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Filippo Gazzola
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Geometric Properties for Parabolic and Elliptic PDE's

GPPEPDEs, Palinuro, Italy, May 2015

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Preface

This monograph contains the contributions from the speakers at the 4th Italian-Japanese workshop on *Geometric Properties for Parabolic and Elliptic PDE's*, which was held in Palinuro (Italy) during the week of 25–29 May 2015. The first three workshops were held in Sendai (Japan, 2009), Cortona (Italy, 2011) and Tokyo (Japan, 2013) and on all the three occasions the proceedings were subsequently published: see, respectively, Discrete Contin. Dyn. Syst. Ser. S 4 (2011), Springer INdAM Ser. 2 (2013) and Kodai Math. J. 37 (2014). Based on the success of the previous workshops and the associated publications, we believe that this monograph will be of great interest for the mathematical community and in particular for researchers studying parabolic and elliptic PDE's.

As would be expected from such a wide topic, the contributions are very diverse. They cover many different fields of current research as follows: nonlinear parabolic and elliptic equations, Hardy–Rellich inequalities, overdetermined problems, optimal transport, anisotropic equations, symmetry problems on isothermic surfaces, dynamic hybrid systems, Littlewood's fourth principle, eigenvalue problems, singular solutions, stability of Delaunay surfaces, non-Archimedean mathematics. In order to guarantee quality, all the papers have been submitted to two referees, chosen among the experts on related topics.

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Estimates for Solutions to Anisotropic Elliptic Equations with Zero Order Term

Angela Alberico, Giuseppina di Blasio and Filomena Feo

Abstract Estimates for solutions to homogeneous Dirichlet problems for a class of elliptic equations with zero order term in the form $L(u) = g(x, u) + f(x)$, where the operator L fulfills an anisotropic elliptic condition, are established. Such estimates are obtained in terms of solutions to suitable problems with radially symmetric data, when no sign conditions on g are required.

Keywords Anisotropic symmetrization · A priori estimate · Anisotropic Dirichlet problems

1 Introduction

We are concerned with a comparison result via symmetrization for solutions to a class of anisotropic Dirichlet problems, whose prototype can be written as follows

$$\begin{cases} \displaystyle\sum_{i=1}^{N} \alpha_i (|\partial_{x_i} u|^{p_i-2} \partial_{x_i} u)_{x_i} = c(x)|u|^{\bar{p}-2}u + f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

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where Ω is a bounded, smooth open subset of $\mathbb{R}^N$, $N \geq 2$, $\alpha_i > 0$ for $i = 1, \ldots, N$, $1 \leq p_1, \ldots, p_N < \infty$ such that their harmonic mean $\bar{p}$ is greater than 1, and f belongs to a suitable Lebesque space.

In the last years anisotropic problems have been largely studied by many authors (see e.g. [6, 10, 15, 20–22, 27]).

We observe that when $p_i = p \neq 2$ for every $i = 1, \ldots, N$, the principal part operator in problem (1) coincides with the so-called pseudo-Laplacian operator, whereas when $p_i = 2$ for every $i = 1, \ldots, N$ the operator coincides with the usual Laplacian.

Symmetrization methods in comparison results for solutions to isotropic elliptic problems with a zero lower order term were used in several papers (see e.g. [5, 18, 19, 28], and bibliography therein). In the quoted papers the authors require either sign assumptions on $c(x)$ or not.

In the same spirit of [19], we are interested in studying the anisotropic problem (1) when no sign assumption is made on $c(x)$. Using symmetrization techniques we want estimate a solution to problem (1) with the solution to an appropriate symmetrized problem which takes into account also of the influence of the zero order term.

More precisely, we compare the decreasing rearrangement u^* of a solution u to problem (1) with the decreasing rearrangement v^* of the solution v to the following symmetrized problem

$$\begin{cases} -\text{div}\left(\Lambda|\nabla v|^{\bar{p}-2}\nabla v\right) = \widehat{c}|v|^{\bar{p}-2}v + f^\bigstar & \text{in } \Omega^\bigstar \\ v = 0 & \text{on } \partial\Omega^\bigstar. \end{cases} \tag{2}$$

In (2) the datum $f^\bigstar$ is the spherically symmetric decreasing rearrangement of f, $\Omega^\bigstar$ is the ball centered at the origin such that $\left|\Omega^\bigstar\right| = |\Omega|$, Λ is a suitable positive constant, $\bar{p}$ is the harmonic mean of $p_1, \ldots, p_N$ and the coefficient $\widehat{c}$ is linked to the rearrangements of the positive and negative part of $c(x)$, i.e. it is linked to $c^+(x) = \max\{c(x), 0\}$ and $c^-(x) = \max\{-c(x), 0\}$. In contrast to the isotropic case not only the domain and the data of problem (2) are symmetrized, but also the ellipticity condition is subject to an appropriate symmetrization. Indeed the operator in problem (2) is the isotropic $\bar{p}$-Laplacian.

Since no sign condition on $c(x)$ is required, in order to assure the existence of a unique nonnegative weak solution v to problem (2), we have to impose a smallness condition on $c^+(x)$, namely,

$$\left\|c^+\right\|_{L^\infty} < \Lambda\, \lambda(\Omega^\bigstar), \tag{3}$$

where $\lambda(\Omega^\bigstar)$ is the first eigenvalue of the Dirichlet problem

$$\begin{cases} -\text{div}\left(|\nabla v|^{\bar{p}-2}\nabla v\right) = \lambda|v|^{\bar{p}-2}v & \text{in } \Omega^\bigstar \\ v = 0 & \text{on } \partial\Omega^\bigstar. \end{cases}$$

If the smallness condition (3) holds, we prove that the following pointwise estimate

$$u^*(s) \le v^*(s)$$

holds in $[0, s_0]$ with $s_0 = \inf\left\{s \in [0, |\Omega|] : \left(c^-\right)_*(s) > 0\right\}$ and the following comparison between concentration

$$\int_0^s \left(u^*(\sigma)\right)^{\bar{p}-1} d\sigma \le \int_0^s \left(v^*(\sigma)\right)^{\bar{p}-1} d\sigma$$

holds in $[s_0, |\Omega|]$.

On the other hand, if (3) is not verified, a comparison result is not assured (see Remark 1).

We emphasize that problem (1) belongs to a larger class of anisotropic problems

$$\begin{cases} -\operatorname{div}(a(x, u, \nabla u)) = F(x, u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where the ellipticity condition is given in term of a N-dimensional Young function $\Phi : \mathbb{R}^N \to [0, +\infty)$, i.e., for *a.e.* $x \in \Omega$,

$$a(x, s, \xi) \cdot \xi \ge \Phi(\xi) \quad \text{for} \quad (s, \xi) \in \mathbb{R} \times \mathbb{R}^N. \tag{4}$$

In this case the anisotropy is governed by a general N-dimensional convex function of the gradient not necessary of polynomial type as in problem (1).

Problems governed by fully anisotropic growth conditions as in (4) have been recently studied in [1–3, 12, 13]. There is also a large number of papers related to a different type of anisotropy (see e.g. [4, 8, 14]).

2 Preliminaries

2.1 Anisotropic Spaces

Let Ω be a bounded, smooth open subset of $\mathbb{R}^N$ with $N \ge 2$, and let $1 \le p_1, \ldots, p_N < \infty$ be N real numbers. Recall that the anisotropic Sobolev space (see e.g. [29]) defined by

$$W^{1,\vec{p}}(\Omega) = \left\{u \in W^{1,1}(\Omega) : \partial_{x_i} u \in L^{p_i}(\Omega), i = 1, \ldots, N\right\}$$

is a Banach space with respect to the norm

$$\|u\|_{W^{1,\vec{p}}(\Omega)} = \|u\|_{L^1(\Omega)} + \sum_{i=1}^{N} \left\|\partial_{x_i} u\right\|_{L^{p_i}(\Omega)}. \tag{5}$$

The space $W_0^{1,\vec{p}}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm (5).

In this anisotropic setting a Poincaré-type inequality holds (see [20]). If $u \in W_0^{1,\vec{p}}(\Omega)$, then for every $q \geq 1$ there exists a constant C_P, depending on q and i, such that

$$\|u\|_{L^q(\Omega)} \leq \; C_P \; \left\|\partial_{x_i} u\right\|_{L^q(\Omega)} \quad \text{for } \; i = 1, \dots, N. \tag{6}$$

Denoted by $\bar{p}$ the harmonic mean of $p_1, \dots, p_N$, i.e.

$$\frac{1}{\bar{p}} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{p_i}, \tag{7}$$

a Sobolev-type inequality tells us that whenever u belongs to $W_0^{1,\vec{p}}(\Omega)$, there exists a constant C_S such that

$$\|u\|_{L^q(\Omega)} \leq C_S \sum_{i=1}^{N} \left\|\partial_{x_i} u\right\|_{L^{p_i}(\Omega)} \tag{8}$$

(see [29]). If in plus $\bar{p} < N$, inequality (8) implies the continuous embedding of the space $W_0^{1,\vec{p}}(\Omega)$ into $L^q(\Omega)$ for every $q \in [1, \bar{p}^*]$. On the other hand, the continuity of the embedding $W_0^{1,\vec{p}}(\Omega) \subset L^{p_+}(\Omega)$ with $p_+ := \max\{p_1, \dots, p_N\}$ relies on inequality (6). It may happen that $\bar{p}^* < p_+$ if the exponents p_i are not closed enough. In this case $p_\infty := \max\{\bar{p}^*, p_+\}$ turns out to be the critical exponent in the anisotropic Sobolev embedding.

2.2 *Symmetrization*

A precise statement of our result requires the use of classical notions of rearrangement and of suitable symmetrization of a N-dimensional Young function introduced by Klimov in [25].

Let u be a measurable function (continued by 0 outside its domain) fulfilling

$$\left|\{x \in \mathbb{R}^N : |u(x)| > t\}\right| < +\infty \quad \text{for } \; t > 0. \tag{9}$$

The *symmetric decreasing rearrangement* of u is the function $u^\star : \mathbb{R}^N \to [0, +\infty[$ satisfying

$$\{x \in \mathbb{R}^N : u^\bigstar(x) > t\} = \{x \in \mathbb{R}^N : |u(x)| > t\}^\bigstar \quad \text{for } t > 0. \tag{10}$$

The *decreasing rearrangement* u^* of u is defined as

$$u^*(s) = \sup\{t > 0 : |\{x \in \mathbb{R}^N : |u(x)| > t\}| > s\} \quad \text{for } s \geq 0. \tag{11}$$

Analogously, if

$$|\{x \in \mathbb{R}^N : |u(x)| < t\}| < +\infty \quad \text{for } t > 0,$$

we define the *symmetric increasing rearrangement* $u_\bigstar$ and the *increasing rearrangement* u_* on replacing ">" by "<" in equalities (10) and (11), respectively.

Moreover, for a.e. $x \in \mathbb{R}^N$,

$$u^\bigstar(x) = u^*(\omega_N |x|^N)$$

and

$$u_\bigstar(x) = u_*(\omega_N |x|^N)$$

with ω_N the measure of the N-dimensional unit ball.

For more details on rearrangements see, for example, [9, 11, 23]. We just recall the following properties of rearrangements which will be useful in the sequel.

Lemma 1 (Proposition 3.6, Chap. 2 of [9]) *Let $f_1(s)$ and $f_2(s)$ be measurable, positive functions such that*

$$\int_0^s f_1(t)\, dt \leq \int_0^s f_2(t)\, dt \quad \textit{for } s \in [0, \alpha].$$

If $f_3 \geq 0$ is a decreasing function, then

$$\int_0^s f_1(t) f_3(t)\, dt \leq \int_0^s f_2(t) f_3(t)\, dt \quad \textit{for } s \in [0, \alpha].$$

Lemma 2 (Theorem 4.6, Chap. 2 of [9]) *If $f_1(s)$ and $f_2(s)$ belong to $L^p(\Omega)$ for $1 \leq p \leq \infty$, and*

$$\int_0^s f_1^*(t)\, dt \leq \int_0^s f_2^*(t)\, dt \quad \textit{for } s \in [0, |\Omega|],$$

then

$$\|f_1\|_{L^p(\Omega)} \leq \|f_2\|_{L^p(\Omega)}.$$

Now we introduce a suitable symmetrization of a N-dimensional Young function. Let $\Phi : \mathbb{R}^N \to [0, +\infty[$ be a N*-dimensional Young function*, namely an even convex function such that

$$\Phi(0) = 0 \qquad \text{and} \qquad \lim_{|\xi|\to+\infty} \Phi(\xi) = +\infty.$$

The Young inequality tells us that

$$\xi \cdot \xi' \le \Phi(\xi) + \Phi_\bullet(\xi') \quad \text{for } \xi, \xi' \in \mathbb{R}^N,$$

where $\Phi_\bullet$ is the *Young conjugate* function of Φ given by

$$\Phi_\bullet(\xi') = \sup\left\{\xi \cdot \xi' - \Phi(\xi) : \xi \in \mathbb{R}^N\right\} \quad \text{for } \xi' \in \mathbb{R}^N.$$

Here, " $\cdot$ " stands for scalar product in $\mathbb{R}^N$. We observe that the function $\Phi_\bullet$ enjoys the same properties as Φ and is a N-dimensional Young function if

$$\lim_{|\xi|\to+\infty} \frac{\Phi(\xi)}{|\xi|} = +\infty.$$

We denote by $\Phi_\blacklozenge : \mathbb{R} \to [0, +\infty[$ the symmetrization of Φ introduced by Klimov in [25]. It is the one-dimensional Young function fulfilling

$$\Phi_\blacklozenge(|\xi|) = \Phi_{\bullet\bigstar\bullet}(\xi) \quad \text{for } \xi \in \mathbb{R}^N,$$

namely it is the composition of Young conjugation, symmetric increasing rearrangement and Young conjugate again. We stress that the functions $\Phi_\blacklozenge$ and $\Phi_\bigstar$ are not equal in general. However, $\Phi_\blacklozenge$ and $\Phi_\bigstar$ are always equivalent, in the sense that constants $k_1 = k_1(n)$ and $k_2 = k_2(n)$ exist such that

$$\Phi_\bigstar(k_1\xi) \le \Phi_\blacklozenge(|\xi|) \le \Phi_\bigstar(k_2\xi) \qquad \text{for } \xi \in \mathbb{R}^N$$

(see [24, Lemma 7]). Moreover $\Phi_\blacklozenge(|\cdot|) = \Phi_\bigstar(\cdot)$ if and only if Φ is radial, i.e. $\Phi = \Phi_\bigstar$.

In this paper we will consider

$$\Phi(\xi) = \sum_{i=1}^{N} \alpha_i |\xi_i|^{p_i} \tag{12}$$

for $\xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^N$ and some $\alpha_i > 0$ for $i = 1, \dots, N$. Easy calculations show that

$$\Phi_\bullet(\xi) = \sum_{i=1}^{N} \frac{|\xi_i|^{p_i'}}{(p_i\alpha_i)^{p_i'/p_i}}$$

for $\xi \in \mathbb{R}^N$, where $p_i' = \frac{p_i}{p_i-1}$ with the usual conventions if $p_i = 1$. In [13] it is proved that

$$\Phi_\blacklozenge(|\xi|) = \Lambda\, |\xi|^{\bar{p}}, \tag{13}$$

where $\bar{p}$ is defined in (7) and

$$\Lambda = \frac{2^{\bar{p}}\,(\bar{p}-1)^{\bar{p}-1}}{\bar{p}^{\bar{p}}}\left[\frac{\prod_{i=1}^{N} p_i^{\frac{1}{p_i}}\,\left(p_i'\right)^{\frac{1}{p_i'}}\,\Gamma(1+1/p_i')}{\omega_N\,\Gamma(1+N/\bar{p}')}\right]^{\frac{\bar{p}}{N}}\left(\prod_{i=1}^{N}\alpha_i^{\frac{1}{p_i}}\right)^{\frac{\bar{p}}{N}} \tag{14}$$

with Γ the Gamma function.

We recall that in the anisotropic setting a *Polya-Szegö principle* holds (see [13]). Let u be a weakly differentiable function in $\mathbb{R}^N$ satisfying (9) and such that $\int_{\mathbb{R}^N}\Phi\,(\nabla u)\,dx < +\infty$, then $u^\star$ is weakly differentiable in $\mathbb{R}^N$ and

$$\int_{\mathbb{R}^N}\Phi_\blacklozenge\left(|\nabla u^\star|\right)dx \le \int_{\mathbb{R}^N}\Phi\,(\nabla u)\,dx\,. \tag{15}$$

3 Comparison Results

We consider the following class of problems

$$\begin{cases} -\operatorname{div}(a(x,u,\nabla u)) = g(x,u) + f(x) & \text{in } \Omega \\ \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{16}$$

where Ω is a bounded, smooth open subset of $\mathbb{R}^N$ with $N \ge 2$, $1 \le p_1,\ldots,p_N < \infty$ such that $\bar{p} > 1$, $a : \Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}^N$ and $g : \Omega\times\mathbb{R}\to\mathbb{R}$ are Carathéodory functions such that for *a.e.* $x\in\Omega$, for every $s\in\mathbb{R}$, $\xi\in\mathbb{R}^N$ and for $j = 1,\ldots,N$

(A1) $\quad a(x,s,\xi)\cdot\xi \ge \sum_{i=1}^{N}\alpha_i\,|\xi_i|^{p_i}$ with $\alpha_i > 0$;

(A2) $\quad \left|a_j(x,s,\xi)\right| \le \beta\left[h(x) + |s|^{\bar{p}} + \sum_{i=1}^{N}|\xi_i|^{p_i}\right]^{\frac{1}{p_j'}}$ with $\beta > 0$, $h \ge 0$, $h\in L^1(\Omega)$;

(A3) $\quad |g(x,s)| \le \gamma\,|s|^{\bar{p}-1}$ with $\gamma > 0$;

(A4) $\quad g(x,s)s \le c(x)\,|s|^{\bar{p}}$ with $c\in L^\infty(\Omega)$.

Moreover on the data f we will make suitable summability assumptions.

Let us recall the definition of a weak solution to problem (16).

Definition 1 A weak solution to problem (16) is a function $u\in W_0^{1,\vec{p}}(\Omega)$ such that

$$\int_\Omega a(x,u,\nabla u)\cdot\nabla\varphi(x)\,dx = \int_\Omega\left[g(x,u)+f(x)\right]\varphi(x)\,dx \quad \forall\varphi\in W_0^{1,\vec{p}}(\Omega). \tag{17}$$

Our aim is to obtain an estimate for solution to problem (16) in term of the solution to a suitable problem which takes into account of the influence of zero order term.

Theorem 1 *Let us suppose that* $f \in L^{\infty}(\Omega)$*,* $f \not\equiv 0$*, and conditions* $(A1)$–$(A4)$ *and (3) hold. Putting*

$$s_0 = \inf \left\{ s \in [0, |\Omega|] : \left(c^{-}\right)_{*}(s) > 0 \right\}, \tag{18}$$

if u *is a weak solution to problem (16), we have*

$$u^{*}(s) \leq v^{*}(s) \quad \textit{for} \quad s \in [0, s_0] \tag{19}$$

and

$$\int_0^s \left(u^{*}(\sigma)\right)^{\bar{p}-1} d\sigma \leq \int_0^s \left(v^{*}(\sigma)\right)^{\bar{p}-1} d\sigma \quad \textit{for} \;\; s \in [s_0, |\Omega|], \tag{20}$$

where $v(x)$ *is the solution to problem*

$$\begin{cases} -\operatorname{div}\left(\Lambda \left|\nabla v\right|^{\bar{p}-2} \nabla v\right) = \widehat{c}\,|v|^{\bar{p}-2} v + f^{\star} & \textit{in } \Omega^{\star} \\ v = 0 & \textit{on } \partial\Omega^{\star} \end{cases} \tag{21}$$

with Λ *and* $\bar{p}$ *defined as in (14) and in (7), respectively, and* $\widehat{c}(x) := \left(c^{+}\right)^{\star}(x) - \left(c^{-}\right)_{\star}(x)$ *for* $x \in \Omega^{\star}$*.*

Proof We split the proof in several steps.
Step 1: We define the functions $u_{\kappa,t} : \Omega \to \mathbb{R}$ as

$$u_{\kappa,t}(x) = \begin{cases} 0 & \text{if } |u(x)| \leq t, \\ (|u(x)| - t)\,\operatorname{sign}(u(x)) & \text{if } t < |u(x)| \leq t + \kappa \\ \kappa \,\operatorname{sign}(u(x)) & \text{if } t + \kappa < |u(x)| \end{cases} \tag{22}$$

for any fixed t and $\kappa > 0$. Observing that $u_{\kappa,t}$ belongs to $W_0^{1,\overrightarrow{p}}(\Omega)$, and $\nabla u_{\kappa,t} = \chi_{\{t<|u|\leq t+\kappa\}} \nabla u$ *a.e.* in Ω, function $u_{\kappa,t}$ can be chosen as test function in (17) and by (A1) we get

$$\begin{aligned} \frac{1}{\kappa}\int_{t<|u|\leq t+\kappa} \Phi(\nabla u)\, dx &\leq \frac{1}{\kappa}\int_{t<|u|\leq t+\kappa} a(x,u,\nabla u)\, dx \\ &= \frac{1}{\kappa}\int_{t<|u|\leq t+\kappa} (g(x,u) + f(x))\,(|u(x)| - t)\operatorname{sign}(u(x))\, dx \\ &\quad + \int_{|u|>t+\kappa} (g(x,u) + f(x))\operatorname{sign}(u(x))\, dx\,. \end{aligned} \tag{23}$$

Arguing as in [13], we apply Polya-Szegö principle (15) to function $u_{\kappa,t}$ continued by 0 outside Ω, obtaining

$$\int_{t<|u|\le t+\kappa} \Phi(\nabla u)\,dx = \int_{\mathbb{R}^N} \Phi(\nabla u_{t,\kappa})\,dx \ge \int_{\mathbb{R}^N} \Phi_\blacklozenge(|\nabla (u_{t,\kappa})^\bigstar|)\,dx$$
$$= \int_{\mathbb{R}^N} \Phi_\blacklozenge(|\nabla (u^\bigstar)_{t,\kappa}|)\,dx = \int_{t<u^\bigstar\le t+\kappa} \Phi_\blacklozenge(|\nabla u^\bigstar|)\,dx,$$

where $(u_{t,\kappa})^\bigstar = (u^\bigstar)_{t,\kappa}$ *a.e.* in $\mathbb{R}^N$ and the function $(u^\bigstar)_{t,\kappa}$ is defined as in (22) with u replaced by $u^\bigstar$.
Since the functions

$$t \longrightarrow \int_{|u|>t} \Phi(\nabla u)\,dx\,, \qquad t \longrightarrow \int_{u^\bigstar>t} \Phi_\blacklozenge(|\nabla u^\bigstar|)\,dx$$

are Lipschitz continuous on $(0, +\infty)$, by (12) and (13), we have

$$-\frac{d}{dt}\int_{u^\bigstar>t} \Lambda|\nabla u^\bigstar|^{\bar p}dx = -\frac{d}{dt}\int_{u^\bigstar>t} \Phi_\blacklozenge(|\nabla u^\bigstar|)\,dx \tag{24}$$
$$\le -\frac{d}{dt}\int_{|u|>t} \Phi(\nabla u)\,dx = -\frac{d}{dt}\int_{|u|>t} \sum_{i=1}^N \alpha_i \left|\frac{\partial u}{\partial x_i}\right|^{p_i} dx\,.$$

Step 2: By (23), (24) and (A4) we get

$$-\frac{d}{dt}\int_{u^\bigstar>t} \Lambda|\nabla u^\bigstar|^{\bar p}dx \le \int_{|u|>t} c\,(x)\,|u\,(x)|^{\bar p-1}\,dx + \int_{|u|>t} |f\,(x)|\,dx \tag{25}$$
$$\le \int_0^{\mu_u(t)} \left[\left(c^+\right)^*(s) - \left(c^-\right)_*(s)\right]\left(u^*(s)\right)^{\bar p-1} ds + \int_0^{\mu_u(t)} f^*(s)\,ds,$$

where $\mu_u(t) = |\{x \in \Omega : |u(x)| > t\}|$ and the last inequality follows by Hardy-Littlewood inequality. By Coarea formula, recalling that the level set of $u^\bigstar$ are balls and using Hölder inequality, we get

$$\left(-\frac{d}{dt}\int_{u^\bigstar>t} |\nabla u^\bigstar|^{\bar p}dx\right)^{1/\bar p} \ge N\omega_N^{1/N}\mu_u\,(t)^{1/N'}\left(-\mu_u'\,(t)\right)^{1/\bar p'} \quad \text{a.e. } t>0. \tag{26}$$

Using (26) in (25) we get

$$\left(N\omega_N^{1/N}\mu_u\,(t)^{1/N'}\left(-\mu_u'\,(t)\right)^{1/\bar p'}\right)^{\bar p} \tag{27}$$
$$\le \frac{1}{\Lambda}\int_0^{\mu_u(t)} \left\{\left[\left(c^+\right)^*(s) - \left(c^-\right)_*(s)\right]\left(u^*(s)\right)^{\bar p-1} + f^*(s)\right\} ds.$$

The last inequality can be rewritten as

$$1 \le \frac{-\mu_u'(t)\ \Lambda^{-\frac{1}{\bar{p}-1}}}{\left(N\omega_N^{1/N}(\mu_u(t))^{1/N'}\right)^{\bar{p}'}}\left[\mathscr{F}(\mu_u(t)) + \mathscr{U}(\mu_u(t))\right]^{1/(\bar{p}-1)} \quad \text{a.e. } t > 0, \tag{28}$$

where

$$\begin{aligned}\widetilde{c}(s) &= \left(c^+\right)^*(s) - \left(c^-\right)_*(s),\\ \mathscr{U}(s) &= \int_0^s \widetilde{c}(t)\left(u^*(t)\right)^{\bar{p}-1} dt,\\ \mathscr{F}(s) &= \int_0^s f^*(t)\, dt\end{aligned} \tag{29}$$

with $s \in [0, |\Omega|]$. An integration gives

$$(-u^*(s))' \le \gamma(s)\left[\mathscr{F}(s) + \mathscr{U}(s)\right]^{1/(\bar{p}-1)} \quad \text{a.e. } s \in (0, |\Omega|), \tag{30}$$

with $\gamma(s) = \Lambda^{-\frac{1}{\bar{p}-1}}\left(N\omega_N^{1/N} s^{1/N'}\right)^{-\bar{p}'}$.

Now let us consider problem (21). The solution v of problem (21) is unique and the symmetry of data assure that $v(x) = v(|x|)$, i.e. v is positive and radially symmetric. Moreover, putting $s = \omega_N|x|^N$ and $\widetilde{v}(s) = v((s/\omega_N)^{1/N})$, we get for all $s \in [0, |\Omega|]$

$$\begin{aligned}&-\Lambda|\widetilde{v}(s)|^{\bar{p}-2}\widetilde{v}'(s)\\ &= \frac{s^{-p/N'}}{(N\omega_N{}^{1/N})^{\bar{p}}}\int_0^s \left\{\left[\left(c^+\right)^*(\sigma) - \left(c^-\right)_*(\sigma)\right]\left(\widetilde{v}^*(\sigma)\right)^{\bar{p}-1} + f^*(\sigma)\right\} d\sigma.\end{aligned}$$

It is possible to show (see Lemma 3.2 of [18]) that the above integral is positive and this assure that $v(x) = v^\bigstar(x)$. By the properties of v we can repeat arguments used to prove (30) replacing all the inequalities by equalities and obtaining

$$(-v^*(s))' = \gamma(s)\left[\mathscr{F}(s) + \mathscr{V}(s)\right]^{1/(\bar{p}-1)} \quad \text{a.e. } s \in (0, |\Omega|), \tag{31}$$

where

$$\mathscr{V}(s) = \int_0^s \widetilde{c}(t)\left(v^*(t)\right)^{\bar{p}-1} dt \qquad \forall s \in [0, |\Omega|]. \tag{32}$$

Step 3: From now on, the proof runs as the proof of Theorem 4.1 in [19]. Here, we give a sketch for the convenience of the reader. Since $f \not\equiv 0$, we claim that

$$\int_0^s (c^-)_*(t)(u^*(t))^{\bar{p}-1}\, dt \le \int_0^s (c^-)_*(t)(v^*(t))^{\bar{p}-1}\, dt \quad \text{for } s \in [s_0, |\Omega|] \tag{33}$$

with s_0 defined in (18). By Lemma 1, we obtain

$$\int_{s_0}^{s} (u^*(t))^{\bar{p}-1}\, dt \le \int_{s_0}^{s} (v^*(t))^{\bar{p}-1}\, dt \quad \text{for } s \in [s_0, |\Omega|]. \tag{34}$$

Then, Lemma 2 assures

$$u^*(s_0) \le v^*(s_0). \tag{35}$$

Since $(c^-)_*(s) = 0$ if $s \in [0, s_0)$, equality (29) becomes $\mathscr{U}(s) = \int_0^s (c^+)^*(t)$ $(u^*(t))^{\bar{p}-1}\, dt$ and equality (32) becomes $\mathscr{V}(s) = \int_0^s (c^+)^*(t)(v^*(t))^{\bar{p}-1}\, dt$, for $s \in [0, s_0]$. Now, we state that the following inequality holds

$$\mathscr{U}(s) \le \mathscr{V}(s) \quad \text{for } s \in [0, s_0]. \tag{36}$$

By estimates (30), (31) and (36) we obtain

$$(-u^*(s))' \le (-v^*(s))' \quad \text{for} s \in [0, s_0]. \tag{37}$$

Thanks to (35) and (37), it follows

$$u^*(s) - v^*(s_0) \le u^*(s) - u^*(s_0) \le v^*(s) - v^*(s_0) \quad \text{for } s \in (0, s_0),$$

namely (19). Finally, (19) and (34) give desiderated inequality (20).

Step 4: In order to complete the proof, we need to prove inequalities (33) and (36). We argue by absurdum, starting from (30) and (31) and using elementary tools of calculus. We remark that it is necessary to distinguish different cases, thus, for more details, we refer the reader to Lemma 4.3 and Lemma 4.4 in [19]. □

Remark 1 We stress that if $\left\|c^+\right\|_{L^\infty} = \left\|\widehat{c}^+\right\|_{L^\infty}$, assumption (3) implies that problem (21) has a unique nonnegative radially decreasing symmetric solution. If (3) is not verified, then, in general, a comparison result can not be expected. Indeed, let us consider the following problem

$$\begin{cases} -\operatorname{div}(\nabla u) = \lambda(\Omega^\star)u + f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{38}$$

with $f \in L^\infty(\Omega)$ and $f > 0$. Problem (38) belongs to the class of problems (16) with $p_i = 2$ for every $i = 1, \ldots, N$, f positive and $c(x) = \lambda(\Omega^\star)$. Note that here $\bar{p} = 2$.

Denoted by $\lambda(\Omega)$ the first eigenvalue of the following Dirichlet problem

$$\begin{cases} -\operatorname{div}(\nabla w) = \lambda w & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega, \end{cases}$$

the well-known Faber-Krahn inequality (see e.g. [7, 23]) says that

$$\lambda(\Omega^{\bigstar}) \leq \lambda(\Omega). \tag{39}$$

If Ω is not a ball, by (39) and by the characterization of eigenvalue, problem (38) has a unique solution. Whereas the uniqueness fails for related symmetrized problem

$$\begin{cases} -\operatorname{div}(\nabla v) = \lambda(\Omega^{\bigstar})v + f^{\bigstar} & \text{in } \Omega^{\bigstar} \\ v = 0 & \text{on } \partial\Omega^{\bigstar}. \end{cases}$$

Remark 2 In the case $c(x) \leq 0$, it is possible to prove a pointwise estimate of the solution u to problem (16). More precisely,

$$u^*(s) \leq w^*(s) \quad \text{for } s \in [0, |\Omega|],$$

where w is the solution to the following problem

$$\begin{cases} -\operatorname{div}\left(\Lambda\, |\nabla w|^{\bar{p}-2}\, \nabla w\right) = f^{\bigstar} & \text{in } \Omega^{\bigstar} \\ w = 0 & \text{on } \partial\Omega^{\bigstar} \end{cases} \tag{40}$$

with Λ and $\bar{p}$ defined as in (14) and in (7), respectively. Indeed, the proof runs as in the proof of Theorem 1 with (28) replaced by

$$1 \leq \frac{-\mu_u'(t)\ \ \Lambda^{-\frac{1}{\bar{p}-1}}}{\left(N\omega_N^{1/N}\,(\mu_u(t))^{1/N'}\right)^{\bar{p}'}} \left[\mathscr{F}(\mu_u(t))\right]^{1/(\bar{p}-1)} \quad \text{a.e. } t > 0.$$

In this case the symmetrized problem (40) does not take into account of the presence of zero order term.

Now we are interested in a slight extension of Theorem 1 when the datum in problem (21) is not the rearrangement of datum f of problem (16), but it is a function that dominates f.

Corollary 1 *Assume the same hypothesis of Theorem 1. Let u be a weak solution to problem (16) and $v(x)$ be the solution to the following problem*

$$\begin{cases} -\operatorname{div}\left(\Lambda\, |\nabla w|^{\bar{p}-2}\, \nabla w\right) = \widehat{c}\, |w|^{\bar{p}-2}\, w + \widetilde{f} & \text{in } \Omega^{\bigstar} \\ w = 0 & \text{on } \partial\Omega^{\bigstar}, \end{cases}$$

where $\widetilde{f} = \widetilde{f}^{\bigstar}$ is a function such that

$$\int_0^s f^*(\sigma)d\sigma \le \int_0^s \widetilde{f}^*(\sigma)\, d\sigma \quad \textit{for} \;\; s \in [0, |\Omega|]\,. \tag{41}$$

Then we have

$$u^*(s) \le w^*(s) \quad \textit{for} \;\; s \in [0, s_0] \tag{42}$$

and

$$\int_0^s \left(u^*(\sigma)\right)^{\bar{p}-1} d\sigma \le \int_0^s \left(w^*(\sigma)\right)^{\bar{p}-1} d\sigma \quad \textit{for} \;\; s \in [s_0, |\Omega|]\,, \tag{43}$$

where s_0 is defined as in (18).

Proof The results follow arguing as in the proof of Theorem 1. In this case, instead of (28), we obtain the following inequality

$$1 \le \frac{-\mu_u'(t)\;\; \Lambda^{\frac{1}{\bar{p}-1}}}{\left(N\omega_N^{1/N}\left(\mu_u(t)\right)^{1/N'}\right)^{\bar{p}'}} \left[\widetilde{\mathscr{F}}\left(\mu_u(t)\right) + \mathscr{U}\left(\mu_u(t)\right)\right]^{1/(\bar{p}-1)} \quad \text{a.e. } t > 0 \tag{44}$$

with $\widetilde{\mathscr{F}}(s) = \int_0^s \widetilde{f}^*(t)\,dt$. Here (44) follows using (41) in (27). □

Now we are interested to obtain a comparison result requiring less summability on f. To this end we have to impose a sign condition on coefficient $c(x)$.

As usual, in order to assure the existence of a weak solution $u \in W_0^{1,\overrightarrow{p}}(\Omega)$ to problem (16) with $f \in L^{(\bar{p}^*)'}(\Omega)$ (see [26]), we consider the additional assumption

(A5) $\left(a(x,s,\xi) - a(x,s,\xi')\right) \cdot \left(\xi - \xi'\right) > 0$ with $\xi \neq \xi'$, a.e. $x \in \Omega$, $\forall s \in \mathbb{R}^N$, $\xi \in \mathbb{R}^N$.

As regards existence, uniqueness and regularity to anisotropic problems we refer also to [6, 15–17] and the bibliography therein.

Corollary 2 *Let us suppose that $f \in L^{(\bar{p}^*)'}(\Omega)$, and conditions (A1)–(A5) and $c(x) \le 0$ hold. If u is a weak solution to problem (16), we have*

$$\int_0^s \left(u^*(\sigma)\right)^{\bar{p}-1} d\sigma \le \int_0^s \left(v^*(\sigma)\right)^{\bar{p}-1} d\sigma \quad \textit{for} \;\; s \in [0, |\Omega|]\,, \tag{45}$$

where v is the weak solution to the following problem

$$\begin{cases} -\text{div}\left(\Lambda\,|\nabla v|^{\bar{p}-2}\,\nabla v\right) = -c_\star\,|v|^{\bar{p}-2}\,v + f^\star & \textit{in } \Omega^\star \\ v = 0 & \textit{on } \partial\Omega^\star \end{cases}$$

with Λ and $\bar{p}$ defined in (14) and in (7), respectively.

Proof The case $f \equiv 0$ is trivial. If $f \not\equiv 0$, arguments similar to those used in Theorem 1 run also for $f \in L^{(\bar{p}^*)'}(\Omega)$ (see also [18]). Indeed, if $c(x) \le 0$, it follows that $\widehat{c}(x) = -\left(c^-\right)_\bigstar(x)$, $s_0 = 0$ and inequality (45) holds for $s \in [0, |\Omega|]$. □

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A Topological Approach to Non-Archimedean Mathematics

Vieri Benci and Lorenzo Luperi Baglini

Abstract Non-Archimedean mathematics (in particular, nonstandard analysis) allows to construct some useful models to study certain phenomena arising in PDE's; for example, it allows to construct generalized solutions of differential equations and variational problems that have no classical solution. In this paper we introduce certain notions of Non-Archimedean mathematics (and of nonstandard analysis) by means of an elementary topological approach; in particular, we construct Non-Archimedean extensions of the reals as appropriate topological completions of $\mathbb{R}$. Our approach is based on the notion of Λ-limit for real functions, and it is called Λ-theory. It can be seen as a topological generalization of the α-theory presented in [6], and as an alternative topological presentation of the ultrapower construction of nonstandard extensions (in the sense of [21]). To motivate the use of Λ-theory for applications we show how to use it to solve a minimization problem of calculus of variations (that does not have classical solutions) by means of a particular family of generalized functions, called ultrafunctions.

Keywords Non-Archimedean mathematics · Nonstandard analysis · Limits of functions · Generalized functions · Ultrafunctions

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1 Introduction

In a previous series of papers [5, 9–14] we have introduced and studied a new family of generalized functions called ultrafunctions and its applications to certain problems in mathematical analysis, including some applications to PDE's in [14]. The development of a rigorous study of (a large class of) PDE's in ultrafunction theory is the object of [15], where we exemplify our approach by studying in detail Burgers' equation. Henceforth, it is our feeling that many problems in PDE's theory could be fruitfully studied by means of the theory of ultrafunctions.

However, one might have the impression that a drawback of our approach is the use of the machinery of NSA, which is not a "common working tool" for most analysts. Even if NSA has already been applied to many different fields of mathematics (such as functional analysis, probability theory, combinatorial number theory, mathematical physics and so on) to obtain important results, the original formalism of Robinson, based on model theory (see e.g. [25]), appears too technical to many researchers, and not directly usable by most mathematicians. Since Robinson's work first appeared, a simpler semantic approach (due to Robinson himself and Elias Zakon) has been developed using the purely set-theoretic notion of superstructure (see [27]); we recall also the pioneering work by Luxemburg (see [23]), where a direct use of ultrapowers was made (see [6, 8] for a complete presentation of alternative simplified approaches to NSA). However, many researcher working in NSA have the feeling that also these technical notions are not needed in order to carry out calculus with actual infinitesimals, as well as to carry out several applications of NSA. As a consequence, there have been many attempts to simplify and popularize NSA by means of simplified presentations. We recall here in particular the approaches of Henson [20], Keisler [21] and Nelson [24]; other attempts have been made by Benci, Di Nasso and Forti with algebraic (see [3, 4, 7, 17]) and topological approaches (see [8, 16]). We also suggest [22] where NSA is introduced in a simplified way suitable for many applications. In our previous papers, we tried to address the same issue by means of Λ-limits (see e.g. [11] for an axiomatic presentation of this approach to NSA). The basic idea of Λ-limits is to present nonstandard objects as limits of standard ones. However, in our previous works the word "limits" was not intended in a topological sense: the "limits" where defined axiomatically and no explicit topology was involved in the constructions.

The main aim of this paper is to show that, actually, Λ-limits can be precisely characterized as topological limits. This approach will be called Λ-theory; it allows to construct a topological approach to NSA (related to but different from the approach of Benci, Di Nasso and Forti in [8, 16]) that, in our opinion, is well-suited for researchers that are not experts in NSA and are interested to use certain Non-Archimedean arguments to study problems in analysis. In fact, it is our feeling that presenting nonstandard constructions and results by means of a topological approach might help such researchers to use them. For example, we construct extensions of the reals (in the sense of NSA) as appropriate topological completions of $\mathbb{R}$.

Λ-theory can be seen as a topological generalization of the α-theory presented in [6]. The idea behind our approach is to embed $\mathbb{R}$ in particular Hausdorff topological spaces in which it is possible to formalize the intuitive idea of hyperreals as topological limits (in a sense that we will make precise in Sect. 2.1) of real functions. From this point of view, our construction of the hyperreals starting from $\mathbb{R}$ shares some features with the construction of $\mathbb{R}$ as the Cauchy completion of $\mathbb{Q}$. We also extend our construction to define a topology on the superstructure $V(\mathbb{R})$ on $\mathbb{R}$, that we use to define Λ-limits of bounded functions defined on $V(\mathbb{R})$. Our construction is substantially equivalent to the ultrapower approach, and we will prove in Sect. 3 that within $\Lambda-$theory it is possible to construct a nonstandard universe in the sense of [21]. To motivate our feeling that Λ-theory can be fruitfully applied to study certain problems in Analysis, in Sect. 4 we apply Λ-theory to solve a minimization problem of calculus of variations that does not have classical solutions.

We want to remark that readers expert in NSA will easily recognize that Λ-theory is essentially equivalent to the ultrapower construction (we prove this fact in Sect. 3). Anyhow, in this paper, we do not assume the knowledge of NSA by the reader.

2 Λ-theory

2.1 The Λ-limit

The only technical notion that we need to develop our approach to Non-Archimedean mathematics is that of ultrafilter:

Definition 1 Let X be a set. An ultrafilter $\mathcal{U}$ on X is a family of subsets of X that has the following properties:

1. $X \in \mathcal{U}, \emptyset \notin \mathcal{U}$;
2. for every $A, B \subseteq X$ if $A \in \mathcal{U}$ and $A \subseteq B$ then $B \in \mathcal{U}$;
3. for every $A, B \in \mathcal{U}$, $A \cap B \in \mathcal{U}$;
4. for every $A \subseteq X$ we have that $A \in \mathcal{U}$ or $A^c \in \mathcal{U}$.

An ultrafiler $\mathcal{U}$ on X is principal if there exists an element $x \in X$ such that $\mathcal{U} = \{A \subseteq X \mid x \in A\}$. An ultrafilter is free if it is not principal. From now on we let $\mathfrak{L}$ be an infinite set equipped with a free ultrafilter $\mathcal{U}$. Every set $Q \in \mathcal{U}$ will be called **qualified set**. We will say that a property P is **eventually** true for the function $\varphi(\lambda)$ if it is true for every λ in a qualified set, namely if there exists $Q \in \mathcal{U}$ such that $P(\varphi(\lambda))$ holds for every $\lambda \in Q$. We let $\Lambda \notin \mathfrak{L}$ and we consider the space $\mathfrak{L} \cup \{\Lambda\}$. We equip $\mathfrak{L} \cup \{\Lambda\}$ with a topology in which the neighborhoods of Λ are of the form $\{\Lambda\} \cup Q,\ \ Q \in \mathcal{U}$. In this sense, one can imagine Λ as being a "point at infinity" for $\mathfrak{L}$ (in this sense, it plays a similar role to that of α in the Alpha-Theory, see [6]). With respect to this topology, the notion of limit of a function at Λ is specified as follows:

Definition 2 Let (X, τ) be a Hausdorff topological space, let $x_0 \in X$ and let $\varphi : \mathfrak{L} \to X$ be a function. We say that x_0 is the Λ-limit of the function φ, and we write

$$\lim_{\lambda \to \Lambda} \varphi(\lambda) = x_0, \tag{1}$$

if for every neighborhood V of x_0 the function φ is eventually in V, namely if there is a qualified set Q such that $\varphi(Q) \subset V$.

Remark 1 We use the notation $\lim_{\lambda \to \Lambda} \varphi(\lambda)$ since, as we already noticed, one may think of $\Lambda \notin \mathfrak{L}$ as a "point at ∞" and to the sets in $\mathscr{U}$ as neighborhoods of Λ; it is conceptually similar to the point ∞ when one considers $\mathbb{R} \cup \{+\infty\}$. We prefer to use the symbol Λ rather than ∞ since one may think of Λ as a function of $\mathscr{U}$, namely $\Lambda = \Lambda(\mathscr{U})$. Thus the explicit mention of Λ is a reminder that the Λ-limit depends on $\mathscr{U}$.

Remark 2 Another way to look at the limit (1) is to consider the Stone-Čech compactification $\beta\mathfrak{L}$ of $\mathfrak{L}$ with the relative topology and to think of $\Lambda \in \beta\mathfrak{L}$ as of a nontrivial element of this compactification.

Limits as given by Eq. (1) will be called $\Lambda-$limits, and we will call Λ-theory the approach to Non-Archimedean mathematics based on the notion of Λ-limit.

Our main result is the following:

Theorem 1 *There exists a Hausdorff topological space* $(\mathbb{R}_{\mathfrak{L}}, \tau)$ *such that*

1. $\mathbb{R}_{\mathfrak{L}} = cl_\tau(\mathfrak{L} \times \mathbb{R})$;
2. $\mathbb{R} \subseteq \mathbb{R}_{\mathfrak{L}}$ *and* $\forall c \in \mathbb{R}$
$$\lim_{\lambda \to \Lambda} (\lambda, c) = c;$$
3. *for every function* $\varphi : \mathfrak{L} \to \mathbb{R}$*, the limit*
$$\lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda))$$
exists in $(\mathbb{R}_{\mathfrak{L}}, \tau)$;
4. *two functions* φ, ψ *are eventually equal if and only if*
$$\lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda)) = \lim_{\lambda \to \Lambda} (\lambda, \psi(\lambda)).$$

Proof We set

$$I = \{\varphi \in \mathfrak{F}(\mathfrak{L}, \mathbb{R}) \mid \varphi(x) = 0 \text{ in a qualified set}\}.$$

It is not difficult to prove that I is a maximal ideal in $\mathfrak{F}(\mathfrak{L}, \mathbb{R})$; then

$$\mathbb{K} := \frac{\mathfrak{F}(\mathfrak{L}, \mathbb{R})}{I}$$

is a field. In the following, we shall identify a real number $c \in \mathbb{R}$ with the equivalence class of the constant function $\varphi : \mathfrak{L} \to \mathbb{R}$ such that $\varphi(\lambda) = c$ for every $\lambda \in \mathfrak{L}$.
We set

$$\mathbb{R}_{\mathfrak{L}} = (\mathfrak{L} \times \mathbb{R}) \cup \mathbb{K}.$$

We equip $\mathbb{R}_{\mathfrak{L}}$ with the following topology τ. A basis for τ is given by

$$b(\tau) = \left\{ N_{\varphi,Q} \mid \varphi \in \mathfrak{F}(\mathfrak{L}, \mathbb{R}), Q \in \mathscr{U} \right\} \cup \mathscr{P}(\mathfrak{L} \times \mathbb{R})$$

where

$$N_{\varphi,Q} := \{(\lambda, \varphi(\lambda)) \mid \lambda \in Q\} \cup \{[\varphi]_I\}$$

is a neighborhood of $[\varphi]_I$ for every $Q \in \mathscr{U}$.

In order to show that $b(\tau)$ is a basis for a topology, we have to show that

$$\forall A, B \in b(\tau)\ \forall x \in A \cap B\ \exists C \in b(\tau) \text{ such that } x \in C \subset A \cap B.$$

Let $A, B \in b(\tau)$. Let $x \in A \cap B$. If $x \notin \mathbb{K}$ then we can just set $C = A \cap B \cap \mathfrak{L} \times \mathbb{R}$, as the topology is discrete on $\mathfrak{L} \times \mathbb{R}$. If $x \in \mathbb{K}$ then there exist $R, S \in \mathscr{U}$ such that $A = N_{\varphi,R}$ and $B = N_{\psi,S}$ with $[\varphi]_I = [\psi]_I = x$. Hence there exists $Q \in \mathscr{U}$ such that

$$\forall \lambda \in Q,\ \varphi(\lambda) = \psi(\lambda).$$

Thus if we set $C := N_{\varphi,R\cap S\cap Q}$ we have that $x \in C \subset A \cap B$.

Let us show that τ is a Hausdorff topology. Clearly it is sufficient to check it for points in $\mathbb{K}$, so let $\xi \neq \zeta \in \mathbb{K}$. Since $\xi \neq \zeta$, there exists $\varphi, \psi \in \mathfrak{F}(\mathfrak{L}, \mathbb{R})$, $Q \in \mathscr{U}$ such that

$$\xi = [\varphi]_I\,,\ \zeta = [\psi]_I \quad \text{and} \quad \forall \lambda \in Q,\ \varphi(\lambda) \neq \psi(\lambda).$$

Therefore

$$N_{\varphi,Q} \cap N_{\psi,Q} = \varnothing.$$

Let us observe that, by construction, for every function $\varphi : \mathfrak{L} \to \mathbb{R}$ we have that

$$\lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda)) = [\varphi]_I\,. \tag{2}$$

In fact, given a neighborhood $N_{\varphi,Q}$ of $[\varphi]_I$, we have that $\{\varphi(\lambda) \mid \lambda \in Q\} \subseteq N_{\varphi,Q}$, so $[\varphi]_I$ is a Λ-limit of the function $(\lambda, \varphi(\lambda))$. Since the space is Hausdorff, the limit is unique, so $\lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda)) = [\varphi]_I$.

Let us prove that $(\mathbb{R}_{\mathfrak{L}}, \tau)$ has the desired properties:

- property (1) follows directly by the definition of τ;
- property (2) follows by the identification of every real number $c \in \mathbb{R}$ with the equivalence class of the constant function $[c]_I$;
- properties (3) and (4) follow by Eq. (2).

Remark 3 In [8, 16], nonstandard extensions are constructed by means of similar, but different, topological considerations based on the choice of the ultrafilter $\mathcal{U}$. However the authors showed (see Theorem 4.5 in [16]) that such extensions are Hausdorff if and only if the ultrafilter $\mathcal{U}$ is Hausdorff (see again [16], Sects. 4 and 6), and in [2] Bartoszynski and Shelah proved that it is consistent with ZFC that there are no Hausdorff ultrafilters. By contrast, in our topological approach the extensions are always constructed inside Hausdorff topological spaces under the much milder request of $\mathcal{U}$ being free. This is possible because we incorporate the set of indices $\mathfrak{L}$ in the space.

Motivated by the philosophical similarity between the properties expressed in Theorem 1 and the construction of $\mathbb{R}$ as the Cauchy completion of $\mathbb{Q}$, we introduce the following definition:

Definition 3 *A Hausdorff topological space* $(\mathbb{R}_{\mathfrak{L}}, \tau)$ that satisfies conditions (1)–(4) of Theorem 1 will be called a $(\mathfrak{L}, \mathcal{U})$**-completion of** $\mathbb{R}$.

2.2 The Hyperreal Field

Let $(\mathbb{R}_{\mathfrak{L}}, \tau)$ be a $(\mathfrak{L}, \mathcal{U})$-completion of $\mathbb{R}$. Let us fix some notation: we will denote by $\mathbb{K}$ the set

$$\mathbb{K} = \left\{ \lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda)) \mid \varphi \in \mathfrak{F}(\mathfrak{L}, \mathbb{R}) \right\}.$$

The aim of this section is to study the basic properties of $\mathbb{K}$.

Proposition 1 $(\mathfrak{L} \times \mathbb{R}) \cap \mathbb{K} = \emptyset$.

Proof Let us suppose by contrast that there exists $\varphi : \mathfrak{L} \to X$ such that

$$\lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda)) = (\lambda_0, r) \in \mathfrak{L} \times \mathbb{R}.$$

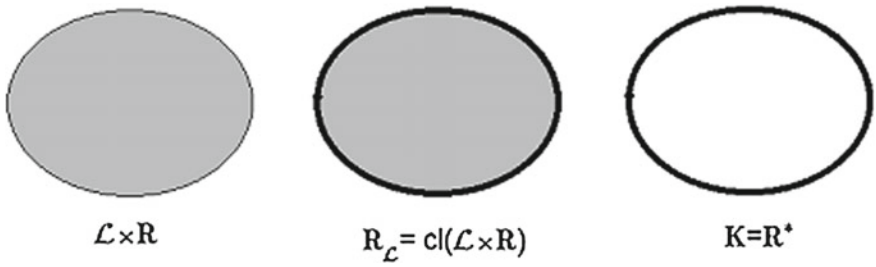
Since $\{(\lambda_0, r)\}$ is open, by definition there exists $Q \in \mathcal{U}$ such that $\forall \lambda \in Q$, $(\lambda, \varphi(\lambda)) = (\lambda_0, r)$. Therefore $Q = \{\lambda_0\}$, and this is absurd since $\mathcal{U}$ is free.

From condition (1) in Theorem 1 we know that $(\mathfrak{L} \times \mathbb{R}) \uplus \mathbb{K} \subseteq \mathbb{R}_{\mathfrak{L}}$. In general, this inclusion might be proper; henceforth we introduce the following definition:

Definition 4 We say that $(\mathbb{R}_{\mathfrak{L}}, \tau)$ is a minimal $(\mathfrak{L}, \mathcal{U})$-completion of $\mathbb{R}$ if $\mathbb{R}_{\mathfrak{L}} = (\mathfrak{L} \times \mathbb{R}) \uplus \mathbb{K}$.

It is immediate to see that any $(\mathfrak{L}, \mathcal{U})$-completion of $\mathbb{R}$ contains a minimal $(\mathfrak{L}, \mathcal{U})$-completion of $\mathbb{R}$, and that any minimal $(\mathfrak{L}, \mathcal{U})$-completion of $\mathbb{R}$ does not properly contain another minimal $(\mathfrak{L}, \mathcal{U})$-completion of $\mathbb{R}$ (and this is what motivates the choice of the name "minimal" for such extensions).

From now on we will be only interested in minimal $(\mathfrak{L}, \mathscr{U})$-completions.



By condition (2) in the definition of $(\mathfrak{L}, \mathscr{U})$-completions it follows that $\mathbb{R} \subseteq \mathbb{K}$. Moreover we have the following result:

Proposition 2 *For every finite subset $F \subseteq \mathbb{R}$, for every function $\varphi : \mathfrak{L} \to F$ we have that*

$$\lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda)) \in F.$$

Proof Let $F = \{x_1, ..., x_n\}$. For every $i \leq n$ let

$$A_i = \{\lambda \in \mathfrak{L} \mid \varphi(\lambda) = x_i\}.$$

Since $\mathscr{U}$ is an ultrafilter, there exists exactly one index $i_0 \leq n$ such that $A_{i_0} \in \mathscr{U}$. Now let $\xi = \lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda))$. Let us suppose that $\xi \neq x_{i_0}$. Let O_1, O_2 be disjoint open sets such that $\xi \in O_1, x_{i_0} \in O_2$. Since x_{i_0} is the limit of the constant function with value x_{i_0}, there exists $B \in \mathscr{U}$ such that

$$\{(\lambda, x_{i_0}) \mid \lambda \in B\} \subseteq O_{2.}$$

Let $C \in \mathscr{U}$ be such that $\{(\lambda, \varphi(\lambda)) \mid \lambda \in C\} \subseteq O_1$. Then by construction we have that

$$\forall \lambda \in A_{i_0} \cap B \cap C \ (\lambda, \varphi(\lambda)) = (\lambda, x_{i_0}) \in O_1 \cap O_{2,}$$

and this is a contradiction since $O_1 \cap O_2 = \emptyset$. Therefore $\lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda)) = x_{i_0} \in F$.

There is a natural way to define sums and products of elements of $\mathbb{K}$:

Definition 5 We set

$$\lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda)) + \lim_{\lambda \to \Lambda} (\lambda, \psi(\lambda)) := \lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda) + \psi(\lambda));$$
$$\lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda)) \cdot \lim_{\lambda \to \Lambda} (\lambda, \psi(\lambda)) := \lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda) \cdot \psi(\lambda)).$$

Theorem 2 $(\mathbb{K}, +, \cdot, 0, 1)$ *is a field which contains* $\mathbb{R}$.

Proof That $\mathbb{R} \subseteq \mathbb{K}$ follows by condition (2) of the definition of $(\mathfrak{L}, \mathscr{U})$-completion. The only non trivial property that we have to prove to show that $\mathbb{K}$ is a field is the existence of a multiplicative inverse for every $x \neq 0$. Let $x \in \mathbb{K}$, $x \neq 0$. Since the topology is Hausdorff and $x \neq 0$, there is a set $Q \in \mathscr{U}$ such that

$$\forall \lambda \in Q, \ \varphi(\lambda) \neq 0.$$

Let $\phi : \mathfrak{L} \to \mathbb{R}$ be defined as follows:

$$\phi(\lambda) = \begin{cases} 1 & \text{if } \lambda \notin Q; \\ \frac{1}{\varphi(\lambda)} & \text{if } \lambda \in Q. \end{cases}$$

Then $\varphi(\lambda) \cdot \phi(\lambda) = 1$ for every $\lambda \in Q$, thus $\lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda)) \cdot \lim_{\lambda \to \Lambda} (\lambda, \phi(\lambda)) = \lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda) \cdot \phi(\lambda)) = 1$, namely

$$x^{-1} := \lim_{\lambda \to \Lambda} (\lambda, \phi(\lambda))$$

is the inverse of x.

The ordering of $\mathbb{R}$ can be extended to $\mathbb{K}$ by setting

$$\lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda)) < \lim_{\lambda \to \Lambda} (\lambda, \psi(\lambda)) \Leftrightarrow \varphi(\lambda) < \psi(\lambda) \text{ eventually,} \tag{3}$$

namely iff $\{(\lambda, \varphi(\lambda) - \psi(\lambda)) \mid \varphi(\lambda) - \psi(\lambda) \geq 0\} \cup [\varphi - \psi]$ is open (i.e. iff $\{\lambda \in \mathfrak{L} \mid \varphi(\lambda) < \psi(\lambda)\}$ is qualified). This ordering is clearly an extension of the ordering relation defined on $\mathbb{R}$ since, for every $x, y \in \mathbb{R}$, if $x \leq y$ and $\varphi_x, \varphi_y : \mathfrak{L} \to \mathbb{R}$ are the constant sequences with values resp. x, y then

$$\{\lambda \in \mathfrak{L} \mid \varphi_x(\lambda) < \varphi_y(\lambda)\} = \mathfrak{L},$$

which is qualified.

Remark 4 Usually, the inclusion $\mathbb{R} \subseteq \mathbb{K}$ is proper: e.g., let $\mathscr{U}$ be a countably incomplete ultrafilter.[1] Let $\langle A_n \mid n \in \mathbb{N} \rangle$ be a family of elements of $\mathscr{U}$ such that $\bigcap_{n \in \mathbb{N}} A_n = \emptyset$, let $B_n = \bigcap_{i \leq n} A_n$ for all $n \in \mathbb{N}$ and let $\phi : \mathfrak{L} \to \mathbb{R}$ be defined as follows: for every $\lambda \in \mathfrak{L}$,

$$\phi(\lambda) = n \Leftrightarrow \lambda \in B_n \setminus B_{n+1.}$$

[1]An ultrafilter $\mathscr{U}$ is countably incomplete if there exists a family $\langle A_n \mid n \in \mathbb{N} \rangle$ of elements of $\mathscr{U}$ such that $\bigcap_{n \in \mathbb{N}} A_n = \emptyset$.

Then $\lim_{\lambda\to\Lambda}\ (\lambda, \phi(\lambda)) \notin \mathbb{R}$: in fact, $\lim_{\lambda\to\Lambda}\ (\lambda, \phi(\lambda)) > n$ for every $n \in \mathbb{N}$ (and so, in particular, this limit is infinite). This holds since, for every $n \in \mathbb{N}$, by construction we have that

$$\{\lambda \in \mathfrak{L} \mid \phi(\lambda) \geq n\} = B_n \in \mathscr{U}.$$

When the inclusion $\mathbb{R} \subseteq \mathbb{K}$ is proper we have that $\mathbb{K}$ is a superreal non Archimedean field.[2] In this case, it will be called a **hyperreal field**. The terminology will be motivated by Corollary 1, where we make precise the relationship (as fields) between the hyperreal field $\mathbb{K}$ and the ultrapower $\mathbb{R}^{\mathfrak{L}}_{\mathscr{U}}$. Let us recall the definition of $\mathbb{R}^{\mathfrak{L}}_{\mathscr{U}}$:

Definition 6 Let $\equiv_{\mathscr{U}}$ be the equivalence relation on $\mathbb{R}^{\mathfrak{L}}$ defined as follows: for every $\varphi, \psi : \mathfrak{L} \to \mathbb{R}$

$$\varphi \equiv_{\mathscr{U}} \psi \Leftrightarrow \{\lambda \in \mathfrak{L} \mid \varphi(\lambda) = \psi(\lambda)\} \in \mathscr{U}.$$

The equivalence class of every function $\varphi : \mathfrak{L} \to \mathbb{R}$ will be denoted by $[\varphi]_{\mathscr{U}}$. The ultrapower $\mathbb{R}^{\mathfrak{L}}_{\mathscr{U}}$ is the quotient $\mathbb{R}^{\mathfrak{L}}/\equiv_{\mathscr{U}}$.

The operations on $\mathbb{R}^{\mathfrak{L}}_{\mathscr{U}}$ are defined componentwise: for every $\varphi, \psi : \mathfrak{L} \to \mathbb{R}$ we set

$$[\varphi]_{\mathscr{U}} + [\psi]_{\mathscr{U}} := [\varphi + \psi]_{\mathscr{U}};\ [\varphi]_{\mathscr{U}} + [\psi]_{\mathscr{U}} := [\varphi \cdot \psi]_{\mathscr{U}}.$$

A well-known result (see e.g. [21]) is that $(\mathbb{R}^{\mathfrak{L}}_{\mathscr{U}}, [0]_{\mathscr{U}}, [1]_{\mathscr{U}}, +, \cdot)$ is a field. Moreover, we have the following:

Corollary 1 *$\mathbb{K}$ and $\mathbb{R}^{\mathfrak{L}}_{\mathscr{U}}$ are isomorphic as fields.*

Proof The isomorphism is given by the map $\Psi : \mathbb{K} \to \mathbb{R}^{\mathfrak{L}}_{\mathscr{U}}$ such that, for every $\varphi : \mathfrak{L} \to \mathbb{R}$,

$$\Psi\left(\lim_{\lambda\to\Lambda}(\lambda, \varphi(\lambda))\right) = [\varphi]_{\mathscr{U}}.$$

Condition (4) in the definition of $(\mathfrak{L}, \mathscr{U})$-completion entails that Ψ is injective, whereas the definition of $\mathbb{K}$ as the set of all possible Λ-limits entails that Ψ is surjective. Since it is immediate to see that Ψ also preserves the operations, we have that it is an isomorphism.

We will strengthen Corollary 1 in Theorem 4. By Corollary 1 it clearly follows that, if the $(\mathfrak{L}, \mathscr{U})$ -completion is minimal, as sets $\mathbb{R}_{\mathfrak{L}} \cong (\mathfrak{L} \times \mathbb{R}) \uplus \mathbb{R}^{\mathfrak{L}}_{\mathscr{U}}$.

Remark 5 Let us note that $((\mathfrak{L} \times \mathbb{R}) \uplus \mathbb{R}^{\mathfrak{L}}_{\mathscr{U}}, \tau)$ is a $(\mathfrak{L}, \mathscr{U})$-completion of $\mathbb{R}$ for different choices of τ. One such choice is the topology $\tau_{\mathscr{U}}$ introduced in the proof of Theorem 1; a different topology can be constructed as follows: let us fix a function φ with $\lim_{\lambda\to\Lambda}(\lambda, \varphi(\lambda)) \notin \mathbb{R}$, a nonempty infinite set $B \notin \mathscr{U}$, a free filter $\mathscr{F}$ on B and

[2] A superreal non Archimedean field is an ordered field that properly contains $\mathbb{R}$.

let us consider the following topology $\widetilde{\tau}$ on $(\mathfrak{L}\times\mathbb{R})\uplus\mathbb{R}_{\mathscr{U}}^{\mathfrak{L}}$: if $\xi\neq\lim_{\lambda\to\Lambda}(\lambda,\varphi(\lambda))$ then a family of open neighborhoods of ξ is

$$\left\{O_{\psi,Q}\mid Q\in\mathscr{U},\,\psi\text{ function with }\xi=\lim_{\lambda\to\Lambda}(\lambda,\psi(\lambda))\right\};$$

if $\xi=\lim_{\lambda\to\Lambda}(\lambda,\varphi(\lambda))$ then a family of open neighborhoods of ξ is

$$\{O_{F,Q}\mid F\in\mathscr{F},\,Q\in\mathscr{U}\}$$

where, for every $F\in\mathscr{F}$, $Q\in\mathscr{U}$ we set

$$O_{F,Q}=O_{\varphi,Q}\cup\{(\lambda,x)\mid\lambda\in F,\,x\in\mathbb{R}\}.$$

By construction, $\left((\mathfrak{L}\times\mathbb{R})\uplus\mathbb{R}_{\mathscr{U}}^{\mathfrak{L}},\widetilde{\tau}\right)$ is a $(\mathfrak{L},\mathscr{U})$-completion of $\mathbb{R}$.

A consequence of Remark 5 is that there are infinitely many topologies τ that make $((\mathfrak{L}\times\mathbb{R})\uplus\mathbb{R}_{\mathscr{U}}^{\mathfrak{L}},\tau)$ a $(\mathfrak{L},\mathscr{U})$-completion of $\mathbb{R}$. However, the topology introduced in the proof of Theorem 1 plays a central role in our approach. For this reason, we introduce the following definition.

Definition 7 Let $(\mathbb{R}_{\mathfrak{L}},\tau)$ be a $(\mathfrak{L},\mathscr{U})$-completion of $\mathbb{R}$. We call **slim topology**, and we denote by $\tau_{\mathscr{U}}$, the topology on $\mathbb{R}_{\mathfrak{L}}$ generated by the family of open sets

$$\left\{N_{\varphi,Q}\mid\varphi\in\mathfrak{F}(\mathfrak{L},\mathbb{R}),\,Q\in\mathscr{U}\right\}\cup\mathscr{P}(\mathfrak{L}\times\mathbb{R})$$

where, for every $\varphi\in\mathfrak{F}(\mathfrak{L},\mathbb{R})$, $Q\in\mathscr{U}$ we set

$$N_{\varphi,Q}:=\{(\lambda,\varphi(\lambda))\mid\lambda\in Q\}\cup\left\{\lim_{\lambda\to\Lambda}(\lambda,\varphi(\lambda))\right\}.$$

Proposition 3 *The slim topology $\tau_{\mathscr{U}}$ is finer than any topology τ that makes $\left((\mathfrak{L}\times\mathbb{R})\uplus\mathbb{R}_{\mathscr{U}}^{\mathfrak{L}},\tau\right)$ a $(\mathfrak{L},\mathscr{U})$-completion of $\mathbb{R}$.*

Proof Let τ be given, let O be an open set in τ and let $x\in O$. If $x\in\mathfrak{L}\times\mathbb{R}$ then $\{x\}$ is an open neighborhood of x in $\tau_{\mathscr{U}}$ contained in O; if $x=\lim_{\lambda\to\Lambda}(\lambda,\varphi(\lambda))$ for some function $\varphi:\mathfrak{L}\to\mathbb{R}$ then let $B\in\mathscr{U}$ be such that $\{(\lambda,\varphi(\lambda))\mid\lambda\in B\}\subseteq O$; therefore, by construction, $O_{\varphi,B}$ is an open neighborhood of x in $\tau_{\mathscr{U}}$ entirely contained in O. This proves that O is an open set in $\tau_{\mathscr{U}}$, therefore $\tau_{\mathscr{U}}$ is finer than τ.

The slim topology can also be characterized in terms of closure of subsets of $(\mathfrak{L}\times\mathbb{R})$:

Proposition 4 *Let* $\left((\mathfrak{L}\times\mathbb{R})\uplus\mathbb{R}^{\mathfrak{L}}_{\mathscr{U}},\tau\right)$ *be a* $(\mathfrak{L},\mathscr{U})$*-completion of* $\mathbb{R}$*. The following facts are equivalent:*

1. $\tau=\tau_{\mathscr{U}}$;
2. *for every set* $B\subseteq(\mathfrak{L}\times\mathbb{R})$ *we have that*

$$cl_\tau(B)=B\cup\left\{\lim_{\lambda\to\Lambda}(\lambda,\varphi(\lambda))\mid\exists A\in\mathscr{U}\ \forall\lambda\in A\ (\lambda,\varphi(\lambda))\in B\right\}.$$

Proof (1) $\Rightarrow$ (2) Let $\varphi:\mathfrak{L}\to\mathbb{R}$, let $B\subseteq(\mathfrak{L}\times\mathbb{R})$ and let $\xi=\lim_{\lambda\to\Lambda}(\lambda,\varphi(\lambda))$. Let $A=\{\lambda\in\mathfrak{L}\mid(\lambda,\varphi(\lambda))\in B\}$. If $A\in\mathscr{U}$ then for every open neighborhood O of ξ we have that $O\cap B\neq\emptyset$ by construction, so $\xi\in cl_{\tau_{\mathscr{U}}}(B)$; if $A\notin\mathscr{U}$ then $O_{\varphi,A}$ is a neighborhood of ξ such that $O_{\varphi,A}\cap B=\emptyset$, therefore $\xi\notin cl_{\tau_{\mathscr{U}}}(B)$.

(2) $\Rightarrow$ (1) Let $A\in\mathscr{U}$, let $\varphi:\mathfrak{L}\to\mathbb{R}$ and let $\xi=\lim_{\lambda\to\Lambda}(\lambda,\varphi(\lambda))$. Let us consider $B=(\mathfrak{L}\times\mathbb{R})\setminus O_{A,\varphi}$. By hypothesis and construction

$$cl_\tau(B)=\left[(\mathfrak{L}\times\mathbb{R})\uplus\mathbb{R}^{\mathfrak{L}}_{\mathscr{U}}\right]\setminus O_{A,\varphi}.$$

Therefore $O_{A,\varphi}$ is open for every $A\in\mathscr{U}$, $\varphi:\mathfrak{L}\to\mathbb{R}$, so τ is finer than $\tau_{\mathscr{U}}$ which, as a consequence of Proposition 3, entails that $\tau=\tau_{\mathscr{U}}$.

Definition 8 We will call $\left((\mathfrak{L}\times\mathbb{R})\uplus\mathbb{R}^{\mathfrak{L}}_{\mathscr{U}},\tau_{\mathscr{U}}\right)$ the **canonical** $(\mathfrak{L},\mathscr{U})$-completion of $\mathbb{R}$.

From the next section on we will work only with the minimal canonical $(\mathfrak{L},\mathscr{U})$-completion of $\mathbb{R}$.

2.3 Natural Extension of Sets and Functions

From now on, $\overline{(\cdot)}$ will denote the closure operator in the canonical $(\mathfrak{L},\mathscr{U})$-completion of $\mathbb{R}$.

Definition 9 For every $E\subseteq\mathbb{R}$ we set

$$E_{\mathfrak{L}}:=\overline{\mathfrak{L}\times E}.$$

A different and related (as we will show in Proposition 5) extension of E is the following:

Definition 10 Given *a set* $E\subset\mathbb{R}$, we set

$$E^*:=\left\{\lim_{\lambda\to\Lambda}(\lambda,\psi(\lambda))\,|\psi(\lambda)\in E\right\};$$

E^* is called the **natural extension** of E.

Let us observe that by property (2) of the definition of $(\mathfrak{L}, \mathcal{U})$-completions it follows that $E \subseteq E^*$. Following the notation introduced in Definition 10, from now on we will denote $\mathbb{K}$ by $\mathbb{R}^*$.

It is easy to modify the proof of Proposition 1 to obtain the following result:

Proposition 5 *For every* $E \subseteq \mathbb{R}$ *we have that* $E_{\mathfrak{L}} = (\mathfrak{L} \times E) \uplus E^*$.

It is also possible to extend functions to $\mathbb{R}_{\mathfrak{L}}$. To this aim, given a function

$$f : A \to B$$

we will denote by

$$f_{\mathfrak{L}} : \mathfrak{L} \times A \to \mathfrak{L} \times B$$

the function defined as follows:

$$f_{\mathfrak{L}}(\lambda, x) = (\lambda, f(x)) .$$

Lemma 1 *For every* $A, B \subseteq \mathbb{R}$, *for every function* $f : A \to B$, f *can be extended to a continuous function*

$$\overline{f_{\mathfrak{L}}} : A_{\mathfrak{L}} \to B_{\mathfrak{L}}.$$

Moreover, the restriction of $\overline{f_{\mathfrak{L}}}$ *to* A *coincides with* f.

Proof The extension of f to $\mathfrak{L} \times A$ is given by $f_{\mathfrak{L}}$. Therefore to get the desired extension to $A_{\mathfrak{L}}$ it is sufficient to extend $f_{\mathfrak{L}}$ on A^*. For every $\varphi \in A^{\mathfrak{L}}$ we set

$$\overline{f_{\mathfrak{L}}}\left(\lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda))\right) = \lim_{\lambda \to \Lambda} (\lambda, f(\varphi(\lambda))) .$$

Let us note that the definition is well posed and that $\overline{f_{\mathfrak{L}}}(\lim_{\lambda \to \Lambda}(\lambda, \varphi(\lambda))) \in B^*$ since, for every $\varphi \in A^{\mathfrak{L}}$, the function $f \circ \varphi \in B^{\mathfrak{L}}$. This extension is continuous: let Ω be a basis open subset of $B_{\mathfrak{L}}$. If $\Omega = \{(\lambda, x)\}$ then

$$\overline{f_{\mathfrak{L}}}^{-1}(\Omega) = \bigcup_{y \in f^{-1}(x)} (\lambda, y),$$

which is open. If $\Omega = N_{\varphi, Q}$ for some $\varphi : \mathfrak{L} \to \mathbb{R}$, $Q \in \mathcal{U}$ then let $\xi \in \overline{f_{\mathfrak{L}}}^{-1}(\Omega)$. If $\xi = (\lambda, x)$ for some $x \in A$ then $\{(\lambda, x)\}$ is a neighborhood of (λ, x) included in $\overline{f_{\mathfrak{L}}}^{-1}(\Omega)$; if $\xi = \lim_{\lambda \to \Lambda}(\lambda, \psi(\lambda))$ then $\overline{f_{\mathfrak{L}}}(\xi) = \lim_{\lambda \to \Lambda}(\lambda, \varphi(\lambda))$, therefore there exists $Q_1 \in \mathcal{U}$ such that $f(\psi(\lambda)) = \varphi(\lambda)$ for all $\lambda \in Q_1$, hence if we set $Q_2 = Q \cap Q_1$ we have that N_{ψ, Q_2} is a neighborhood of ξ included in $\overline{f_{\mathfrak{L}}}^{-1}(\Omega)$, thus $\overline{f_{\mathfrak{L}}}^{-1}(\Omega)$ is open, and this proves that $\overline{f_{\mathfrak{L}}}$ is continuous.

