$ and therefore in $\mathcal{S}'$, see Lemma 2.5.
Step 2. Let $q = \infty$ and suppose $p = 1$. Let $\psi, \psi^* \in C_0^\infty$ be functions such that $\psi(\xi) = 1$, $|\xi| \le 1$, $\psi(\xi) = 0$ if $|\xi| > 3/2$ and $\psi^*(\xi) = 1$, $|\xi| \le 6$. Then checking the Fourier support of the product $S^k f\, S^k g$ and using linearity of $\mathcal{F}$ we conclude

$$\Big\langle S^k f\, S^k g - S^j f\, S^j g, \varphi \Big\rangle = \Big\langle S^k f\, S^k g - S^j f\, S^j g, \mathcal{F}^{-1}[(\psi^*(2^k\xi) - \psi^*(2^j\xi))\mathcal{F}\varphi(\xi)](\,\cdot\,)\Big\rangle .$$

For brevity we put

$$h_1 := S^k f\, S^k g - S^j f\, S^j g \qquad \text{and} \qquad h_2 := \mathcal{F}^{-1}[(\psi^*(2^k\xi) - \psi^*(2^j\xi))\mathcal{F}\varphi(\xi)](\,\cdot\,)\,.$$

h_1, h_2 are smooth functions with compactly supported Fourier transform. Hence,

$$h_1 = \sum_{k \in I_1} \square_k h_1 \qquad \text{and} \qquad h_2 = \sum_{k\in I_2} \square_k h_2\,,$$

where I_1, I_2 are finite subsets of $\mathbb{Z}^n$. This allows us to rewrite $\Big\langle S^k f\, S^k g - S^j f\, S^j g, \varphi\Big\rangle$ as follows

$$\begin{aligned}\Big\langle S^k f\, S^k g - S^j f\, S^j g, \varphi \Big\rangle &= \sum_{k\in I_1}\sum_{\ell \in I_2} \int \square_k h_1(x)\, \square_\ell h_2(x)\, dx \\ &= \sum_{\ell\in I_2}\ \sum_{k\in I_1:\ Q_k\cap Q_\ell \neq \emptyset} \int \square_k h_1(x)\, \square_\ell h_2(x)\, dx\,.\end{aligned}$$

Application of Hölder's inequality yields

$$\begin{aligned}\Big|\Big\langle S^k f\, S^k g - S^j f\, S^j g, \varphi \Big\rangle\Big| &\le 2^n \sup_{k\in\mathbb{Z}^n} \langle k\rangle^s \,\|\, \square_k h_1 \,\|_{L_{p_1}} \Big(\sum_{\ell\in\mathbb{Z}^n} \langle \ell\rangle^{-s}\,\|\,\square_\ell h_2\,\|_{L_{p_2}}\Big) \\ &\le 2^n \,\|\, h_1 \,\|_{M^s_{p_1,\infty}}\, \|\, h_2\,\|_{M^{-s}_{p_2,1}}\,. \end{aligned} \tag{3.8}$$

By means of Lemma 3.12 and (3.7) we know that

$$\begin{aligned}\|\, h_1\,\|_{M^s_{p_1,\infty}} &= \|\, S^k f\, S^k g - S^j f\, S^j g\,\|_{M^s_{p_1,\infty}} \\ &\le c_1 \sup_{j\in\mathbb{N}_0} \|\, S^j g\,\|_{M^s_{p_2,q}}\, \|\, S^j f\,\|_{M^{|s|}_{p_1,1}} \\ &\le c_2 \,\|\, g\,\|_{M^s_{p_2,q}}\, \|\, f\,\|_{M^{|s|}_{p_1,1}}\,.\end{aligned}$$

On the other hand, if $j \le k$, a standard Fourier multiplier argument yields

$$\begin{aligned}\|\, h_2\,\|_{M^{-s}_{p_2,1}} &= \|\, \mathcal{F}^{-1}[(\psi^*(2^k\xi) - \psi^*(2^j\xi))\mathcal{F}\varphi(\xi)](\,\cdot\,)\,\|_{M^{-s}_{p_2,1}} \\ &\le C \sum_{A\,2^j \le |\ell| \le B\,2^k} \langle \ell\rangle^{-s}\, \|\, \square_\ell \varphi\,\|_{L_{p_2}}\end{aligned}$$

for appropriate positive constants A, B, C independent of j, k and φ. Since $\varphi \in \mathcal{S} \subset M^{-s}_{p_2,1}$ we conclude that the right-hand side tends to 0 if $j \to \infty$. This finally proves

$$\left|\left\langle S^k f\, S^k g - S^j f\, S^j g, \varphi\right\rangle\right| < \varepsilon \qquad \text{if} \quad j, k \ge j_0(\varepsilon)\,.$$

Hence $(S^k f\, S^k g)_k$ is weakly convergent in $\mathcal{S}'$. Now, Lemma 3.12 yields the claim also for $q = \infty$.
Step 3. Let $q = \infty$ and suppose $1 < p \le \infty$. We employ (3.8) with $p_1 = \infty$ and $p_2 = 1$ and afterwards Proposition 2.8. It follows

$$\begin{aligned}\left|\left\langle S^k f\, S^k g - S^j f\, S^j g, \varphi\right\rangle\right| &\le 2^n \, \| \, h_1 \, \|_{M^s_{\infty,\infty}} \, \| \, h_2 \, \|_{M^{-s}_{1,1}} \\ &\le c_1 \, \| \, h_1 \, \|_{M^s_{p,q}} \, \| \, h_2 \, \|_{M^{-s}_{1,1}}\,.\end{aligned}$$

Now we can argue as in Step 2. ∎

Remark 3.19 For a partial result concerning Theorem 3.18 we refer to Feichtinger [7].

3.4 One Example

We consider the Dirac δ distribution. Since

$$\mathcal{F}\delta(\xi) = (2\pi)^{-n/2}\,, \qquad \xi \in \mathbb{R}^n\,,$$

it is easily seen that $\delta \in M^0_{p,\infty}$ for all p. Also not difficult to see is that $M^0_{1,\infty}$ is the smallest space of type $M^s_{p,q}$ to which δ belongs to. Theorem 3.18 yields

$$\| \, f \;\cdot\; \delta \, \|_{M^0_{p,\infty}} \le c \, \| \, \delta \, \|_{M^0_{p,\infty}} \, \| \, f \, \|_{M^0_{\infty,1}}$$

with some c independent of $f \in M^0_{\infty,1}$. With other words, we can multiply δ with a modulation space $M^s_{p,q}$ if this space is embedded into C_{ub}, see Corollary 2.10. This looks reasonable.

3.5 The Second Method

Finally, we would like to investigate also the cases $\min(s_1, s_2) \le n/q'$. For dealing with this special situation we turn to a different method which will allow a better localization in the Fourier image. Therefore we shall work with the frequency-uniform decomposition $(\sigma_k)_k$. Recall that $\operatorname{supp}\sigma_k \subset Q_k := \{\xi \in \mathbb{R}^n : -1 \le \xi_i - k_i \le 1,\ i = 1, \dots, n\}$. For brevity we put

$$f_k(x) := \mathcal{F}^{-1}[\sigma_k(\xi)\mathcal{F}f(\xi)](x)\,, \qquad x \in \mathbb{R}^n\,, \quad k \in \mathbb{Z}^n\,.$$

Then, at least formally, we have the following representation of the product $f \cdot g$ as

$$f \cdot g = \sum_{k,l \in \mathbb{Z}^n} f_k \cdot g_l.$$

In what follows we shall study bounds for related partial sums.

Lemma 3.20 *Let* $1 \le p_1, p_2 \le \infty$, $1 < q \le \infty$ *and* $s_0, s_1, s_2 \in \mathbb{R}$. *Define* p *by* $\frac{1}{p} := \frac{1}{p_1} + \frac{1}{p_2}$. *If* $p \in [1, \infty]$, $0 \le s_0 \le \min(s_1, s_2)$ *and* $s_2 + s_1 - s_0 > n/q'$, *then there exists a constant* c *such that*

$$\| \sum_{k,l \in \mathbb{Z}^n} f_k \cdot g_l \|_{M^{s_0}_{p,q}} \le c\, \|f\|_{M^{s_1}_{p_1,q}}\, \|g\|_{M^{s_2}_{p_2,q}}$$

holds for all $f, g \in \mathcal{S}'$ *such that* $\operatorname{supp}\mathcal{F}f$ *and* $\operatorname{supp}\mathcal{F}g$ *are compact. The constant* c *is independent from* $\operatorname{supp}\mathcal{F}f$ *and* $\operatorname{supp}\mathcal{F}g$, *respectively.*

Proof Later on, we shall use the same strategy of proof as below in slightly different situations. For this reason and later use we shall take care of all constants showing up in our estimates below.

Step 1. Preparations. Determining the Fourier support of $f_j \cdot g_l$ we see that

$$\begin{aligned} \operatorname{supp}\mathcal{F}(f_j \cdot g_l) &= \operatorname{supp}(\mathcal{F}f_j * \mathcal{F}g_l) \\ &\subset \{\xi \in \mathbb{R}^n : j_i + l_i - 2 \le \xi_i \le j_i + l_i + 2,\ i = 1, \dots, n\}. \end{aligned}$$

Hence, the term $\mathcal{F}^{-1}(\sigma_k\mathcal{F}(f_j \cdot g_l))$ vanishes if $\|k - (j + l)\|_\infty \ge 3$. In addition, since $\operatorname{supp}\mathcal{F}f$ and $\operatorname{supp}\mathcal{F}g$ are compact, the sum $\sum_{j,l \in \mathbb{Z}^n} f_j \cdot g_l$ is a finite sum. We obtain

$$\begin{aligned} \sigma_k\mathcal{F}(f \cdot g) = \sigma_k\mathcal{F}\Big(\sum_{j,l \in \mathbb{Z}^n} f_j \cdot g_l\Big) &= \sigma_k\mathcal{F}\Big(\sum_{\substack{j,l \in \mathbb{Z}^n,\\ k_i - 3 < j_i + l_i < k_i + 3,\\ i = 1,\dots,n}} f_j \cdot g_l\Big) \\ &\overset{[r=j+l]}{=} \sum_{\substack{r \in \mathbb{Z}^n,\\ k_i - 3 < r_i < k_i + 3,\\ i = 1,\dots,n}} \sum_{l \in \mathbb{Z}^n} \sigma_k\mathcal{F}\big(f_{r-l} \cdot g_l\big)\,. \end{aligned}$$