Finally, $\overline{f_{\mathfrak{L}}}$ restricted to A coincides with f since, for every $a \in A$, by definition

$$\overline{f_{\mathfrak{L}}}(a) = \overline{f_{\mathfrak{L}}}\left(\lim_{\lambda \to \Lambda} (\lambda, a)\right) = \lim_{\lambda \to \Lambda} (\lambda, f(a)) = f(a).$$

Lemma 1 entails that the following definition is well posed:

Definition 11 Given a function

$$f : A \to B$$

the restriction of $\overline{f_{\mathfrak{L}}}$ to A^* is called the **natural extension** of f and it will be denoted by

$$f^* : A^* \to B^*.$$

In particular, $f^*(a) = f(a)$ for every $a \in A$.

2.4 *The Λ-limit in $V_\infty(\mathbb{R})$*

In this section we want to extend the notion of Λ-limit to a wider family of functions. To do that, we have to introduce the notion of superstructure on a set (see also [21]):

Definition 12 Let E be an infinite set. The superstructure on E is the set

$$V_\infty(E) = \bigcup_{n \in \mathbb{N}} V_n(E),$$

where the sets $V_n(E)$ are defined by induction by setting

$$V_0(E) = E$$

and, for every $n \in \mathbb{N}$,

$$V_{n+1}(E) = V_n(E) \cup \mathscr{P}(V_n(E)).$$

Here $\mathscr{P}(E)$ denotes the power set of E. Identifying the couples with the Kuratowski pairs and the functions and the relations with their graphs, it follows that $V_\infty(E)$ contains almost every usual mathematical object that can be constructed starting with E; in particular, $V_\infty(\mathbb{R})$ contains almost every usual mathematical object of analysis.

Sometimes, following e.g. [21], we will refer to

$$\mathbb{U} := V_\infty(\mathbb{R})$$

as to the **standard universe**. A mathematical entity (number, set, function or relation) is said to be **standard** if it belongs to $\mathbb{U}$.

Now we want to formally define the Λ-limit of $(\lambda, \varphi(\lambda))$ where $\varphi(\lambda)$ is any bounded function of mathematical objects in $V_\infty(\mathbb{R})$ (a function $\varphi : \mathfrak{L} \to V_\infty(\mathbb{R})$ is called bounded if there exists n such that $\forall\lambda \in \mathfrak{L},\ \varphi(\lambda) \in V_n(\mathbb{R})$). To this aim, let us consider a function

$$\varphi : \mathfrak{L} \to V_n(\mathbb{R}). \tag{4}$$

We will define $\lim_{\lambda\to\Lambda}(\lambda, \varphi(\lambda))$ by induction on n.

Definition 13 For $n = 0$, $\lim_{\lambda\to\Lambda}(\lambda, \varphi(\lambda))$ exists by Theorem 1; so by induction we may assume that the limit is defined for $n - 1$ and we define it for the function (4) as follows:

$$\lim_{\lambda\to\Lambda}(\lambda, \varphi(\lambda)) = \left\{ \lim_{\lambda\to\Lambda}(\lambda, \psi(\lambda)) \mid \psi : \mathfrak{L} \to V_{n-1}(\mathbb{R}) \text{ and } \forall\lambda \in \mathfrak{L},\ \psi(\lambda) \in \varphi(\lambda) \right\}.$$

Clearly $\lim_{\lambda\to\Lambda}(\lambda, \varphi(\lambda))$ is a well defined set in $V_\infty(\mathbb{R}^*)$.

Definition 14 A mathematical entity (number, set, function or relation) which is the Λ-limit of a function is called **internal**.

Notice that $V_\infty(\mathbb{R}^*)$ contains sets which are not internal.

Example 1 Each real number is standard and internal. However the set of real numbers $\mathbb{R} \in V_\infty(\mathbb{R}^*)$ is standard, but not internal. In order to see this let us suppose that there is a function $\varphi : \mathfrak{L} \to V_1(\mathbb{R})$ such that $\mathbb{R} = \lim_{\lambda\to\Lambda}(\lambda, \varphi(\lambda))$. Therefore, by definition, we would have

$$\mathbb{R} = \left\{ \lim_{\lambda\to\Lambda}(\lambda, \psi(\lambda)) \mid \psi : \mathfrak{L} \to \mathbb{R} \text{ and } \forall\lambda \in \mathfrak{L},\ \psi(\lambda) \in \varphi(\lambda) \right\}.$$

In particular, for every constant $c \in \mathbb{R}$ we have that $c \in \varphi(\lambda)$; therefore, $\varphi(\lambda) = \mathbb{R}$ for every $\lambda \in \mathfrak{L}$, and this is absurd because

$$\lim_{\lambda\to\Lambda}(\lambda, \mathbb{R}) = \mathbb{R}^*,$$

and (except trivial cases) $\mathbb{R}^*$ properly includes $\mathbb{R}$. Let us explicitly observe that (except trivial cases), while for every $c \in \mathbb{R}$ the function $\lambda \to (\lambda, c)$ converges to c, given $A \in V_n(\mathbb{R})$, for $n \geq 1$ the function $\lambda \to (\lambda, A)$ converges to a proper superset of A.

Definition 15 A mathematical entity (number, set, function or relation) which is not internal is called **external.**

As it is given, the definition of limit given by Definition 13 is not related to any topology. Thus a question arises naturally: is there a topological Hausdorff space such that the limit given by Definition 13 is the topological limit of a function?

The answer is affirmative, and it is a consequence of the possibility to topologize the set

$$\mathbb{U}_{\mathfrak{L}} = [\mathfrak{L} \times V_\infty(\mathbb{R})] \uplus V_\infty(\mathbb{R}^*).$$

To topologize $\mathbb{U}_{\mathfrak{L}}$ we take as open sets:

- every subset of $\mathfrak{L} \times V_\infty(\mathbb{R})$;
- $\{x\}$ for every $x \in V_\infty(\mathbb{R}^*)$ that is external;
- $N_{\varphi,Q} := \{(\lambda, \varphi(\lambda)) \,|\, \lambda \in Q\} \cup \{x\}$ for every x internal such that φ is a bounded sequence with

$$x = \lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda))\,.$$

We let $\sigma_{\mathscr{U}}$ be the topology on $\mathbb{U}_{\mathfrak{L}}$ generated by these open sets. It is clear that this topology is Hausdorff and that the Λ-limit is a limit in this topology.

The set

$$\mathbb{U}_{\mathfrak{L}} = [\mathfrak{L} \times V_\infty(\mathbb{R})] \cup V_\infty(\mathbb{R}^*)$$

will be called the **expanded universe**. Let us note that, by construction, $\mathbb{U}_{\mathfrak{L}} \subseteq V_\infty(\mathbb{R}_{\mathfrak{L}})$.

The results about extensions of subsets of $\mathbb{R}$ and of functions $f : A \to B$, $A, B \subseteq \mathbb{R}$, can be generalized to our new general setting. Since a function f can be identified with its graph then the natural extension of a function is defined by the above definition. Moreover we have the following result, that can be proved as Lemma 1:

Theorem 3 *For every sets $E, F \in V_\infty(\mathbb{R})$ and for every function $f : E \to F$ the* ***natural extension*** *of f is a continuous function*

$$f^* : E^* \to F^*,$$

and for every function $\varphi : \mathfrak{L} \to E$ we have that

$$\lim_{\lambda \to \Lambda} f(\lambda, \varphi(\lambda)) = f^*\left(\lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda))\right).$$

3 Comparison Between Λ-theory and Ultrapowers

3.1 Λ-theory and Nonstandard Universes

It should be evident to any reader with a background in NSA that Λ -theory (when restricted to minimal canonical extensions) is closely related to ultrapowers (which, from a purely logical point of view, are even easier to define). In this section we want to detail the relationship between Λ-theory and NSA. We will show that $\mathbb{U}_{\mathfrak{L}}$ contains a nonstandard universe in the sense of Keisler [21]. We recall the main definitions of [21].

Definition 16 A **superstructure embedding** is a one to one mapping $*$ of $V_\infty(\mathbb{R})$ into another superstructure $V_\infty(\mathbb{S})$ such that

1. $\mathbb{R}$ is a proper subset of $\mathbb{S}$, $r^* = r$ for all $r \in \mathbb{R}$, and $\mathbb{R}^* = \mathbb{S}$;
2. for $x, y \in V_\infty(\mathbb{R})$, $x \in y$ if and only if $x^* \in y^*$.

To avoid confusion, in this section we will use the letter $\mathbb{K}$ to denote the Non-Archimedean field constructed in Sect. 2.2, while $\mathbb{R}^*$ will be used as in Definition 16.

Let us denote by $\mathscr{L}$ a formal language relative to a first order predicate logic with the equality symbol, a binary relation symbol $\in$, and a constant symbol for each element in $V_\infty(\mathbb{R})$. We recall that a sentence $p \in \mathscr{L}$ is bounded if every quantifier in p is bounded (see e.g. [21]). The notion of bounded sequence allows to define the notion of nonstandard universe.

Definition 17 A **nonstandard universe** is a superstructure embedding $* : V_\infty(\mathbb{R}) \to V_\infty(\mathbb{R}^*)$ which satisfies Leibniz' Principle, which is the property that states that for each bounded sentence $p \in \mathscr{L}$, p is true in $V_\infty(\mathbb{R})$ if and only if p^* is true[3] in $V_\infty(\mathbb{R}^*)$.

Definition 18 We let $* : V_\infty(\mathbb{R}) \to V_\infty(\mathbb{K})$ be the map defined as follows: for every element $x \in V_\infty(\mathbb{R})$ we set

$$x^* = \lim_{\lambda \to \Lambda} (\lambda, x).$$

Remark 6 Following Keisler (see [21]), in Definition 17 we have called nonstandard universe just the superstructure embedding; however, in our approach, probably, it would be more appropriate to call nonstandard universe the set $V_\infty(\mathbb{K})$; in this case the global picture would be the following one: the extended universe

$$\mathbb{U}_{\mathfrak{L}} = [\mathfrak{L} \times V_\infty(\mathbb{R})] \uplus V_\infty(\mathbb{K})$$

contains pairs (λ, x) and elements of the nonstandard universe $V_\infty(\mathbb{K})$; the latter contains the following objects:

- standard elements, namely objects $x \in V_\infty(\mathbb{R}) \subset V_\infty(\mathbb{K})$;
- nonstandard elements, namely objects $x \in V_\infty(\mathbb{K}) \setminus V_\infty(\mathbb{R})$;
- hyperimages, namely objects x such that there exists $y \in V_\infty(\mathbb{R})$ with $x = y^*$;
- internal objects, namely Λ-limits of bounded functions;
- external objects.

To give some examples: 7, $\mathbb{R}$, $\mathscr{P}(\mathbb{R} \times \mathscr{P}(\mathbb{R}))$ are all standard elements; 7 is also an hyperimage, while $\mathbb{R}$, $\mathscr{P}(\mathbb{R} \times \mathscr{P}(\mathbb{R}))$ are not; $\mathbb{K}$, $\mathscr{P}(\mathbb{R})^*$ and $\lim_{\lambda \to \Lambda}(\lambda, \varphi(\lambda))$

[3] p^* is the bounded sentence obtained by changing every constant symbol $c \in V_\infty(\mathbb{R})$ that appears in p with c^*.

for every $\varphi : \mathfrak{L} \to \mathbb{R}$ which is not eventually constant are nonstandard elements, and they are all internal; $\mathbb{R}$ and $\mathbb{K}\backslash\mathbb{R}$ are external objects.

An interesting class of internal objects, particularly important for our applications to PDEs, is that of hyperfinite objects[4]:

Definition 19 An object $\xi \in V_\infty(\mathbb{K})$ is hyperfinite if there exists a natural number n and a bounded function $\varphi : \mathfrak{L} \to \mathscr{P}_{fin}(V_n(\mathbb{R}))$ such that $\xi = \lim_{\lambda\to\Lambda}(\lambda, \varphi(\lambda))$.

Hyperfinite objects are the analogue, in the universe $V_\infty(\mathbb{K})$, of finite objects in $V_\infty(\mathbb{R})$. The notion of hyperfinite object will be used in Sect. 4 to show some applications of Λ-theory.

To detail the relationship between Λ-theory and nonstandard universes in the sense of Keisler we need to specify how we interpret formulas in $V_\infty(\mathbb{K})$[5]:

Definition 20 Let $p(x_1, \dots, x_n) \in \mathscr{L}$ be a bounded formula having $x_1, \dots, x_n$ as its only free variables. Let $\xi_1 = \lim_{\lambda\to\Lambda}(\lambda, \varphi_1(\lambda)), \dots, \xi_n = \lim_{\lambda\to\Lambda}(\lambda, \varphi_n(\lambda))$. We say that $p^*(\xi_1, \dots, \xi_n)$ holds in $V_\infty(\mathbb{K})$ iff $p(\varphi_1(\lambda), \dots, \varphi_n(\lambda))$ is eventually true in $V_\infty(\mathbb{R})$, namely iff

$$\{(\lambda, (\varphi_1(\lambda), \dots, \varphi_n(\lambda)) \mid p(\varphi_1(\lambda), \dots, \varphi_n(\lambda)) \text{ holds in } V_\infty(\mathbb{R})\} \cup \{(\xi_1, \dots, \xi_n)\}$$

is open in $\sigma_{\mathscr{U}}$.

Theorem 4 *Let $*$ be defined as in Definition 18; then*

$$(V_\infty(\mathbb{R}), V_\infty(\mathbb{K}), *)$$

is a nonstandard universe.

Proof That $* : V_\infty(\mathbb{R}) \to V_\infty(\mathbb{K})$ is a superstructure embedding follows clearly from the definitions.

Moreover, for every bounded formula $p(x_1, \dots, x_n) \in \mathscr{L}$ having $x_1, \dots, x_n$ as its only free variables, for every $\xi_1 = \lim_{\lambda\to\Lambda}(\lambda, \varphi_1(\lambda)), \dots, \xi_n = \lim_{\lambda\to\Lambda}(\lambda, \varphi_n(\lambda))$, we have that

$$\begin{gathered} p(\xi_1, \dots, \xi_n) \text{ holds in } V_\infty(\mathbb{K}) \Leftrightarrow \\ \{(\lambda, (\varphi_1(\lambda), \dots, \varphi_n(\lambda)) \mid p(\varphi_1(\lambda), \dots, \varphi_n(\lambda)) \text{ holds in } V_\infty(\mathbb{R})\} \cup \{(\xi_1, \dots, \xi_n)\} \\ \text{is open in } \sigma_{\mathscr{U}} \Leftrightarrow \\ \{\lambda \in \mathscr{L} \mid p(\varphi_1(\lambda), \dots, \varphi_n(\lambda)) \text{ holds in } V_\infty(\mathbb{R})\} \in \mathscr{U} \Leftrightarrow \\ p([\varphi_1], \dots, [\varphi_n]) \text{ holds in } \mathbb{R}^{\mathfrak{L}}_{\mathscr{U}}. \end{gathered}$$

This equivalence can be used to easily prove the transfer property for $* : V_\infty(\mathbb{R}) \to V_\infty(\mathbb{K})$ by induction on the complexity of formulas.

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[4]See e.g. [1], where many different applications of hyperfinite objects and other nonstandard tools are developed.

[5]Once again, it should be evident to readers expert in NSA that our definition is precisely analogous to the one that is given for ultrapowers.

3.2 General Remarks

Theorem 4 makes precise the intuition that the topological approach to Non-Archimedean mathematics given by Λ-theory is closely related with NSA as presented by Keisler in [21]. As we said in the introduction, we think of Λ-theory as a way to present to a non-expert reader many basic ideas of NSA in a more familiar language. Nevertheless, we think that from a philosophical point of view point there are some differences between Λ-theory and the ultrapower approach:

1. in Λ-theory we assume the existence of a unique mathematical universe $\mathbb{U}_{\mathfrak{L}} \subset V_\infty(\mathfrak{L} \cup \mathbb{K})$. Inside this universe there are entities that do not appear in traditional mathematics but that can be obtained as limits of traditional objects, namely the internal elements. Moreover, there are also external objects, and some of them are objects of traditional mathematics (e.g., $\mathbb{R}$);
2. in NSA the primitive concept is that of hyperimage, the other concepts (e.g., the concept of internal object) are derived by that one; in Λ -theory, the primitive concept is that of Λ-limit, while the concept of hyperimage is derived by the limit. So, within Λ-theory the notion of internal object (being defined as a Λ-limit) is more primitive than that of hyperimage;
3. the construction of the hyperreal field in our approach has a topological "flavour" which is similar to other constructions in traditional mathematics. In fact, e.g. whitin our approach the construction of $\mathbb{R}^*$ as "set of limits of functions with values in $\mathfrak{L} \times \mathbb{R}$" has some similarities with the construction of $\mathbb{R}$ as set of limits of Cauchy sequences with values in $\mathbb{Q}$.

4 Generalized Solutions

In many circumstances, the notion of function is not sufficient to the needs of a theory and it is necessary to extend it. Many different constructions have been considered in the literature to deal with this problem, both with standard (for example, Colombeau's Theory, see e.g. [19] and references therein for a complete presentation of the theory and [18] and reference therein for some new developments of the theory with applications to generalized ODE's) and nonstandard techniques (see e.g. [26]). In this section we want to apply Λ-theory to construct spaces of generalized functions called ultrafunctions (see also [5, 9–14], and to use them to study a simple class of problems in calculus of variations. As we are going to show, ultrafunctions are constructed by means of a particular version of the hyperfinite approach which can be naturally introduced by means of Λ-theory.

In this section we will use the following shorthand notation: for every bounded function $\varphi : \mathfrak{L} \to V_\infty(\mathbb{R})$ we let

$$\lim_{\lambda \uparrow \Lambda} \varphi(\lambda) := \lim_{\lambda \to \Lambda} (\lambda, \varphi(\lambda)) .$$

4.1 Ultrafunctions

Let N be a natural number, let Ω be a set in $\mathbb{R}^N$ and let $V(\Omega)$ be a function vector space. We want to define the space of ultrafunctions generated by $V(\Omega)$. We assume that

$$\mathfrak{L} = \mathscr{P}_{fin}(V(\Omega)),$$

and we let $\mathscr{U}$ be a fine ultrafilter[6] on $\mathfrak{L}$. For any $\lambda \in \mathfrak{L}$, we set

$$V_\lambda(\Omega) = Span\{\lambda \cap V(\Omega)\}.$$

Let us note that, by construction, $V_\lambda(\Omega)$ is a finite dimensional vector subspace of $V(\Omega)$.

Definition 21 Given the function space $V(\Omega)$ we set

$$V_\Lambda(\Omega) := \lim_{\lambda \uparrow \Lambda} V_\lambda(\Omega) = \left\{ \lim_{\lambda \uparrow \Lambda} u_\lambda \mid u_\lambda \in V_\lambda(\Omega) \right\}.$$

$V_\Lambda(\Omega)$ will be called the **space of ultrafunctions** generated by $V(\Omega)$.

Given any vector space of functions $V(\Omega)$, we have the following three properties:

1. the ultrafunctions in $V_\Lambda(\Omega)$ are Λ-limits of functions valued in $V(\Omega)$, so they are all internal functions;
2. the space of ultrafunctions $V_\Lambda(\Omega)$ is a vector space of hyperfinite dimension;
3. if we identify a function f with its natural extension f^* then $V_\Lambda(\Omega)$ includes $V(\Omega)$, hence we have that

$$V(\Omega) \subset V_\Lambda(\Omega) \subset V(\Omega)^*.$$

Remark 7 Notice that the natural extension f^* of a function f is an ultrafunction if and only if $f \in V(\Omega)$.

Proof The proof of this result is trivial.[7]

Ultrafunctions can be used to give generalized solutions to some problems in the calculus of variations (see e.g. [11]). Usually this kind of problems have a "natural space" where to look for solutions: the appropriate function space has to be a space in which the problem is well posed and (relatively) easy to solve. For a very large class of problems the natural space is a Sobolev space. However, many times even the best candidates to be natural spaces are inadequate to study the problem, since there is no solution in them. So the choice of the appropriate function space is part of the problem

[6]Let us recall that an ultrafilter $\mathscr{U}$ on $\mathfrak{L}$ is fine if for every $\lambda \in \mathfrak{L}$ the set $\{\mu \in \mathfrak{L} \mid \mu \subseteq \lambda\} \in \mathscr{U}$. We also point out that, for more complicated applications, it would be better to take $\mathfrak{L} = \mathscr{P}_{fin}(V_\infty(\mathbb{R}))$.

[7]Any interested reader can find it in [10].

itself; this choice is somewhat arbitrary and it might depend on the final goals. In the framework of ultrafunctions this situation persists. The general rule is: choose the "natural space" $V(\Omega)$ and look for a generalized solution in $V_\Lambda(\Omega)$. For many applications, an hypothesis[8] that we need to assume is that $D(\Omega) \subseteq V(\Omega) \subseteq L^2(\Omega)$. In this case, since $V_\Lambda(\Omega) \subset \left[L^2(\Omega)\right]^*$, we can equip $V_\Lambda(\Omega)$ with the following scalar product:

$$(u, v) = \int^* u(x)v(x)\,dx, \tag{5}$$

where $\int^*$ is the natural extension of the Lebesgue integral considered as a functional

$$\int : L^1(\Omega) \to \mathbb{R}.$$

The norm[9] of an ultrafunction will be given by

$$\|u\| = \left(\int^* |u(x)|^2\,dx\right)^{\frac{1}{2}}.$$

Moreover, using the inner product (5), we can identify $L^2(\Omega)$ with a subset of $V'(\Omega)$ and hence $\left[L^2(\Omega)\right]^*$ with a subset of $\left[V'(\Omega)\right]^*$; in this case, $\forall f \in \left[L^2(\Omega)\right]^*$, we let $\widetilde{f}$ be the unique ultrafunction such that, $\forall v \in V_\Lambda(\Omega)$,

$$\int^* \widetilde{f}(x)v(x)\,dx = \int^* f(x)v(x)\,dx,$$

namely we associate to every $f \in L^2(\Omega)^*$ the function $\widetilde{f} = P_\Lambda(f)$, where

$$P_\Lambda : \left[L^2(\Omega)\right]^* \to V_\Lambda(\Omega)$$

is the orthogonal projection.

Remark 8 There are a few different ways to prove the existence of an orthogonal projection of $L^2(\Omega)^*$ on $V_\Lambda(\Omega)$. For example, consider, for every $\lambda \in \mathfrak{L}$, the orthogonal projection $P_\lambda : L^2(\Omega) \to V_\lambda(\Omega)$. Let $F := \lim_{\lambda \uparrow \Lambda} P_\lambda$. It is immediate to see that $F : L^2(\Omega)^* \to V_\Lambda(\Omega)$ is an orthogonal projection.

Let us note that the key property to associate an ultrafunction to every function in $\left[L^2(\Omega)\right]^*$ is that $\left[L^2(\Omega)\right]^*$ can be identified with a subset of $\left[V'(\Omega)\right]^*$. Therefore, using a similar idea, it is also possible to extend a large class of operators:

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[8] E.g., in [12] a (slightly modified) version of this hypothesis is used to construct an embedding of the space of distributions in a particular algebra of functions constructed by means of ultrafunctions.

[9] Let us observe that both the scalar product and the norm take values in $\mathbb{R}^*$.

Definition 22 Given an operator

$$\mathscr{A} : V(\Omega) \to V'(\Omega),$$

we can extend it to an operator

$$\widetilde{\mathscr{A}} : V_\Lambda(\Omega) \to V_\Lambda(\Omega)$$

in the following way: given an ultrafunction u, $\mathscr{A}_\Lambda(u)$ is the unique ultrafunction such that

$$\forall v \in V_\Lambda(\Omega), \int^* \widetilde{\mathscr{A}}(u) v dx = \int^* \mathscr{A}^*(u) v dx;$$

namely

$$\widetilde{\mathscr{A}} = P_\Lambda \circ \mathscr{A}^*,$$

where P_Λ is the canonical projection.

This association can be used, e.g., to define the derivative of an ultrafunction, by setting

$$Du := \widetilde{\partial} u = P_\Lambda(\partial^* u)$$

for every ultrafunction $u \in V_\Lambda(\Omega) \cap \mathscr{C}^1(\Omega)^*$.

4.2 Applications to Calculus of Variations

To give an example of application of ultrafunctions to calculus of variations, we will show the ultrafunction interpretation of the Lavrentiev phenomenon. Let us consider the following problem: minimize the functional

$$J_0(u) = \int_0^1 \left[\left(|\nabla u|^2 - 1 \right)^2 + |u|^2 \right] dx$$

in the function space $\mathscr{C}_0^1(\Omega) = \mathscr{C}^1(\Omega) \cap \mathscr{C}_0(\overline{\Omega})$. We assume Ω to be bounded to avoid problems of summability.[10]

It is not difficult to realize that any minimizing sequence u_n converges uniformly to 0 and that $J_0(u_n) \to 0$, but $J_0(0) > 0$ for any $u \in \mathscr{C}_0^1(0, 1)$. Hence there is no minimizer in $\mathscr{C}_0^1(\Omega)$.

[10] This example has already been studied in greater detail in [11].

On the contrary, it is possible to show that this problem has a minimizer in the space of ultrafunctions

$$V_0^1(\Omega) = \left[\mathscr{C}^1(\Omega) \cap \mathscr{C}_0(\overline{\Omega})\right]_\Lambda .$$

In $V_0^1(\Omega)$ our problem becomes

$$\text{find } v \in V_0^1(\Omega) \text{ s.t. } \widetilde{J}_0(v) = \min_{u \in V_0^1(\Omega)} \widetilde{J}_0(u). \tag{P}$$

To solve (P), let us prove the following "ultrafunction version" of an existence result for minimizers of coercive continuous operators; the proof is based on a variant of Faedo-Galerkin method.

Theorem 5 *Let $V(\Omega) \subseteq L^2(\Omega)$ be a vector space and let*

$$J : V(\Omega) \to \mathbb{R}$$

be an operator continuous and coercive on finite dimensional spaces. Then the operator

$$\widetilde{J} : V_\Lambda(\Omega) \to \mathbb{R}^*$$

has a minimum point. If J itself has a minimizer u, then u^* is a minimizer of $\widetilde{J}$.

Proof Take $\lambda \in \mathfrak{L}$; since the operator

$$J|_{V_\lambda} : V_\lambda(\Omega) \longrightarrow \mathbb{R}$$

is continuous and coercive, it has a minimizer; namely

$$\exists u_\lambda \in V_\lambda \; \forall v \in V_\lambda \; J(u_\lambda) \leq J(v).$$

We set

$$u_\Lambda = \lim_{\lambda \uparrow \Lambda} u_\lambda.$$

We show that u_Λ is a minimizer of $\widetilde{J}$. Let $v \in V_\Lambda(\Omega)$. Let us suppose that $v = \lim_{\lambda \uparrow \Lambda} v_\lambda$; then by construction

$$\forall \lambda \in \mathfrak{L} \; J(u_\lambda) \leq J(v_\lambda),$$

therefore

$$\widetilde{J}(u_\Lambda) \leq \widetilde{J}(v).$$

If J itself has a minimizer $\overline{u}$, then u_λ is eventually equal to $\overline{u}$ and hence $u_\Lambda = \overline{u}^*$.

As a consequence, problem (P) has a solution, since the functional J_0 satisfies the hypothesis of Theorem 5. So there exists an ultrafunction $u \in V_0^1(\Omega)$ that minimizes $\widetilde{J}_0$. Moreover, it can be represented as the Λ-limit of a function of minimizers of the approximate problems on the spaces $\left[\mathscr{C}^1(\Omega) \cap \mathscr{C}_0(\overline{\Omega})\right]_\lambda$. By using this characterization, it is also possible to derive some qualitative properties of u, e.g. it is not difficult to show that, $\forall x \in (0,1)^*$, the minimizer $u_\Lambda(x) \sim 0$ and that $\widetilde{J}_0(u_\Lambda)$ is a positive infinitesimal.

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A Note on an Overdetermined Problem for the Capacitary Potential

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Abstract We consider an overdetermined problem arising in potential theory for the capacitary potential and we prove a radial symmetry result.

Keywords Overdetermined boundary value problems · Electrostatic potential · Symmetry · Capacity

1 Introduction

In this note we deal with an overdetermined problem for the electrostatic potential. The electrostatic capacity of a bounded set $\Omega \subset \mathbb{R}^n$, $n \geq 3$, is defined by

$$\mathrm{Cap}(\Omega) = \inf \left\{ \int_{\mathbb{R}^n} |Dv|^2 dx \ : \ v \in C_c^\infty(\mathbb{R}^n)\,, \ v(x) \geq 1 \quad \forall x \in \Omega \right\}, \qquad (1)$$

where $C_c^\infty(\mathbb{R}^n)$ denotes the set of C^∞ functions having compact support. It is well-known that it can be equivalently obtained via the asymptotic expansion of the so-called *electrostatic potential* of Ω (or *capacitary function* of Ω), i.e.

$$\mathrm{Cap}(\Omega) = (n-2)\omega_n \lim_{|x| \to \infty} u(x)|x|^{n-2}\,, \qquad (2)$$

where ω_n denotes the surface area of the unit sphere in $\mathbb{R}^n$, and u is a minimizer of problem (1) and hence it satisfies

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$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n \setminus \overline{\Omega}\,, \\ u = 1 & \text{on } \partial\Omega\,, \\ \lim_{|x|\to+\infty} u(x) = 0\,. \end{cases} \tag{3}$$

We mention that the electrostatic potential u represents the potential energy of the electrical field induced by the conductor Ω, normalized so that the voltage difference between $\partial\Omega$ and infinity is one, and hence $\mathrm{Cap}(\Omega)$ represents the total electric charge needed to induce the potential u (see for instance [9]).

A classical question in potential theory is the study of symmetry properties for problem (3). More precisely, one imposes an extra assumption to Problem (3) and studies how such an overdetermination reflects on the domain Ω. In particular, one can ask whether certain geometric properties of the constraint are inherited by the domain Ω. In this respect, a typical problem is the so-called Serrin's exterior problem, where one assumes that

$$|Du| = c \quad \text{on } \partial\Omega\,, \tag{4}$$

where c is a positive constant, and one proves that a solution to (3) and (4) exists if and only if the domain Ω is a ball. This result has been established in [10] by using the method of moving planes. Other similar problems and related results can be found in [2, 3, 7, 11, 13, 14].

In this note we discuss two kinds of overdetermining conditions involving the mean curvature $H_{\partial\Omega}$ of $\partial\Omega$ (that is the average of the principal curvatures of $\partial\Omega$). More precisely, we prove the following theorem.

Theorem 1 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary of class $C^{3,\alpha}$ and let u be the solution of* (3). *If u and Ω are such that either*

$$\int_{\partial\Omega} |Du|^2 \left[H_{\partial\Omega} - \frac{|Du|}{n-2} \right] d\mathscr{H}^{n-1} \leq 0, \tag{5}$$

or

$$\int_{\partial\Omega} |Du|^2 \left[(n-1)H_{\partial\Omega} - \frac{n|Du|}{2(n-2)} \right] d\mathscr{H}^{N-1} \leq \frac{(n-2)^3}{2}\omega_n \left(\frac{\mathrm{Cap}(\Omega)}{(n-2)\omega_n} \right)^{\frac{n-4}{n-2}}, \tag{6}$$

then Ω is a ball and u is radially symmetric.

We mention that in the case that constraint (5) holds, Theorem 1 was already proven in [1]. Indeed, in [1, Theorem1.1] the authors prove the symmetry result by using a conformal reformulation of the problem and by proving the rotational symmetry via a splitting argument. In this respect, we give a different proof of this theorem.

Our approach is very simple and use a chain of integral identities and a basic inequality for symmetric elementary functions (known as Newton's inequality), as in the spirit of [4–6]. More precisely, by considering the auxiliary problem for the function

$$v = u^{-\frac{2}{n-2}} ,$$

where u solves (3), we prove that v must be quadratic, and hence the capacitary function u has radial symmetry. This approach is very flexible and it has been extended to more general settings [2, 3].

It is interesting to notice that from the proof of Theorem 1 (see Step 1 in Sect. 3) we immediately obtain the following lower bound for the capacity, for $n = 3$:

$$\mathrm{Cap}(\Omega) \int_{\partial\Omega} |Du|^2 \left[4H_{\partial\Omega} - 3|Du|\right] d\mathcal{H}^2 \geq 4\pi . \tag{7}$$

This lower bound is optimal, in the sense that the equality sign is attained when Ω is a ball.

2 Preliminaries

We use the following notation. Let $A = (a_{ij})$ be a real $n \times n$ symmetric matrix. We denote by $S_k(A)$, $k \in \{1, \dots, n\}$, the sum of all the principal minors of A of order k, so that $S_1(A) = tr(A)$ and $S_n(A) = det(A)$. Denoting by

$$S^k_{ij}(A) = \frac{\partial}{\partial a_{ij}} S_k(A),$$

it holds

$$S_k(A) = \frac{1}{k} S^k_{ij}(A) a_{ij},$$

where here and later the Einstein summation convention is applied. In particular for $k = 2$

$$S^2_{ij}(A) = \frac{\partial}{\partial a_{ij}} S_2(A) = \begin{cases} -a_{ji} & i \neq j \\ \sum_{k \neq i} a_{kk} & i = j . \end{cases}$$

Notice that $S_k(A)$ are the k-th elementary symmetric function of the eigenvalues of A; so that

$$S_k(A) = S_k(\lambda_1, \dots, \lambda_n) = \sum_{1 \leq i_1 < \dots < i_k \leq n} \lambda_{i_1} \cdot \ldots \cdot \lambda_{i_n},$$

where λ_i are the eigenvalues of the matrix A.

When $A = D^2 v$ it was proven in [12] that

$$S_k(D^2 v) = \frac{1}{k} \mathrm{div}(S^k_{ij}(D^2 v) v_j) , \tag{8}$$

which follows from the fact that the vector $(S^k_{i1}(D^2v), \dots, S^k_{in}(D^2v))$ is divergence free for $i = 1, \dots, n$, i.e.

$$\frac{\partial}{\partial x_j} S^k_{ij}(D^2v) = 0, \quad i = 1, \dots, n.$$

In particular, for $k = 2$ we have

$$S_2(D^2v) = \frac{1}{2} S^2_{ij}(D^2v)v_{ij} = \frac{1}{2}\mathrm{div}\Big(S^2_{ij}(D^2v)v_j\Big),$$

where

$$S^2_{ij}(D^2v) = \frac{\partial}{\partial v_{ij}} S_2(D^2v) = \begin{cases} -v_{ji} & i \neq j \\ \Delta v - v_{ii} & i = j\,. \end{cases}$$

Notice that if $L_t = \{v > t\}$ is a super level set of v, then

$$|Dv|^2 \Delta v = (n-1) H_{\partial L_t} |Dv|^3 + v_i v_{ij} v_j \quad \text{on } \partial L_t\,, \tag{9}$$

so that, if ∂L_t is oriented such that $\nu = Dv/|Dv|$, then

$$S^2_{ij}(D^2v) v_i v_j = (n-1) H_{\partial L_t} |Dv|^3 \quad \text{on } \partial L_t\,. \tag{10}$$

Two crucial ingredients for the proof of Theorem 1 are contained in next lemmas.

Lemma 1 (Newton Inequality) *Let A be a symmetric matrix in $\mathbb{R}^{n\times n}$; it holds*

$$S_2(A) \le \frac{n-1}{2n} \mathsf{Tr}(A)^2\,. \tag{11}$$

Moreover, if $\mathsf{Tr}(A) \neq 0$ *and equality holds in (11), then*

$$A = \frac{\mathsf{Tr}(A)}{n} I\,.$$

Lemma 2 *For any smooth positive function v and $\gamma \in \mathbb{R}$ we have the following identity:*

$$\begin{aligned} &2v^\gamma S_2(D^2v) = \\ &= \mathrm{div}\Big(\frac{\gamma}{2} v^{\gamma-1}|Dv|^2 Dv + v^\gamma S^2_{ij}(D^2v)v_i\Big) - \frac{3}{2}\gamma v^{\gamma-1}|Dv|^2\Delta v - \frac{\gamma(\gamma-1)}{2} v^{\gamma-2}|Dv|^4\,. \end{aligned} \tag{12}$$

Proof We notice that for $\gamma = 0$ (12) is just the definition of S_2 and then we may assume $\gamma \neq 0$. Identity (12) immediately follows from the following two identities:

$$\mathrm{div}(v^\gamma S^2_{ij}(D^2v)v_i) = 2v^\gamma S_2(D^2v) + \gamma v^{\gamma-1} S^2_{ij}(D^2v) v_i v_j\,, \tag{13}$$

and

$$v^{\gamma-1}S^2_{ij}(D^2v)v_iv_j = \frac{3}{2}v^{\gamma-1}|Dv|^2\Delta v + \frac{\gamma-1}{2}v^{\gamma-2}|Dv|^4 - \frac{1}{2}\mathrm{div}(v^{\gamma-1}|Dv|^2Dv)\,. \tag{14}$$

Identity (13) is readily obtained from $\gamma v^{\gamma-1}v_i = (v^\gamma)_i$, (8) and

$$S_2(D^2v) = \frac{1}{2}S^2_{ij}(D^2v)v_{ij} = \frac{1}{2}\mathrm{div}(S^2_{ij}(D^2v)v_i)\,.$$

To prove (14) we notice that, since

$$S^2_{ij}(D^2v)v_iv_j = |Dv|^2\Delta v - v_iv_jv_{ij}\,,$$

we have that

$$\begin{aligned}
v^{\gamma-1}S^2_{ij}(D^2v)v_iv_j &= v^{\gamma-1}|Dv|^2\Delta v - v^{\gamma-1}v_iv_jv_{ij}\\
&= v^{\gamma-1}|Dv|^2\Delta v + \frac{1}{2}\big[-\mathrm{div}(v^{\gamma-1}|Dv|^2Dv) + (\gamma-1)v^{\gamma-2}|Dv|^4\\
&\quad + v^{\gamma-1}|Dv|^2\Delta v\big]\\
&= \frac{3}{2}v^{\gamma-1}|Dv|^2\Delta v + \frac{\gamma-1}{2}v^{\gamma-2}|Dv|^4 - \frac{1}{2}\mathrm{div}(v^{\gamma-1}|Dv|^2Dv)\,,
\end{aligned}$$

which gives (14).

We conclude this section by recalling some well-known properties of the capacitary potential (see [9]) which will be useful for the proof of Theorem 1:

$$\begin{aligned}
u &= \frac{\mathrm{Cap}(\Omega)}{(n-2)\omega_n}|x|^{2-n} + o(|x|^{2-n})\,,\\
u_i &= -\frac{\mathrm{Cap}(\Omega)}{\omega_n}|x|^{-n}x_i + o(|x|^{1-n})\,,\\
u_{ij} &= \frac{\mathrm{Cap}(\Omega)}{\omega_n}|x|^{-n}\left(n\frac{x_ix_j}{|x|^2} - \delta_{ij}\right) + o(|x|^{-n})\,,
\end{aligned} \tag{15}$$

as $|x| \to +\infty$.

3 Proof of Theorem 1

Step 1. We prove that the reverse inequality holds in (5) and (6). More precisely, we shall prove that if u is a solution of (3), then it satisfies

$$\int_{\partial\Omega} |Du|^2\left(H_{\partial\Omega} - \frac{1}{n-2}\frac{|Du|}{u}\right)d\mathscr{H}^{N-1} \geq 0\,, \tag{16}$$

and

$$\int_{\partial\Omega} |Du|^2 \left((n-1)H_{\partial\Omega} - \frac{n}{2(n-2)} \frac{|Du|}{u} \right) d\mathscr{H}^{N-1} \geq \frac{(n-2)^3}{2} \omega_n \left(\frac{\mathrm{Cap}(\Omega)}{(n-2)\omega_n} \right)^{\frac{n-4}{n-2}} . \tag{17}$$

The proof of (16) and (17) is based on Lemma 2 and the Newton Inequality (11) applied to the Hessian matrix of the function $v = u^{-\frac{2}{n-2}}$. Notice that v solves

$$\begin{cases} \Delta v = \frac{n}{2} \frac{|Dv|^2}{v} & \text{in } \mathbb{R}^n \setminus \overline{\Omega}, \\ v = 1 & \text{on } \partial\Omega, \\ v \to \infty & \text{as } |x| \to +\infty. \end{cases} \tag{18}$$

Moreover, it follows from (15) that v satisfies

$$\begin{aligned} v &= \left(\frac{\mathrm{Cap}(\Omega)}{(n-2)\omega_n} \right)^{-\frac{2}{n-2}} |x|^2 + o(|x|^2) , \\ v_i &= 2 \left(\frac{\mathrm{Cap}(\Omega)}{(n-2)\omega_n} \right)^{-\frac{2}{n-2}} x_i + o(|x|) , \\ v_{ij} &= 2 \left(\frac{\mathrm{Cap}(\Omega)}{(n-2)\omega_n} \right)^{-\frac{2}{n-2}} \delta_{ij} + o(1) , \end{aligned} \tag{19}$$

as $|x| \to +\infty$. We notice that, since $\partial\Omega$ is of class $C^{3,\alpha}$, by Schauder's theory we have that $u, v \in C^{3,\alpha}(\mathbb{R}^n \setminus \Omega)$ (see [8]).

We are ready to give the proof of (16) and (17). Let γ be a fixed parameter to be chosen later and apply (12) to the solution v of (18). From (11) we have that

$$\begin{aligned} & v^\gamma \frac{n-1}{n} (\Delta v)^2 \geq \\ & \geq \mathrm{div}(v^\gamma S^2_{ij}(D^2 v) v_i) + \frac{\gamma}{2} \mathrm{div}(v^{\gamma-1} |Dv|^2 Dv) - \frac{3}{2} \gamma v^{\gamma-1} |Dv|^2 \Delta v - \frac{\gamma}{2} (\gamma - 1) v^{\gamma-2} |Dv|^4 . \end{aligned}$$

Since v satisfies (18), we obtain that

$$\mathrm{div}(v^\gamma S^2_{ij}(D^2 v) v_i) + \frac{\gamma}{2} \mathrm{div}(v^{\gamma-1} |Dv|^2 Dv) \leq |Dv|^4 v^{\gamma-2} \left(\frac{n}{4}(n-1) - \frac{\gamma}{2}(1-\gamma) + \frac{3}{2}\gamma \frac{n}{2} \right) . \tag{20}$$

Now, we make our choiche of γ so that the right hand side of the above inequality vanishes. This is achieved for $\gamma_1 = 1 - n$ and $\gamma_2 = -n/2$. Hence, by choosing $\gamma = \gamma_i$, $i = 1, 2$, we obtain that v satisfies the following inequality in $\mathbb{R}^n \setminus \Omega$:

$$\mathrm{div}(v^\gamma S^2_{ij}(D^2 v) v_i) + \frac{\gamma}{2} \mathrm{div}(v^{\gamma-1} |Dv|^2 Dv) \leq 0.$$

Let $R > 0$ be such that $\overline{\Omega} \subset B_R$. Since $v \in C^{3,\alpha}(\mathbb{R}^n \setminus \Omega)$, we can integrate the last inequality over $B_R \setminus \Omega$ and apply the divergence theorem. Notice that $Dv \neq 0$ on $\partial\Omega$ and $\nu = Dv/|Dv|$ on $\partial\Omega$, so from (10) we have that

$$\int_{\partial\Omega} \left(v^\gamma (n-1) H_{\partial\Omega} |Dv|^2 + \frac{\gamma}{2} v^{\gamma-1} |Dv|^3 \right) d\mathscr{H}^{N-1} \geq$$
$$\geq \int_{\partial B_R} \left(v^\gamma S^2_{ij}(D^2 v) v_i \nu^j_{B_R} + \frac{\gamma}{2} v^{\gamma-1} |Dv|^2 v_j \nu^j_{B_R} \right) d\mathscr{H}^{N-1}, \tag{21}$$

where ν_{B_R} denotes the outer unit normal vector to B_R. Now we notice that if $\gamma = \gamma_1$, then (19) implies that

$$\lim_{R\to\infty} \int_{\partial B_R} v^{\gamma_1} S^2_{ij}(D^2 v) v_i \nu^j_{B_R} + \frac{\gamma_1}{2} v^{\gamma_1 - 1} |Dv|^2 v_i \nu^i_{B_R} = 0, \tag{22}$$

while if $\gamma = \gamma_2$ then (19) yields

$$\lim_{R\to\infty} \int_{\partial B_R} v^{\gamma_2} S^2_{ij}(D^2 v) v_i \nu^j_{B_R} + \frac{\gamma_2}{2} v^{\gamma_2 - 1} |Dv|^2 v_i \nu^i_{B_R} = 2(n-2)\omega_n \left(\frac{\operatorname{Cap}(\Omega)}{(n-2)\omega_n} \right)^{\frac{n-4}{n-2}}, \tag{23}$$

since ∂B_R is asymptotically a level set of v. Indeed the superlevel sets of v are convex sets which recover the whole $\mathbb{R}^N$.

By using the fact that $v = 1$ on Ω and coupling (21) and (22), we obtain

$$\int_{\partial\Omega} |Dv|^2 \left(H_{\partial\Omega} - \frac{1}{2} \frac{|Dv|}{v} \right) \geq 0,$$

while from (21) and (23) we find

$$\int_{\partial\Omega} |Dv|^2 \left((n-1) H_{\partial\Omega} - \frac{n}{4} \frac{|Dv|}{v} \right) \geq 2(n-2)\omega_n \left(\frac{\operatorname{Cap}(\Omega)}{(n-2)\omega_n} \right)^{\frac{n-4}{n-2}}.$$

By recalling that $v = u^{-\frac{2}{n-2}}$, from the last two inequalities we immediately obtain (16) and (17).

Step 2. From Step 1 we have that the equality sign holds in (5) and (6). This means that the equality sign holds in Newton inequality, which implies that for every $x \in \mathbb{R}^n \setminus \Omega$ there exists a constant $\lambda(x)$ such that

$$D^2 v(x) = \lambda(x) Id.$$

It is easy to see that λ must be constant. Indeed, let $i \in \{1, \ldots, n\}$ be fixed and chose any $j \neq i$; we have that

$$\partial_{x_i} \lambda(x) = \partial_{x_i} u_{x_j x_j} = \partial_{x_j} u_{x_j x_i} = 0,$$

which implies that λ is constant. Hence,

$$D^2 v = c Id \,. \tag{24}$$

From (18) we find that $|Dv|$ is constant on every level surface of v. In particular, $|Dv|$ is constant on $\partial\Omega$ and hence from (9) and (24) we find that $H_{\partial\Omega}$ is constant and by using Alexandrov Theorem we conclude that Ω is a ball. The proof is complete.

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A Note on Some Poincaré Inequalities on Convex Sets by Optimal Transport Methods

Lorenzo Brasco and Filippo Santambrogio

Abstract We show that a class of Poincaré-Wirtinger inequalities on bounded convex sets can be obtained by means of the dynamical formulation of Optimal Transport. This is a consequence of a more general result valid for convex sets, possibly unbounded.

Keywords Poincaré inequalities · Wasserstein distances

AMS Subject Classification 39B62 · 46E35

1 Introduction

1.1 Overview

Let $1 < p < \infty$ and $0 < r < \infty$. For an open set $\Omega \subset \mathbb{R}^N$, we introduce the Sobolev spaces

$$\mathrm{W}^{1,p}_r(\Omega) := \left\{\phi \in L^r(\Omega) \, : \, \nabla\phi \in L^p(\Omega; \mathbb{R}^N)\right\},$$

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and

$$\ddot{\mathrm{W}}^{1,p}_r(\Omega) := \left\{\phi \in \mathrm{W}^{1,p}_r(\Omega) \ : \ \int_\Omega |\phi|^{r-1}\,\phi\,dx = 0\right\}.$$

In the particular case $r = p$, we will omit to indicate it and simply write $\mathrm{W}^{1,p}(\Omega)$ and $\ddot{\mathrm{W}}^{1,p}(\Omega)$.

The aim of this note is to prove some functional inequalities for the space $\ddot{\mathrm{W}}^{1,p}_r(\Omega)$, by means of Optimal Transport techniques. The use of Optimal Transport to prove functional and geometric inequalities is nowadays classical. We are not concerned here with geometric inequalities, thus we only refer to Sects. 2.5.3 and 7.4.2 of [22] for a brief discussion on the subject (in particular on the isoperimetric and the Brunn-Minkowski inequalities). As for functional inequalities obtained via Optimal Transport techniques, which is the main concern of this paper, after the fundamental paper [7] the literature on the subject is now quite rich. In addition to [7], we encourage the reader to look in details into the papers [3, 6, 13, 14, 18], for example.

It is useful to observe that most of these papers use the geometric properties of the optimal transport map as a tool to obtain a clever change-of-variable. This is indeed the case for the transport-based proof of the isoperimetric, Sobolev and Gagliardo-Nirenberg inequalities. We could say that they are based on the "statical" version of Optimal Transport problems.

On the contrary, the proof that we propose here is based on the "dynamical" counterpart of Optimal Transport (the so-called *Benamou-Brenier formula*, see [5]) and on *displacement convexity* considerations, see [17]. In this respect, it can be more suitably compared to the transport-based proof of the Brunn-Minkowski inequality.

It is also useful to remark that while the above cited papers deal with functional inequalities which are invariant for the transformation $\phi \mapsto |\phi|$, such as Sobolev and Gagliardo-Nirenberg ones, this is not the case here. Indeed, if a function ϕ belongs to our space $\ddot{\mathrm{W}}^{1,p}_r(\Omega)$, then $|\phi| \notin \ddot{\mathrm{W}}^{1,p}_r(\Omega)$. Thus, in order to prove our main result (see Theorem 1 below), we can not reduce to the case of positive functions and then use an optimal transport to transform any positive function ϕ into an extremal of the relevant functional inequality, as in [7]. Roughly speaking, what we do is to perform an optimal transport between the positive and negative parts ϕ_+ and ϕ_- (suitably renormalized).

Our proof has some points in common with the one presented by Rajala in [21], which is valid in general metric measure spaces under *Ricci curvature conditions*. Indeed, it is well-known that Ricci curvature conditions are linked to the displacement convexity of suitable functionals (see for instance the work [12] by Lott and Villani, to which [21] is inspired). However, even if the result of [21, Theorem 1.1] holds in a much more general setting, we stress that the tools used in [21] are not the same as ours. Moreover, the result of [21] only concerns with Poincaré inequalities on balls in the case $q = 1$ (with our notation below).

1.2 Main Result

In order to neatly present the main result, we first need to recall some basic definitions and notations.

We indicate by $\mathscr{P}(\Omega)$ the set of all Borel probability measures over Ω. Then for $1 \le m < \infty$, we define

$$\mathscr{P}_m(\Omega) = \left\{ \mu \in \mathscr{P}(\Omega) \, : \, \int_\Omega |x|^m \, d\mu < \infty \right\}, \tag{1}$$

i.e. the set of probability measure over Ω with finite moment of order m. For every $\mu, \nu \in \mathscr{P}_m(\Omega)$ their $m-$*Wasserstein distance* is defined through the optimal transport problem

$$W_m(\mu, \nu) = \left(\min_{\gamma \in \Pi(\mu,\nu)} \int_{\Omega \times \Omega} |x - y|^m \, d\gamma \right)^{\frac{1}{m}}.$$

Here $\Pi(\mu, \nu) \subset \mathscr{P}(\Omega \times \Omega)$ is the set of *transport plans*, i.e. the probability measures on the product space $\Omega \times \Omega$ such that

$$\gamma(A \times \Omega) = \mu(A) \qquad \gamma(\Omega \times B) = \nu(B), \qquad \text{for every } A, B \subset \Omega \text{ Borel sets.}$$

In what follows, we will note by $\mathscr{L}^N$ the N-dimensional Lebesgue measure. For a function $f \in L^1$, the writing

$$\mu = f \cdot \mathscr{L}^N,$$

will indicate the Radon measure which is absolutely continuous with respect to $\mathscr{L}^N$ and whose Radon-Nikodym derivative is given by f.

In this note we prove the following scaling invariant inequality, which is valid for general convex sets.

Theorem 1 *Let $1 < p < \infty$ and $1 < q < p$. Let $\Omega \subset \mathbb{R}^N$ be an open convex set. For every $\phi \in \ddot{\mathrm{W}}^{1,p}_{q-1}(\Omega)$ such that*

$$\int_\Omega |x|^{\frac{p}{p-q}} \, |\phi|^{q-1} \, dx < \infty, \tag{2}$$

we define the two probability measures $\rho_0, \rho_1 \in \mathscr{P}_{p/(p-q)}(\Omega)$

$$\rho_0 = \frac{|\phi|^{q-2}\,\phi_+}{\displaystyle\int_\Omega |\phi|^{q-2}\,\phi_+ \, dx} \cdot \mathscr{L}^N \qquad \text{and} \qquad \rho_1 = \frac{|\phi|^{q-2}\,\phi_-}{\displaystyle\int_\Omega |\phi|^{q-2}\,\phi_- \, dx} \cdot \mathscr{L}^N.$$

Then there holds

$$\left(\int_{\Omega} |\phi|^q \, dx\right)^{p-q+1} \leq \frac{\left(W_{\frac{p}{p-q}}(\rho_0, \rho_1)\right)^p}{2^{p-1}} \int_{\Omega} |\nabla \phi|^p \, dx \left(\int_{\Omega} |\phi|^{q-1} \, dx\right)^{p-q}. \quad (3)$$

The proof of this result is postponed to Sect. 3. We point out that inequality (3) in turn implies a handful of Poincaré-type inequalities with explicit constants. The reader is invited to jump directly to Sect. 4 in order to discover them. In particular, as a corollary we can obtain a lower bound for the first non-trivial Neumann eigenvalue of the p-Laplacian, see Corollary 4. This can be seen as a weak version of the *Payne-Weinberger inequality* (see [4, 9, 19]): though the explicit constant we get is not optimal, we believe the method of proof to be of independent interest.

Remark 1 We point out that the hypothesis $\phi \in L^q(\Omega)$ *is not needed* in Theorem 1. Rather, inequality (3) permits to show that on a convex set, functions in $\ddot{\mathrm{W}}^{1,p}_{q-1}(\Omega)$ verifying (2) are automatically in $L^q(\Omega)$.

2 Preliminaries

2.1 An Embedding Result

We will need some basic inequalities for Sobolev spaces in bounded sets. The proofs are standard, but we give them for the reader's convenience. The values of the constants appearing in the inequalities below will have no bearing in what follows.

Lemma 1 *Let $1 < p < \infty$ and let $\Omega \subset \mathbb{R}^N$ be an open connected and bounded set, with Lipschitz boundary. Then for every $\phi \in \mathrm{W}^{1,p}(\Omega)$ such that the set*

$$A_\phi := \{x \in \Omega \,:\, |\phi(x)| = 0\},$$

has positive measure, we have

$$\int_{\Omega} |\phi|^p \, dx \leq C \, \frac{|\Omega|}{|A_\phi|} \int_{\Omega} |\nabla \phi|^p \, dx, \quad (4)$$

for some $C = C(N, p, \Omega) > 0$.

Proof The proof is an adaptation of that of [10, Theorem 3.16]. We first observe that if we indicate by $\overline{\phi}_\Omega$ the mean of ϕ over Ω, then

$$|A_\phi| \, |\overline{\phi}_\Omega|^p = \int_{A_\phi} |\overline{\phi}_\Omega|^p \, dx = \int_{A_\phi} |\phi - \overline{\phi}_\Omega|^p \, dx \leq \int_{\Omega} |\phi - \overline{\phi}_\Omega|^p \, dx.$$

By using this information, with elementary manipulations we then get

$$\int_\Omega |\phi|^p\,dx \le 2^{p-1}\int_\Omega |\phi - \overline{\phi}_\Omega|^p\,dx + 2^{p-1}\,\frac{|\Omega|}{|A_\phi|}\int_\Omega |\phi - \overline{\phi}_\Omega|^p\,dx.$$

We can conclude by applying Poincaré inequality for functions with vanishing mean, see for example [10, Theorem 3.14]. □

The next interpolation inequality for the Sobolev space $W_r^{1,p}(\Omega)$ will be useful.

Lemma 2 *Let $1 < p < \infty$ and $0 < r < p$. Let $\Omega \subset \mathbb{R}^N$ be an open connected and bounded set, with Lipschitz boundary. Then $W_r^{1,p}(\Omega) \subset L^p(\Omega)$. More precisely, for every $\phi \in W_r^{1,p}(\Omega)$ we have*

$$\int_\Omega |\phi|^p\,dx \le C\int_\Omega |\nabla\phi|^p\,dx + C\left(\int_\Omega |\phi|^r\,dx\right)^{\frac{p}{r}},$$

for some $C = C(N, p, \Omega) > 0$.

Proof Given $\phi \in W_r^{1,p}(\Omega)$, for every $t > 0$ and $M > 0$ we define

$$\phi_t(x) = (|\phi(x)| - t)_+ \qquad \text{and} \qquad \phi_{t,M}(x) = \min\{\phi_t(x),\ M\}.$$

The function $\phi_{t,M}$ belongs to $W^{1,p}(\Omega)$, if we set $A_{t,M} = \{x \in \Omega\ :\ \phi_{t,M}(x) = 0\}$ then by Chebyshev's inequality

$$|\Omega \setminus A_{t,M}| := \big|\{x \in \Omega\ :\ \phi_{t,M}(x) \ne 0\}\big| \le \frac{1}{t^r}\int_\Omega |\phi|^r\,dx. \tag{5}$$

From (4) we get

$$\int_\Omega |\phi_{t,M}|^p\,dx \le C\,\frac{|\Omega|}{|A_{t,M}|}\int_\Omega |\nabla\phi_{t,M}|^p\,dx,$$

and observe that from (5)

$$\frac{|\Omega|}{|A_{t,M}|} = \frac{|\Omega|}{|\Omega| - |\Omega \setminus A_{t,M}|} \le 2, \qquad \text{if we choose } t = \left(\frac{2}{|\Omega|}\right)^{1/r}\|\phi\|_{L^r(\Omega)}.$$

We thus obtain

$$\int_\Omega |\phi_{t,M}|^p\,dx \le 2\,C\int_\Omega |\nabla\phi|^p\,dx.$$

It is now possible to take the limit as M goes to ∞, thus getting by Fatou's Lemma

$$\int_\Omega |\phi_t|^p\,dx \le 2\,C\int_\Omega |\nabla\phi|^p\,dx.$$

By recalling the choice of t and observing that $|\phi| \le t + \phi_t$, we get the desired conclusion. □

2.2 Some Tools from Optimal Transport

We recall a couple of standard results in Optimal Transport, that will be needed for the proof of the main result. For more details, the reader is invited to refer to classical monographs such as [2] or [23], or to the more recent one [22].

Definition 1 The *m-Wasserstein space over Ω* is the set $\mathscr{P}_m(\Omega)$ defined in (1), equipped with the Wasserstein distance W_m. This metric space will be denoted by $\mathbb{W}_m(\Omega)$.

The first important tool we need is a characterization of geodesics in the Wasserstein space. This is essentially a refined version of the celebrated Benamou-Brenier formula, firstly introduced in [5]. The proof can be found in [22, Theorem 5.14 and Proposition 5.30].

Proposition 1 (Wasserstein geodesics) *Let $1 < m < \infty$ and let $\Omega \subset \mathbb{R}^N$ be an open bounded convex set. Let $\rho_0, \rho_1 \in \mathbb{W}_m(\Omega)$, then there exist an absolutely continuous curve $(\mu_t)_{t\in[0,1]}$ in the Wasserstein space $\mathbb{W}_m(\Omega)$ and a vector field $\mathbf{v}_t \in L^m(\Omega;\mu_t)$ such that*

- $\mu_0 = \rho_0$ *and* $\mu_1 = \rho_1$*;*
- *the continuity equation*

$$\begin{cases} \partial_t \mu_t + \operatorname{div}(\mathbf{v}_t\, \mu_t) = 0, & \textit{in } \Omega, \\ \langle \mathbf{v}_t, \nu_\Omega\rangle = 0, & \textit{on } \partial\Omega \end{cases}$$

holds in distributional sense, i.e. for every $\phi \in C^1([0,1]\times\overline{\Omega})$ there holds

$$\int_0^1 \int_\Omega \partial_t \phi \, d\mu_t\, dt + \int_0^1 \int_\Omega \langle \nabla\phi, \mathbf{v}_t\rangle \, d\mu_t\, dt = \int_\Omega \phi(1,\cdot)\, d\rho_1 - \int_\Omega \phi(0,\cdot)\, d\rho_0;$$

- *we have*

$$\left(\int_0^1 \|\mathbf{v}_t\|^m_{L^m(\Omega;\mu_t)}\, dt\right)^{\frac{1}{m}} = W_m(\rho_0,\rho_1).$$

The other expedient result from Optimal Transport we need is the following convexity property of L^q norms. For $m = 2$, this is a particular instance of a result by McCann, see [17]. The proof can be found, for example, in [22, Theorem 7.28].

Proposition 2 (Geodesic convexity of L^q norms) *Let $1 < m < \infty$ and let $\Omega \subset \mathbb{R}^N$ be an open bounded convex set. Let $\rho_0 = f_0 \cdot \mathscr{L}^N$ and $\rho_1 = f_1 \cdot \mathscr{L}^N$ be two* *probability measures on Ω, such that $f_0, f_1 \in L^q(\Omega)$ for some $1 \leq q \leq \infty$. If $(\mu_t)_{t\in[0,1]} \subset \mathbb{W}_m(\Omega)$ is the curve of Proposition 1, then we have*

$$\mu_t = f_t \cdot \mathscr{L}^N \quad \textit{and} \quad \|f_t\|_{L^q(\Omega)} \leq \Big((1-t)\,\|f_0\|^q_{L^q(\Omega)} + t\,\|f_1\|^q_{L^q(\Omega)}\Big)^{\frac{1}{q}}, \quad t \in [0,1].$$

3 Proof of the Main Result

3.1 An Expedient Estimate

We first need the following preliminary result. The idea of the proof is similar to that of [11, Proposition 2.6] and [15, Lemma 3.5], though the final outcome is different. We also cite the short unpublished note [20] containing interesting uniform estimates on these topics.

Lemma 3 *Let $1 < q < p < \infty$ and let $\Omega \subset \mathbb{R}^N$ be an open and bounded convex set. For every $\phi \in \mathrm{W}^{1,p}(\Omega)$ and every $f_0, f_1 \in L^{q'}(\Omega)$ such that*

$$\int_\Omega f_0\,dx = \int_\Omega f_1\,dx = 1, \qquad f_0, f_1 \geq 0,$$

we have

$$\int_\Omega \phi\,(f_1 - f_0)\,dx \leq W_{\frac{p}{p-q}}(\rho_0, \rho_1)\,\|\nabla\phi\|_{L^p(\Omega)} \left(\frac{\|f_0\|^{q'}_{L^{q'}(\Omega)} + \|f_1\|^{q'}_{L^{q'}(\Omega)}}{2}\right)^{\frac{q-1}{p}}, \tag{6}$$

where

$$\rho_i = f_i \cdot \mathscr{L}^N, \qquad i = 0, 1,$$

Proof Let us first suppose that $\phi \in C^1(\overline{\Omega})$. In this case we clearly have $C^1(\overline{\Omega}) \subset \mathrm{W}^{1,p}(\Omega)$.

For notational simplicity we set $r := p/(p-q)$. Then, by using Propositions 1 and 2 with $\rho_0 = f_0 \cdot \mathscr{L}^N$ and $\rho_1 = f_1 \cdot \mathscr{L}^N$ and observing that ϕ does not depend on t, with the previous notation we can infer

$$\begin{aligned}
\int_\Omega \phi\,(f_1 - f_0)\,dx &= \int_0^1 \int_\Omega \langle \nabla\phi, \mathbf{v}_t\rangle\, f_t\,dx\,dt \\
&\leq \left(\int_0^1 \int_\Omega |\nabla\phi|^{\frac{p}{q}}\, f_t\,dx\,dt\right)^{\frac{q}{p}} \left(\int_0^1 \int_\Omega |\mathbf{v}_t|^r\, f_t\,dx\,dt\right)^{\frac{1}{r}} \\
&\leq \left(\int_0^1 \int_\Omega |\nabla\phi|^p\,dx\,dt\right)^{\frac{1}{p}} \left(\int_0^1 \|f_t\|^{q'}_{L^{q'}(\Omega)}\,dt\right)^{\frac{q-1}{p}} W_r(\rho_0, \rho_1),
\end{aligned}$$

Observe that the last term is finite, since $f_t \in L^{q'}(\Omega)$ and its $L^{q'}$ norm is integrable in time, thanks to Proposition 2.

Since ϕ does not depend on t, from the previous estimate we get in particular

$$\int_\Omega \phi\,(f_1 - f_0)\,dx \le W_r(\rho_0, \rho_1)\,\|\nabla\phi\|_{L^p(\Omega)} \left(\int_0^1 \|f_t\|^{q'}_{L^{q'}(\Omega)}\,dt\right)^{\frac{q-1}{p}}.$$

We now observe that by Proposition 2

$$\int_0^1 \|f_t\|^{q'}_{L^{q'}(\Omega)}\,dt \le \int_0^1 \left[\|f_0\|^{q'}_{L^{q'}(\Omega)} + t\left(\|f_1\|^{q'}_{L^{q'}(\Omega)} - \|f_0\|^{q'}_{L^{q'}(\Omega)}\right)\right] dt$$
$$= \frac{\|f_0\|^{q'}_{L^{q'}(\Omega)} + \|f_1\|^{q'}_{L^{q'}(\Omega)}}{2}.$$

thus we obtain the desired estimate (6), for $\phi \in C^1(\overline{\Omega})$.

Finally, we get the general case by using that for a convex set $C^1(\overline{\Omega})$ is dense in $\mathrm{W}^{1,p}(\Omega)$, see [16, Theorem 1, Sect. 1.1.6]. □

3.2 *Proof of Theorem 1*

We divide the proof in two steps: we first prove the inequality for bounded convex sets and then consider the general case. For notational simplicity, we set again $r := p/(p-q)$.

Bounded convex sets. Let $\phi \in \ddot{\mathrm{W}}^{1,p}_{q-1}(\Omega) \setminus \{0\}$, the hypothesis $\int_\Omega |\phi|^{q-2}\,\phi = 0$ implies

$$\int_\Omega |\phi|^{q-1}\,dx = 2\int_\Omega |\phi|^{q-2}\,\phi_+\,dx = 2\int_\Omega |\phi|^{q-2}\,\phi_-\,dx. \tag{7}$$

By Lemma 2, we have $\phi \in \mathrm{W}^{1,p}(\Omega)$ as well, thus we can now apply (6) with the choices

$$\rho_1 = f_1 \cdot \mathscr{L}^N := \frac{|\phi|^{q-2}\,\phi_+}{\displaystyle\int_\Omega |\phi|^{q-2}\,\phi_+\,dx} \cdot \mathscr{L}^N \text{ and } \rho_0 = f_0 \cdot \mathscr{L}^N = \frac{|\phi|^{q-2}\,\phi_-}{\displaystyle\int_\Omega |\phi|^{q-2}\,\phi_-\,dx} \cdot \mathscr{L}^N.$$

For the left-hand side of (6), by using (7) we get

$$\int_\Omega \phi\,(f_1 - f_0)\,dx = 2\,\frac{\displaystyle\int_\Omega |\phi|^q\,dx}{\displaystyle\int_\Omega |\phi|^{q-1}\,dx}.$$

For the right-hand side of (6), we observe that again by (7) and using that

$$|\phi|^{q-2}\,\phi_+ = \phi_+^{q-1}, \qquad |\phi|^{q-2}\,\phi_- = \phi_-^{q-1},$$

we get

$$\begin{aligned}\|f_0\|^{q'}_{L^{q'}(\Omega)} + \|f_1\|^{q'}_{L^{q'}(\Omega)} &= \frac{\displaystyle\int_\Omega \left(|\phi|^{q-2}\,\phi_-\right)^{\frac{q}{q-1}}\,dx}{\left(\displaystyle\int_\Omega |\phi|^{q-2}\,\phi_-\,dx\right)^{\frac{q}{q-1}}} + \frac{\displaystyle\int_\Omega \left(|\phi|^{q-2}\,\phi_+\right)^{\frac{q}{q-1}}\,dx}{\left(\displaystyle\int_\Omega |\phi|^{q-2}\,\phi_+\,dx\right)^{\frac{q}{q-1}}}\\ &= 2^{\frac{q}{q-1}}\,\frac{\displaystyle\int_\Omega |\phi|^q\,dx}{\left(\displaystyle\int_\Omega |\phi|^{q-1}\,dx\right)^{\frac{q}{q-1}}}.\end{aligned}$$

Then from (6) we finally obtain

$$\frac{\displaystyle\int_\Omega |\phi|^q\,dx}{\displaystyle\int_\Omega |\phi|^{q-1}\,dx} \le \frac{W_r(\rho_0,\rho_1)}{2^{\frac{p-1}{p}}}\left(\int_\Omega |\nabla\phi|^p\,dx\right)^{\frac{1}{p}} \frac{\left(\displaystyle\int_\Omega |\phi|^q\,dx\right)^{\frac{q-1}{p}}}{\left(\displaystyle\int_\Omega |\phi|^{q-1}\,dx\right)^{\frac{q}{p}}}.$$

After a simplification, this proves the desired inequality (3) when Ω is a bounded set.

General convex sets. Let us now assume that Ω is a generic open convex set and $\phi \in \ddot{\mathrm{W}}^{1,p}_{q-1}(\Omega)\setminus\{0\}$. We can suppose that the origin belongs to Ω, then for $k \in \mathbb{N}\setminus\{0\}$ we define

$$\Omega_k = \{x \in \Omega \,:\, |x| < k\} \qquad \text{and} \qquad \delta_k = \left(\frac{\displaystyle\int_{\Omega_k} |\phi_+|^{q-1}\,dx}{\displaystyle\int_{\Omega_k} |\phi_-|^{q-1}\,dx}\right)^{1/(q-1)}.$$

Note that, at least for k large, δ_k is well-defined, since

$$\lim_{k\to\infty}\int_{\Omega_k} |\phi_-|^{q-1}\,dx = \int_\Omega |\phi_-|^{q-1}\,dx,$$

and the last quantity is strictly positive, since $\phi \not\equiv 0$.

The function $\phi_k = \phi_+ - \delta_k\,\phi_-$ belongs to $\ddot{\mathrm{W}}^{1,p}_{q-1}(\Omega_k)$, by construction. Moreover, since $\phi \in \ddot{\mathrm{W}}^{1,p}_{q-1}(\Omega)$, we have

$$\lim_{k\to\infty} \delta_k = 1. \tag{8}$$

We also set

$$\rho_{1,k} := \frac{|\phi_k|^{q-2}\,(\phi_k)_+}{\displaystyle\int_{\Omega_k} |\phi_k|^{q-2}\,(\phi_k)_+\,dx} \cdot \mathscr{L}^N = \frac{|\phi|^{q-2}\,\phi_+}{\displaystyle\int_{\Omega_k} |\phi|^{q-2}\,\phi_+\,dx} \cdot \mathscr{L}^N$$

and

$$\rho_{0,k} := \frac{|\phi_k|^{q-2}\,(\phi_k)_-}{\displaystyle\int_{\Omega_k} |\phi_k|^{q-2}\,(\phi_k)_-\,dx} \cdot \mathscr{L}^N = \frac{|\phi|^{q-2}\,\phi_-}{\displaystyle\int_{\Omega_k} |\phi|^{q-2}\,\phi_-\,dx} \cdot \mathscr{L}^N.$$

Since Ω_k is convex and bounded, from the previous step we obtain

$$\left(\int_{\Omega_k} |\phi_k|^q\,dx\right)^{p-q+1} \le \frac{\big(W_r(\rho_{0,k},\rho_{1,k})\big)^p}{2^{p-1}} \int_{\Omega_k} |\nabla\phi_k|^p\,dx \left(\int_{\Omega_k} |\phi_k|^{q-1}\,dx\right)^{p-q}. \tag{9}$$

We now observe that

$$\lim_{k\to\infty} W_r(\rho_{0,k},\rho_{1,k}) = W_r(\rho_0,\rho_1).$$

Indeed, it is enough to remark that we have $\rho_{i,k} \to \rho_i$ in $\mathbb{W}_r(\Omega)$ for $i = 0, 1$. This follows from the fact that the convergence in $\mathbb{W}_r$ is equivalent to the weak convergence plus the convergence of the moments of order r (see for instance [22, Theorem 5.11]). Both conditions are easily seen to hold true here.

Moreover, by construction we have

$$|\phi_k|^{q-1} \cdot 1_{\Omega_k} \le (\max\{1,\delta_k\})^{q-1}\,|\phi|^{q-1} \cdot 1_{\Omega},$$

and

$$|\nabla\phi_k|^p \cdot 1_{\Omega_k} \le (\max\{1,\delta_k\})^p\,|\nabla\phi|^p \cdot 1_{\Omega}.$$

If we use (8), we can pass to the limit as k goes to ∞ in (9), by using the Dominated Convergence Theorem on the right-hand side and Fatou's Lemma on the left-hand side. This finally gives (3) for a generic function $\phi \in \ddot{\mathrm{W}}^{1,p}_{q-1}(\Omega)$.

4 Some Consequences

In this section, we discuss some functional inequalities which are contained in nuce in Theorem 1.

4.1 General Convex Sets

We start with the following inequality, valid for general convex sets. We observe again that it is not necessary to assume $\phi \in L^q(\Omega)$.

Corollary 1 *Let* $1 < p < \infty$ *and* $1 < q < p$. *Let* $\Omega \subset \mathbb{R}^N$ *be an open convex set. For every* $\phi \in \ddot{\mathrm{W}}^{1,p}_{q-1}(\Omega)$ *such that*

$$\int_\Omega |x|^{\frac{p}{p-q}} |\phi|^{q-1} \, dx < \infty,$$

we have

$$\left(\int_\Omega |\phi|^q \, dx\right)^{p-q+1} \leq 2 \left(\inf_{x_0\in\Omega} \int_\Omega |x - x_0|^{\frac{p}{p-q}} |\phi|^{q-1} \, dx\right)^{p-q} \int_\Omega |\nabla \phi|^p \, dx. \tag{10}$$

Proof Let ϕ be a function as in the statement. We use the notations of Theorem 1 and take $\gamma_{opt} \in \Pi(\rho_0, \rho_1)$ an optimal transport plan for $W_r(\rho_0, \rho_1)$ (where, as usual, $r = p/(p - q)$). By using the triangle inequality and the definition of transport plan, we get

$$\begin{aligned} W_r(\rho_0, \rho_1) &\leq \left(\int_{\Omega\times\Omega} |x - x_0|^r \, d\gamma_{opt}\right)^{1/r} + \left(\int_{\Omega\times\Omega} |y - x_0|^r \, d\gamma_{opt}\right)^{1/r} \\ &= \left(\int_\Omega |x - x_0|^r \, d\rho_0\right)^{1/r} + \left(\int_\Omega |y - x_0|^r \, d\rho_1\right)^{1/r}, \end{aligned}$$

for every $x_0 \in \Omega$. By using concavity of the map $\tau \mapsto \tau^{1/r}$, this in turn gives

$$\begin{aligned} W_r(\rho_0, \rho_1) &\leq 2^{\frac{q}{p}} \left(\int_\Omega |x - x_0|^r \, (d\rho_0 + d\rho_1)\right)^{1/r} \\ &= 2 \left(\int_\Omega |x - x_0|^r \, |\phi|^{q-1} \, dx\right)^{1/r} \left(\int_\Omega |\phi|^{q-1} \, dx\right)^{\frac{q-p}{p}}, \end{aligned}$$

where we used again (7), by assumption. By using this estimate in (3) and appealing to the arbitrariness of $x_0 \in \Omega$, we get the desired result. □

4.2 Bounded Convex Sets

In this case, Theorem 1 implies some known inequalities, with explicit constants depending on simple geometric quantities and p only.