Consequently

$$\big\| \mathcal{F}^{-1}\big(\sigma_k \mathcal{F}(f\cdot g)\big)\big\|_{L_p} \le \sum_{\substack{r\in\mathbb{Z}^n,\\ k_i-3<r_i<k_i+3,\\ i=1,\ldots,n}} \sum_{l\in\mathbb{Z}^n} \|\mathcal{F}^{-1}\big(\sigma_k \mathcal{F}(f_{r-l}\cdot g_l)\big)\|_{L_p}$$

$$\overset{[t=r-k]}{=} \sum_{\substack{t\in\mathbb{Z}^n,\\ -3<t_i<3,\\ i=1,\ldots,n}} \sum_{l\in\mathbb{Z}^n} \|\mathcal{F}^{-1}\big(\sigma_k \mathcal{F}(f_{t-(l-k)}\cdot g_l)\big)\|_{L_p}.$$

Step 2. Norm estimates. These preparations yield the following estimates

$$\Big(\sum_{k\in\mathbb{Z}^n} \langle k\rangle^{s_0 q} \|\mathcal{F}^{-1}\big(\sigma_k \mathcal{F}(f\cdot g)\big)\|_{L_p}^q\Big)^{\frac{1}{q}}$$

$$\le \Bigg(\sum_{k\in\mathbb{Z}^n} \langle k\rangle^{s_0 q}\Bigg[\sum_{\substack{t\in\mathbb{Z}^n,\\ -3<t_i<3,\\ i=1,\ldots,n}} \sum_{l\in\mathbb{Z}^n} \|\mathcal{F}^{-1}\big(\sigma_k \mathcal{F}(f_{t-(l-k)}\cdot g_l)\big)\|_{L_p}\Bigg]^q\Bigg)^{\frac{1}{q}}$$

$$\le \sum_{\substack{t\in\mathbb{Z}^n,\\ -3<t_i<3,\\ i=1,\ldots,n}} \Bigg(\sum_{k\in\mathbb{Z}^n} \langle k\rangle^{s_0 q}\Bigg[\sum_{l\in\mathbb{Z}^n} \|\mathcal{F}^{-1}\big(\sigma_k \mathcal{F}(f_{t-(l-k)}\cdot g_l)\big)\|_{L_p}\Bigg]^q\Bigg)^{\frac{1}{q}}.$$

Observe

$$\begin{aligned}\|\mathcal{F}^{-1}\big(\sigma_k \mathcal{F}(f_{t-(l-k)}\cdot g_l)\big)\|_{L^p} &= (2\pi)^{-n/2}\|(\mathcal{F}^{-1}\sigma_k) * (f_{t-(l-k)}\cdot g_l)\,\|_{L_p}\\ &\le (2\pi)^{-n/2}\|\mathcal{F}^{-1}\sigma_k\,\|_{L^1}\,\|\,f_{t-(l-k)}\cdot g_l\,\|_{L_p}\\ &= (2\pi)^{-n/2}\|\mathcal{F}^{-1}\sigma_0\,\|_{L^1}\,\|\,f_{t-(l-k)}\cdot g_l\,\|_{L_p}\,,\end{aligned}$$

where we used Young's inequality. We put $c_1 := (2\pi)^{-n/2}\|\mathcal{F}^{-1}\sigma_0\,\|_{L_1}$. This implies

$$\Big(\sum_{k\in\mathbb{Z}^n} \langle k\rangle^{s_0 q} \|\mathcal{F}^{-1}\big(\sigma_k \mathcal{F}(f\cdot g)\big)\|_{L_p}^q\Big)^{\frac{1}{q}}$$

$$\le c_1 \sum_{\substack{t\in\mathbb{Z}^n,\\ -3<t_i<3,\\ i=1,\ldots,n}} \Bigg(\sum_{k\in\mathbb{Z}^n} \langle k\rangle^{s_0 q}\Bigg[\sum_{l\in\mathbb{Z}^n} \|\,f_{t-(l-k)}\cdot g_l\|_{L_p}\Bigg]^q\Bigg)^{\frac{1}{q}}.$$

We continue by using Hölder's inequality to get

$$\Big(\sum_{k\in\mathbb{Z}^n}\langle k\rangle^{s_0 q}\|\mathcal{F}^{-1}\big(\sigma_k\mathcal{F}(f\cdot g)\big)\|_{L_p}^q\Big)^{\frac{1}{q}}$$

$$\leq c_2 \max_{\substack{t\in\mathbb{Z}^n,\\ -3<t_i<3,\\ i=1,\dots,n}} \left(\sum_{k\in\mathbb{Z}^n}\langle k\rangle^{s_0 q}\left[\sum_{l\in\mathbb{Z}^n}\|f_{t-(l-k)}\|_{L_{p_1}}\|g_l\|_{L_{p_2}}\right]^q\right)^{\frac{1}{q}}$$

with $c_2 := c_1\, 5^n$. Since $s_0 \geq 0$ elementary calculations yield

$$\langle k\rangle^{s_0}\Big[\sum_{l\in\mathbb{Z}^n}\|f_{t-(l-k)}\|_{L_{p_1}}\ \|g_l\|_{L_{p_2}}\Big]$$

$$\leq 2^{s_0}\sum_{\substack{l\in\mathbb{Z}^n,\\ |l|\leq|l-k|}}\langle k-l\rangle^{s_0}\|f_{t-(l-k)}\|_{L_{p_1}}\|g_l\|_{L_{p_2}}$$

$$+2^{s_0}\sum_{\substack{l\in\mathbb{Z}^n,\\ |l-k|\leq|l|}}\|f_{t-(l-k)}\|_{L_{p_1}}\langle l\rangle^{s_0}\|g_l\|_{L_{p_2}}\,.$$

Both parts of this right-hand side will be estimated separately. We put

$$S_{1,t,k} := \sum_{\substack{l\in\mathbb{Z}^n,\\ |l|\leq|l-k|}}\langle k-l\rangle^{s_1}\|f_{t-(l-k)}\|_{L_{p_1}}\langle l\rangle^{s_2}\|g_l\|_{L_{p_2}}\ \langle k-l\rangle^{s_0-s_1}\ \langle l\rangle^{-s_2}\,;$$

$$S_{2,t,k} := \sum_{\substack{l\in\mathbb{Z}^n,\\ |l-k|\leq|l|}}\langle k-l\rangle^{s_1}\ \|f_{t-(l-k)}\|_{L_{p_1}}\langle l\rangle^{s_2}\ \|g_l\|_{L_{p_2}}\ \langle k-l\rangle^{-s_1}\ \langle l\rangle^{s_0-s_2}\,.$$

With $\frac{1}{q}+\frac{1}{q'}=1$ we find

$$S_{1,t,k} \overset{[j=l-k]}{=} \sum_{\substack{j\in\mathbb{Z}^n,\\ |j+k|\leq|j|}}\langle j\rangle^{s_0}\|f_{t-j}\|_{L_{p_1}}\|g_{j+k}\|_{L_{p_2}}$$

$$\leq \Big(\sum_{\substack{j\in\mathbb{Z}^n,\\ |j+k|\leq|j|}}(\langle j\rangle^{s_1}\|f_{t-j}\|_{L_{p_1}}\langle j+k\rangle^{s_2}\|g_{j+k}\|_{L_{p_2}})^q\Big)^{1/q}$$

$$\times\ \Big(\sum_{\substack{j\in\mathbb{Z}^n,\\ |j+k|\leq|j|}}(\langle j\rangle^{s_0-s_1}\ \langle j+k\rangle^{-s_2})^{q'}\Big)^{\frac{1}{q'}}\,.$$

Substep 2.1. Our assumptions $s_0 \le s_1$, $s_2 \ge 0$ and $s_1 + s_2 - s_0 > n/q'$ imply

$$\Big(\sum_{\substack{j\in\mathbb{Z}^n,\\ |j+k|\le|j|}} \Big| \langle j\rangle^{s_0-s_1} \langle j+k\rangle^{-s_2} \Big|^{q'} \Big)^{\frac{1}{q'}} \le \Big(\sum_{m\in\mathbb{Z}^n} \langle m\rangle^{(s_0-s_1-s_2)q'} \Big)^{\frac{1}{q'}} =: c_3 < \infty\,.$$

Inserting this in our previous estimate we obtain

$$\Big(\sum_{k\in\mathbb{Z}^n} S^q_{1,t,k}\Big)^{1/q} \le c_3 \Big(\sum_{k\in\mathbb{Z}^n} \sum_{\substack{j\in\mathbb{Z}^n,\\ |j+k|\le|j|}} \langle j\rangle^{s_1 q} \| f_{t-j}\|^q_{L^{p_1}} \langle j+k\rangle^{s_2 q} \| g_{j+k}\|^q_{L^{p_2}} \Big)^{1/q}$$
$$\le c_3 \Big(\sum_{j\in\mathbb{Z}^n} \langle j\rangle^{s_1 q} \| f_{t-j}\|^q_{L^{p_1}} \sum_{k\in\mathbb{Z}^n} \langle j+k\rangle^{s_2 q} \| g_{j+k}\|^q_{L^{p_2}} \Big)^{\frac{1}{q}}\,.$$

Because of $1+|j|^2 \le 1+8n+|j-t|^2$ we know

$$\max_{\substack{t\in\mathbb{Z}^n,\\ -3<t_i<3,\\ i=1,\ldots,n}} \sup_{j\in\mathbb{Z}^n} \frac{\langle j\rangle^{s_1}}{\langle j-t\rangle^{s_1}} \le (1+8n)^{s_1/2} =: c_4 < \infty\,.$$

This implies

$$\Big(\sum_{k\in\mathbb{Z}^n} S^q_{1,t,k}\Big)^{1/q} \le c_3\, c_4\, \|g\|_{M^{s_2}_{p_2,q}} \|f\|_{M^{s_1}_{p_1,q}}\,, \tag{3.9}$$

where c_3, c_4 are independent of f, g and t.

Substep 2.2. Because of $0 \le s_0 \le s_1$, $s_0 \le s_2$ and $s_1 + s_2 - s_0 > n/q'$ we conclude

$$\Big(\sum_{\substack{l\in\mathbb{Z}^n,\\ |l-k|\le|l|}} \Big| \langle k-l\rangle^{-s_1} \langle l\rangle^{s_0-s_2} \Big|^{q'} \Big)^{\frac{1}{q'}} \le \Big(\sum_{m\in\mathbb{Z}^n} \langle m\rangle^{(s_0-s_1-s_2)q'} \Big)^{\frac{1}{q'}} =: c_5 < \infty\,.$$

This leads to the estimate

$$\Big(\sum_{k\in\mathbb{Z}^n} S^q_{2,t,k}\Big)^{1/q} \le c_5 c_6\, \|g\|_{M^{s_2}_{p_2,q}} \|f\|_{M^{s_1}_{p_1,q}} \tag{3.10}$$

with some constants c_6 independent from f and g. Combining the inequalities (3.9) and (3.10) we have proved the claim. ∎

Remark 3.21 Some basic ideas of the above proof are taken over from [5], see also [23].