Corollary 2 (Nash-type inequality) *Let $1 < p < \infty$ and $1 < q < p$. Let $\Omega \subset \mathbb{R}^N$ be an open and bounded convex set. Then for every $\phi \in \ddot{\mathrm{W}}^{1,p}_{q-1}(\Omega)$*

$$\left(\int_\Omega |\phi|^q\,dx\right)^{p-q+1} \le \frac{\operatorname{diam}(\Omega)^p}{2^{p-1}} \int_\Omega |\nabla\phi|^p\,dx \left(\int_\Omega |\phi|^{q-1}\,dx\right)^{p-q}. \tag{11}$$

Proof In order to prove (11), it is sufficient to observe that for a bounded set we have

$$W_r(\rho_0, \rho_1) \le \operatorname{diam}(\Omega).$$

If we spend this information in (3), we can then conclude. □

Corollary 3 (Poincaré-Wirtinger inequality) *Let $1 < p < \infty$ and $1 < q < p$. Let $\Omega \subset \mathbb{R}^N$ be an open and bounded convex set. Then for every $\phi \in \mathrm{W}^{1,p}_{q-1}(\Omega)$, there holds*

$$\min_{t\in\mathbb{R}} \left(\int_\Omega |\phi - t|^q\,dx\right)^{\frac{p}{q}} \le \frac{\operatorname{diam}(\Omega)^p}{2^{p-1}} |\Omega|^{\frac{p}{q}-1} \int_\Omega |\nabla\phi|^p\,dx\,. \tag{12}$$

Proof Let $\phi \in \mathrm{W}^{1,p}_{q-1}(\Omega)$, by Lemma 2 we know in particular that $\phi \in L^q(\Omega)$. Then we can define the unique minimizer t_q of

$$t \mapsto \left(\int_\Omega |\phi - t|^q\,dx\right)^{\frac{p}{q}}.$$

By minimality, we have

$$\int_\Omega |\phi - t_q|^{q-2}\,(\phi - t_q)\,dx = 0.$$

Thus the function $\phi - t_q$ belongs to $\ddot{\mathrm{W}}^{1,p}_{q-1}(\Omega)$. We just need to observe that since $\phi - t_q \in L^q(\Omega)$, then

$$\left(\int_\Omega |\phi - t_q|^{q-1}\,dx\right)^{p-q} \le |\Omega|^{\frac{p-q}{q}} \left(\int_\Omega |\phi - t_q|^q\,dx\right)^{\frac{p-q}{q}\,(q-1)}.$$

By using this in (11) for the function $\phi - t_q$, we get the conclusion. □

Remark 2 Observe that the constant in (12) degenerates to 0 as the measure $|\Omega|$ gets smaller and smaller. This behaviour is optimal, as one may easily verify. Indeed, by taking $n \in \mathbb{N} \setminus \{0\}$ and

$$\Omega_n = \left[-\frac{1}{2}, \frac{1}{2}\right] \times \left[0, \frac{1}{n}\right] \times \cdots \times \left[0, \frac{1}{n}\right] \qquad \text{and} \qquad \phi(x) = x_1, \tag{13}$$

we have

$$\frac{\min_{t\in\mathbb{R}} \left(\int_{\Omega_n} |\phi - t|^q \, dx\right)^{\frac{p}{q}}}{\int_{\Omega_n} |\nabla\phi|^p \, dx} = \frac{\left(\int_{\Omega_n} |\phi|^q \, dx\right)^{\frac{p}{q}}}{\int_{\Omega_n} |\nabla\phi|^p \, dx} \simeq \left(\frac{1}{n}\right)^{(N-1)\,\frac{p-q}{q}} = |\Omega_n|^{\frac{p-q}{q}}.$$

We conclude the paper with an application to spectral problems. Let $1 < p < \infty$, for every $\Omega \subset \mathbb{R}^N$ open and bounded set we introduce its *first non-trivial Neumann eigenvalue of the p-Laplacian*, i.e.

$$\mu(\Omega; p) := \inf_{\phi\in W^{1,p}(\Omega)\setminus\{0\}} \left\{ \frac{\int_\Omega |\nabla\phi|^p \, dx}{\int_\Omega |\phi|^p \, dx} \; : \; \int_\Omega |\phi|^{p-2}\,\phi \, dx = 0 \right\}.$$

The terminology is justified by the fact that for a connected set with Lipschitz boundary, the constant $\mu(\Omega; p)$ is attained and coincides with the smallest number different from 0 such that the Neumann boundary value problem

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\,\nabla u) = \mu\,|u|^{p-2}\,u, & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu_\Omega} = 0, & \text{on } \partial\Omega \end{cases}$$

admits non-trivial weak solutions. We then have the following result, which corresponds to the limit case $q = p$ of Theorem 1.

Corollary 4 (Payne-Weinberger type estimate) *Let $1 < p < \infty$ and let $\Omega \subset \mathbb{R}^N$ be an open and bounded convex set. We have the lower bound*

$$\left(\frac{2^{\frac{p-1}{p}}}{\mathrm{diam}(\Omega)}\right)^p \le \mu(\Omega; p). \tag{14}$$

Proof We take $\phi \in W^{1,p}(\Omega) \setminus \{0\}$ such that $\int_\Omega |\phi|^{p-2}\,\phi\,dx = 0$. Then we have

$$\min_{t\in\mathbb{R}} \int_\Omega |\phi - t|^p \, dx = \int_\Omega |\phi|^p \, dx. \tag{15}$$

For $1 < q < p$, we take $t_q \in \mathbb{R}$ to be the unique minimizer of

$$t \mapsto \left(\int_\Omega |\phi - t|^q \, dx\right)^{\frac{p}{q}}.$$

By minimality of t_q and Minkowski inequality, we have

$$t_q\,|\Omega|^{\frac{1}{q}} - \left(\int_\Omega |\phi|^q\,dx\right)^{\frac{1}{q}} \le \left(\int_\Omega |\phi - t_q|^q\,dx\right)^{\frac{1}{q}} \le \left(\int_\Omega |\phi|^q\,dx\right)^{\frac{1}{q}}.$$

This shows that $\{t_q\}_{q<p}$ is bounded, thus if we take the limit as q goes to p, then t_q converges (up to a subsequence) to some $\bar{t}$. By passing to the limit in (12) we get

$$\int_\Omega |\phi - \bar{t}|^p\,dx \le \frac{\mathrm{diam}(\Omega)^p}{2^{p-1}} \int_\Omega |\nabla\phi|^p\,dx.$$

By keeping into account (15), we get the desired conclusion. □

Remark 3 As mentioned in the Introduction, the constant appearing in the left-hand side of (14) is not sharp. Indeed, the sharp lower bound is known to be

$$\left(\frac{\pi_p}{\mathrm{diam}(\Omega)}\right)^p < \mu(\Omega; p), \qquad \text{where } \pi_p = 2\pi\,\frac{(p-1)^{\frac{1}{p}}}{p\,\sin\left(\dfrac{\pi}{p}\right)}, \tag{16}$$

as proved by Payne and Weinberger in [19] for $p = 2$ (see also [4]). The general case $p \neq 2$ has been proved in [8, 9]. We recall that (16) is sharp in the following sense: for every convex set Ω the inequality in (16) is strict and it becomes asymptotically an equality along the sequence of sets $\{\Omega_n\}_{n\in\mathbb{N}}$ in (13).

In the limit case $p = 1$, a related result can be found in [1].

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Analyticity and Criticality Results for the Eigenvalues of the Biharmonic Operator

Davide Buoso

Abstract We consider the eigenvalues of the biharmonic operator subject to several homogeneous boundary conditions (Dirichlet, Neumann, Navier, Steklov). We show that simple eigenvalues and elementary symmetric functions of multiple eigenvalues are real analytic, and provide Hadamard-type formulas for the corresponding shape derivatives. After recalling the known results in shape optimization, we prove that balls are always critical domains under volume constraint.

Keywords Biharmonic operator · Boundary value problems · Steklov · Plates · Eigenvalues · Perturbations · Hadamard formulas · Isovolumetric perturbations · Shape criticality

1 Introduction

In this paper we consider eigenvalue problems for the biharmonic operator subject to several homogeneous boundary conditions in bounded domains Ω in $\mathbb{R}^N$, $N \geq 2$. Note that such problems arise in the study of vibrating plates within the so-called Kirchhoff-Love model (see e.g., [43]). In particular, we consider the following equation

$$\Delta^2 u - \tau \Delta u = \lambda u, \text{ in } \Omega, \tag{1}$$

where τ is a non-negative constant related to the lateral tension of the plate. As for the boundary conditions, we are interested in Dirichlet boundary conditions

$$u = \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega, \tag{2}$$

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which are related to clamped plates, Navier boundary conditions

$$u = (1-\sigma)\frac{\partial^2 u}{\partial \nu^2} + \sigma \Delta u = 0 \text{ on } \partial\Omega, \tag{3}$$

which are related to hinged plates, and Neumann boundary conditions

$$(1-\sigma)\frac{\partial^2 u}{\partial \nu^2} + \sigma\Delta u = \tau\frac{\partial u}{\partial \nu} - \frac{\partial \Delta u}{\partial \nu} - (1-\sigma)\mathrm{div}_{\partial\Omega}\left(\nu^t D^2 u\right)_{\partial\Omega} = 0 \text{ on } \partial\Omega, \tag{4}$$

which are related to free plates. Note that σ denotes the Poisson ratio of the material, typically $0 \le \sigma \le 0.5$. We recall that the conditions (4) have been known for long time only in dimension $N = 2$ (see e.g., [28, 47]), while the general case first appeared in [21] (see also [22]). We recall here that, given a vector function f, its tangential component is defined as $f_{\partial\Omega} = f - (f \cdot \nu)\nu$, and the tangential divergence operator is $\mathrm{div}_{\partial\Omega} f = \mathrm{div} f - \frac{\partial f}{\partial \nu} \cdot \nu$.

We also consider Steklov-type problems for the biharmonic operator. Note that the first one to appear was the following

$$\begin{cases} \Delta^2 u = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \\ \Delta u = \lambda \frac{\partial u}{\partial \nu}, & \text{on } \partial\Omega, \end{cases} \tag{5}$$

and it was introduced in [32]. Problem (5) has proved itself to be quite strange with respect to other Laplacian-related eigenvalue problem, at least concerning shape optimization results. In fact, differently from the classical Steklov problem where the interesting problem is the maximization of eigenvalues under volume constraint, here one searches for minimizers and, strikingly, the ball is not the optimal shape for the first eigenvalue (at least in dimension $N = 2$), as shown in [33]. Nevertheless, in [7] the authors can prove that, among all convex domain of fixed measure there exists a minimizer, but nothing is known about the optimal shape, or if the convexity assumption can be relaxed. We also refer to [2, 5, 6] for other results on problem (5).

Another Steklov problem for the biharmonic operator which has appeared very recently in [15] (see also [16]) is the following

$$\begin{cases} \Delta^2 u - \tau\Delta u = 0, & \text{in } \Omega, \\ \frac{\partial^2 u}{\partial \nu^2} = 0, & \text{on } \partial\Omega, \\ \tau\frac{\partial u}{\partial \nu} - \frac{\partial \Delta u}{\partial \nu} - \mathrm{div}_{\partial\Omega}\left(\nu^t D^2 u\right)_{\partial\Omega} = \lambda u, & \text{on } \partial\Omega. \end{cases} \tag{6}$$

In contrast with problem (5), problem (6) presents several spectral features resembling those of the Steklov-Laplacian. As shown in [15], problem (6) can be viewed as a limiting Neumann problem via mass concentration arguments (cf. [38]), and moreover, for any fixed $\tau > 0$, the maximizer of the first positive eigenvalue among all bounded smooth domains is the ball.

In this paper we are interested in analyticity properties of the eigenvalues of problems (1)–(6). This type of analysis was first done by Lamberti and Lanza de Cristoforis in [35], where they study regularity properties of the elementary symmetric functions of the eigenvalues of the Laplace operator subject to Dirichlet boundary conditions. Note that in general, when dealing with eigenvalues splitting from a multiple eigenvalue, bifurcation phenomena may occur, and the use of symmetric functions of the eigenvalues permits to bypass such situations. The techniques in [35] were later used to treat other types of boundary conditions (see [34, 37]) and even other operators (see [9, 13, 14]). As for the biharmonic operator, this kind of analysis has been already carried out in several specific cases, see [11, 12, 15, 16]. Our aim here is to treat those cases altogether in order to give a general overview.

After proving that the elementary symmetric functions of the eigenvalues are analytic upon domain perturbations, we compute their shape differential. Following the lines of [36], by means of the Lagrange Multiplier Theorem, we can show that the ball is a critical domain under volume constraint for any of the elementary symmetric functions of the eigenvalues of problems (1)–(6). We observe that, regarding problem (5), such a result was already obtained in [7] but only for the first eigenvalue. We remark that the question of criticality of domains is strictly related with shape optimization problems, where the minimizing (resp. maximizing) domain has to be found in a class of fixed volume ones. This type of problems for the eigenvalues of the biharmonic operator have been solved only in very specific cases, the optimal domain for the first eigenvalue being the ball (see [3, 15, 21, 22, 39]). As we have said above, for problem (5) the ball has been proved not to be the minimizer, nevertheless it still is a critical domain (cf. Theorem 6).

The paper is organized as follows. Section 2 is devoted to some preliminaries. In Sect. 3 we examine the problem of shape differentiability of the eigenvalues. We consider problem (10) in $\phi(\Omega)$ and pull it back to Ω, where ϕ belongs to a suitable class of diffeomorphisms. We also derive Hadamard-type formulas for the elementary symmetric functions of the eigenvalues. In Sect. 4 we consider the problem of finding critical points for such functions under volume constraint. We provide a characterization for the critical domains, and show that, for all the problems considered, balls are critical domains for all the elementary symmetric functions of the eigenvalues. Finally, in Sect. 5 we prove some technical results.

2 Preliminaries

Let $N \in \mathbb{N}$, $N \geq 2$, and let Ω be a bounded open set in $\mathbb{R}^N$ of class C^1. By $H^k(\Omega)$, $k \in \mathbb{N}$, we denote the Sobolev space of functions in $L^2(\Omega)$ with derivatives up to order k in $L^2(\Omega)$, and by $H_0^k(\Omega)$ we denote the closure in $H^k(\Omega)$ of the space of C^∞-functions with compact support in Ω.

Let also $\tau \geq 0$, $-\frac{1}{N-1} < \sigma < 1$. We consider the following bilinear form on $H^2(\Omega)$

$$P = (1-\sigma)M + \sigma B + \tau L, \tag{7}$$

where

$$M[u][v] = \int_\Omega D^2u : D^2v dx, \quad B[u][v] = \int_\Omega \Delta u \Delta v dx,$$

and

$$L[u][v] = \int_\Omega \nabla u \cdot \nabla v dx,$$

for any $u, v \in H^2(\Omega)$, where we denote by $D^2u : D^2v$ the Frobenius product $D^2u : D^2v = \sum_{i,j=1}^N \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial^2 v}{\partial x_i \partial x_j}$. We also consider the following bilinear forms on $H^2(\Omega)$

$$J_1[u][v] = \int_\Omega uv dx, \quad J_2[u][v] = \int_{\partial\Omega} \frac{\partial u}{\partial \nu}\frac{\partial v}{\partial \nu} d\sigma, \quad J_3[u][v] = \int_{\partial\Omega} uv d\sigma,$$

for any $u, v \in H^2(\Omega)$, where we denote by ν the unit outer normal vector to $\partial\Omega$, and by $d\sigma$ the area element.

Using this notation, problems (1)–(4) can be stated in the following weak form

$$P[u][v] = \lambda J_1[u][v], \ \forall v \in V(\Omega),$$

where $V(\Omega)$ is either $H_0^2(\Omega)$ (for the Dirichlet problem), or $H^2(\Omega) \cap H_0^1(\Omega)$ (for the Navier problem), or $H^2(\Omega)$ (for the Neumann problem). Here and in the sequel the bilinear forms defined on $V(\Omega)$ will be understood also as linear operators acting from $V(\Omega)$ to its dual.

As for Steklov-type problems, we shall consider their generalizations according to the definition of P. In particular, regarding problem (5), we consider the following generalization

$$\begin{cases} \Delta^2 u - \tau \Delta u = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \\ (1-\sigma)\frac{\partial^2 u}{\partial \nu^2} + \sigma \Delta u = \lambda \frac{\partial u}{\partial \nu}, & \text{on } \partial\Omega, \end{cases} \tag{8}$$

whose weak formulation is

$$P[u][v] = \lambda J_2[u][v], \ \forall v \in H^2(\Omega) \cap H_0^1(\Omega).$$

We also consider the following generalization of problem (6)

$$\begin{cases} \Delta^2 u - \tau \Delta u = 0, & \text{in } \Omega, \\ (1-\sigma)\frac{\partial^2 u}{\partial \nu^2} + \sigma \Delta u = 0, & \text{on } \partial\Omega, \\ \tau \frac{\partial u}{\partial \nu} - \frac{\partial \Delta u}{\partial \nu} - (1-\sigma)\mathrm{div}_{\partial\Omega}\left(\nu^t D^2 u\right)_{\partial\Omega} = \lambda u, & \text{on } \partial\Omega, \end{cases} \tag{9}$$

whose weak formulation is

$$P[u][v] = \lambda J_3[u][v], \ \forall v \in H^2(\Omega).$$

Using a unified notation, we can therefore write all the problems we are considering as

$$P[u][v] = \lambda J_i[u][v], \ \forall v \in V(\Omega), \tag{10}$$

where:

- for the Dirichlet problem (1), (2) we set $i = 1$, $V(\Omega) = H_0^2(\Omega)$;
- for the Navier problem (1), (3) we set $i = 1$, $V(\Omega) = H^2(\Omega) \cap H_0^1(\Omega)$;
- for the Neumann problem (1), (4) we set $i = 1$, $V(\Omega) = H^2(\Omega)$;
- for the Steklov problem (8) we set $i = 2$, $V(\Omega) = H^2(\Omega) \cap H_0^1(\Omega)$;
- for the Steklov problem (9) we set $i = 3$, $V(\Omega) = H^2(\Omega)$.

It is clear that both the Neumann problem (1), (4) and the Steklov problem (9) have non-trivial kernel. In particular, if $\tau > 0$ then both kernels are given by the constant functions, while if $\tau = 0$ the kernels have dimension $N + 1$ including also the coordinate functions $x_1, \ldots, x_N$ (cf. [15, Theorem 3.8]). For this reason, we will restrict our attention to the case $\tau > 0$ and consider instead $V(\Omega) = H^2(\Omega)/\mathbb{R}$ for problems (1), (4) and (9) (the case $\tau = 0$ being similar).

With this choice of the space $V(\Omega)$, it is possible to show that the bilinear form P defines a scalar product on $V(\Omega)$ which is equivalent to the standard one. We shall therefore consider $V(\Omega)$ as endowed with such a scalar product.

It is easily seen that P, considered as an operator acting from $V(\Omega)$ to its dual, is a linear homeomorphism. In particular, we can define

$$T_i = P^{(-1)} \circ J_i, \tag{11}$$

for $i = 1, 2, 3$. We have the following

Theorem 1 *Let* $-\frac{1}{N-1} < \sigma < 1$, $\tau > 0$. *Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1. The operator T_i defined in (11) is a non-negative compact selfadjoint operator on the Hilbert space $V(\Omega)$. Its spectrum is discrete and consists of a decreasing sequence of positive eigenvalues of finite multiplicity converging to zero. Moreover, the equation $T_i u = \mu u$ is satisfied for some $u \in V(\Omega)$, $\mu > 0$ if and only if Eq. (10) is satisfied with $0 \neq \lambda = \mu^{-1}$ for any $v \in V(\Omega)$. In particular, the eigenvalues of problem (10) can be arranged in a diverging sequence*

$$0 < \lambda_1[\Omega] \leq \lambda_2[\Omega] \leq \cdots \leq \lambda_k[\Omega] \leq \cdots,$$

where all the eigenvalues are repeated according to their multiplicity, and the following variational characterization holds

$$\lambda_k[\Omega] = \min_{\substack{E \le V(\Omega) \\ \dim E = k}} \max_{\substack{u \in E \\ J_i[u][u] \neq 0}} \frac{P[u][u]}{J_i[u][u]}.$$

Proof For the selfadjointness, it suffices to observe that

$$< T_i[u], v >=< P^{(-1)} \circ J_i[u], v >= P[P^{(-1)} \circ J_i[u]][v] = J_i[u][v],$$

for any $u, v \in V(\Omega)$. For the compactness, just observe that the operator J_i is compact. The remaining statements are straightforward.

Remark 2 As we have said in Sect. 1, in applications $0 \le \sigma \le 0.5$. However, there are examples of materials with high or negative Poisson ratio, namely ($N = 2$)

$$-1 < \sigma < 1.$$

In general dimension we choose $-\frac{1}{N-1} < \sigma < 1$. This is due to the fact that, thanks to the inequality

$$|D^2u|^2 \ge \frac{1}{N}(\Delta u)^2 \; \forall u \in H^2(\Omega),$$

then, for σ in that range, the operator P turns out to be coercive. We also remark that, following the arguments in [15, Sect. 3], the Steklov problem (9) can be seen as a limiting Neumann problem of the type (1), (4) with a mass distribution concentrating to the boundary.

We note that problem (5) is obtained for $\sigma = 1$, which is out of our range. Under some additional regularity assumptions, for instance $\Omega \in C^2$ (see e.g., [7] for general conditions), then the operator becomes coercive and all the results here and in the sequel apply as well. The same remains true also for the Navier problem (1), (3), which for $\sigma = 1$ reads

$$\begin{cases} \Delta^2 u - \tau \Delta u = \lambda u, & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega, \end{cases}$$

which has been extensively studied in the case $\tau = 0$ (we refer to [4, 27, 41, 45] and the references therein).

The situation is instead completely different in the case of Neumann boundary conditions with $\sigma = 1$, namely

$$\begin{cases} \Delta^2 u - \tau \Delta u = \lambda u, & \text{in } \Omega, \\ \Delta u = \frac{\partial \Delta u}{\partial \nu} = 0, & \text{on } \partial\Omega. \end{cases} \tag{12}$$

It is easy to see that problem (12) has an infinite dimensional kernel since all harmonic functions belong to the eigenspace associated with the eigenvalue $\lambda = 0$. In particular, the boundary conditions do not satisfy the complementing conditions, see [1, 27]. We refer to [42] for considerations on the spectrum of problem (12).

3 Analyticity of the Eigenvalues and Hadamard Formulas

The study of the dependence of the eigenvalues of elliptic operators on the domain has nowadays become a classical problem in the field of perturbation theory. Shape continuity of the eigenvalues has been known for long time [23], and can also be improved to Hölder or Lipschitz continuity using stability estimates as in [8, 17–20]. However, while the continuity holds for all the eigenvalues, only the simple ones enjoy an analytic dependence (see e.g., [30]). On the other hand, when the eigenvalue is multiple, bifurcation phenomena occur, so that, if the perturbation is parametrized by one real variable, then the eigenvalues are described by suitable analytic branches (cf. [44, Theorem 1]). Unfortunately, if the family of perturbations is not parametrized by one real variable, one cannot expect the eigenvalues to split into analytic branches anymore. In this case, the use of elementary symmetric functions of the eigenvalues (see [35, 37]) has the advantage of bypassing splitting phenomena, in fact such functions turn out to be analytic.

To this end, we shall consider problem (10) in a family of open sets parametrized by suitable diffeomorphisms ϕ defined on a bounded open set Ω in $\mathbb{R}^N$ of class C^1. Namely, we set

$$\mathscr{A}_\Omega = \left\{ \phi \in C^2(\overline{\Omega}\,;\mathbb{R}^N) : \inf_{\substack{x_1,x_2\in\overline{\Omega}\\ x_1\neq x_2}} \frac{|\phi(x_1)-\phi(x_2)|}{|x_1-x_2|} > 0 \right\},$$

where $C^2(\overline{\Omega}\,;\mathbb{R}^N)$ denotes the space of all functions from $\overline{\Omega}$ to $\mathbb{R}^N$ of class C^2. Note that if $\phi \in \mathscr{A}_\Omega$ then ϕ is injective, Lipschitz continuous and $\inf_{\overline{\Omega}} |\det \nabla\phi| > 0$. Moreover, $\phi(\Omega)$ is a bounded open set of class C^1 and the inverse map $\phi^{(-1)}$ belongs to $\mathscr{A}_{\phi(\Omega)}$. Thus it is natural to consider problem (10) on $\phi(\Omega)$ and study the dependence of $\lambda_k[\phi(\Omega)]$ on $\phi \in \mathscr{A}_\Omega$. To do so, we endow the space $C^2(\overline{\Omega}\,;\mathbb{R}^N)$ with its usual norm. Note that $\mathscr{A}_\Omega$ is an open set in $C^2(\overline{\Omega}\,;\mathbb{R}^N)$, see [35, Lemma 3.11]. Thus, it makes sense to study differentiability and analyticity properties of the maps $\phi \mapsto \lambda_k[\phi(\Omega)]$ defined for $\phi \in \mathscr{A}_\Omega$. For simplicity, we write $\lambda_k[\phi]$ instead of $\lambda_k[\phi(\Omega)]$. We fix a finite set of indexes $F \subset \mathbb{N}$ and we consider those maps $\phi \in \mathscr{A}_\Omega$ for which the eigenvalues with indexes in F do not coincide with eigenvalues with indexes not in F; namely we set

$$\mathscr{A}_{F,\Omega} = \{\phi \in \mathscr{A}_\Omega : \ \lambda_k[\phi] \neq \lambda_l[\phi],\ \forall\, k \in F,\ l \in \mathbb{N}\setminus F\}.$$

It is also convenient to consider those maps $\phi \in \mathscr{A}_{F,\Omega}$ such that all the eigenvalues with index in F coincide and set

$$\Theta_{F,\Omega} = \left\{\phi \in \mathscr{A}_{F,\Omega} : \ \lambda_{k_1}[\phi] = \lambda_{k_2}[\phi],\ \ \forall\, k_1, k_2 \in F\right\}.$$

For $\phi \in \mathscr{A}_{F,\Omega}$, the elementary symmetric functions of the eigenvalues with index in F are defined by

$$\Lambda_{F,s}[\phi] = \sum_{\substack{k_1,\dots,k_s \in F \\ k_1 < \dots < k_s}} \lambda_{k_1}[\phi] \cdots \lambda_{k_s}[\phi], \quad s = 1, \dots, |F|.$$

We have the following

Theorem 3 *Let Ω be a bounded open set in $\mathbb{R}^N$ of class C^1, $\tau > 0$, $-\frac{1}{N-1} < \sigma < 1$, and F be a finite set in $\mathbb{N}$. The set $\mathscr{A}_{F,\Omega}$ is open in $\mathscr{A}_\Omega$, and the real-valued maps $\Lambda_{F,s}$ are real-analytic on $\mathscr{A}_{F,\Omega}$, for all $s = 1, \dots, |F|$. Moreover, if $\tilde{\phi} \in \Theta_{F,\Omega}$ is such that the eigenvalues $\lambda_k[\tilde{\phi}]$ assume the common value $\lambda_F[\tilde{\phi}]$ for all $k \in F$, and $\tilde{\phi}(\Omega)$ is of class C^4 then the Fréchet differential of the map $\Lambda_{F,s}$ at the point $\tilde{\phi}$ is delivered by the formula*

$$d|_{\phi=\tilde{\phi}}(\Lambda_{F,s})[\psi] = \lambda_F^s[\tilde{\phi}] \binom{|F|-1}{s-1} \sum_{l=1}^{|F|} \int_{\partial\tilde{\phi}(\Omega)} G(v_l)(\psi \circ \tilde{\phi}^{(-1)}) \cdot \nu d\sigma, \qquad (13)$$

for all $\psi \in C^2(\overline{\Omega}; \mathbb{R}^N)$, where $\{v_l\}_{l \in F}$ is an orthonormal basis in $V(\tilde{\phi}(\Omega))$ of the eigenspace associated with $\lambda_F[\tilde{\phi}]$ (the orthonormality being taken with respect to (7)), and:

- $G(v) = -\left(\frac{\partial^2 v}{\partial \nu^2}\right)^2$ *for the Dirichlet problem;*
- $G(v) = (1-\sigma)|D^2 v|^2 + \sigma(\Delta v)^2 + \tau|\nabla v|^2 - \lambda_F[\tilde{\phi}(\Omega)]v^2$ *for the Neumann problem;*
- $G(v) = 2\frac{\partial v}{\partial \nu}\left(\frac{\partial \Delta v}{\partial \nu} + (1-\sigma)\mathrm{div}_{\partial\tilde{\phi}(\Omega)}(\nu \cdot D^2 v)_{\partial\tilde{\phi}(\Omega)}\right) + (1-\sigma)|D^2 v|^2 + \sigma(\Delta v)^2 + \tau\left(\frac{\partial v}{\partial \nu}\right)^2$ *for the Navier problem;*
- $G(v) = 2\frac{\partial v}{\partial \nu}\left(\frac{\partial \Delta v}{\partial \nu} + (1-\sigma)\mathrm{div}_{\partial\tilde{\phi}(\Omega)}(\nu \cdot D^2 v)_{\partial\tilde{\phi}(\Omega)}\right) + (1-\sigma)|D^2 v|^2 + \sigma(\Delta v)^2 + \tau\left(\frac{\partial v}{\partial \nu}\right)^2 - \lambda_F[\tilde{\phi}(\Omega)]K\left(\frac{\partial v}{\partial \nu}\right)^2 - \lambda_F[\tilde{\phi}(\Omega)]\frac{\partial}{\partial \nu}\left(\frac{\partial v}{\partial \nu}\right)^2$ *for the Steklov problem (8);*
- $G(v) = (1-\sigma)|D^2 v|^2 + \sigma(\Delta v)^2 + \tau|\nabla v|^2 - \lambda_F[\tilde{\phi}(\Omega)]Kv^2 - \lambda_F[\tilde{\phi}(\Omega)]\frac{\partial (v)^2}{\partial \nu}$ *for the Steklov problem (9),*

where by K we denote the mean curvature of $\partial\tilde{\phi}(\Omega)$.

Proof For the proof of the first part of the theorem we refer to [11, Theorem 3.1] (see also [35]). Concerning formula (13), we start by recalling that, for $\phi \in \mathscr{A}_\Omega$, we have that the pull-back of the operator M is defined by

$$M[u][v] = \int_\Omega D^2(u \circ \phi^{-1}) : D^2(v \circ \phi^{-1})|\det D\phi| dx,$$

for any $u, v \in V(\Omega)$, and similarly also for B, L, P, J_i and T_i. We have

$$\mathrm{d}|_{\phi=\tilde{\phi}}\Lambda_{F,s}[\psi] = -\lambda_F^{s+1}[\tilde{\phi}]\binom{|F|-1}{s-1}\sum_{l\in F} P_{\tilde{\phi}}\Big[\mathrm{d}|_{\phi=\tilde{\phi}}T_{i,\phi}[\psi][u_l]\Big]\Big[u_l\Big]$$

for all $\psi \in C^2(\bar{\Omega}\,;\mathbb{R}^N)$ (cf. [35, proof of Theorem 3.38]), where $u_l = v_l \circ \tilde{\phi}$ for all $l \in F$. Note also that by standard regularity theory (see e.g., [27, Theorem 2.20]) $v_l \in H^4(\tilde{\phi}(\Omega))$ for all $l \in F$.

By standard calculus we have

$$P_{\tilde{\phi}}\Big[\mathrm{d}|_{\phi=\tilde{\phi}}T_{i,\phi}[\psi][u_l]\Big]\Big[u_l\Big] = d|_{\phi=\tilde{\phi}}J_{i,\phi}[\psi][u_l][u_l] \;\; - \lambda_F^{-1}[\tilde{\phi}]d|_{\phi=\tilde{\phi}}P_{\phi}[\psi][u_l][u_l].$$

Applying Lemmas 7 and 8, we obtain formula (13).

We note that formula (13) for the Dirichlet problem was already obtained in [40] using different techniques, but only for simple eigenvalues. We also remark that some of the specific cases of formula (13) were already obtained in [11, 12, 14–16]. In particular, in [12] it was derived a different formula for the Navier problem, which was later shown to be equivalent to the one proposed here (see [8, 14]).

4 Shape Optimization and Isovolumetric Perturbations

Important shape optimization problems for the eigenvalues of elliptic operators were already addressed in [43], where the author claims that among all bounded domains in $\mathbb{R}^2$ of given area, the ball minimizes the first eigenvalue of the Dirichlet Laplacian and Bilaplacian, namely

$$\lambda_1(\Omega) \geq \lambda_1(\Omega^*), \tag{14}$$

where Ω^* is a ball with $|\Omega| = |\Omega^*|$. He does not provide any proof of inequality (14), claiming it trivial for physical evidence. The actual proof of inequality (14), in the case of the Dirichlet Laplacian, is due to Faber [26] and Krahn [31], and was then followed by similar inequalities for the eigenvalues of the Laplace operator subject to other boundary conditions. We refer to [29] for an extensive discussion on the topic.

On the other hand, in the case of the biharmonic operator inequality (14) is instead still an open problem and is known as the Rayleigh conjecture. It has been proved only in low dimension, namely $N = 2, 3$, by Nadirashvili [39] and Ashbaugh and Benguria [3] improving an argument due to Talenti [46]. Unfortunately, such an argument does not seem to work in higher dimension. Regarding other boundary conditions, similar inequalities have been derived for the Neumann problem (1), (4) and for the Steklov problem (6), also in quantitative form (see [10, 15, 16, 21, 22]), showing that in such cases the ball is actually the maximizer. The Steklov problem

(5) is instead the only one in which the ball has been shown not to be the optimizer for the first eigenvalue (cf. [33]).

Here we consider the following shape optimization problems for the symmetric functions of the eigenvalues

$$\min_{V[\phi]=\text{const}} \Lambda_{F,s}[\phi] \quad \text{or} \quad \max_{V[\phi]=\text{const}} \Lambda_{F,s}[\phi], \tag{15}$$

where V is the real valued function defined on $\mathscr{A}_\Omega$ which takes $\phi \in \mathscr{A}_\Omega$ to $V[\phi] = |\phi(\Omega)|$. Note that if $\tilde{\phi} \in \mathscr{A}_\Omega$ is a minimizer or maximizer in (15) then $\tilde{\phi}$ is a critical domain transformation for the map $\phi \mapsto \Lambda_{F,s}[\phi]$ subject to volume constraint, i.e.,

$$\text{Ker } \mathrm{d}|_{\phi=\tilde{\phi}} V \subset \text{Ker } \mathrm{d}|_{\phi=\tilde{\phi}} \Lambda_{F,s}.$$

The following theorem provides a characterization of all critical domain transformations ϕ (see also [11–13, 15, 36]).

Theorem 4 *Let Ω be a bounded open set in $\mathbb{R}^N$ of class C^1, and F be a finite subset of $\mathbb{N}$. Assume that $\tilde{\phi} \in \Theta_{F,\Omega}$ is such that $\tilde{\phi}(\Omega)$ is of class C^4 and that the eigenvalues $\lambda_j[\tilde{\phi}]$ have the common value $\lambda_F[\tilde{\phi}]$ for all $j \in F$. Let $\{v_l\}_{l\in F}$ be an orthornormal basis in $V(\tilde{\phi}(\Omega))$ of the eigenspace corresponding to $\lambda_F[\tilde{\phi}]$ (the orthonormality being taken with respect to (7)). Then $\tilde{\phi}$ is a critical domain transformation for any of the functions $\Lambda_{F,s}$, $s = 1, \dots, |F|$, with volume constraint if and only if there exists $c \in \mathbb{R}$ such that*

$$\sum_{l=1}^{|F|} G(v_l) = c, \quad \text{on } \partial\tilde{\phi}(\Omega). \tag{16}$$

Proof The proof is a straightforward application of Lagrange Multipliers Theorem combined with formula (13).

As we have said, balls play a relevant role in the shape optimization of the eigenvalues of the Laplace and biharmonic operators. Hence we need to analyze in more details the behavior of the eigenfunctions on balls. We have the following result (cf. [15, Lemma 4.22]).

Theorem 5 *Let B be a ball in $\mathbb{R}^N$ centered at zero, and let λ be an eigenvalue of problem (10) in B. Let F be the subset of $\mathbb{N}$ of all j such that the jth eigenvalue of problem (10) in B coincides with λ. Let $v_1, \dots, v_{|F|}$ be an orthonormal basis of the eigenspace associated with the eigenvalue λ, where the orthonormality is taken with respect to the scalar product in $V(B)$. Then*

$$\sum_{j=1}^{|F|} v_j^2, \ \sum_{j=1}^{|F|} |\nabla v_j|^2, \ \sum_{j=1}^{|F|} |\Delta v_j|^2, \ \sum_{j=1}^{|F|} |D^2 v_j|^2$$

are radial functions.

Proof First of all, note that by standard regularity theory (see e.g., [1, 27]), the functions $v_j \in C^\infty(\overline{B})$ for all $j \in F$.

Let $O_N(\mathbb{R})$ denote the group of orthogonal linear transformations in $\mathbb{R}^N$. Since the operators P and J_i, $i = 1, 2, 3$ are invariant under rotations, then $v_k \circ R$, where $R \in O_N(\mathbb{R})$, is still an eigenfunction with eigenvalue λ; moreover, $\{v_j \circ R : j = 1, \dots, |F|\}$ is another orthonormal basis for the eigenspace associate with λ. Since both $\{v_j : j = 1, \dots, |F|\}$ and $\{v_j \circ R : j = 1, \dots, |F|\}$ are orthonormal bases, then there exists $A[R] \in O_N(\mathbb{R})$ with matrix $(A_{ij}[R])_{i,j=1,\dots,|F|}$ such that

$$v_j = \sum_{l=1}^{|F|} A_{jl}[R] v_l \circ R. \tag{17}$$

This implies that

$$\sum_{j=1}^{|F|} v_j^2 = \sum_{j=1}^{|F|} (v_j \circ R)^2,$$

from which we get that $\sum_{j=1}^{|F|} v_j^2$ is radial. Moreover, using standard calculus and (17), we get

$$\sum_{j=1}^{|F|} |\nabla v_j|^2 = \sum_{j,l_1,l_2=1}^{|F|} A_{jl_1}[R] A_{jl_2}[R] \left(\nabla v_{l_1} \circ R\right) \cdot \left(\nabla v_{l_2} \circ R\right) = \sum_{l=1}^{|F|} |\nabla v_l \circ R|^2.$$

Similarly,

$$\sum_{j=1}^{|F|} |\Delta v_j|^2 = \sum_{j=1}^{|F|} |\Delta v_j \circ R|^2.$$

On the other hand,

$$\begin{aligned} D^2 v_j \cdot D^2 v_j &= \sum_{l_1,l_2=1}^{|F|} A_{jl_1}[R] A_{jl_2}[R] R^t \cdot (D^2 v_{l_1} \circ R) \cdot R \cdot R^t \cdot (D^2 v_{l_2} \circ R) \cdot R \\ &= \sum_{l_1,l_2=1}^{|F|} A_{jl_1}[R] A_{jl_2}[R] R^t \cdot (D^2 v_{l_1} \circ R) \cdot (D^2 v_{l_2} \circ R) \cdot R, \end{aligned}$$

therefore

$$|D^2 v_j|^2 = \operatorname{tr}(D^2 v_j \cdot D^2 v_j) = \sum_{l_1,l_2=1}^{|F|} A_{jl_1}[R] A_{jl_2}[R] (D^2 v_{l_1} \circ R) : (D^2 v_{l_2} \circ R),$$

from which we get

$$\sum_{j=1}^{|F|} |D^2 v_j|^2 = \sum_{j=1}^{|F|} |D^2 v_j \circ R|^2.$$

For rotation invariant operators such as the Laplace or the biharmonic operator, it is easy to show that any eigenfunction associated with a simple eigenvalue is radial. Theorem 5 tells us that, when dealing with a multiple eigenvalue, we cannot consider the eigenfunctions alone, but we have to consider the whole eigenspace. In particular, this is very useful when coupled with condition (16).

Theorem 6 *Let Ω be a domain in $\mathbb{R}^N$ of class C^1. Let $\tilde{\phi} \in \mathscr{A}_\Omega$ be such that $\tilde{\phi}(\Omega)$ is a ball. Let $\tilde{\lambda}$ be an eigenvalue of problem (10) in $\tilde{\phi}(\Omega)$, and let F be the set of $j \in \mathbb{N}$ such that $\lambda_j[\tilde{\phi}] = \tilde{\lambda}$. Then $\tilde{\phi}$ is a critical point for $\Lambda_{F,s}$ under volume constraint, for all $s = 1, \dots, |F|$.*

Proof Thanks to Theorem 5, it remains to prove that, for problems (1), (3) and (8), the function

$$\sum_{j=1}^{|F|} \frac{\partial v_j}{\partial \nu} \left(\frac{\partial \Delta v_j}{\partial \nu} + (1-\sigma)\mathrm{div}_{\partial\tilde{\phi}(\Omega)}(\nu \cdot D^2 v_j)_{\partial\tilde{\phi}(\Omega)} \right)$$

is a radial function. In particular, here $V(\tilde{\phi}(\Omega)) = H^2(\tilde{\phi}(\Omega)) \cap H_0^1(\tilde{\phi}(\Omega))$, hence

$$\frac{\partial}{\partial \nu} \nabla v_j = \nabla \frac{\partial v_j}{\partial \nu},$$

from which we get

$$\mathrm{div}_{\partial\tilde{\phi}(\Omega)}(\nu \cdot D^2 v_j)_{\partial\tilde{\phi}(\Omega)} = \Delta_{\partial\tilde{\phi}(\Omega)} \left(\frac{\partial v_j}{\partial \nu} \right)$$

on $\partial\tilde{\phi}(\Omega)$. Therefore

$$\begin{aligned} \sum_{j=1}^{|F|} \frac{\partial v_j}{\partial \nu} \mathrm{div}_{\partial\tilde{\phi}(\Omega)}(\nu \cdot D^2 v_j)_{\partial\tilde{\phi}(\Omega)} &= \sum_{j=1}^{|F|} \frac{\partial v_j}{\partial \nu} \Delta_{\partial\tilde{\phi}(\Omega)} \left(\frac{\partial v_j}{\partial \nu} \right) \\ &= \frac{1}{2} \Delta_{\partial\tilde{\phi}(\Omega)} \left(\sum_{j=1}^{|F|} \left(\frac{\partial v_j}{\partial \nu} \right)^2 \right) - \sum_{j=1}^{|F|} \left| \nabla_{\partial\tilde{\phi}(\Omega)} \frac{\partial v_j}{\partial \nu} \right|^2, \end{aligned}$$

where the two summands on the right-hand side can be shown to be constant on $\partial\tilde{\phi}(\Omega)$ following the lines of the proof of Theorem 5.

On the other hand,

$$\sum_{j=1}^{|F|} \nabla v_j \cdot \nabla \Delta v_j = \frac{1}{8}\Delta^2 \left(\sum_{j=1}^{|F|} v_j^2 \right) - \frac{\lambda}{4}\sum_{j=1}^{|F|} v_j^2 - \frac{1}{4}\sum_{j=1}^{|F|} (\Delta v_j)^2 - \frac{1}{2}\sum_{j=1}^{|F|} |D^2 v_j|^2,$$

from which we deduce that $\sum_{j=1}^{|F|} \frac{\partial v_j}{\partial \nu}\frac{\partial \Delta v_j}{\partial \nu}$ is constant on $\partial\tilde{\phi}(\Omega)$.

In general, balls are expected to be the extremizer of problems of the type (15) only when the first eigenvalue is involved (see e.g., [29]), and even in this case problem (5) shows up as a counterexample. Nevertheless, thanks to Theorem 6, we know that balls still are critical domains for all the eigenvalues. It also would be interesting to characterize the family of open sets $\tilde{\phi}(\Omega)$ such that condition (16) is satisfied. The only result in this direction is due to Dalmasso [24], who proved that the ball is the only domain satisfying condition (16) for the first eigenvalue of the biharmonic operator subject to Dirichlet boundary conditions under the additional hypotesis that the first eigenfunction does not change sign.

5 Some Technical Lemmas

In this section we prove two lemmas that has been used in the proof of Theorem 3.

Lemma 7 *Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1, and let $\tilde{\phi} \in \mathscr{A}_\Omega$ be such that $\tilde{\phi}(\Omega)$ is of class C^2. Let $u_1, u_2 \in H^2(\Omega)$ be such that $v_1 = u_1 \circ \tilde{\phi}^{-1}$, $v_2 = u_2 \circ \tilde{\phi}^{-1} \in H^4(\tilde{\phi}(\Omega))$. Then*

$$\begin{aligned}
d|_{\phi=\tilde{\phi}} M_\phi[\psi][u_1][u_2] &= \int_{\partial\tilde{\phi}(\Omega)} (D^2 v_1 : D^2 v_2)\zeta \cdot \nu d\sigma \\
&+ \int_{\partial\tilde{\phi}(\Omega)} \left(\mathrm{div}_{\partial\tilde{\phi}(\Omega)}(\nu \cdot D^2 v_1)_{\partial\tilde{\phi}(\Omega)} \nabla v_2 + \mathrm{div}_{\partial\tilde{\phi}(\Omega)}(\nu \cdot D^2 v_2)_{\partial\tilde{\phi}(\Omega)} \nabla v_1 \right) \cdot \zeta d\sigma \\
&+ \int_{\partial\tilde{\phi}(\Omega)} \left(\frac{\partial \Delta v_1}{\partial \nu}\nabla v_2 + \frac{\partial \Delta v_2}{\partial \nu}\nabla v_1 \right) \cdot \zeta d\sigma - \int_{\tilde{\phi}(\Omega)} \left(\Delta^2 v_1 \nabla v_2 + \Delta^2 v_2 \nabla v_1 \right) \cdot \zeta d\sigma \\
&\qquad - \int_{\partial\tilde{\phi}(\Omega)} \left(\frac{\partial^2 v_1}{\partial \nu^2}\nabla v_2 + \frac{\partial^2 v_2}{\partial \nu^2}\nabla v_1 \right) \cdot \frac{\partial \zeta}{\partial \nu} d\sigma \\
&\qquad - \int_{\partial\tilde{\phi}(\Omega)} \left(\frac{\partial^2 v_1}{\partial \nu^2}\frac{\partial}{\partial \nu}\nabla v_2 + \frac{\partial^2 v_2}{\partial \nu^2}\frac{\partial}{\partial \nu}\nabla v_1 \right) \cdot \zeta d\sigma, \quad (18)
\end{aligned}$$

$$
\begin{aligned}
d|_{\phi=\tilde{\phi}} B_\phi[\psi][u_1][u_2] &= \int_{\partial\tilde{\phi}(\Omega)} \Delta v_1 \Delta v_2 \zeta \cdot \nu d\sigma \\
&+ \int_{\partial\tilde{\phi}(\Omega)} \left(\frac{\partial \Delta v_1}{\partial \nu} \nabla v_2 + \frac{\partial \Delta v_2}{\partial \nu} \nabla v_1 \right) \cdot \zeta d\sigma - \int_{\tilde{\phi}(\Omega)} \left(\Delta^2 v_1 \nabla v_2 + \Delta^2 v_2 \nabla v_1 \right) \cdot \zeta d\sigma \\
&- \int_{\partial\tilde{\phi}(\Omega)} (\Delta v_1 \nabla v_2 + \Delta v_2 \nabla v_1) \cdot \frac{\partial \zeta}{\partial \nu} d\sigma \\
&- \int_{\partial\tilde{\phi}(\Omega)} \left(\Delta v_1 \frac{\partial}{\partial \nu} \nabla v_2 + \Delta v_2 \frac{\partial}{\partial \nu} \nabla v_1 \right) \cdot \zeta d\sigma, \quad (19)
\end{aligned}
$$

and

$$
\begin{aligned}
d|_{\phi=\tilde{\phi}} L_\phi[\psi][u_1][u_2] &= - \int_{\partial\tilde{\phi}(\Omega)} \nabla v_1 \cdot \nabla v_2 \zeta \cdot \nu d\sigma \\
&+ \int_{\partial\tilde{\phi}(\Omega)} \left(\frac{\partial v_1}{\partial \nu} \nabla v_2 + \frac{\partial v_2}{\partial \nu} \nabla v_1 \right) \cdot \zeta d\sigma - \int_{\tilde{\phi}(\Omega)} (\Delta v_1 \nabla v_2 + \Delta v_2 \nabla v_1) \cdot \zeta d\sigma, \quad (20)
\end{aligned}
$$

for all $\psi \in C^2(\overline{\Omega}; \mathbb{R}^N)$*, where* $\zeta = \psi \circ \tilde{\phi}^{-1}$.

Proof First of all, we observe that the proof of (18) and of (20) can be done following that of [15, Lemma 4.4] (we also refer to [8, Lemmas 2.4 and 2.6]). As for (19), we have (see also [8, Lemma 2.5])

$$
\begin{aligned}
d|_{\phi=\tilde{\phi}} B_\phi[\psi][u_1][u_2] \\
&= \int_\Omega (d|_{\phi=\tilde{\phi}} \Delta(u_1 \circ \phi^{-1}) \circ \phi)[\psi](\Delta(u_2 \circ \tilde{\phi}^{-1}) \circ \tilde{\phi}) |\det D\tilde{\phi}| dx \\
&+ \int_\Omega (\Delta(u_1 \circ \tilde{\phi}^{-1}) \circ \tilde{\phi})(d|_{\phi=\tilde{\phi}} \Delta(u_2 \circ \phi^{-1}) \circ \phi)[\psi]| \det D\tilde{\phi}| dx \\
&+ \int_\Omega (\Delta(u_1 \circ \tilde{\phi}^{-1}) \circ \tilde{\phi})(\Delta(u_2 \circ \tilde{\phi}^{-1}) \circ \tilde{\phi}) d|_{\phi=\tilde{\phi}} |\det D\phi|[\psi] dx,
\end{aligned} \quad (21)
$$

and we note that, by the equality

$$
\left[\left(\mathrm{d}|_{\phi=\tilde{\phi}} (\det \nabla \phi) [\psi] \right) \circ \tilde{\phi}^{(-1)} \right] \det \nabla \tilde{\phi}^{(-1)} = \operatorname{div} \left(\psi \circ \tilde{\phi}^{(-1)} \right), \quad (22)
$$

the last summand in (21) equals

$$
\int_{\tilde{\phi}(\Omega)} \Delta v_1 \Delta v_2 \operatorname{div} \zeta dy.
$$

We have

$$
\begin{aligned}
\int_{\tilde{\phi}(\Omega)} \frac{\partial^2 v_1}{\partial y_r \partial y_s} \frac{\partial \zeta_r}{\partial y_s} \Delta v_2 dy &= \int_{\partial\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_s} \frac{\partial \zeta_r}{\partial y_s} \nu_r \Delta v_2 d\sigma \\
&- \int_{\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_s} \frac{\partial \mathrm{div}\zeta}{\partial y_s} \Delta v_2 dy - \int_{\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_s} \frac{\partial \zeta_r}{\partial y_s} \frac{\partial \Delta v_2}{\partial y_r} dy \\
&= \int_{\partial\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_s} \frac{\partial \zeta_r}{\partial y_s} \nu_r \Delta v_2 d\sigma - \int_{\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_s} \frac{\partial \zeta_r}{\partial y_s} \frac{\partial \Delta v_2}{\partial y_r} dy \\
&- \int_{\partial\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial \nu} \Delta v_2 \mathrm{div}\zeta d\sigma + \int_{\tilde{\phi}(\Omega)} \Delta v_1 \Delta v_2 \mathrm{div}\zeta dy \\
&+ \int_{\tilde{\phi}(\Omega)} \nabla v_1 \cdot \nabla \Delta v_2 \mathrm{div}\zeta dy, \quad (23)
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_s} \Delta \zeta_s \Delta v_2 dy &= \int_{\partial\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_s} \frac{\partial \zeta_s}{\partial \nu} \Delta v_2 d\sigma \\
&- \int_{\tilde{\phi}(\Omega)} \frac{\partial^2 v_1}{\partial y_i \partial y_s} \frac{\partial \zeta_s}{\partial y_i} \Delta v_2 dy - \int_{\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_s} \frac{\partial \zeta_s}{\partial y_i} \frac{\partial \Delta v_2}{\partial y_i} dy \\
&= \int_{\partial\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_s} \frac{\partial \zeta_s}{\partial \nu} \Delta v_2 d\sigma - \int_{\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_s} \frac{\partial \zeta_s}{\partial y_i} \frac{\partial \Delta v_2}{\partial y_i} dy \\
&- \int_{\partial\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_i} \frac{\partial \zeta_s}{\partial y_i} \nu_s \Delta v_2 d\sigma + \int_{\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_i} \frac{\partial \mathrm{div}\zeta}{\partial y_i} \Delta v_2 dy \\
&+ \int_{\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_i} \frac{\partial \zeta_s}{\partial y_i} \frac{\partial \Delta v_2}{\partial y_s} dy = \int_{\partial\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_s} \frac{\partial \zeta_s}{\partial \nu} \Delta v_2 d\sigma \\
&- \int_{\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_s} \frac{\partial \zeta_s}{\partial y_i} \frac{\partial \Delta v_2}{\partial y_i} dy - \int_{\partial\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_i} \frac{\partial \zeta_s}{\partial y_i} \nu_s \Delta v_2 d\sigma \\
&+ \int_{\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_i} \frac{\partial \zeta_s}{\partial y_i} \frac{\partial \Delta v_2}{\partial y_s} dy + \int_{\partial\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial \nu} \Delta v_2 \mathrm{div}\zeta d\sigma \\
&- \int_{\tilde{\phi}(\Omega)} \Delta v_1 \Delta v_2 \mathrm{div}\zeta dy - \int_{\tilde{\phi}(\Omega)} \nabla v_1 \cdot \nabla \Delta v_2 \mathrm{div}\zeta dy. \quad (24)
\end{aligned}
$$

Combining (21), (23), and (24) we get

$$
\begin{aligned}
d|_{\phi=\tilde{\phi}} B_\phi[\psi][u_1][u_2] = &- \int_{\partial\tilde{\phi}(\Omega)} \left(\frac{\partial v_1}{\partial y_s} \Delta v_2 + \frac{\partial v_2}{\partial y_s} \Delta v_1 \right) \frac{\partial \zeta_r}{\partial y_s} \nu_r d\sigma \\
&+ \int_{\tilde{\phi}(\Omega)} \left(\frac{\partial v_1}{\partial y_s} \frac{\partial \Delta v_2}{\partial y_r} + \frac{\partial v_2}{\partial y_s} \frac{\partial \Delta v_1}{\partial y_r} \right) \frac{\partial \zeta_r}{\partial y_s} dy \\
&+ \int_{\partial\tilde{\phi}(\Omega)} \left(\frac{\partial v_1}{\partial \nu} \Delta v_2 + \frac{\partial v_2}{\partial \nu} \Delta v_1 \right) \mathrm{div}\zeta d\sigma \\
&- \int_{\tilde{\phi}(\Omega)} \Delta v_1 \Delta v_2 \mathrm{div}\zeta dy - \int_{\tilde{\phi}(\Omega)} (\nabla v_1 \cdot \nabla \Delta v_2 + \nabla v_2 \cdot \nabla \Delta v_1) \mathrm{div}\zeta dy \\
&- \int_{\partial\tilde{\phi}(\Omega)} \left(\frac{\partial v_1}{\partial y_s} \Delta v_2 + \frac{\partial v_2}{\partial y_s} \Delta v_1 \right) \frac{\partial \zeta_s}{\partial \nu} d\sigma \\
&+ \int_{\tilde{\phi}(\Omega)} \left(\frac{\partial v_1}{\partial y_s} \frac{\partial \Delta v_2}{\partial y_i} + \frac{\partial v_2}{\partial y_s} \frac{\partial \Delta v_1}{\partial y_i} \right) \frac{\partial \zeta_s}{\partial y_i} dy. \quad (25)
\end{aligned}
$$

The last summand in the right-hand side of (25) equals

$$
\begin{aligned}
&\int_{\partial\tilde{\phi}(\Omega)} \left(\frac{\partial \Delta v_1}{\partial \nu} \nabla v_2 + \frac{\partial \Delta v_2}{\partial \nu} \nabla v_1 \right) \cdot \zeta d\sigma - \int_{\tilde{\phi}(\Omega)} \left(\Delta^2 v_1 \nabla v_2 + \Delta^2 v_2 \nabla v_1 \right) \cdot \zeta dy \\
&\quad - \int_{\tilde{\phi}(\Omega)} \left(\frac{\partial^2 v_1}{\partial y_i \partial y_s} \frac{\partial \Delta v_2}{\partial y_i} + \frac{\partial^2 v_2}{\partial y_i \partial y_s} \frac{\partial \Delta v_1}{\partial y_i} \right) \zeta_s dy \\
&= \int_{\partial\tilde{\phi}(\Omega)} \left(\frac{\partial \Delta v_1}{\partial \nu} \nabla v_2 + \frac{\partial \Delta v_2}{\partial \nu} \nabla v_1 \right) \cdot \zeta d\sigma - \int_{\tilde{\phi}(\Omega)} \left(\Delta^2 v_1 \nabla v_2 + \Delta^2 v_2 \nabla v_1 \right) \cdot \zeta dy \\
&\quad - \int_{\partial\tilde{\phi}(\Omega)} (\nabla v_1 \cdot \nabla \Delta v_2 + \nabla v_2 \cdot \nabla \Delta v_1) \zeta \cdot \nu d\sigma \\
&\quad + \int_{\tilde{\phi}(\Omega)} (\nabla v_1 \cdot \nabla \Delta v_2 + \nabla v_2 \cdot \nabla \Delta v_1) \mathrm{div}\zeta dy \\
&\quad + \int_{\tilde{\phi}(\Omega)} \left(\frac{\partial v_1}{\partial y_i} \frac{\partial^2 \Delta v_2}{\partial y_i \partial y_s} + \frac{\partial v_2}{\partial y_i} \frac{\partial^2 \Delta v_1}{\partial y_i \partial y_s} \right) \zeta_s dy,
\end{aligned}
$$

while the second one equals

$$
\begin{aligned}
&\int_{\partial\tilde{\phi}(\Omega)} \left(\frac{\partial v_1}{\partial \nu} \nabla \Delta v_2 + \frac{\partial v_2}{\partial \nu} \nabla \Delta v_1 \right) \cdot \zeta d\sigma \\
&\quad - \int_{\tilde{\phi}(\Omega)} \left(\frac{\partial v_1}{\partial y_s} \frac{\partial^2 \Delta v_2}{\partial y_r \partial y_s} + \frac{\partial v_2}{\partial y_s} \frac{\partial^2 \Delta v_1}{\partial y_r \partial y_s} \right) \zeta_r dy \\
&\quad - \int_{\partial\tilde{\phi}(\Omega)} \Delta v_1 \Delta v_2 \zeta \cdot \nu d\sigma + \int_{\tilde{\phi}(\Omega)} \Delta v_1 \Delta v_2 \mathrm{div}\zeta dy.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
d|_{\phi=\tilde{\phi}} B_\phi[\psi][u_1][u_2] = &- \int_{\partial\tilde{\phi}(\Omega)} \left(\frac{\partial v_1}{\partial y_s} \Delta v_2 + \frac{\partial v_2}{\partial y_s} \Delta v_1 \right) \frac{\partial \zeta_r}{\partial y_s} \nu_r d\sigma \\
&+ \int_{\partial\tilde{\phi}(\Omega)} \left(\frac{\partial v_1}{\partial \nu} \nabla \Delta v_2 + \frac{\partial v_2}{\partial \nu} \nabla \Delta v_1 \right) \cdot \zeta d\sigma \\
&+ \int_{\partial\tilde{\phi}(\Omega)} \left(\frac{\partial v_1}{\partial \nu} \Delta v_2 + \frac{\partial v_2}{\partial \nu} \Delta v_1 \right) \operatorname{div} \zeta d\sigma \\
- \int_{\partial\tilde{\phi}(\Omega)} \Delta v_1 \Delta v_2 \zeta \cdot \nu d\sigma &- \int_{\partial\tilde{\phi}(\Omega)} \left(\frac{\partial v_1}{\partial y_s} \Delta v_2 + \frac{\partial v_2}{\partial y_s} \Delta v_1 \right) \frac{\partial \zeta_s}{\partial \nu} d\sigma \\
&+ \int_{\partial\tilde{\phi}(\Omega)} \left(\frac{\partial \Delta v_1}{\partial \nu} \nabla v_2 + \frac{\partial \Delta v_2}{\partial \nu} \nabla v_1 \right) \cdot \zeta d\sigma \\
&- \int_{\tilde{\phi}(\Omega)} \left(\Delta^2 v_1 \nabla v_2 + \Delta^2 v_2 \nabla v_1 \right) \cdot \zeta dy \\
&- \int_{\partial\tilde{\phi}(\Omega)} \left(\nabla v_1 \cdot \nabla \Delta v_2 + \nabla v_2 \cdot \nabla \Delta v_1 \right) \zeta \cdot \nu d\sigma.
\end{aligned} \tag{26}
$$

The first summand in (26) equals

$$
\begin{aligned}
&- \int_{\partial\tilde{\phi}(\Omega)} \left(\frac{\partial v_1}{\partial \nu} \Delta v_2 + \frac{\partial v_2}{\partial \nu} \Delta v_1 \right) \frac{\partial \zeta_r}{\partial \nu} \nu_r d\sigma \\
&+ \int_{\partial\tilde{\phi}(\Omega)} \left(\nabla_{\partial\tilde{\phi}(\Omega)} v_1 \cdot \nabla_{\partial\tilde{\phi}(\Omega)} \Delta v_2 + \nabla_{\partial\tilde{\phi}(\Omega)} v_2 \cdot \nabla_{\partial\tilde{\phi}(\Omega)} \Delta v_1 \right) \zeta \cdot \nu d\sigma \\
&+ \int_{\partial\tilde{\phi}(\Omega)} \left(\Delta v_1 \Delta_{\partial\tilde{\phi}(\Omega)} v_2 + \Delta v_2 \Delta_{\partial\tilde{\phi}(\Omega)} v_1 \right) \zeta \cdot \nu d\sigma \\
&+ \int_{\partial\tilde{\phi}(\Omega)} \left(\Delta v_1 \nabla_{\partial\tilde{\phi}(\Omega)} v_2 + \Delta v_1 \nabla_{\partial\tilde{\phi}(\Omega)} v_1 \right) \cdot (\nabla_{\partial\tilde{\phi}(\Omega)} \nu_r) \zeta_r d\sigma,
\end{aligned}
$$

while the second one equals

$$
\begin{aligned}
&\int_{\partial\tilde{\phi}(\Omega)} \left(\frac{\partial v_1}{\partial \nu} \frac{\partial \Delta v_2}{\partial \nu} + \frac{\partial v_2}{\partial \nu} \frac{\partial \Delta v_1}{\partial \nu} \right) \zeta \cdot \nu d\sigma \\
&+ \int_{\partial\tilde{\phi}(\Omega)} K \left(\frac{\partial v_1}{\partial \nu} \Delta v_2 + \frac{\partial v_2}{\partial \nu} \Delta v_1 \right) \zeta \cdot \nu d\sigma \\
&- \int_{\partial\tilde{\phi}(\Omega)} \left(\frac{\partial v_1}{\partial \nu} \Delta v_2 + \frac{\partial v_2}{\partial \nu} \Delta v_1 \right) \operatorname{div}_{\partial\tilde{\phi}(\Omega)} \zeta d\sigma \\
&- \int_{\partial\tilde{\phi}(\Omega)} \left(\Delta v_1 \nabla_{\partial\tilde{\phi}(\Omega)} \frac{\partial v_2}{\partial \nu} + \Delta v_2 \nabla_{\partial\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial \nu} \right) \cdot \zeta d\sigma,
\end{aligned}
$$

where K denotes the mean curvature of $\partial\tilde{\phi}(\Omega)$. Therefore the first three terms in the right-hand side of (26) equal

$$\int_{\partial\tilde{\phi}(\Omega)} (\nabla v_1 \cdot \nabla \Delta v_2 + \nabla v_2 \cdot \nabla \Delta v_1)\, \zeta \cdot \nu d\sigma$$
$$- 2\int_{\partial\tilde{\phi}(\Omega)} \Delta v_1 \Delta v_2 \zeta \cdot \nu d\sigma - \int_{\partial\tilde{\phi}(\Omega)} \left(\Delta v_1 \frac{\partial^2 v_2}{\partial \nu^2} + \Delta v_2 \frac{\partial^2 v_1}{\partial \nu^2} \right) \zeta \cdot \nu d\sigma$$
$$+ \int_{\partial\tilde{\phi}(\Omega)} \left(\Delta v_1 \nabla_{\partial\tilde{\phi}(\Omega)} v_2 + \Delta v_1 \nabla_{\partial\tilde{\phi}(\Omega)} v_1 \right) \cdot (\nabla_{\partial\tilde{\phi}(\Omega)} \nu_r) \zeta_r d\sigma$$
$$- \int_{\partial\tilde{\phi}(\Omega)} \left(\Delta v_1 \nabla_{\partial\tilde{\phi}(\Omega)} \frac{\partial v_2}{\partial \nu} + \Delta v_2 \nabla_{\partial\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial \nu} \right) \cdot \zeta d\sigma. \quad (27)$$

Now note that summing the third and the fifth terms in (27) we get

$$- \int_{\partial\tilde{\phi}(\Omega)} \left(\Delta v_1 \nabla \frac{\partial v_2}{\partial \nu} + \Delta v_2 \nabla \frac{\partial v_1}{\partial \nu} \right) \cdot \zeta d\sigma$$
$$= - \int_{\partial\tilde{\phi}(\Omega)} \left(\Delta v_1 \frac{\partial}{\partial \nu} \nabla v_2 + \Delta v_2 \frac{\partial}{\partial \nu} \nabla v_1 \right) \cdot \zeta d\sigma$$
$$- \int_{\partial\tilde{\phi}(\Omega)} \left(\Delta v_1 \nabla_{\partial\tilde{\phi}(\Omega)} v_2 + \Delta v_2 \nabla_{\partial\tilde{\phi}(\Omega)} v_1 \right) \cdot (\nabla_{\partial\tilde{\phi}(\Omega)} \nu_r) \zeta_r d\sigma. \quad (28)$$

Using (26), (27) and (28), we finally get formula (19).

Lemma 8 *Let Ω be a bounded domain in $\mathbb{R}^N$ of class C^1, and let $\tilde{\phi} \in \mathcal{A}_\Omega$ be such that $\tilde{\phi}(\Omega)$ is of class C^2. Let $u_1, u_2 \in H^2(\Omega)$ be such that $v_1 = u_1 \circ \tilde{\phi}^{-1}$, $v_2 = u_2 \circ \tilde{\phi}^{-1} \in H^4(\tilde{\phi}(\Omega))$. Then*

$$d|_{\phi=\tilde{\phi}} J_{1,\phi}[\psi][u_1][u_2] = \int_{\tilde{\phi}(\Omega)} v_1 v_2 \mathrm{div}\zeta dy, \quad (29)$$

$$d|_{\phi=\tilde{\phi}} J_{2,\phi}[\psi][u_1][u_2] = \int_{\partial\tilde{\phi}(\Omega)} \left(K \frac{\partial v_1}{\partial \nu} \frac{\partial v_2}{\partial \nu} + \frac{\partial}{\partial \nu} (\frac{\partial v_1}{\partial \nu} \frac{\partial v_2}{\partial \nu}) \right) \zeta \cdot \nu d\sigma$$
$$- \int_{\partial\tilde{\phi}(\Omega)} \nabla (\frac{\partial v_1}{\partial \nu} \frac{\partial v_2}{\partial \nu}) \cdot \mu d\sigma - 2\int_{\partial\tilde{\phi}(\Omega)} \nabla (\frac{\partial v_1}{\partial \nu} \frac{\partial v_2}{\partial \nu}) \frac{\partial \zeta}{\partial \nu} \cdot \nu d\sigma, \quad (30)$$

and

$$d|_{\phi=\tilde{\phi}} J_{3,\phi}[\psi][u_1][u_2] = \int_{\partial\tilde{\phi}(\Omega)} \left(K v_1 v_2 + \frac{\partial}{\partial \nu} (v_1 v_2) \right) \zeta \cdot \nu d\sigma - \int_{\partial\tilde{\phi}(\Omega)} \nabla (v_1 v_2) \cdot \mu d\sigma, \quad (31)$$

for all $\psi \in C^2(\overline{\Omega}; \mathbb{R}^N)$, where $\zeta = \psi \circ \tilde{\phi}^{-1}$ and K is the mean curvature on $\partial\tilde{\phi}(\Omega)$.

Proof Formula (29) is immediate from (22), while for formula (31) we refer to [34, Lemma 3.3].

Regarding formula (30), we start observing that

$$J_2[u_1][u_2] = \int_{\partial\Omega} \nabla u_1 \cdot \nabla u_2 d\sigma,$$

since $\nabla u = \frac{\partial u}{\partial \nu}\nu$ for any $u \in H^2(\Omega) \cap H^1_0(\Omega)$. Hence

$$J_{2,\phi}[u_1][u_2] = \int_{\partial\Omega} \left(\nabla u_1 \cdot \nabla(\phi^{-1})\right) \cdot \left(\nabla u_2 \cdot \nabla(\phi^{-1})\right) |\nu \cdot \nabla(\phi^{-1})|| \det \nabla\phi| d\sigma,$$

from which we get

$$\begin{aligned} d|_{\phi=\tilde{\phi}} J_{2,\phi}[\psi][u_1][u_2] &= -\int_{\partial\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial y_r}\left(\frac{\partial \zeta_r}{\partial y_s} + \frac{\partial \zeta_s}{\partial y_r}\right)\frac{\partial v_2}{\partial y_s} d\sigma + \int_{\partial\tilde{\phi}(\Omega)} \nabla v_1 \cdot \nabla v_2 \mathrm{div}\, \zeta d\sigma \\ &- \int_{\partial\tilde{\phi}(\Omega)} \nabla v_1 \cdot \nabla v_2 \frac{\partial \zeta}{\partial \nu} \cdot \nu d\sigma = -2\int_{\partial\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial \nu}\frac{\partial v_2}{\partial \nu}\frac{\partial \zeta}{\partial \nu} \cdot \nu d\sigma + \int_{\partial\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial \nu}\frac{\partial v_2}{\partial \nu} \mathrm{div}_{\partial\tilde{\phi}(\Omega)} \zeta d\sigma. \end{aligned} \tag{32}$$

Using the Tangential Green's Formula (cf. [25, Sect. 8.5]) we have

$$\begin{aligned} \int_{\partial\tilde{\phi}(\Omega)} \frac{\partial v_1}{\partial \nu}\frac{\partial v_2}{\partial \nu} \mathrm{div}_{\partial\tilde{\phi}(\Omega)} \zeta d\sigma = \int_{\partial\tilde{\phi}(\Omega)} K \frac{\partial v_1}{\partial \nu}\frac{\partial v_2}{\partial \nu} \zeta \cdot \nu d\sigma \\ - \int_{\partial\tilde{\phi}(\Omega)} \nabla_{\partial\tilde{\phi}(\Omega)} \left(\frac{\partial v_1}{\partial \nu}\frac{\partial v_2}{\partial \nu}\right) \cdot \zeta d\sigma, \end{aligned} \tag{33}$$

where $\nabla_{\partial\tilde{\phi}(\Omega)}$ is the tangential gradient. Combining (32) and (33) we obtain (30).

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A Remark on an Overdetermined Problem in Riemannian Geometry

Giulio Ciraolo and Luigi Vezzoni

Abstract Let (M, g) be a Riemannian manifold with a distinguished point O and assume that the geodesic distance d from O is an isoparametric function. Let $\Omega \subset M$ be a bounded domain, with $O \in \Omega$, and consider the problem $\Delta_p u = -1$ in Ω with $u = 0$ on $\partial\Omega$, where Δ_p is the p-Laplacian of g. We prove that if the normal derivative $\partial_\nu u$ of u along the boundary of Ω is a function of d satisfying suitable conditions, then Ω must be a geodesic ball. In particular, our result applies to open balls of $\mathbb{R}^n$ equipped with a rotationally symmetric metric of the form $g = dt^2 + \rho^2(t)\, g_S$, where g_S is the standard metric of the sphere.

Keywords Overdetermined PDE · Comparison principle · Riemannian Geometry · Rotationally symmetric spaces · Isoparametric functions

1 Introduction

In this note we consider an overdetermined problem in Riemannian Geometry. An overdetermined problem usually consists in a partial differential equation with “too many” prescribed boundary conditions. Typically, these kinds of problems are not well-posed and the existence of a solution imposes strong restrictions on the shape of the domain where the problem is defined. Consequently, the research in overdetermined problems usually consists in classifying all the possible domains where the problem is well-posed. A central result in this context was obtained by Serrin in his seminal paper [20]. The today known *Serrin's overdetermined problem* consists in the torsion problem

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$$\begin{cases} \Delta u = -1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, together with a constant Neumann condition at the boundary $\partial\Omega$:

$$u_\nu = const \quad \text{on } \partial\Omega. \tag{2}$$

In [20] Serrin proved that (1) and (2) admits a solution if and only if Ω is a ball whose radius, in view of the divergence theorem, is determined by the constant in condition (2); moreover the solution u is radially symmetric. Other proofs and generalizations of Serrin's theorem can be found for instance in [2, 3, 24].

A related problem was considered by Greco in [12], where it is investigated the following question:

Given a point $O \in \mathbb{R}^n$, $n \geq 2$, which overdetermined conditions force Ω to be a ball about O?

The main result in [12], in its simplest form, is the following:

Let Ω be a bounded domain in $\mathbb{R}^n$ containing the fixed point O and assume that there exists a solution u of (1) in Ω satisfying $\partial_\nu u = c|x|$ at every $x \in \partial\Omega$ for some constant c. Then Ω is a ball centered at O.

The assumption $O \in \Omega$ in the statement above cannot be dropped in general, as pointed out in some examples in [12]. For some related results we refer to [1, 13].