Of course the above method of proof works as well for $q = 1$. But all spaces $M^s_{p,1}$, $s \ge 0$, are algebras.

Theorem 3.22 *Let* $1 \le p, p_1, p_2 \le \infty$ *and* $s_0, s_1, s_2 \in \mathbb{R}$. *Let* $1/p \le (1/p_1) + (1/p_2)$, $1 < q \le \infty$, $0 \le s_0 \le \min(s_1, s_2)$ *and* $s_1 + s_2 - s_0 > n/q'$. *There exists a constant c such that*

$$\| f \cdot g \|_{M^{s_0}_{p,q}} \le c \, \| f \|_{M^{s_1}_{p_1,q}} \, \| g \|_{M^{s_2}_{p_2,q}}$$

holds for all $f \in M^{s_1}_{p_1,q}$ *and all* $g \in M^{s_2}_{p_2,q}$.

Proof We only comment on the case $1/p = (1/p_1) + (1/p_2)$, see Corollary 2.7. It will be enough to prove the weak convergence of $(S^k f \cdot S^k g)_k$ in $\mathcal{S}'$. The claimed estimate will then follow from Lemma 3.20. We employ the method and the notation used in proof of Theorem 3.18 (Steps 2 and 3). There we have proved

$$\Big| \Big\langle S^k f \, S^k g - S^j f \, S^j g, \varphi \Big\rangle \Big| \le c_1 \, \| \, h_1 \, \|_{M^{s_0}_{p,q}} \, \| \, h_2 \, \|_{M^{-s_0}_{1,1}}$$

with c_1 independent of f, g, k and j. By means of Lemma 3.20 we know the uniform boundedness of $\| \, h_1 \, \|_{M^{s_0}_{p,q}}$ in k and j. The estimate of $\| \, h_2 \, \|_{M^{-s_0}_{1,1}}$ can be done as above. It follows

$$\| \, h_2 \, \|_{M^{-s_0}_{1,1}} \le \varepsilon$$

if $j, k \ge j_0(\varepsilon)$. This guarantees the weak convergence of $(S^k f \cdot S^k g)_k$ in $\mathcal{S}'$. ∎

Our sufficient conditions are not far away from necessary conditions.

Lemma 3.23 *Let* $1 \le p_1, p_2, p, q \le \infty$ *and* $s_0, s_1, s_2 \in \mathbb{R}$. *Suppose that there exists a constant c such that*

$$\| f \cdot g \|_{M^{s_0}_{p,q}} \le c \, \| f \|_{M^{s_1}_{p_1,q}} \, \| g \|_{M^{s_2}_{p_2,q}} \tag{3.11}$$

holds for all $f, g \in \mathcal{S}$.
(i) *It follows* $s_0 \le \min(s_1, s_2)$, $s_1 + s_2 \ge 0$ *and* $s_1 + s_2 - s_0 \ge n/q'$.
(ii) *If* $1 \le p_2 = p < \infty$ *and* $1 \le q < \infty$, *then either* $q = 1$ *and* $s_1 \ge 0$ *or* $1 < q < \infty$ *and* $s_1 > n/q'$.

Proof Part (ii) is an immediate consequence of Lemma 3.7. Concerning the proof of (i) we shall work with the same test functions as used in Step 2 of the proof of Corollary 2.10, see (2.1).
Step 1. We choose $a_k := \delta_{k,\ell}$, $k \in \mathbb{Z}^n$, for a fixed given $\ell \in \mathbb{Z}^n$ and put $b_k := \delta_{k,0}$, $k \in \mathbb{Z}^n$. Then we define

$$f(x) := \psi(x) \, e^{i\ell x} \qquad \text{and} \qquad g(x) := \psi(x) \, .$$

We obtain

$$\| f \|_{M^{s_1}_{p_1,q}} \cdot \| g \|_{M^{s_2}_{p_2,q}} = \| \psi \|_{L_{p_1}} \, \| \psi \|_{L_{p_2}} \, \langle \ell \rangle^{s_1}$$

as well as

$$\| f \cdot g\|_{M^{s_0}_{p,q}} = \|\psi^2\|_{L_p} \, \langle \ell\rangle^{s_0} \, .$$

Hence, (3.11) implies $s_0 \le s_1$. Interchanging the roles of f and g leads to the conclusion $s_0 \le s_2$.
Step 2. Let $\ell \in \mathbb{Z}^n$ be fixed. We choose $a_k := \delta_{k,\ell}$, $k \in \mathbb{Z}^n$, and $b_k := \delta_{k,-\ell}$, $k \in \mathbb{Z}^n$. Then we define

$$f(x) := \psi(x)\, e^{i\ell x} \qquad \text{and} \qquad g(x) := \psi(x)\, e^{-i\ell x} \, .$$

It follows

$$\| f\|_{M^{s_1}_{p_1,q}} \cdot \|g\|_{M^{s_2}_{p_2,q}} = \|\psi\|_{L_{p_1}} \, \|\psi\|_{L_{p_2}} \, \langle \ell\rangle^{s_1+s_2}$$

as well as

$$\| f \cdot g\|_{M^{s_0}_{p,q}} = \|\psi^2\|_{L_p} \, .$$

Hence, (3.11) implies $s_1 + s_2 \ge 0$.
Step 3. Let $\varepsilon_1, \varepsilon_2 \ge 0$. These two numbers will be chosen such that

$$\min(s_1 + \varepsilon_1 + n/q, s_2 + \varepsilon_2 + n/q) > 0 \qquad \text{and} \qquad s_0 + \varepsilon_2 + \varepsilon_1 + n > 0 \, .$$

We choose $a_k := \langle k\rangle^{\varepsilon_1}$, $k \in \mathbb{Z}^n$, and $b_k := \langle k\rangle^{\varepsilon_2}$, $k \in \mathbb{Z}^n$. Then we define

$$f(x) := \psi(x) \sum_{\|k\|_\infty \le N} a_k\, e^{ikx} \qquad \text{and} \qquad g(x) := \psi(x) \sum_{\|k\|_\infty \le N} b_k\, e^{ikx} \, .$$

By means of the same arguments as used in Substep 3.1 of the proof of Theorem 3.5, we conclude

$$\| f\|_{M^{s_1}_{p_1,q}} \asymp N^{s_1+\varepsilon_1+n/q} \qquad \text{and} \qquad \|g\|_{M^{s_2}_{p_2,q}} \asymp N^{s_2+\varepsilon_2+n/q} \, .$$

In addition, we have

$$\begin{aligned}
\| f \cdot g\|_{M^{s_0}_{p,q}} & \asymp \Bigg(\sum_{\|m\|_\infty \le 2N} \langle m\rangle^{s_0 q} \Big| \sum_{\substack{k:\ \|k\|_\infty \le N \\ \|m-k\|_\infty \le N}} a_k\, b_{m-k}\Big|^q \Bigg)^{1/q} \\
& \ge \frac{1}{2n} \Bigg(\sum_{\|m\|_\infty \le N} \langle m\rangle^{(s_0+\varepsilon_2)q} \Big| \sum_{k:\ \|k\|_\infty \le \|m\|_\infty/2} \langle k\rangle^{\varepsilon_1}\Big|^q \Bigg)^{1/q} \\
& \ge C_1 \Bigg(\sum_{\|m\|_\infty \le N} \langle m\rangle^{(s_0+\varepsilon_2+\varepsilon_1+n)q} \Bigg)^{1/q} \\
& \ge C_2\, N^{s_0+\varepsilon_2+\varepsilon_1+n+n/q}
\end{aligned}$$

for some C_1, C_2 independent of N, see Substep 3.2 of the proof of Theorem 3.5. The inequality (3.11) yields

$$s_0 + \varepsilon_2 + \varepsilon_1 + n + n/q \le s_1 + \varepsilon_1 + n/q + s_2 + \varepsilon_2 + n/q$$

which proves the claim. ■

The duality argument used in the proof of Lemma 3.7 allows to treat the case $s_0 < 0$.

Theorem 3.24 *Let* $1 \le p, p_1, p_2 \le \infty$ *and* $s_0, s_1, s_2 \in \mathbb{R}$. *Let* $1/p \le (1/p_1) + (1/p_2)$, $1 \le q < \infty$, $s_0 \le s_2 \le 0$, $0 \le s_1 + s_2$ *and* $s_1 + s_2 - s_0 > n/q$. *There exists a constant c such that*

$$\| f \cdot g \|_{M^{s_0}_{p,q}} \le c \, \| f \|_{M^{s_1}_{p_1,q'}} \, \| g \|_{M^{s_2}_{p_2,q}}$$

holds for all $f \in M^{s_1}_{p_1,q'}$ *and all* $g \in M^{s_2}_{p_2,q}$.

Remark 3.25 Theorems 3.18 and 3.24 have some overlap.

3.6 Some Further Remarks to the Literature

Here, we recall results of Iwabuchi [13] and Toft et al. [32]. As Cordero and Nicola [6] also Iwabuchi considered the more general situation $M^{s_1}_{p_1,q_1} \cdot M^{s_2}_{p_2,q_2} \hookrightarrow M^{s_0}_{p,q}$. This greater flexibility with respect to the tripel q, q_1, q_2 allows to treat cases not covered by Theorems 3.22, 3.24.

Proposition 3.26 (Iwabuchi [13])
Let $1 \le p, p_1, p_2 \le \infty$, $1 < q, q_1, q_2 < \infty$ *and* $0 < s_0 < n/q$.
(i) *If* $q \ge q_1$,

$$\frac{1}{p} \le \frac{1}{p_1} + \frac{1}{p_2} \quad \text{and} \quad 1 + \frac{1}{q} - \Big(\frac{1}{q_1} + \frac{1}{q_2}\Big) = \frac{s_0}{n}\,, \tag{3.12}$$

then there exists a constant c such that

$$\| f \cdot g \|_{M^{-s_0}_{p,q}} \le c \, \| f \|_{M^{0}_{p_1,q_1}} \, \| g \|_{M^{0}_{p_2,q_2}}$$

holds for all $f \in M^0_{p_1,q_1}$ *and all* $g \in M^0_{p_2,q_2}$.
(ii) *Assume* $q \ge \max(q_1, q_2)$ *and* (3.12). *Then, there exists a constant c such that*

$$\| f \cdot g \|_{M^{s_0}_{p,q}} \le c \, \| f \|_{M^{s_0}_{p_1,q_1}} \, \| g \|_{M^{s_0}_{p_2,q_2}}$$

holds for all $f \in M^{s_0}_{p_1,q_1}$ *and all* $g \in M^{s_0}_{p_2,q_2}$.

Remark 3.27 Let us take $q = q_1 = q_2$. Then (3.12) reads as $s_0 = n/q'$. In combination with $0 < s_0 < n/q$ this yields $1 < q < 2$. Hence, (i) reads as

$$\| f \cdot g \|_{M^{-n/q'}_{p,q}} \le c \, \| f \|_{M^0_{p_1,q}} \, \| g \|_{M^0_{p_2,q}} \, ,$$

whereas (ii) gives

$$\| f \cdot g \|_{M^{n/q'}_{p,q}} \le c \, \| f \|_{M^{n/q'}_{p_1,q_1}} \, \| g \|_{M^{n/q'}_{p_2,q_2}} \, .$$

Toft et al. [32] also consider the situation $M^{s_1}_{p_1,q_1} \cdot M^{s_2}_{p_2,q_2} \hookrightarrow M^{s_0}_{p,q}$. Recall, $\mathring{M}^{s_0}_{p,q}$ denotes the closure of $\mathcal{S}$ in $M^{s_0}_{p,q}$.

Proposition 3.28 (Toft et al. [32])
Let $1 \le p, p_1, p_2, q, q_1, q_2 \le \infty$ *and* $s_0, s_1, s_2 \in \mathbb{R}$.
(i) *We suppose*

(a) $1 + \frac{1}{p} - \frac{1}{p_1} - \frac{1}{p_2} \le 1$;
(b) $0 \le 1 + \frac{1}{q} - \frac{1}{q_1} - \frac{1}{q_2} \le 1/2$;
(c) $s_0 \le \min(s_1, s_2)$;
(d) $s_1 + s_2 \ge 0$;
(e) $s_1 + s_2 - s_0 - n\left(1 + \frac{1}{q} - \frac{1}{q_1} - \frac{1}{q_2}\right) \ge 0$;
(f) $s_1 + s_2 - s_0 - n\left(1 + \frac{1}{q} - \frac{1}{q_1} - \frac{1}{q_2}\right) > 0$ *if* $1 + \frac{1}{q} - \frac{1}{q_1} - \frac{1}{q_2} > 0$ *and either* s_1 *or* s_2 *or* $-s_0$ *equals* $n\left(1 + \frac{1}{q} - \frac{1}{q_1} - \frac{1}{q_2}\right)$.

Then there exists a constant c such that

$$\| f \cdot g \|_{M^{s_0}_{p,q}} \le c \, \| f \|_{M^{s_1}_{p_1,q_1}} \, \| g \|_{M^{s_2}_{p_2,q_2}} \tag{3.13}$$

holds for all $f \in \mathring{M}^{s_1}_{p_1,q_1}$ *and all* $g \in \mathring{M}^{s_2}_{p_2,q_2}$.
(ii) *If* (3.13) *holds for all* $f, g \in \mathcal{S}$, *then (c), (d) and (e) follow.*

Remark 3.29 Again we consider the case $q = q_1 = q_2$. Then (b) implies $1 \le q \le 2$ and (e) reads as $s_1 + s_2 - s_0 - n/q' \ge 0$. Hence, if we restrict us to $1 < q \le 2$, Proposition 3.28 is slightly more general than Theorem 3.22 and Theorem 3.24. However, for our purpose, see the next section on composition of functions, Theorem 3.22 is already sufficient. Let us mention that Proposition 4.5 below, which is nothing but a modification of Lemma 3.20, is of central importance for the applications to composition operators we have in mind.

3.7 An Important Special Case

We consider $M^s_{2,2}$. A simple argument, based on the frequency-uniform decomposition yields $M^s_{2,2} = H^s$ in the sense of equivalent norms, see Remark 2.3. For these Sobolev spaces H^s almost all is known.

- H^s is an algebra with respect to pointwise multiplication if and only if $s > n/2$, see Strichartz [28], Triebel [33, 2.8] or [25, Theorem 4.6.4/1]. This coincides with Theorem 3.5.
- Let E be a Banach space of functions. By $M(E)$ we denote the set of all pointwise multipliers of E, i.e., the set of all f such that T_f, defined as $T_f(g) = f \cdot g$, maps E into E. We equip $M(E)$ with the norm $\|f\|_{M(E)} := \|T_f\|_{\mathcal{L}(E)}$. For a description of $M(H^s)$ one needs the classes $H^{s,\ell oc}$. Here $H^{s,\ell oc}$ denotes the collection of all distributions $f \in \mathcal{S}'$ such that $f \cdot \varphi \in H^s$ for all $\varphi \in C_0^\infty$. In case $s > n/2$ it holds

$$M(H^s) = \Big\{ f \in H^{s,\ell oc} : \quad \|f\|^*_{M(H^s)} := \sup_{\lambda \in \mathbb{R}^n} \|\psi(\cdot - \lambda)\, f\|_{H^s} < \infty \Big\}$$

  in the sense of equivalent norms. Here ψ is a smooth nontrivial cut-off function supported around the origin. For all this we refer to Strichartz [28].
- In case $0 \le s < n/2$ also characterizations of $M(H^s)$ are known, this time more complicated, based on capacities. For all details we refer to the monograph of Maz'ya and Shaposnikova [20, Theorem 3.2.2, pp. 86].
- Now we concentrate on the situation described in Theorem 3.22 in case $0 < s < \frac{n}{2}$. As it is well-known, there exists a constant c such that

$$\|\, f \cdot g\,\|_{H^{2s-n/2}} \le c\, \|f\|_{H^s}\, \|g\|_{H^s}$$

  holds for all $f, g \in H^s$, see e.g., [25, Theorem 4.5.2]. In Theorem 3.22 we proved that for any $\varepsilon > 0$ there exists a constant c_ε such that

$$\|\, f \cdot g\,\|_{M^{2s-n/2-\varepsilon}_{1,2}} \le c_\varepsilon\, \|f\|_{H^s}\, \|g\|_{H^s}$$

  holds for all $f, g \in H^s$. We conjecture that $M^{2s-n/2-\varepsilon}_{1,2}$ and $H^{2s-n/2}$ are incomparable.

4 Composition of Functions

There are some attempts to investigate composition of functions in the framework of modulation spaces, i.e., we consider the operator

$$T_f : \quad g \mapsto f \circ g, \qquad g \in M^s_{p,q}, \tag{4.1}$$

and ask for mapping properties. Of course, we used the symbol T_f before with a different meaning, but we hope that will not cause problems. Within Sect. 4 T_f will have the meaning as in (4.1). Based on pointwise multiplication one can treat f to be a polynomial or even the more general case of f being an entire function.

4.1 Polynomials

We consider the case

$$f(z) := \sum_{\ell=1}^{m} a_\ell \, z^\ell \,, \qquad z \in \mathbb{C}\,,$$

where $m \in \mathbb{N}$, $m \geq 2$, and $a_\ell \in \mathbb{C}$, $\ell = 1, \ldots, m$. For brevity we denote the associated composition operator by T_m. In addition we need the abbreviation

$$t_m(s) := s + (m-1)(s - n/q')\,, \qquad m = 2, 3, \ldots\,.$$

Theorem 4.1 *Let* $1 \leq p, q \leq \infty$ *and* $m \in \mathbb{N}$, $m \geq 2$.
(i) *Let either* $s \geq 0$ *and* $q = 1$ *or* $s > n/q'$. *Then* T_m *maps* $M^s_{p,q}$ *into itself. There exists a constant c such that*

$$\| \, T_m g \, \|_{M^s_{p,q}} \leq c \, \| \, g \, \|_{M^s_{p,q}} \sum_{\ell=1}^{m} |a_\ell| \, \| \, g \, \|^{\ell-1}_{M^0_{\infty,1}}$$

holds for all $g \in M^s_{p,q}$.
(ii) *Let* $1 < q \leq \infty$, $0 < s \leq n/q'$ *and* $t_m(s) > 0$. *If* $p \in [m, \infty]$ *and* $t < t_m(s)$, *then there exists a constant c such that*

$$\| \, T_m g \, \|_{M^t_{p/m,q}} \leq c \sum_{\ell=1}^{m} |a_\ell| \, \|g\|^{\ell}_{M^s_{p,q}}$$

holds for all $g \in M^s_{p,q}$.
(iii) *Let* $q = 1$ *and* $s \geq 0$. *If* $p \in [m, \infty]$, *then there exists a constant c such that*

$$\| \, T_m g \, \|_{M^s_{p/m,1}} \leq c \sum_{\ell=1}^{m} |a_\ell| \, \|g\|^{\ell}_{M^s_{p,1}}$$

holds for all $g \in M^s_{p,1}$.

Proof Step 1. Both parts, (i) and (ii), can be proved by induction based on Theorem 3.5 or Theorem 3.22. We concentrate on the proof of (ii). Let $m = 2$. Then by assumption $t_2(s) = 2s - n/q' > 0$. Hence, we may apply Theorem 3.22 with $p_1 = p_2 = p$ and $s_1 = s_2$ and obtain

$$\| \, g^2 \, \|_{M^t_{p/2,q}} \leq c \, \|g\|^2_{M^s_{p,q}}$$

for any $t < 2s - n/q' = t_2(s)$. Now we assume that part (ii) is correct for all natural numbers in the interval $[2, m]$. We split the product g^{m+1} into the two factors g^m

and g. By assumption $g^m \in M^t_{p/m,q}$ for any $t < t_m(s)$. We put $s_1 = t = t_m(s) - \varepsilon$, $s_2 = s$, $p_1 = p/m$ and $p_2 = p$, where we assume that $\varepsilon > 0$ is sufficiently small. This guarantees

$$s_1 + s_2 - \frac{n}{q'} = s + (m-1)\Big(s - \frac{n}{q'}\Big) - \varepsilon + s - \frac{n}{q'} = t_{m+1}(s) - \varepsilon > 0\,.$$

Hence, we may choose s_0 by

$$s_0 < \min(s_1, s_2, t_{m+1}(s) - \varepsilon) = t_{m+1}(s) - \varepsilon\,.$$

Since $\varepsilon > 0$ is arbitrary, any value $< t_{m+1}(s)$ becomes admissible for s_0. An application of Theorem 3.22 yields

$$\|\, g^m \cdot g \,\|_{M^{s_0}_{p/(m+1),q}} \le c \,\|\, g_m \,\|_{M^{t_m-\varepsilon}_{p/m,q}} \, \|g\|_{M^s_{p,q}}\,.$$

Step 2. Part (iii) is an immediate consequence of Lemma 3.10. ■

Remark 4.2 For the case $s = 0$ we refer to Cordero, Nicola [6], Toft [30] and Guo et al. [11].