The problem posed by Greco [12] still makes sense in the Riemannian setting, where $\mathbb{R}^n$ is replaced by a smooth n-dimensional manifold M, the Euclidean metric by a Riemannian metric g on M and Euclidean balls by geodesic balls about a fixed point O. In the present paper, we generalize some of the results in [12] to the Riemannian setting by assuming that the distance function from the fixed point O is isoparametric, i.e. it is of class C^2 in $M \setminus \{O\}$ and there exists a continuous function η such that $\Delta d = \eta(d)$ in $M \setminus \{O\}$ (see [23]). A related result can be found in [6].

We shall use the following notation: B_r denotes the geodesic ball of radius r centered at O; $|B_r|$ and $|\partial B_r|$ are the volume and the perimeter of B_r, respectively,

$$\Phi(r) := \left(\frac{|B_r|}{|\partial B_r|} \right)^{\frac{1}{p-1}}, \tag{3}$$

and Δ_p is the p-Laplacian operator. Notice that $\Phi(r)$ is exactly the value of the interior normal derivative at the boundary of the solution to the problem

$$\begin{cases} \Delta_p v = -1 & \text{in } B_r, \\ v = 0 & \text{on } \partial B_r, \end{cases} \tag{4}$$

which is constant if one assumes that the distance function is isoparametric (see Lemma 1).

Our main result is the following.

Theorem 1 *Let (M, g) be a Riemannian manifold and assume that the distance function d from a fixed point O is isoparametric. Let $\Omega \subset M$ be a bounded domain with $O \in \Omega$ and let Φ be given by (3). Assume that there exists a solution u to*

$$\begin{cases} \Delta_p u = -1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \partial_\nu u = f \circ d & \text{on } \partial\Omega, \end{cases} \tag{5}$$

with $p > 1$, where f satisfies one of the following two conditions:

(i) the function $f(t)/\Phi(t)$ is monotone nondecreasing in $(0, \operatorname{diam}\Omega)$;
(ii) there exists $R > 0$ such that $f(R)/\Phi(R) = 1$, $f(r)/\Phi(r) > 1$ for $r > R$, and $f(r)/\Phi(r) < 1$ for $r < R$.

Then Ω is a geodesic ball centered at O. Moreover, in case (ii) we have that $\Omega = B_R$.

Our proof relies on comparison principles. More specifically, our approach is based on the comparison of the solution in Ω with the solution in a ball about O.

As far as we know, few overdetermined problems have been studied in a Riemannian setting (see [4, 6–9]), where classical tools for proving symmetry like the method of moving planes can not be employed (at least in a standard way). Our approach is close to the one in [6] where, by using comparison principles, the authors prove that if there exists a lower bounded nonconstant function u which is p-harmonic $(1 < p < n)$ in a punctured domain and such that u and u_ν are constant on $\partial\Omega$, then u is radial and $\partial\Omega$ is a geodesic sphere.

2 The Euclidean Case

In this preliminary section we consider the basic case when the Riemannian manifold is the Euclidean space, O is the origin of $\mathbb{R}^n$ and the p-Laplacian is the usual Laplacian operator (i.e. $p = 2$).

Let Ω be a bounded domain in the Euclidean space containing the origin O and consider the overdetermined problem

$$\begin{cases} \Delta u = -1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \partial_\nu u(x) = f(|x|) & \text{on } \partial\Omega, \end{cases} \tag{6}$$

where ν denotes the normal inward to $\partial\Omega$ and $f : (0, +\infty) \to (0, +\infty)$ is a continuous function satisfying certain conditions which we specify later. Since both Dirichlet and Neumann boundary conditions are imposed on $\partial\Omega$, problem (6) is not well-posed unless f and Ω satisfy some compatibility conditions. Our goal is to consider conditions on f which imply that Ω is a ball centered at the origin.

The scheme that we have in mind is the following. Let B_{r_0} be the largest ball contained in Ω centered at the origin and let B_{r_1} be the smallest ball containing Ω centered at the origin. If we denote by v^r the solution to

$$\begin{cases} \Delta v = -1 & \text{in } B_r, \\ v = 0 & \text{on } \partial B_r, \end{cases}$$

then we have that

$$v^r(x) = \frac{r^2 - |x|^2}{2n},$$

and hence $\partial_\nu v^r$ is constant on ∂B_r and is given by

$$\partial_\nu v^r = \frac{|B_r|}{|\partial B_r|} = \frac{r}{n} \quad \text{on } \partial B_r.$$

By comparison principle, we have that $v^{r_0} \leq u$ in B_{r_0} and $v^{r_1} \geq u$ in Ω and, by looking at the normal derivatives at the tangency points between Ω and B_{r_0} and B_{r_1}, we readily obtain that

$$\frac{r_0}{n} \leq f(r_0) \quad \text{and} \quad f(r_1) \leq \frac{r_1}{n}. \tag{7}$$

Now, let

$$F(t) = \frac{nf(t)}{t}.$$

We have the following results. The first one is essentially contained in [12].

- *Case 1. If $F(t)$ is monotone nondecreasing then Ω is a ball centered at the origin.*
  Indeed, if $F(r_0) < F(r_1)$ then (7) immediately implies that $r_0 = r_1$ and then Ω is a ball. If $F(r_0) = F(r_1)$, from (7) we have that

  $$\partial_\nu u(x) = \frac{|x|}{n}$$

  for every $x \in \partial\Omega$. Hence, the function $w = u - v^{r_0}$ satisfies

  $$\begin{cases} \Delta w = 0 & \text{in } B_{r_0}, \\ w \geq 0 & \text{in } \partial B_{r_0}, \\ \partial_\nu w(p) = 0, \end{cases}$$

  where $p \in \partial\Omega$ is a tangency point between B_{r_0} and Ω. From Hopf's boundary point Lemma we obtain that $w \equiv 0$ in $\overline{B}_{r_0}$ and hence that Ω is a ball.

- *Case 2. If there exists R such that*

$$F(R) = 1, \quad F(r) > 1 \text{ if } r > R, \quad F(R) < 1 \text{ if } r < R, \tag{8}$$

then Ω is the ball of radius R centered at the origin.

This case is simpler then the previous one. Indeed, from (7) we have that $F(r_0) \geq 1$ and $F(r_1) \leq 1$, which imply that $r_0 \geq R$ and $r_1 \leq R$. Since $r_0 \leq r_1$, we conclude.

3 Riemannian Setting

In this section we study the overdetermined problem (6) in a Riemannian manifold.

Let (M, g) be an dimensional Riemannian manifold with a fixed point O. We recall that given a C^2 map $u\colon M \to \mathbb{R}$ its Laplacian is defined by

$$\Delta u = \operatorname{div}(Du),$$

where Du is the gradient of u (which in the Riemannian setting is defined as the g-dual of the differential of u) and div is the divergence. Δu writes in local coordinates $(x^1, \dots, x^n)$ as

$$\Delta u = \frac{1}{\sqrt{|g|}} \partial_{x^j} \left(g^{jk} \sqrt{|g|} \partial_{x^k} u \right).$$

We further recall that the p-Laplacian operator on a Riemannian manifold is defined by

$$\Delta_p u = \operatorname{div}(|Du|^{p-2} Du), \quad p > 1.$$

In this context a function $u\colon M \to \mathbb{R}$ is called *radial* if can be written as $u = f \circ d$ for some real function f. Radial functions are usually studied in polar coordinates. Here we recall that if $0 < \delta$ is less than the injectivity radius of M at O, then $\exp_O$ induces polar coordinates $(r, q) \in (0, \delta) \times S^{n-1}$ on B_δ induced by the usual polar coordinates on the tangent space to M at O. If $u = f \circ d$ is a radial function, then its Laplacian in polar coordinates takes the following expression

$$\Delta u = \partial^2_{rr} u + \left(\frac{n-1}{r} + \frac{\partial_r \det(d \exp_O)}{\det(d \exp_O)} \right) \partial_r u$$

(see e.g. [19]). In particular, if $u = d$ we get

$$\Delta d = \frac{n-1}{r} + \frac{\partial_r \det(d \exp_O)}{\det(d \exp_O)}.$$

We say that the distance function from the fixed point O is *isoparametric* if it is of class C^2 in $M \setminus \{O\}$ and there exists a continuous function η such that $\Delta d = \eta(d)$ in $M \setminus \{O\}$ (see [23]). We notice that assuming d isoparametric in the geodesic ball B_δ is equivalent to assume that the quantity

$$\frac{\partial_r \det(d \exp_O)}{\det(d \exp_O)}$$

defines a function of the distance itself. Moreover, we remark that

$$\lim_{d \to 0} \Delta d = +\infty. \tag{9}$$

The following lemma gives the explicit behaviour of the solution to (4) in a geodesic ball. This will be used in the proof of Theorem 1 when we will compare the solution to (5) to solutions to (4) in suitable geodesic balls.

Lemma 1 *Let (M, g) be a Riemannian manifold and assume that the distance d from a point $O \in M$ is isoparametric. Let v^r be the solution to*

$$\begin{cases} \Delta_p v^r = -1 & \text{in } B_r, \\ v^r = 0 & \text{on } \partial B_r. \end{cases} \tag{10}$$

Then v^r is a function which depends only on the distance from the origin and is given by

$$v^r(x) = \int_{|x|}^{r} e^{\frac{1}{p-1}\int_t^r \eta(s)ds} \left[\frac{|B_r|}{|\partial B_r|} - \int_t^r e^{-\int_\tau^r \eta(s)ds} d\tau \right]^{\frac{1}{p-1}} dt. \tag{11}$$

In particular, $\partial_\nu v^r$ is constant on ∂B_r and is given by

$$\partial_\nu v^r = \left(\frac{|B_r|}{|\partial B_r|} \right)^{\frac{1}{p-1}}. \tag{12}$$

Proof We look for a solution of the form $v^r(x) = V(d(x))$, where V is a decreasing function. Since d is isoparametric, with $\Delta d = \eta(d)$, and the gradient of the distance function has unit norm in $M \setminus \{O\}$, then we have that V satisfies

$$|V'(t)|^{p-2} \left[(p-1)V''(t) + \eta(t)V'(t) \right] = -1,$$

and hence

$$\frac{d}{dt}|V'(t)|^{p-1} + \eta(t)|V'(t)|^{p-1} = 1.$$

Therefore

$$|V'(t)|^{p-1} = e^{\int_t^r \eta(s)ds}\left[|V'(r)|^{p-1} - \int_t^r e^{-\int_\tau^r \eta(s)ds}d\tau\right].$$

Since we are looking for a solution in B_r which depends only on d, from (10) and the divergence theorem we have that $|V'(r)|^{p-1} = |B_r|/|\partial B_r|$ and hence

$$V'(t) = -e^{\frac{1}{p-1}\int_t^r \eta(s)ds}\left[\frac{|B_r|}{|\partial B_r|} - \int_t^r e^{-\int_\tau^r \eta(s)ds}d\tau\right]^{\frac{1}{p-1}},$$

and the expression for v^r follows. From (9) we have that $V'(0) = 0$ and then $v^r \in C^{1,\alpha}$ in B_r and satisfies (10).

We are ready to prove Theorem 1.

Proof of Theorem 1. We firstly give some remarks on the regularity of the solution. From elliptic regularity theory we have that $u \in C^{1,\alpha}(\Omega)$ (see [5, 16, 21]) and $u \in C^{2,\alpha}$ in a neighborhood of any point where $|\nabla u| \neq 0$ (see [11]). About the regularity at the boundary, we notice that we have by assumption that $|\nabla u| \neq 0$ on $\partial\Omega$ and hence $|\nabla u| \neq 0$ in a tubular neighborhood of $\partial\Omega$. From [10, 22] we obtain that $\partial\Omega$ is of class C^2 and from [17] we have that $u \in C^{1,\alpha}(\overline{\Omega})$.

Now, we observe that $u > 0$ in Ω. Indeed, the boundary condition in (5) implies that $u > 0$ in a neighborhood of $\partial\Omega$. If $u = 0$ at some interior point of Ω, then the strong maximum principle (see [18]) implies that $u \equiv 0$ in Ω, which gives a contradiction. Hence, $u > 0$ in Ω.

We define r_0 and r_1 as follows

$$r_0 = \sup\{r > 0 : \; B_r \subset \Omega\} \quad \text{and} \quad r_1 = \inf\{r > 0 : \; \Omega \subset B_r\},$$

and we denote by x_i a tangency points between ∂B_{r_i} and $\partial\Omega$, for $i = 0, 1$.

As in the Euclidean case in Sect. 2, the proof is based on the comparison between u and the solutions of the p-torsion problem in B_{r_0} and B_{r_1}. Since $B_{r_0} \subseteq \Omega \subseteq B_{r_1}$, by the weak comparison principle (see [14, 18]) we have that $v^{r_0} \leq u$ in B_{r_0} and $u \leq v^{r_1}$ in Ω, where v^{r_0} and v^{r_1} are given by (11).

Since x_i is a tangency point between ∂B_{r_i} and $\partial\Omega$, the inward normal vectors to ∂B_{r_i} and to $\partial\Omega$ at x_i agree and $d(x_i) = r_i$ for $i = 0, 1$. Moreover, $v^{r_i}(x_i) = u(x_i) = 0$, and by comparison we have that

$$\Phi(r_0) = \partial_\nu v^{r_0}(x_0) \leq \partial_\nu u(x_0) \quad \text{and} \quad \partial_\nu u(x_1) \leq \partial_\nu v^{r_1}(x_1) = \Phi(r_1),$$

where Φ and $\partial_\nu v^r$ are given by (3) and (12), respectively, and hence

$$1 \leq \frac{f(r_0)}{\Phi(r_0)} \quad \text{and} \quad \frac{f(r_1)}{\Phi(r_1)} \leq 1. \tag{13}$$

If we assume that case (ii) in the assertion of the theorem occurs, then (13) implies that $r_1 \le R \le r_0$, and hence $r_0 = r_1 = R$.

In case (i), we have that (13) implies

$$\frac{f(r)}{\Phi(r)} = 1 \quad \text{for every } r_0 \le r \le r_1,$$

and hence

$$\partial_\nu u(x) = \Phi(|x|) \quad \text{for every } x \in \partial\Omega.$$

In particular, we have that

$$\partial_\nu u(x_0) = \partial_\nu v^{r_0}(x_0). \tag{14}$$

Since $\partial_\nu u(x_0) > 0$, there exists $\rho > 0$ such that $|\nabla u| \neq 0$ in $B_\rho(x_0) \cap \Omega$. By choosing $\rho < r_0$ we also have that $|\nabla v^{r_0}| \neq 0$ in $W := B_\rho(x_0) \cap B_{r_0}$. By standard elliptic regularity theory, we have that u and v^{r_0} are classical solutions of $\Delta_p u = -1$ in W and the difference $u - v^{r_0}$ is nonnegative and satisfies a linear uniformly elliptic equation in W:

$$\begin{cases} L(u - v^{r_0}) = 0 \quad \text{and} \quad u - v^{r_0} \ge 0 \quad \text{in } W, \\ \partial_\nu(u - v^{r_0})(x_0) = 0. \end{cases}$$

By Hopf's Lemma (see [15]) we have that $u = v^{r_0}$ in W. In particular, we obtain that $u = 0$ in $\partial B_\rho(x_0) \cap B_{r_0}$, which implies that ∂B_{r_0} and $\partial\Omega$ coincide in a open neighborhood of x_0. In particular, we have proved that the set of tangency points between $\partial\Omega$ and ∂B_{r_0} is open. Moreover, by construction we have that the set of tangency points between $\partial\Omega$ and ∂B_{r_0} is given by $\partial\Omega \cap \partial B_{r_0}$, which is clearly a closed set. Hence, we have proved that $\partial\Omega \cap \partial B_{r_0}$ is both closed and open, and hence we have that $\partial\Omega = \partial B_{r_0}$, i.e. Ω is a ball.

4 Examples

Theorem 1 can be applied to open balls in $\mathbb{R}^n$ equipped with a rotationally symmetric metric. More precisely, let $\bar r \in \mathbb{R} \cup \infty$ be fixed and consider the open ball $B_{\bar r}$ centered at the origin O of $\mathbb{R}^n$ of radius $\bar r$ equipped with a Riemannian metric g which in polar coordinates reads as

$$g = dt^2 + \rho^2 g_S,$$

where $\rho\colon [0, \bar r) \to \mathbb{R}$ is as smooth function such that

$$\rho(0) = 0, \quad \rho(t) > 0,$$

for every $t \in [0, \bar{r})$ and g_S is the standard metric on the unitary $(n-1)$-dimensional sphere S^{n-1}. In this setting the geodesic distance d of a generic point $p \in B_{\bar{r}}$ from O is given by the Euclidean norm of p, since $t \mapsto tp$ is a minimal geodesic connecting the origin to the point p for $t \in [0, \bar{r})$. Moreover, if $u : B_{\bar{r}} \to \mathbb{R}$ is a smooth radial function, then its Laplacian with respect to g takes the following expression

$$\Delta u = \partial^2_{tt} u + (n-1)\frac{\rho'}{\rho}\partial_t u,$$

and consequently

$$\Delta d(x) = (n-1)\frac{\rho'(d(x))}{\rho(d(x))} =: \eta(d(x)),$$

which shows that d is isoparametric. Notice that in this setting the statement of Theorem 1 implies that Ω is an Euclidean ball, since geodesic balls centered at O are exactly the Euclidean balls.

Rotationally symmetric spaces include space form models as particular cases: the Euclidean space, the Hyperbolic space and the unitary sphere, where the function ρ takes the following expression:

- $\rho(t) = t$ in the Euclidean case;
- $\rho(t) = \sinh t$ in the Hyperbolic case;
- $\rho(t) = \sin t$ in the spheric case.

Note that the map v^r in Lemma 1 in the Euclidean case takes the following expression

$$v^r(x) = \left(\frac{p-1}{p}\right)\frac{r^{\frac{p}{p-1}} - |x|^{\frac{p}{p-1}}}{n^{\frac{1}{p-1}}}.$$

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A Note on the Scale Invariant Structure of Critical Hardy Inequalities

Norisuke Ioku and Michinori Ishiwata

Abstract We investigate the scale-invariant structure of the critical Hardy inequality in a unit ball under the power-type scaling. First we consider the remainder term of the critical Hardy inequality which is characterized by the ratio with or the distance from the "virtual minimizer" for the associated variational problem. We also focus on the scale invariance property of the inequality under power-type scaling and investigate the iterated scaling structure of remainder terms. Finally, we give a relation between the usual scaling enjoyed by the classical Hardy inequality and the power-type scaling via the transformation introduced by Horiuchi and Kumlin. As a by-product, we give a relationship between the Moser sequences and the Talenti functions.

Keywords Hardy's inequality · Scale invariance · Remainder term · Talenti functions · Moser sequences

1 Introduction

Let $n \in \mathbb{N}$ with $n \geq 2$. The following classical Hardy inequality appears in the vast amount of fields in the mathematical analysis:

$$\left(\frac{n-p}{p}\right)^p \int_{\mathbb{R}^n} \frac{|u(x)|^p}{|x|^p} dx \leq \int_{\mathbb{R}^n} |\nabla u(x)|^p dx, \quad u \in W^{1,p}(\mathbb{R}^n), \tag{1}$$

where $1 \leq p < n$. It is well known that (1) and the Pólya-Szegö principle yield a continuous embedding $W^{1,p}(\mathbb{R}^n) \subset L^{p^*,p}(\mathbb{R}^n)$. Here $L^{p^*,p}(\mathbb{R}^n)$ is the Lorentz space endowed with the norm

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$$\|u\|_{L^{p^*,p}(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} \left(|x|^{\frac{n}{p^*}} u^\sharp(x) \right)^p \frac{dx}{|x|^n} \right)^{\frac{1}{p}} = \left(\int_{\mathbb{R}^n} \frac{\left| u^\sharp(x) \right|^p}{|x|^p} dx \right)^{\frac{1}{p}},$$

where $u^\sharp$ is the Schwarz symmetrization of u and p^* is the Sobolev exponent defined by $p^* = \frac{np}{n-p}$. Note that the embedding $W^{1,p}(\mathbb{R}^n) \subset L^{p^*,p}(\mathbb{R}^n)$ also gives a well-known Sobolev embedding $W^{1,p}(\mathbb{R}^n) \subset L^{p^*}(\mathbb{R}^n)$ since $L^{p^*,p}(\mathbb{R}^n) \subset L^{p^*,p^*}(\mathbb{R}^n) = L^{p^*}(\mathbb{R}^n)$. Furthermore, the embedding $W^{1,p}(\mathbb{R}^n) \subset L^{p^*,p}(\mathbb{R}^n)$ is best possible in the class of rearrangement invariant spaces (see [14], see also [5, Chap. 2]), where a Banach function space X is said to be a rearrangement invariant space if $\|u\|_X = \|v\|_X$ whenever $u^\sharp = v^\sharp$. Typical examples of such spaces are Lebesgue spaces, Lorentz spaces, and Orlicz spaces.

Another important feature of (1) is the invariance under the standard scaling. Namely, for every $u \in W^{1,p}(\mathbb{R}^n)$, the standard scaling

$$u_\lambda(x) := \lambda^{\frac{n-p}{p}} u(\lambda x), \quad \lambda > 0 \tag{2}$$

does not change the both side of the inequality (1) for every $\lambda > 0$. Moreover, it is known that the constant $\left(\frac{n-p}{p}\right)^p$ in the left hand side is best possible, namely, there holds

$$\left(\frac{n-p}{p}\right)^p = \inf_{u \in W^{1,p}(\mathbb{R}^n)\setminus\{0\}} \frac{\int_{\mathbb{R}^n} |\nabla u(x)|^p dx}{\int_{\mathbb{R}^n} \frac{|u(x)|^p}{|x|^p} dx}, \tag{3}$$

and no $W^{1,p}(\mathbb{R}^n)$-function attain the best constant $\left(\frac{n-p}{p}\right)^p$, while a function $U(x) = |x|^{-\frac{n-p}{p}}$ is a "virtual minimizer" in the sense that though $U \notin W^{1,p}(\mathbb{R}^n)$, a sequence (u_n) with $u_n(x) = \big(\min\{U(x), U(1/n)\} - U(n)\big) \chi_{B_n(0)}(x)$, where $\chi_{B_n(0)}(\cdot)$ is the characteristic function on the ball centered at the origin with radius n, is a minimizing sequence for the Rayleigh quotient defined by (3).

This absence of extremal functions implies the possibility of the validity of more sharp inequality, particularly the existence of the other terms in the left hand side in (1). Indeed, in [10], Cianchi and Ferone give the following type of the Hardy inequality with a *remainder term*, namely, for $1 < p < n$, there exists a constant $C > 0$ satisfying

$$\left(\frac{n-p}{p}\right)^p \int_{\mathbb{R}^n} \frac{|u(x)|^p}{|x|^p} dx \left(1 + C \inf_{a \in \mathbb{R}} \frac{\sup_{x \in \mathbb{R}^n} \frac{(u - aU)^\sharp(x)}{U(x)}}{\|u\|_{L^{p^*,p}}} \right) \le \int_{\mathbb{R}^n} |\nabla u(x)|^p dx, \; u \in W^{1,p}(\mathbb{R}^n). \tag{4}$$

For more details on the remainder terms for the classical Hardy inequality, see e.g. [6, 10, 11, 17, 24] and references therein.

Recall that (1) requires the assumption $p < n$. The following version of Hardy's inequality is known for the case $p = n$:

$$\left(\frac{n-1}{n}\right)^n \int_{B_1} \frac{|u(x)|^n}{|x|^n \left(\log \frac{e}{|x|}\right)^n} dx \le \int_{B_1} |\nabla u(x)|^n dx, \quad u \in W_0^{1,n}(B_1), \tag{5}$$

where B_1 denotes the n-dimensional unit ball centered at the origin. There exists a large amount of literature on applications, generalizations, and improvements of (5) (see e.g. [1, 4, 10–16, 22] and references therein). The important feature of (5) is the monotonicity of the potential function $\frac{1}{|x|^n \left(\log \frac{e}{|x|}\right)^n}$ for $0 < |x| < 1$ and (5) is usually proved by the aid of the symmetrization argument which requires this monotonicity. The particular interest for (5) comes from the fact that, similarly to (1), (5) also yields a best-possible embeddings of Sobolev spaces in the framework of rearrangement invariant spaces. In this respect, (5) has been considered as a natural extension of (1) to the critical case $p = n$ and is called as a critical Hardy inequality. On the other hand, it seems that there is no scale-invariant structure for (5) and this is the main difference between (1) and (5).

For the case $p = n$, there is another version of the Hardy inequality slightly different from (5). In [20], Leray introduced the following inequality

$$\frac{1}{4} \int_{B_1} \frac{|u(x)|^2}{|x|^2 \left(\log \frac{1}{|x|}\right)^2} dx \le \int_{B_1} |\nabla u(x)|^2 dx, \quad u \in W_0^{1,2}(B_1),$$

where B_1 denotes the two-dimensional unit ball centered at the origin. By the same manner as in [20], one can also obtain the following inequality on the n-dimensional unit ball B_1:

$$\left(\frac{n-1}{n}\right)^n \int_{B_1} \frac{|u(x)|^n}{|x|^n \left(\log \frac{1}{|x|}\right)^n} dx \le \int_{B_1} |\nabla u(x)|^n dx, \quad u \in W_0^{1,n}(B_1). \tag{6}$$

The main difference between (6) and (5) is the monotonicity of the potential function, that is, the potential function $\frac{1}{|x|^n \left(\log \frac{1}{|x|}\right)^n}$ in (6) is non-monotone while that in (5) is monotone as is mentioned before. Moreover, there is a significant difference between (5) and (6) on the scaling property. Indeed, recently the authors pointed out in [19] that (6) has a scale invariance property under the following power-type scaling:

$$u_\lambda(x) := \lambda^{-\frac{n-1}{n}} u(|x|^{\lambda-1} x), \quad \lambda > 0, \tag{7}$$

while (5) is not invariant under (7). The scaling (7) for radial functions is introduced by Cassani, Ruf and Tarsi in [8] to consider the attainability of the best constant in the Alvino inequality (see [2])

$$\sup_{x\in B_1} \frac{|u^\sharp(x)|}{\left(\log \frac{1}{|x|}\right)^{\frac{1}{n'}}} \le \pi^{-\frac{1}{2}} n^{-\frac{1}{n}} \left(\Gamma\left(1+\frac{n}{2}\right)\right)^{\frac{1}{n}} \left(\int_{B_1} |\nabla u|^n dx\right)^{\frac{1}{n}}, \ u \in W_0^{1,n}(B_1), \quad (8)$$

which is known to be the critical case of Sobolev embeddings since it implies the optimal embedding of $W_0^{1,n}(B_1)$ into Orlicz spaces (see [9, Example 1] and [5, Lemma 2.8]). Indeed, the scaling (7) for radial functions coincides with the scaling in [8]. Moreover, it is proved in [19] that the constant $\left(\frac{n-1}{n}\right)^n$ in the left hand side of (6) is sharp and is never attained in $W_0^{1,n}(B_1)$. Moreover, it is proved in [19] that if there is an extremal function $\tilde{U}$ which attain the sharp constant, then

$$\tilde{U}(x) = \left(\log \frac{1}{|x|}\right)^{\frac{n-1}{n}} \quad (9)$$

should follow by the simplicity of the first eigenvalue (if exists) and the scale invariance of (6) under the power-type scaling (7). Note that $\tilde{U}$ cannot be a real minimizer since $\|\nabla \tilde{U}\|_{L^n(B_1)} = \infty$ and we will call $\tilde{U}$ a "virtual minimizer", which plays a role of a function $U(x) = r^{-\frac{N-p}{p}}$ for the classical Hardy inequality (1).

In this paper, we first consider the existence of the remainder terms for (6) from the view-point of scale invariant structure under the power-type scaling (7). To this end, we introduce a generalization of inequality (6) which has a scale invariance under

$$u_\lambda(x) := \lambda^{-\frac{p-1}{p}} u(|x|^{\lambda-1}x), \quad \lambda > 0, \quad (10)$$

where $1 < p < \infty$. Now we define a function space to state a generalization of (6).

Definition 1 Let $1 \le p < \infty$. We define a weighted Lebesgue norm as

$$\|u\|_{L^p(B_1,|x|^{p-n})} := \left(\int_{B_1} |u(x)|^p |x|^{p-n} dx\right)^{\frac{1}{p}},$$

and $W_{0,0}^{1,p}(B_1)$ denotes the completion of $C_0^\infty(B_1 \setminus \{0\})$ with respect to the norm $\|\nabla \cdot\|_{L^p(B_1,|x|^{p-n})}$.

Remark 1 The space $W_{0,0}^{1,n}(B_1)$ coincides with the completion of $C_0^\infty(B_1)$ by $\|\nabla \cdot\|_{L^n(B_1)}$ since $C_0^\infty(B_1) \subset W_{0,0}^{1,n}(B_1)$ (see [18, Proposition 2.1]). Particularly, $C_0^\infty(B_1 \setminus \{0\})$ is densely contained in $W_0^{1,n}(B_1)$. See [18] for more details of the completion of $C_0^\infty(B_1 \setminus \{0\})$ by weighted Lebesgue norms.

We have the following Hardy type inequality for $W_{0,0}^{1,p}(B_1)$ functions:

Proposition 1 *Let $n \geq 2$ and $1 < p < \infty$. Then there holds*

$$\int_{B_1} |x|^{p-n} \left| \frac{x}{|x|} \cdot \nabla u(x) \right|^p dx \geq \left(\frac{p-1}{p} \right)^p \int_{B_1} \frac{|u(x)|^p}{|x|^n \left(\log \frac{1}{|x|} \right)^p} dx, \quad u \in W_{0,0}^{1,p}(B_1). \tag{11}$$

By the same methods as in [19], one can show the following facts: the constant $\left(\frac{p-1}{p}\right)^p$ is optimal, there is no extremal function which attains $\left(\frac{p-1}{p}\right)^p$, and $\left(\log \frac{1}{|x|}\right)^{\frac{p-1}{p}}$ is the virtual minimizer associated with (11). The absence of extremals suggests us the existence of a remainder term. Indeed, our first result involves the improvement of Proposition 1 with remainder terms characterized by the ratio with or the distance from the virtual minimizer. Let

$$\varepsilon(u) := \int_{B_1} |x|^{p-n} \left| \frac{x}{|x|} \cdot \nabla u \right|^p dx - \left(\frac{p-1}{p} \right)^p \int_{B_1} \frac{|u(x)|^p}{|x|^n \left(\log \frac{1}{|x|} \right)^p} dx.$$

Our first main result reads as follows:

Theorem 1 *For $n \geq 2$ and $1 < p < \infty$, there exists $C_{p,n} > 0$ depending only on p and n which satisfies*

$$\varepsilon(u) \geq C_{p,n} \int_{B_1} |x|^{p-n} \left(\log \frac{1}{|x|} \right)^{p-1} \left| \frac{x}{|x|} \cdot \nabla \left(\frac{u(x)}{\left(\log \frac{1}{|x|} \right)^{\frac{p-1}{p}}} \right) \right|^p dx, \quad u \in W_{0,0}^{1,p}(B_1) \tag{12}$$

for $2 \leq p < \infty$ and

$$\varepsilon(u)^{\frac{p}{2}} \left(\int_{B_1} |x|^{p-n} \left| \frac{x}{|x|} \cdot \nabla u(x) \right|^p dx \right)^{\frac{2-p}{2}}$$
$$\geq C_{p,n} \int_{B_1} |x|^{p-n} \left(\log \frac{1}{|x|} \right)^{p-1} \left| \frac{x}{|x|} \cdot \nabla \left(\frac{u(x)}{\left(\log \frac{1}{|x|} \right)^{\frac{p-1}{p}}} \right) \right|^p dx, \; u \in W_{0,0}^{1,p}(B_1) \tag{13}$$

for $1 < p < 2$.

We also have a remainder term given by a distance from virtual minimizers. Some studies on this direction can be found in [3] for the Sobolev inequality and in [10] for Hardy inequalities (1) and (5). Let

$$d(f, g) := \sup_{0<r<1} \frac{|f(r) - g(r)|}{\left(\log \frac{1}{r} \right)^{\frac{p-1}{p}}}, \qquad \tilde{u}(r) := \int_{\mathbb{S}^{n-1}} u(r, \theta) d\theta.$$

Theorem 2 *Let $n \geq 2$ and $1 < p < \infty$. Define $\tilde{p} = \max\{p^2, 2p\}$. Then there exists $C_{p,n} > 0$ such that the inequality*

$$\varepsilon(u) \geq C_{p,n} \left(\int_{B_1} \frac{|u(x)|^p}{|x|^n \left(\log \frac{1}{|x|}\right)^p} dx \right) \left[\inf_{a>0} \frac{d\left(\tilde{u}(r), a \left(\log \frac{1}{r}\right)^{\frac{p-1}{p}} \right)}{\left(\int_{B_1} |x|^{p-n} \left| \frac{x}{|x|} \cdot \nabla u(x) \right|^p dx \right)^{\frac{1}{p}}} \right]^{\tilde{p}}$$

holds for every $u \in W_{0,0}^{1,p}(B_1)$.

Remark that the remainder terms in Theorems 1 and 2 are invariant under the power-type scaling (7). Moreover, Proposition 1 can be proved from Theorem 1 directly by neglecting the remainder term. The proof of Theorems 1 and 2 will be given in Sect. 2.

In Sect. 3, we revisit the results in [17, 18] which are concerned with the higher-order remainder terms with iterated logarithmic potential function. We will clarify the self-iterated structure of the remainder terms, namely, we show that the invariance property under the power-type scaling (7) naturally leads a remainder term with $k+1$-th iterated logarithmic function from that with k-th iterated logarithmic function.

Finally, we consider the equivalence between the standard scaling (2) and the power-type scaling (7) under the transformation

$$B_1(0) \setminus \{0\} \ni x \mapsto y = \left(\log \frac{1}{|x|}\right)^{-p} \frac{x}{|x|} \in \mathbb{R}^n \setminus \{0\} \tag{14}$$

and the associated transformation of functions:

$$T_p : C_0^\infty(\mathbb{R}^n \setminus \{0\}) \to C_0^\infty(B_1(0) \setminus \{0\}) \; ; \; T_p u(x) = u\left(\left(\log \frac{1}{|x|}\right)^{-p} \frac{x}{|x|} \right).$$

These transformations are introduced by Horiuchi and Kumlin in [18, Definitions 3.1 and 3.2].

In Sect. 4, we derive the Alvino inequality

$$\sup_{x \in B_1} \frac{|u^\sharp(x)|}{\left(\log \frac{1}{|x|}\right)^{\frac{1}{n'}}} \leq \pi^{-\frac{1}{2}} n^{-\frac{1}{n}} \left(\Gamma\left(1 + \frac{n}{2}\right) \right)^{\frac{1}{n}} \left(\int_{B_1} |\nabla u|^n dx \right)^{\frac{1}{n}}, \quad u \in W_0^{1,n}(B_1)$$

by applying the transformation above to the classical Sobolev inequality and taking a simple limiting procedure. As a by-product, we reveal a relation between Moser sequences which are minimizers associated with the Alvino inequality and Talenti functions which are minimizers associated with the classical Sobolev inequality. These facts are treated in Sect. 4.

2 Proofs of Theorem 1 and Theorem 2

In this section, we give the proof of Theorems 1 and 2. Let $u \in W_{0,0}^{1,p}(B_1)$. It suffices to prove the theorems for $C_0^\infty(B_1 \setminus \{0\})$ functions by the standard density argument associated with the continuous embedding $C_0^\infty(B_1 \setminus \{0\}) \subset W_{0,0}^{1,p}(B_1)$ from the definition of $W_{0,0}^{1,p}(B_1)$. Let us define the partial derivative with respect to radius $|x|$ as $\partial_r u(x) := \frac{x}{|x|} \cdot \nabla u(x)$ for the sake of simplicity of the notation.

Proof of Theorem 1. First we consider the case $2 \le p < \infty$. Let $u \in C_0^\infty(B_1 \setminus \{0\})$ and $v(x) := \frac{u(x)}{\left(\log \frac{1}{|x|}\right)^{\frac{p-1}{p}}}$. Clearly there holds $v \in C_0^\infty(B_1 \setminus \{0\})$ and

$$\partial_r u(x) = \partial_r v(x) \left(\log \frac{1}{|x|}\right)^{\frac{p-1}{p}} + \frac{p-1}{p} \frac{v(x)}{|x|} \left(\log \frac{1}{|x|}\right)^{-\frac{1}{p}}.$$

We now apply the following elementary inequality

$$|a+b|^p \ge |a|^p + p|a|^{p-2}ab + c_p|b|^p, \quad p \ge 2, \quad a, b \in \mathbb{R} \tag{15}$$

(see e.g. Frank-Seiringer [16, p. 3415]). Taking

$$a = \frac{p-1}{p} \frac{v(x)}{|x|} \left(\log \frac{1}{|x|}\right)^{-\frac{1}{p}}, \quad b = \partial_r v(x) \left(\log \frac{1}{|x|}\right)^{\frac{p-1}{p}}, \tag{16}$$

we have

$$\begin{aligned} |x|^{p-n}|\partial_r u|^p \ge &\left(\frac{p-1}{p}\right)^p \frac{|u(x)|^p}{|x|^n \left(\log \frac{1}{|x|}\right)^p} \\ &+ \left(\frac{p-1}{p}\right)^{p-1} \frac{\partial_r(|v|^p)}{|x|^{n-1}} + c_p|x|^{p-n}\left(\log \frac{1}{r}\right)^{p-1} |\partial_r v|^p. \end{aligned} \tag{17}$$

Integrating both sides of (17) on $B_1(0)$ and applying

$$\int_{B_1} \frac{\partial_r(|v|^p)}{|x|^{n-1}} dx = \int_0^1 \int_{\mathbb{S}^{n-1}} \partial_r(|v|^p) d\omega dr = 0, \tag{18}$$

we obtain the desired estimate.

Next we consider the case $1 < p < 2$. In this case, we apply the following inequality instead of (15):

$$|a+b|^p \ge |a|^p + p|a|^{p-2}ab + C\frac{|b|^2}{(|a|+|b|)^{2-p}}, \quad 1 < p < 2, \quad a, b \in \mathbb{R} \tag{19}$$

(see e.g. Lindqvist [21, Lemma 4.2]). Applying (19) with a, b as in (16), we have

$$|x|^{p-n}|\partial_r u|^p \geq \left(\frac{p-1}{p}\right)^p \frac{|u(x)|^p}{|x|^n \left(\log \frac{1}{|x|}\right)^p} + \left(\frac{p-1}{p}\right)^{p-1} \frac{\partial_r(|v|^p)}{|x|^{n-1}} + C|x|^{p-n} \frac{\left|\left(\log \frac{1}{|x|}\right)^{\frac{p-1}{p}} \partial_r v\right|^2}{\left(|\partial_r u| + \frac{p-1}{p}\left(\log \frac{1}{|x|}\right)^{-\frac{1}{p}} \frac{|v(x)|}{|x|}\right)^{2-p}}. \tag{20}$$

Integrating it over B_1 and using (18), we obtain

$$\varepsilon(u) \geq C \int_{B_1} |x|^{p-n} \frac{\left|\left(\log \frac{1}{|x|}\right)^{\frac{p-1}{p}} \partial_r v\right|^2}{\left(|\partial_r u| + \frac{p-1}{p}\left(\log \frac{1}{|x|}\right)^{-\frac{1}{p}} \frac{|v(x)|}{|x|}\right)^{2-p}} dx. \tag{21}$$

Now Hölder's inequality leads

$$\begin{aligned} &\int_{B_1} |x|^{p-n} \left(\log \frac{1}{|x|}\right)^{p-1} |\partial_r v|^p dx \\ &\leq \left(\int_{B_1} |x|^{p-n} \frac{\left|\left(\log \frac{1}{|x|}\right)^{\frac{p-1}{p}} \partial_r v\right|^2}{\left|\partial_r u| + \frac{p-1}{p}\left(\log \frac{1}{|x|}\right)^{-\frac{1}{p}} \frac{|v(x)|}{|x|}\right|^{2-p}} dx\right)^{\frac{p}{2}} \\ &\qquad \times \left(\int_{B_1} |x|^{p-n} \left||\partial_r u| + \frac{p-1}{p}\left(\log \frac{1}{|x|}\right)^{-\frac{1}{p}} \frac{|v(x)|}{|x|}\right|^p dx\right)^{\frac{2-p}{2}}. \end{aligned} \tag{22}$$

It follows from $\varepsilon(u) > 0$ by (21) that

$$\int_{B_1} |x|^{p-n} \left||\partial_r u| + \frac{p-1}{p}\left(\log \frac{1}{|x|}\right)^{-\frac{1}{p}} \frac{|v(x)|}{|x|}\right|^p dx \leq 2^p \int_{B_1} |x|^{p-n}|\partial_r u|^p dx. \tag{23}$$

Applying (21) and (23) to (22), we obtain Theorem 1 for the case $1 < p < 2$. □

Proof of Theorem 2. We follow the idea of Cianchi-Ferone [10]. Let $u \in C_0^\infty(B_1 \setminus \{0\})$. For $\alpha > 0$, we define $A := \left\{r \in (0,1) : \left|\int_{\mathbb{S}^{n-1}} v(r,\theta) d\theta\right| > \varepsilon(u)^\alpha\right\}$, and

$$v(x) := \frac{u(x)}{\left(\log \frac{1}{|x|}\right)^{\frac{p-1}{p}}}, \quad \tilde{v}(r) := \int_{\mathbb{S}^{n-1}} v(r,\theta) d\theta.$$

We choose a suitable $\alpha > 0$ later. First we assume that

$$\|\partial_r u\|_{L^p(B_1,|x|^{p-n})} := \left(\int_{B_1} |x|^{p-n} |\partial_r u|^p dx \right)^{\frac{1}{p}} = 1. \tag{24}$$

Under this normalization, the inequalities in Theorem 1 can be summarized as

$$\varepsilon(u)^{\min\left\{1, \frac{p}{2}\right\}} \geq C_{p,n} \int_{B_1} |x|^{p-n} \left(\log \frac{1}{|x|} \right)^{p-1} |\partial_r v(x)|^p \, dx, \ u \in C_0^\infty(B_1 \setminus \{0\}), \tag{25}$$

where $1 < p < \infty$ and $C_{p,n}$ is a positive constant depending only on p and n.

We give an estimate on $|\tilde{v}(r) - \varepsilon(u)^\alpha|$. If $r \notin A$, clearly we have

$$|\tilde{v}(r) - \varepsilon(u)^\alpha| \leq 2\varepsilon(u)^\alpha. \tag{26}$$

Now assume that $r \in A$. Note that $\tilde{v}$ is continuous on $(0, 1)$ and A is open since $u \in C_0^\infty(B_1)$. Particularly, there exist $a, b \in (0, 1)$ such that $r \in (a, b) \subset A$ with $\tilde{v}(a) = \tilde{v}(b) = \varepsilon(u)^\alpha$. Hence by the fundamental theorem of calculus and Hölder's inequality, we obtain

$$\begin{aligned} |\tilde{v}(r) - \varepsilon(u)^\alpha| &\leq \int_a^r |\tilde{v}'(s)| ds \\ &\leq \left(\int_0^1 |\tilde{v}'(s)|^p s^{p-1} \left(\log \frac{1}{s} \right)^{p-1} \right)^{\frac{1}{p}} \left(\int_a^r \frac{1}{s \left(\log \frac{1}{s} \right)} ds \right)^{\frac{p-1}{p}}. \end{aligned} \tag{27}$$

Minkowski's inequality yields

$$\int_0^1 \left|\tilde{v}'(s)\right|^p s^{p-1} \left(\log \frac{1}{s} \right)^{p-1} ds \leq \int_{B_1} |x|^{p-n} \left(\log \frac{1}{|x|} \right)^{p-1} |\partial_r v|^p dx. \tag{28}$$

Combining (28) and (25) with (27), we have

$$\begin{aligned} |\tilde{v}(r) - \varepsilon(u)^\alpha| &\leq C\varepsilon(u)^{\min\left\{\frac{1}{p}, \frac{1}{2}\right\}} \left(\int_a^r \frac{|\tilde{v}(s)|^p}{\varepsilon(u)^{\alpha p}} \frac{1}{s \left(\log \frac{1}{s} \right)} ds \right)^{\frac{p-1}{p}} \\ &\leq C\varepsilon(u)^{\min\left\{\frac{1}{p}, \frac{1}{2}\right\} - \alpha(p-1)}, \end{aligned} \tag{29}$$

where in the first line we have used the relation $\dfrac{|\tilde{v}(s)|^p}{\varepsilon(u)^{\alpha p}} \geq 1$ which comes from the fact $(a, r) \subset A$ and in the last inequality we have used (24) together with the fact $\varepsilon(u) \geq 0$ by Theorem 1. Therefore, if we choose

$$\alpha = \min\left\{\frac{1}{p}, \frac{1}{2}\right\} - \alpha(p-1), \quad \text{namely,} \quad \alpha = \min\left\{\frac{1}{p^2}, \frac{1}{2p}\right\},$$

then there exists a positive constant C such that

$$C\,|\tilde{v}(r)-\varepsilon(u)^{\alpha}|\le \varepsilon(u)^{\alpha},\quad r\in(0,1),$$

which implies

$$C\sup_{r\in(0,1)}\left|\frac{\tilde{u}(r)-\varepsilon(u)^{\min\left\{\frac{1}{p^2},\frac{1}{2p}\right\}}\left(\log\frac{1}{r}\right)^{\frac{p-1}{p}}}{\left(\log\frac{1}{r}\right)^{\frac{p-1}{p}}}\right|^{\max\{p^2,2p\}}\le\varepsilon(u).$$

This proves Theorem 2 under the normalization $\|\partial_r u\|_{L^p(B_1,|x|^{p-n})}=1$. Finally we remove this restriction. The inequality above with u replaced by $\frac{u}{\|\partial_r u\|_{L^p(B_1,|x|^{p-n})}}$ gives

$$C\sup_{r\in(0,1)}\left|\frac{\frac{\tilde{u}(r)}{\|\partial_r u\|_{L^p(B_1,|x|^{p-n})}}-\frac{\varepsilon(u)^{\min\left\{\frac{1}{p^2},\frac{1}{2p}\right\}}}{\|\partial_r u\|^{\frac{1}{p}}_{L^p(B_1,|x|^{p-n})}}\left(\log\frac{1}{r}\right)^{\frac{p-1}{p}}}{\left(\log\frac{1}{r}\right)^{\frac{p-1}{p}}}\right|^{\max\{p^2,2p\}}\le\frac{\varepsilon(u)}{\|\partial_r u\|^p_{L^p(B_1,|x|^{p-n})}},$$

whence follows the desired estimate. □

3 On the Higher Order Remainder Terms of the Critical Hardy Inequality via the Scale Invariant Structure

In this section, we discuss the existence of higher order remainder terms for the critical Hardy inequality from the view-point of the invariance under the power-type scaling (2). For simplicity we consider the problem in a two-dimensional unit ball $B_1=B_1(0)\subset\mathbb{R}^2$. It is easy to see that $\int_{B_1}\left|\frac{x}{|x|}\cdot\nabla u\right|^2dx$ is invariant under the power-type scaling (7) with $n=2$, namely,

$$u_{\lambda}(x)=\lambda^{-\frac{1}{2}}v(|x|^{\lambda-1}x),\quad \lambda>0,\quad u\in W_0^{1,2}(B_1). \tag{30}$$

First we see that the potential function $V(x)=\frac{1}{|x|^2\left(\log\frac{1}{|x|}\right)^2}$ is naturally arises from the invariance of the associated functional under the power-type scaling (30).

Proposition 2 *Let $V:B_1\to\mathbb{R}$ be a locally integrable function on B_1 with $V\not\equiv 0$. Assume that V satisfies*

$$\int_{B_1}V(x)|u(x)|^2dx=\int_{B_1}V(x)|u_{\lambda}(x)|^2dx,\quad \lambda>0,\quad u\in W_0^{1,2}(B_1). \tag{31}$$

Then there holds

$$V(x) = \theta\left(\frac{x}{|x|}\right)\frac{1}{|x|^2\left(\log\frac{1}{|x|}\right)^2}, \quad x \in B_1 \setminus \{0\}$$

for some function θ defined on $\mathbb{S}^1$.

Proof of Proposition 2. Let B_1 be the two dimensional unit ball centered at the origin. Since $C_0^\infty(B_1 \setminus \{0\})$ is dense in $W_0^{1,2}(B_1)$ (see Remark 1), it suffices to prove the proposition for $C_0^\infty(B_1 \setminus \{0\})$ functions. Let $u \in C_0^\infty(B_1 \setminus \{0\})$ and $\lambda > 0$. By the assumption (31), there holds

$$\int_{B_1} V(x)|u(x)|^2 dx = \int_{B_1} V(x)|u_\lambda(x)|^2 dx.$$

The change of variable from x to $y = |x|^{\lambda-1}x$ yields

$$\int_{B_1} V(x)|u(x)|^2 dx = \int_{B_1} V(|y|^{1/\lambda-1}y)\lambda^{-2}|u(y)|^2|y|^{-2+2/\lambda}dy. \tag{32}$$

Since (32) holds for every $u \in C_0^\infty(B_1 \setminus \{0\})$, we obtain by du Bois-Reymond's lemma that

$$V(a) = |a|^{\frac{2}{\lambda}-2}V(|a|^{\frac{1}{\lambda}-1}a)\lambda^{-2}, \quad \lambda > 0, \quad a \in B_1 \setminus \{0\}. \tag{33}$$

Fix $x \in B_1 \setminus \{0\}$ and choose $a = \frac{x}{2|x|}$, $\lambda = \log\frac{1}{|a|} / \log\frac{1}{|x|}$, so that $|a|^{\frac{1}{\lambda}-1}a = x$. Then (33) yields

$$|x|^2\left(\log\frac{1}{|x|}\right)^2 V(x) = |a|^2\left(\log\frac{1}{|a|}\right)^2 V(a) = \frac{1}{4}(\log 2)^2\, V\left(\frac{x}{2|x|}\right).$$

Remark that the function in the right hand side only depends on $\frac{x}{|x|} \in \mathbb{S}^1$. Taking

$$\theta\left(\frac{x}{|x|}\right) = \frac{1}{4}(\log 2)^2\, V\left(\frac{x}{2|x|}\right),$$

we obtain the conclusion. □

In [13], Detalla, Horiuchi and Ando showed the existence of the remainder terms with iterated logarithmic functions. To state their result, we introduce notation. For $k \in \mathbb{N}$, we define a iterated exponential function by $e_0 := 1, \quad e_{k+1} := e^{e_k}$ and the iterated logarithmic function $\log_k$ by

$$\log_0(r) := r \quad \text{for} \quad r > 0, \qquad \log_{k+1}(r) := \log(\log_k r) \quad \text{for} \quad r > e_{k-1}.$$

Proposition 3 ([13]) *For any $k \in \mathbb{N}$, there holds*

$$\int_{B_1} |\nabla u(x)|^2 dx \geq \frac{1}{4} \sum_{j=1}^{k} \int_{B_1} \frac{|u(x)|^2}{|x|^2 \prod_{i=1}^{j} \left(\log_i \frac{e_k}{|x|}\right)^2} dx, \quad u \in W_0^{1,2}(B_1).$$

They also proved that the constant $\frac{1}{4}$ in the right hand side is best possible for every $k \in \mathbb{N}$. It should be noted that, for every $j \in \mathbb{N}$, the potential function

$$x \mapsto |x|^{-2} \prod_{i=1}^{j} \left(\log_i \frac{e_k}{|x|}\right)^{-2}$$

is monotone decreasing. Indeed, the proof in [13] relies on the rearrangement method which requires this monotonicity of the potential function essentially. Our next result involves the existence of the higher order expansion of a remainder term for a critical Hardy inequality with a non-monotone potential function. Our argument relies on the iterated structure of remainder terms from the view-point of the scale invariance of the problem under power-type scaling (30).

Theorem 3 *For $k \in \mathbb{N}$, there holds*

$$\int_{B_1} \left| \frac{x}{|x|} \cdot \nabla u(x) \right|^2 dx \geq \frac{1}{4} \sum_{j=1}^{k} \int_{B_1} \frac{|u(x)|^2}{|x|^2 \prod_{i=1}^{j} \left(\log_i \frac{1}{|x|}\right)^2} dx, \quad u \in W_0^{1,2}(B_{1/e_{k-1}}).$$

By using a change of variable $x \mapsto \frac{x}{e_{k-1}}$, one can also obtain the following corollary:

Corollary 1 *For $k \in \mathbb{N}$, there holds*

$$\int_{B_1} \left| \frac{x}{|x|} \cdot \nabla u(x) \right|^2 dx \geq \frac{1}{4} \sum_{j=1}^{k} \int_{B_1} \frac{|u(x)|^2}{|x|^2 \prod_{i=1}^{j} \left(\log_i \frac{e_{k-1}}{|x|}\right)^2} dx, \quad u \in W_0^{1,2}(B_1).$$

Note that Proposition 3 follows from Corollary 1 since $e_{k-1} < e_k$. Moreover, the potential function in Corollary 1 is non-monotone while that in Proposition 3 is a monotone.

For the proof of Theorem 3, we need two lemmas. Let

$$w(x) := \left(\log \frac{1}{|x|}\right)^{\frac{1}{2}} \tag{34}$$

be a virtual minimizer associated with the critical Hardy inequality in $B_1 \subset \mathbb{R}^2$ (see (9)). One can check easily that w has a self similarity under the scaling (30), i.e., we have

$$w(x) = \lambda^{-\frac{1}{2}} w(|x|^{\lambda-1}x), \quad \lambda > 0, \quad x \in B_1 \setminus \{0\}.$$

The first lemma shows that one can obtain a remainder term in terms of a self-similar function w:

Lemma 1 *There holds*

$$\int_{B_1} \left| \frac{x}{|x|} \cdot \nabla u \right|^2 dx - \frac{1}{4} \int_{B_1} \frac{|u(x)|^2}{|x|^2 \left(\log \frac{1}{|x|} \right)^2} dx = \int_{B_1} \left(\log \frac{1}{|x|} \right) \left| \frac{x}{|x|} \cdot \nabla \left(\frac{u(x)}{w(x)} \right) \right|^2 dx$$

for $u \in W_0^{1,2}(B_1)$.

The similar relation is obtained in [12], but we prove it for reader's convenience.

Proof of Lemma 1. By the standard density argument, it suffices to prove the lemma for functions in $C_0^\infty(B_1 \setminus \{0\})$ (see Remark 1). Let $u \in C_0^\infty(B_1 \setminus \{0\})$ and define $v(x) := \frac{u(x)}{w(x)}$ for $x \in B_1$. We note $v \in C_0^\infty(B_1 \setminus \{0\})$ since $u \in C_0^\infty(B_1 \setminus \{0\})$. By Leibnitz's rule, we have

$$\begin{aligned} \left| \frac{x}{|x|} \cdot \nabla u(x) \right|^2 = &\left| \frac{x}{|x|} \cdot \nabla v(x) \right|^2 \left(\log \frac{1}{|x|} \right) - \frac{1}{|x|} v(x) \left(\frac{x}{|x|} \cdot \nabla v(x) \right) \\ &+ \frac{1}{4} \frac{1}{|x|^2 \left(\log \frac{1}{|x|} \right)^2} |u(x)|^2. \end{aligned} \tag{35}$$

It is easy to see that

$$\int_{B_1} \frac{1}{|x|} v(x) \left(\frac{x}{|x|} \cdot \nabla v(x) \right) dx = \int_0^{2\pi} \int_0^1 \frac{\partial}{\partial r} (v(r,\theta))^2 dr d\theta = 0$$

since $v \in C_0^\infty(B_1 \setminus \{0\})$. Thus by integrating (35) over B_1, we obtain

$$\int_{B_1} \left| \frac{x}{|x|} \cdot \nabla u(x) \right|^2 dx - \frac{1}{4} \int_{B_1} \frac{|u(x)|^2}{|x|^2 \left(\log \frac{1}{|x|} \right)^2} dx = \int_{B_1} \log \frac{1}{|x|} \left| \frac{x}{|x|} \cdot \nabla v(x) \right|^2 dx.$$

This completes the proof. □

To state the next lemma, we need the following transformation from $C_0^\infty(\mathbb{R}^n \setminus \{0\})$ to $C_0^\infty(B_1(0) \setminus \{0\})$ introduced by Horiuchi and Kumlin [18, Definition 3.1 and 3.2]:

$$T_p : C_0^\infty(\mathbb{R}^n \setminus \{0\}) \to C_0^\infty(B_1(0) \setminus \{0\}) \; ; \; T_p u(x) = u\left(\left(\log \frac{1}{|x|} \right)^{-p} \frac{x}{|x|} \right). \tag{36}$$

They introduced this transformation in connection with the analysis of the attainability of best constants of critical Caffarelli-Kohn-Nirenberg inequalities. The following lemma will give the iterated structure of the higher order remainder terms for the critical Hardy inequality:

Lemma 2 *Let $k \in \mathbb{N}$ and $0 < R \leq 1/e_{k-1}$. Then there holds*

$$\int_{B_R} \left(\prod_{j=1}^{k} \log_j \frac{1}{|x|} \right) \left| \frac{x}{|x|} \cdot \nabla v(x) \right|^2 dx = \int_{B_{\left(\log \frac{1}{R}\right)^{-1}}} \left(\prod_{j=1}^{k-1} \log_j \frac{1}{|x|} \right) \left| \frac{x}{|x|} \cdot \nabla T_1^{-1}(v)(x) \right|^2 dx$$

for $v \in W_0^{1,2}(B_R)$, where T_1^{-1} is the inverse transformation of T_1 in (36) given by

$$T_1^{-1} v(x) := v \left(\exp\left(-\frac{1}{|x|} \right) \frac{x}{|x|} \right), \quad x \in B_{\left(\log \frac{1}{R}\right)^{-1}}.$$

Proof of Lemma 2. Let $k \in \mathbb{N}$, $0 < R \leq 1/e_{k-1}$, and $v \in C_0^\infty(B_R \setminus \{0\})$. Remark that $T_1^{-1}(v) \in C_0^\infty(B_{\left(\log \frac{1}{R}\right)^{-1}} \setminus \{0\})$. Let $y = \left(\log \frac{1}{|x|} \right)^{-1} \frac{x}{|x|}$ for $x \in B_R \setminus \{0\}$. Then the definition of T_1^{-1} gives

$$\begin{aligned} \frac{x}{|x|} \cdot \nabla v(x) &= \frac{y}{|y|} \cdot \left(\frac{\partial(y_1, \cdots, y_n)}{\partial(x_1, \cdots, x_n)} \right) \nabla (T_1^{-1} v)(y) \\ &= e^{\frac{1}{|y|}} |y|^2 \frac{y}{|y|} \cdot \nabla (T_1^{-1} v)(y), \end{aligned}$$

which together with

$$\log_j \frac{1}{|x|} = \log_{j-1} \frac{1}{|y|}, \quad j \in \{1, \cdots, k\}, \qquad \det \left(\frac{\partial(y_1, \cdots, y_n)}{\partial(x_1, \cdots, x_n)} \right) = \frac{1}{|y|^3} e^{-\frac{2}{|y|}}$$

yields

$$\begin{aligned} &\int_{B_R} \left(\prod_{j=1}^{k} \log_j \frac{1}{|x|} \right) \left| \frac{x}{|x|} \cdot \nabla v(x) \right|^2 dx \\ &= \int_{B_{\left(\log \frac{1}{R}\right)^{-1}}} \frac{1}{|y|} \left(\prod_{j=1}^{k-1} \log_j \frac{1}{|y|} \right) \left| \frac{y}{|y|} \cdot \nabla T_1^{-1} v(y) \right|^2 |y|^4 e^{\frac{2}{|y|}} \frac{1}{|y|^3} e^{-\frac{2}{|y|}} dy \\ &= \int_{B_{\left(\log \frac{1}{R}\right)^{-1}}} \left(\prod_{j=1}^{k-1} \log_j \frac{1}{|y|} \right) \left| \frac{y}{|y|} \cdot \nabla T_1^{-1}(v)(y) \right|^2 dy. \end{aligned}$$

This proves the conclusion for $C_0^\infty(B_R \setminus \{0\})$-functions. By the standard density argument (see Remark 1), we obtain the conclusion. $\square$

Proof of Theorem 3. Let $k \in \mathbb{N}$. We will prove

$$\int_{B_1} \left| \frac{x}{|x|} \cdot \nabla u(x) \right|^2 dx = \frac{1}{4} \sum_{j=1}^{k} \int_{B_1} \frac{|u(x)|^2}{|x|^2 \prod_{i=1}^{j} \left(\log_i \frac{1}{|x|} \right)^2} dx + \int_{B_1} \left(\prod_{j=1}^{k} \log_j \frac{1}{|x|} \right) \left| \frac{x}{|x|} \cdot \nabla \left(\frac{u(x)}{\prod_{j=1}^{k} \left(\log_j \frac{1}{|x|} \right)^{\frac{1}{2}}} \right) \right|^2 dx \tag{37}$$

for $u \in C_0^\infty(B_{1/e_{k-1}} \setminus \{0\})$. Clearly the conclusion follows from the above equality by neglecting the last term in (37) together with the standard density argument. A similar equation to (37) can be found in [12], however our derivation of the equation is different and preserves the scale invariant structure. We now prove (37) by the induction argument. Lemma 1 shows us that (37) holds for $k = 1$. Assume that (37) holds for $k \in \mathbb{N}$. Let $u \in C_0^\infty(B_{1/e_k} \setminus \{0\})$ and define $v = \frac{u(x)}{\prod_{j=1}^{k} \left(\log_j \frac{1}{|x|} \right)^{\frac{1}{2}}}$ for the sake of simplicity of the notation. Then, applying Lemma 2 with $R = 1/e_k \leq 1/e_{k-1}$ k-times, we have

$$\int_{B_1} \left(\prod_{j=1}^{k} \log_j \frac{1}{|x|} \right) \left| \frac{x}{|x|} \cdot \nabla v \right|^2 dx = \int_{B_1} \left| \frac{y}{|y|} \cdot \nabla \left((T_1^{-1})^{(k)} v \right) \right|^2 dy,$$

where $(T_1^{-1})^{(k)}$ is the k-times composition of T_1^{-1}. Noting the fact $(T_1^{-1})^{(k)} v \in C_0^\infty(B_1 \setminus \{0\})$ which follows from $u \in C_0^\infty(B_{1/e^k} \setminus \{0\})$, we obtain from Lemma 1 for $(T_1^{-1})^{(k)} v$ that

$$\int_{B_1} \left| \frac{y}{|y|} \cdot \nabla (T_1^{-1})^{(k)} v(y) \right|^2 dy - \frac{1}{4} \int_{B_1} \frac{\left| (T_1^{-1})^{(k)} v(y) \right|^2}{|y|^2 \left(\log \frac{1}{|y|} \right)^2} dy = \int_{B_1} \log \frac{1}{|y|} \left| \frac{y}{|y|} \cdot \nabla \left(\frac{(T_1^{-1})^{(k)} v(y)}{\left(\log \frac{1}{|y|} \right)^{\frac{1}{2}}} \right) \right|^2 dy.$$

Now we remark

$$\frac{y}{|y|} \cdot \nabla (T_1^{-1})^{(k)} v(y) = |x| \left(\prod_{j=1}^{k} \log_j \frac{1}{|x|} \right) \log_k \frac{1}{|x|} \frac{x}{|x|} \cdot \nabla v(x),$$
$$\det \left(\frac{\partial y}{\partial x} \right) = \frac{1}{|x|^2 \prod_{j=1}^{k} \log_j \frac{1}{|x|} \left(\log_k \frac{1}{|x|} \right)^2}. \tag{38}$$

Indeed, let $\phi \in C^1(\mathbb{R})$ be a injective function and put $y = \phi(|x|)\frac{x}{|x|}$ for $x \in \mathbb{R}^n$. Then, for $1 \le i, j \le n$, we see

$$\begin{aligned} \frac{\partial y_i}{\partial x_j} &= \frac{\phi(|x|)}{|x|}\left(\delta_{ij} + \left(\frac{|x|}{\phi(|x|)}\phi'(|x|) - 1\right)\frac{x_i x_j}{|x|^2}\right), \\ \det\left(\frac{\partial y}{\partial x}\right) &= \left(\frac{\phi(|x|)}{|x|}\right)^{n-1}\phi'(|x|). \end{aligned} \tag{39}$$

Let $v \in C^1(\mathbb{R}^n)$ and define $\tilde{v}(y) := v(x)$. Then we have

$$\begin{aligned} \frac{x}{|x|}\cdot\nabla v(x) &= \frac{\phi(|x|)}{|x|}\left(\sum_{i=1}^n\sum_{j=1}^n \delta_{ij}\frac{y_i}{|y|}\partial_j\tilde{v} + \left(\frac{|x|}{\phi(|x|)}\phi'(|x|) - 1\right)\frac{y_i^2 y_j \partial_j \tilde{v}}{|y|^3}\right) \\ &= \phi'(|x|)\frac{y}{|y|}\cdot\nabla\tilde{v}(y). \end{aligned} \tag{40}$$

By taking $\phi(|x|) = \left(\log_k \frac{1}{|x|}\right)^{-1}$ in (39) and (40), we have (38). The relation (38) yields

$$\begin{aligned} &\int_{B_1} \log\frac{1}{|y|}\left|\frac{y}{|y|}\cdot\nabla\left(\frac{(T_1^{-1})^{(k)}v(y)}{\left(\log\frac{1}{|y|}\right)^{\frac{1}{2}}}\right)\right|^2 dy \\ &\qquad = \int_{B_1}\prod_{j=1}^{k+1}\log_j\frac{1}{|x|}\left|\frac{x}{|x|}\cdot\nabla\left(\frac{u(x)}{\left(\prod_{j=1}^{k+1}\log_j\frac{1}{|x|}\right)^{\frac{1}{2}}}\right)\right|^2 dx \end{aligned} \tag{41}$$

and

$$\begin{aligned} \frac{1}{4}\int_{B_1}\frac{\left|(T_1^{-1})^{(k)}v(y)\right|^2}{|y|^2\left(\log\frac{1}{|y|}\right)^2}dy &= \frac{1}{4}\int_{B_1}\left(\log_k\frac{1}{|x|}\right)^2\frac{|v(x)|^2}{\left(\log_{k+1}\frac{1}{|x|}\right)^2}\det\left(\frac{\partial y}{\partial x}\right)dx \\ &= \frac{1}{4}\int_{B_1}\frac{|u(x)|^2}{|x|^2\prod_{j=1}^{k+1}\left(\log_j\frac{1}{|x|}\right)^2}dx. \end{aligned} \tag{42}$$

Combining (38), (41), and (42), we obtain the relation (37) with $k+1 \in \mathbb{N}$ for $u \in C_0^\infty(B_{1/e_k}\setminus\{0\})$. This completes the proof of Theorem 3. □

Remark 2 A similar expansion as in Proposition 2 for higher dimensions $n \ge 3$ is also known (see [17]). However, to the best of our knowledge, the sharp constant with iterated logarithmic functions seems to be unknown for $n \geq 3$. The methods in the proof of Theorem 3 also works for higher dimensions, but it seems that the sharp constant for iterated logarithmic functions does not follow from such arguments.

4 The Relation Between the Sobolev Inequality and the Alvino Inequality via the Transformation of Horiuchi-Kumlin

As is mentioned in Sect. 1, the classical Hardy inequality (1) enjoys the invariance under the standard scaling (2). It is also well known that the Sobolev inequality

$$\left(\int_{\mathbb{R}^n} |v(x)|^{q^*} dx \right)^{\frac{1}{q^*}} \leq C_q \left(\int_{\mathbb{R}^n} |\nabla v(x)|^q \, dx \right)^{\frac{1}{q}}, \quad v \in W^{1,q}(\mathbb{R}^n), \tag{43}$$

where

$$1 \leq q < n, \quad C_q = \pi^{-\frac{1}{2}} n^{-\frac{1}{q}} \left(\frac{q-1}{n-q} \right)^{1-\frac{1}{q}} \left(\frac{\Gamma\left(1+\frac{n}{2}\right) \Gamma(n)}{\Gamma\left(\frac{n}{q}\right) \Gamma\left(1+n-\frac{n}{q}\right)} \right)^{\frac{1}{n}},$$

has the same invariance under the standard scaling (2). We recall that the inequality above yields the optimal embedding of $W^{1,q}(\mathbb{R}^n)$ into the Orlicz space (see [9, Example 1]), the constant C_q is sharp and is characterized by the Talenti functions (see [26]) with parameters $a, b > 0$:

$$v_{q,a,b}(x) = \left(a + (b|x|)^{\frac{q}{q-1}} \right)^{1-\frac{n}{q}} \tag{44}$$

since $v_{q,a,b}$ can be approximated by functions in $C_0^\infty(\mathbb{R}^n \setminus \{0\})$ in the $\|\nabla \cdot \|_{L^q}$ norm although it does not belong to $W^{1,q}(\mathbb{R}^n)$. We also recall another type of the critical inequality introduced by Alvino in [2]:

$$\sup_{x \in B_1} \frac{|u^\sharp(x)|}{\left(\log \frac{1}{|x|}\right)^{\frac{1}{n'}}} \leq \pi^{-\frac{1}{2}} n^{-\frac{1}{n}} \left(\Gamma\left(1+\frac{n}{2}\right)\right)^{\frac{1}{n}} \left(\int_{B_1} |\nabla u|^n dx \right)^{\frac{1}{n}}, \quad u \in W_0^{1,n}(B_1), \tag{45}$$

where again the constant in the right hand side is optimal. It is considered as an another version of the critical Sobolev inequality since (45) also yields the optimal embedding of $W_0^{1,n}(B_1)$ in the framework of Orlicz spaces (see [9, Example 1] and [5, Lemma 2.8]) which is similar to the classical case $1 \leq q < n$. In [8], Cassani, Ruf and Tarsi proved that the optimal constant in (45) is attained by the modified Moser sequence (m_b) with a parameter $b > 0$ given by

$$m_b(x) = \begin{cases} \omega_{n-1}^{-\frac{1}{n}} b^{\frac{1}{n}-1}, & |x| \le e^{-\frac{1}{b}}, \\ \omega_{n-1}^{-\frac{1}{n}} b^{\frac{1}{n}} \log \dfrac{1}{|x|}, & e^{-\frac{1}{b}} < |x| \le 1, \end{cases} \tag{46}$$

where ω_{n-1} is the surface measure of the unit sphere. The sequence (m_b) is essentially introduced by Moser [23] (see also [7, 8, 25]) to find the optimal exponent for the Trudinger-Moser inequality. It is worth mentioning that the similarity between the Sobolev inequality and the classical Hardy inequality concerning the scale invariance is also observed in the critical case $p = n$. Indeed, both of the critical Hardy inequality (6) and the Alvino inequality (45) have the same invariance under the power-type scaling (7). In this section, we show that the transformation T_p of Horiuchi-Kumlin defined by (36) gives a direct relation between the standard scaling (2) and the power-type scaling (7) and, as a by product, we will give in Sect. 4.2 the Alvino inequality from the Sobolev inequality. Somewhat surprisingly, these arguments also yields a Moser sequence from the Talenti functions.

4.1 The Horiuchi-Kumlin Transformation Revisited

Horiuchi-Kumlin showed in [18] that the critical Hardy inequality can be reduced to the classical Hardy inequality by using (36). Indeed, by combining Lemmas 3.2 and 3.3 in [18], for $n \in \mathbb{N}$ with $n \ge 2$, $1 < q < \infty$, and $s > -n + q$, one can obtain that the following inequalities (I) and (II) are equivalent via the transformation (36) with $p = \frac{q-1}{s+n-q}$ and its inverse transformation:

$$\begin{aligned} &\text{(I)} \quad \left(\frac{s+n-q}{q}\right)^q \int_{\mathbb{R}^n} |y|^{s-q} |u(y)|^q dy \le \int_{\mathbb{R}^n} |y|^s \left| \frac{y}{|y|} \cdot \nabla u \right|^q dy, \quad u \in C_0^\infty(\mathbb{R}^n \setminus \{0\}), \\ &\text{(II)} \quad \left(\frac{q-1}{q}\right)^q \int_{B_1} \frac{|v(x)|^q}{|x|^n \left(\log \frac{1}{|x|}\right)^q} dx \le \int_{B_1} |x|^{q-n} \left| \frac{x}{|x|} \cdot \nabla v \right|^q dx, \quad v \in C_0^\infty(B_1 \setminus \{0\}). \end{aligned} \tag{47}$$

It is easy to see that the inequality (I) in (47) with $s = 0$ is a classical Hardy inequality and (II) in (47) follows from (I) by the transformation (36) with $p = \frac{q-1}{n-q}$. In addition, taking the limit $q \to n$ in the inequality (II) in (47), we have the critical Hardy inequality

$$\left(\frac{n-1}{n}\right)^n \int_{B_1} \frac{|v(x)|^n}{|x|^n \left(\log \frac{1}{|x|}\right)^n} dx \le \int_{B_1} \left| \frac{x}{|x|} \cdot \nabla v \right|^n dx, \quad v \in C_0^\infty(B_1 \setminus \{0\}).$$

In this subsection, we revisit the transformation (36) in connection with the scale invariance of the classical Hardy inequality in $\mathbb{R}^n$ and the critical Hardy inequality in

B_1. We first show that the scale invariance of the classical Hardy inequality induces the invariance of the critical Hardy inequality under the power-type scaling by the transformation (36):

Proposition 4 *The inequality (I) in* (47) *is invariant under the standard scaling*

$$D_\lambda(u)(x) := u(\lambda x), \quad u : \mathbb{R}^n \to \mathbb{R}, \;\; \lambda > 0$$

and the inequality (II) in (47) *is invariant under the power-type scaling*

$$S_\lambda v(x) := v(|x|^{\lambda-1}x), \quad v : B_1(0) \to \mathbb{R}, \;\; \lambda > 0.$$

Moreover, there exists a following relation between D_λ and S_λ:

$$T_p\left(D_{\lambda^{-p}}u\right)(x) = S_\lambda(T_p u)(x), \quad u : \mathbb{R}^n \to \mathbb{R}, \quad \lambda > 0, \quad 0 < p < \infty.$$

Proof of Proposition 4. The first and the second assertion can be obtained by direct calculations. Indeed, by (39), the Jacobian matrix of a change of a variable $x \mapsto y = \left(\log \frac{1}{|x|}\right)^{-p} \frac{x}{|x|}$ is given by

$$\begin{aligned}\left(\frac{\partial(x_1, \cdots, x_n)}{\partial(y_1, \cdots, y_n)}\right) &= \frac{e^{-n|y|^{-\frac{1}{p}}}}{|y|^n}\left(\delta_{ij} + \left(\frac{1}{p}|y|^{-\frac{1}{p}} - 1\right)\frac{y_i y_j}{|y|^2}\right)_{1\le i,j\le n} \\ &= |x|\left(\log\frac{1}{|x|}\right)^p\left(\delta_{ij} + \left(\frac{1}{p}\log\frac{1}{|x|} - 1\right)\frac{x_i x_j}{|x|^2}\right)_{1\le i,j\le n},\end{aligned} \tag{48}$$

where δ_{ij} is the Kronecker delta. Therefore, it holds that

$$\det\left(\frac{\partial(y_1, \cdots, y_n)}{\partial(x_1, \cdots, x_n)}\right) = p\left(\log\frac{1}{|x|}\right)^{-pn-1}\frac{1}{|x|^n}. \tag{49}$$

These relations imply the first and the second assertions.
Also we have

$$T_p(D_{\lambda^{-p}}u)(x) = u\left(\lambda^{-p}\left(\log\frac{1}{|x|}\right)^{-p}\frac{x}{|x|}\right) = (T_p u)(|x|^{\lambda-1}x),$$

which implies the last assertion. $\square$

4.2 On the Talenti Functions and the Moser Sequences

In this subsection, we consider what kind of information can be obtained from the argument in Sect. 4.1 if we replace the classical Hardy inequality by the Sobolev inequality. First we introduce an inequality defined in B_1 which is equivalent to the Sobolev inequality in $\mathbb{R}^n$ via the Horiuchi-Kumlin transformation (36).