4.2 Entire Functions

We consider the case of f being an entire analytic function on $\mathbb{C}$, i.e.,

$$f(z) := \sum_{\ell=0}^{\infty} a_\ell\, z^\ell\,, \qquad z \in \mathbb{C}\,,$$

where $a_\ell \in \mathbb{C}$, $\ell \in \mathbb{N}_0$. Clearly, we need to assume $f(0) = a_0 = 0$. Otherwise $T_f g$ will not have global integrability properties. Let

$$f_0(r) := \sum_{\ell=1}^{\infty} |a_\ell|\, r^\ell\,, \qquad r > 0\,.$$

Theorem 4.3 *Let $1 \le p, q \le \infty$ and let either $s \ge 0$ and $q = 1$ or $s > n/q'$. Let f be an entire function satisfying $f(0) = 0$. Then T_f maps $M^s_{p,q}$ into itself. There exist two constants a, b, independent of f, such that*

$$\|\, T_f g \,\|_{M^s_{p,q}} \le a\, f_0(b \,\|\, g \,\|_{M^s_{p,q}})$$

holds for all $g \in M^s_{p,q}$.

Proof The constant c in Theorem 4.1 (i) depends on m. To clarify the dependence on m we proceed by induction. Let c_1 be the best constant in the inequality

$$\| g_1 \cdot g_2 \|_{M^s_{p,q}} \le c_1 \, (\| g_1 \|_{M^s_{p,q}} \, \| g_2 \|_{M^0_{\infty,1}} + \| g_2 \|_{M^s_{p,q}} \, \| g_1 \|_{M^0_{\infty,1}}) \, , \tag{4.2}$$

see Lemma 3.3. Further, let c_2 be the best constant in the inequality

$$\| g_1 \cdot g_2 \|_{M^0_{\infty,1}} \le c_2 \, \| g_1 \|_{M^0_{\infty,1}} \, \| g_2 \|_{M^0_{\infty,1}} \, , \tag{4.3}$$

see also Lemma 3.3. By c_3 we denote $\max(1, c_1, c_2)$. Our induction hypothesis consists in: the inequality

$$\| g^m \|_{M^s_{p,q}} \le c_3^{m-1} \, m \, \| g \|_{M^s_{p,q}} \| g \|^{m-1}_{M^0_{\infty,1}}$$

holds for all $g \in M^s_{p,q}$ and all $m \ge 2$. This follows easily from (4.2) and (4.3). Next we need the best constant, denoted by c_4, in the inequality

$$\| g \|_{M^0_{\infty,1}} \le c_4 \, \| g \|_{M^s_{p,q}} \, , \qquad g \in M^s_{p,q} \, .$$

This proves that

$$\| g^m \|_{M^s_{p,q}} \le c_3^{m-1} \, m \, c_4^{m-1} \, \| g \|^m_{M^s_{p,q}} \tag{4.4}$$

holds for all $g \in M^s_{p,q}$ and all $m \ge 2$. Hence

$$\begin{aligned} \| T_f g \|_{M^s_{p,q}} &\le \sum_{m=1}^{\infty} |a_m| c_3^{m-1} \, m \, c_4^{m-1} \, \| g \|^m_{M^s_{p,q}} \\ &= \frac{1}{c_3 \, c_4} \sum_{m=1}^{\infty} |a_m| \, m \, (c_3 \, c_4 \, \| g \|_{M^s_{p,q}})^m \, . \end{aligned}$$

Since

$$\sup_{m \in \mathbb{N}} m^{1/m} = 3^{1/3}$$

the claimed estimate follows. ∎

Remark 4.4 Theorem 4.3 is essentially known, see e.g., Sugimoto et al. [29] or Bhimani [1].

4.3 One Example

The following example has been considered at various places. Let $f(z) := e^z - 1$, $z \in \mathbb{C}$. For appropriate constants $a, b > 0$ it follows that

$$\| e^g - 1 \|_{M^s_{p,q}} \le a\, e^{b \| g \|_{M^s_{p,q}}} \tag{4.5}$$

holds for all $g \in M^s_{p,q}$.

It will be essential for our approach to non-analytic composition results that we can improve this estimate.

4.4 Non-analytic Superposition Operators

There is a famous classical result by Katznelson [17] (in the periodic case) and by Helson, Kahane, Katznelson, Rudin [12] (nonperiodic case) which says that only analytic functions operate on the Wiener algebra $\mathcal{A}$. More exactly, the operator $T_f : u \mapsto f(u)$ maps $\mathcal{A}$ into $\mathcal{A}$ if and only if $f(0) = 0$ and f is analytic. Here, $\mathcal{A}$ is the collection of all $u \in C$ such that $\mathcal{F}u \in L_1$. Moreover, a similar result is obtained for particular standard modulation spaces. Bhimani and Ratnakumar [2], see also Bhimani [1], proved that T_f maps $M_{1,1}$ into $M_{1,1}$ if and only if $f(0) = 0$ and f is analytic. Therefore, the existence of non-analytic superposition results for weighted modulation spaces is a priori not so clear.

We shall concentrate on the algebra case. Our first aim consists in deriving a better estimate than (4.5).

To proceed we need some preparations. An essential tool in proving our main result will be a certain subalgebra property. Therefore, we consider the following decomposition of the phase space. Let $R > 0$ and $\epsilon = (\epsilon_1, \dots, \epsilon_n)$ be fixed with $\epsilon_j \in \{0, 1\}$, $j = 1, \dots, n$. Then a decomposition of $\mathbb{R}^n$ into $(2^n + 1)$ parts is given by

$$P_R := \{\xi \in \mathbb{R}^n : |\xi_j| \le R,\, j = 1, \dots, n\}$$

and

$$P_R(\epsilon) := \{\xi \in \mathbb{R}^n : \operatorname{sign}(\xi_j) = (-1)^{\epsilon_j},\ j = 1, \dots, n\} \setminus P_R.$$

For given $p, q, s, \epsilon = (\epsilon_1, \dots, \epsilon_n)$ and $R > 0$ we introduce the spaces

$$M^s_{p,q}(\epsilon, R) := \{f \in M^s_{p,q} : \operatorname{supp} \mathcal{F} f \subset P_R(\epsilon)\}\,.$$

Proposition 4.5 *Let $1 \le p_1, p_2 \le \infty$, $1 < q \le \infty$ and $s_0, s_1, s_2 \in \mathbb{R}$. Define p by $\frac{1}{p} := \frac{1}{p_1} + \frac{1}{p_2}$. Let $R > 2$. If $p \in [1, \infty]$, $s_0 \le \min(s_1, s_2)$, $s_1, s_2 \ge 0$ and $s_1 + s_2 - s_0 > n/q'$, then there exists a constant c such that*

$$\| f \cdot g \|_{M^{s_0}_{p,q}} \le c\, (R-2)^{-[(s_1+s_2-s_0)-n/q']} \, \| f \|_{M^{s_1}_{p_1,q}} \, \| g \|_{M^{s_2}_{p_2,q}}$$

holds for all $f \in M^{s_1}_{p_1,q}(\epsilon, R)$ and all $g \in M^{s_2}_{p_2,q}(\epsilon, R)$. The constant c is independent from $R > 2$ and ϵ.

Proof In order to show the subalgebra property we follow the same steps as in the proof of Lemma 3.20. We start with some almost trivial observations. Let $f \in M^s_{p_1,q}(\epsilon, R)$ and $g \in M^s_{p_2,q}(\epsilon, R)$. By

$$\mathrm{supp}\,(\mathcal{F}f * \mathcal{F}g) \subset \{\xi + \eta : \xi \in \mathrm{supp}\,\mathcal{F}f, \eta \in \mathrm{supp}\,\mathcal{F}g\}$$

we have $\mathrm{supp}\,\mathcal{F}(fg) \subset P_R(\epsilon)$. Let

$$P^*_R(\epsilon) := \Big\{k \in \mathbb{Z}^n : \quad \|k\|_\infty > R - 1\,, \quad \mathrm{sign}\,(k_j) = (-1)^{\epsilon_j},\ j = 1, \ldots, n\Big\}.$$

Hence, if $\mathrm{supp}\,\sigma_k \cap P_R(\epsilon) \neq \emptyset$, then $k \in P^*_R(\epsilon)$ follows. Now we continue as in proof of Lemma 3.20, Step 2, and obtain

$$\Big(\sum_{k\in P^*_R(\epsilon)} \langle k\rangle^{s_0 q}\, \|\mathcal{F}^{-1}\big(\sigma_k \mathcal{F}(f\cdot g)\big)\|^q_{L_p}\Big)^{\frac{1}{q}}$$
$$\le \sum_{\substack{t\in\mathbb{Z}^n,\\ -3<t_i<3,\\ i=1,\ldots,n}} \Bigg(\sum_{k\in P^*_R(\epsilon)} \langle k\rangle^{s_0 q}\Big[\sum_{\substack{l\in\mathbb{Z}^n:\\ t-l+k,l\in P^*_R(\epsilon)}} \|\mathcal{F}^{-1}\big(\sigma_k \mathcal{F}(f_{t-(l-k)}\cdot g_l)\big)\|_{L_p}\Big]^q\Bigg)^{\frac{1}{q}}.$$

This implies

$$\Big(\sum_{k\in P^*_R(\epsilon)} \langle k\rangle^{s_0 q}\, \|\mathcal{F}^{-1}\big(\sigma_k \mathcal{F}(f\cdot g)\big)\|^q_{L_p}\Big)^{\frac{1}{q}}$$
$$\le c_1 \sum_{\substack{t\in\mathbb{Z}^n,\\ -3<t_i<3,\\ i=1,\ldots,n}} \Bigg(\sum_{k\in P^*_R(\epsilon)} \langle k\rangle^{s_0 q}\Big[\sum_{\substack{l\in\mathbb{Z}^n:\\ t-l+k,l\in P^*_R(\epsilon)}} \|f_{t-(l-k)}\cdot g_l\|_{L_p}\Big]^q\Bigg)^{\frac{1}{q}}$$
$$\le c_2 \max_{\substack{t\in\mathbb{Z}^n,\\ -3<t_i<3,\\ i=1,\ldots,n}} \Bigg(\sum_{k\in P^*_R(\epsilon)} \langle k\rangle^{s_0 q}\Big[\sum_{\substack{l\in\mathbb{Z}^n:\\ t-l+k,l\in P^*_R(\epsilon)}} \|f_{t-(l-k)}\|_{L_{p_1}}\|g_l\|_{L_{p_2}}\Big]^q\Bigg)^{\frac{1}{q}}$$

with c_2 and c_1 as above. We put

$$S_{1,t,k} := \sum_{\substack{l\in\mathbb{Z}^n:\ t-l+k,l\in P^*_R(\epsilon),\\ |l|\le|l-k|}} \langle k-l\rangle^{s_1}\, \|f_{t-(l-k)}\|_{L_{p_1}}\, \langle l\rangle^{s_2}\, \|g_l\|_{L_{p_2}}\, \langle k-l\rangle^{s_0-s_1}\, \langle l\rangle^{-s_2}\,;$$

$$S_{2,t,k} := \sum_{\substack{l\in\mathbb{Z}^n:\ t-l+k,l\in P^*_R(\epsilon),\\ |l-k|\le|l|}} \langle k-l\rangle^{s_1}\, \|f_{t-(l-k)}\|_{L_{p_1}}\, \langle l\rangle^{s_2}\, \|g_l\|_{L_{p_2}}\, \langle k-l\rangle^{-s_1}\, \langle l\rangle^{s_0-s_2}\,.$$

Hölder's inequality leads to

$$S_{1,t,k} \le \Big(\sum_{\substack{j\in\mathbb{Z}^n,\\ |j+k|\le|j|}} (\langle j\rangle^{s_1} \| f_{t-j}\|_{L_{p_1}} \langle j+k\rangle^{s_2} \| g_{j+k}\|_{L_{p_2}})^q \Big)^{1/q}$$

$$\times \Big(\sum_{\substack{j\in\mathbb{Z}^n : t-j,\, j+k\in P_R^*(\epsilon)\\ |j+k|\le|j|}} (\langle j\rangle^{s_0-s_1} \langle j+k\rangle^{-s_2})^{q'} \Big)^{\frac{1}{q'}} .$$

Our assumptions $s_0 \le s_1,\ s_2 \ge 0$ and $s_1+s_2-s_0 > n/q'$ and $j+k \in P_R^*(\epsilon)$ imply

$$\begin{aligned}
\Big(\sum_{\substack{j\in\mathbb{Z}^n,\\ |j+k|\le|j|}} \Big|(\langle j\rangle^{s_0-s_1} \langle j+k\rangle^{-s_2})\Big|^{q'} \Big)^{\frac{1}{q'}} &\le \Big(\sum_{m\in P_R^*(\epsilon)} \langle m\rangle^{(s_0-s_1-s_2)q'} \Big)^{\frac{1}{q'}} \\
&\le \Big(2^{-n} \int_{\|x\|_\infty > R-2} (1+|x|^2)^{(s_0-s_1-s_2)q'/2}\, dx \Big)^{1/q'} \\
&\le \Big(2^{-n} \int_{|x|>R-2} |x|^{(s_0-s_1-s_2)q'}\, dx \Big)^{1/q'} \\
&\le \Big(\frac{2^{-n}}{(s_1+s_2-s_0)q'-n} \Big)^{1/q'} (R-2)^{-[(s_1+s_2-s_0)-n/q']} .
\end{aligned}$$

With $c_3 := \Big(\frac{2^{-n}}{(s_1+s_2-s_0)q'-n} \Big)^{1/q'}$ we insert this in our previous estimate and obtain

$$\begin{aligned}
\Big(\sum_{k\in\mathbb{Z}^n} S_{1,t,k}^q \Big)^{1/q} &\le c_3\, (R-2)^{-[(s_1+s_2-s_0)-n/q']} \\
&\quad \times \Big(\sum_{k\in\mathbb{Z}^n} \sum_{\substack{j\in\mathbb{Z}^n,\\ |j+k|\le|j|}} \langle j\rangle^{s_1 q} \| f_{t-j}\|_{L^{p_1}}^q \langle j+k\rangle^{s_2 q} \| g_{j+k}\|_{L^{p_2}}^q \Big)^{1/q} \\
&\le c_3\, c_4\, (R-2)^{-[(s_1+s_2-s_0)-n/q']} \|g\|_{M^{s_2}_{p_2,q}} \|f\|_{M^{s_1}_{p_1,q}} .
\end{aligned}$$

Here, c_3, c_4 are independent of f, g, ϵ and R. For the second sum the estimate

$$\Big(\sum_{k\in\mathbb{Z}^n} S_{2,t,k}^q \Big)^{1/q} \le c_5\, (R-2)^{-[(s_1+s_2-s_0)-n/q']} \|g\|_{M^{s_2}_{p_2,q}} \|f\|_{M^{s_1}_{p_1,q}}$$

follows by analogous computations. The proof is complete. ■

Of course, the above arguments have a counterpart in case $q' = \infty$.

Proposition 4.6 *Let* $1 \le p_1, p_2 \le \infty$, $q = 1$ *and* $s_1, s_2 \in \mathbb{R}$. *Define* p *by* $\frac{1}{p} := \frac{1}{p_1} + \frac{1}{p_2}$. *Let* $R > 2$. *If* $p \in [1, \infty]$, $s_1, s_2 \ge 0$ *and* $s_0 := \min(s_1, s_2)$, *then there exists a constant* c *such that*

$$\| f \cdot g \|_{M^{s_0}_{p,1}} \le c\,(R-2)^{-(s_1+s_2-s_0)}\, \| f \|_{M^{s_1}_{p_1,1}}\, \|g\|_{M^{s_2}_{p_2,1}}$$

holds for all $f \in M^{s_1}_{p_1,1}(\epsilon, R)$ *and all* $g \in M^{s_2}_{p_2,1}(\epsilon, R)$. *The constant* c *is independent from* $R > 2$ *and* ϵ.

As a consequence of Nikol'kij's inequality, see Lemma 2.6, Proposotion 4.5 (with $s_0 = s_1 = s_2$ and $p_1 = p$, $p_2 = \infty$) and Corollary 2.7 we obtain the following.

Proposition 4.7 *Let* $1 \le p \le \infty$ *and* $R > 2$.
(i) *Let* $1 < q \le \infty$ *and* $s > n/q'$. *Then there exists a constant* c *such that*

$$\| f \cdot g \|_{M^{s}_{p,q}} \le c\,(R-2)^{-(s-n/q')}\, \| f \|_{M^{s}_{p,q}}\, \|g\|_{M^{s}_{p,q}}$$

holds for all $f, g \in M^s_{p,q}(\epsilon, R)$. *The constant* c *is independent from* $R > 2$ *and* ϵ.
(ii) *Let* $q = 1$ *and* $s \ge 0$. *Then there exists a constant* c *such that*

$$\| f \cdot g \|_{M^{s}_{p,1}} \le c\,(R-2)^{-s}\, \| f \|_{M^{s}_{p,1}}\, \|g\|_{M^{s}_{p,1}}$$

holds for all $f, g \in M^s_{p,1}(\epsilon, R)$. *The constant* c *is independent from* $R > 2$ *and* ϵ.

Note that in the following, we assume every function to be real-valued unless it is explicitly stated that complex-valued functions are allowed. To make this more clear we switch from $g \in M^s_{p,q}$ to $u \in M^s_{p,q}$.

Next we have to recall some assertions from harmonic analysis. The first one concerns a standard estimate of Fourier multipliers, see e.g., [33, Theorem1.5.2].

Lemma 4.8 *Let* $1 \le r \le \infty$ *and assume that* $s > n/2$. *Then there exists a constant* $c > 0$ *such that*

$$\|\mathcal{F}^{-1}[\phi\,\mathcal{F}g](\,\cdot\,)\|_{L_r} \le c\, \|\phi\|_{H^s}\, \|g\|_{L_r}$$

holds for all $g \in L_r$ *and all* $\phi \in H^s$.

The next lemma is taken from [5].

Lemma 4.9 *Let* $N \in \mathbb{N}$ *and suppose* $a_1, a_2, \ldots, a_N$ *to be complex numbers. Then, it holds*

$$a_1 \cdot a_2 \cdot \ldots \cdot a_N - 1 = \sum_{l=1}^{N} \sum_{\substack{j=(j_1,\ldots,j_l),\\ 0\le j_1<\ldots<j_l\le N}} (a_{j_1} - 1) \cdot \ldots \cdot (a_{j_l} - 1).$$

In our approach the next estimate will be fundamental.

Proposition 4.10 *Let $1 < p < \infty$, $1 \le q \le \infty$ and $s > n/q'$. Then there exists a positive constant C such that*

$$\|e^{iu} - 1\|_{M^s_{p,q}} \le C\, \|u\|_{M^s_{p,q}} \, \Big(1 + \|u\|_{M^s_{p,q}}\Big)^{(s+n/q)(1+\frac{1}{s-n/q'})}$$

holds for all real-valued $u \in M^s_{p,q}$.

Proof This proof follows ideas developed in [5], but see also [23].
Step 1. Let u be a nontrivial function in $M^s_{p,q}$ satisfying $\operatorname{supp} \mathcal{F}u \subset P_R$ for some $R \ge 2$.

First we consider the Taylor expansion

$$e^{iu} - 1 = \sum_{l=1}^{r} \frac{(iu)^l}{l!} + \sum_{l=r+1}^{\infty} \frac{(iu)^l}{l!}$$

resulting in the norm estimate

$$\|e^{iu} - 1\|_{M^s_{p,q}} \le \Big\| \sum_{l=1}^{r} \frac{(iu)^l}{l!} \Big\|_{M^s_{p,q}} + \Big\| \sum_{l=r+1}^{\infty} \frac{(iu)^l}{l!} \Big\|_{M^s_{p,q}} .$$

For brevity we put

$$S_1 := \Big\| \sum_{l=1}^{r} \frac{(iu)^l}{l!} \Big\|_{M^s_{p,q}} \qquad \text{and} \qquad S_2 := \Big\| \sum_{l=r+1}^{\infty} \frac{(iu)^l}{l!} \Big\|_{M^s_{p,q}} .$$

The natural number r will be chosen later on. Next we employ the algebra property, in particular the estimate (4.4) with $C_1 := 2\, c_3\, c_4$. We obtain

$$S_2 \le \sum_{l=r+1}^{\infty} \frac{1}{l!} \|u^l\|_{M^s_{p,q}} \le \frac{1}{C_1} \sum_{l=r+1}^{\infty} \frac{(C_1\, \|u\|_{M^s_{p,q}})^l}{l!} .$$

Now we choose r as a function of $\|u\|_{M^s_{p,q}}$ and distinguish two cases:

1. $C_1\, \|u\|_{M^s_{p,q}} > 1$. Assume that

$$3\, C_1\, \|u\|_{M^s_{p,q}} \le r \le 3\, C_1\, \|u\|_{M^s_{p,q}} + 1 \tag{4.6}$$

and recall Stirling's formula $l! = \Gamma(l+1) \ge l^l e^{-l} \sqrt{2\pi l}$. Thus, we get

$$\sum_{l=r+1}^{\infty} \frac{(C_1 \|u\|_{M^s_{p,q}})^l}{l!} \le \sum_{l=r+1}^{\infty} \Big(\frac{r}{l}\Big)^l \Big(\frac{e}{3}\Big)^l \frac{1}{\sqrt{2\pi l}}$$
$$\le \sum_{l=r+1}^{\infty} \Big(\frac{e}{3}\Big)^l \le \frac{3}{3-e}.$$