Theorem 4 *Let* $1 < q < n$ *and* $\frac{1}{q^*} = \frac{1}{q} - \frac{1}{n}$. *Then the following inequalities are equivalent:*

$$\text{(I)} \quad \left(\int_{B_1} \frac{|u(x)|^{q^*}}{|x|^n \left(\log \frac{1}{|x|}\right)^{pn+1}} dx \right)^{\frac{1}{q^*}}$$

$$\leq C_q p^{\frac{1}{n}} \left(\int_{B_1} |x|^{q-n} \left(\log \frac{1}{|x|}\right)^{pq-pn-1} \left| |\nabla u|^2 - |\partial_r u|^2 + \left(\frac{1}{p} \log \frac{1}{|x|}\right)^2 |\partial_r u|^2 \right|^{\frac{q}{2}} dx \right)^{\frac{1}{q}},$$

$$u \in C_0^\infty(B_1 \setminus \{0\}),$$

$$\text{(II)} \quad \left(\int_{\mathbb{R}^n} |v(y)|^{q^*} dy \right)^{\frac{1}{q^*}} \leq C_q \left(\int_{\mathbb{R}^n} |\nabla v|^q \, dy \right)^{\frac{1}{q}}, \quad v \in C_0^\infty(\mathbb{R}^n \setminus \{0\}),$$

where C_q *is the Sobolev best constant given by*

$$C_q = \pi^{-\frac{1}{2}} n^{-\frac{1}{q}} \left(\frac{q-1}{n-q}\right)^{1-\frac{1}{q}} \left(\frac{\Gamma\left(1+\frac{n}{2}\right)\Gamma(n)}{\Gamma\left(\frac{n}{q}\right)\Gamma\left(1+n-\frac{n}{q}\right)} \right)^{\frac{1}{n}}.$$

Proof of Theorem 4. First we derive (I) by assuming (II). Let $p > 0$, $u \in C_0^\infty(B_1 \setminus \{0\})$ and $v(y) := T_p^{-1} u(y)$ for $y \in \mathbb{R}^n \setminus \{0\}$. Then v satisfies (II) since $v \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$. By (48) and (49), it holds that

$$\left(\int_{\mathbb{R}^n} |v(y)|^{q^*} dy \right)^{\frac{1}{q^*}} = p^{\frac{1}{q^*}} \left(\int_{B_1} \frac{|u(x)|^{q^*}}{|x|^n \left(\log \frac{1}{|x|}\right)^{pn+1}} dx \right)^{\frac{1}{q^*}}$$

and

$$\left(\int_{\mathbb{R}^n} |\nabla v|^q \, dy \right)^{\frac{1}{q}}$$

$$= \left(\int_{B_1} p^1 |x|^{q-n} \left(\log \frac{1}{|x|}\right)^{pq-pn-1} \left| \left(\delta_{ij} + \left(\frac{1}{p} \log \frac{1}{|x|} - 1\right) \frac{x_i x_j}{|x|^2} \right)_{1 \leq i,j \leq n} \nabla u \right|^q dx \right)^{\frac{1}{q}}.$$

Direct calculations show

$$\left| \left(\delta_{ij} + \left(\frac{1}{p} \log \frac{1}{|x|} - 1 \right) \frac{x_i x_j}{|x|^2} \right)_{1 \le i,j \le n} \nabla u(x) \right|^2 = |\nabla u(x)|^2 + \left(\frac{1}{p^2} \left(\log \frac{1}{|x|} \right)^2 - 1 \right) |\partial_r u(x)|^2$$

for all $x \in B_1$, and we obtain the inequality (I). One can prove (II) from (I) by the same argument with the transformation T_p. □

Remark 3 The inequalities in Theorem 4 also have the same scale invariant structure as in Proposition 4.

By using the inequality (I) in Theorem 4, one can derive Alvino's inequality

$$\sup_{x \in B_1} \frac{|u^\sharp(x)|}{\left(\log \frac{1}{|x|} \right)^{\frac{1}{n'}}} \le \pi^{-\frac{1}{2}} n^{-\frac{1}{n}} \left(\Gamma \left(1 + \frac{n}{2} \right) \right)^{\frac{1}{n}} \left(\int_{B_1} |\nabla u|^n dx \right)^{\frac{1}{n}}, \quad u \in W_0^{1,n}(B_1) \quad (50)$$

from Sobolev's inequality (II) in Theorem 4.

Let us prove this assertion. We first assume that u is a radially symmetric function belonging to $C_0^\infty(B_1 \setminus \{0\})$. Note that $|\nabla u(x)| = |\partial_r u(x)|$ for all $x \in B_1$ when u is radially symmetric. Theorem 4 with $p = \frac{q-1}{n-q}$ leads

$$\left(\int_{B_1} \frac{|u(x)|^{q^*}}{|x|^n \left(\log \frac{1}{|x|} \right)^{\frac{q^*}{n'}}} dx \right)^{\frac{1}{q^*}}$$

$$\le \pi^{-\frac{1}{2}} n^{-\frac{1}{q}} \left(\frac{n-q}{q-1} \right)^{\frac{n-q}{nq}} \left(\frac{\Gamma\left(1 + \frac{n}{2}\right) \Gamma(n)}{\Gamma\left(\frac{n}{q}\right) \Gamma\left(1 + n - \frac{n}{q}\right)} \right)^{\frac{1}{n}} \left(\int_{B_1} |x|^{q-n} |\nabla u|^q \, dx \right)^{\frac{1}{q}}$$

by virtue of the Sobolev inequality

$$\left(\int_{\mathbb{R}^n} |v(y)|^{q^*} dy \right)^{\frac{1}{q^*}} \le C_q \left(\int_{\mathbb{R}^n} |\nabla v|^q \, dy \right)^{\frac{1}{q}}.$$

It remains to prove

$$\lim_{q \uparrow n} \left(\int_{B_1} \frac{|u(x)|^{q^*}}{|x|^n \left(\log \frac{1}{|x|} \right)^{\frac{q^*}{n'}}} dx \right)^{\frac{1}{q^*}} = \sup_{x \in B_1} \frac{|u(x)|}{\left(\log \frac{1}{|x|} \right)^{\frac{1}{n}}}, \quad u \in C_0^\infty(B_1 \setminus \{0\}). \quad (51)$$

Take $\varepsilon > 0$ and consider

$$A_\varepsilon := \left\{ x \in B_1 \;\middle|\; \frac{|u(x)|}{\left(\log \frac{1}{|x|}\right)^{\frac{1}{n'}}} > \sup_{x \in B_1} \frac{|u(x)|}{\left(\log \frac{1}{|x|}\right)^{\frac{1}{n'}}} - \varepsilon \right\}.$$

The fact $u \in C_0^\infty(B_1 \setminus \{0\})$ implies that A_ε has positive measure and $B_\delta(0) \cap A_\varepsilon = \phi$ holds for some $\delta > 0$. Therefore we have the positivity and the finiteness of $\int_{A_\varepsilon} \frac{dx}{|x|^n}$, thus

$$\left(\int_{B_1} \frac{|u(x)|^{q^*}}{|x|^n \left(\log \frac{1}{|x|}\right)^{\frac{q^*}{n'}}} dx \right)^{1/q^*} > \left(\int_{A_\varepsilon} \frac{dx}{|x|^n} \right)^{1/q^*} \left(\sup_{x \in B_1} \frac{|u(x)|}{\left(\log \frac{1}{|x|}\right)^{\frac{1}{n'}}} - \varepsilon \right) \tag{52}$$

follows. Hence it holds that

$$\liminf_{q \uparrow n} \left(\int_{B_1} \frac{|u(x)|^{q^*}}{|x|^n \left(\log \frac{1}{|x|}\right)^{\frac{q^*}{n'}}} dx \right)^{1/q^*} \geq \sup_{x \in B_1} \frac{|u(x)|}{\left(\log \frac{1}{|x|}\right)^{\frac{1}{n'}}} - \varepsilon, \quad \varepsilon > 0. \tag{53}$$

On the other hand, since $B_\delta(0) \cap \operatorname{supp} u = \phi$ for sufficiently small $\delta > 0$, $\frac{1}{|x|^n}$ is integrable in $\operatorname{supp} u$. Hence we have

$$\left(\int_{B_1} \frac{|u(x)|^{q^*}}{|x|^n \left(\log \frac{1}{|x|}\right)^{\frac{q^*}{n'}}} dx \right)^{1/q^*} \leq \left(\int_{\operatorname{supp} u} \frac{dx}{|x|^n} \right)^{1/q^*} \left(\sup_{x \in B_1} \frac{|u(x)|}{\left(\log \frac{1}{|x|}\right)^{\frac{1}{n'}}} \right),$$

which yields

$$\limsup_{q \uparrow n} \left(\int_{B_1} \frac{|u(x)|^{q^*}}{|x|^n \left(\log \frac{1}{|x|}\right)^{\frac{q^*}{n'}}} dx \right)^{1/q^*} \leq \left(\sup_{x \in B_1} \frac{|u(x)|}{\left(\log \frac{1}{|x|}\right)^{\frac{1}{n'}}} \right).$$

This together with (53) yields (51), hence

$$\sup_{x \in B_1} \frac{|u(x)|}{\left(\log \frac{1}{|x|}\right)^{\frac{1}{n'}}} \leq \pi^{-\frac{1}{2}} n^{-\frac{1}{n}} \left(\Gamma \left(1 + \frac{n}{2} \right) \right)^{\frac{1}{n}} \left(\int_{B_1} |\nabla u|^n dx \right)^{\frac{1}{n}} \tag{54}$$

holds for a radially symmetric function u in $C_0^\infty(B_1 \setminus \{0\})$. We now assume $u \in W_0^{1,n}(B_1)$. Since $u^\sharp$ is radially symmetric function belonging to $W_0^{1,n}(B_1)$, one can construct radially symmetric functions $\{u_n\} \subset C_0^\infty(B_1 \setminus \{0\})$ satisfying $u_n \to u^\sharp$ in $W^{1,n}(B_1)$ by following the proof of the density of $C_0^\infty(B_1 \setminus \{0\})$ in $W_0^{1,n}(B_1)$ (see Sect. 8.1 in [18]). Therefore, applying (54) to u_n and taking $n \to \infty$ together with the Pólya-Szegö principle, we obtain the Alvino inequality (50).

Remark 4 The argument above implies that the best constant in the inequality (I) in Theorem 4 is characterized by the function which is a composition of the Talenti function and the transformation $T_{\frac{q}{n-q}}$:

$$u_{q,a,b}(x) = \left(a + \left(b \log \frac{1}{|x|} \right)^{-\frac{q}{n-q}} \right)^{1-\frac{n}{q}} = \frac{b \log \frac{1}{|x|}}{\left(a \left(b \log \frac{1}{|x|} \right)^{\frac{q}{n-q}} + 1 \right)^{1-\frac{n}{q}}}, \quad a, b > 0 \tag{55}$$

since the sharp constant of the inequality (II) in Theorem 4 is characterized by the Talenti functions

$$v_{q,a,b}(y) = \left(a + (b|y|)^{\frac{q}{q-1}} \right)^{1-\frac{n}{q}}, \quad a, b > 0.$$

Recall that the Talenti function does not belong to $C_0^\infty(\mathbb{R}^n \setminus \{0\})$ but can be approximated by functions in $C_0^\infty(\mathbb{R}^n \setminus \{0\})$ in the $\|\nabla \cdot\|_{L^q}$ norm. Now taking the limit $q \uparrow n$ in (55) by noting

$$\lim_{q \uparrow n} \left(a \left(b \log \frac{1}{|x|} \right)^{\frac{q}{n-q}} + 1 \right)^{\frac{n-q}{q}} = \begin{cases} b \log \dfrac{1}{|x|}, & |x| < e^{-\frac{1}{b}}, \\ 1, & |x| \geq e^{-\frac{1}{b}}, \end{cases}$$

we obtain

$$u_b(x) := \lim_{q \uparrow n} u_{q,a,b}(x) = \begin{cases} 1, & |x| < e^{-\frac{1}{b}}, \\ b \log \dfrac{1}{|x|}, & e^{-\frac{1}{b}} \leq |x| \leq 1. \end{cases} \tag{56}$$

Finally, normalizing (56) by $\|\nabla u_b\|_{L^n}$, we obtain the modified Moser sequence (m_b) with a parameter $b > 0$ defined in (46). From this view-point, the modified Moser sequence arises naturally from the Talenti functions via the Horiuchi-Kumlin transformation (36) and the limit procedure $q \uparrow n$ which preserves the scale invariant structure in the sense of Proposition 4.

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Stability Analysis of Delaunay Surfaces as Steady States for the Surface Diffusion Equation

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Abstract The stability of steady states for the surface diffusion equation will be studied. In the axisymmetric setting, steady states are the Delaunay surfaces, which are the axisymmetric constant mean curvature surfaces. We consider a linearized stability of these surfaces and derive criteria of the stability by investigating the sign of eigenvalues corresponding to the linearized problem.

Keywords Surface diffusion equation · Delaunay surfaces · Stability

1 Introduction

Let $\Gamma_t \subset \mathbb{R}^3$ be a moving surface with respect to time t governed by the geometric evolution law

$$V = -\Delta_{\Gamma_t} H \;\text{ on }\; \Gamma_t, \tag{1}$$

where V is the normal velocity of Γ_t, H is the mean curvature of Γ_t, and Δ_{Γ_t} is the Laplace-Beltrami operator on Γ_t. In our sign convention, the mean curvature H for spheres with outer unit normal is negative. Equation (1) is called surface diffusion equation. The surface diffusion equation (1) was first derived by Mullins [12] to model the motion of interfaces in the case that the motion of interfaces is governed purely by mass diffusion within the interfaces. (For simplicity, we choose 1 as the diffusion constant.) Also, Cahn and Taylor [14] showed that (1) is the H^{-1}-gradient flow of the area functional of Γ_t, so that this geometric evolution equation has a variational structure that the area of the surface decreases whereas the volume of the region enclosed by the surface is preserved.

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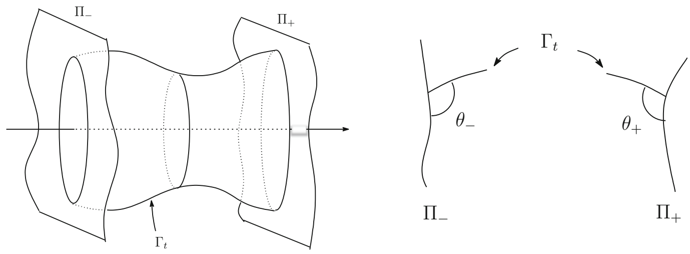


Fig. 1 Setting of (2)

In this paper, we consider the following problem. For $\phi_\pm : \mathbb{R}_+ \to \mathbb{R}$, set

$$\Pi_\pm = \{(\phi_\pm(|\boldsymbol{\eta}|), \boldsymbol{\eta})^T \mid \boldsymbol{\eta} \in \mathbb{R}^2\},$$
$$\Omega = \{(x, \boldsymbol{\eta})^T \mid \phi_-(|\boldsymbol{\eta}|) \le x \le \phi_+(|\boldsymbol{\eta}|),\ \boldsymbol{\eta} \in \mathbb{R}^2\}.$$

Note that $\partial\Omega = \Pi_-$ or Π_+. Let us assume that $\Gamma_t \subset \Omega$ and the motion of Γ_t is governed by

$$\begin{cases} V = -\Delta_{\Gamma_t} H \ \text{ on } \ \Gamma_t, \\ (N_{\Gamma_t}, N_{\Pi_\pm})_{\mathbb{R}^3} = \cos\theta_\pm \ \text{ on } \ \Gamma_t \cap \Pi_\pm, \\ (\nabla_{\Gamma_t} H, \nu_\pm)_{\mathbb{R}^3} = 0 \ \text{ on } \ \Gamma_t \cap \Pi_\pm, \\ \Gamma_t|_{t=0} = \Gamma_0. \end{cases} \tag{2}$$

Here, N_{Γ_t} and $N_{\Pi_\pm}$ are the outer unit normals to Γ_t and $\Pi_\pm$ $(= \partial\Omega)$, respectively, and $\nu_\pm$ are the outer unit co-normals to $\partial\Gamma_t$ on $\Gamma_t \cap \Pi_\pm$. The problem (2) are obtained as the H^{-1}-gradient flow of the capillary energy

$$\text{Area}\,[\Gamma_t] + \mu_+\text{Area}\,[\Sigma_{t,+}] + \mu_-\text{Area}\,[\Sigma_{t,-}],$$

where $\Sigma_{t,\pm}$ are the part of $\Pi_\pm$ with the boundary $\partial\Sigma_{t,\pm} = \Gamma_t \cap \Pi_\pm$. Note that contact angles $\theta_\pm$ are given by $\cos\theta_\pm = \mu_\pm$ (Fig. 1).

Let Γ_* be the steady states for (2) and H_* be the mean curvature of Γ_*. Then H_* satisfies

$$\begin{cases} \Delta_{\Gamma_*} H_* = 0 \ \text{ on } \ \Gamma_*, \\ (\nabla_{\Gamma_*} H_*, \nu_\pm)_{\mathbb{R}^3} = 0 \ \text{ on } \ \Gamma_* \cap \Pi_\pm. \end{cases}$$

This implies

$$\|\nabla_{\Gamma_*} H_*\|^2_{L^2(\Gamma_*)} = 0,$$

so that we see that the steady states of (2) are the constant mean curvature surfaces (CMC surfaces). In this paper, we consider the axisymmetric CMC surfaces, which are called Delaunay surfaces, as the steady states Γ_*, and analyze the eigenvaule problem corresponding to the linearized problem for (2) under axisymmetric perturbations in order to derive the criteria for the stability of Γ_* in an axisymmetric direction.

Let us introduce the results on the stability of Delaunay surfaces as the variational problem for the capillary energy. In the case that $\Pi_\pm$ are the parallel planes and $\theta_\pm = \pi/2$, Athanassenas [1] and Vogel [15] proved that cylinders are stable if $d \le r\pi$ and unstable if $d > r\pi$, where d and r are the length and the radius of a cylinder, respectively. Moreover, they showed that unduloids are unstable. Vogel [16] also studied the case of general contact angles. In the case that $\Pi_\pm$ are spheres, Vogel [17] analyzed the stability for convex unduloids and nodoids by using a method which allows to consider asymmetric perturbations. In [18, 19], Vogel also showed some partial results for this case. Recently, Fel and Rubinstein [8, 13] derived the precise criteria of the stability by means of analysis based on the Weierstrass representation of the 2nd variation for the capillary energy under axisymmetric perturbations. In [8] they studied the stability for catenoids and cylinders in the case that $\Pi_\pm$ are spheres, paraboloids, catenoids, ellipsoids, and that $\Pi_\pm$ are sphere and plane, and in [13] they analyzed the stability for unduloids and nodoids in the case that $\Pi_\pm$ are spheres. Our results give criteria for the stability of Delaunay surfaces by a different approach from that of Fel and Rubinstein, and generalizes their results in the point that $\Pi_\pm$ are more general surfaces.

We also introduce the results on dynamics of moving surfaces governed by (1). Bernoff, Bertozzi, and Witelski [4] studied the stability of the axisymmetric equilibria for (1). For cylinders and spheres, they got the spectrum of the linearized operator in a periodic setting and analyzed the stability of them. Moreover, with the help of numerical computation, they showed that unduloids form an unstable branch of solutions that bifurcates subcritically from the cylinder and reconnects to the sphere in a singular limit involving a change of topology. LeCrone and Simonett [11] proved the well-posedness for the axisymmetric surface diffusion equation with periodic boundary conditions and that the family of cylinders with the radius bigger than a critical radius are asymptotically, exponentially stable under a large class of nonlinear perturbations which maintain the same axis of symmetry. In addition, they rigorously showed the existence of branches of bifurcating equilibria which intersect the family of cylinders at critical radii. We also refer Athanassenas [2] and Athanassenas and Kandanaarachchi [3]. Under an axisymmetric setting, they studied the convergence to a CMC surface (in [3], to a hemisphere or a sphere) of evolving hypersurfaces governed by volume-preserving mean curvature flow which has the same variational structure as (1).

2 Delaunay Surfaces and the Linearized Problem

Let Γ_* be a axisymmetric steady state of (2) and set

$$\Gamma_* = \{(x_*(s), y_*(s)\cos\zeta, y_*(s)\sin\zeta)^T \mid s \in [0, d], \zeta \in [0, 2\pi]\},$$

where s is the arc-length parameter of a generating curve $(x_*(s), y_*(s))^T$. We refer to the following theorem on the representation of Delaunay surfaces.

Theorem 2.1 ([10]) *Let H_* be a constant satisfying $H_* \neq 0$ (assuming $H_* < 0$). Then a generating curve $(x_*(s), y_*(s))$ of an axisymmetric surface with a constant mean curvature H_* is represented by*

$$x_*(s) = \int_0^s \frac{1 - B\sin(2H_*(\sigma - \tau))}{\sqrt{1 + B^2 - 2B\sin(2H_*(\sigma - \tau))}}\,d\sigma,$$
$$y_*(s) = -\frac{1}{2H_*}\sqrt{1 + B^2 - 2B\sin(2H_*(s - \tau))},$$

where $B \geq 0$ is a constant.

Theorem 2.1 implies cylinders ($B = 0$), unduloids ($0 < B < 1$), spheres ($B = 1$), and nodoids ($B > 1$). In this paper, we only consider cylinders (see Fig. 2) and unduloids (see Fig. 3).

For the steady states $\Gamma_* = \{(x_*(s), y_*(s)\cos\zeta, y_*(s)\sin\zeta)^T \mid 0 \leq s \leq d, \zeta \in [0, 2\pi]\}$, which are cylinders or unduloids, set

$$\Phi_*(s,\zeta) = (x_*(s), y_*(s)\cos\zeta, y_*(s)\sin\zeta)^T.$$

Recall $\Omega = \{(x, \boldsymbol{\eta})^T \mid \phi_-(|\boldsymbol{\eta}|) \leq x \leq \phi_+(|\boldsymbol{\eta}|),\ \boldsymbol{\eta} \in \mathbb{R}^2\}$ and define

$$\gamma(s,\rho) = \gamma_-(\rho) + \frac{s}{d}\{\gamma_+(\rho) - \gamma_-(\rho)\},$$

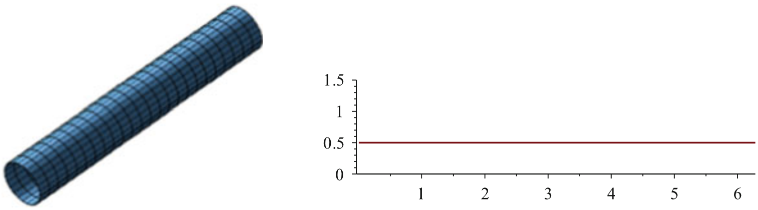


Fig. 2 (*Left*) cylinder ($H_* \neq 0,\ B = 0$), (*Right*) a generating curve of a cylinder

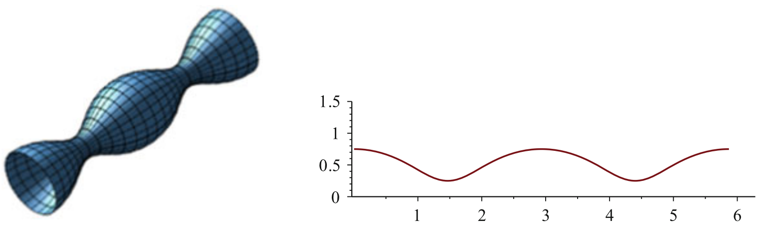


Fig. 3 (*Left*) unduloid ($H_* \neq 0,\ 0 < B < 1$), (*Right*) a generating curve of a unduloid

where

$$\gamma_-(\rho) = \min\{s \mid \Phi_*(s,\zeta) + \rho N_*(s,\zeta) \in \Omega\},$$
$$\gamma_+(\rho) = \max\{s \mid \Phi_*(s,\zeta) + \rho N_*(s,\zeta) \in \Omega\}.$$

Note that $\gamma_-(0) = 0, \gamma_+(0) = d, \gamma(s,0) = s, \gamma(0,\rho) = \gamma_-(\rho)$, and $\gamma(d,\rho) = \gamma_+(\rho)$. In addition, set

$$\Psi(s,\zeta,\rho) = \Phi_*(\gamma(s,\rho),\zeta) + \rho N_*(\gamma(s,\xi),\zeta)$$

and define for $v : [0,d] \times [0,T] \to [-\varepsilon,\varepsilon]$, $(s,t) \mapsto v(s,t)$

$$\Phi(s,\zeta,t) = \Psi(s,\zeta,v(s,t)).$$

Then we give an axisymmetric perturbation Γ_t from Γ_* by

$$\Gamma_t = \{\Phi(s,\zeta,t) \mid s \in [0,d],\ \zeta \in [0,2\pi],\ t \in [0,T]\}.$$

This implies the nonlinear problem

$$\begin{cases} V(v_t, v, \partial_s v) = -\Delta(v,\partial_s v) H(v,\partial_s v,\partial_s^2 v) \text{ for } (s,t) \in [0,d] \times [0,T], \\ (N(v,\partial_s v), N_{\Pi_\pm}(v))_{\mathbb{R}^3} = \cos\theta_\pm \text{ at } s = 0, d,\ t \in [0,T], \\ (\nabla_{\Gamma_t} H(v,\partial_s v,\partial_s^2 v), \nu_\pm(v,\partial_s v))_{\mathbb{R}^3} = 0 \text{ at } s = 0, d,\ t \in [0,T], \end{cases} \tag{3}$$

where $N_{\Pi_\pm}$ are given by

$$N_{\Pi_\pm} = \mp \frac{(-1, \dot{\phi}_\pm(|\boldsymbol{\eta}|)\boldsymbol{\xi})^T}{\sqrt{1 + (\dot{\phi}_\pm(|\boldsymbol{\eta}|))^2}} \quad \left(\boldsymbol{\xi} = \frac{\boldsymbol{\eta}}{|\boldsymbol{\eta}|} \in S^1\right).$$

Linearizing (3), we obtain

$$\begin{cases} v_t = -\frac{1}{2}\Delta_{\Gamma_*} L[v], \\ \partial_s v \pm (\kappa_{\Pi_\pm} \csc\theta_\pm - \kappa_{\Gamma_*} \cot\theta_\pm) v = 0, \\ \partial_s L[v] = 0, \end{cases} \tag{4}$$

where $L[v] = \Delta_{\Gamma_*} v + |A_*|^2 v$ with

$$\Delta_{\Gamma_*} = \frac{1}{y_*}\left\{\partial_s(y_*\partial_s) + \frac{1}{y_*}\partial_\zeta^2\right\}, \quad |A_*|^2 = (-x_*'' y_*' + x_*' y_*'')^2 + \left(\frac{x_*'}{y_*}\right)^2$$

and

$$\kappa_{\Pi_\pm} = \pm \frac{\ddot{\phi}_\pm(y_*)}{\{1 + (\dot{\phi}_\pm(y_*))^2\}^{3/2}}, \quad \kappa_{\Gamma_*} = -x_*'' y_*' + x_*' y_*''.$$

Taking account of the fact that v is independent of ζ, we have

$$\Delta_{\Gamma_*} v = \frac{1}{y_*} \{\partial_s (y_* \partial_s v)\}.$$

Remark 2.1 In this paper, we consider only axisymmetric perturbations. According to Vogel [15, Lemma 2.1], the Schwarz symmetrization works in the setting that $\Pi_\pm$ are parallel planes, so that it is proved that the capillary energy for axisymmetric perturbations is smaller than that for asymmetric perturbations. Thus, in such case, it is seems that our analysis is enough to discuss the stability. But, in the case that $\Pi_\pm$ are not parallel planes, the Schwarz symmetrization does not work well. At present, we are not sure whether an analysis under axisymmetric perturbations is enough to study the stability. This means that there is a possibility that steady states are unstable in an asymmetric direction even if they are (linearly) stable in axisymmetric directions.

3 Gradient Flow Structure for the Linearized Problem

Let consider the gradient flow structure for the linearized problem (4). Set

$$\mathcal{E} = \left\{ v \in H^1(\Gamma_*) \,\Big|\, \int_0^d v\, y_* \, ds = 0 \right\},$$
$$\mathcal{X} = \left\{ v \in (H^1(\Gamma_*))^* \,\big|\, \langle v, 1 \rangle = 0 \rangle \right\},$$

where $(H^1(\Gamma_*))^*$ is the duality space of $H^1(\Gamma_*)$ and $\langle \cdot, \cdot \rangle$ is the duality pairing $(H^1(\Gamma_*))^*$ and $H^1(\Gamma_*)$.

Definition 3.1 Let $\zeta \in \mathcal{X}$. Then we say that $u_\zeta \in \mathcal{E}$ is a weak solution of

$$\begin{cases} -\Delta_{\Gamma_*} u_\zeta = \zeta \text{ for } \sigma \in (0, d), \\ \partial_s u_\zeta = 0 \qquad \text{at } \sigma = 0, d \end{cases} \tag{5}$$

if and only if u_ζ satisfies

$$\int_0^d \partial_s u_\zeta \partial_s \phi \, y_* ds = \langle \zeta, \phi \rangle \quad (\phi \in \mathcal{E}). \tag{6}$$

Definition 3.2 Let $\zeta \in \mathcal{X}$. Then we say that $v \in H^3(\Gamma_*)$ with

$$\int_0^d v\, y_* ds = 0$$

is a weak solution of

$$\begin{cases} -\Delta_{\Gamma_*} L[v] = \zeta, \\ \partial_s v \pm (\kappa_{\Pi_\pm} \csc\theta_\pm - \kappa_{\Gamma_*} \cot\theta_\pm) v = 0, \\ \partial_s L[v] = 0 \end{cases} \tag{7}$$

if and only if v satisfies

$$\int_0^d \partial_s L[v]\, \partial_s \phi\, y_* ds = \langle \zeta, \phi \rangle \quad (\phi \in \mathcal{E}) \tag{8}$$

and the boundary conditions

$$\partial_s v \pm (\kappa_{\Pi_\pm} \csc\theta_\pm - \kappa_{\Gamma_*} \cot\theta_\pm) v = 0 \quad \text{at} \quad s = 0, d. \tag{9}$$

Now, we define the symmetric bilinear form

$$\begin{aligned} I[v_1, v_2] = &\int_0^d \left\{ \partial_s v_1 \partial_s v_2 - |A_*|^2 v_1 v_2 \right\} y_*\, ds \\ &+ y_* (\kappa_{\Pi_+} \csc\theta_+ - \kappa_{\Gamma_*} \cot\theta_+) v_1 v_2 \big|_{s=d} \\ &+ y_* (\kappa_{\Pi_-} \csc\theta_- - \kappa_{\Gamma_*} \cot\theta_-) v_1 v_2 \big|_{s=0}. \end{aligned}$$

and the H^{-1}-inner product

$$(v_1, v_2)_{-1} = \int_0^d \partial_s u_{v_1} \partial_s u_{v_2}\, y_* ds,$$

where u_{v_i} is a weak solution of (5) for $v_i \in \mathcal{X}$. Then we obtain the following theorem.

Theorem 3.1 *Let $\zeta \in \mathcal{X}$ and $v \in \mathcal{E}$. Then the following (i) and (ii) are equivalent.*

(i) $v \in H^3(\Gamma_*)$ *and* v *is a weak solution of* (7);
(ii) v *satisfies*

$$-I[v, \phi] = (\zeta, \phi)_{-1} \quad (\phi \in \mathcal{E}). \tag{10}$$

Applying a similar argument to that of [6, 9], we can prove this theorem. Thus we omit the proof.

4 Eigenvalue Problem

Let us consider the eigenvalue problem for (4).

$$\begin{cases} -\Delta_{\Gamma_*} L[w] = \lambda w \quad \text{for} \ \ s \in (0, d), \\ \partial_s w \pm (\kappa_{\Pi_\pm} \csc\theta_\pm - \kappa_{\Gamma_*} \cot\theta_\pm) w = 0 \quad \text{at} \ \ s = 0, d, \\ \partial_s L[w] = 0 \quad \text{at} \ \ s = 0, d. \end{cases} \tag{11}$$

Set

$$\mathcal{D}(\mathcal{A}) = \left\{ w \in H^3(\Gamma_*) \,\middle|\, w \text{ satisfies (9) and } \int_0^d w\, y_*\, ds = 0 \right\}$$

and define the linear operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \to \mathcal{X}$ by

$$\langle \mathcal{A}w, \phi \rangle = \int_0^d \partial_s L[w]\, \partial_s \phi\, y_* ds \quad (w \in \mathcal{D}(\mathcal{A}),\ \phi \in \mathcal{E}).$$

Then, by the definition of $\mathcal{A}$ and Theorem 3.1, we obtain

$$(\mathcal{A}w, \phi)_{-1} = -I[w, \phi] \quad (\phi \in \mathcal{E}).$$

This easily implies that $\mathcal{A}$ is symmetric with respect to the inner product $(\cdot, \cdot)_{-1}$. Then we have the following theorem.

Theorem 4.1 *$\mathcal{A}$ is self-adjoint with respect to $(\cdot\,, \cdot)_{-1}$.*

Applying a similar argument to that of [6, 9], we can prove this theorem. Thus we omit the proof.

Lemma 4.1 *Let $\{\lambda_n\}_{n\in\mathbb{N}}$ be the eigenvalues of $\mathcal{A}$ with $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots$. Then the following properties hold.*

(i) For $n \in \mathbb{N}$, $n \geq 2$,

$$\lambda_1 = -\inf_{w\in\mathcal{E}\setminus\{0\}} \frac{I[w, w]}{(w, w)_{-1}}, \qquad \lambda_n = -\sup_{\mathcal{W}\in\Sigma_{n-1}} \inf_{w\in\mathcal{W}^\perp\setminus\{0\}} \frac{I[w, w]}{(w, w)_{-1}}.$$

Here, Σ_n is the class of subspaces of $\mathcal{E}$ with dimension n and $\mathcal{W}^\perp$ is the orthogonal subspace of $\mathcal{W}$ with respect to H^{-1}-inner product.

(ii) The eigenvalues of $\mathcal{A}$ depend continuously on $\kappa_{\Pi_\pm} \csc\theta_\pm$, $\kappa_{\Gamma_{\rho_*}} \cot\theta_\pm$, and d, and are monotone decreasing with respect to $\kappa_{\Pi_\pm} \csc\theta_\pm$.

Applying a similar argument to [5, Chap. VI] and using Theorem 3.1, we can prove this lemma. Thus we omit the proof.

5 Stability Analysis

We say that the steady states Γ_* is linearly stable if and only if all of eigenvalues of $\mathcal{A}$ are negative, that is, the maximal eigenvalue λ_1 is negative. In order to analyze the stability of Γ_*, we first introduce a weighted Wirtinger inequality shown by Farroni, Giova, and Ricciardi [7] (We introduce only the case of $p = 2$ in [7].)

Lemma 5.1 ([7]) *Let $N \geq 1$ and $T > 0$ and let $\rho : [0, T] \to \mathbb{R}$ be a measurable function such that $1 \leq \rho(t) \leq L$. Then the inequality*

$$\int_0^T |\boldsymbol{u}(t)|^2 \rho(t)\, dt \leq C_b \int_0^T |\boldsymbol{u}'(t)|^2 \rho(t)\, dt \tag{12}$$

holds for $\boldsymbol{u} \in H_0^1(0, T; \mathbb{R}^N)$, where

$$C_b = \left(\frac{T}{4 \arctan L^{-1/2}} \right)^2 .$$

Remark 5.1 Set

$$\alpha_* := \min_{s \in [0,d]} y_*(s)\ (> 0), \quad \beta_* := \max_{s \in [0,d]} y_*(s),$$
$$C_{*,d} := \left(\frac{d}{4 \arctan \sqrt{\alpha_*/\beta_*}} \right)^2 . \tag{13}$$

In the case of cylinders, y_* is represented by

$$y_*(s) = -\frac{1}{2H_*} \ \ (H_* < 0),$$

so that we see $\alpha_* = \beta_* = -1/\,(2H_*)$. This gives

$$C_{*,d} = \left(\frac{d}{\pi} \right)^2 .$$

In the case of unduloids, y_* is represented by

$$y_*(s) = -\frac{1}{2H_*} \sqrt{1 + B^2 - 2B \sin(2H_*(s - \tau))} \ \ (H_* < 0,\ 0 < B < 1),$$

so that we have $\alpha_* \geq -(1 - B)/\,(2H_*)$, $\beta_* \leq -(1 + B)/\,(2H_*)$. This implies $\alpha_*/\beta_* \geq (1 - B)/(1 + B)$. Thus we obtain

$$C_{*,d} \leq \left(\frac{d}{4 \arctan \sqrt{(1 - B)/(1 + B)}} \right)^2 .$$

Set

$$M := \max_{s\in[0,d]} |A_*|, \quad \Lambda_\pm := \kappa_{\Pi_\pm} \csc\theta_\pm - \kappa_{\Gamma_*} \cot\theta_\pm.$$

Lemma 5.2 *Assume that* $Md < \dfrac{\pi}{\sqrt{C_{*,\pi}+\varepsilon}}$ *for* $\varepsilon > 0$*, where* $C_{*,\pi}$ *is given by* (13) *with* $d = \pi$*. Then there exists an* $m > 0$ *such that*

$$I[w, w] > 0 \quad (w \in \mathcal{E} \setminus \{0\})$$

provided that $\Lambda_-, \Lambda_+ > m$*.*

Proof By the definition of I, we have

$$\begin{aligned} I[w, w] &= \int_0^d \left\{|\partial_s w|^2 - |A_*|^2 w^2\right\} y_*\, ds + y_* \Lambda_+ w^2\big|_{s=d} + y_* \Lambda_- w^2\big|_{s=0} \\ &\geq \int_0^d |\partial_s w|^2\, y_* ds - M^2 \int_0^d w^2\, y_*\, ds + y_* \Lambda_+ w^2\big|_{s=d} + y_* \Lambda_- w^2\big|_{s=0}. \end{aligned}$$

Set

$$I_M[w, w] := \int_0^d |\partial_s w|^2\, y_* ds - M^2 \int_0^d w^2\, y_*\, ds + y_* \Lambda_+ w^2\big|_{s=d} + y_* \Lambda_- w^2\big|_{s=0}.$$

Then we should prove that there exists an $m > 0$ such that $I_M[w, w] > 0\,(w \in \mathcal{E} \setminus \{0\})$ provided that $\Lambda_-, \Lambda_+ > m$. Applying the change of variable

$$\tilde{w}(\sigma) = w\Big(\frac{d}{\pi}\sigma\Big), \quad \tilde{y}_*(\sigma) = \frac{1}{\alpha_*} y_*\Big(\frac{d}{\pi}\sigma\Big)$$

and taking account of $Md < \dfrac{\pi}{\sqrt{C_{*,\pi}+\varepsilon}}$, we obtain

$$\begin{aligned} &I_M[w, w] \\ &= \frac{\pi\alpha_*}{d}\left\{\int_0^\pi |\partial_\sigma \tilde{w}|^2\, \tilde{y}_* d\sigma - \Big(\frac{Md}{\pi}\Big)^2 \int_0^\pi \tilde{w}^2\, \tilde{y}_* d\sigma + \frac{d}{\pi}\big(\tilde{y}_* \Lambda_+ \tilde{w}^2|_{\sigma=\pi} + \tilde{y}_* \Lambda_- \tilde{w}^2|_{\sigma=0}\big)\right\} \\ &\geq \frac{\pi\alpha_*}{d}\left\{\int_0^\pi |\partial_\sigma \tilde{w}|^2\, \tilde{y}_* d\sigma - \frac{1}{C_{*,\pi}+\varepsilon} \int_0^\pi \tilde{w}^2\, \tilde{y}_* d\sigma + \frac{d}{\pi}\big(\tilde{y}_* \Lambda_+ \tilde{w}^2|_{\sigma=\pi} + \tilde{y}_* \Lambda_- \tilde{w}^2|_{\sigma=0}\big)\right\}. \end{aligned}$$

Setting

$$\tilde{\mathcal{E}} = \left\{\tilde{w} \in H^1(0,\pi) \,\Big|\, \int_0^\pi \tilde{w}\, \tilde{y}_*\, d\sigma = 0\right\},$$

we should prove that for $\tilde{w} \in \tilde{\mathcal{E}} \setminus \{0\}$ there exists an $m > 0$ such that

$$\|\partial_\sigma \tilde{w}\|^2_{L^2(0,\pi;\tilde{y}_*)} - \frac{1}{C_{*,\pi}+\varepsilon} \|\tilde{w}\|^2_{L^2(0,\pi;\tilde{y}_*)} + \tilde{m}\big(\tilde{y}_* \tilde{w}^2\big|_{\sigma=\pi} + \tilde{y}_* \tilde{w}^2\big|_{\sigma=0}\big) > 0.$$

Let us apply a contradiction argument. Suppose that there are no $m > 0$ such that the above inequality is fulfilled. This gives that for any $n \in \mathbb{N}$ there exists $\tilde{w}_n \in \tilde{\mathcal{E}} \setminus \{0\}$ such that

$$\|\partial_\sigma \tilde{w}_n\|^2_{L^2(0,\pi;\tilde{y}_*)} - \frac{1}{C_{*,\pi} + \varepsilon}\|\tilde{w}_n\|^2_{L^2(0,\pi;\tilde{y}_*)} + n\left(\tilde{y}_* \tilde{w}_n^2\big|_{\sigma=\pi} + \tilde{y}_* \tilde{w}_n^2\big|_{\sigma=0}\right) \le 0.$$

Without loss of generality, we can take $\|\tilde{w}_n\|_{L^2(0,\pi;\tilde{y}_*)} = 1$. Then we have

$$\|\partial_\sigma \tilde{w}_n\|^2_{L^2} \le \frac{1}{C_{*,\pi} + \varepsilon}.$$

Thus there exists a subsequence $\{\tilde{w}_{n_k}\} \subset \{\tilde{w}_n\}$ such that

$$\begin{aligned} &\partial_\sigma \tilde{w}_{n_k} \to \partial_\sigma \tilde{w} \ \text{ weakly in } L^2(0,\pi;\tilde{y}_*),\\ &\tilde{w}_{n_k} \to \tilde{w} \ \text{ strongly in } L^2(0,\pi;\tilde{y}_*),\\ &\tilde{w}_{n_k} \to \tilde{w} \ \text{ strongly in } C([0,\pi]). \end{aligned}$$

By the lower semicontinuity of the L^2-norm under the weak convergence, we are led to

$$\|\partial_\sigma \tilde{w}\|^2_{L^2(0,\pi;\tilde{y}_*)} \le \frac{1}{C_{*,\pi} + \varepsilon} = \frac{1}{C_{*,\pi} + \varepsilon}\|\tilde{w}\|^2_{L^2(0,\pi;\tilde{y}_*)}. \tag{14}$$

On the other hand, we see

$$\tilde{y}_* \tilde{w}_n^2\big|_{\sigma=\pi} + \tilde{y}_* \tilde{w}_n^2\big|_{\sigma=0} \le \frac{1}{n\,(C_{*,\pi} + \varepsilon)}.$$

Since $\tilde{y}_* > 0$, this implies

$$\tilde{w}^2\big|_{\sigma=\pi} = \tilde{w}^2\big|_{\sigma=0} = 0\,.$$

Applying Lemma 5.1 with $N = 1$, $T = \pi$, and a weight $\tilde{y}_*$, we have

$$\|\tilde{w}\|^2_{L^2(0,\pi;\tilde{y}_*)} \le C_{*,\pi}\|\partial_\sigma \tilde{w}\|^2_{L^2(0,\pi;\tilde{y}_*)}.$$

This contradicts (14). Consequently, we obtain the desired result. □

Set

$$K_\pm := \kappa_{\Pi_\pm} \csc\theta_\pm.$$

It follows from Lemma 5.2 that there exists $m > 0$ such that the maximal eigenvalue λ_1 is non-positive provided that $K_-, K_+ > m$. That is, all of eigenvalues are non-positive in such case. The eigenvalues depend continuously on the parameters. Thus we investigate the condition that the zero is an eigenvalue for the eigenvalue problem (11). To do it, we should solve

$$\Delta_{\Gamma_*} L[w] = 0, \tag{15}$$
$$\partial_s w \pm (K_\pm - \kappa_{\Gamma_*} \cot\theta_\pm) w = 0, \tag{16}$$
$$\partial_s L[w] = 0. \tag{17}$$

By (15) and (17), we have

$$\|\partial_s L[w]\|^2_{L^2(\Gamma_*)} = 0.$$

This implies

$$L[w] = \text{constant}.$$

Thus we can get the fundamental solutions of the boundary value problem (15) and (17) if we solve

$$L[w] = 0, \quad L[w] = \beta\,(\neq 0). \tag{18}$$

Let $w_1,\ w_2$ be solutions of $L[v] = 0$ and let w_3 be a solution of $L[v] = \beta$. Then a solution of the boundary value problem (15) and (17) is represented by

$$w(s) = c_1 w_1(s) + c_2 w_2(s) + c_3 w_3(s).$$

Deriving the condition that this w is a non-trivial solution under (16) and

$$\int_0^d v\, y_* ds = 0,$$

it is the condition that the zero is an eigenvalue for the eigenvalue problem (11). That is, the zero is an eigenvalue if and only if the parameters satisfy

$$\begin{vmatrix} w_1'(0) - \Lambda_- w_1(0) & w_2'(0) - \Lambda_- w_2(0) & w_3'(0) - \Lambda_- w_3(0) \\ w_1'(d) + \Lambda_+ w_1(d) & w_2'(d) + \Lambda_+ w_2(d) & w_3'(d) + \Lambda_+ w_3(d) \\ \int_0^d w_1\, y_* ds & \int_0^d w_2\, y_* ds & \int_0^d w_3\, y_* ds \end{vmatrix} = 0, \tag{19}$$

where $\Lambda_\pm := K_\pm - \kappa_{\Gamma_*} \cot\theta_\pm$. Then, setting

$$\boldsymbol{w}(s) = (w_1(s), w_2(s), w_3(s))^T, \quad \boldsymbol{I}(d) = \left(\int_0^d w_1\, y_* ds, \int_0^d w_2\, y_* ds, \int_0^d w_3\, y_* ds\right)^T,$$

(19) is equivalent to

$$\begin{aligned} &-(\boldsymbol{w}(0) \times \boldsymbol{w}(d), \boldsymbol{I}(d))_{\mathbb{R}^3} \Lambda_- \Lambda_+ - (\boldsymbol{w}(0) \times \boldsymbol{w}'(d), \boldsymbol{I}(d))_{\mathbb{R}^3} \Lambda_- \\ &\quad + (\boldsymbol{w}'(0) \times \boldsymbol{w}(d), \boldsymbol{I}(d))_{\mathbb{R}^3} \Lambda_+ + (\boldsymbol{w}'(0) \times \boldsymbol{w}'(d), \boldsymbol{I}(d))_{\mathbb{R}^3} = 0. \end{aligned}$$

Moreover, we rewrite it as

$$A^{\boldsymbol{w}} K_- K_+ + B_-^{\boldsymbol{w}} K_- + B_+^{\boldsymbol{w}} K_+ + C^{\boldsymbol{w}} = 0, \tag{20}$$

where

$$\begin{aligned}
A^{\boldsymbol{w}} &= -(\boldsymbol{w}(0) \times \boldsymbol{w}(d), \boldsymbol{I}(d))_{\mathbb{R}^3},\\
B_-^{\boldsymbol{w}} &= -(\boldsymbol{w}(0) \times \boldsymbol{w}'(d), \boldsymbol{I}(d))_{\mathbb{R}^3} + (\boldsymbol{w}(0) \times \boldsymbol{w}(d), \boldsymbol{I}(d))_{\mathbb{R}^3} \kappa_*(d) \cot\theta_+,\\
B_+^{\boldsymbol{w}} &= (\boldsymbol{w}'(0) \times \boldsymbol{w}(d), \boldsymbol{I}(d))_{\mathbb{R}^3} + (\boldsymbol{w}(0) \times \boldsymbol{w}(d), \boldsymbol{I}(d))_{\mathbb{R}^3} \kappa_*(0) \cot\theta_-,\\
C^{\boldsymbol{w}} &= (\boldsymbol{w}'(0) \times \boldsymbol{w}'(d), \boldsymbol{I}(d))_{\mathbb{R}^3} - (\boldsymbol{w}(0) \times \boldsymbol{w}(d), \boldsymbol{I}(d))_{\mathbb{R}^3} \kappa_*(d)\kappa_*(0) \cot\theta_+ \cot\theta_-.
\end{aligned}$$

Then we have the following three representations of (20).
Case I: $A^{\boldsymbol{w}} \neq 0$ and $B_-^{\boldsymbol{w}} B_+^{\boldsymbol{w}} - A^{\boldsymbol{w}} C^{\boldsymbol{w}} \neq 0$.

$$(20) \quad \Leftrightarrow \quad K_+ = -\frac{B_-^{\boldsymbol{w}}}{A^{\boldsymbol{w}}} + \frac{\dfrac{B_-^{\boldsymbol{w}} B_+^{\boldsymbol{w}} - A^{\boldsymbol{w}} C^{\boldsymbol{w}}}{(A^{\boldsymbol{w}})^2}}{K_- - \left(-\dfrac{B_+^{\boldsymbol{w}}}{A^{\boldsymbol{w}}}\right)}.$$

Case II: $A^{\boldsymbol{w}} \neq 0$ and $B_-^{\boldsymbol{w}} B_+^{\boldsymbol{w}} - A^{\boldsymbol{w}} C^{\boldsymbol{w}} = 0$.

$$(20) \quad \Leftrightarrow \quad \left\{K_- - \left(-\frac{B_+^{\boldsymbol{w}}}{A^{\boldsymbol{w}}}\right)\right\}\left\{K_+ - \left(-\frac{B_-^{\boldsymbol{w}}}{A^{\boldsymbol{w}}}\right)\right\} = 0.$$

Case III: $A^{\boldsymbol{w}} = 0$.

$$(20) \quad \Leftrightarrow \quad B_-^{\boldsymbol{w}} K_- + B_+^{\boldsymbol{w}} K_+ + C^{\boldsymbol{w}} = 0.$$

If we can prove that $-B_+^{\boldsymbol{w}}/A^{\boldsymbol{w}}$ and $-B_-^{\boldsymbol{w}}/A^{\boldsymbol{w}}$ are monotone increasing in d, then we expect two type of figures for (20) in (K_-, d, K_+)-coordinate space. If Case I and Case III arise, (20) is represented as the figure of a type of (a) (see Fig. 4). On the other hand, if all of cases arise, (20) is represented as the figure of a type of (b) (see Fig. 5). In the following subsections, we check the above for the case that Γ_* are cylinders and unduloids.

5.1 Cylinders

In the case that $\Gamma_* = \{(x_*(s), y_*(s)\cos\zeta, y_*(s)\sin\zeta)^T \mid s \in [0, d],\ \zeta \in [0, 2\pi]\}$ is a cylinder with a constant mean curvature H_* (< 0), we remember

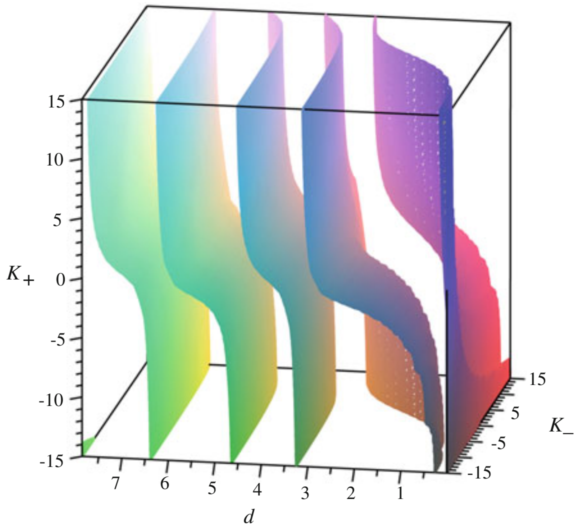


Fig. 4 **a** Case I + Case III

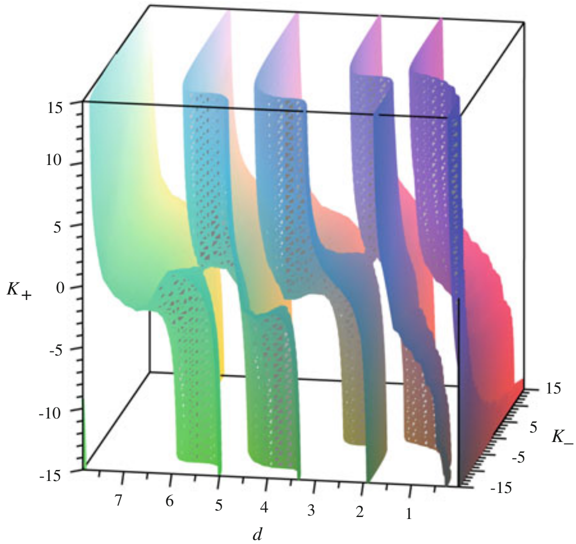


Fig. 5 **b** Case I + Case II + Case III

$$x_*(s) = s, \quad y_*(s) = -\frac{1}{2H_*} \, (=: r > 0).$$

In this case, $L[w]$ is given by

$$L[w] = \partial_s^2 w + \frac{1}{r^2} w$$

and the boundary condition (16) is given by

$$\partial_s w \pm K_\pm w = 0 \quad \text{at} \quad s = 0, d. \tag{21}$$

$L[w] = 0$ and $L[w] = 1/r^2$ (we choose $1/r^2$ as β in (18)) are linear second order ordinary differential equations with constant coefficients. Solving them, we obtain

$$w_1(s) = \cos\left(\frac{s}{r}\right), \quad w_2(s) = \sin\left(\frac{s}{r}\right), \quad w_3(s) = 1. \tag{22}$$

Then we are led to the following lemma.

Lemma 5.3 *Zero is an eigenvalue of the eigenvalue problem* (11) *if and only if parameters satisfy*

$$A^c(r,d)K_+K_- + B^c(r,d)(K_+ + K_-) + C^c(r,d) = 0, \tag{23}$$

where

$$A^c(r,d) = 2 - 2\cos\left(\frac{d}{r}\right) - \frac{d}{r}\sin\left(\frac{d}{r}\right),$$
$$B^c(r,d) = \frac{1}{r}\left\{\sin\left(\frac{d}{r}\right) - \frac{d}{r}\cos\left(\frac{d}{r}\right)\right\},$$
$$C^c(r,d) = \frac{d}{r^3}\sin\left(\frac{d}{r}\right).$$

Moreover, the multiplicity of the zero eigenvalue is one.

Proof Substituting (22) for (20) and calculating it, we get (23). In addition, by the help with Maple 17, we can obtain that the rank of the 3×3-matrix in (19) is two if the parameters satisfy (23). This leads us to the fact that the multiplicity of the zero eigenvalue is one. □

Let us investigate d satisfying $A^c(r,d) = 0$ for each $r > 0$. Differentiating $A^c(r,d)$ with respect to d, we obtain

$$\partial_d A^c(r,d) = \frac{1}{r}\left\{\sin\left(\frac{d}{r}\right) - \frac{d}{r}\cos\left(\frac{d}{r}\right)\right\} = \frac{1}{r}\left\{\tan\left(\frac{d}{r}\right) - \frac{d}{r}\right\}\cos\left(\frac{d}{r}\right)$$
$$\left(\text{if } d \neq \left(m\pi + \frac{\pi}{2}\right)r,\ m \in \mathbb{N} \cup \{0\}\right).$$

Thus we see that $d \in \left(m\pi r, \left(m\pi + \pi/2\right)r\right)$ $(m \in \mathbb{N})$ satisfying $\tan(d/r) = d/r$ implies $\partial_d A^c(r,d) = 0$. Let $q_m \in \left(m\pi r, \left(m\pi + \pi/2\right)r\right)$ $(m \in \mathbb{N})$ be the values of d fulfilling the above. Then it follows that

$$\partial_d A^c(r,d) \begin{cases} < 0 \ (d \in (q_{2\ell-1}, q_{2\ell})), \\ > 0 \ (d \in (q_{2\ell}, q_{2\ell+1})) \end{cases}$$

for $\ell \in \mathbb{N}$. Moreover, we have

$$A^c(r, (2\ell-1)\pi r) = 4 > 0, \quad A^c(r, 2\ell\pi r) = 0 \quad (\ell \in \mathbb{N}),$$

so that, taking account of $(2\ell-1)\pi r < q_{2\ell-1} < 2\ell\pi r < q_{2\ell}$, we are led to

$$A^c(r, q_{2\ell-1}) > 0, \quad A^c(r, q_{2\ell}) < 0 \quad (\ell \in \mathbb{N}).$$

Hence, for each $r > 0$, there exist $p_m\ (= p_m(r)) \in (q_m, q_{m+1})\ (m \in \mathbb{N})$ such that $A^c(r, p_m) = 0$. In particular, we get $p_{2\ell-1} = 2\ell\pi r\ (\ell \in \mathbb{N})$. In addition, we obtain for $d \neq p_m$

$$\{B^c(r,d)\}^2 - A^c(r,d)C^c(r,d) = \frac{1}{r^2}\left\{\frac{d}{r} - \sin\left(\frac{d}{r}\right)\right\}^2 > 0 \quad (d > 0).$$

Consequently, we see that, for the case that Γ_* are cylinders, only Case I and Case III arise in the representations of (23). Also we can observe for $d \neq p_m$

$$\frac{\partial}{\partial d}\left(-\frac{B^c(r,d)}{A^c(r,d)}\right) = \frac{1}{c^2\{A^c(r,d)\}^2}\left\{\frac{d}{c} - \sin\left(\frac{d}{c}\right)\right\}^2 > 0 \quad (d > 0),$$

so that $-B^c(r,d)/A^c(r,d)$ is monotone increasing in d for each $r > 0$. Thus we have the figure of the type (a) in (K_-, d, K_+)-coordinate space (see Fig. 4). Then we are led to the following theorem.

Theorem 5.1 *Set $A^c(r,d)$ such that $A^c(r,d) > 0$ for $d \in (0, p_1)$. If r, d, K_-, K_+ satisfy*

$$A^c(r,d)K_-K_+ + B^c(r,d)(K_- + K_+) + C^c(r,d) > 0 \quad \text{and} \quad d < p_1 = 2\pi r,$$

then there exists a pair of (K_-, K_+) such that cylinders are linearly stable for an axisymmetric perturbation. Moreover, if $d \geq 2\pi r$, then there are no pairs of (K_-, K_+) such that cylinders are stable.

Proof By Lemma 5.2, we obtain that there exist $m > 0$ such that $\lambda_1 \leq 0$ provided that $K_-, K_+ > m$. On the other hand, By Lemma 5.3, zero is eigenvalue if and only if c, d, K_-, K_+ satisfy (23). Thus it follows from the continuity with respect to r, d, K_-, K_+ and the monotonicity with respect to K_-, K_+ that $\lambda_1 < 0$ if r, d, K_-, K_+ fulfill

$$A^c(r,d)K_-K_+ + B^c(r,d)(K_- + K_+) + C^c(r,d) > 0 \quad \text{and} \quad d < p_1 = 2\pi r.$$

That is, cylinders are linearly stable. On the other hand, if $d \geq 2\pi r$, at least $\lambda_1 > 0$, so that any pairs of (K_-, K_+) does not imply that cylinders are stable. □

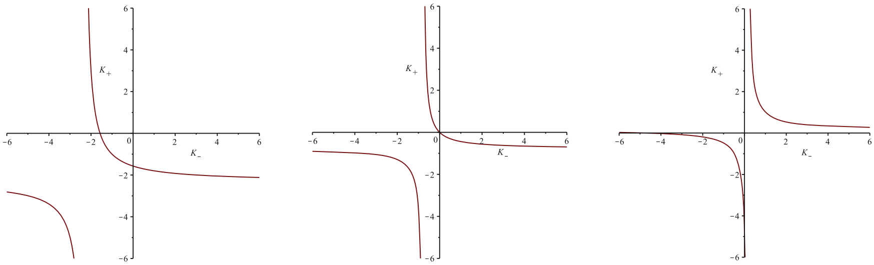


Fig. 6 (*Left*) $d = \pi/2,\ r = 1$, (*Middle*) $d = \pi,\ r = 1$ (*Right*) $d = 3\pi/2,\ r = 1$

Let us compare the above results with those of Athanassenas [1] and Vogel [15]. In [1, 15], they obtain, for the case that $\Pi_\pm$ are parallel planes and $\theta_\pm = \pi/2$, that cylinders are stable if $0 < d/r < \pi$ and cylinders are unstable if $d/r > \pi$. In our results, the case that $\Pi_\pm$ are parallel planes and $\theta_\pm = \pi/2$ corresponds to $(K_-, K_+) = (0, 0)$. Checking $(K_-, K_+) = (0, 0)$ by Fig. 6, we see that our results are consistent with the result of [1, 15].

5.2 Unduloids

In the case that $\Gamma_* = \{(x_*(s), y_*(s)\cos\zeta, y_*(s)\sin\zeta)^T \mid s \in [0, d],\ \zeta \in [0, 2\pi]\}$ is a unduloid with a constant mean curvature H_* (< 0), we remember

$$
\begin{aligned}
x_*(s) &= \int_0^s \frac{1 - B\sin(2H_*(s-\tau))}{\sqrt{1 + B^2 - 2B\sin(2H_*(s-\tau))}}\, d\sigma, \\
y_*(s) &= -\frac{1}{2H_*}\sqrt{1 + B^2 - 2B\sin(2H_*(s-\tau))},
\end{aligned}
$$

where $B \in (0, 1)$. In this case, $L[w]$ and (16) are given by

$$
\begin{aligned}
&L[w] = \partial_s^2 w + |A_*|^2 w, \\
&\partial_s w \pm (K_\pm + \kappa_{\Gamma_*})w = 0 \quad \text{at} \quad s = 0, d,
\end{aligned} \tag{24}
$$

where

$$
\begin{aligned}
|A_*|^2 &= \frac{4H_*^2\{B^2(B - \sin(2H_*(s-\tau))^2 + (1 - B\sin(2H_*(s-\tau))^2\}}{(1 + B^2 - 2B\sin(2H_*(s-\tau)))^2}, \\
\kappa_* &= \frac{2BH_*(B - \sin(2H_*(s-\tau))}{1 + B^2 - 2B\sin(2H_*(s-\tau))}.
\end{aligned}
$$

$L[w] = 0$ and $L[w] = 1$ (we choose 1 as β in (18)) are linear second order ordinary differential equation with variable coefficients. Solving them, we obtain

$$\begin{cases} w_1(s) = \dfrac{\cos(2H_*(s-\tau))}{\sqrt{1+B^2-2B\sin(2H_*(s-\tau))}}, \\ w_2(s) = \sin(2H_*(s-\tau)) + 2H_* \left\{ \dfrac{1+B^2}{2} I_1(s;B) - \dfrac{1}{2} I_2(s;B) \right\}, \\ w_3(s) = \dfrac{1}{4H_*^2} + \dfrac{B}{2H_*} I_1(s;B)\, w_1(s), \end{cases} \tag{25}$$

where

$$I_1(s;B) = \int_0^s \frac{1}{\sqrt{1+B^2-2B\sin(2H_*(\sigma-\tau))}}\, d\sigma,$$
$$I_2(s;B) = \int_0^s \sqrt{1+B^2-2B\sin(2H_*(\sigma-\tau))}\, d\sigma.$$

Here, set

$$\hat{H}_* = -H_*\,(>0), \quad \alpha = \hat{H}_*\tau + \frac{\pi}{4}$$

and assume $\alpha \in [-\pi/4, \pi/2)$. For $-\pi/2 + m\pi < \hat{H}_* s - \alpha < -\pi/2 + (m+1)\pi$ $(m \in \mathbb{N} \cup \{0\})$, $I_1(s;B)$ and $I_2(s;B)$ are given by

$$I_1(s;B) = \frac{1}{\hat{H}_*(1+B)} \big\{ 2mK(k) + (-1)^m F(\sin(\hat{H}_* s - \alpha);k) - F(\sin(-\alpha);k) \big\},$$
$$I_2(s;B) = \frac{1+B}{\hat{H}_*} \big\{ 2mE(k) + (-1)^m E(\sin(\hat{H}_* s - \alpha);k) - E(\sin(-\alpha);k) \big\},$$

where $k = 2\sqrt{B}/(1+B)$, $K(k)$ and $E(k)$ are 1st and 2nd complete elliptic integral, $F(s;k)$ and $E(s;k)$ are 1st snd 2nd incomplete elliptic integral. Then we obtain the following lemma.

Lemma 5.4 *Zero is an eigenvalue of the eigenvalue problem* (11) *if and only if parameters satisfy*

$$A^u(H_*, B, d, \tau) K_- K_+ + B_-^u(H_*, B, d, \tau, \theta_+) K_- + B_+^u(H_*, B, d, \tau, \theta_-) K_+ + C^u(H_*, B, d, \tau, \theta_+, \theta_-) = 0. \tag{26}$$

Proof Substituting (25) for (20) and calculating it, we get (26). □

The precise forms of A^u, B_-^u, B_+^u, and C^u are obtained by Maple 17. The representation of $B_\pm^u$ and C^u is very complicated, so that we show only the form of A^u:

$$
\begin{aligned}
&A^u(H_*, B, d, \tau) \\
&= \frac{1}{8H_*^3 PQ}\Big\{H_*^2(1-B^2)^2 I_1^2 \cos(2H_*\tau)\cos(2H_*(d-\tau)) \\
&\quad + 3H_*^2 I_2^2 \cos(2H_*\tau)\cos(2H_*(d-\tau)) - 4H_*^2(1+B^2)I_1 I_2 \cos(2H_*\tau)\cos(2H_*(d-\tau)) \\
&\quad + 2H_*(1+B^2)I_1\big(P\sin(2H_*\tau)\cos(2H_*(\tau)) + Q\cos(2H_*\tau)\sin(2H_*(d-\tau))\big) \\
&\quad + 4H_* B I_1\big(P\cos(2H_*(d-\tau)) - Q\cos(2H_*\tau)\big) \\
&\quad - 4H_* I_2\big(P\sin(2H_*\tau)\cos(2H_*(d-\tau)) + Q\cos(2H_*\tau)\sin(2H_*(d-\tau))\big) \\
&\quad + 2PQ\big(1+\sin(2H\tau)\sin(2H_*(d-\tau))\big) - (P^2+Q^2)\cos(2H_*\tau)\cos(2H_*(d-\tau))\Big\},
\end{aligned}
$$

where

$$
P = \sqrt{1+B^2+2B\sin(2H_*\tau)}, \quad Q = \sqrt{1+B^2-2B\sin(2H_*(d-\tau))}.
$$

Then, by the help with Maple 17, we derive for $A^u(H_*, B, d, \tau) \neq 0$

$$
\begin{aligned}
&B_-^u(H_*, B, d, \tau, \theta_+)B_+^u(H_*, B, d, \tau, \theta_-) - A^u(H_*, B, d, \tau)C^u(H_*, B, d, \tau, \theta_+, \theta_-) \\
&= \frac{1}{16H_*^4 PQ}\Big[H_*\big\{(1+B^2)(1+\sin(2H_*\tau)\sin(2H_*(d-\tau))) - (P^2+Q^2)\big\}I_1 \\
&\quad + H_*\big(3-\sin(2H_*(d-\tau))\sin(2H_*\tau)\big)I_2 \\
&\quad - P\cos(2H_*\tau)\sin(2H_*(d-\tau)) - Q\sin(2H_*\tau)\cos(2H_*(d-\tau))\Big]^2 \geq 0.
\end{aligned}
$$

Let us consider the stability of unduloids under the following setting:

$$
H_* = -1, \quad B = 0.6, \quad \theta_\pm = \frac{\pi}{2}.
$$

Moreover, we divide into the following two cases:

$$
\text{Case (U-1)}: \tau = \frac{\pi}{4} \text{ (see Fig. 7), Case (U-2)}: \tau = -\frac{\pi}{4} \text{ (see Fig. 8)}
$$

• **Case (U-1)**

First, let us derive $d > 0$ satisfying $A^u\big(-1, B, d, \pi/4\big) = 0$ for each $B \in (0, 1)$. Since $d = \pi/2 + m\pi$ $(m \in \mathbb{N} \cup \{0\})$ does not satisfy $A^u\big(-1, B, d, \pi/4\big) = 0$, we assume $d \neq \pi/2 + m\pi$. Then it follows

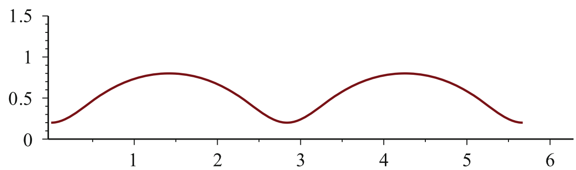


Fig. 7 Case (U-1): generating *curve*

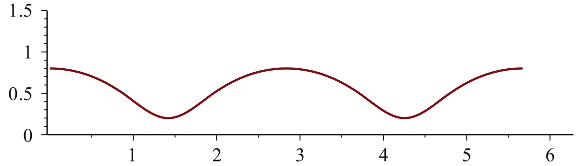


Fig. 8 Case (U-2): generating *curve*

$$A^u\Big(-1, B, d, \frac{\pi}{4}\Big) = -\frac{\sin(d)\cos(d)}{2Q_{\frac{\pi}{4}}(d; B)}\big\{(1-B)^2 I_{1,\frac{\pi}{4}}(d; B) - 2I_{2,\frac{\pi}{4}}(d; B) + Q_{\frac{\pi}{4}}(d; B)\tan(d)\big\},$$

where $Q_{\frac{\pi}{4}}(d; B) = \sqrt{1 + B^2 - 2B\cos(2d)}$ and

$$I_{1,\frac{\pi}{4}}(d; B) = \int_0^d \frac{1}{\sqrt{1 + B^2 - 2B\cos(2\sigma)}}\, d\sigma,$$
$$I_{2,\frac{\pi}{4}}(d; B) = \int_0^d \sqrt{1 + B^2 - 2B\cos(2\sigma)}\, d\sigma.$$

We easily obtain that $d = m\pi\ (m \in \mathbb{N})$ satisfy $A^u\big(-1, B, d, \pi/4\big) = 0$ for $B \in (0, 1)$. In addition, setting

$$f_{\frac{\pi}{4}}(d; B) = (1-B)^2 I_{1,\frac{\pi}{4}}(d; B) - 2I_{2,\frac{\pi}{4}}(d; B) + Q_{\frac{\pi}{4}}(d; B)\tan(d)$$

and differentiating $f_{\frac{\pi}{4}}$ with respect to d for $d \in (m\pi, (m+1)\pi)$ and $d \neq m\pi + \pi/2\ (m \in \mathbb{N} \cup \{0\})$, we have

$$\partial_d f_{\frac{\pi}{4}}(d; B) = \frac{\big\{(1-B)^2 + 4B\sin^2(d)\big\}\tan^2(d)}{\sqrt{(1+B)^2 - 4B\cos^2(d)}} > 0.$$

Thus $f_{\frac{\pi}{4}}(d; B)$ is strictly monotone increasing in d for $d \in (m\pi, (m+1)\pi)$ and $d \neq m\pi + \pi/2\ (m \in \mathbb{N} \cup \{0\})$. Also we obtain for $\ell \in \mathbb{N} \cup \{0\}$

$$f_{\frac{\pi}{4}}(\ell\pi; B) = 2\ell(1+B)\big\{(1-k^2)K(k) - 2E(k)\big\},$$

where $k = 2\sqrt{B}/(1+B)$, which gives $1 - k^2 = (1-B)^2/(1+B)^2$. Taking account of $k \in (0, 1)$, we see

$$(1-k^2)K(k) - 2E(k) < -\frac{\pi}{2} < 0.$$

This implies for $B \in (0, 1)$

$$f_{\frac{\pi}{4}}(\ell\pi; B) < 0 \quad (\ell = m,\ m+1,\ m \in \mathbb{N}).$$

Since we have

$$\lim_{d\to m\pi+\frac{\pi}{2}-0} f_{\frac{\pi}{4}}(d;B)=\infty, \qquad \lim_{d\to m\pi+\frac{\pi}{2}+0} f_{\frac{\pi}{4}}(d;B)=-\infty,$$

there exists a unique $d_m(B)\in\big(m\pi, m\pi+\pi/2\big)$ ($m\in\mathbb{N}$) such that $f_{\frac{\pi}{4}}(d_m(B);B)=0$ for each $B\in(0,1)$. Thus it follows that $A^u\big(-1,B,d,\pi/4\big)=0$ at $d=m\pi$ and $d=d_m(B)$ for each $B\in(0,1)$.