2. $C_1 \|u\|_{M^s_{p,q}} \le 1$. It follows

$$\sum_{l=r+1}^{\infty} \frac{(C_1 \|u\|_{M^s_{p,q}})^l}{l!} \le C_1 \|u\|_{M^s_{p,q}} \sum_{l=1}^{\infty} \frac{1}{l!} \le C_1\, e\, \|u\|_{M^s_{p,q}}.$$

Both together can be summarized as

$$S_2 \le C_2 \|u\|_{M^s_{p,q}}\,, \qquad C_2 := \max\Big(e, \frac{3}{C_1(3-e)}\Big).$$

To estimate S_1 we check the support of $\mathcal{F}u^\ell$ and find

$$S_1 = \Big\| \sum_{l=1}^{r} \frac{(iu)^l}{l!} \Big\|_{M^s_{p,q}} = \Big(\sum_{k\in\mathbb{Z}^n} \langle k\rangle^{sq} \Big\| \Box_k\Big(\sum_{l=1}^{r} \frac{(iu)^l}{l!}\Big)\Big\|_{L_p}^q \Big)^{\frac{1}{q}}$$
$$= \Big(\sum_{\substack{k\in\mathbb{Z}^n,\\ -Rr-1<k_i<Rr+1,\\ i=1,\dots,n}} \langle k\rangle^{sq} \Big\| \Box_k\Big(\sum_{l=1}^{r} \frac{(iu)^l}{l!}\Big)\Big\|_{L_p}^q \Big)^{\frac{1}{q}}$$
$$\le \Big(\sum_{\substack{k\in\mathbb{Z}^n,\\ -Rr-1<k_i<Rr+1,\\ i=1,\dots,n}} \langle k\rangle^{sq} \|\Box_k(e^{iu}-1)\|_{L_p}^q \Big)^{\frac{1}{q}} + S_2\,.$$

Concerning S_2 we proceed as above. To estimate the first part we observe that

$$C_3 := \sup_{k\in\mathbb{Z}^n} \|\,\sigma_k\,\|_{H^t} = \|\,\sigma_0\,\|_{H^t} < \infty\,,$$

see Lemma 4.8. Furthermore, cos, sin are Lipschitz continuous and consequently we get

$$\|\Box_k(e^{iu}-1)\|_{L_p} \le C_3\, \|e^{iu}-1\|_{L_p}$$
$$\le C_3\, (\|\cos u - \cos 0\|_{L_p} + \|\sin u - \sin 0\|_{L_p})$$
$$\le 2\, C_3\, \|u - 0\|_{L_p}\,.$$

This implies

$$\Big(\sum_{\substack{k\in\mathbb{Z}^n,\\ -Rr-1<k_i<Rr+1,\\ i=1,\dots,n}} \langle k\rangle^{sq} \, \|\Box_k(e^{iu}-1)\|_{L_p}^q \Big)^{\frac{1}{q}} \le 2\, C_3 \, \|\, u\,\|_{L_p} \Big(\sum_{\substack{k\in\mathbb{Z}^n,\\ -Rr-1<k_i<Rr+1,\\ i=1,\dots,n}} \langle k\rangle^{sq} \Big)^{\frac{1}{q}} .$$

Clearly,

$$\begin{aligned} \sum_{\substack{k\in\mathbb{Z}^n,\\ -Rr-1<k_i<Rr+1,\\ i=1,\dots,n}} \langle k\rangle^{sq} &\le \int_{\|\,x\,\|_\infty<Rr+1} \langle x\rangle^{sq}\, dx \\ &\le \int_{|x|<\sqrt{n}(Rr+1)} \langle x\rangle^{sq}\, dx \\ &\le 2\, \frac{\pi^{n/2}}{\Gamma(n/2)} \int_0^{\sqrt{n}(Rr+1)} (1+\tau)^{n-1+sq}\, d\tau \\ &\le 2\, \frac{\pi^{n/2}}{\Gamma(n/2)}\, \frac{1}{n+sq}\, (\sqrt{n}(Rr+2))^{n+sq} \, . \end{aligned}$$

To simplify notation we define

$$C_4 := \Big(2\, \frac{\pi^{n/2}}{\Gamma(n/2)}\, \frac{1}{n+sq}\, \sqrt{n}^{n+sq} \Big)^{1/q} .$$

In addition we shall use in case $1 < q \le \infty$

$$\|u\|_{L^p} \le C_5\, \|u\|_{M^s_{p,q}} , \qquad C_5 := \Big(\sum_{k\in\mathbb{Z}^n} \langle k\rangle^{-sq'} \Big)^{1/q'}$$

which follows from Hölder's inequality and in case $q = 1$

$$\|u\|_{L^p} \le \|u\|_{M^s_{p,1}}$$

as a consequence of triangle inequality. Summarizing we have found

$$\|e^{iu}-1\|_{M^s_{p,q}} \le \Big(2\, C_2 + 2\, \max(C_5, 1)\, C_4\, C_3\, (Rr+2)^{s+n/q} \Big)\, \|u\|_{M^s_{p,q}} .$$

Next we apply (4.6) which results in

$$\|e^{iu}-1\|_{M^s_{p,q}} \le C_6\,\|u\|_{M^s_{p,q}}\Big(1+R\,\|u\|_{M^s_{p,q}}\Big)^{s+n/q}\,, \qquad (4.7)$$

valid for all $u \in M^s_{p,q}$ satisfying supp $\mathcal{F}u \subset P_R$ and with positive constant C_6 not depending on u and $R \ge 2$.
Step 2. This time we consider $u \in M^s_{p,q}$ without any restriction on the Fourier support. Here we need the restriction $1 < p < \infty$. For those p the characteristic functions χ of cubes are Fourier multipliers in L^p by the famous Riesz Theorem and therefore also in $M^s_{p,q}$. In addition we shall make use of the fact that the norm of the operator $f \mapsto \mathcal{F}^{-1}\chi\,\mathcal{F}f$ does not depend on the size of the cube. Below we shall denote this norm by $C_7 = C_7(p)$. We refer to Lizorkin [19] for all details. For decomposing u on the phase space we introduce functions $\chi_{R,\epsilon}$ and χ_R, that is, the characteristic functions of the sets $P_R(\epsilon)$ and P_R, respectively. By defining

$$u_\epsilon(x) = \mathcal{F}^{-1}[\chi_{R,\epsilon}(\xi)\,\mathcal{F}u(\xi)](x), \qquad x \in \mathbb{R}^n\,,$$
$$u_0(x) = \mathcal{F}^{-1}[\chi_R(\xi)\,\mathcal{F}u(\xi)](x)\,, \qquad x \in \mathbb{R}^n\,,$$

we can rewrite u as

$$u(x) = u_0(x) + \sum_{\epsilon \in I} u_\epsilon(x), \qquad (4.8)$$

where I is the set of all $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ with $\epsilon_j \in \{0,1\}$, $j = 1, \ldots, n$. Hence

$$\|u\|_{M^s_{p,q}} \le \|u_0\|_{M^s_{p,q}} + \sum_{\epsilon\in I} \|u_\epsilon\|_{M^s_{p,q}}$$

and

$$\max\Big(\|u_0\|_{M^s_{p,q}}, \|u_\epsilon\|_{M^s_{p,q}}\Big) \le C_7\,\|\,u\,\|_{M^s_{p,q}}\,.$$

Due to the representation (4.8) and using an appropriate enumeration Lemma 4.9 leads to

$$e^{iu} - 1 = \sum_{l=1}^{2^n+1} \sum_{0\le j_1<\ldots<j_l\le 2^n} (e^{iu_{j_1}}-1)\cdot\ldots\cdot(e^{iu_{j_l}}-1)\,.$$

The algebra property, in particular the estimate (4.4) with $C_1 := 2\,c_3\,c_4$, yields

$$\|e^{iu}-1\|_{M^s_{p,q}} \le \sum_{l=1}^{2^n+1} C_1^{l-1} \sum_{0\le j_1<\ldots<j_l\le 2^n} \|e^{iu_{j_1}}-1\|_{M^s_{p,q}}\cdot\ldots\cdot\|e^{iu_{j_l}}-1\|_{M^s_{p,q}}\,. \qquad (4.9)$$

By Proposition 4.7 and (4.7) it follows

$$\|e^{iu_{j_k}}-1\|_{M^s_{p,q}} = \Big\| \sum_{l=1}^\infty \frac{(iu_{j_k})^l}{l!} \Big\|_{M^s_{p,q}} \le \frac{R^{s-n/q'}}{c} \Big(e^{c\,\|u_{j_k}\|_{M^s_{p,q}}/R^{s-n/q'}} - 1\Big)$$
$$\le \frac{(R-2)^{s-n/q'}}{c} \Big(e^{c\,C_7\,\|u\|_{M^s_{p,q}}/(R-2)^{s-n/q'}} - 1\Big), \qquad (4.10)$$

as well as

$$\|e^{iu_0}-1\|_{M^s_{p,q}} \le C_6\, C_7\, \|u\|_{M^s_{p,q}} \Big(1 + R\, C_7 \|u\|_{M^s_{p,q}}\Big)^{s+n/q}, \qquad (4.11)$$

where we used the Fourier multiplier assertion mentioned at the beginning of this step. The final step in our proof is to choose the number R as a function of $\|u\|_{M^s_{p,q}}$ such that (4.10) and (4.11) will be approximately of the same size.
Substep 2.1. Let $\|u\|_{M^s_{p,q}} \le 1$. We choose $R = 3$. Then (4.9) combined with (4.10) and (4.11) results in the estimate

$$\|e^{iu}-1\|_{M^s_{p,q}} \le C_8\, \|u\|_{M^s_{p,q}},$$

where C_8 does not depend on u.
Substep 2.2. Let $\|u\|_{M^s_{p,q}} > 1$. We choose $R \ge 3$ such that

$$(R-2)^{s-n/q'} = \|u\|_{M^s_{p,q}}.$$

Now (4.9), combined with (4.10) and (4.11), results in

$$\|e^{iu}-1\|_{M^s_{p,q}} \le C_9\, \|u\|_{M^s_{p,q}} \Big(1 + \|u\|_{M^s_{p,q}}\Big)^{(s+n/q)(1+\frac{1}{s-n/q'})}, \qquad (4.12)$$

with a constant C_9 independent of u. ■

Remark 4.11 The restriction of p to the interval $(1,\infty)$ is caused by our decomposition technique, see Step 2 of the preceding proof. We do not know whether Proposition 4.10 extends to $p = 1$ and/or $p = \infty$.

Next, we need again a technical lemma.

Lemma 4.12 *Let* $1 < p < \infty$, $1 \le q \le \infty$ *and* $s > n/q'$.
(i) *The mapping* $u \mapsto e^{iu} - 1$ *is locally Lipschitz continuous (considered as a mapping of* $M^s_{p,q}$ *into* $M^s_{p,q}$*).*
(ii) *Assume* $u \in M^s_{p,q}$ *to be fixed and define a function* $g : \mathbb{R} \mapsto M^s_{p,q}$ *by* $g(\xi) = e^{iu(x)\xi} - 1$. *Then the function* g *is continuous.*

Proof Local Lipschitz continuity follows from the identity

$$e^{iu} - e^{iv} = (e^{iv} - 1)\,(e^{i(u-v)} - 1) + (e^{i(u-v)} - 1)\,, \tag{4.13}$$

the algebra property of $M^s_{p,q}$ and Proposition 4.10.

To prove the continuity of g we also employ the identity (4.13). The claim follows by using the algebra property and Proposition 4.10. ∎

Now we are in position to prove the main result of this section.

Theorem 4.13 *Let $1 < p < \infty$, $1 \le q \le \infty$ and $s > n/q'$. Let μ be a complex measure on $\mathbb{R}$ such that*

$$L := \int_{-\infty}^{\infty} (1 + |\xi|)^{1+(s+n/q)(1+\frac{1}{s-n/q'})}\, d|\mu|(\xi) < \infty \tag{4.14}$$

and such that $\mu(\mathbb{R}) = 0$. Furthermore, assume that the function f is the inverse Fourier transform of μ. Then f is a continuous function and the composition operator $T_f : u \mapsto f \circ u$ maps $M^s_{p,q}$ into $M^s_{p,q}$.

Proof Equation (4.14) yields $\int_{\mathbb{R}^n} d|\mu|(\xi) < \infty$. Thus, μ is a finite measure and $\mu(\mathbb{R}) = 0$ makes sense. Now we define the inverse Fourier transform of μ

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^n} e^{i\xi t}\, d\mu(\xi).$$

Moreover, since

$$(s + n/q)\Big(1 + \frac{1}{s - n/q'}\Big) > n$$

we conclude that $\int_{\mathbb{R}} |(i\xi)^j|\, d|\mu|(\xi) < \infty$, $j = 1, \dots, n+1$, which implies $f \in C^{n+1}$. Due to $\mu(\mathbb{R}) = 0$ we can also write f as follows:

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (e^{i\xi t} - 1)\, d\mu(\xi).$$

Since μ is a complex measure we can split it up into real part μ_r and imaginary part μ_i, where each of them is a signed measure. Without loss of generality we proceed our computations only with the positive real measure μ_r^+. For all measurable sets E we have $\mu_r^+(E) \le |\mu|(E)$.

Let $u \in M^s_{p,q}$ and define the function $g(\xi) = e^{iu(x)\xi} - 1$ analogously to Lemma 4.12. Then g is Bochner integrable because of its continuity and taking into account that the measure μ_r^+ is finite. Therefore we obtain the Bochner integral

$$\int_{-\infty}^{\infty} \big(e^{iu(x)\xi} - 1\big)\, d\mu_r^+(\xi) = \int_{-\infty}^{\infty} g(\xi)\, d\mu_r^+(\xi)$$

with values in $M^s_{p,q}$. By applying Minkowski inequality it follows

$$\Big\| \int_{-\infty}^{\infty} \big(e^{iu(\cdot)\xi} - 1\big)\, d\mu_r^+(\xi) \Big\|_{M^s_{p,q}} \le \int_{-\infty}^{\infty} \big\| e^{iu(\cdot)\xi} - 1 \big\|_{M^s_{p,q}}\, d|\mu|(\xi).$$

Using the abbreviation $\|u\| := \|u\|_{M^s_{p,q}}$, Proposition 4.10 together with equation (4.14) yields

$$\begin{aligned}
&\int_{|\xi|\|u\|\ge 1} \big\| e^{iu(\cdot)\xi} - 1 \big\|_{M^s_{p,q}}\, d|\mu|(\xi) \\
&\qquad \le C' \, \| u \|_{M^s_{p,q}}^{1+(s+n/q)(1+\frac{1}{s-n/q'})} \int_{|\xi|\|u\|\ge 1} |\xi|^{1+(s+n/q)(1+\frac{1}{s-n/q'})}\, d|\mu|(\xi) \\
&\qquad < \infty .
\end{aligned}$$

In a similar way the remaining part $|\xi| \le 1/\|u\|$ of the integral can be treated. The same estimates also hold for the measures μ_r^-, μ_i^+ and μ_i^-. Thus, the result is obtained by

$$\begin{aligned}
&\|\sqrt{2\pi}\, f(u(x))\|_{M^s_{p,q}} \\
&= \Big\| \int_{-\infty}^{\infty} g(\xi)\, d\mu_r^+ - \int_{-\infty}^{\infty} g(\xi)\, d\mu_r^- + i \int_{-\infty}^{\infty} g(\xi)\, d\mu_i^+ - i \int_{-\infty}^{\infty} g(\xi)\, d\mu_i^- \Big\|_{M^s_{p,q}} \\
&\le \int_{-\infty}^{\infty} \|g(\xi)\|_{M^s_{p,q}}\, d|\mu_r^+| + \int_{-\infty}^{\infty} \|g(\xi)\|_{M^s_{p,q}}\, d|\mu_r^-| \\
&\qquad + \int_{-\infty}^{\infty} \|g(\xi)\|_{M^s_{p,q}}\, d|\mu_i^+| + \int_{-\infty}^{\infty} \|g(\xi)\|_{M^s_{p,q}}\, d|\mu_i^-| ,
\end{aligned}$$

where every integral on the right-hand side is finite. Thus, the statement is proved. ∎

A bit more transparent sufficient conditions can be obtained by using Szasz theorem, see Peetre [22, pp. 9–11] and [27, Proposition 1.7.5]. By $B^s_{p,q}(\mathbb{R})$ we denote the Besov spaces on $\mathbb{R}$, see e.g., [33] or [25] for details.

Lemma 4.14 *Let $t \ge 0$ and suppose $f \in B^{t+1/2}_{2,1}(\mathbb{R})$. Then the Fourier transform of f is a regular distribution and*

$$\int_{-\infty}^{\infty} (1 + |\xi|^2)^{t/2} |\mathcal{F} f(\xi)|\, d\xi \le c\, \| f \|_{B^{t+1/2}_{2,1}(\mathbb{R})}$$

follows with some c independent of f.

Based on Lemma 4.14 and Theorem 4.13 one obtains the next result.

Corollary 4.15 *Let $1 < p < \infty$, $1 \le q \le \infty$ and $s > n/q'$. Let $f \in B^t_{2,1}(\mathbb{R})$ for some*

$$t \geq \frac{3}{2} + (s + n/q)\left(1 + \frac{1}{s - n/q'}\right)$$

and suppose $f(0) = 0$. *Then the composition operator* $T_f : u \mapsto f \circ u$ *maps real-valued functions in* $M^s_{p,q}$ *boundedly into* $M^s_{p,q}$.

Proof Boundedness of T_f follows from Proposition 4.10, the proof of Theorem 4.13 and Lemma 4.14. ■

Remark 4.16 Let $t > 0$ be given. A function $f : \mathbb{R} \to \mathbb{R}$, m-times continuously differentiable, compactly supported and satisfying $f^m \in \operatorname{Lip}\alpha$ for some $\alpha \in (0, 1]$, belongs to $B^t_{2,1}(\mathbb{R})$ if $t < m + \alpha$.

4.5 One Example

Ruzhansky, Sugimoto, and Wang [26] suggested to study the operator T_α associated to $f_\alpha(t) := t\,|t|^\alpha$, $t \in \mathbb{R}$, with $\alpha > 0$. This function belongs locally to the Besov space $B^{\alpha+1+1/p}_{p,\infty}(\mathbb{R})$, $1 \leq p \leq \infty$, see [25, Lemma 2.3.1/1] for a related case. Let $\psi \in C_0^\infty(\mathbb{R})$ be a smooth cut-off function such that $\psi(x) = 1$ if $|x| \leq 1$. Then the function

$$\tilde{f}_{\alpha,\lambda}(t) := \psi(t/\lambda) \cdot f_\alpha(t), \qquad t \in \mathbb{R},$$

belongs to $B^{\alpha+1+1/p}_{p,\infty}$ for any p, $1 \leq p \leq \infty$, and any $\lambda > 0$. Applying Corollary 4.15 and

$$u(x)\,|u(x)|^\alpha = \tilde{f}_{\alpha,\lambda}(u(x)), \qquad x \in \mathbb{R}^n, \quad \lambda := \|u\|_{L_\infty},$$

we find the following.

Corollary 4.17 *Let* $1 < p < \infty$, $1 \leq q \leq \infty$ *and* $s > n/q'$. *Let* α *be a positive real number such that*

$$(s + n/q)\left(1 + \frac{1}{s - n/q'}\right) < \alpha\,.$$

Then the composition operator $T_\alpha : \ u \mapsto u\,|u|^\alpha$ *maps real-valued functions in* $M^s_{p,q}$ *boundedly into* $M^s_{p,q}$.

4.6 The Special Case $p = q = 2$

Finally, we will have a look onto the special case $M^s_{2,2} = H^s$, $s > n/2$. In Bourdaud, Moussai, S. [4] the set of functions f such that $T_f : g \mapsto f \circ g$ maps H^s into itself has been characterized.

Proposition 4.18 *Let* $s > \frac{1}{2}\max(n, 3)$. *For a Borel measurable function* $f : \mathbb{R} \to \mathbb{R}$ *the composition operator* T_f *acts on* H^s *if and only if* $f(0) = 0$ *and* $f \in H^{s,\ell oc}(\mathbb{R})$.

Concerning our example T_α treated above this yields the following: T_α maps H^s into itself if and only if $\alpha > s - 3/2$ (instead of $\alpha > s + \frac{n}{2} + \frac{s+n/2}{s-n/2}$ as required in Corollary 4.17).

Corollary 4.15 and Corollary 4.17 may be understood as first results about sufficient conditions, not more.

4.7 A Final Remark

The method employed here has been used before in connection with composition operators on Gevrey-modulation spaces and modulation spaces of ultradifferentiable functions, see Bourdaud [3], Bourdaud et al. [5], Reich et al. [5], and Reich [24], for Hörmander-type spaces $B_{p,k}$ we refer to Jornet and Oliaro [16]. It would be desirable to develop this method more systematically.

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