Second, we analyze $B^u_- B^u_+ - A^u C^u$. In the case (U-1), we see

$$\begin{aligned} &B^u_- B^u_+ - A^u C^u\\ &=\frac{(\text{const.})}{Q_{\frac{\pi}{4}}}\cdot\Big\{(1-B)^2(1+\cos(2d))I_{1,\frac{\pi}{4}}-(3+\cos(2d))I_{2,\frac{\pi}{4}}+Q_{\frac{\pi}{4}}\sin(2d)\Big\}^2. \end{aligned}$$

Let us investigate whether there exist d such that $B^u_- B^u_+ - A^u C^u=0$ for each $B\in(0,1)$. Set

$$g_{\frac{\pi}{4}}(d;B)=(1-B)^2(1+\cos(2d))I_{1,\frac{\pi}{4}}-(3+\cos(2d))I_{2,\frac{\pi}{4}}+Q_{\frac{\pi}{4}}\sin(2d).$$

Differentiating g with respect to d, we have

$$\partial_d\, g_{\frac{\pi}{4}}(d;B)=-2\{(1-B)^2 I_{1,\frac{\pi}{4}}-I_{2,\frac{\pi}{4}}\}\sin(2d)-2Q_{\frac{\pi}{4}}(1-\cos(2d)).$$

We easily obtain that $d=m\pi$ ($m\in\mathbb{N}$) implies $\partial_d\, g_{\frac{\pi}{4}}(d;B)=0$ for $B\in(0,1)$. Set

$$\varphi_{\frac{\pi}{4}}(d;B)=(1-B)^2 I_{1,\frac{\pi}{4}}-I_{2,\frac{\pi}{4}}+Q_{\frac{\pi}{4}}\frac{1-\cos(2d)}{\sin(2d)}$$

for $d\in(m\pi,(m+1)\pi)$ and $d\neq m\pi+\pi/2$ ($m\in\mathbb{N}\cup\{0\}$). Differentiating $\varphi_{\frac{\pi}{4}}$ with respect to d, we get

$$\partial_d\,\varphi_{\frac{\pi}{4}}(d;B)=2Q_{\frac{\pi}{4}}\frac{1-\cos(2d)}{\sin^2(2d)}>0.$$

Thus $\varphi_{\frac{\pi}{4}}(d;B)$ is strictly monotone increasing in d for $d\in(m\pi,(m+1)\pi)$ and $d\neq m\pi+\pi/2$ ($m\in\mathbb{N}\cup\{0\}$). Since we have

$$\begin{aligned} &\varphi_{\frac{\pi}{4}}(\ell\pi;B)=2\ell(1+B)\Big\{(1-k^2)K(k)-E(k)\Big\}<0,\\ &\lim_{d\to m\pi+\frac{\pi}{2}-0}\varphi_{\frac{\pi}{4}}(d;B)=\infty, \qquad \lim_{d\to m\pi+\frac{\pi}{2}+0}\varphi_{\frac{\pi}{4}}(d;B)=-\infty, \end{aligned}$$

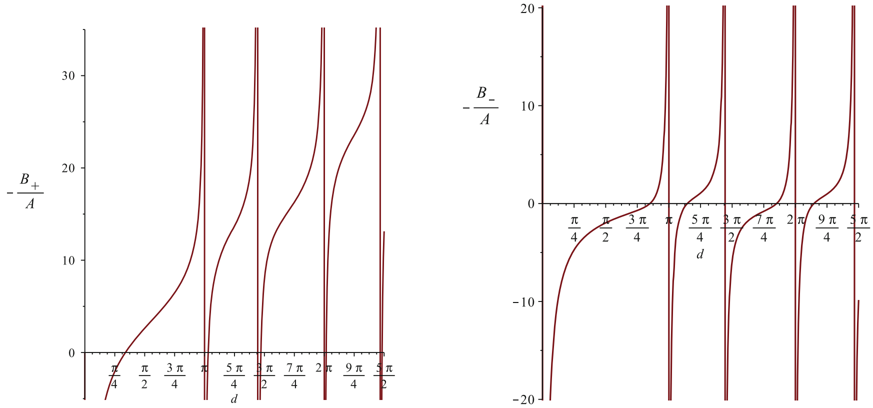


Fig. 9 (*Left*) $K_- = -\dfrac{B_+^u}{A^u}$, (*Right*) $K_+ = -\dfrac{B_-^u}{A^u}$

there exists a unique $\hat{d}_m(B) \in \big(m\pi, m\pi + \dfrac{\pi}{2}\big)$ $(m \in \mathbb{N})$ such that $\varphi_{\frac{\pi}{4}}(\hat{d}_m(B)) = 0$. Hence we get

$$\partial_d\, g_{\frac{\pi}{4}}(d; B) \begin{cases} > 0 \ (d \in (m\pi, \hat{d}_m(B))), \\ < 0 \ (d \in (\hat{d}_m(B), (m+1)\pi)). \end{cases}$$

for $B \in (0, 1)$, so that $g_{\frac{\pi}{4}}$ is the local maximum at $d = \hat{d}_m(B)$. Since we obtain $g_{\frac{\pi}{4}}(d; B) < g_{\frac{\pi}{4}}(0; B) = 0$ for $d \in (0, \pi]$ and $g_{\frac{\pi}{4}}(d; B) \le g_{\frac{\pi}{4}}(\hat{d}_m(B); B) = -2I_{2,\frac{\pi}{4}}(\hat{d}_m(B); B) < 0$ for $d \in [m\pi, (m+1)\pi]$, there are no $d > 0$ such that $B_-^u B_+^u - A^u C^u = 0$.

Consequently, in the case of (U-1), only Case I and Case Ⅲ arise in the representations of (26). Also, by the help with Maple 17, we can draw the graph of $K_- = -B_+^u/A$ and $K_+ = -B_-^u/A$ for $B = 0.6$ (see Fig. 9). By these figures, we expect that $-B_+^u/A^u$ and $-B_-^u/A^u$ are monotone increasing in d. (Unfortunately, at present, this property can not be proved analytically because of the complexity of $-B_+^u/A^u$ and $-B_-^u/A^u$.) Thus we have the figure of the type (a) in (K_-, d, K_+)-coordinate space (see Fig. 4). In the case (U-1), we are led to the following theorem.

Theorem 5.2 *In the case (U-1), set $A^u(d)$ such that $A^u(d) > 0$ for $d \in (0, \pi)$. If d, K_-, K_+ satisfy*

$$A^u(d)K_-K_+ + B_-^u(d)K_- + B_+^u(d)K_+ + C^u(d) > 0 \quad \text{and} \quad d < \pi,$$

then there exists a pair of (K_-, K_+) such that unduloids are linearly stable for an axisymmetric perturbation. Moreover, if $d \ge \pi$, then there are no pairs of (K_-, K_+) such that unduloids are stable.

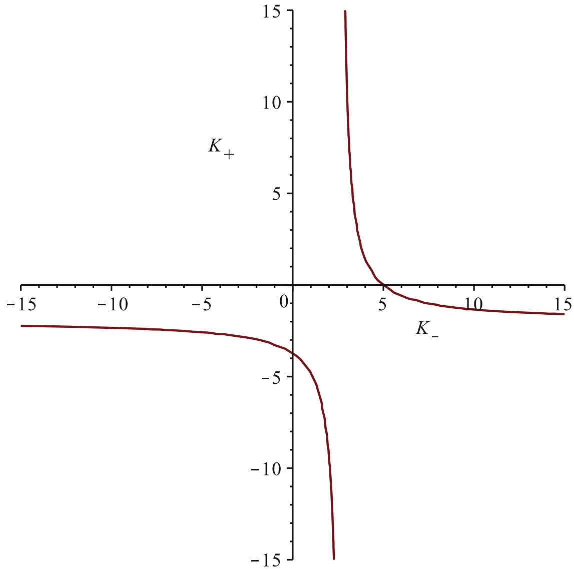


Fig. 10 Equation (26) in the case of (U-1) with $d = \dfrac{\pi}{2}$

Let us compare with the results of [1, 15]. According to [1, 15], in the case that $\Pi_\pm$ are parallel planes and $\theta_\pm = \pi/2$, unduloids are unstable. In the case of (U-1), the situation that $\Pi_\pm$ are parallel planes and $\theta_\pm = \pi/2$ is included in the case of $d = m\pi/2\,(m \in \mathbb{N})$. The fact that $\Pi_\pm$ are parallel planes corresponds to $(K_-, K_+) = (0, 0)$. It follows from Theorem 5.2 that there are no pairs of (K_-, K_+) such that unduloids are stable provided that $d \geq \pi$. Thus it is sufficient to check only the case of $d = \pi/2$. In the case of $d = \pi/2$, we obtain the hyperbola in Fig. 10. According to Fig. 10, $(K_-, K_+) = (0, 0)$ is included in the region of parameters that unduloids are unstable. That is, our results are consistent with the results of [1, 15].

- **Case (U-2)**

First, let us derive $d > 0$ satisfying $A^u\bigl(-1, B, d, -\pi/4\bigr) = 0$ for each $B \in (0, 1)$. Since $d = \pi/2 + m\pi$ $(m \in \mathbb{N} \cup \{0\})$ does not satisfy $A^u\bigl(-1, B, d, -\pi/4\bigr) = 0$, we assume $d \neq \pi/2 + m\pi$. Then it follows

$$
\begin{aligned}
&A^u\left(-1, B, d, -\frac{\pi}{4}\right) \\
&= -\frac{\sin(d)\cos(d)}{2Q_{-\frac{\pi}{4}}(d; B)}\Bigl\{(1+B)^2 I_{1,-\frac{\pi}{4}}(d; B) - 2I_{2,-\frac{\pi}{4}}(d; B) + Q_{-\frac{\pi}{4}}(d; B)\tan(d)\Bigr\},
\end{aligned}
$$

where $Q_{-\frac{\pi}{4}}(d;B) = \sqrt{1+B^2+2B\cos(2d)}$ and

$$I_{1,-\frac{\pi}{4}}(d;B) = \int_0^d \frac{1}{\sqrt{1+B^2+2B\cos(2\sigma)}}\,d\sigma,$$
$$I_{2,-\frac{\pi}{4}}(d;B) = \int_0^d \sqrt{1+B^2+2B\cos(2\sigma)}\,d\sigma.$$

We easily obtain that $d = m\pi$ ($m \in \mathbb{N}$) implies $A^u\bigl(-1, B, d, -\pi/4\bigr) = 0$ for $B \in (0,1)$. Set

$$f_{-\frac{\pi}{4}}(d;B) = (1+B)^2 I_{1,-\frac{\pi}{4}}(d;B) - 2I_{2,-\frac{\pi}{4}}(d;B) + Q_{-\frac{\pi}{4}}(d;B)\tan(d).$$

Differentiating $f_{-\frac{\pi}{4}}$ with respect to d for $d \in (m\pi, (m+1)\pi)$ and $d \neq m\pi + \pi/2$ ($m \in \mathbb{N}\cup\{0\}$), we have

$$\partial_d f_{-\frac{\pi}{4}}(d;B) = \frac{\bigl\{(1-B)^2 + 4B\cos^2(d)\bigr\}\tan^2(d)}{\sqrt{(1-B)^2+4B\cos^2(d)}} > 0.$$

Thus $f_{-\frac{\pi}{4}}(d;B)$ is strictly monotone increasing in d for $d \in (m\pi, (m+1)\pi)$ and $d \neq m\pi + \pi/2$ ($m \in \mathbb{N}\cup\{0\}$). Also we obtain for $\ell \in \mathbb{N}\cup\{0\}$

$$f_{-\frac{\pi}{4}}(\ell\pi;B) = 2m(1+B)\bigl\{K(k) - 2E(k)\bigr\},$$

where $k = 2\sqrt{B}/(1+B)$. Then we see that there exists $k_c \in (0,1)$ such that

$$K(k) - 2E(k) \begin{cases} < 0 \ (k \in (0, k_c)), \\ = 0 \ (k = k_c), \\ > 0 \ (k \in (k_c, 1)). \end{cases}$$

According to Maple 17, $k_c \approx 0.9089$. Hence there exists $B_c \in (0,1)$ such that

$$f_{-\frac{\pi}{4}}(\ell\pi;B) \begin{cases} < 0 \ (B \in (0, B_c)), \\ = 0 \ (B = B_c), \\ > 0 \ (B \in (B_c, 1)). \end{cases}$$

According to Maple 17, $B_c \approx 0.4114$. Also we have for each $B \in (0,1)$

$$\lim_{d\to m\pi+\frac{\pi}{2}-0} f_{-\frac{\pi}{4}}(d;B) = \infty, \quad \lim_{d\to m\pi+\frac{\pi}{2}+0} f_{-\frac{\pi}{4}}(d;B) = -\infty,$$

so that, if $B \in (0, B_c)$, there exists a unique $d_m(B) \in \bigl(m\pi, m\pi+\pi/2\bigr)$ ($m \in \mathbb{N}$) such that $f_{-\frac{\pi}{4}}(d_m(B);B) = 0$, and if $B \in (B_c, 1)$, there exists a unique $\tilde{d}_m(B) \in \bigl(m\pi + \pi/2, (m+1)\pi\bigr)$ ($m \in \mathbb{N}\cup\{0\}$) such that $f_{-\frac{\pi}{4}}(\tilde{d}_m(B);B) = 0$. Thus it follows that

$A^u\big(-1, B, d, -\pi/4\big) = 0$ at $d = m\pi$, $d_m(B)$ for each $B \in (0, B_c)$ and at $d = m\pi$, $\tilde{d}_m(B)$ for each $B \in (B_c, 1)$.

Second, we analyze $B^u_- B^u_+ - A^u C^u$. In the case (U-2), we see

$$\begin{aligned} &B^u_- B^u_+ - A^u C^u \\ &= \frac{\text{(const.)}}{Q_{-\frac{\pi}{4}}} \cdot \big\{(1+B)^2(1+\cos(2d))I_{1,-\frac{\pi}{4}} - (3+\cos(2d))I_{2,-\frac{\pi}{4}} + Q_{-\frac{\pi}{4}}\sin(2d)\big\}^2. \end{aligned}$$

Let us investigate whether there exists d such that $B^u_- B^u_+ - A^u C^u = 0$ for each $B \in (0, 1)$. Set

$$g_{-\frac{\pi}{4}}(d; B) = (1+B)^2(1+\cos(2d))I_{1,-\frac{\pi}{4}} - (3+\cos(2d))I_{2,-\frac{\pi}{4}} + Q_{-\frac{\pi}{4}}\sin(2d).$$

Differentiating g with respect to d, we have

$$\partial_d\, g_{-\frac{\pi}{4}}(d; B) = -2\{(1+B)^2 I_{1,-\frac{\pi}{4}} - I_{2,-\frac{\pi}{4}}\}\sin(2d) - 2Q_{-\frac{\pi}{4}}(1-\cos(2d)).$$

We easily obtain that $d = m\pi$ $(m \in \mathbb{N})$ implies $\partial_d\, g_{-\frac{\pi}{4}}(d; B) = 0$ for $B \in (0, 1)$. Set

$$\varphi_{-\frac{\pi}{4}}(d; B) = (1+B)^2 I_{1,-\frac{\pi}{4}} - I_{2,-\frac{\pi}{4}} + Q_{-\frac{\pi}{4}}\frac{1-\cos(2d)}{\sin(2d)}$$

for $d \in (m\pi, (m+1)\pi)$ and $d \neq m\pi + \pi/2\, (m \in \mathbb{N} \cup \{0\})$. Differentiating $\varphi_{-\frac{\pi}{4}}$ with respect to d, we get

$$\partial_d\, \varphi_{-\frac{\pi}{4}}(d; B) = 2Q_{-\frac{\pi}{4}}\frac{1-\cos(2d)}{\sin^2(2d)} > 0.$$

Thus $\varphi_{-\frac{\pi}{4}}(d; B)$ is strictly monotone increasing in d for $d \in (m\pi, (m+1)\pi)$ and $d \neq m\pi + \pi/2\, (m \in \mathbb{N} \cup \{0\})$. Since we have

$$\begin{aligned} &\varphi_{-\frac{\pi}{4}}(\ell\pi; B) = 2\ell(1+B)\big\{K(k) - E(k)\big\} > 0, \\ &\lim_{d \to m\pi + \frac{\pi}{2} - 0} \varphi_{-\frac{\pi}{4}}(d; B) = \infty, \qquad \lim_{d \to m\pi + \frac{\pi}{2} + 0} \varphi_{-\frac{\pi}{4}}(d; B) = -\infty, \end{aligned}$$

there exists a unique $\hat{d}_m(B) \in \big(m\pi + \pi/2, (m+1)\pi\big)\, (m \in \mathbb{N} \cup \{0\})$ such that $\varphi_{-\frac{\pi}{4}}(\hat{d}_m(B)) = 0$. Thus it follows that

$$\partial_d\, g_{-\frac{\pi}{4}}(d; B) \begin{cases} < 0 \ (d \in (m\pi, \hat{d}_m(B))), \\ > 0 \ (d \in (\hat{d}_m(B), (m+1)\pi)) \end{cases}$$

for $B \in (0, 1)$, so that $g_{-\frac{\pi}{4}}$ is the local minimum at $d = \hat{d}_m(B)$ and is the local maximum at $d = m\pi$. Since

$$g_{-\frac{\pi}{4}}(\ell\pi; B) = 4\ell(1+B)\{K(k) - 2E(k)\} \begin{cases} < 0 \ (B \in (0, B_c)), \\ = 0 \ (B = B_c), \\ > 0 \ (B \in (B_c, 1)), \end{cases}$$

$$g_{-\frac{\pi}{4}}\left(\ell\pi + \frac{\pi}{2}; B\right) = -4\ell(1+B)E(k) < 0,$$

$$g_{-\frac{\pi}{4}}(\hat{d}_m(B); B) = -2I_{2,-\frac{\pi}{4}}(\hat{d}_m(B); B) < 0,$$

there exist d such that $B^u_- B^u_+ - A^u C^u = 0$ for $B \in [B_c, 1)$. Taking account of $0.6 \in (B_c, 1)$, we see that for $m \in \mathbb{N}$ there exist $p_m \in (\hat{d}_{m-1}(B), m\pi)$ and $q_m \in \big(m\pi, m\pi + \pi/2\big)$ such that $B^u_- B^u_+ - A^u C^u = 0$.

Consequently, in the case of (U-2), all of cases (that is, Case I, Case II, Case III) arise in the representations of (26). Also, by the help with Maple 17, we can draw the graph of $K_- = -B^u_+/A$ and $K_+ = -B^u_-/A^u$ for $B = 0.6$ (see Fig. 11). By these figures, we expect that $-B^u_+/A^u$ and $-B^u_-/A^u$ are monotone increasing in d. Thus we have the figure of the type (b) in (K_-, d, K_+)-coordinate space (see Fig. 5). In the case (U-2), we are led to the following theorem.

Theorem 5.3 *In the case (U-2), set $A^u(d)$ such that $A^u(d) > 0$ for $d \in (0, \hat{d}_0)$. If d, K_-, K_+ satisfy*

$$A^u(d)K_-K_+ + B^u_-(d)K_- + B^u_+(d)K_+ + C^u(d) > 0 \quad \text{and} \quad d < \hat{d}_0,$$

then there exists a pair of (K_-, K_+) such that unduloids are linearly stable for an axisymmetric perturbation. Moreover, if $d \geq \hat{d}_0$, then there are no pairs of (K_-, K_+) such that unduloids are stable.

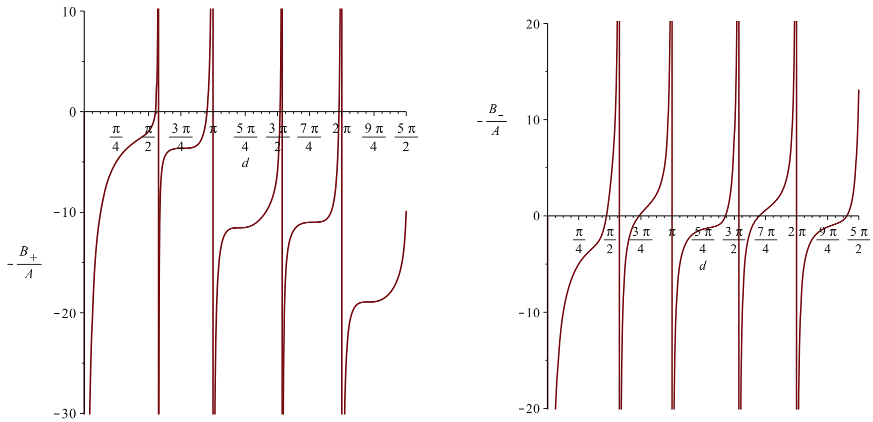
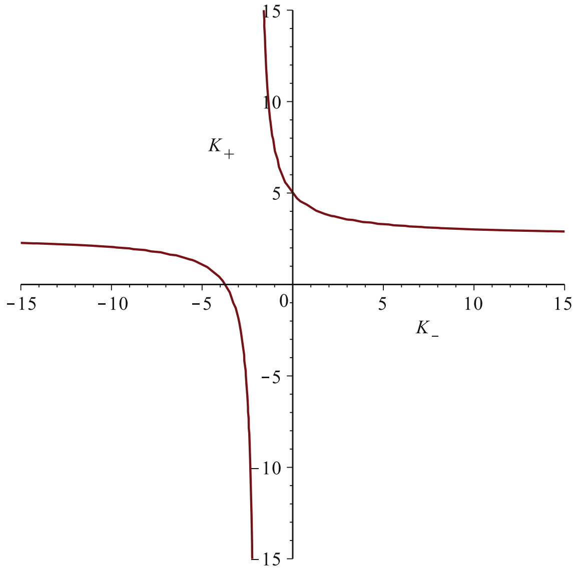


Fig. 11 (*Left*) $K_- = -\dfrac{B^u_+}{A^u}$, (*Right*) $K_+ = -\dfrac{B^u_-}{A^u}$

Fig. 12 Equation (26) in the case of (U-2) with $d = \dfrac{\pi}{2}$



Let us compare with the results of [1, 15]. In the case of (U-2), the situation that $\Pi_\pm$ are parallel planes and $\theta_\pm = \pi/2$ is also included in the case of $d = m\pi/2$ ($m \in \mathbb{N}$), and the fact that $\Pi_\pm$ are parallel planes corresponds to $(K_-, K_+) = (0, 0)$. It follows from Theorem 5.3 that there are no pairs of (K_-, K_+) such that unduloids are stable provided that $d \geq \hat{d}_0$ with $\hat{d}_0 \in (\pi/2, \pi)$. Thus it is sufficient to check only the case of $d = \pi/2$. In the case of $d = \pi/2$, we obtain the hyperbola in Fig. 12. According to Fig. 12, $(K_-, K_+) = (0, 0)$ is included in the region of parameters that unduloids are unstable. That is, our results are consistent with the results of [1, 15].

Remark 5.2 In [8, 13] Fel and Rubinstein obtained that precise criteria of the stability of Delaunay surfaces in the case that $\Pi_\pm$ are not parallel planes. It seems that their results have the intersection with our results, but at present it has not been checked. It will be mentioned in the coming paper.

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Littlewood's Fourth Principle

Rolando Magnanini and Giorgio Poggesi

Abstract In real analysis, Littlewood's three principles are known as heuristics that help teach the essentials of measure theory and reveal the analogies between the concepts of topological space and continuous function on one side and those of measurable space and measurable function on the other one. They are based on important and rigorous statements, such as Lusin's and Egoroff-Severini's theorems, and have ingenious and elegant proofs. We shall comment on those theorems and show how their proofs can possibly be made simpler by introducing a *fourth principle*. These alternative proofs make even more manifest those analogies and show that Egoroff-Severini's theorem can be considered as the natural generalization of the classical Dini's monotone convergence theorem.

Keywords Measurable finctions · Egorov's theorem · Lusin's theorem

1 Introduction

John Edenson Littlewood (9 June 1885–6 September 1977) was a British mathematician. In 1944, he wrote an influential textbook, *Lectures on the Theory of Functions* ([5]), in which he proposed three principles as guides for working in real analysis; these are heuristics to help teach the essentials of measure theory, as Littlewood himself wrote in [5]:

> The extent of knowledge [of real analysis] required is nothing like so great as is sometimes supposed. There are three principles, roughly expressible in the following terms: every (measurable) set is nearly a finite sum of intervals; every function (of class L^λ) is nearly continuous; every convergent sequence is nearly uniformly convergent. Most of the results

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> of the present section are fairly intuitive applications of these ideas, and the student armed with them should be equal to most occasions when real variable theory is called for. If one of the principles would be the obvious means to settle a problem if it were "quite" true, it is natural to ask if the "nearly" is near enough, and for a problem that is actually soluble it generally is.

To benefit our further discussion, we shall express Littlewood's principles and their rigorous statements in forms that are slightly different from those originally stated.

The first principle descends directly from the very definition of Lebesgue measurability of a set.

First Principle *Every measurable set is nearly closed.*

The second principle relates the measurability of a function to the more familiar property of continuity.

Second Principle *Every measurable function is nearly continuous.*

The third principle connects the pointwise convergence of a sequence of functions to the standard concept of uniform convergence.

Third Principle *Every sequence of measurable functions that converges pointwise almost everywhere is nearly uniformly convergent.*

These principles are based on important theorems that give a rigorous meaning to the term "nearly". We shall recall these in the next section along with their ingenious proofs that give a taste of the standard arguments used in real analysis.

In Sect. 3, we will discuss a *fourth principle* that associates the concept of finiteness of a function to that of its boundedness.

Fourth Principle *Every measurable function that is finite almost everywhere is nearly bounded.*

In the mathematical literature (see [1, 2, 5, 7, 9, 10, 12]), the proof of the second principle is based on the third; it can be easily seen that the fourth principle can be derived from the second.

However, we shall see that the fourth principle can also be proved independently; this fact makes possible a proof of the second principle *without* appealing for the third, that itself can be derived from the second, by a totally new proof based on *Dini's monotone convergence theorem*.

As in [5], to make our discussion as simple as possible, we shall consider the Lebesgue measure m for the real line $\mathbb{R}$.

2 The Three Principles

We recall the definitions of *inner* and *outer measure* of a set $E \subseteq \mathbb{R}$: they are

$$m_i(E) = \sup\{|K| : K \text{ is compact and } K \subseteq E\},$$
$$m_e(E) = \inf\{|A| : A \text{ is open and } A \supseteq E\},$$

where the number $|K|$ is the infimum of the total lengths of all the finite unions of open intervals that contain K; accordingly, $|A|$ is the supremum of the total lengths of all the finite unions of closed intervals contained in A. It always holds that $m_i(E) \le m_e(E)$. The set E is (Lebesgue) measurable if and only if $m_i(E) = m_e(E)$; when this is the case, the *measure* of E is $m(E) = m_i(E) = m_e(E)$; thus $m(E) \in [0, \infty]$ and it can be proved that m is a measure on the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}$.

By the properties of the supremum, it is easily seen that, for any pair of subsets E and F of $\mathbb{R}$, $m_e(E \cup F) \le m_e(E) + m_e(F)$ and $m_e(E) \le m_e(F)$ if $E \subseteq F$.

The first principle is a condition for the measurability of subsets of $\mathbb{R}$.

Theorem 1 (First Principle) *Let $E \subset \mathbb{R}$ be a set of finite outer measure.*

Then, E is measurable if and only if for every $\varepsilon > 0$ there exist two sets K and F, with K closed (compact), $K \cup F = E$ and $m_e(F) < \varepsilon$.

This is what is meant by *nearly closed.*

Proof If E is measurable, for any $\varepsilon > 0$ we can find a compact set $K \subseteq E$ and an open set $A \supseteq E$ such that

$$m(K) > m(E) - \varepsilon/2 \quad \text{and} \quad m(A) < m(E) + \varepsilon/2.$$

The set $A \setminus K$ is open and contains $E \setminus K$. Thus, by setting $F = E \setminus K$, we have $E = K \cup F$ and

$$m_e(F) \le m(A) - m(K) < \varepsilon.$$

Viceversa, for every $\varepsilon > 0$ we have:

$$m_e(E) = m_e(K \cup F) \le m_e(K) + m_e(F) < m(K) + \varepsilon \le m_i(E) + \varepsilon.$$

Since ε is arbitrary, then $m_e(E) \le m_i(E)$. □

The second and third principles concern measurable functions from (measurable) subsets of $\mathbb{R}$ to the *extended real line* $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$, that is functions are allowed to have values $+\infty$ and $-\infty$.

Let $f : E \to \overline{\mathbb{R}}$ be a function defined on a measurable subset E of $\mathbb{R}$. We say that f is *measurable* if the *level sets* defined by

$$L(f, t) = \{x \in E : f(x) > t\}$$

are measurable subsets of E for every $t \in \mathbb{R}$. It is easy to verify that if we replace $L(f,t)$ by $L^*(f,t) = \{x \in E : f(x) \geq t\}$ we have an equivalent definition.

Since the countable union and intersection of measurable sets are measurable, it is not hard to show that the pointwise infimum and supremum of a sequence of measurable functions $f_n : E \to \overline{\mathbb{R}}$ are measurable functions as well as the function defined for any $x \in E$ by

$$\limsup_{n\to\infty} f_n(x) = \inf_{k\geq 1} \sup_{n\geq k} f_n(x).$$

Since the countable union of sets of measure zero has measure zero and the difference between E and any set of measure zero is measurable, the same definitions and conclusions hold even if the functions f and f_n are defined *almost everywhere* (denoted for short by a.e.), that is if the subsets of E in which they are not defined have measure zero. In the same spirit, we say that a function or a sequence of functions satisfies a given property *a.e.* in E, if that property holds with the exception of a subset of measure zero.

As already mentioned, the third principle is needed to prove the second and is known as *Egoroff's theorem* or *Egoroff-Severini's theorem*.[1]

Theorem 2 (Third Principle; Egoroff-Severini) *Let $E \subset \mathbb{R}$ be a measurable set with finite measure and let $f : E \to \overline{\mathbb{R}}$ be measurable and finite a.e. in E.*

The sequence of measurable functions $f_n : E \to \overline{\mathbb{R}}$ converges a.e. to f in E for $n \to \infty$ if and only if, for every $\varepsilon > 0$, there exists a closed set $K \subseteq E$ such that $m(E \setminus K) < \varepsilon$ and f_n converges uniformly to f on K.

This is what we mean by *nearly uniformly convergent*.

Proof If $f_n \to f$ a.e. in E as $n \to \infty$, the subset of E in which $f_n \to f$ pointwise has the same measure as E; hence, without loss of generality, we can assume that $f_n(x)$ converges to $f(x)$ for every $x \in E$.

Consider the functions defined by

$$g_n(x) = \sup_{k\geq n} |f_k(x) - f(x)|, \quad x \in E \tag{1}$$

and the sets

$$E_{n,m} = \left\{ x \in E : g_n(x) < \frac{1}{m} \right\} \quad \text{for } n, m \in \mathbb{N}. \tag{2}$$

[1]Dmitri Egoroff, a Russian physicist and geometer and Carlo Severini, an Italian mathematician, published independent proofs of this theorem in 1910 and 1911 (see [3, 11]); Severini's assumptions are more restrictive. Severini's result is not very well-known, since it is hidden in a paper on orthogonal polynomials, published in Italian.

Observe that, if $x \in E$, then $g_n(x) \to 0$ as $n \to \infty$ and hence for any $m \in \mathbb{N}$

$$E = \bigcup_{n=1}^{\infty} E_{n,m}.$$

As $E_{n,m}$ is increasing with n, the monotone convergence theorem implies that $m(E_{n,m})$ converges to $m(E)$ for $n \to \infty$ and for any $m \in \mathbb{N}$. Thus, for every $\varepsilon > 0$ and $m \in \mathbb{N}$, there exists an index $\nu = \nu(\varepsilon, m)$ such that $m(E \setminus E_{\nu,m}) < \varepsilon/2^{m+1}$.

The measure of the set $F = \bigcup_{m=1}^{\infty} (E \setminus E_{\nu,m})$ is arbitrary small, in fact

$$m(F) \leq \sum_{m=1}^{\infty} m(E \setminus E_{\nu,m}) < \varepsilon/2.$$

Also, since $E \setminus F$ is measurable, by Theorem 1 there exists a compact set $K \subseteq E \setminus F$ such that $m(E \setminus F) - m(K) < \varepsilon/2$, and hence

$$m(E \setminus K) = m(E \setminus F) + m(F) - m(K) < \varepsilon.$$

Since $K \subseteq E \setminus F = \bigcap_{m=1}^{\infty} E_{\nu(\varepsilon,m),m}$ we have that

$$|f_n(x) - f(x)| < \frac{1}{m} \quad \text{for any} \quad x \in K \quad \text{and} \quad n \geq \nu(\varepsilon, m),$$

by the definitions of $E_{\nu,m}$ and g_n; this means that f_n converges uniformly to f on K as $n \to \infty$.

Viceversa, if for every $\varepsilon > 0$ there is a closed set $K \subseteq E$ with $m(E \setminus K) < \varepsilon$ and $f_n \to f$ uniformly on K, then by choosing $\varepsilon = 1/m$ we can say that there is a closed set $K_m \subseteq E$ such that $f_n \to f$ uniformly on K_m and $m(E \setminus K_m) < 1/m$.

Therefore, $f_n(x) \to f(x)$ for any x in the set $F = \bigcup_{m=1}^{\infty} K_m$ and

$$m(E \setminus F) = m\Big(\bigcap_{m=1}^{\infty} (E \setminus K_m)\Big) \leq m(E \setminus K_m) < \frac{1}{m} \quad \text{for any} \quad m \in \mathbb{N},$$

which implies that $m(E \setminus F) = 0$. Thus, $f_n \to f$ a.e. in E as $n \to \infty$. □

The second principle corresponds to *Lusin's theorem* (see [6]),[2] that we state here in a form similar to Theorems 1 and 2.

[2] N.N. Lusin or Luzin was a student of Egoroff. For biographical notes on Egoroff and Lusin see [4].

Theorem 3 (Second Principle; Lusin) *Let $E \subset \mathbb{R}$ be a measurable set with finite measure and let $f : E \to \overline{\mathbb{R}}$ be finite a.e. in E.*

Then, f is measurable in E if and only if, for every $\varepsilon > 0$, there exists a closed set $K \subseteq E$ such that $m(E \setminus K) < \varepsilon$ and the restriction of f to K is continuous.

This is what we mean by *nearly continuous*.

The proof of Lusin's theorem is done by approximation by simple functions. A *simple function* is a measurable function that has a finite number of real values. If $c_1, \ldots, c_n$ are the *distinct* values of a simple function s, then s can be conveniently represented as

$$s = \sum_{j=1}^{n} c_j \mathscr{X}_{E_j},$$

where $\mathscr{X}_{E_j}$ is the characteristic function of the set $E_j = \{x \in E : s(x) = c_j\}$. Notice that the E_j's form a covering of E of pairwise disjoint measurable sets.

Simple functions play a crucial role in real analysis; this is mainly due to the following result of which we shall omit the proof.

Theorem 4 (Approximation by Simple Functions, [9, 10]) *Let $E \subseteq \mathbb{R}$ be a measurable set and let $f : E \to [0, +\infty]$ be a measurable function.*

Then, there exists an increasing sequence of non-negative simple functions s_n that converges pointwise to f in E for $n \to \infty$.

Moreover, if f is bounded, then s_n converges to f uniformly in E.

We can now give the proof of Lusin's theorem.

Proof Any measurable function f can be decomposed as $f = f^+ - f^-$, where $f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$ are measurable and non-negative functions. Thus, we can always suppose that f is non-negative and hence, by Theorem 4, it can be approximated pointwise by a sequence of simple functions.

We first prove that a simple function s is nearly continuous. Since the sets E_j defining s are measurable, if we fix $\varepsilon > 0$ we can find closed subsets K_j of E_j such that $m(E_j \setminus K_j) < \varepsilon/n$ for $j = 1, \ldots, n$. The union K of the sets K_j is also a closed set and, since the E_j's cover E, we have that $m(E \setminus K) < \varepsilon$. Since the closed sets K_j are pairwise disjoint (as the E_j's are pairwise disjoint) and s is constant on K_j for all $j = 1, \ldots, n$, we conclude that s is continuous in K.

Now, if f is measurable and non-negative, let s_n be a sequence of simple functions that converges pointwise to f and fix an $\varepsilon > 0$.

As the s_n's are nearly continuous, for any natural number n, there exists a closed set $K_n \subseteq E$ such that $m(E \setminus K_n) < \varepsilon/2^{n+1}$ and s_n is continuous in K_n. By Theorem 2, there exists a closed set $K_0 \subseteq E$ such that $m(E \setminus K_0) < \varepsilon/2$ and s_n converges uniformly to f in K_0 as $n \to \infty$. Thus, in the set

$$K = \bigcap_{n=0}^{\infty} K_n$$

the functions s_n are all continuous and converge uniformly to f. Therefore f is continuous in K and

$$m(E \setminus K) = m\Big(\bigcup_{n=0}^{\infty}(E \setminus K_n)\Big) \le \sum_{n=0}^{\infty} m(E \setminus K_n) < \varepsilon.$$

Viceversa, if f is nearly continuous, fix an $\varepsilon > 0$ and let K be a closed subset of E such that $m(E \setminus K) < \varepsilon$ and f is continuous in K. For any $t \in \mathbb{R}$, we have:

$$L^*(f, t) = \{x \in K : f(x) \ge t\} \cup \{x \in E \setminus K : f(x) \ge t\}.$$

The former set in this decomposition is closed, as the restriction of f to K is continuous, while the latter is clearly a subset of $E \setminus K$ and hence its outer measure must be less than ε. By Theorem 1, $L^*(f, t)$ is measurable (for any $t \in \mathbb{R}$), which means that f is measurable. □

3 The Fourth Principle

We shall now present alternative proofs of Theorems 2 and 3. They are based on a fourth principle, that corresponds to the following theorem.

Theorem 5 (Fourth Principle) *Let $E \subset \mathbb{R}$ be a measurable set with finite measure and let $f : E \to \overline{\mathbb{R}}$ be a measurable function.*

Then, f is finite a.e. in E if and only if, for every $\varepsilon > 0$, there exists a closed set $K \subseteq E$ such that $m(E \setminus K) < \varepsilon$ and f is bounded on K.

This is what we mean by *nearly bounded*.

Proof If f is finite a.e., we have that

$$m(\{x \in E : |f(x)| = \infty\}) = 0.$$

As f is measurable, $|f|$ is also measurable and so are the sets

$$L(|f|, n) = \{x \in E : |f(x)| > n\}, \ n \in \mathbb{N}.$$

Observe that the sequence of sets $L(|f|, n)$ is decreasing and

$$\bigcap_{n=1}^{\infty} L(|f|, n) = \{x \in E : |f(x)| = \infty\}.$$

As $m(L(|f|,1)) \leq m(E) < \infty$, we can apply the (downward) monotone convergence theorem and infer that

$$\lim_{n\to\infty} m(L(|f|,n)) = m(\{x \in E : |f(x)| = \infty\}) = 0.$$

Thus, if we fix $\varepsilon > 0$, there is an $n_\varepsilon \in \mathbb{N}$ such that $m(L(|f|,n_\varepsilon)) < \frac{\varepsilon}{2}$. Also, we can find a closed subset K of the measurable set $E \setminus L(|f|,n_\varepsilon)$ such that $m(E \setminus L(|f|,n_\varepsilon)) - m(K) < \frac{\varepsilon}{2}$. Finally, since $K \subseteq E \setminus L(|f|,n_\varepsilon)$, $|f|$ is obviously bounded by n_ε on K and

$$m(E \setminus K) = m(E \setminus L(|f|,n_\varepsilon)) + m(L(|f|,n_\varepsilon) \setminus K) < \varepsilon.$$

Viceversa, if f is nearly bounded, then for any $n \in \mathbb{N}$ there exists a closed set $K_n \subseteq E$ such that $m(E \setminus K_n) < 1/n$ and f is bounded (and hence finite) in K_n. Thus, $\{x \in E : |f(x)| = \infty\} \subseteq E \setminus K_n$ for any $n \in \mathbb{N}$, and hence

$$m(\{x \in E : |f(x)| = \infty\}) \leq \lim_{n\to\infty} m(E \setminus K_n) = 0,$$

that is f is finite a.e. □

Remark 6 Notice that this theorem can also be derived from Theorem 3. In fact, without loss of generality, the closed set K provided by Theorem 3 can be taken to be compact and hence, f is surely bounded on K, being continuous on a compact set.

More importantly for our aims, Theorem 5 enables us to prove Theorem 3 *without* using Theorem 2.

Alternative proof of Theorem 3 The proof runs similarly to that presented in Sect. 2. If f is measurable, without loss of generality, we can assume that f is non-negative and hence f can be approximated pointwise by a sequence of simple functions s_n, which we know are nearly continuous. Thus, for any $\varepsilon > 0$, we can still construct the sequence of closed subsets K_n of E such that $m(E \setminus K_n) < \varepsilon/2^{n+1}$ and s_n is continuous in K_n.

Now, as f is finite a.e., Theorem 5 implies that it is nearly bounded, that is we can find a closed subset K_0 of E in which f is bounded and $m(E \setminus K_0) < \varepsilon/2$. We apply the second part of the Theorem 4 and infer that s_n converges uniformly to f in K_0. As seen before, we conclude that f is continuous in the intersection K of all the K_n's, because in K it is the uniform limit of the sequence of continuous functions s_n. As before $m(E \setminus K) < \varepsilon$.

The reverse implication remains unchanged. □

In order to give our alternative proof of Theorem 2, we need to recall a classical result for sequences of continuous functions.

Theorem 7 (Dini) *Let K be a compact subset of $\mathbb{R}$ and let be given a sequence of continuous functions $f_n : K \to \mathbb{R}$ that converges pointwise and monotonically in K to a function $f : K \to \mathbb{R}$.*

If f is also continuous, then f_n converges uniformly to f.

Proof We shall prove the theorem when f_n is monotonically increasing.

For each $n \in \mathbb{N}$, set $h_n = f - f_n$; as $n \to \infty$ the continuous functions h_n decrease pointwise to 0 on K.

Fix $\varepsilon > 0$. The sets $A_n = \{x \in K : h_n(x) < \varepsilon\}$ are relatively open in K, since the h_n's are continuous; also, $A_n \subseteq A_{n+1}$ for every $n \in \mathbb{N}$, since the h_n's decrease; finally, the A_n's cover K, since the h_n converge pointwise to 0.

By compactness, K is then covered by a finite number m of the A_n's, which means that $A_m = K$ for some $m \in \mathbb{N}$. This implies that $|f(x) - f_n(x)| < \varepsilon$ for all $n \geq m$ and $x \in K$, as desired. □

Remark 8 The conclusion of Theorem 7 still holds true if we assume that the sequence of f_n's is increasing (respectively decreasing) and f and all the f_n's are lower (respectively upper) semicontinuous. (We say that f is lower semicontinuous if the level sets $L_+(f, t)$ are open for every $t \in \mathbb{R}$; f is upper semicontinuous if $-f$ is lower semicontinuous.)

Now, Theorem 2 can be proved by appealing to Theorems 3 and 7.

Alternative proof of Theorem 2 As in the classical proof of this theorem, we can always assume that $f_n(x) \to f(x)$ for every $x \in E$.

Consider the functions and sets defined in (1) and (2), respectively. We shall first show that there exists an $\nu \in \mathbb{N}$ such that g_n is nearly bounded for every $n \geq \nu$. In fact, as already observed, since $g_n \to 0$ pointwise in E as $n \to \infty$, we have that

$$E = \bigcup_{n=1}^{\infty} E_{n,1},$$

and the $E_{n,1}$'s increase with n. Hence, if we fix $\varepsilon > 0$, there is a $\nu \in \mathbb{N}$ such that $m(E \setminus E_\nu) < \varepsilon/2$. Since E_ν is measurable, by Theorem 1 we can find a closed subset K of E_ν such that $m(E_\nu \setminus K) < \varepsilon/2$.

Therefore, $m(E \setminus K) < \varepsilon$ and for every $n \geq \nu$

$$0 \leq g_n(x) \leq g_\nu(x) < 1, \quad \text{for any } \ x \in K.$$

Now, being g_n nearly bounded in E for every $n \geq \nu$, the alternative proof of Theorem 3 implies that g_n is nearly continuous in E, that is for every $n \geq \nu$ there exists a closed subset K_n of E such that $m(E \setminus K_n) < \varepsilon/2^{n-\nu+1}$ and g_n is continuous on K_n. The set

$$K = \bigcap_{n=\nu}^{\infty} K_n$$

is closed, $m(E \setminus K) < \varepsilon$ and on K the functions g_n are continuous for any $n \geq \nu$ and monotonically decrease to 0 as $n \to \infty$.

By Theorem 7, the g_n's converge to 0 uniformly on K. This means that the f_n's converge to f uniformly on K as $n \to \infty$.

The reverse implication remains unchanged. □

Remark 9 Egoroff's theorem can be considered, in a sense, as the *natural* substitute of Dini's theorem, in case the monotonicity assumption is removed. In fact, notice that the sequence of the g_n's defined in (1) is decreasing; however, the g_n's are in general no longer upper semicontinuous (they are only lower semicontinuous) and Dini's theorem (even in the form described in Remark 8) cannot be applied. In spite of that, the g_n's *remain* measurable if the f_n's are so.

Of course, all the proofs presented in Sects. 2 and 3 still work if we replace the real line $\mathbb{R}$ by an Euclidean space of any dimension.

Theorems 2, 3 and 5 can also be extended to general measure spaces not necessarily endowed with a topology: the intersted reader can refer to [1, 8, 9, 12].

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The Phragmèn-Lindelöf Theorem for a Fully Nonlinear Elliptic Problem with a Dynamical Boundary Condition

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Abstract The Phragmén-Lindelöf theorem is established for viscosity solutions of fully nonlinear second order elliptic equations in a half space of $\mathbb{R}^n$ with a dynamical boundary condition.

Keywords Phragmén-Lindelöf theorem · Nonllinear elliptic equations · Viscosity solutions

1 Introduction

The maximum principle is one of the most important properties of the solutions of the elliptic boundary value problems in bounded domains, whereas it does not necessarily hold in unbounded domains. The Phragmén-Lindelöf theorem ensures that the maximum principle holds for the elliptic boundary value problems in unbounded domains under the restriction of the growth rate of the solutions at the space infinity and it has been studied in many papers. For example, the following holds:

- Let $u \in C^2(\mathbb{R}^n_+) \cap C(\overline{\mathbb{R}^n_+})$ satisfy

$$-\Delta u \le 0 \quad \text{in } \mathbb{R}^n_+, \qquad u \le 0 \quad \text{on } \partial\mathbb{R}^n_+,$$

  where $n \ge 2$ and $\mathbb{R}^n_+ := \{(x', x_n) : x' \in \mathbb{R}^{n-1},\ x_n > 0\}$. Then $u \le 0$ in $\mathbb{R}^n_+$ provided that

$$\limsup_{R\to\infty} R^{-1} \sup_{|x|=R} u(x) \le 0.$$

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This was proved by Gilbarg [17] in $n = 2$ and by Hopf [19] in $n \geq 3$. The Phragmén-Lindelöf theorem for more general second-order uniformly elliptic operator $Lu = -a_{ij} D_{ij} u$ has been studied by Oddson [30] in $n = 2$ and by Miller [29] in $n \geq 3$. For further details on the Phragmén-Lindelöf theorem, we refer to [7, 21, 24, 27, 31, 32].

In this paper we consider a fully nonlinear elliptic equation with a dynamical boundary condition

$$F(x, t, Du, D^2 u) = 0 \quad \text{in } \mathbb{R}^n_+ \times (0, T], \qquad \partial_t u + \partial_\nu u = 0 \quad \text{on } \partial\mathbb{R}^n_+ \times (0, T], \tag{1}$$

and establish a Phragmén-Lindelöf theorem for viscosity solutions of (1) under suitable structure conditions of F. Here $\partial_\nu := -\partial/\partial x_n$, $D := (\partial/\partial x_1, \dots, \partial/\partial x_n)$, $D^2 := (\partial^2/\partial x_i \partial x_j)_{i,j=1,\dots,n}$ and $0 < T < \infty$,

The boundary condition from (1) describes thermal contact with a perfect conductor or diffusion of solute from a well-stirred fluid or vapour (see e.g., [8]). For a list of other mathematical models where dynamical boundary conditions occur, we refer to [3]. The elliptic problems with the dynamical boundary conditions have been studied in many paper, e.g., [1–3, 5, 9–16, 18, 20, 22, 23, 25, 26, 28, 32–34] and references therein. Among others, in [14], the authors considered the following nonlinear elliptic problem with the dynamical boundary condition

$$\begin{cases} -\Delta u = f(u), & x \in \mathbb{R}^n_+, \ t > 0, \\ \partial_t u + \partial_\nu u = 0, & x \in \partial\mathbb{R}^n_+, \ t > 0, \\ u(x, 0) = \varphi(x'), & x = (x', 0) \in \partial\mathbb{R}^n_+, \end{cases} \tag{2}$$

where φ is a nonnegative measurable function in $\mathbb{R}^{n-1}$ and $f \in C(\mathbb{R})$. In particular, they proved the following theorem:

Theorem 1 *Assume that f is a nondecreasing continuous function in $\mathbb{R}$ such that $f(0) \geq 0$. Let φ be a nonnegative measurable function in $\mathbb{R}^{n-1}$. Then the following statements are equivalent:*

(a) *Problem (2) has a local-in-time nonnegative solution;*
(b) *Problem (2) has a global-in-time nonnegative solution;*
(c) *Problem*

$$-\Delta v = f(v) \quad \text{in} \quad \mathbb{R}^n_+, \qquad v = \varphi \quad \text{on} \quad \partial\mathbb{R}^n_+, \tag{3}$$

has a nonnegative solution.

Furthermore, if $u = u(x, t)$ and $v = v(x)$ are minimal nonnegative solutions of (2) and (3), respectively, then

$$u(x', x_n, t) = v(x', x_n + t)$$

for almost all $x' \in \mathbb{R}^{n-1}$ and all $x_n \geq 0$ and $t > 0$.

The following Phragmén-Lindelöf theorem was a crucial ingredient of the proof of Theorem 1. See [14, Theorem 3.1].

Theorem 2 *Let $T > 0$ and let $u = u(x,t)$ satisfy*

$$u(\cdot,t) \in C^2(\mathbb{R}^n_+) \cap C^1(\overline{\mathbb{R}^n_+}) \quad \text{for any} \quad t \in (0,T],$$
$$u \in C(\overline{\mathbb{R}^n_+} \times (0,T]), \quad \partial_t u \in C(\partial\mathbb{R}^n_+ \times (0,T]),$$

and

$$-\Delta u \leq 0 \quad \text{in} \quad \mathbb{R}^n_+ \times (0,T], \qquad \partial_t u + \partial_\nu u \leq 0 \quad \text{on} \quad \mathbb{R}^n_+ \times (0,T], \tag{4}$$

where $n \geq 2$. Assume that

$$\limsup_{t\to+0} \sup_{|x'|\leq R} u(x',0,t) \leq 0 \quad \text{for any } R > 0, \tag{5}$$

$$\liminf_{R\to\infty} \sup_{|x|=R,\, t\in(0,T]} \frac{u(x,t)}{1+x_n} \leq 0. \tag{6}$$

Then $u \leq 0$ in $\mathbb{R}^n_+ \times (0,T]$.

The purposes of this paper are to generalize Theorem 2 to more general elliptic equations in unbounded domains and to establish Phragmén-Lindelöf theorems for viscosity solutions of fully nonlinear elliptic equations with the dynamical boundary condition. In particular, for problem (4), we obtain the following two theorems by our main results (see Theorems 5 and 7).

Theorem 3 *Let $T > 0$ and $n \geq 2$. Let $u = u(x,t) \in C(\overline{\mathbb{R}^n_+} \times (0,T])$ be a viscosity solution of (4) satisfying (5) and (6). Then $u \leq 0$ in $\overline{\mathbb{R}^n_+} \times (0,T]$.*

Theorem 4 *Let $T > 0$ and $n \geq 2$. Let $u = u(x,t) \in C(\overline{\mathbb{R}^n_+} \times (0,T])$ be a viscosity solution of (4) such that*

$$L := \sup_{(x,t)\in\partial\mathbb{R}^n_+\times(0,T]} u(x,t) < \infty \qquad \text{and} \qquad \lim_{t\to+0} \sup_{x\in\partial\mathbb{R}^n_+} u(x,t) \leq 0.$$

Then there exists a constant $\alpha > 0$ such that, if

$$\limsup_{x\in\mathbb{R}^n_+,\, |x|\to\infty} \frac{u(x,t)}{(1+|x|)^\alpha} = 0 \qquad \text{and} \qquad \limsup_{x'\in\mathbb{R}^{n-1},|x'|\to\infty} u(x',0,t) < L$$

for $t \in (0,T]$, then $u \leq 0$ in $\overline{\mathbb{R}^n_+} \times (0,T]$.

Theorem 4 actually weakens the assumption in Theorem 2 on the growth rate of the solution $u = u(x',x_n,t)$ of (4) as $|x'| \to \infty$.

The rest of this paper is organized as follows. In Sect. 2 we introduce the definition of viscosity solutions of (1) and recall some preliminary lemmas on fully nonlinear elliptic equations. In Sect. 3 we prove the Phragmén-Lindelöf theorem for a viscosity solution of problem (1) under assumptions (5) and (6). In Sect. 4 we present a Aleksandrov-Bakelman-Pucci type estimates on viscosity subsolutions of fully nonlinear PDEs with dynamical boundary condition under a different growth condition. Furthermore, we show another Phragmén-Lindelöf theorem for problem (1).

2 Preliminaries

In this section we introduce some notation and define viscosity solutions of (1). Furthermore, we recall some preliminary lemmas on fully nonlinear elliptic equations.

Let Ω be an unbounded domain in $\mathbb{R}^n$ and $0 < T < \infty$. Set

$$\Omega_T := \Omega \times (0, T], \qquad \overline{\Omega}_T := \overline{\Omega} \times (0, T], \qquad \partial\Omega_T := \partial\Omega \times (0, T].$$

Let $B_R(x) := \{y \in \mathbb{R}^n : |y - x| < R\}$ for $x \in \mathbb{R}^n$ and $R > 0$. We denote by $\mathbb{S}^n$ the set of real symmetric matrices of order n. For any $0 < \lambda \le \Lambda$, we define the Pucci operators $\mathscr{P}^{\pm}_{\lambda,\Lambda}$ by

$$\begin{aligned}
&\mathscr{P}^{-}_{\lambda,\Lambda}(M) := \min\{-\mathrm{trace}(AM) : \lambda I \le A \le \Lambda I \text{ for } A \in \mathbb{S}^n\},\\
&\mathscr{P}^{+}_{\lambda,\Lambda}(M) = -\mathscr{P}^{-}_{\lambda,\Lambda}(-M),
\end{aligned}$$

for $M \in \mathbb{S}^n$, where I is the identity matrix in $\mathbb{R}^n$. In particular,

$$\mathscr{P}^{\pm}_{1,1}(M) = -\mathrm{trace}\,(M), \quad M \in \mathbb{S}^n,$$

which corresponds to the Laplace operator $-\Delta$.

Let $F = F(z, p, M)$ be a continuous function in $\overline{\Omega}_T \times \mathbb{R}^n \times \mathbb{S}^n$ and assume the following:

(F1) F is degenerate elliptic, that is

$$F(z, p, M) \le F(z, p, N)$$

for all $z = (x, t) \in \Omega_T$, $p \in \mathbb{R}^n$ and $M, N \in \mathbb{S}^n$ with $M \ge N$;

(F2) There exist positive constants λ and Λ with $\lambda \le \Lambda$ such that

$$\mathscr{P}^{-}_{\lambda,\Lambda}(M) - b(x)|p'| \le F(z, p, M)$$

for all $z = (x, t) \in \Omega_T$, $p = (p', p_n) \in \mathbb{R}^n$ and $M \in \mathbb{S}^n$. Here b is a continuous function in Ω such that

$$b(x) \le b_0/(1 + |x|^2)^{1/2} \quad \text{in} \quad \Omega$$

for some constant $b_0 > 0$.

We define viscosity subsolutions, viscosity supersolutions and viscosity solutions of (1).

Definition 1 Let Ω be a domain in $\mathbb{R}^n$ and $0 < T < \infty$. Let $u \in C(\overline{\Omega_T})$. Then u is said to be a viscosity subsolution of $F(x, t, Du, D^2u) = 0$ in Ω_T if

(i) $F(z, D\phi(z), D^2\phi(z)) \le 0$ holds for $z \in \Omega_T$ and $\phi \in C^2(\Omega_T)$ whenever z is a local maximum point of $u - \phi$.

Furthermore, u is said to be a viscosity subsolution of (1) if u satisfies (i) and

(ii) $\min\{F(z, D\phi(z), D^2\phi(z)),\ \phi_t(z) + \partial_\nu\phi(z))\} \le 0$ holds for $z \in \partial\Omega_T$ and $\phi \in C^2(\overline{\Omega}_T)$ whenever z is a local maximum point of $u - \phi$.

On the other hand, u is said to be a viscosity supersolution of $F(x, t, Du, D^2u) = 0$ in Ω_T if

(iii) $F(z, D\phi(z), D^2\phi(z)) \ge 0$ holds for $z \in \Omega_T$ and $\phi \in C^2(\Omega_T)$ whenever z is a local minimum point of $u - \phi$.

Moreover, u is said to be a viscosity supersolution of (1) if u satisfies (iii) and

(iv) $\min\{F(z, D\phi(z), D^2\phi(z)),\ \phi_t(z) + \partial_\nu\phi(z))\} \ge 0$ holds for $z \in \partial\Omega_T$ and $\phi \in C^2(\overline{\Omega}_T)$ whenever z is a local minimum point of $u - \phi$.

In addition, u is said to be a viscosity solution of (1) if it is both a viscosity subsolution and a viscosity supersolution of (1).

We often say that u is a viscosity solution of

$$F(x, t, Du, D^2u) \le 0 \quad \text{in} \quad \Omega_T, \quad \partial_t u + \partial_\nu u \le 0 \quad \text{on} \quad \partial\Omega_T$$

if it is a viscosity subsolution of (1).

We recall two lemmas on viscosity solutions for fully nonlinear elliptic problems. Lemma 1 follows from the same argument as in the proof of [7, Lemma 1].

Lemma 1 *Let* $0 < T < \infty$ *and* $F = F(z, p, M)$ *be a continuous function in* $\overline{\Omega}_T \times \mathbb{R}^n \times \mathbb{S}^n$ *and assume* (F1) *and* (F2). *Let* w *be a viscosity solution of*

$$F(x, t, Dw, D^2w) \le f(x) \quad in \quad \Omega_T,$$

where $f \in C(\Omega)$. *If* $\xi \in C^2(\Omega)$ *is such that*

$$\xi(x) > 0, \quad \frac{|D\xi|}{\xi}(x) \le k_1(x), \quad \frac{|D^2\xi|}{\xi}(x) \le k_2(x) \quad in \quad \Omega$$

for some continuous positive functions k_1 *and* k_2*, then* $u := w/\xi$ *is a viscosity solution of*

$$\mathscr{P}^{-}_{\lambda,\Lambda}(D^2u) - \gamma_1(x)|Du| - \gamma_2(x)u^+ \le \frac{f(x)}{\xi(x)} \quad in \quad \Omega_T,$$

where $u^+ := \max\{u, 0\}$, $\gamma_1(x) = h_1k_1(x) + b(x)$ *and* $\gamma_2(x) = h_2k_2(x) + k_1(x)b(x)$ *with positive constants* h_1 *and* h_2.

The second lemma is concerned with the Aleksandrov-Bakelman-Pucci maximum principle for elliptic equations in unbounded domain and it follows from [24, Theorem 3.6] with

$$R_0 = 1, \quad \eta = 1, \quad \sigma = \tau = \frac{1}{2}, \quad R_y = |y|.$$

See also [7].

Lemma 2 *Let* $w \in C(\Omega)$ *be a viscosity subsolution bounded from above of*

$$\mathscr{P}^{-}_{\lambda,\Lambda}(D^2u) - b(x)|Du| \le f(x) \quad in \quad \Omega$$

with $f \in C(\Omega) \cap L^n_+(\Omega)$*, satisfying*

$$\sup_{y\in\Omega,|y|>1} |y|\|f\|_{L^n(A_y\cap\Omega)} < \infty.$$

Then there exists positive constant $C = C(n, \lambda, \Lambda)$ *and* $\varepsilon = \varepsilon(n)$ *such that*

$$\sup_{\Omega} w \le \sup_{\partial\Omega} w^+(x) + C \sup_{y\in\Omega,|y|>1} |y|\|f\|_{L^n(A_y\cap\Omega)} + C \sup_{y\in\Omega,|y|\le1} \|f\|_{L^n(B_y\cap\Omega)}.$$

Here $A_y = B_{2|y|}(0) \setminus B_{\varepsilon|y|}(0)$ *and* $B_y = B_{2|y|}(0)$ *for* $y \in \Omega$.

3 Phragmén-Lindelöf Theorem I

We focus on the case $\Omega = \mathbb{R}^n_+$ and prove a Phragmén-Lindelöf theorem for fully nonlinear elliptic boundary value problem (1). In this section we prove the following theorem, which is one of the main results of this paper and which is a generalization of Theorem 2 to viscosity solutions of (1).

Theorem 5 *Let* $T > 0$ *and* $F = F(z, p, M)$ *be a continuous function in* $\overline{\mathbb{R}^n_+} \times (0, T] \times \mathbb{R}^n \times \mathbb{S}^n$ *and assume* (F1) *and* (F2)*. Let* $u = u(x, t) \in C(\overline{\mathbb{R}^n_+} \times (0, T])$ *be a viscosity subsolution of* (1) *satisfying* (5) *and* (6)*. Then* $u \le 0$ *in* $\overline{\mathbb{R}^n_+} \times (0, T]$.

Proof Let $w(x,t) := e^{-t}u(x,t)/(1+x_n)$ for $(x,t) \in \overline{\mathbb{R}^n_+} \times (0,T]$. Then w satisfies

$$\mathscr{P}^-(D^2 w) - \frac{2\Lambda}{1+x_n}|Dw| - b(x)|D'w| \le 0 \quad \text{in} \quad \mathbb{R}^n_+ \times (0,T],$$
$$\partial_t w + \partial_\nu w \le 0 \quad \text{on} \quad \partial\mathbb{R}^n_+ \times (0,T],$$

in the sense of viscosity solutions.

Let $\varepsilon > 0$. By (6) we can find $\{R_k\}_{k=1}^\infty$ such that

$$\lim_{k\to\infty} R_k = \infty \qquad \text{and} \qquad \lim_{k\to\infty} \sup_{|x|=R_k, t\in(0,T]} w \le \varepsilon.$$

Suppose that

$$L_k := \sup_{x\in\mathbb{R}^n_+\cap B(0,R_k), t\in(0,T]} w > \varepsilon$$

for some k, and then we will get a contradiction. Then it follows from the maximum principle that

$$L_k = \sup_{[\partial\mathbb{R}^n_+\cap B(0,R_k)]\times(0,T]} w > \varepsilon.$$

Furthermore, by (5) we can find $(x_*, t_*) \in [\partial\mathbb{R}^n_+ \cap B(0,R_k)] \times (0,T]$ such that

$$w(x_*, t_*) = L_k > 0.$$

On the other hand, by the Hopf lemma (c.f. [4]), we have

$$\limsup_{h\to 0} \frac{w(x_* - h\nu, t_*) - w(x_*, t_*)}{h} = -2A < 0 \quad \text{for some } A > 0,$$

which implies that

$$w(x_* - h\nu, t_*) < L_k - Ah$$

for small $h > 0$.

Let ϕ be a test function such that

$$\phi(x,t) = \gamma|x' - x'_*|^2 - Ax_n^2 - \sigma x_n + |t - t_*|^2$$

where $\gamma = A/2\Lambda(n-1)$ and $\sigma = A/4\Lambda$. Then we see that $(x_*, t_*) \in \partial\mathbb{R}^n_+ \times (0,T]$ be a local maximum point of $u - \phi$. Therefore we obtain

$$\mathscr{P}^-_{\lambda,\Lambda}(D^2\phi(x_*, t_*)) - \frac{2\Lambda}{1+(x_{*,n})}|D\phi(x_*, t_*)| - b(x_*)|D'\phi(x_*, t_*)| = A - \frac{A}{2} > 0,$$
$$\partial_t\phi(x_*, t_*) + \partial_\nu\phi(x_*, t_*) = 0 + \frac{A}{4} > 0.$$

This contradicts the definition of viscosity solutions at the boundary, which means that $L_k \le \varepsilon$ for any $k > 0$. Since ε is arbitrary and $\lim_{k\to\infty} R_k = \infty$, we see that $u \le 0$ in $\overline{\mathbb{R}^n_+} \times (0, T]$. Thus Theorem 5 follows. □

Proof of Theorem 3. Theorem 3 follows from Theorem 7 in the case

$$F(z, p, M) = -\mathrm{trace}\, M$$

for $z \in \mathbb{R}^n_+ \times (0, T]$, $p \in \mathbb{R}^n$ and $M \in \mathbb{S}^n$. □

4 Phragmén-Lindelöf Theorem II

In this section we establish another Phragmén-Lindelöf theorem for problem (1). It stands on a different growth assumption.

We first prove the following theorem on the boundedness of viscosity solutions of (1).

Theorem 6 *Let $T > 0$. Let $F = F(z, p, M)$ be a continuous function in $\overline{\mathbb{R}^n_+} \times (0, T] \times \mathbb{R}^n \times \mathbb{S}^n$ and assume* (F1) *and* (F2). *Let $u = u(x, t) \in C(\overline{\mathbb{R}^n_+} \times (0, T])$ be a viscosity subsolution of* (1) *such that*

$$L := \sup_{\partial\mathbb{R}^n_+ \times (0,T]} u(x, t) < +\infty.$$

Then there exists a positive constant α such that, if

$$\limsup_{x\in\mathbb{R}^n_+,\, |x|\to\infty} \frac{u(x, t)}{(1 + |x|)^\alpha} = 0 \quad \text{for } t \in (0, T],$$

then $u \le L$ in $\overline{\mathbb{R}^n_+} \times (0, T]$.

Proof Let $t > 0$. By Definition 1 and (F2), we see that the function $\tilde{u}(x, t) := u(x, t) - L$ is a viscosity subsolution of

$$\mathscr{P}^-_{\lambda,\Lambda}(D^2\tilde{u}) - b(x)|D'\tilde{u}| = 0 \quad \text{in } \mathbb{R}^n_+ \times (0, T], \quad \partial_t\tilde{u} + \partial_\nu\tilde{u} = 0 \text{ on } \partial\mathbb{R}^n_+ \times (0, T].$$

Let α be a positive constant to be chosen later and set

$$\xi(x) = (1 + |x|^2)^{\frac{\alpha}{2}}.$$

The function $v = \tilde{u}/\xi$ is bounded from above and obviously

$$\sup_{x\in\partial\mathbb{R}^n_+ \times (0,T]} v \le 0. \tag{7}$$

By Lemma 1 we obtain

$$0 \geq \frac{1}{\xi(x)}\left(\mathscr{P}^-(D^2\tilde{u}) - b(x)|D'\tilde{u}|\right) \geq \mathscr{P}^-(D^2 v) - \gamma_1(x)|Dv| - \alpha\gamma_2(x)v^+ \quad \text{in} \quad \mathbb{R}^n_+ \times (0, T],$$

$$0 \geq \frac{1}{\xi(x)}\left(\partial_t \tilde{u} - \partial_{x_n}\tilde{u}\right) = \partial_t v + \partial_\nu v \quad \text{on} \quad \partial\mathbb{R}^n_+ \times (0, T],$$

in the sense of viscosity solutions, where

$$\gamma_1(x) = \frac{c_0 + b_0}{2(1+|x|^2)^{1/2}} \quad \text{and} \quad \gamma_2(x) = \frac{c_1 + b_0}{1+|x|^2}$$

for some positive constants c_0 and c_1. Applying Lemma 2 with $f^+ = \alpha\gamma_2 v^+$, we can find positive constants $C = C(n, \lambda, \Lambda)$ and $\varepsilon = \varepsilon(n)$ such that

$$\begin{aligned}\sup_{x\in\mathbb{R}^n_+} v(x,t) \leq \sup_{x\in\partial\mathbb{R}^n_+} v(x,t)^+ &+ C \sup_{y\in\mathbb{R}^n_+, |y|>1} |y| \|\alpha\gamma_2 v^+\|_{L^n(A_y\cap\mathbb{R}^n_+)} \\ &+ C \sup_{y\in\mathbb{R}^n_+, |y|\leq 1} \|\alpha\gamma_2 v^+\|_{L^n(B_y\cap\mathbb{R}^n_+)}\end{aligned} \tag{8}$$

for all $t \in (0, T]$, where $A_y = B_{2|y|}(0) \setminus B_{\varepsilon|y|}(0)$ and $B_y = B_{2|y|}(0)$ for $y \in \mathbb{R}^n_+$. Furthermore, there exist positive constants k_1, k_2 and k_3, depending only on n, such that

$$\begin{aligned}|y|\|\alpha\gamma_2 v^+(t)\|_{L^n(A_y\cap\mathbb{R}^n_+)} &\leq \alpha|y| \sup_{\mathbb{R}^n_+} v^+(t) \|\gamma_2\|_{L^n(A_y)} \\ &\leq k_1\alpha|y| \sup_{\mathbb{R}^n_+} v^+(t) \left(\int_{\varepsilon|y|}^{|y|} \frac{r^{n-1}}{(1+r^2)^n} dr\right)^{1/n} \\ &\leq k_2\alpha \sup_{\mathbb{R}^n_+} v^+(t) \quad \text{for} \quad |y| > 1, \\ \|\alpha\gamma_2 v^+\|_{L^n(B_y\cap\mathbb{R}^n_+)} &\leq \alpha \sup_{\mathbb{R}^n_+} v^+(t) \|\gamma_2\|_{L^n(B_y)} \\ &\leq \alpha \sup_{\mathbb{R}^n_+} v^+(t) \left(\int_0^{|y|} \frac{r^{n-1}}{(1+r^2)^n} dr\right)^{1/n} \\ &\leq k_3\alpha \sup_{\mathbb{R}^n_+} v^+(t) \quad \text{for} \quad |y| \leq 1.\end{aligned}$$

These together with (8) imply that

$$\sup_{\mathbb{R}^n_+\times(0,T]} v \leq \sup_{\partial\mathbb{R}^n_+\times(0,T]} v^+ + \alpha K \sup_{\mathbb{R}^n_+\times(0,T]} v^+. \tag{9}$$

for some positive constant K depending only on λ, Λ and n. By (7) and (9) we take a sufficiently small $\alpha > 0$ such that $\alpha K < 1$ and we obtain

$$\sup_{\mathbb{R}^n_+\times(0,T]} v(x,t) \le 0.$$

This means that

$$\sup_{\mathbb{R}^n_+\times(0,T]} u(x,t) \le L,$$

and the proof is complete. □

Now we are ready to state and prove the following Phragmén-Lindelöf theorem for problem (1).

Theorem 7 *Let $T > 0$. Let $F = F(z, p, M)$ be a continuous function in $\overline{\mathbb{R}^n_+} \times (0, T] \times \mathbb{R}^n \times \mathbb{S}^n$ and assume (F1) and (F2). Let $u = u(x, t) \in C(\overline{\mathbb{R}^n_+} \times (0, T])$ be a viscosity subsolution of (1) such that such that*

$$L := \sup_{(x,t)\in\partial\mathbb{R}^n_+\times(0,T]} u(x,t) < \infty \quad \text{and} \quad \lim_{t\to+0} \sup_{x\in\partial\mathbb{R}^n_+} u(x,t) \le 0. \tag{10}$$

Then there exists a constant $\alpha > 0$ such that, if

$$\limsup_{x\in\mathbb{R}^n_+,\,|x|\to\infty} \frac{u(x,t)}{(1+|x|)^\alpha} = 0 \quad \text{and} \quad \limsup_{x'\in\mathbb{R}^{n-1},|x'|\to\infty} u(x',0,t) < L \tag{11}$$

for $t \in (0, T]$, then $u \le 0$ in $\overline{\mathbb{R}^n_+} \times (0, T]$.

Proof It follows from (11) that

$$\limsup_{x\in\mathbb{R}^n_+,\,|x|\to\infty} \frac{u(x,t)}{(1+|x|)^\alpha} = 0 \quad \text{for} \quad t \in (0,T]. \tag{12}$$

Then, by (10) we apply Theorem 6 to obtain

$$u \le L \quad \text{in} \quad \overline{\mathbb{R}^n_+} \times (0,T].$$

So it suffices to consider the case $L > 0$. Then it follows from (10) that

$$L = \sup_{\partial\mathbb{R}^n_+\times(0,T]} u(x,t) = \sup_{\overline{\mathbb{R}^n_+}\times(0,T]} u > 0. \tag{13}$$

Let $w(x,t) := e^{-t}u(x,t)/(1+x_n)$ for $(x,t) \in \overline{\mathbb{R}^n_+} \times (0,T]$. By (10) and (11) we see that

$$w(x,t) < L \quad \text{in} \quad \mathbb{R}^n_+ \times (0,T] \quad \text{and} \quad \limsup_{t\to+0} \sup_{x\in\partial\mathbb{R}^n_+} w(x,t) \le 0. \tag{14}$$

Furthermore, w satisfies

$$\mathscr{P}^-(D^2w) - \frac{2\Lambda}{1+x_n}|Dw| - b(x)|D'w| \leq 0 \quad \text{in } \Omega_T, \quad \partial_t w + \partial_\nu w \leq 0 \quad \text{on } \partial\Omega_T.$$

in the sense of viscosity solutions.

Since it follows from (11) that

$$\limsup_{|x'|\to\infty} u(x', 0, t) = 0,$$

by (11), (13) and (14), we can find $(x_*, t_*) \in \partial\mathbb{R}^n_+ \times (0, T]$ such that $w(x_*, t_*) = L$. Then it follows from the Hopf lemma that

$$\limsup_{h\to 0} \frac{w(x_* - \nu h, t_*) - w(x_*, t_*)}{h} = -2A < 0$$

for some constant $A > 0$. This implies that

$$w(x_* - \nu h, t_*) < L - Ah$$

for sufficiently small $h > 0$.

Let ϕ be a test function as in Theorem 5 such that

$$\phi(x, t) = \gamma|x' - x'_*|^2 - Ax_n^2 - \sigma x_n + |t - t_*|^2,$$

where $\gamma = A/2\Lambda(n-1)$ and $\sigma = A/4\Lambda$. Then we easily see that $(x_*, t_*) \in \partial\mathbb{R}^n_+ \times (0, T]$ be a local maximum point of $u - \phi$. Therefore we obtain

$$\mathscr{P}^-_{\lambda,\Lambda}(D^2\phi(x_*, t_*)) - \frac{2\Lambda}{1+(x_{*,n})}|D\phi(x_*, t_*)| - b(x_*)|D'\phi(x_*, t_*)| = A - \frac{A}{2} > 0,$$

$$\partial_t\phi(x_*, t_*) + \partial_\nu\phi(x_*, t_*) = 0 + \frac{A}{4} > 0.$$

This contradicts the definition of viscosity solutions at the boundary. Therefore we see that $L \leq 0$ and Theorem 7 follows from (12). □

Similarly to Sect. 3, Theorem 4 follows from Theorem 3.

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Entire Solutions to Generalized Parabolic k-Hessian Equations

Saori Nakamori and Kazuhiro Takimoto

Abstract In this paper, we deal with entire solutions to the generalized parabolic k-Hessian equation of the form $u_t = \mu(F_k(D^2u)^{1/k})$ in $\mathbb{R}^n \times (-\infty, 0]$. We prove that for $1 \le k \le n$, any strictly convex-monotone solution $u = u(x,t) \in C^{4,2}(\mathbb{R}^n \times (-\infty, 0])$ to $u_t = \mu(F_k(D^2u)^{1/k})$ in $\mathbb{R}^n \times (-\infty, 0]$ must be a linear function of t plus a quadratic polynomial of x, under some assumptions on $\mu : (0, \infty) \to \mathbb{R}$ and some growth conditions on u.

Keywords Entire solution · Fully nonlinear equation · Generalized parabolic k-Hessian equation · Pogorelov type lemma

1 Introduction

This paper is a sequel to [18].

The characterization of entire solutions to PDEs has been extensively studied in the literature. For example, the following is known as Liouville's theorem for harmonic functions; *If* $u \in C^2(\mathbb{R}^n)$ *is a harmonic function in* $\mathbb{R}^n$ (*i.e., a solution to Laplace equation* $\Delta u = 0$ *in* $\mathbb{R}^n$) *which is bounded in* $\mathbb{R}^n$, *then u is a constant function.* Using this theorem, one can see that any convex solution to Poisson equation $\Delta u = 1$ in $\mathbb{R}^n$ must be a quadratic polynomial.

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For the minimal surface equation in $\mathbb{R}^2$, Bernstein [2] proved the following theorem about a hundred years ago; *If* $f = f(x, y) \in C^2(\mathbb{R}^2)$ *is a solution to the minimal surface equation*

$$(1 + f_y)^2 f_{xx} - 2 f_x f_y f_{xy} + (1 + f_x^2) f_{yy} = 0 \quad in\ \mathbb{R}^2,$$

then f *is an affine function of* x *and* y.

Here we list some results concerning this type of theorems for *fully nonlinear* equations. First, for Monge-Ampère equation, the following theorem is known. We denote by $D^2u = (D_{ij}u)_{\substack{1\le i\le n \\ 1\le j\le n}}$ the Hessian matrix of u, where $D_{ij}u = \partial^2 u/\partial x_i \partial x_j$.

Theorem 1.1 *Let* $u \in C^4(\mathbb{R}^n)$ *be a convex solution to*

$$\det D^2u = 1 \quad in\ \mathbb{R}^n. \tag{1}$$

Then u is a quadratic polynomial.

This theorem was proved by Jörgens [15] for $n = 2$, by Calabi [6] for $n \le 5$, and by Pogorelov [20] for arbitrary $n \ge 2$ (see also [7] for a simpler proof). We note that Bernstein's theorem for the minimal surface equation stated above can also be proved via Theorem 1.1 for $n = 2$.

Moreover, Caffarelli [3] proved that Theorem 1.1 holds also for viscosity solutions (see also [4]). Jian and Wang [14] obtained Bernstein type result for a certain Monge-Ampère equation in the half space $\mathbb{R}^n_+$.

We note that the convexity assumption in Theorem 1.1 is quite natural, since Monge-Ampère operator $\det D^2u$ is degenerate elliptic for convex functions so that we usually seek solutions in the class of convex functions when we deal with Monge-Ampère equation.

Later, Bao et al. [1] extended this result to the so-called *k-Hessian equation* of the form

$$F_k(D^2u) = 1 \quad \text{in } \mathbb{R}^n, \tag{2}$$

for $1 \le k \le n$. Here $F_k(D^2u)$ is defined by

$$F_k(D^2u) = S_k(\lambda_1, \dots, \lambda_n), \tag{3}$$

where, for a C^2 function u, $\lambda_1, \dots, \lambda_n$ denote the eigenvalues of D^2u, and S_k denotes the k-th elementary symmetric function, that is

$$S_k(\lambda_1, \dots, \lambda_n) = \sum \lambda_{i_1} \cdots \lambda_{i_k},$$

where the sum is taken over all increasing k-tuples, $1 \le i_1 < \cdots < i_k \le n$.

Laplace operator Δu and Monge-Ampère operator $\det D^2u$ correspond respectively to the special cases $k = 1$ and $k = n$ in (3). Thus one can say that the class of k-Hessian equations includes important PDEs which arise in physics and geometry.

Here we remark that (3) is a linear operator for $k = 1$ while it is a fully nonlinear operator for $k \geq 2$. It is much harder to study the intermediate case $2 \leq k \leq n - 1$. Though, there are a number of papers concerning the analysis of k-Hessian equation, such as the solvability of the Dirichlet problem. See, for example, [5, 9, 12, 21–26].

We state Bernstein type theorem for k-Hessian equation of the form (2), which was proved by Bao et al. [1].

Theorem 1.2 *Let $1 \leq k \leq n$ and $u \in C^4(\mathbb{R}^n)$ be a strictly convex solution to (2). Suppose that there exist constants $A, B > 0$ such that for all $x \in \mathbb{R}^n$,*

$$u(x) \geq A|x|^2 - B. \tag{4}$$

Then u is a quadratic polynomial.

In this theorem, for the case $k = n$ which corresponds to Monge-Ampère equation (1), the assumption (4) can be removed, due to Theorem 1.1. Furthermore, as mentioned before, one can remove the assumption (4) for the case $k = 1$ which corresponds to Poisson equation $\Delta u = 1$. Bernstein type theorems were also obtained for other elliptic fully nonlinear equations, such as (n, k)-Hessian quotient equation [1] and the special Lagrangian equation [30].

Next, Gutiérrez and Huang [11] extended Theorem 1.1 to the parabolic analogue of Monge-Ampère equation

$$-u_t \det D^2u = 1 \quad \text{in } \mathbb{R}^n \times (-\infty, 0]. \tag{5}$$

Here D^2u means the matrix of second partial derivatives with respect to the space variables $x = (x_1, \dots, x_n)$. This type of equation was firstly proposed by Krylov [16].

The function $u = u(x, t) : \mathbb{R}^n \times (-\infty, 0] \to \mathbb{R}$ is said to be *convex-monotone* if it is convex in x and non-increasing in t. Gutiérrez and Huang [11] proved the following Bernstein type theorem for (5).

Theorem 1.3 *Let $u \in C^{4,2}(\mathbb{R}^n \times (-\infty, 0])$ be a convex-monotone solution to (5). Suppose that there exist constants $m_1 \geq m_2 > 0$ such that for all $(x, t) \in \mathbb{R}^n \times (-\infty, 0]$,*

$$-m_1 \leq u_t(x, t) \leq -m_2.$$

Then u has the form $u(x, t) = -mt + p(x)$ where $m > 0$ is a constant and p is a quadratic polynomial.

We note that Xiong and Bao [29] have obtained Bernstein type theorems for more general parabolic Monge-Ampère equations, such as $u_t = (\det D^2u)^{1/n}$ and $u_t = \log \det D^2u$. As far as we know, Bernstein type theorems for parabolic fully nonlinear equations were known only for the parabolic Monge-Ampère type equations.

Recently, in the paper [18] we dealt with the parabolic analogue of k-Hessian equation of the following form

$$-u_t F_k(D^2u) = 1 \quad \text{in } \mathbb{R}^n \times (-\infty, 0], \tag{6}$$

for $1 \leq k \leq n$, and obtained Bernstein type theorem for (6). Here $F_k(D^2u)$ is the k-Hessian operator defined in (3). For the special case $k = n$, (6) reduces to (5) which is the parabolic Monge-Ampère equation.

However, there are different parabolic analogues of k-Hessian equation which have been studied in the literature.

Ivochkina and Ladyzhenskaya [13] have studied the solvability of the first initial boundary value problem for

$$-u_t + F_k(D^2u)^{\frac{1}{k}} = \psi.$$

Wang [28] considered a following version of parabolic equation,

$$-u_t + \log F_k(D^2u) = \psi.$$

For the case $k = n$, this equation reduces to

$$-u_t + \log \det D^2u = \psi,$$

which was studied by G. Wang and W. Wang [27]. Moreover,

$$S_k(-u_t, \lambda_1, \ldots, \lambda_n) = \psi,$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of D^2u, i.e., $-u_t F_{k-1}(D^2u) + F_k(D^2u) = \psi$, was considered in [17].

We would like to get Bernstein type theorems for as many PDEs as possible, but it seems hard to deal with PDEs above one by one to obtain Bernstein type theorems. The aim of this paper is to obtain Bernstein type theorems for more general parabolic k-Hessian equations (see (7) in Sect. 2), which includes the particular parabolic k-Hessian equation (6) studied in our previous work [18].

This paper is constructed as follows. In Sect. 2, we state our main result and give some examples of PDEs for which one can get Bernstein type theorem. In Sect. 3, we prove Pogorelov type lemma, which is crucial in our argument. Section 4 is devoted to the proof of the main result.

2 Main Theorem

In this paper, we shall consider parabolic k-Hessian equation of the form

$$u_t = \mu\left(F_k(D^2u)^{\frac{1}{k}}\right) \quad \text{in } \mathbb{R}^n \times (-\infty, 0], \tag{7}$$

where $\mu : (0, \infty) \to \mathbb{R}$ is a function. We state our main result, which is a Bernstein type theorem for (7). Here, the function $u = u(x, t) : \mathbb{R}^n \times (-\infty, 0] \to \mathbb{R}$ is said to be *strictly convex-monotone* if u is strictly convex in x and decreasing in t.

Theorem 2.1 *Let $\mu \in C^2(0, \infty)$, $1 \le k \le n$ and $u \in C^{4,2}(\mathbb{R}^n \times (-\infty, 0])$ be a strictly convex-monotone solution to (7). We suppose that μ and u satisfy the following conditions.*

(A) *For all $s \in (0, \infty)$, $\mu'(s) > 0$ and $\mu''(s) \le 0$.*
(B) *There exist constants $m_1 \ge m_2 > 0$ such that*

$$\lim_{s\to+0} \mu(s) < -m_1 \le -m_2 < \lim_{s\to\infty} \mu(s) \tag{8}$$

and that for all $(x, t) \in \mathbb{R}^n \times (-\infty, 0]$,

$$-m_1 \le u_t(x, t) \le -m_2. \tag{9}$$

(C) *There exist constants $A, B > 0$ such that for all $x \in \mathbb{R}^n$, $u(x, 0) \ge A|x|^2 - B$.*

Then, u has the form $u(x, t) = -mt + p(x)$ where $m > 0$ is a constant and p is a quadratic polynomial.

Remark 2.1 We denote by $\mathbb{S}^{n\times n}$ and $\mathbb{S}^{n\times n}_+$, the set of all symmetric $n \times n$ matrices and that of all non-negative definite symmetric $n \times n$ matrices, respectively. Set $\widetilde{F}(M) = \mu(F_k(M)^{1/k})$ for $M = (m_{ij})_{\substack{1\le i\le n\\ 1\le j\le n}} \in \mathbb{S}^{n\times n}$. Then the condition (A) guarantees that $\widetilde{F}$ is concave in $\mathbb{S}^{n\times n}_+$. Indeed, if we define $f(M) = F_k(M)^{1/k}$, then easy calculation shows that

$$\widetilde{F}_{ij,rs} = \mu'' f_{ij} f_{rs} + \mu' f_{ij,rs},$$

where we write $f_{ij} = \partial f/\partial m_{ij}$ and $f_{ij,rs} = \partial^2 f/\partial m_{ij}\partial m_{rs}$, which yields that for all $\xi = (\xi_{ij})_{\substack{1\le i\le n\\ 1\le j\le n}} \in \mathbb{R}^{n\times n}$,

$$\widetilde{F}_{ij,rs}\xi_{ij}\xi_{rs} = \mu''(f_{ij}\xi_{ij})^2 + \mu' f_{ij,rs}\xi_{ij}\xi_{rs} \le 0,$$

due to the concavity of f in $\mathbb{S}^{n\times n}_+$ (see [5]). Also, (A) implies that $\widetilde{F}$ is non-decreasing in $\mathbb{S}^{n\times n}_+$, which means that for $M, N \in \mathbb{S}^{n\times n}_+$ with $M \le N$ it holds that $\widetilde{F}(M) \le \widetilde{F}(N)$.

Example 2.1 (1) If we set $\mu(s) = -s^{-k}$, then the Eq. (7) reduces to (6), and we can see that Bernstein type theorem for (6) holds, which was obtained previously by the authors [18].

(2) If we set $\mu(s) = -1/s$, then we can obtain Bernstein type theorem for

$$-u_t F_k(D^2u)^{\frac{1}{k}} = 1 \quad \text{in } \mathbb{R}^n \times (-\infty, 0].$$

(3) If we set $\mu(s) = k \log s$, then we can obtain Bernstein type theorem for the following equation

$$u_t = \log F_k(D^2u) \quad \text{in } \mathbb{R}^n \times (-\infty, 0].$$

It should be noted that in this case the condition (B) can be replaced by the boundedness of u_t in $\mathbb{R}^n \times (-\infty, 0]$. Therefore, u needs not to be decreasing in t, while u must be strictly convex in x. Indeed, considering $v(x, t) = u(x, t) - ct$ for sufficiently large $c > 0$ and setting $\mu(s) = k \log s - c$, we get the desired result.

(4) For the following version of parabolic k-Hessian equation

$$u_t = F_k(D^2u)^{\frac{1}{k}} \quad \text{in } \mathbb{R}^n \times (-\infty, 0], \tag{10}$$

we can also obtain Bernstein type theorem by using Theorem 2.1. We remark that for $k = 1$, (10) reduces to the heat equation $u_t = \Delta u$ which is well-known. Here we state Bernstein type theorem for (10).

Corollary 2.2 *Let $1 \le k \le n$ and $u \in C^{4,2}(\mathbb{R}^n \times (-\infty, 0])$ be a solution to* (10) *which is strictly convex in x. We assume* (C) *in Theorem 2.1 and the condition given below are satisfied.*

(D) *There exist constants $c_1 \ge c_2 > 0$ such that for all $(x, t) \in \mathbb{R}^n \times (-\infty, 0]$, $c_1 \le u_t(x, t) \le c_2$.*

Then, u has the form $u(x, t) = Ct + p(x)$ where $C > 0$ is a constant and p is a quadratic polynomial.

Proof We set $v(x, t) = u(x, t) - (c_2 + 1)t$. Then it follows from (C) and (D) that $v \in C^{4,2}(\mathbb{R}^n \times (-\infty, 0])$ is a strictly convex-monotone solution to

$$v_t = F_k(D^2v)^{\frac{1}{k}} - (c_2 + 1) \quad \text{in } \mathbb{R}^n \times (-\infty, 0],$$

and satisfies $-(c_2 - c_1 + 1) \le v_t \le -1$ in $\mathbb{R}^n \times (-\infty, 0]$ and $v(x, 0) \ge A|x|^2 - B$ for all $x \in \mathbb{R}^n$. Applying Theorem 2.1 for $\mu(s) = s - (c_2 + 1)$, we are done. □

Remark 2.2 It is known that the k-Hessian operator F_k is degenerate elliptic for *k-convex functions* (see [5] for the proof). The class of k-convex functions is strictly wider than that of convex functions for $1 \le k < n$. Hence, when we study k-Hessian equation, it is natural to seek solutions in the class of k-convex functions, rather than in the class of convex functions. But we do not know whether Theorems 1.2 and 2.1 remain true if one replaces strict convexity by strict k-convexity for the case $1 \le k < n$.

3 Pogorelov Type Lemma

We introduce some notation. First, for $D \subset \mathbb{R}^n \times (-\infty, 0]$ and $t \le 0$, $D(t)$ is defined by

$$D(t) = \{x \in \mathbb{R}^n \mid (x, t) \in D\}.$$

Let $D \subset \mathbb{R}^n \times (-\infty, 0]$ be a bounded set and $t_0 = \inf\{t \le 0 \mid D(t) \ne \emptyset\}$. The parabolic boundary $\partial_p D$ of D is defined by

$$\partial_p D = \left(\overline{D(t_0)} \times \{t_0\}\right) \cup \bigcup_{t \le 0} \left(\partial D(t) \times \{t\}\right),$$

where we denote by $\overline{D(t_0)}$ and $\partial D(t)$, the closure of $D(t_0)$ and the boundary of $D(t)$ in $\mathbb{R}^n$ respectively. We say that a domain $D \subset \mathbb{R}^n \times (-\infty, 0]$ is a *bowl-shaped* domain if $D(t)$ is convex for each $t \in (-\infty, 0]$ and $D(t_1) \subset D(t_2)$ for $t_1 \le t_2 \le 0$.

Next, for $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $1 \le m \le n$, we define

$$S_{m;i_1 i_2 \ldots i_j}(\lambda) = \begin{cases} S_m(\widehat{\lambda}) & \text{if } i_r \ne i_s \text{ for any } 1 \le r < s \le j, \\ 0 & \text{otherwise,} \end{cases}$$

where $\widehat{\lambda} = (\widehat{\lambda}_1, \ldots, \widehat{\lambda}_n)$ is defined by

$$\widehat{\lambda_i} = \begin{cases} 0 & \text{if } i = i_r \text{ for some } 1 \le r \le j, \\ \lambda_i & \text{otherwise.} \end{cases}$$

We note that

$$\frac{\partial}{\partial \lambda_i} S_m(\lambda) = S_{m-1;i}(\lambda)$$

for $i = 1, \ldots, n$.

In this section, we shall prove Pogolerov type lemma for parabolic k-Hessian equation. This is an analogue of the result of Pogorelov [19], who derived interior C^2-estimates of a solution from its C^1-estimates for Monge-Ampère equation.

Lemma 3.1 *Let $\mu \in C^2(0, \infty)$ which satisfies* (A) *in Theorem 2.1, $1 \le k \le n$, D be a bounded bowl-shaped domain in $\mathbb{R}^n \times (-\infty, 0]$ and $u \in C^{4,2}(\overline{D})$ be a strictly convex-monotone solution to $u_t = \mu(F_k(D^2u)^{1/k})$ in D with $u = 0$ on $\partial_p D$. We* *suppose that there exist constants $m_1 \geq m_2 > 0$ such that (8) holds and that u satisfies (9) for all $(x,t) \in D$. Then there exists a constant $C = C(n, k, m_1, m_2, \mu, \|u\|_{C^1(D)})$ such that*

$$\sup_{(x,t)\in D} |u(x,t)|^4 |D^2 u(x,t)| \leq C. \tag{11}$$

Proof The idea of the proof is adapted from that of [8] (see also [18]). From now on, we denote $D_i u$ by u_i, $D_{ij}u$ by u_{ij}, $D_i u_t$ by u_{it}, and so on.

The function u satisfies

$$-u_t + \mu(f(D^2 u)) = 0 \quad \text{in } \overline{D}, \tag{12}$$

where $f(M) = F_k(M)^{1/k}$. Differentiating (12) with respect to x_γ $(\gamma = 1, \ldots, n)$ yields that

$$-u_{\gamma t} + \mu'(f(D^2 u)) f_{ij} u_{ij\gamma} = 0 \quad \text{in } \overline{D}. \tag{13}$$

Differentiating (13) with respect to x_γ, we obtain that

$$\begin{aligned} -u_{\gamma\gamma t} + \mu''(f(D^2u))(f_{ij}u_{ij\gamma})^2 + \mu'(f(D^2u)) f_{ij} u_{ij\gamma\gamma} \\ + \mu'(f(D^2u)) f_{ij,rs} u_{ij\gamma} u_{rs\gamma} = 0 \quad \text{in } \overline{D}. \end{aligned} \tag{14}$$

As we have done in our previous paper [18], we consider the auxiliary function

$$\Psi(x,t;\xi) = (-u(x,t))^4 \varphi\left(\frac{|Du(x,t)|^2}{2}\right) D_{\xi\xi} u(x,t), \quad (x,t) \in \overline{D},\ |\xi| = 1,$$

where $\varphi(s) = (1 - s/M)^{-1/8}$ and $M = 2\sup_{(x,t)\in D} |Du(x,t)|^2$. We note that $u \leq 0$ in $\overline{D}$. Then one can take a point $(x_0, t_0) \in \overline{D} \setminus \partial_p D$ and a unit vector $\xi_0 \in \mathbb{R}^n$ which satisfy

$$\Psi(x_0, t_0; \xi_0) = \max\{\Psi(x,t;\xi) \mid (x,t) \in \overline{D},\ |\xi| = 1\}.$$

Rotating the coordinates appropriately, we may take $\xi_0 = e_1$ and $D^2u(x_0,t_0)$ is diagonal with $u_{11}(x_0,t_0) \geq u_{22}(x_0,t_0) \geq \cdots \geq u_{nn}(x_0,t_0) > 0$. Then $\Psi = \Psi(x,t;e_1) = (-u(x,t))^4 \varphi(|Du(x,t)|^2/2) u_{11}(x,t)$ attains its maximum at (x_0,t_0). It is enough to consider the case $\lambda_1 = u_{11}(x_0,t_0) \geq 1$.

Since Ψ attains its maximum at (x_0,t_0), direct calculation gives

$$(\log \Psi)_i = \frac{4u_i}{u} + \frac{\varphi_i}{\varphi} + \frac{u_{11i}}{u_{11}} = 0, \tag{15}$$

$$(\log \Psi)_{ii} = 4\left(\frac{u_{ii}}{u} - \frac{u_i^2}{u^2}\right) + \frac{\varphi_{ii}}{\varphi} - \frac{\varphi_i^2}{\varphi^2} + \frac{u_{11ii}}{u_{11}} - \frac{u_{11i}^2}{u_{11}^2} \leq 0, \tag{16}$$

$$(\log \Psi)_t = \frac{4u_t}{u} + \frac{\varphi_t}{\varphi} + \frac{u_{11t}}{u_{11}} \geq 0, \tag{17}$$

$$\varphi_i = \varphi'\left(\frac{|Du|^2}{2}\right) u_i u_{ii}, \tag{18}$$

$$\varphi_{ii} = \varphi''\left(\frac{|Du|^2}{2}\right) u_i^2 u_{ii}^2 + \varphi'\left(\frac{|Du|^2}{2}\right)\left(u_{ii}^2 + \sum_{j=1}^{n} u_j u_{iij}\right), \tag{19}$$

$$\varphi_t = \varphi'\left(\frac{|Du|^2}{2}\right) \sum_{j=1}^{n} u_j u_{jt} \tag{20}$$

at (x_0, t_0), for $i = 1, \ldots, n$.

Here we claim that the following inequality holds:

$$f_{ij,rs}(D^2u) u_{ij\gamma} u_{rs\gamma} \le -\frac{1}{k} \sum_{i,j=1}^{n} S_k(\lambda)^{\frac{1}{k}-1} S_{k-2;ij}(\lambda) u_{ij\gamma}^2 \tag{21}$$

at (x_0, t_0), where $\lambda = (\lambda_1, \ldots, \lambda_n) = (u_{11}(x_0, t_0), \ldots, u_{nn}(x_0, t_0))$.

Indeed, since $D^2u(x_0, t_0)$ is diagonal, we have

$$f_{ij}(D^2u) = \begin{cases} \frac{1}{k} S_k(\lambda)^{\frac{1}{k}-1} S_{k-1;i}(\lambda) & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases} \tag{22}$$

$$f_{ij,rs}(D^2u) = \begin{cases} \frac{1}{k} S_k(\lambda)^{\frac{1}{k}-1} S_{k-2;ir}(\lambda) & \\ \quad + \frac{1}{k}\left(\frac{1}{k} - 1\right) S_k(\lambda)^{\frac{1}{k}-2} S_{k-1;i}(\lambda) S_{k-1;r}(\lambda) & \text{if } i = j, r = s, \\ -\frac{1}{k} S_k(\lambda)^{\frac{1}{k}-1} S_{k-2;ij}(\lambda) & \text{if } i = s \ne r = j, \\ 0 & \text{otherwise} \end{cases} \tag{23}$$

at (x_0, t_0) (see [8, Sect. 4]). Combining (23) and $S_{k-2;ii}(\lambda) = 0$ for $i = 1, \ldots, n$ yields that

$$\begin{aligned} & f_{ij,rs}(D^2u) u_{ij\gamma} u_{rs\gamma} \\ &= \sum_{i,j=1}^{n} \left[\frac{1}{k} S_k(\lambda)^{\frac{1}{k}-1} S_{k-2;ij}(\lambda) + \frac{1}{k}\left(\frac{1}{k} - 1\right) S_k(\lambda)^{\frac{1}{k}-2} S_{k-1;i}(\lambda) S_{k-1;j}(\lambda)\right] u_{ii\gamma} u_{jj\gamma} \\ &\quad - \sum_{i,j=1}^{n} \frac{1}{k} S_k(\lambda)^{\frac{1}{k}-1} S_{k-2;ij}(\lambda) u_{ij\gamma}^2 \\ &= \sum_{i,j=1}^{n} \frac{\partial^2}{\partial\lambda_i \partial\lambda_j} S_k(\lambda)^{\frac{1}{k}} u_{ii\gamma} u_{jj\gamma} - \frac{1}{k} \sum_{i,j=1}^{n} S_k(\lambda)^{\frac{1}{k}-1} S_{k-2;ij}(\lambda) u_{ij\gamma}^2 \end{aligned}$$

$$\leq -\frac{1}{k}\sum_{i,j=1}^{n} S_k(\lambda)^{\frac{1}{k}-1} S_{k-2;ij}(\lambda) u_{ij\gamma}^2.$$

In the last inequality, we use the concavity of $S_k(\cdot)^{1/k}$ in $\{(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \mid \lambda_1 > 0, \ldots, \lambda_n > 0\}$. Hence (21) holds.

Letting $\gamma = 1$ in (14) and using (21) and (A), we obtain that at (x_0, t_0)

$$-u_{11t} + \mu' f_{ii} u_{11ii} \geq \frac{1}{k}\mu' \sum_{i,j=1}^{n} S_k(\lambda)^{\frac{1}{k}-1} S_{k-2;ij}(\lambda) u_{1ij}^2. \tag{24}$$

Let L be the linearized operator of (12) at (x_0, t_0):

$$L = -D_t + \mu'(f(D^2u(x_0, t_0))) f_{ij}(D^2u(x_0, t_0)) D_{ij}.$$

By (16), (17) and (A), we obtain that

$$\begin{aligned} L(\log \Psi) = &-\left(\frac{4u_t}{u} + \frac{\varphi_t}{\varphi} + \frac{u_{11t}}{u_{11}}\right) \\ &+ \mu' f_{ii}\left(4\left(\frac{u_{ii}}{u} - \frac{u_i^2}{u^2}\right) + \frac{\varphi_{ii}}{\varphi} - \frac{\varphi_i^2}{\varphi^2} + \frac{u_{11ii}}{u_{11}} - \frac{u_{11i}^2}{u_{11}^2}\right) \leq 0 \end{aligned} \tag{25}$$

at (x_0, t_0). By using (13), (19), (20) and (24), it holds from (25) that

$$\begin{aligned} &-\frac{4u_t}{u} + \frac{1}{k}\mu' \sum_{i,j=1}^{n} S_k(\lambda)^{\frac{1}{k}-1} S_{k-2;ij}(\lambda) \frac{u_{1ij}^2}{u_{11}} \\ &+ \mu' f_{ii}\left(4\left(\frac{u_{ii}}{u} - \frac{u_i^2}{u^2}\right) + \frac{\varphi''}{\varphi}u_i^2 u_{ii}^2 + \frac{\varphi'}{\varphi}u_{ii}^2 - \frac{\varphi'^2}{\varphi^2}u_i^2 u_{ii}^2 - \frac{u_{11i}^2}{u_{11}^2}\right) \leq 0 \end{aligned} \tag{26}$$

at (x_0, t_0).

Now we split into two cases.

(i) $u_{kk} \geq K u_{11}$, *where* $K > 0$ *is a small constant to be determined later.*

By (15) and (18), we have

$$\frac{u_{11i}^2}{u_{11}^2} = \left(\frac{4u_i}{u} + \frac{\varphi_i}{\varphi}\right)^2 \leq 2\left(\frac{16u_i^2}{u^2} + \frac{\varphi'^2 u_i^2 u_{ii}^2}{\varphi^2}\right) \tag{27}$$

at (x_0, t_0). We note that the second term of the left hand side of (26) is non-negative, due to (A) and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$. By this fact and (27), we obtain that

$$-\frac{4u_t}{u} + \mu' f_{ii}\left(4\left(\frac{u_{ii}}{u} - \frac{9u_i^2}{u^2}\right) + \left(\frac{\varphi''}{\varphi} - \frac{3\varphi'^2}{\varphi^2}\right)u_i^2 u_{ii}^2 + \frac{\varphi'}{\varphi}u_{ii}^2\right) \leq 0 \tag{28}$$

at (x_0, t_0). Since

$$\sum_{i=1}^{n} f_{ii}u_{ii}^2 > f_{kk}u_{kk}^2 \geq \theta_1 \sum_{i=1}^{n} f_{ii}u_{11}^2$$

holds for some constant $\theta_1 > 0$ (cf. [8]), $\varphi''/\varphi - 3\varphi'^2/\varphi^2 \geq 0$ and $\varphi'/\varphi \geq c_1$ for some positive constant c_1, we get by (28) that at (x_0, t_0)

$$-\frac{4u_t}{u} + c_1\theta_1\mu' \sum_{i=1}^{n} f_{ii}u_{11}^2 + \frac{4\mu' f(D^2u)}{u} - \frac{C}{u^2}\mu' \sum_{i=1}^{n} f_{ii} \leq 0.$$

Here we used the equality $\sum_{i=1}^n f_{ii}(D^2u)u_{ii} = f(D^2u)$ at (x_0, t_0) which is implied by the 1-homogeneity of f. Since m_1 and m_2 satisfy (8), it follows from (A) that the inverse image $\mu^{-1}([-m_1, -m_2])$ is a closed interval, that is,

$$\mu^{-1}([-m_1, -m_2]) = [r_1, r_2] \quad \text{for some positive constants } r_1 \text{ and } r_2. \tag{29}$$

Therefore, it holds that

$$f(D^2u) = \mu^{-1}(u_t) \in [r_1, r_2] \quad \text{in } D \tag{30}$$

since u satisfies (9) in D, which implies that $\mu'(f(D^2u)) \in [\mu'(r_2), \mu'(r_1)]$ (we note that μ' is non-increasing due to (A)). Therefore we obtain that at (x_0, t_0)

$$\frac{4(m_1 + r_2\mu'(r_1))}{u} + c_1\theta_1\mu'(r_2) \sum_{i=1}^{n} f_{ii}u_{11}^2 - \frac{C}{u^2}\mu'(r_1) \sum_{i=1}^{n} f_{ii} \leq 0. \tag{31}$$

It holds that at (x_0, t_0)

$$\begin{aligned}\sum_{i=1}^{n} f_{ii}(D^2u) &= \sum_{i=1}^{n} \frac{1}{k} F_k(D^2u)^{\frac{1}{k}-1} \frac{\partial F_k}{\partial m_{ii}}(D^2u) \\ &= \frac{1}{k}\left(\mu^{-1}(u_t)\right)^{1-k} \sum_{i=1}^{n} S_{k-1;i}(\lambda) \leq C u_{11}^{k-1},\end{aligned} \tag{32}$$

and that

$$\begin{aligned}\sum_{i=1}^{n} f_{ii}(D^2u) &\geq f_{nn}(D^2u)\\ &= \frac{1}{k}F_k(D^2u)^{\frac{1}{k}-1}\frac{\partial F_k}{\partial m_{nn}}(D^2u)\\ &\geq \frac{1}{k}\left(\mu^{-1}(u_t)\right)^{1-k}\theta_2 u_{11}\cdots u_{k-1,k-1} \geq c_2 u_{11}^{k-1}, \qquad (33)\end{aligned}$$

for some constants $\theta_2, c_2 > 0$, since $u_{ii} \geq K u_{11}$ for $i = 1, \ldots, k-1$ by the hypothesis. By (31)–(33), we have

$$\theta u_{11}^{k+1} + \frac{C}{u} - \frac{C}{u^2}u_{11}^{k-1} \leq 0 \qquad (34)$$

at (x_0, t_0), for some constant $\theta > 0$. Multiplying $(-u)^8\varphi^2 u_{11}^{-(k-1)}/\theta$ by (34), we see that

$$(-u)^8\varphi^2 u_{11}^2 \leq C\frac{(-u)^7\varphi^2}{u_{11}^{k-1}} + C(-u)^6\varphi^2 \leq C(-u)^7\varphi^2 + C(-u)^6\varphi^2,$$

from which follows that $\Psi^2 \leq C(n, k, m_1, m_2, \mu, \|u\|_{C^1(D)})$ at (x_0, t_0).
(ii) *$u_{kk} \leq K u_{11}$, that is, $u_{jj} \leq K u_{11}$ for $j = k, k+1, \ldots, n$.*
By (15),

$$\frac{u_{111}}{u_{11}} = -\left(\frac{\varphi_1}{\varphi} + \frac{4u_1}{u}\right), \quad \frac{u_i}{u} = -\frac{1}{4}\left(\frac{\varphi_i}{\varphi} + \frac{u_{11i}}{u_{11}}\right), \quad i = 2, \ldots, n \qquad (35)$$

at (x_0, t_0). Combining (26) and (35) yields that

$$\begin{aligned}0 \geq\ & \mu' f_{11}\left(4\left(\frac{u_{11}}{u} - \frac{u_1^2}{u^2}\right) + \frac{\varphi''}{\varphi}u_1^2u_{11}^2 + \frac{\varphi'}{\varphi}u_{11}^2 - \frac{\varphi'^2}{\varphi^2}u_1^2u_{11}^2 - \left(\frac{\varphi_1}{\varphi} + \frac{4u_1}{u}\right)^2\right)\\ &+ \mu'\sum_{i=2}^{n} f_{ii}\left(\frac{4u_{ii}}{u} - \frac{1}{4}\left(\frac{\varphi_i}{\varphi} + \frac{u_{11i}}{u_{11}}\right)^2 + \frac{\varphi''}{\varphi}u_i^2u_{ii}^2 + \frac{\varphi'}{\varphi}u_{ii}^2 - \frac{\varphi'^2}{\varphi^2}u_i^2u_{ii}^2 - \frac{u_{11i}^2}{u_{11}^2}\right)\\ &+ \frac{1}{k}\mu'\sum_{i,j=1}^{n} S_k(\lambda)^{\frac{1}{k}-1}S_{k-2;ij}(\lambda)\frac{u_{1ij}^2}{u_{11}} - \frac{4u_t}{u}\end{aligned}$$

$$
\begin{aligned}
&\geq \mu' \left[\sum_{i=1}^{n} f_{ii} \left(\frac{4u_{ii}}{u} + \left(\frac{\varphi''}{\varphi} - \frac{3\varphi'^2}{\varphi^2} \right) u_i^2 u_{ii}^2 + \frac{\varphi'}{\varphi} u_{ii}^2 \right) - 36 f_{11} \frac{u_1^2}{u^2} \right] \\
&\quad + \mu' \left[-\frac{3}{2} \sum_{i=2}^{n} f_{ii} \frac{u_{11i}^2}{u_{11}^2} + \frac{1}{k} \sum_{i,j=1}^{n} S_k(\lambda)^{\frac{1}{k}-1} S_{k-2;ij}(\lambda) \frac{u_{1ij}^2}{u_{11}} \right] - \frac{4u_t}{u} \\
&=: \mu' I_1 + \mu' I_2 - \frac{4u_t}{u}
\end{aligned} \tag{36}
$$

at (x_0, t_0). I_1 can be estimated from below as

$$
I_1 \geq \frac{4\mu^{-1}(u_t)}{u} + \theta_1 f_{11} u_{11}^2 - \frac{C}{u^2} f_{11} \geq \frac{1}{2} \theta_1 f_{11} u_{11}^2 + \frac{4\mu^{-1}(u_t)}{u}, \tag{37}
$$

provided $u(x_0, t_0)^2 u_{11}(x_0, t_0)^2 \geq 2C/\theta_1$. If $u(x_0, t_0)^2 u_{11}(x_0, t_0)^2 < 2C/\theta_1$, then (11) is obvious. Hence we may assume $u(x_0, t_0)^2 u_{11}(x_0, t_0)^2 \geq 2C/\theta_1$ hereafter. As in [18, (3.23)], I_2 can be also estimated from below as

$$
\begin{aligned}
I_2 &\geq -\frac{3}{2} \cdot \frac{1}{k} S_k(\lambda)^{\frac{1}{k}-1} \sum_{i=2}^{n} S_{k-1;i}(\lambda) \frac{u_{11i}^2}{u_{11}^2} + 2 \cdot \frac{1}{k} S_k(\lambda)^{\frac{1}{k}-1} \sum_{i=2}^{n} S_{k-2;1i}(\lambda) \frac{u_{11i}^2}{u_{11}} \\
&= \frac{2}{k} S_k(\lambda)^{\frac{1}{k}-1} \left(\sum_{i=2}^{n} \left(S_{k-2;1i}(\lambda) - \frac{3}{4} \frac{S_{k-1;i}(\lambda)}{\lambda_1} \right) \frac{u_{11i}^2}{\lambda_1} \right) \geq 0
\end{aligned} \tag{38}
$$

by using (22) and $\lambda_1 S_{k-2;1i}(\lambda) \geq 3S_{k-1;i}(\lambda)/4$, provided $K > 0$ is sufficiently small (see [8, Lemma 3.1]). Therefore, (30), (36)–(38) and (A) yield that

$$
f_{11} u_{11}^2 \leq \frac{2}{\theta_1} \left(\frac{4u_t}{u\mu'(f(D^2u))} - \frac{4\mu^{-1}(u_t)}{u} \right) \leq -\frac{C}{u} \tag{39}
$$

at (x_0, t_0). On the other hand, by the fact $\lambda_1 S_{k-1;1}(\lambda) \geq \theta_3 S_k(\lambda)$ for some constant $\theta_3 > 0$ (see [8, Lemma 3.1]), we see that

$$
f_{11} u_{11}^2 = \frac{1}{k} S_k(\lambda)^{\frac{1}{k}-1} S_{k-1;1}(\lambda) \lambda_1^2 \geq \frac{\theta_3}{k} S_k(\lambda)^{\frac{1}{k}} \lambda_1 = \frac{\theta_3}{k} \mu^{-1}(u_t) u_{11} \tag{40}
$$

at (x_0, t_0). Combining (39) and (40), we have

$$
u_{11} \leq -\frac{C}{u} \tag{41}
$$

at (x_0, t_0). Multiplying $(-u)^4 \varphi$ by (41), we get

$$
(-u)^4 \varphi u_{11} \leq C(-u)^3 \varphi,
$$

from which follows that $\Psi \leq C(n, k, m_1, m_2, \mu, \|u\|_{C^1(D)})$ at (x_0, t_0).

Therefore, $(-u)^4|D^2u|$ can be estimated from above by some constant C, so that (11) holds. □

4 Proof of Theorem 2.1

The strategy of the proof of Theorem 2.1 is similar to that of [18, Theorem 2.1], but we shall explain it here for the sake of completeness.

Before giving a proof of Theorem 2.1, we introduce some notation. For a subset $D \subset \mathbb{R}^n \times (-\infty, 0]$, a function $v : D \to \mathbb{R}$ and a constant $\alpha \in (0, 1)$, α-Hölder seminorm of v over D is denoted by

$$[v]_{\alpha,D} = \sup_{\substack{(x,t),(y,s)\in D,\\ (x,t)\neq(y,s)}} \frac{|v(x,t) - v(y,s)|}{(|x-y|^2 + |t-s|)^{\frac{\alpha}{2}}}.$$

We begin the proof of Theorem 2.1. Let $u \in C^{4,2}(\mathbb{R}^n \times (-\infty, 0])$ be a strictly convex-monotone solution to (6), which satisfies the growth conditions (B) and (C). We may assume without loss of generality that $u(0,0) = 0$, $Du(0,0) = 0$, by considering $u(x,t) - u(0,0) - Du(0,0)\cdot x$ instead of $u(x,t)$. Then it can be seen by (C) that there exists another constant $\widetilde{A} > 0$ such that $u(x,0) \geq \widetilde{A}|x|^2$ for all $x \in \mathbb{R}^n$.

Let $R > 0$ be fixed. We define $v(x,t)(= v_R(x,t)) = u(Rx, R^2t)/R^2$. Then v satisfies $v_t(x,t) = u_t(Rx, R^2t)$ and $v_{ij}(x,t) = u_{ij}(Rx, R^2t)$, so that $v \in C^{4,2}(\mathbb{R}^n \times (-\infty, 0])$ is also a strictly convex-monotone classical solution to

$$v_t = \mu\left(F_k(D^2v)^{\frac{1}{k}}\right) \quad \text{in } \mathbb{R}^n \times (-\infty, 0]. \tag{42}$$

Moreover, v satisfies the following.

(B)' $-m_1 \leq v_t(x,t) \leq -m_2$ for all $(x,t) \in \mathbb{R}^n \times (-\infty, 0]$.
(C)' $v(x,0) \geq \widetilde{A}|x|^2$ for all $x \in \mathbb{R}^n$.

First, we shall obtain the local gradient estimate of the solution v. For $q > 0$, we set

$$\Omega_q = \{(x,t) \in \mathbb{R}^n \times (-\infty, 0] \mid v(x,t) < \widetilde{A}q\}.$$

It follows from (B)', (C)' and the strict convex-monotonicity of v that Ω_q is a bounded bowl-shaped domain and that

$$\Omega_q(t) \subset \Omega_q(0) \subset B(0, \sqrt{q}). \tag{43}$$

We prove the following estimate for $|Dv|$.

Lemma 4.1 *Let v and Ω_q be defined as above. Then there exists a constant $C > 0$, independent of q and R, such that for all $(x,t) \in \Omega_q$,*

$$|Dv(x,t)| \le C\sqrt{q}. \tag{44}$$

Proof We note that $v(x,t)$ is strictly convex in x.

By applying Aleksandrov's maximum principle (cf. [10]) to $v(\cdot,t) - \widetilde{A}q$ in $\Omega_q(t)$, we see that

$$|v(x_0,t) - \widetilde{A}q|^n \le C\left(\operatorname{diam}\Omega_q(t)\right)^{n-1} \operatorname{dist}(x_0, \partial\Omega_q(t))|Dv(\Omega_q(t))| \tag{45}$$

for any point $(x_0,t) \in \Omega_q$. Using (29), (42), (43), (45), (B)' and $(F_k(M)/\binom{n}{k})^{1/k} \ge F_n(M)^{1/n} = (\det M)^{1/n}$ for all $M \in \mathbb{S}_+^{n\times n}$ due to Newton-Maclaurin's inequality, we obtain that

$$\begin{aligned} |v(x_0,t) - \widetilde{A}q|^n &\le C(2\sqrt{q})^{n-1} \operatorname{dist}(x_0, \partial\Omega_q(t)) \int_{\Omega_q(t)} \det D^2 v(x,t)\, dx \\ &\le Cq^{\frac{n-1}{2}} \operatorname{dist}(x_0, \partial\Omega_q(t)) \int_{\Omega_q(t)} F_k(D^2 v(x,t))^{\frac{n}{k}}\, dx \\ &= Cq^{\frac{n-1}{2}} \operatorname{dist}(x_0, \partial\Omega_q(t)) \int_{\Omega_q(t)} \mu^{-1}(v_t(x,t))^n\, dx \\ &\le Cq^{\frac{n-1}{2}} \operatorname{dist}(x_0, \partial\Omega_q(t)) \cdot r_2^n |B(0,\sqrt{q})| \\ &= Cq^{n-\frac{1}{2}} \operatorname{dist}(x_0, \partial\Omega_q(t)), \end{aligned}$$

which implies that

$$|v(x_0,t) - \widetilde{A}q| \le Cq^{1-\frac{1}{2n}} \operatorname{dist}(x_0, \partial\Omega_q(t))^{\frac{1}{n}}.$$

Therefore, for all $x_0 \in \Omega_{q/2}(t)$,

$$\widetilde{A}q - \frac{1}{2}\widetilde{A}q \le \widetilde{A}q - v(x_0,t) \le Cq^{1-\frac{1}{2n}} \operatorname{dist}(x_0, \partial\Omega_q(t))^{\frac{1}{n}}.$$

Hence,

$$\operatorname{dist}(\Omega_{\frac{q}{2}}(t), \partial\Omega_q(t)) \ge C\sqrt{q}.$$

From this inequality and the convexity of v with respect to x, it follows that for $(x,t) \in \Omega_{q/2}$,

$$|Dv(x,t)| \le \frac{\widetilde{A}q - \widetilde{A}\frac{q}{2}}{\operatorname{dist}(\Omega_{\frac{q}{2}}(t), \partial\Omega_q(t))} \le \frac{\frac{\widetilde{A}q}{2}}{C\sqrt{q}} = C\sqrt{q}.$$

Replacing q by $2q$, we obtain the desired estimate (44). □

Especially, it holds that the estimate $|Dv(x,t)| \le C$ for all $(x,t) \in \Omega_1$, in which C is independent of R. By applying Lemma 3.1 to the function $v(x,t) - \widetilde{A}$ in Ω_1, we see that

$$\sup_{(x,t)\in\Omega_1} \left(\widetilde{A} - v(x,t)\right)^4 |D^2 v(x,t)| \le C.$$

This implies that for $(x,t) \in \Omega_{1/2}$,

$$|D^2 v(x,t)| \le \frac{C}{\left(\widetilde{A} - v(x,t)\right)^4} \le \frac{C}{\left(\widetilde{A} - \frac{\widetilde{A}}{2}\right)^4} = C, \tag{46}$$

so that we have the local estimate of $|D^2 v|$.

The next lemma can be proved by the same argument in [18, Lemma 4.3].

Lemma 4.2 *There exists a constant $C > 0$, independent of R, such that*

$$\mathrm{dist}(\Omega_{\frac{1}{8}}, \partial_p \Omega_{\frac{1}{2}}) \ge C.$$

The next task is to obtain local Hölder estimates of $D^2 v$ and v_t. We use the following Evans-Krylov type theorem (see [11]).

Theorem 4.3 *Let D and D' be bounded bowl-shaped domains which satisfy $D' \subset D$ and $\mathrm{dist}(D', \partial_p D) > 0$, and u be a $C^{4,2}(D)$ solution to the equation*

$$G(u_t, D^2 u) = 0$$

in D, where $G = G(q, M)$ is defined for all $(q, M) \in \mathbb{R} \times \mathbb{S}^{n\times n}$ with $G(\cdot, M) \in C^1(\mathbb{R})$ for each $M \in \mathbb{S}^{n\times n}$, and $G \in C^2(\mathbb{R} \times X)$ for some $X \subset \mathbb{S}^{n\times n}$ which is a neighborhood of $D^2 u(D)$. Suppose that the following hold.

(i) G is uniformly parabolic, i.e., there exist positive constants λ and Λ such that

$$-\Lambda \le G_q(q, M) \le -\lambda, \tag{47}$$

$$\lambda \|N - M\| \le G(q, N) - G(q, M) \le \Lambda \|N - M\|, \tag{48}$$

for all $q \in \mathbb{R}$ and $M, N \in \mathbb{S}^{n\times n}$ with $M \le N$.

(ii) G is concave in M.

If $\|u\|_{C^{2,1}(D)} \le K$, then there exist positive constants C depending on λ, Λ, n, K, D, D' and $G(0,0)$, and $\alpha \in (0,1)$ depending on λ, Λ and n such that

$$\|u\|_{C^{2+\alpha, 1+\frac{\alpha}{2}}(D')} \le C.$$

We set $G(q, M) = -q + \mu(f(M))$ for $(q, M) \in [-m_1, -m_2] \times X$, where

$$X = \{M = (m_{ij}) \in \mathbb{S}_+^{n\times n} \mid r_1 \le f(M) \le r_2,\ |m_{ij}| \le C \ \text{ for } i, j = 1, \ldots, n\},$$

in which C is a constant appeared in (46).

It is trivial that (47) holds for $0 < \lambda \le 1 \le \Lambda$. We can also see that (48) in Theorem 4.3 holds in $[-m_1, -m_2] \times X$ for some constants $\lambda, \Lambda > 0$, due to [5]. Finally, as we have mentioned in Remark 2.1, (ii) in Theorem 4.3 holds.

One can extend G in $\mathbb{R} \times \mathbb{S}^{n\times n}$ so that G satisfies (i) and (ii) in Theorem 4.3, possibly for different positive constants λ and Λ. Then it holds that v is a solution to $G(v_t, D^2 v) = 0$ in $\Omega_{1/2}$. Using Lemma 4.2 and Theorem 4.3, we obtain that there exists $\alpha \in (0, 1)$ such that

$$\|v\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\Omega_{\frac{1}{8}})} \le C.$$

Therefore it follows that $[D_{ij}v]_{\alpha,\Omega_{1/8}} \le C$ for $i, j = 1, \ldots, n$ and $[v_t]_{\alpha,\Omega_{1/8}} \le C$. We note that the constant C does not depend on R.

By substituting $v(x, t) = u(Rx, R^2 t)/R^2$, we have

$$[D_{ij}u]_{\alpha,\left\{u(x,t)<\frac{\tilde{A}}{8}R^2\right\}} \le CR^{-\alpha},$$

and

$$[u_t]_{\alpha,\left\{u(x,t)<\frac{\tilde{A}}{8}R^2\right\}} \le CR^{-\alpha}$$

for any $R > 0$. This implies that $[D_{ij}u]_{\alpha,\Omega} = 0$, and $[u_t]_{\alpha,\Omega} = 0$ for any bounded subset Ω of $\mathbb{R}^n \times (-\infty, 0]$. Hence $D_{ij}u$ and u_t are constants in $\mathbb{R}^n \times (-\infty, 0]$, so that we obtain that u has the form $u(x, t) = -mt + p(x)$ where $m > 0$ is a constant and p is a quadratic polynomial.

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Dynamical Aspects of a Hybrid System Describing Intermittent Androgen Suppression Therapy of Prostate Cancer

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Abstract We consider a mathematical model describing Intermittent Androgen Suppression therapy (IAS therapy) of prostate cancer. The system has a hybrid structure, i.e., the system consists of two different systems by the medium of an unknown binary function denoting the treatment state. In this paper, we shall prove that the hybrid system has a unique solution with the property that the binary function keeps on changing its value. In the clinical point of view, the result asserts that one can plan the IAS therapy for each prostate cancer patient, provided that the tumor satisfies a certain condition.

Keywords Parabolic comparison principle · Indirectly controlled parameter

1 Introduction

Prostate cancer is one of the diseases of male. By the fact that prostate cells proliferate by a male hormone so-called androgen, it is expected that prostate tumors are sensitive to androgen suppression. Huggins and Hodges [10] demonstrated the validity of the androgen deprivation. Since then, the hormonal therapy has been a major therapy of prostate cancer. The therapy aims to induce apoptosis of prostate cancer cells under the androgen suppressed condition. For instance, the androgen suppressed condition can be kept by medicating a patient continuously [22], and the therapy is called Continuous Androgen Suppression therapy (CAS therapy). However, during several years of the CAS therapy, the relapse of prostate tumor often occurs. More precisely, the relapse means that the prostate tumor mutates to androgen independent tumor. Then the CAS therapy is not effective in treating the tumor [5]. The fact was also verified mathematically by [13, 14]. It is known that there exist Androgen-

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Dependent cells (AD cells) and Androgen-Independent cells (AI cells) in prostate tumors. AI cells are considered as one of the causes of the relapse. For, AD cells can not proliferate under the androgen suppressed condition, whereas AI cells are not sensitive to androgen suppression and can still proliferate under the androgen poor condition [2, 18]. Thus the relapse of prostate tumors is caused by progression to androgen independent cancer due to emergence of AI cells.

In order to prevent or delay the relapse of prostate tumors, Intermittent Androgen Suppression therapy (IAS therapy) was proposed and has been studied clinically by many researchers (e.g., see [1, 3, 19], and the references therein). In contrast to the CAS therapy, the IAS therapy does not aim to exterminate prostate cancer. We mention the typical feature of the clinical phenomenon. Since prostate cancer cells produce large amount of Prostate-Specific Antigen, the PSA is regarded as a good biomarker of prostate cancer [21], and the plan of IAS therapy is based on the level:

(F) In the IAS therapy, the medication is stopped when the serum PSA level falls enough, and resumed when the serum PSA level rises enough.

Indeed, if one can *optimally* plan the IAS therapy, then the size of tumor remains in an appropriate range by way of on and off of the medication. In order to comprehend qualitative property of prostate tumors under the IAS therapy, several mathematical models were proposed and have been studied in the mathematical literature, for instance, ODE models ([9, 11, 12, 20], and references therein) and PDE models [8, 15, 23–25]. Due to (F), an unknown binary function, denoting the treatment state, appears in the models. The discontinuity of the binary function is the difficulty in mathematical analysis on the models. To the best of our knowledge, there is no result dealing with switching phenomena of the binary function in the PDE models.

The purpose of this paper is to prove the existence of a solution with the switching property for the PDE model introduced by Tao et al. [23]:

$$\begin{cases} \dfrac{da}{dt}(t) = -\gamma(a(t)-a_*) - \gamma a_* S(t) & \text{in } \mathbb{R}_+, \\ \partial_t u(\rho,t) - \mathscr{L}(v,R)u(\rho,t) = F_u(u(\rho,t),w(\rho,t),a(t)) & \text{in } I_\infty, \\ \partial_t w(\rho,t) - \mathscr{L}(v,R)w(\rho,t) = F_w(u(\rho,t),w(\rho,t),a(t)) & \text{in } I_\infty, \\ v(\rho,t) = \dfrac{1}{\rho^2}\displaystyle\int_0^\rho F_v(u(r,t),w(r,t),a(t))r^2\,dr & \text{in } I_\infty, \\ \dfrac{dR}{dt}(t) = v(1,t)R(t) & \text{in } \mathbb{R}_+, \\ S(t) = \begin{cases} 0 \to 1 & \text{when} \quad R(t) = r_1 \quad \text{and} \quad R'(t) > 0, \\ 1 \to 0 & \text{when} \quad R(t) = r_0 \quad \text{and} \quad R'(t) < 0, \end{cases} & \text{in } \mathbb{R}_+, \\ \partial_\rho u(\rho,t)|_{\rho\in\{0,1\}} = \partial_\rho w(\rho,t)|_{\rho\in\{0,1\}} = 0, \quad v(0,t) = 0, & \text{in } \mathbb{R}_+, \\ (a,u,w,R,S)|_{t=0} = (a_0, u_0(\rho), w_0(\rho), R_0, S_0) & \text{in } I, \end{cases} \tag{IAS}$$

where $I = (0, 1)$, $\mathbb{R}_+ = \{t \in \mathbb{R} \,|\, t > 0\}$, $I_\infty = I \times \mathbb{R}_+$, and

$$\mathscr{L}(v, R)\varphi = \frac{D}{R(t)^2}\frac{1}{\rho^2}\partial_\rho[\rho^2\partial_\rho\varphi] + \rho v(1, t)\partial_\rho\varphi - \frac{1}{\rho^2}\partial_\rho[\rho^2 v(\rho, t)\varphi], \tag{1}$$

$$F_u = f_1(a)u - c_1uw, \quad F_w = f_2(a)w - c_2uw, \quad F_v = F_u + F_w. \tag{2}$$

The unknowns a, u, w, v, R, and S denote respectively the androgen concentration, the volume fraction of AD cells, the volume fraction of AI cells, the advection velocity of the cancer cells, the radius of the tumor, and the treatment state. Here $S = 0$ and $S = 1$ correspond to the medication state and the non-medication state, respectively. The authors of [23] assumed that the prostate tumor is radially symmetric and densely packed by AD and AI cells. Moreover they regarded the tumor as a three dimensional sphere. Thus the unknowns u, w, and v are radially symmetric functions defined on the unit ball $B_1 = \{x \in \mathbb{R}^3 \mid |x| < 1\}$, i.e., $\rho = |x|$. The unknown $S(t)$ is governed by $R(t)$, for they formulated the serum PSA level as the radius of the tumor. Although the condition on S in (IAS) is a concise form, the precise form is expressed as follows: $S(t) \in \{0, 1\}$ and

$$S(t) = \begin{cases} \{0,1\} \setminus \lim\limits_{\tau\uparrow t} S(\tau) & \text{if } \begin{cases} \lim\limits_{\tau\uparrow t} R'(\tau) > 0, \ \lim\limits_{\tau\uparrow t} R(\tau) = r_1, \text{ and } \lim\limits_{\tau\uparrow t} S(\tau) = 0, \\ \lim\limits_{\tau\uparrow t} R'(\tau) < 0, \ \lim\limits_{\tau\uparrow t} R(\tau) = r_0, \text{ and } \lim\limits_{\tau\uparrow t} S(\tau) = 1, \end{cases} \\ \lim\limits_{\tau\uparrow t} S(\tau) & \text{otherwise.} \end{cases}$$

The parameters $a_*, \gamma, c_1, c_2, r_0$, and r_1 denote the normal androgen concentration, the reaction velocity, the effective competition coefficient from AD to AI cells, and from AI to AD cells, the lower and upper thresholds, respectively. The given functions $f_1 : [0, a_*] \to \mathbb{R}$ and $f_2 : [0, a_*] \to \mathbb{R}$ describe the net growth rate of AD and AI cells, respectively. Although the typical form of f_i were given by [23], we deal with general f_i satisfying several conditions, which are stated later.

In [23], it was shown that, for each initial data $u_0 \in W_p^2(I)$, there exists a short time solution $u \in W_p^{2,1}(I \times (0, T))$ of (IAS). However, the result is not sufficient to construct a "*switching solution*". For, if (u, w, v, R, a, S) is a switching solution of (IAS), then (IAS) must be solvable at least locally in time for each "initial data" $(u, w, R, a, S)|_{\{t=t_j\}}$, where t_j is a switching time. Nevertheless, the result in [23] does not ensure the solvability.

The existence of switching solutions of (IAS) is a mathematically outstanding question. We are interested in the following mathematical problem:

Problem 1.1 Does there exist a switching solution of (IAS) with appropriate thresholds $0 < r_0 < r_1 < \infty$? Moreover, what is the dynamical aspect of the solution?

We consider the initial data $(u_0, w_0, R_0, a_0, S_0)$ satisfying the following:

$$\begin{cases} u_0, w_0 \in C^{2+\alpha}(B_1), \ \partial_\rho u_0(\rho)|_{\rho\in\{0,1\}} = \partial_\rho w_0(\rho)|_{\rho\in\{0,1\}} = 0, \\ u_0 \geq 0, \ \ w_0 \geq 0, \ \ u_0 + w_0 \equiv 1, \ \ R_0 > 0, \ \ 0 < a_0 < a_*, \ \ S_0 \in \{0, 1\}, \end{cases} \tag{3}$$

where $\alpha \in (0, 1)$. Let f_1 and f_2 satisfy

$$\begin{cases} f_1(a_*) > 0, \quad f_1(0) < 0, \quad f_1 \in C^1([0, a_*]), \quad f_1' > 0 \quad \text{in} \quad [0, a_*], \\ f_2(0) > 0, \quad f_2 \in C^1([0, a_*]), \quad f_2' \leq 0 \quad \text{in} \quad [0, a_*]. \end{cases} \tag{A0}$$

We note that (A0) is a natural assumption in the clinical point of view, and typical f_1 and f_2, which were given in [23], also satisfy (A0). In order to comprehend the role of f_i and c_i, we classify asymptotic behavior of non-switching solutions of (IAS) in terms of f_i and c_i under (A0) (see Theorems 3.2–3.5). Following the results obtained by Theorems 3.2–3.5, we impose (A0) and the following assumptions on f_i and c_i:

$$f_1(a_*) - f_2(a_*) - c_1 > 0; \tag{A1}$$

$$f_1(0) - f_2(0) + c_2 > 0. \tag{A2}$$

From now on, let $Q_T := B_1 \times (0, T)$. We denote by $C^{2\kappa+\alpha,\kappa+\beta}(Q_T)$ the Hölder space on Q_T, where $\kappa \in \mathbb{N} \cup \{0\}$, $0 < \alpha < 1$, and $0 < \beta < 1$ (for the precise definition, see [16]).

Then we give an affirmative answer to Problem 1.1:

Theorem 1.1 *Let f_i and c_i satisfy* (A0)–(A2). *Let $(u_0, w_0, R_0, a_0, S_0)$ satisfy* (3), *$u_0 > 0$ in $\overline{B_1}$, and $S_0 = 0$. Then, there exists a pair (r_0, r_1) with $0 < r_0 < r_1 < \infty$ such that the system* (IAS) *has a unique solution (u, w, v, R, a, S) in the class*

$$u, w \in C^{2+\alpha,1+\alpha/2}(Q_\infty), \quad v \in C^{1+\alpha,\alpha/2}([0, 1) \times \mathbb{R}_+) \cap C^1([0, 1) \times \mathbb{R}_+),$$
$$R \in C^1(\mathbb{R}_+), \quad a \in C^{0,1}(\mathbb{R}_+).$$

Moreover, the following hold:

(i) *There exists a strictly monotone increasing divergent sequence $\{t_j\}_{j=0}^\infty$ with $t_0 = 0$ such that $a \in C^1((t_j, t_{j+1}))$ and*

$$S(t) = \begin{cases} 0 & \text{in} \quad [t_{2j}, t_{2j+1}), \\ 1 & \text{in} \quad [t_{2j+1}, t_{2j+2}), \end{cases} \qquad \text{for any} \quad j \in \mathbb{N} \cup \{0\};$$

(ii) *There exist positive constants $C_1 < C_2$ such that*

$$C_1 \leq R(t) \leq C_2 \quad \text{for any} \quad t \geq 0.$$

We mention the mathematical contributions of Theorem 1.1 and a feature of the system (IAS). The system is composed of two different systems (S0) and (S1) by the medium of the binary function $S(t)$, where (S0) and (S1) respectively denote (IAS) with $S(t) \equiv 0$ and $S(t) \equiv 1$. Generally the system with such structure is called *hybrid system*. Regarding (S0), the assumption (A1) implies that $R(t)$ diverges to infinity as $t \to \infty$ (see Theorem 3.4). On the other hand, regarding (S1), we can show that the assumption (A2) implies the following: (i) $R(t)$ diverges to infinity as $t \to \infty$ if u_0 is sufficiently small (Theorem 3.2); (ii) $R(t)$ converges to 0 as $t \to \infty$ if u_0 is sufficiently close to 1 (Theorem 3.3). It is natural to ask whether a solution $R(t)$ of (IAS) is bounded or not. One of the contributions of the present paper is to show how to determine thresholds $0 < r_0 < r_1 < \infty$ such that (IAS) with the thresholds has a bounded solution with infinite opportunities of switching. Furthermore, due to the discontinuity of $S(t)$, it is expected that the switching solution is not so smooth. However, Theorem 1.1 indicates that the switching solution gains its regularity with the aid of the "indirectly controlled parameter" $a(t)$. The other contribution of this paper is to mathematically clarify the immanent structure of the hybrid system (IAS).

We mention the clinical contribution of Theorem 1.1. Although one can expect that the system (IAS) gives us how to optimally plan the IAS therapy for each prostate cancer patient, it is not trivial matter. To do so, first we have to prove the existence of admissible thresholds for each patient. Moreover, if the admissible threshold is not unique, then we investigate the optimality of the admissible thresholds. Here, we say that the thresholds is admissible for a prostate cancer patient, if for the initial data (IAS) with the thresholds has a switching solution. Although [23] indicated that the problem, even the existence, is difficult to analyze mathematically, they numerically showed that (i) the IAS therapy fails for unsuitable thresholds, more precisely, the radius of tumor diverges to infinity after several times of switching opportunities, and while, (ii) the IAS therapy succeeds for suitable thresholds, i.e., the radius of tumor remains in a bounded range by way of infinitely many times of switching opportunities. One of the clinical contribution of Theorem 1.1 is to prove the existence of admissible thresholds for each patients, provided that (A0)–(A2) are fulfilled. Moreover, Theorem 1.1 also implies that the IAS therapy has an advantage over the CAS therapy for some patients. Indeed, Theorem 3.2 gives an instance showing a failure of the CAS therapy, whereas Theorem 1.1 asserts that the patient can be treated successfully by the IAS therapy. The fact is an example that switching strategy under the IAS therapy is able to be a successful strategy. On the other hand, the pair of admissible thresholds given by Theorem 1.1 is not uniquely determined. Thus, in order to optimally plan the IAS therapy, we have to investigate its optimality. However the optimality of the admissible thresholds is an outstanding problem.

The paper is organized as follows: In Sect. 2, we give a modified system of (IAS) and reduce the system to a simple hybrid system. Making use of the modified system, we prove the short time existence of the solution to (IAS). In Sect. 3, we show the existence of the non-switching solution of (IAS) for any finite time. Moreover, we classify the asymptotic behaviors of the non-switching solutions in terms of f_i and c_i. In Sect. 4, we prove Theorem 1.1, i.e., we show the existence of a switching solution of (IAS) and give its property.

2 Short Time Existence

The main purpose of this section is to show the short time existence of the solution of (IAS). As [23] mentioned, it is difficult to prove that (IAS) has a short time solution in the Hölder space (see Remark 4.1 in [23]). The difficulty rises from the singularity of v/ρ at $\rho = 0$. Indeed, the singularity prevents us from applying the Schauder estimate. To overcome the difficulty, first we consider a modified hybrid system. More precisely, we replace the "boundary condition"

$$v(0,t) = 0 \quad \text{in} \quad \mathbb{R}_+ \tag{4}$$

by

$$\frac{v(\rho,t)}{\rho}\Big|_{\rho=0} = \frac{1}{3} F_v(u(0,t), w(0,t), a(t)) \quad \text{in} \quad \mathbb{R}_+.$$

Then the modified hybrid system is expressed as follows:

$$\begin{cases} \dfrac{da}{dt}(t) = -\gamma(a(t)-a_*) - \gamma a_* S(t) & \text{in} \ \ \mathbb{R}_+, \\ \partial_t u(\rho,t) - \mathscr{L}(v,R)u(\rho,t) = F_u(u(\rho,t), w(\rho,t), a(t)) & \text{in} \ \ I_\infty, \\ \partial_t w(\rho,t) - \mathscr{L}(v,R)w(\rho,t) = F_w(u(\rho,t), w(\rho,t), a(t)) & \text{in} \ \ I_\infty, \\ v(\rho,t) = \dfrac{1}{\rho^2}\displaystyle\int_0^\rho F_v(u(r,t), w(r,t), a(t)) r^2\, dr & \text{in} \ \ I_\infty, \\ \dfrac{dR}{dt}(t) = v(1,t)R(t) & \text{in} \ \ \mathbb{R}_+, \\ S(t) = \begin{cases} 0 \to 1 \ \ \text{when} \ \ R(t) = r_1 \ \ \text{and} \ \ R'(t) > 0, \\ 1 \to 0 \ \ \text{when} \ \ R(t) = r_0 \ \ \text{and} \ \ R'(t) < 0, \end{cases} & \text{in} \ \ \mathbb{R}_+, \\ \partial_\rho u(\rho,t)|_{\rho\in\{0,1\}} = \partial_\rho w(\rho,t)|_{\rho\in\{0,1\}} = 0 & \text{in} \ \ \mathbb{R}_+, \\ \dfrac{v(\rho,t)}{\rho}\Big|_{\rho=0} = \dfrac{1}{3} F_v(u(0,t), w(0,t), a(t)) & \text{in} \ \ \mathbb{R}_+, \\ (a,u,w,R,S)|_{t=0} = (a_0, u_0(\rho), w_0(\rho), R_0, S_0) & \text{in} \ \ I. \end{cases} \tag{mIAS}$$

To begin with, we show that $u + w$ is invariant under (mIAS).

Lemma 2.1 *Let $(u_0, w_0, R_0, a_0, S_0)$ be an initial data satisfying (3). Assume that (u, w, v, R, a, S) is a solution of* (mIAS) *with $u, w \in C^{2+\alpha, 1+\alpha/2}(Q_T)$ and $S(t) \equiv S_0$ in $[0, T)$. Then $u + w \equiv 1$ in $B_1 \times [0, T)$.*

Proof Setting $V := u + w$, we reduce (mIAS) to the following system:

$$
\begin{cases}
\dfrac{da}{dt}(t) = -\gamma(a(t) - a_*) - \gamma a_* S_0 & \text{in } \mathbb{R}_+, \\
\partial_t V(\rho, t) - \mathscr{L}(v, R)V(\rho, t) = \dfrac{1}{\rho^2}\partial_\rho[\rho^2 v(\rho, t)] & \text{in } I_\infty, \\
v(\rho, t) = \dfrac{1}{\rho^2}\displaystyle\int_0^\rho F_v(u(r, t), w(r, t), a(t))r^2\, dr & \text{in } I_\infty, \\
\dfrac{dR}{dt}(t) = v(1, t)R(t) & \text{in } \mathbb{R}_+, \\
\partial_\rho V(\rho, t)|_{\rho\in\{0,1\}} = 0, \ \dfrac{v(\rho, t)}{\rho}\Big|_{\rho=0} = \dfrac{1}{3}F_v(u(0, t), w(0, t), a(t)), & \text{in } \mathbb{R}_+, \\
V(\rho, 0) = 1, \quad a(0) = a_0, \quad R(0) = R_0, & \text{in } I.
\end{cases}
\tag{5}
$$

In the derivation of the second equation in (5), we used the fact $F_u + F_w = F_v$ and the equation on v. We shall prove that $V \equiv 1$ in $B_1 \times [\,0, T\,)$. The second equation in (5) is written as

$$
\partial_t V = \frac{D}{R(t)^2}\Delta_x V - \frac{x}{\rho}\cdot\nabla_x\{v(V-1)\} + v(1, t)x\cdot\nabla_x V - \frac{2}{\rho}v(V-1) \tag{6}
$$

in terms of the three-dimensional Cartesian coordinates, where $\rho = |x|$. In what follows, we use ∇ and Δ instead of ∇_x and Δ_x, respectively, if there is no fear of confusion. First, we observe from (6) that

$$
\begin{aligned}
\frac{d}{dt}\|V-1\|^2_{L^2(B_1)} &= -\frac{2D}{R(t)^2}\|\nabla(V-1)\|^2_{L^2(B_1)} - 2\int_{B_1}(V-1)\frac{x}{\rho}\cdot\nabla\{v(V-1)\}\, dx \\
&\quad + 2\int_{B_1}(V-1)v(1, t)x\cdot\nabla V\, dx - 2\int_{B_1}\frac{v}{\rho}(V-1)^2\, dx =: J_1 + J_2 + J_3 + J_4.
\end{aligned}
$$

We start with an estimate of J_1. Since it follows from the third and fourth equations in (5) that

$$
R(t) = R_0 \exp\Big[\int_0^t v(1, s) ds\Big] \le R_0 e^{\kappa T},
$$

we have

$$
J_1 \le -\frac{2D}{R_0^2 e^{2\kappa T}}\|\nabla(V-1)\|^2_{L^2(B_1)},
$$

where κ is a positive constant given by

$$
3\kappa := \|f_1(a)u + f_2(a)w - (c_1 + c_2)uw\|_{L^\infty(Q_T)}.
$$

We turn to J_2. By the relation

$$\partial_\rho v = -2\frac{v}{\rho} + f_1(a(t))u + f_2(a(t))w - (c_1 + c_2)uw,$$

the integral J_2 is reduced to

$$J_2 = 4\int_{B_1} \frac{v}{\rho}|V-1|^2\,dx - 2\int_{B_1}(V-1)\frac{v}{\rho}x\cdot\nabla V\,dx$$
$$-2\int_{B_1}\{f_1(a(t))u + f_2(a(t))w - (c_1+c_2)uw\}|V-1|^2\,dx.$$

Observing that

$$\left|\ \frac{v(\rho,t)}{\rho}\ \right| \le \frac{1}{\rho^3}\int_0^\rho |f_1(a(t))u + f_2(a(t))w - (c_1+c_2)uw|r^2\,dr \le \kappa,$$

and using Hölder's inequality and Young's inequality, we find

$$|J_2| \le \varepsilon\|\nabla(V-1)\|^2_{L^2(B_1)} + C(\varepsilon)\|V-1\|^2_{L^2(B_1)}.$$

Regarding J_3 and J_4, the same argument as in the estimate of J_2 asserts that

$$|J_3| \le \varepsilon\|\nabla(V-1)\|^2_{L^2(B_1)} + C(\varepsilon)\|V-1\|^2_{L^2(B_1)},\quad |J_4| \le 2\kappa\|V-1\|^2_{L^2(B_1)}.$$

Thus, letting $\varepsilon > 0$ small enough, we obtain

$$\frac{d}{dt}\|V-1\|^2_{L^2(B_1)} \le C\|V-1\|^2_{L^2(B_1)}. \tag{7}$$

Since $V(\cdot,0) = 1$, applying Gronwall's inequality to (7), we obtain the conclusion. □

Here we reduce the system (mIAS) to the following hybrid system:

$$\begin{cases} \dfrac{da}{dt}(t) = -\gamma(a(t)-a_*) - \gamma a_* S(t) & \text{in } \mathbb{R}_+, \\ \partial_t u(\rho,t) - \mathscr{L}'(v,R)u(\rho,t) = P(u(\rho,t),a(t)) & \text{in } I_\infty, \\ v(\rho,t) = \dfrac{1}{\rho^2}\displaystyle\int_0^\rho F(u(r,t),a(t))r^2\,dr & \text{in } I_\infty, \\ \dfrac{dR}{dt}(t) = v(1,t)R(t) & \text{in } \mathbb{R}_+, \\ S(t) = \begin{cases} 0 \to 1 & \text{when } R(t) = r_1 \text{ and } R'(t) > 0, \\ 1 \to 0 & \text{when } R(t) = r_0 \text{ and } R'(t) < 0, \end{cases} & \text{in } \mathbb{R}_+, \\ \partial_\rho u(\rho,t)\big|_{\rho\in\{0,1\}} = 0, \quad \dfrac{v(\rho,t)}{\rho}\Big|_{\rho=0} = \dfrac{1}{3}F(u(0,t),a(t)), & \text{in } \mathbb{R}_+, \\ a(0) = a_0,\ u(\rho,0) = u_0(\rho),\ R(0) = R_0,\ S(0) = S_0, & \text{in } I, \end{cases} \tag{P}$$

where

$$\mathscr{L}'(v, R)\varphi = \frac{D}{R(t)^2}\frac{1}{\rho^2}\partial_\rho[\rho^2\partial_\rho\varphi] - [v(\rho, t) - \rho v(1, t)]\,\partial_\rho\varphi, \tag{8}$$
$$P(u, a) = \{f_1(a) - f_2(a) - c_1 + (c_1 + c_2)u\}\,u(1 - u),$$
$$F(u, a) = f_1(a)u + \{f_2(a) - (c_1 + c_2)u\}\,(1 - u).$$

The reduction is justified as follows:

Lemma 2.2 *The system* (mIAS) *is equivalent to* (P).

Proof If (u, w, v, R, a, S) satisfies (mIAS), then Lemma 2.1 implies that $u + w \equiv 1$. Using $w = 1 - u$, we can reduce (mIAS) to (P). On the other hand, if (u, v, R, a, S) satisfies (P), then, setting $w := 1 - u$, we obtain (mIAS) from (P). □

In order to prove the short time existence of a solution to (mIAS), we first consider the following system, which is formally derived from (P) provided $S(t) \equiv S_0$.

$$\begin{cases} \dfrac{da}{dt}(t) = -\gamma(a(t) - a_*) - \gamma a_* S_0 & \text{in } \mathbb{R}_+, \\ \partial_t u(\rho, t) - \mathscr{L}'(v, R)u(\rho, t) = P(u(\rho, t), a(t)) & \text{in } I_\infty, \\ v(\rho, t) = \dfrac{1}{\rho^2}\displaystyle\int_0^\rho F(u(r, t), a(t))r^2\,dr & \text{in } I_\infty, \\ \dfrac{dR}{dt}(t) = v(1, t)R(t) & \text{in } \mathbb{R}_+, \\ \partial_\rho u(\rho, t)\big|_{\rho\in\{0,1\}} = 0, \quad \dfrac{v(\rho, t)}{\rho}\Big|_{\rho=0} = \dfrac{1}{3}F(u(0, t), a(t)), & \text{in } \mathbb{R}_+, \\ a(0) = a_0, \;\; u(\rho, 0) = u_0(\rho), \;\; R(0) = R_0, & \text{in } I. \end{cases} \tag{PS$_0$}$$

Lemma 2.3 *Let* (u_0, R_0, a_0, S_0) *satisfy* (3)*. Then there exists* $T > 0$ *such that the system* (PS$_0$) *has a unique solution* (u, v, R, a) *in the class*

$$C^{2+\alpha,1+\frac{\alpha}{2}}(Q_T)\times(C^{1+\alpha,\frac{\alpha}{2}}([0,1)\times(0,T))\cap C^1([0,1)\times(0,T)))\times(C^1((0,T)))^2.$$

Proof We shall prove Lemma 2.3 by the contraction mapping principle. Let us define a metric space $(X_M, \|\cdot\|_X)$ as follows:

$$X_M = \{u \in C^{\alpha,\frac{\alpha}{2}}(Q_T) \mid u(x, t) = u(|x|, t), u|_{t=0} = u_0, \|u\|_X \le M\},$$

where $\|u\|_X = \|u\|_{C^{\alpha,\alpha/2}(Q_T)}$. We will take the constants $T > 0$ and $M > 0$ appropriately, later.

Step 1: *We shall construct a mapping* $\Psi : X_M \to X_M$. Let $u \in X_M$. For $u(\rho, t)$, let us define $(v(\rho, t), R(t))$ as the solution of the following system:

$$\begin{cases} v(\rho,t) = \dfrac{1}{\rho^2}\displaystyle\int_0^\rho F(u(r,t),a(t))r^2\,dr & \text{in} \quad I \times [\,0,T\,), \\ \dfrac{dR}{dt}(t) = v(1,t)R(t) & \text{in} \quad (\,0,T\,), \\ \dfrac{v(\rho,t)}{\rho}\Big|_{\rho=0} = \dfrac{1}{3}F(u(0,t),a(t)) & \text{in} \quad [\,0,T\,), \\ R(0) = R_0. \end{cases} \tag{9}$$

For (v, R) defined by (9), let $\tilde{u}(x,t) = \tilde{u}(|x|,t) = \tilde{u}(\rho,t)$ denote the solution of

$$\begin{cases} \partial_t\tilde{u}(\rho,t) - \mathscr{L}'(v,R)\tilde{u}(\rho,t) = P(u(\rho,t),a(t)) & \text{in} \quad I \times (\,0,T\,), \\ \partial_\rho\tilde{u}(0,t) = \partial_\rho\tilde{u}(1,t) = 0 & \text{in} \quad (\,0,T\,), \\ \tilde{u}(\rho,0) = u_0(\rho) & \text{in} \quad I. \end{cases} \tag{10}$$

If we consider the problem as an initial-boundary value problem for a one dimensional parabolic equation, the parabolic equation has a singularity at $\rho = 0$. In order to eliminate the singularity, we rewrite the problem in terms of the three dimensional Cartesian coordinate as follows:

$$\begin{cases} \partial_t\tilde{u}(|x|,t) + \left[\dfrac{v(|x|,t)}{|x|} - v(1,t)\right] x\cdot\nabla\tilde{u}(|x|,t) & \\ \qquad = \dfrac{D}{R(t)^2}\Delta\tilde{u}(|x|,t) + P(u(|x|,t),a(t)) & \text{in} \quad Q_T, \\ \partial_\rho\tilde{u}(0,t) = \partial_\rho\tilde{u}(1,t) = 0 & \text{in} \quad (\,0,T\,), \\ \tilde{u}(|x|,0) = u_0(|x|) & \text{in} \quad B_1. \end{cases} \tag{11}$$

We prove that $\tilde{u} \in X_M$ by applying the Schauder estimate to (11). Since $u \in X_M$, it is clear that $F(u,a) \in C^{\alpha,\alpha/2}(Q_T)$, $P(u,a) \in C^{\alpha,\alpha/2}(Q_T)$, and

$$v(1,t) = \int_0^1 F(u(r,t),a(t))r^2\,dr \in C^{\frac{\alpha}{2}}((\,0,T\,)). \tag{12}$$

Moreover, since $R(t) > 0$ in $[\,0,T\,)$, the fact (12) implies $1/R(t)^2 \in C^{\alpha/2}((\,0,T\,))$. In the following, we will show

$$\mathscr{V}(\rho,t) := \frac{v(\rho,t)}{\rho} \in C^{\alpha,\frac{\alpha}{2}}(Q_T). \tag{13}$$

(i) Let us fix $\rho \in (\,0,1\,)$ arbitrarily. Since now $\mathscr{V}$ satisfies, for any $0 < t < s < T$,

$$\mathscr{V}(\rho,s) - \mathscr{V}(\rho,t) = \frac{1}{\rho^3}\int_0^\rho \{F(u(r,s),a(s)) - F(u(r,t),a(t))\}r^2\,dr, \tag{14}$$

we estimate the integrand. It follows from $u \in X_M$ that

$$\begin{aligned}
&|F(u(r,s),a(s)) - F(u(r,t),a(t))| \\
&\leq C(M)\left\{|u(r,s)-u(r,t)| + \sum_{i=1}^{2}|f_i(a(s)) - f_i(a(t))|\right\} \\
&\leq C(M)\left\{M|s-t|^{\frac{\alpha}{2}} + \sum_{i=1}^{2}|f_i(a(s)) - f_i(a(t))|\right\}.
\end{aligned} \tag{15}$$

Furthermore, the mean value theorem implies

$$|f_i(a(s)) - f_i(a(t))| \leq C|s-t| \quad \text{for} \quad i = 1, 2,$$

where $C = C(f_i, a_*, \gamma)$. Combining the estimate with (15), we find

$$|F(u(r,s),a(s)) - F(u(r,t),a(t))| \leq C(M)|s-t|^{\frac{\alpha}{2}}.$$

Consequently, we deduce from (14) that

$$|\mathscr{V}(\rho,s) - \mathscr{V}(\rho,t)| \leq C(M)|s-t|^{\frac{\alpha}{2}}.$$

(ii) Let $\rho = 0$. Then by the same argument as in (i), we see that

$$|\mathscr{V}(0,s) - \mathscr{V}(0,t)| = \frac{1}{3}|F(u(0,s),a(s)) - F(u(0,t),a(t))| \leq C(M)|s-t|^{\frac{\alpha}{2}}$$

for any $0 < t < s < T$.

(iii) Fix $0 < t < T$ arbitrarily. Then, for any $0 < \rho < \sigma < 1$, it holds that

$$\begin{aligned}
\mathscr{V}(\sigma,t) - \mathscr{V}(\rho,t) &= \{\mathscr{V}(\sigma,t) - \mathscr{V}(0,t)\} - \{\mathscr{V}(\rho,t) - \mathscr{V}(0,t)\} \\
&= \frac{1}{\sigma^3}\int_\rho^\sigma \{F(u(r,t),a(t)) - F(u(0,t),a(t))\}r^2\,dr \\
&\quad + \left(\frac{1}{\sigma^3} - \frac{1}{\rho^3}\right)\int_0^\rho \{F(u(r,t),a(t)) - F(u(0,t),a(t))\}r^2\,dr.
\end{aligned}$$

Since $u \in X_M$, we observe that

$$|F(u(\rho,t),a(t)) - F(u(0,t),a(t))| \leq C(M)|u(\rho,t) - u(0,t)| \leq C(M)\rho^\alpha.$$

Therefore we obtain

$$\begin{aligned}
|\mathscr{V}(\sigma,t) - \mathscr{V}(\rho,t)| &\leq C(M)\frac{1}{\sigma^3}\int_\rho^\sigma r^{2+\alpha}\,dr + C(M)\left|\frac{\rho^3-\sigma^3}{\sigma^3\rho^3}\right|\int_0^\rho r^{2+\alpha}\,dr \\
&\leq C(M)\left|\frac{\sigma^3-\rho^3}{\sigma^{3-\alpha}}\right| \leq C(M)|\sigma-\rho|^\alpha.
\end{aligned}$$

(iv) Let us fix $0 < t < T$ arbitrarily. The same argument as in (iii) implies that

$$|\mathscr{V}(\rho, t) - \mathscr{V}(0, t)| \le C(M)\frac{1}{\rho^3}\int_0^\rho r^{2+\alpha}\,dr \le C(M)\rho^\alpha \quad \text{for any} \quad \rho \in (0, 1).$$

From (i)–(iv), we conclude (13). Hence, by virtue of (11) we can apply the Schauder estimate (Theorem 5.3, [16]) to (10):

$$\|\tilde{u}\|_{C^{2+\alpha,1+\alpha/2}(Q_T)} \le C\left(\|P\|_X + \|u_0\|_{C^{2+\alpha}(B_1)}\right) \le C(M) + C\|u_0\|_{C^{2+\alpha}(B_1)}.$$

On the other hand, it follows from the mean value theorem that

$$\|\tilde{u} - u_0\|_X \le \max\left\{T, T^{1-\frac{\alpha}{2}}\right\}\|\tilde{u}\|_{C^{2+\alpha,1+\alpha/2}(Q_T)}. \tag{16}$$

Therefore, for $T < 1$, we obtain

$$\begin{aligned}\|\tilde{u}\|_X &\le T^{1-\frac{\alpha}{2}}\|\tilde{u}\|_{C^{2+\alpha,1+\alpha/2}(Q_T)} + \|u_0\|_{C^{2+\alpha}(B_1)}\\ &\le T^{1-\frac{\alpha}{2}}\{C(M) + C\|u_0\|_{C^{2+\alpha}(B_1)}\} + \|u_0\|_{C^{2+\alpha}(B_1)}.\end{aligned}$$

Consequently, for $M := 1 + \|u_0\|_{C^{2+\alpha}(B_1)}$, setting $T < 1$ small enough as

$$T^{1-\frac{\alpha}{2}}\{C(M) + C\|u_0\|_{C^{2+\alpha}(B_1)}\} < 1, \tag{17}$$

we deduce that $\tilde{u} \in X_M$. We define a mapping $\Psi : X_M \to X_M$ as $\Psi(u) = \tilde{u}$.

Step 2: *We show that Ψ is a contraction mapping.* Let $u_i \in X_M$. We denote by $(v_i(\rho, t), R_i(t))$ the solution of (9) with $u = u_i$, where $i = 1, 2$. For $\tilde{u}_i := \Psi(u_i)$, set $U := \tilde{u}_1 - \tilde{u}_2$. By a simple calculation, we see that U satisfies

$$\begin{cases}\partial_t U(\rho, t) - \mathscr{L}'(v_2, R_2)U(\rho, t) = G(u_1, u_2) & \text{in} \quad I \times (0, T),\\ \partial_\rho U(0, t) = \partial_\rho U(1, t) = 0 & \text{in} \quad (0, T),\\ U(\rho, 0) = 0 & \text{in} \quad I,\end{cases}$$

where $G(u_1, u_2)$ is given by

$$G(u_1, u_2) = \{\mathscr{L}'(v_1, R_1) - \mathscr{L}'(v_2, R_2)\}\tilde{u}_1 + \{P(u_1) - P(u_2)\}.$$

Adopting a similar argument as in Step 1, we find $G(u_1, u_2) \in C^{\alpha,\alpha/2}(Q_T)$ and

$$\|G(u_1, u_2)\|_X \le C(T, u_0, R_0)\|u_1 - u_2\|_X.$$

Then the Schauder estimate asserts that

$$\|U\|_{C^{2+\alpha,1+\alpha/2}(Q_T)} \le C(T, u_0, R_0)\|u_1 - u_2\|_X.$$

By the fact that $U(|x|, 0) = 0$ in B_1 and a similar argument as in (16), it holds that

$$\|\Psi(u_1) - \Psi(u_2)\|_X = \|U\|_X \leq T^{1-\frac{\alpha}{2}} \|U\|_{C^{2+\alpha,1+\alpha/2}(Q_T)} \leq T^{1-\frac{\alpha}{2}} C \|u_1 - u_2\|_X,$$

where $C = C(T, u_0, R_0)$. Thus, letting T small enough as $T^{1-\alpha/2}C < 1$, we conclude that Ψ is a contraction mapping. Then Banach's fixed point theorem indicates that there exists $u \in X_M$ uniquely such that $\Psi(u) = u$. By the definition of Ψ, u is a unique solution of (PS_0) on $[0, T)$. Moreover, we infer from the above argument that $u \in C^{2+\alpha,1+\alpha/2}(Q_T)$.

Finally we prove that $v \in C^{1+\alpha,\alpha/2}([0, 1) \times (0, T)) \cap C^1([0, 1) \times (0, T))$. By a direct calculation, we have $v \in C([0, T); H^1(I))$. Combining the fact with the Sobolev embedding theorem $H^1(I) \hookrightarrow C^{0,1/2}(\bar{I})$, we obtain $v \in C([0, T); C^{0,1/2}(\bar{I}))$, in particular $v \in C(\bar{I} \times [0, T))$. Thus it follows from the continuity that

$$v(0, t) = \lim_{\rho \downarrow 0} v(\rho, t) = 0 \quad \text{for any} \quad t \in [0, T). \tag{18}$$

Then, along the same line as in [23], we see that $v \in C^1([0, 1) \times (0, T))$. Moreover, applying the same argument as in (13) to

$$\partial_\rho v(\rho, t) = \begin{cases} -\dfrac{2}{\rho^3} \displaystyle\int_0^\rho F(u(r, t), a(t)) r^2\, dr + F(u(\rho, t), a(t)) & \text{if} \quad \rho > 0, \\ \dfrac{1}{3} F(u(0, t), a(t)) & \text{if} \quad \rho = 0, \end{cases}$$

we find $v \in C^{1+\alpha,\alpha/2}([0, 1) \times (0, T))$. This completes the proof. □

Theorem 2.1 *Let* $(u_0, w_0, R_0, a_0, S_0)$ *satisfy* (3). *Then there exists* $T > 0$ *such that the system* (IAS) *has a unique solution* (u, w, v, R, a, S) *with* $S(t) \equiv S_0$ *in* $[0, T)$ *in the class*

$$\begin{cases} u, w \in C^{2+\alpha,1+\frac{\alpha}{2}}(Q_T), \quad R, a \in C^1((0, T)), \\ v \in C^{1+\alpha,\frac{\alpha}{2}}([0, 1) \times (0, T)) \cap C^1([0, 1) \times (0, T)). \end{cases} \tag{19}$$

Proof Let (u, v, R, a) be the solution of (PS_0). According to Lemma 2.3, we see that the solution (u, v, R, a) belongs to the class

$$C^{2+\alpha,1+\frac{\alpha}{2}}(Q_T) \times (C^{1+\alpha,\frac{\alpha}{2}}([0, 1) \times (0, T)) \cap C^1([0, 1) \times (0, T))) \times (C^1((0, T)))^2$$

for some $T > 0$. To begin with, we prove the existence of a short time solution to (P). If there exists $T_1 \in (0, T]$ such that $R(t) \equiv R_0$ in $[0, T_1)$, then (u, v, R, a, S) with $S(t) \equiv S_0$ is a solution of (P), for the fact that $dR/dt = 0$ in $(0, T_1)$ implies that $S(t)$ does not switch in $(0, T_1)$. On the other hand, if there exists no such T_1, there exists $T_2 \in (0, T]$ such that $R(t) \notin \{r_0, r_1\}$ in $(0, T_2)$, for $R(t)$ is continuous. Then it is clear that (u, v, R, a, S) with $S(t) \equiv S_0$ satisfies (P) in $(0, T_2)$. Thus we see that (u, v, R, a, S) with $S(t) \equiv S_0$ is a solution of (P) in $(0, T^*)$ for some $T^* \in (0, T]$.

We show the uniqueness. Let $(u_1, v_1, R_1, a_1, S_1) \neq (u_2, v_2, R_2, a_2, S_2)$ be solutions of (P) satisfying (19). Along the same line as above, we see that $S_1(t) = S_2(t) = S_0$ in $[0, \widetilde{T})$ for some $\widetilde{T} \in (0, T^*]$. Then the uniqueness of the solution of (PS_0) leads a contradiction.

Thanks to Lemma 2.2, we observe that (mIAS) has a unique solution. Moreover, it follows from (18) that the solution satisfies (IAS). Finally we show the uniqueness of solutions of (IAS). Suppose that $(u_i, w_i, v_i, R_i, a_i, S_0)$ are solutions of (IAS) in the class (19), where $i = 1, 2$. Then, by the proof of Lemma 2.2, we observe that (IAS) is reduced to (P) replaced the condition on v/ρ by (4). It is clear that $a_1(t) = a_2(t)$ in $[0, T)$. Set $U := u_1 - u_2$. Then it follows from Step 2 in the proof of Lemma 2.3 that

$$\|U\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(Q_T)} \leq C\|U\|_{C^{\alpha,\frac{\alpha}{2}}(Q_T)}. \tag{20}$$

Moreover, we find

$$\|U\|_{C^{\alpha,\frac{\alpha}{2}}(Q_T)} \leq T^{1-\frac{\alpha}{2}}\|U\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(Q_T)} \leq CT^{1-\frac{\alpha}{2}}\|U\|_{C^{\alpha,\frac{\alpha}{2}}(Q_T)}. \tag{21}$$

Letting T be small enough such that $CT^{1-\alpha/2} < 1$, we observe from (21) that $\|U\|_{C^{\alpha,\alpha/2}} = 0$. Combining the fact with (20), we obtain the conclusion. □

In order to prove $u, w \in [0, 1]$ in $B_1 \times [0, T)$, we apply a parabolic comparison principle to (IAS). Using (u, v, R, a, S), which is the solution of (P) in Q_T constructed by Theorem 2.1, we define the operator

$$\mathscr{P}_i : C^{2,1}(B_1 \times (0, T)) \cap C(\overline{B_1} \times [0, T)) \to C(B_1 \times (0, T))$$

as follows:

$$\mathscr{P}_1 z := \partial_t z - \mathscr{L}'(v, R)z - P(z, a), \quad \mathscr{P}_2 z := \partial_t z - \mathscr{L}'(v, R)z + P(1 - z, a).$$

Regarding the operator $\mathscr{P}_i$, the following parabolic comparison principle holds:

Lemma 2.4 *Assume that $z, \zeta \in C^{2,1}(B_1 \times (0, T)) \cap C(\overline{B_1} \times [0, T))$ satisfy*

$$\begin{cases} \mathscr{P}_i z \geq \mathscr{P}_i \zeta & \text{in} \quad B_1 \times (0, T), \\ \partial_\nu z \geq \partial_\nu \zeta & \text{on} \quad \partial B_1 \times (0, T), \\ z \geq \zeta & \text{in} \quad \overline{B_1} \times \{t = 0\}. \end{cases}$$

Then $z \geq \zeta$ in $\overline{B_1} \times [0, T)$.

Proof Since the proof of Lemma 2.3 implies that the coefficients in the operator $\mathscr{L}'(v, R)$ are bounded, we can prove Lemma 2.4 along the standard argument (e.g., see [4, 17]). □

By virtue of Lemma 2.4, one can verify $0 \le u \le 1$ and $0 \le w \le 1$:

Lemma 2.5 *Let* (u, w, v, R, a, S) *be a solution of* (IAS) *obtained by Theorem 2.1. Then,* $0 \le u \le 1$ *and* $0 \le w \le 1$ *in* $\overline{B_1} \times [0, T)$.

We close this section with a property of certain quantities of u and w.

Lemma 2.6 *Let us define*

$$\begin{cases} U(t) := 4\pi R^3(t)\displaystyle\int_0^1 u(\rho, t)\rho^2\, d\rho, \\ W(t) := 4\pi R^3(t)\displaystyle\int_0^1 w(\rho, t)\rho^2\, d\rho, \end{cases} \qquad \begin{cases} V_1(t) := \displaystyle\int_0^1 u(\rho, t)\rho^2\, d\rho, \\ V_2(t) := \displaystyle\int_0^1 w(\rho, t)\rho^2\, d\rho. \end{cases}$$

Then U, W, V_1, *and* V_2 *satisfy*

$$\frac{dU}{dt}(t) = 4\pi R^3(t)\int_0^1 c_1 u(\rho, t)^2\rho^2 d\rho + \{f_1(a(t)) - c_1\}U(t), \tag{22}$$

$$\frac{dW}{dt}(t) = 4\pi R^3(t)\int_0^1 c_2 w(\rho, t)^2\rho^2 d\rho + \{f_2(a(t)) - c_2\}W(t), \tag{23}$$

$$\frac{dV_1}{dt}(t) \begin{cases} \le g(a(t))V_1(t) + 3\{-g(a(t)) + c_1 + c_2\}V_1(t)^2, \\ \ge \{g(a(t)) - c_1\}V_1(t) - 3\{g(a(t)) - c_1\}V_1(t)^2, \end{cases} \tag{24}$$

$$\frac{dV_2}{dt}(t) \begin{cases} \le -g(a(t))V_2(t) + 3\{g(a(t)) + c_1 + c_2\}V_2(t)^2, \\ \ge -\{g(a(t)) + c_2\}V_2(t) + 3\{g(a(t)) + c_2\}V_2(t)^2, \end{cases} \tag{25}$$

respectively, where g *is a function defined by*

$$g(z) := f_1(z) - f_2(z). \tag{26}$$

Proof The Eqs. (22) and (23) were obtained by [23]. We shall show (24) and (25). It follows from Jensen's inequality and Lemma 2.5 that

$$3V_1(t)^2 \le \int_0^1 u(\rho, t)^2\rho^2\, d\rho \le V_1(t), \quad 3V_2(t)^2 \le \int_0^1 w(\rho, t)^2\rho^2\, d\rho \le V_2(t). \tag{27}$$

Combining (27) with the same argument as in [23], we obtain the conclusion. □

Remark 2.1 The function g denotes the difference of net *growth* rate of AD cells and AI cells. We employ the notation frequently in the rest of the paper.

3 Asymptotic Behavior of Non-switching Solutions

We devote this section to investigating the asymptotic behavior of "non-switching" solutions of (IAS). To begin with, we shall show the long time existence of the non-switching solutions of (IAS).

Theorem 3.1 *Let* $(u_0, w_0, R_0, a_0, S_0)$ *satisfy* (3) *and* $S_0 = 1$. *Then the system* (IAS) *with* $r_0 = 0$ *has a unique solution* (u, w, v, R, a, S) *with* $S(t) \equiv 1$ *in* $[\,0, \infty\,)$ *in the class*

$$\begin{cases} u, w \in C^{2+\alpha, 1+\frac{\alpha}{2}}(Q_\infty), \quad R, a \in C^1(\mathbb{R}_+), \\ v \in C^{1+\alpha, \frac{\alpha}{2}}([\,0, 1\,) \times \mathbb{R}_+)) \cap C^1([\,0, 1\,) \times \mathbb{R}_+). \end{cases}$$

Proof It follows from Theorem 2.1 that (IAS) with $r_0 = 0$ has a unique solution with $S(t) \equiv 1$ in Q_T for some $T > 0$. Since

$$R(t) = R_0 \exp\Big[\int_0^t v(1, s)\, ds \Big],$$

we observe from the continuity of the solution that $R(t)$ is positive, i.e. $S(t) \equiv 1$, while the solution exists. Thus, by a standard argument (e.g., see [6]), we prove that the solution can be extended beyond for any $T > 0$. Indeed, if there exists $\widetilde{T} > 0$ such that the solution can not be extended beyond $\widetilde{T}$, then the proof of Theorem 2.1 implies that

$$\|u(\cdot, t)\|_{C^{2+\alpha}(B_1)} \to \infty \quad \text{as} \quad t \uparrow \widetilde{T}. \tag{28}$$

On the other hand, since u is a solution of (PS_0) on $[\,0, \widetilde{T}\,)$, it holds that

$$\|u(\cdot, t)\|_{C^{2+\alpha}(B_1)} \le \|u\|_{C^{2+\alpha, 1+\alpha/2}_{x,t}(Q_{\widetilde{T}})} \le C(C(\widetilde{T}) + \|u_0\|_{C^{2+\alpha}(B_1)}). \tag{29}$$

Since (29) contradicts (28), we obtain the conclusion. □

Remark 3.1 The system (IAS) with $r_0 = 0$ and $S_0 = 1$ describes a tumor growth under the CAS therapy.

Corollary 3.1 *Let* $(u_0, w_0, R_0, a_0, S_0)$ *satisfy* (3) *and* $S_0 = 0$. *Then the system* (IAS) *with* $r_1 = \infty$ *has a unique solution* (u, w, v, R, a, S) *with* $S(t) \equiv 0$ *in* $[\,0, \infty\,)$ *in the class*

$$\begin{cases} u, w \in C^{2+\alpha, 1+\frac{\alpha}{2}}(Q_\infty), \quad R, a \in C^1(\mathbb{R}_+), \\ v \in C^{1+\alpha, \frac{\alpha}{2}}([\,0, 1\,) \times \mathbb{R}_+)) \cap C^1([\,0, 1\,) \times \mathbb{R}_+). \end{cases}$$

In the following, we classify the asymptotic behavior of non-switching solutions obtained by Theorem 3.1 and Corollary 3.1. Recalling Lemma 2.2 and Theorem 2.1, we may consider (P) instead of (IAS).

If u_0 is trivial, i.e., $u_0 \equiv 0$ or $u_0 \equiv 1$, then Lemma 2.4 asserts that u is also trivial in Q_T. Thus it is sufficient to consider the initial data (u_0, R_0, a_0, S_0) satisfying

$$\begin{cases} u_0 \in C^{2+\alpha}(B_1), \quad \partial_\rho u_0(0) = \partial_\rho u_0(1) = 0, \quad 0 \le u_0 \le 1, \\ u_0(\rho) \not\equiv 0, \quad u_0(\rho) \not\equiv 1, \quad 0 < a_0 < a_*, \quad R_0 > 0, \quad S_0 \in \{0, 1\}, \end{cases} \tag{IC}$$

where $0 < \alpha < 1$. Regarding f_i and c_i, we assume (A0) throughout this section.

From now on, for a function $h : [\,0, a_*\,] \to \mathbb{R}$, we define $\|h\|_\infty$ by

$$\|h\|_\infty := \sup_{z \in [\,0, a_*\,]} |h(z)|. \tag{30}$$

First we consider the asymptotic behavior of solutions to (P) with $S \equiv 1$.

Theorem 3.2 *Let $r_0 = 0$. Let (u_0, R_0, a_0, S_0) satisfy* (IC) *and $S_0 = 1$. Assume that either of two assumptions holds*:

(i) $g(0) + c_2 < 0$;

(ii) $g(0) + c_2 > 0$ *and*

$$\int_0^1 u_0(\rho)\rho^2 \, d\rho < \frac{1}{3}\frac{-g(0)}{-g(0) + c_1 + c_2} \exp\Big[-\frac{a_0}{\gamma}\|g'\|_\infty \Big]. \tag{31}$$

Then the solution (u, v, R, a, S) of (P) *satisfies $R(t) \to \infty$ as $t \to \infty$.*

Proof To begin with, we note that $S(t) \equiv 1$ under (P) with $r_0 = 0$ and $S_0 = 1$.

We prove the case (i). Since $S \equiv 1$ yields the monotonicity of $a(t)$, especially that of $f_i(a(t))$, from the assumptions (A0) and (i), we find $s_1 > 0$ such that

$$f_1(a(t)) < 0, \quad f_2(a(t)) > 0, \quad -g(a(t)) - c_2 > 0 \quad \text{for any} \quad t \ge s_1.$$

Recalling that $u_0 \not\equiv 1$ yields $V_2(t) > 0$ for any $t \ge 0$ and setting $\widetilde{V}_2(t) := 1/V_2(t)$, we observe from (25) that

$$\frac{d\widetilde{V}_2}{dt}(t) \le \{g(a(t)) + c_2\}\widetilde{V}_2(t) - 3\{g(a(t)) + c_2\}. \tag{32}$$

Applying Gronwall's inequality to (32), we have

$$\widetilde{V}_2(t) \le 3 + \big(\widetilde{V}_2(0) - 3\big) \exp\Big[\int_0^t \{g(a(s)) + c_2\} \, ds \Big].$$

Since

$$\begin{aligned} \int_0^t \{g(a(s)) + c_2\} \, ds &= \int_0^{s_1} \{g(a(s)) + c_2\} \, ds + \int_{s_1}^t \{g(a(s)) + c_2\} \, ds \\ &\le (g(a_0) + c_2)s_1 - \{-g(a(s_1)) - c_2\}(t - s_1) \to -\infty \quad \text{as} \quad t \to \infty, \end{aligned}$$

one can verify that $\limsup_{t\to\infty} \widetilde{V}_2(t) \le 3$. On the other hand, since $w \le 1$ yields $\widetilde{V}_2(t) \ge 3$ in $[0, \infty)$, we find $\liminf_{t\to\infty} \widetilde{V}_2(t) \ge 3$. Thus we have $\lim_{t\to\infty} \widetilde{V}_2(t) = 3$ and then

$$\|w(\cdot, t) - 1\|_{L^\infty(B_1)} \to 0 \quad \text{as} \quad t \to \infty. \tag{33}$$

By way of $u + w \equiv 1$, it follows from (33) that for any ε with

$$0 < \varepsilon < \frac{f_2(s_1)}{-g(0) + c_1 + c_2}, \tag{34}$$

there exists $T_1 > s_1$ such that

$$\|u(\cdot, t)\|_{L^\infty(B_1)} < \varepsilon \quad \text{for any} \quad t > T_1. \tag{35}$$

In what follows, let $t > T_1$. Since R satisfies

$$R(t) = R_0 \exp\Big[\int_0^{T_1} v(1, s)\, ds \Big] \exp\Big[\int_{T_1}^{t} v(1, s)\, ds \Big], \tag{36}$$

it is sufficient to estimate the integrals in the right-hand side of (36). We observe from the continuity of $v(1, \cdot)$ that

$$\int_0^{T_1} v(1, s)\, ds \ge -CT_1$$

for some $C > 0$. Moreover, we obtain

$$\begin{aligned} \int_{T_1}^{t} v(1, s)\, ds &= \int_{T_1}^{t} \int_0^1 F(u(\rho, s), a(s))\rho^2\, d\rho\, ds \\ &\ge \int_{T_1}^{t} \int_0^1 [-\{-g(a(t)) + c_1 + c_2\}u + f_2(a(T_1))]\rho^2\, d\rho\, ds \\ &\ge \frac{1}{3}\{-(-g(0) + c_1 + c_2)\varepsilon + f_2(a(s_1))\}(t - T_1). \end{aligned}$$

Hence, it follows from (34) and (35) that $\liminf_{t\to\infty} R(t) = \infty$.

Next we turn to the case (ii). By the assumption (A0) and the monotonicity of $f_i(a(\cdot))$, there exists $s_2 \ge 0$ such that

$$f_2(a(t)) > 0, \quad g(a(t)) < 0, \quad \text{for any} \quad t \ge s_2.$$

Recalling $V_1(t) > 0$ in $[0, \infty)$ and setting $\widetilde{V}_1(t) := 1/V_1(t)$, we reduce (24) to

$$\frac{d\widetilde{V}_1}{dt}(t) \geq -g(a(t))\widetilde{V}_1 - 3\{-g(a(t)) + c_1 + c_2\}.$$

Since it follows from the same argument as in (i) that

$$\widetilde{V}_1(t) \geq e^{-\int_0^t g(a(s))\,ds} \left[3(g(0) - c_1 - c_2) \int_0^t e^{\int_0^s g(a(\tau))\,d\tau} ds + \widetilde{V}_1(0) \right], \tag{37}$$

we estimate the integral in the right-hand side of (37). Noting that $a(\cdot)$ is monotone decreasing, we use the change of variable $a(s) = z$, and then

$$\begin{aligned} \int_0^t g(a(s))\,ds &= -\frac{1}{\gamma} \int_{a_0}^{a(t)} \frac{g(z)}{z}\,dz = -\frac{1}{\gamma} \int_{a_0}^{a(t)} \left[\frac{g(0)}{z} + g'(\tilde{z}) \right] dz \\ &\leq -\frac{g(0)}{\gamma} \log \frac{a(t)}{a_0} + \frac{a_0}{\gamma} \|g'\|_\infty, \end{aligned} \tag{38}$$

where $\tilde{z} \in (0, a_0)$. Combining (37) with (38), we obtain

$$\begin{aligned} \widetilde{V}_1(t) \geq &\left(\frac{a(t)}{a_0} \right)^{\frac{g(0)}{\gamma}} \left[3(g(0) - c_1 - c_2) \int_0^\infty \left(\frac{a_0}{a(s)} \right)^{\frac{g(0)}{\gamma}} ds + \widetilde{V}_1(0) e^{-\frac{a_0}{\gamma}\|g'\|_\infty} \right] \\ \geq &\left(\frac{a(t)}{a_0} \right)^{\frac{g(0)}{\gamma}} \left[\frac{-3(-g(0) + c_1 + c_2)}{-g(0)} + \widetilde{V}_1(0) e^{-\frac{a_0}{\gamma}\|g'\|_\infty} \right]. \end{aligned}$$

Under (A0) and (31), the inequality implies that $\widetilde{V}_1 \to \infty$ as $t \to \infty$, i.e., $V_1(t) \to 0$ as $t \to \infty$. Thus for any ε with

$$0 < \varepsilon < \frac{f_2(a(s_2))}{-g(0) + c_1 + c_2}, \tag{39}$$

there exists $T_2 \geq s_2$ such that

$$\|u(\cdot, t)\|_{L^\infty(B_1)} < \varepsilon \quad \text{for any} \quad t > T_2. \tag{40}$$

By virtue of (39) and (40), we have

$$\begin{aligned} \int_{T_2}^t v(1, s)\,ds &\geq \int_{T_2}^t \int_0^1 \{-(-g(0) + c_1 + c_2)u + f_2(a(T_2))\}\rho^2\,d\rho\,ds \\ &\geq \frac{1}{3}\{-(-g(0) + c_1 + c_2)\varepsilon + f_2(a(s_2))\}(t - T_2). \end{aligned}$$

Thus we see that $\liminf_{t\to\infty} R(t) = \infty$ along the same line as in (i). □

Next we give the asymptotic behavior of solutions to (P) with $r_0 = 0$ and $S_0 = 1$.

Theorem 3.3 *Let* $r_0 = 0$. *Let* (u_0, R_0, a_0, S_0) *satisfy* (IC) *and* $S_0 = 1$. *Assume that*

$$g(0) + c_2 > 0 \tag{41}$$

and

$$\min_{\rho \in [0,1]} u_0(\rho) > 1 - \frac{g(0) + c_2}{g(0) + c_1 + 2c_2}. \tag{42}$$

Then the solution (u, v, R, a, S) *of* (P) *satisfies* $R(t) \to 0$ *as* $t \to \infty$.

Proof Recalling that $S \equiv 1$ under (P) with $r_0 = 0$ and $S_0 = 1$, and using (A0), we find $s_3 \geq 0$ such that

$$f_1(a(t)) < 0 \quad \text{for any} \quad t \geq s_3. \tag{43}$$

Let $\overline{w}$ be the solution of the following initial value problem:

$$\begin{cases} \dfrac{d\overline{w}}{dt}(t) = -\{g(a(t)) + c_2\}\overline{w}(t) + \{g(a(t)) + c_1 + 2c_2\}\overline{w}(t)^2, \\ \overline{w}(0) = 1 - \displaystyle\min_{\rho \in [0,1]} u_0(\rho). \end{cases}$$

Then Lemma 2.4 asserts that

$$0 \leq w(\rho, t) \leq \overline{w}(t) \quad \text{for any} \quad (\rho, t) \in [0, 1] \times [0, \infty), \tag{44}$$

i.e., $\overline{w}$ is a supersolution of w. Since $w_0 \not\equiv 0$, the relation (44) implies $\overline{w}(t) > 0$ for any $t \geq 0$. Setting $\omega := 1/\overline{w}$, we see that ω is expressed by

$$\omega = e^{\int_0^t \{g(a(s))+c_2\}\, ds} \left[-\int_0^t \{g(a(s)) + c_1 + 2c_2\} e^{-\int_0^s \{g(a(\tau))+c_2\}\, d\tau} ds + \frac{1}{\overline{w}(0)} \right].$$

Here we have

$$\begin{aligned} &\int_0^t \{g(a(s)) + c_1 + 2c_2\} e^{-\int_0^s \{g(a(\tau))+c_2\}\, d\tau}\, ds \\ &= \int_0^t \{g(a(s)) + c_2\} e^{-\int_0^s \{g(a(\tau))+c_2\}\, d\tau} ds + (c_1 + c_2) \int_0^t e^{-\int_0^s \{g(a(\tau))+c_2\}\, d\tau} ds \\ &\leq -e^{-\int_0^t \{g(a(\tau))+c_2\}\, d\tau} + 1 + (c_1 + c_2) \int_0^t e^{-\{g(0)+c_2\}s}\, ds \\ &\leq 1 + \frac{c_1 + c_2}{g(0) + c_2} \left(1 - e^{-\{g(0)+c_2\}t}\right) \leq \frac{g(0) + c_1 + 2c_2}{g(0) + c_2}. \end{aligned}$$

Since it follows from (41) that

$$\liminf_{t \to \infty} \exp \left[\int_0^t \{g(a(s)) + c_2\}\, ds \right] \geq \liminf_{t \to \infty} \exp\left[(g(0) + c_2)t\right] = \infty,$$

we observe from (42) that $\lim_{t\to\infty}\omega(t)=\infty$, i.e., $\lim_{t\to\infty}\overline{w}(t)=0$, where we used the positivity of $\overline{w}$. With the aid of (44), for any ε with

$$0<\varepsilon<\frac{-f_1(a(s_3))}{-g(0)}, \tag{45}$$

there exists $T_3>s_3$ such that

$$\|w(\cdot,t)\|_{L^\infty(B_1)}<\varepsilon \quad \text{for any} \quad t>T_3.$$

Recalling $u=1-w$ and using the same argument as in the proof of Theorem 3.2 (i), we can verify that

$$\begin{aligned}R(t)&\le R_0e^{CT_3}e^{\int_{T_3}^t v(1,s)\,ds}\le R_0e^{CT_3}\exp\Big[\int_{T_3}^t\{-g(a(s))w+f_1(a(s))\}\,ds\Big]\\&\le R_0e^{CT_3}\exp\Big[\frac{1}{3}\{-g(0)\varepsilon+f_1(s_3)\}(t-T_3)\Big].\end{aligned}$$

Then (45) yields $\limsup_{t\to\infty}R(t)=0$. □

We turn to the case of (P) with $r_1=\infty$ and $S_0=0$. We note that (P) with $r_1=\infty$ and $S_0=0$ describes the behavior of prostate tumor under non-medication.

Theorem 3.4 *Let* $r_1=\infty$. *Let* (u_0,R_0,a_0,S_0) *satisfy* (IC) *and* $S_0=0$. *We suppose that one of the following assumptions holds*:
(i) $f_1(a_*)-c_1>0$; (ii) $f_2(a_*)-c_2>0$; (iii) $g(a_*)-c_1>0$;
(iv) $-g(a_*)+c_1>0$, $f_2(a_*)>0$, *and*

$$\max_{\rho\in[0,1]}u_0(\rho)<\frac{-g(a_*)+c_1}{-g(a_*)+2c_1+c_2}; \tag{46}$$

(v) $g(a_*)+c_2>0$ *and*

$$\min_{\rho\in[0,1]}u_0(\rho)>1-\frac{g(a_*)+c_2}{g(a_*)+c_1+2c_2}\exp\Big[-\frac{a_*}{\gamma}\|g'\|_\infty\Big].$$

Then the solution (u,v,R,a,S) *of* (P) *satisfies* $R(t)\to\infty$ *as* $t\to\infty$.

Proof We prove the case (i). Remark that $S\equiv 0$ yields the monotonicity of $a(t)$, especially that of $f_i(a(t))$. Under the assumption (i), we find $s_4\ge 0$ such that $f_1(a(t))-c_1>0$ for any $t\ge s_4$. Since it follows from (22) that

$$\frac{dU}{dt}\ge\{f_1(a(t))-c_1\}U(t) \quad \text{for any} \quad t\ge 0,$$

making use of Gronwall's inequality and the monotonicity of $f_1(a(\cdot))$, we find

$$\begin{aligned} U(t) &\geq U(s_4)\exp\Big[\int_{s_4}^{t}\{f_1(a(s))-c_1\}\,ds\Big] \\ &\geq U(s_4)\exp\left[\{f_1(a(s_4))-c_1\}(t-s_4)\right] \quad \text{for any} \quad t\geq s_4. \end{aligned}$$

Consequently we see that

$$\liminf_{t\to\infty}\frac{4}{3}\pi R^3(t)=\liminf_{t\to\infty}\{U(t)+W(t)\}\geq\liminf_{t\to\infty}U(t)=\infty.$$

Regarding the other cases, we obtain the conclusion along the same line as in the proof of Theorem 3.2. □

By the same argument as in the proof of Theorem 3.3, we obtain the following:

Theorem 3.5 *Let $r_1=\infty$. Let (u_0, R_0, a_0, S_0) satisfy* (IC) *and $S_0=0$. Assume that*

$$-g(a_*)+c_1>0, \quad f_2(a_*)<0, \tag{47}$$

and (46)*. Then the solution (u, v, R, a, S) of* (P) *satisfies $R(t)\to 0$ as $t\to\infty$.*

4 Proof of the Main Theorem

The purpose of this section is to prove the existence of a switching solution of (IAS) and investigate its property under the assumption (A0)–(A2). Here we note that (A1) and (A2) are written as $g(a_*)-c_1>0$ and $g(0)+c_2>0$, respectively, where g was defined by (26). For this purpose, we may deal with (P) instead of (IAS), for the solution of (P) constructed in Sect. 2 also satisfies (IAS). In the following, we fix (u_0, R_0, a_0, S_0) satisfying (IC), $u_0>0$, and $S_0=0$, arbitrarily.

To begin with, we shall study the behavior of solutions of (P) with $S\equiv 0$. More precisely, for each "initial data" $(\tilde{u}_0, \widetilde{R}_0, \tilde{a}_0)$, we consider the following system:

$$\begin{cases} \dfrac{d\tilde{a}}{dt}(t)=-\gamma(\tilde{a}(t)-a_*) & \text{in} \quad \mathbb{R}_+, \\ \partial_t\tilde{u}(\rho,t)-\mathscr{L}'(\tilde{v},\tilde{R})\tilde{u}(\rho,t)=P(\tilde{u}(\rho,t),\tilde{a}(t)) & \text{in} \quad I_\infty, \\ \tilde{v}(\rho,t)=\dfrac{1}{\rho^2}\displaystyle\int_0^{\rho}F(\tilde{u}(r,t),\tilde{a}(t))r^2\,dr & \text{in} \quad I_\infty, \\ \dfrac{d\widetilde{R}}{dt}(t)=\tilde{v}(1,t)\widetilde{R}(t) & \text{in} \quad \mathbb{R}_+, \\ \partial_\rho\tilde{u}(\rho,t)\big|_{\rho\in\{0,1\}}=0, \quad \dfrac{\tilde{v}(\rho,t)}{\rho}\Big|_{\rho=0}=\dfrac{1}{3}F(\tilde{u}(0,t),\tilde{a}(t)), & \text{in} \quad \mathbb{R}_+, \\ \tilde{a}(0)=\tilde{a}_0, \;\; \tilde{u}(\rho,0)=\tilde{u}_0(\rho), \;\; \widetilde{R}(0)=\widetilde{R}_0, & \text{in} \quad I, \end{cases} \tag{P0}$$

where the operator $\mathscr{L}'$ was defined by (8). We characterize the time variable in terms of the solution $\tilde{a}(\cdot)$ to (P0). Recalling that f_1 is monotone, we define a function $\tau_0 : (\,0,\, f_1(a_*) - f_1(\tilde{a}_0)\,] \to [\,0, \infty\,)$ as

$$\tau_0(\varepsilon) = \tilde{a}^{-1}(f_1^{-1}(f_1(a_*) - \varepsilon)), \tag{48}$$

where $\tilde{a}^{-1}$ and f_1^{-1} denote the inverse functions of $\tilde{a}$ and f_1, respectively. Note that, since $\tilde{a}(t) \uparrow a_*$ as $t \to \infty$, $\varepsilon \downarrow 0$ is equivalent to $\tau_0(\varepsilon) \to \infty$.

From now on, we will follow the notation $\|\cdot\|_\infty$ defined in (30).

Lemma 4.1 *Assume that there exist constants* $A \in (\,0, 1\,)$ *and* $\kappa \in (\,0, a_*\,)$ *such that* $(\tilde{u}_0, \widetilde{R}_0, \tilde{a}_0)$ *satisfies* (IC) *and the following*:

$$\min_{\rho \in [\,0,1\,]} \tilde{u}_0(\rho) \geq A; \tag{49}$$

$$\tilde{a}_0 \leq \kappa. \tag{50}$$

Then there exists a strictly monotone increasing continuous function

$$\Gamma_0(\varepsilon; A, \kappa) : (\,0,\, f_1(a_*) - f_1(0)\,] \to \mathbb{R}_+$$

with $\Gamma_0(\varepsilon; A, \kappa) \downarrow 0$ *as* $\varepsilon \downarrow 0$ *such that the solution of* (P0) *satisfies*

$$\|\tilde{u}(\cdot, \tau_0(\varepsilon)) - 1\|_{L^\infty(B_1)} \leq \Gamma_0(\varepsilon; A, \kappa) \quad in \quad (\,0,\, f_1(a_*) - f_1(\tilde{a}_0)\,].$$

Proof Let us consider

$$\begin{cases} \dfrac{d\overline{w}}{dt} = -(g(\tilde{a}(t)) - c_1)(1 - \overline{w})\overline{w}, \\ \overline{w}(0) = 1 - \displaystyle\min_{\rho \in [\,0,1\,]} \tilde{u}_0(\rho). \end{cases} \tag{51}$$

By way of Lemma 2.4, one can easily verify that $\overline{w}$ is a supersolution of $1 - \tilde{u}$. Solving (51) and setting $t = \tau_0(\varepsilon)$, we find

$$\omega(\tau_0(\varepsilon)) = 1 + (\omega(0) - 1) \exp\Big[\int_0^{\tau_0(\varepsilon)} \{g(\tilde{a}(s)) - c_1\}\, ds\,\Big],$$

where $\omega = 1/\overline{w}$. From the change of variable $\tilde{a}(s) = z$, we have

$$\int_0^{\tau_0(\varepsilon)} \{g(\tilde{a}(s)) - c_1\} ds = -\frac{1}{\gamma} \int_{\tilde{a}_0}^{\tilde{a}(\tau_0(\varepsilon))} \Big[\, \frac{g(z) - g(a_*)}{z - a_*} + \frac{g(a_*) - c_1}{z - a_*}\,\Big] dz \tag{52}$$

$$\geq -\frac{\tilde{a}(\tau_0(\varepsilon)) - \tilde{a}_0}{\gamma} \|g'\|_\infty + \frac{g(a_*) - c_1}{\gamma} \log \frac{a_* - \tilde{a}_0}{a_* - \tilde{a}(\tau_0(\varepsilon))}$$

$$\geq -\frac{a_*}{\gamma} \|g'\|_\infty + \frac{g(a_*) - c_1}{\gamma} \log \frac{a_* - \tilde{a}_0}{a_* - \tilde{a}(\tau_0(\varepsilon))},$$

where we used (A1) in the last inequality. Therefore, using (49) and (50), we define the required function $\Gamma_0(\varepsilon; A, \kappa)$ as follows:

$$\overline{w}(\tau_0(\varepsilon)) \leq \Big[\, 1 + Ae^{-\frac{a_*}{\gamma}\|g'\|_\infty} \Big(\frac{a_* - \kappa}{a_* - f_1^{-1}(f_1(a_*) - \varepsilon)} \Big)^{\frac{g(a_*)-c_1}{\gamma}} \Big]^{-1} =: \Gamma_0(\varepsilon; A, \kappa).$$

This completes the proof. □

Lemma 4.2 *Under the same assumption as in Lemma 4.1, there exists a constant* $\varepsilon_1 \in (\,0,\, f_1(a_*)\,)$*, independent of* $(\tilde{u}_0, \widetilde{R}_0, \tilde{a}_0)$*, such that the solution of* (P0) *satisfies*

$$\frac{d\widetilde{R}}{dt}(\tau_0(\varepsilon)) > 0 \quad \textit{for any} \quad \varepsilon \in (\,0, \varepsilon_1\,].$$

Proof Since $d\widetilde{R}/dt$ is written by

$$\frac{d\widetilde{R}}{dt}(t) = \widetilde{R}(t)\tilde{v}(1,t) = \widetilde{R}(t) \int_0^1 F(\tilde{u}(\rho,t), \tilde{a}(t))\rho^2\, d\rho, \tag{53}$$

we observe that the sign of $d\widetilde{R}/dt$ is determined by that of the integral in (53). In particular, we focus on the sign of F. From $\partial_z^2 F(z,\alpha) = 2(c_1 + c_2) > 0$, we find

$$\begin{aligned} F(z,\alpha) &> F(1,\alpha) + \partial_z F(1,\alpha)(z-1) \\ &\geq F(1,\alpha) + \partial_z F(1,a_*)(z-1) =: y(z;\alpha) \quad \text{in} \quad [\,0,1\,) \times [\,0, a_*\,], \end{aligned} \tag{54}$$

where we used the monotonicity of $\partial_z F(1,\alpha) = g(\alpha) - c_1 - c_2$ in the second inequality. Here, noting the positivity of $\partial_z F(1,a_*)$, we denote by $z_0(\alpha)$ the zero point of $y(z;\alpha)$ given by

$$z_0(\alpha) = \frac{-F(1,\alpha) + \partial_z F(1,a_*)}{\partial_z F(1,a_*)}.$$

Since (48) yields that

$$F(1, \tilde{a}(\tau_0(\varepsilon))) = f_1(a_*) - \varepsilon > 0 \quad \text{for any} \quad \varepsilon \in (\,0,\, f_1(a_*)\,), \tag{55}$$

we see that

$$z_0(\tilde{a}(\tau_0(\varepsilon))) < 1, \quad y(1, \tilde{a}(\tau_0(\varepsilon))) > 0, \quad \text{for all} \quad \varepsilon \in (\,0,\, f_1(a_*)\,). \tag{56}$$

Then, for each $\varepsilon \in (\,0,\, f_1(a_*)\,)$, we observe from (56) that $y(z, \tilde{a}(\tau_0(\varepsilon))) \geq 0$ for all $z \in [\, z_0(\tilde{a}(\tau_0(\varepsilon))), 1\,]$. Combining the fact with (54)–(55), we infer that

$$F(z, \tilde{a}(\tau_0(\varepsilon))) > 0 \;\text{ for all }\; z \in [\, z_0(\tilde{a}(\tau_0(\varepsilon))), 1\,], \;\text{ if }\; \varepsilon \in (\,0,\, f_1(a_*)\,). \tag{57}$$

In order to complete the proof of Lemma 4.2, it is sufficient to prove the claim: there exists a constant $\varepsilon_1 \in (0, f_1(a_*))$, independent of $(\tilde{u}_0, \widetilde{R}_0, \tilde{a}_0)$, such that the solution $(\tilde{u}, \tilde{v}, \widetilde{R}, \tilde{a})$ of (P0) satisfies

$$\min_{\rho \in [0,1]} \tilde{u}(\rho, \tilde{a}(\tau_0(\varepsilon))) \geq z_0(\tilde{a}(\tau_0(\varepsilon))) \quad \text{for any} \quad \varepsilon \in (0, \varepsilon_1].$$

Indeed, combining the claim with (57), we clearly obtain the conclusion. We shall show the claim by way of Lemma 4.1. Since $z_0(\tilde{a}(\tau_0(f_1(a_*))) = 1$ and

$$z_0(\tilde{a}(\tau_0(\varepsilon))) \downarrow z_0(a_*) < 1, \quad 1 - \varGamma_0(\varepsilon; A, \kappa) \uparrow 1, \quad \text{as} \quad \varepsilon \downarrow 0,$$

from the monotonicity of $z_0(\tilde{a}(\tau_0(\varepsilon)))$ and $1 - \varGamma_0(\varepsilon; A, \kappa)$, we find a constant $\tilde{\varepsilon}_1 \in (0, f_1(a_*))$ uniquely, independent of $(\tilde{u}_0, \widetilde{R}_0, \tilde{a}_0)$, such that

$$1 - \varGamma_0(\varepsilon; A, \kappa) \geq z_0(\tilde{a}(\tau_0(\varepsilon))) \quad \text{for any} \quad \varepsilon \in (0, \tilde{\varepsilon}_1]. \tag{58}$$

Recalling (50) implies that $f_1(\kappa) \geq f_1(\tilde{a}_0)$ and setting $\varepsilon_1 := \min\{\tilde{\varepsilon}_1, f_1(a_*) - f_1(\kappa)\}$, we observe from (58) and Lemma 4.1 that

$$\min_{\rho \in [0,1]} \tilde{u}(\rho, \tilde{a}(\tau_0(\varepsilon))) \geq 1 - \varGamma_0(\varepsilon; A, \kappa) \geq z_0(\tilde{a}(\tau_0(\varepsilon))) \quad \text{for any} \quad \varepsilon \in (0, \varepsilon_1].$$

Then the claim holds true and we have completed the proof. □

Lemma 4.3 *Let* $(\tilde{u}_0, \widetilde{R}_0, \tilde{a}_0) = (u_0, R_0, a_0)$. *Then there exist monotone decreasing functions* M^- *and* M^+ *defined on* $(0, f_1(a_*) - f_1(0)]$ *such that the solution of* (P0) *satisfies*

$$R_0 \exp M^-(\varepsilon) \leq \widetilde{R}(\tau_0(\varepsilon)) \leq R_0 \exp M^+(\varepsilon) \quad \textit{in} \quad (0, f_1(a_*) - f_1(a_0)], \tag{59}$$

where the second inequality is strict for any $\varepsilon \in (0, f_1(a_*) - f_1(a_0))$. *Moreover,* M^- *and* M^+ *satisfy the following*:

$$-\infty < M^-(\varepsilon) \leq M^+(\varepsilon) < \infty \quad \textit{in} \quad (0, f_1(a_*) - f_1(0)]; \tag{60}$$

$$\lim_{\varepsilon \downarrow 0} M^-(\varepsilon) = \infty. \tag{61}$$

Proof Since $\widetilde{R}(\tau_0(\varepsilon))$ is given by

$$\widetilde{R}(\tau_0(\varepsilon)) = R_0 \exp\Big[\int_0^{\tau_0(\varepsilon)} \tilde{v}(1, s)\, ds \Big] \quad \text{in} \quad (0, f_1(a_*) - f_1(a_0)], \tag{62}$$

we will estimate the integral in (62). To this aim, setting $\tilde{w} = 1 - \tilde{u}$, we decompose the integral as follows:

$$\int_0^{\tau_0(\varepsilon)} \tilde{v}(1,s)\,ds = (c_1 + c_2)\int_0^{\tau_0(\varepsilon)}\int_0^1 \tilde{w}^2\rho^2\,d\rho ds \tag{63}$$
$$- \int_0^{\tau_0(\varepsilon)}\int_0^1 \big[g(\tilde{a}(s)) + c_1 + c_2\big]\tilde{w}\rho^2\,d\rho ds + \frac{1}{3}\int_0^{\tau_0(\varepsilon)} f_1(\tilde{a}(s))\,ds =: I_1 + I_2 + I_3.$$

First we construct M^-. Regarding I_1, it follows from Jensen's inequality that

$$I_1 \geq \frac{c_1 + c_2}{3}\int_0^{\tau_0(\varepsilon)}\left(\int_0^1 \tilde{w}\rho^2\,d\rho\right)^2 ds =: \frac{c_1 + c_2}{27}\int_0^{\tau_0(\varepsilon)} \mathscr{W}(s)^2\,ds. \tag{64}$$

Employing a differential inequality in (25), we see that $\mathscr{W}$ satisfies

$$\mathscr{W}(s) \geq \frac{1}{1 + \left(\frac{1}{\mathscr{W}(0)} - 1\right)\exp\left[\int_0^s \{g(\tilde{a}(\tau)) + c_2\}\,d\tau\right]}. \tag{65}$$

Furthermore, the same argument as in (52) yields

$$\int_0^s \{g(\tilde{a}(\tau)) + c_2\}\,d\tau \leq \frac{g(a_*) + c_2}{\gamma}\log \mathscr{T}_{a_0}(\tilde{a}(s)), \tag{66}$$

where

$$\mathscr{T}_{z_1}(z_2) := \frac{a_* - z_1}{a_* - z_2}. \tag{67}$$

Hence, combining (64) with (65)–(66), we have

$$I_1 \geq \frac{c_1 + c_2}{27}\int_0^{\tau_0(\varepsilon)}\left[\,1 + \left[\,\frac{1}{\mathscr{W}(0)} - 1\,\right]\mathscr{T}_{a_0}(\tilde{a}(s))^{\frac{g(a_*)+c_2}{\gamma}}\,\right]^{-2} ds =: I_{11}.$$

Changing the variable

$$\eta = 1 + \left(\frac{1}{\mathscr{W}(0)} - 1\right)\mathscr{T}_{a_0}(\tilde{a}(s))^{\frac{g(a_*)+c_2}{\gamma}}$$

and setting

$$\eta_0 := \frac{1}{\mathscr{W}(0)}, \quad \eta_\varepsilon := 1 + \left(\frac{1}{\mathscr{W}(0)} - 1\right)\mathscr{T}_{a_0}(\tilde{a}(\tau_0(\varepsilon)))^{\frac{g(a_*)+c_2}{\gamma}},$$

we can define $M_1^- : (\,0,\, f_1(a_*) - f_1(a_0)\,] \to \mathbb{R}$ as follows:

$$\begin{aligned}
I_{11} &= C_1 \int_{\eta_0}^{\eta_\varepsilon} \frac{d\eta}{(\eta-1)\eta^2} \geq C_1 \left[\log \frac{\eta_0(\eta_\varepsilon - 1)}{\eta_\varepsilon(\eta_0 - 1)} - \frac{1}{\eta_0} \right] \\
&= -\,C_1 \left[\log \left[\mathscr{U}_0 + (1-\mathscr{U}_0)\mathscr{T}_{a_0}(\tilde{a}(\tau_0(\varepsilon)))^{-\frac{g(a_*)+c_2}{\gamma}} \right] + 1 - \mathscr{U}_0 \right] \\
&\geq -\,C_1 \left[\log \left[1 + (1-\mathscr{U}_0)\mathscr{T}_{\kappa_0}(\tilde{a}(\tau_0(\varepsilon)))^{-\frac{g(a_*)+c_2}{\gamma}} \right] + 1 - \mathscr{U}_0 \right] =: M_1^-(\varepsilon),
\end{aligned}$$

where $C_1 = (c_1 + c_2)/(27(g(a_*) + c_2))$, and

$$\mathscr{U}(s) := 3 \int_0^1 \tilde{u}(\rho, s)\, \rho^2\, d\rho, \quad \mathscr{U}_0 := \mathscr{U}(0), \quad \kappa_0 := \max\{a_0,\, f_1^{-1}(0)\}. \tag{68}$$

Regarding I_2, it follows from $\tilde{w} = 1 - \tilde{u}$ that

$$I_2 \geq -\frac{g(a_*) + c_1 + c_2}{3} \int_0^{\tau_0(\varepsilon)} \{1 - \mathscr{U}(s)\}\, ds.$$

Using (24) and the same calculation as in (66), we have

$$\mathscr{U}(s) \geq \left[1 + (\mathscr{U}_0^{-1} - 1)\mathscr{T}_{a_0}(\tilde{a}(s))^{-\frac{g(a_*)-c_1}{\gamma}} \right]^{-1}.$$

Then, by the same argument as in the derivation of M_1^-, we obtain

$$\begin{aligned}
I_2 &\geq -\frac{g(a_*) + c_1 + c_2}{3} \int_0^{\tau_0(\varepsilon)} \frac{\left(\mathscr{U}_0^{-1} - 1\right) \mathscr{T}_{a_0}(\tilde{a}(s))^{-\frac{g(a_*)-c_1}{\gamma}}}{1 + (\mathscr{U}_0^{-1} - 1)\mathscr{T}_{a_0}(\tilde{a}(s))^{-\frac{g(a_*)-c_1}{\gamma}}}\, ds \\
&= \frac{1}{3} \frac{g(a_*) + c_1 + c_2}{g(a_*) - c_1} \log \left[\mathscr{U}_0 + (1 - \mathscr{U}_0)\mathscr{T}_{a_0}(\tilde{a}(\tau_0(\varepsilon)))^{-\frac{g(a_*)-c_1}{\gamma}} \right] \\
&\geq \frac{1}{3} \frac{g(a_*) + c_1 + c_2}{g(a_*) - c_1} \log \mathscr{U}_0 =: M_2^-(\varepsilon).
\end{aligned}$$

It follows from the same argument as in (52) that

$$\begin{aligned}
I_3 &= \frac{f_1(a_*)}{3\gamma} \log \mathscr{T}_{a_0}(\tilde{a}(\tau_0(\varepsilon))) - \frac{1}{3\gamma} \int_{a_0}^{\tilde{a}(\tau_0(\varepsilon))} f'(\tilde{z})\, dz \\
&\geq \frac{f_1(a_*)}{3\gamma} \log \mathscr{T}_{\kappa_0}(\tilde{a}(\tau_0(\varepsilon))) - \frac{a_*}{3\gamma} \| f_1' \|_\infty =: M_3^-(\varepsilon),
\end{aligned} \tag{69}$$

where $\tilde{z} \in (a_0, a_*)$. Setting $M^-(\varepsilon) = \sum_{i=1}^3 M_i^-(\varepsilon)$ and recalling (48), we see that M^- is well-defined on $(\,0,\, f_1(a_*) - f_1(a_0)\,]$.

We shall derive M^+. Since $\tilde{w} = 1 - \tilde{u} \le 1$, the same argument as in M_2^- yields

$$\begin{aligned} I_1 &\le (c_1 + c_2) \int_0^{\tau_0(\varepsilon)} \int_0^1 \tilde{w}\rho^2 \, d\rho ds = \frac{c_1 + c_2}{3} \int_0^{\tau_0(\varepsilon)} \{1 - \mathscr{U}(s)\} \, ds \\ &\le -\frac{1}{3} \frac{c_1 + c_2}{g(a_*) - c_1} \log \left[\mathscr{U}_0 + (1 - \mathscr{U}_0) \mathscr{T}_{a_0}(\tilde{a}(\tau_0(\varepsilon)))^{-\frac{g(a_*) - c_1}{\gamma}} \right] \\ &\le -\frac{1}{3} \frac{c_1 + c_2}{g(a_*) - c_1} \log \mathscr{U}_0 =: M_1^+(\varepsilon). \end{aligned}$$

Regarding I_2, we have

$$I_2 \le -\frac{g(0) + c_1 + c_2}{3} \int_0^{\tau_0(\varepsilon)} \mathscr{W}(s) \, ds \le 0 =: M_2^+(\varepsilon).$$

Eliminating the negative term from the first line in (69), we find

$$I_3 \le \frac{f_1(a_*)}{3\gamma} \log \mathscr{T}_{a_0}(\tilde{a}(\tau_0(\varepsilon))) \le \frac{f_1(a_*)}{3\gamma} \log \mathscr{T}_0(\tilde{a}(\tau_0(\varepsilon))) =: M_3^+(\varepsilon),$$

where the first inequality is followed from the monotonicity of f_1, and it is strict for any $\varepsilon \in (0, f_1(a_*) - f_1(a_0))$. Setting $M^+(\varepsilon) := \sum_{i=1}^3 M_i^+(\varepsilon)$, we observe that $M^+(\varepsilon)$ is well-defined on $(0, f_1(a_*) - f_1(a_0)]$.

From the definition of M^- and M^+, we see that (59) and (61) hold true. Moreover, thanks to $\tilde{a}(\tau_0(\varepsilon)) = f_1^{-1}(f_1(a_*) - \varepsilon)$, we infer that M^- and M^+ can be extended on $(0, f_1(a_*) - f_1(0)]$ and (60) holds. This completes the proof. □

Lemma 4.4 *Let* $M^\pm : (0, f_1(a_*) - f_1(0)] \to \mathbb{R}$ *be the functions constructed by Lemma* 4.3. *Let* $(\tilde{u}_0, \widetilde{R}_0, \tilde{a}_0)$ *satisfy*

$$\int_0^1 \tilde{u}_0(\rho)\rho^2 \, d\rho \ge \int_0^1 u_0(\rho)\rho^2 \, d\rho, \tag{70}$$

$$\tilde{a}_0 \le \kappa_0, \tag{71}$$

and (IC), *where* κ_0 *is defined by* (68). *Then the solution of* (P0) *satisfies*

$$\widetilde{R}_0 \exp M^-(\varepsilon) \le \widetilde{R}(\tau_0(\varepsilon)) \le \widetilde{R}_0 \exp M^+(\varepsilon) \quad in \quad (0, f_1(a_*) - f_1(\tilde{a}_0)], \tag{72}$$

where the second inequality is strict for any $\varepsilon \in (0, f_1(a_*) - f_1(\tilde{a}_0))$.

Proof In the same manner as in the proof of Lemma 4.3, we see that (59) replaced (M^-, M^+, a_0) by $(\tilde{M}^-, \tilde{M}^+, \tilde{a}_0)$ holds true, where $\tilde{M}^-$ and $\tilde{M}^+$ are respectively determined by M^- and M^+, replaced (u_0, a_0) by $(\tilde{u}_0, \tilde{a}_0)$. Since (70) and (71) imply that

$$\tilde{\mathscr{U}}_0 := 3\int_0^1 \tilde{u}_0(\rho)\rho^2\,d\rho \geq 3\int_0^1 u_0(\rho)\rho^2\,d\rho = \mathscr{U}_0$$

and

$$\mathscr{T}_{\kappa_0}(\alpha) \leq \mathscr{T}_{\tilde{a}_0}(\alpha) \leq \mathscr{T}_0(\alpha) \quad \text{for any} \quad \alpha \in [\,0, a_*\,],$$

we find

$$\tilde{M}^+(\varepsilon) \leq M^+(\varepsilon), \quad \tilde{M}^-(\varepsilon) \geq M^-(\varepsilon), \quad \text{in} \quad (\,0,\, f_1(a_*) - f_1(0)\,].$$

Thus we obtain (72). □

In order to investigate the behavior of solutions of (P) with $S \equiv 1$, for each "initial data" $(\tilde{u}_0, \widetilde{R}_0, \tilde{a}_0)$, we consider the following system:

$$\begin{cases} \dfrac{d\tilde{a}}{dt}(t) = -\gamma\tilde{a}(t) & \text{in} \quad \mathbb{R}_+, \\ \partial_t\tilde{u}(\rho,t) - \mathscr{L}'(\tilde{v},\tilde{R})\tilde{u}(\rho,t) = P(\tilde{u}(\rho,t),\tilde{a}(t)) & \text{in} \quad I_\infty, \\ \tilde{v}(\rho,t) = \dfrac{1}{\rho^2}\displaystyle\int_0^\rho F(\tilde{u}(r,t),\tilde{a}(t))r^2\,dr & \text{in} \quad I_\infty, \\ \dfrac{d\widetilde{R}}{dt}(t) = \tilde{v}(1,t)\widetilde{R}(t) & \text{in} \quad \mathbb{R}_+, \\ \partial_\rho\tilde{u}(\rho,t)\big|_{\rho\in\{0,1\}} = 0, \quad \dfrac{\tilde{v}(\rho,t)}{\rho}\Big|_{\rho=0} = \dfrac{1}{3}F(\tilde{u}(0,t),\tilde{a}(t)), & \text{in} \quad \mathbb{R}_+, \\ \tilde{a}(0) = \tilde{a}_0, \;\; \tilde{u}(\rho,0) = \tilde{u}_0(\rho), \;\; \widetilde{R}(0) = \widetilde{R}_0, & \text{in} \quad I. \end{cases} \tag{P1}$$

We characterize the time variable in terms of the solution $\tilde{a}(\cdot)$ to (P1). Following the same manner as in (48) and recalling the monotonicity of f_1, we define a function $\tau_1 : (\,0,\, f_1(\tilde{a}_0) - f_1(0)\,] \to [\,0, \infty\,)$ as

$$\tau_1(\delta) = \tilde{a}^{-1}(f_1^{-1}(f_1(0)+\delta)). \tag{73}$$

Since $\tilde{a}(t) \downarrow 0$ as $t \to \infty$, $\delta \downarrow 0$ is equivalent to $\tau_1(\delta) \to \infty$.

Lemma 4.5 *Let* $(\tilde{u}_0, \widetilde{R}_0, \tilde{a}_0)$ *satisfy* (IC) *and*

$$\min_{\rho\in[\,0,1\,]} \tilde{u}_0(\rho) > 1 - \frac{g(0)+c_2}{g(0)+c_1+2c_2} =: 1 - C_g. \tag{74}$$

Then the solution $(\tilde{u}, \tilde{v}, \widetilde{R}, \tilde{a})$ *of* (P1) *satisfies*

$$\min_{\rho\in[\,0,1\,]} \tilde{u}(\rho,\tau_1(\delta)) \geq \min_{\rho\in[\,0,1\,]} \tilde{u}_0(\rho) \quad \textit{in} \quad (\,0,\, f_1(\tilde{a}_0) - f_1(0)\,].$$

Proof Recalling that (A2) and (74) respectively correspond to (41) and (42), we can construct the supersolution $\overline{w}$ of $\tilde{w} = 1 - \tilde{u}$ along the same argument as in the proof of Theorem 3.3. Using the change of variable $\tilde{a}(t) = z$, we have

$$\overline{w}(\tau_1(\delta)) \leq \left[\frac{1}{C_g} + \left[\frac{1}{\overline{w}(0)} - \frac{1}{C_g} \right] \left[\frac{\tilde{a}_0}{f_1^{-1}(f_1(0) + \delta)} \right]^{\frac{g(0)+c_2}{\gamma}} \right]^{-1} =: \Gamma_1(\delta)$$

for any $\delta \in (\, 0,\ f_1(\tilde{a}_0) - f_1(0)\,]$. Then $\underline{u} := 1 - \Gamma_1$ is a subsolution of $\tilde{u}$. In particular, the monotonicity of $\Gamma_1(\cdot)$ gives us the conclusion. □

Next we construct an analogue of Lemma 4.2 for (P1). To this aim, we note that

$$F(z, \alpha) = (c_1 + c_2) \left(z - K^*(\alpha)\right)^2 - (c_1 + c_2) K^*(\alpha) + f_2(\alpha),$$

where

$$K^*(\alpha) := \frac{-g(\alpha) + c_1 + c_2}{2(c_1 + c_2)}. \tag{75}$$

Lemma 4.6 *Let* $(\tilde{u}_0, \widetilde{R}_0, \tilde{a}_0)$ *satisfy* (IC), (74), *and the following*:

$$\min_{\rho \in [0,1]} \tilde{u}_0(\rho) \geq K^*(0); \tag{76}$$

$$\tilde{a}_0 > f_1^{-1}(0). \tag{77}$$

Then the solution $(\tilde{u}, \tilde{v}, \widetilde{R}, \tilde{a})$ *of* (P1) *satisfies*

$$\frac{d\widetilde{R}}{dt}(\tau_1(\delta)) < 0 \quad \textit{for any} \quad \delta \in (\, 0, -f_1(0)\,).$$

Proof In order to verify the sign of $d\widetilde{R}/dt$, we use a similar way in Lemma 4.2, i.e., focus on the sign of $F(\tilde{u}, \tilde{a})$. First we note that (77) is equivalent to $f_1(\tilde{a}_0) > 0$. Recalling the relation $(\, 0, -f_1(0)\,) \subset (\, 0,\ f_1(\tilde{a}_0) - f_1(0)\,]$, we find

$$F(1, \tilde{a}(\tau_1(\delta))) = f_1(\tilde{a}(\tau_1(\delta))) = f_1(0) + \delta < 0 \quad \text{in} \quad (\, 0, -f_1(0)\,). \tag{78}$$

Since (A2) implies $K^*(0) < 1$, the monotonicity of $K^*(\cdot)$ and (78) asserts that

$$F(z, \alpha) < 0 \quad \text{for any} \quad z \in [\, K^*(0), 1\,] \times (\, 0, -f_1(0)\,). \tag{79}$$

By virtue of (74), we can apply Lemma 4.5 to the solution $\tilde{u}$ and then (76) implies that

$$\min_{\rho\in[0,1]} \tilde{u}(\rho, \tau_1(\delta)) \ge \min_{\rho\in[0,1]} \tilde{u}_0(\rho) \ge K^*(0) \quad \text{for any} \quad \delta \in (0, f_1(\tilde{a}_0) - f_1(0)]. \tag{80}$$

Therefore we have completed the proof. □

Lemma 4.7 *Let* $(\tilde{u}_0, \widetilde{R}_0, \tilde{a}_0)$ *satisfy*

$$\min_{\rho\in[0,1]} \tilde{u}_0(\rho) \ge 1 - \frac{1}{2}C_g, \tag{81}$$

$$\tilde{a}_0 \ge f_1^{-1}(0), \tag{82}$$

and (IC). *Then there exist monotone increasing functions* L^- *and* L^+ *defined on the interval* $(0, f_1(a_*) - f_1(0)]$, *independent of* $(\tilde{u}_0, \widetilde{R}_0, \tilde{a}_0)$, *such that the solution of* (P1) *satisfies*

$$\widetilde{R}_0 \exp L^-(\delta) \le \widetilde{R}(\tau_1(\delta)) \le \widetilde{R}_0 \exp L^+(\delta) \quad in \quad (0, f_1(\tilde{a}_0) - f_1(0)], \tag{83}$$

in particular, the first inequality in (83) *is strict in* $(0, f_1(\tilde{a}_0) - f_1(0))$. *Moreover,* L^- *and* L^+ *satisfy the following*:

$$-\infty < L^-(\delta) \le L^+(\delta) < \infty \quad in \quad (0, f_1(a_*) - f_1(0)]; \tag{84}$$

$$\lim_{\delta\downarrow 0} L^+(\delta) = -\infty. \tag{85}$$

Proof Along the same line as in the proof of Lemma 4.3, we will estimate the following:

$$I_1 + I_2 + I_3 := (c_1 + c_2)\int_0^{\tau_1(\delta)} \int_0^1 \tilde{w}^2\rho^2\, d\rho \\ - \int_0^{\tau_1(\delta)} \int_0^1 (g(\tilde{a}(s)) + c_1 + c_2)\tilde{w}\rho^2\, d\rho ds + \frac{1}{3}\int_0^{\tau_1(\delta)} f_1(\tilde{a}(s))\, ds,$$

where $\tilde{w} = 1 - \tilde{u}$. First, since $I_1 \ge 0$, we set $L_1^-(\delta) \equiv 0$. Using the supersolution $\overline{w}$ of $\tilde{w}$ constructed in the proof of Theorem 3.3 and its estimate, we observe from (81) that

$$I_2 \ge -\frac{g(a_*) + c_1 + c_2}{3}\int_0^{\tau_1(\delta)} \overline{w}(s)\, ds \tag{86}$$

$$\ge -\frac{g(a_*) + c_1 + c_2}{3} C_g \int_0^{\tau_1(\delta)} \exp\left[-\int_0^s \{g(\tilde{a}(\tau')) + c_2\}\, d\tau'\right] ds.$$

Since the change of variable $\tilde{a}(\tau') = s'$ yields

$$-\int_0^s \{g(\tilde{a}(\tau')) + c_2\}\, d\tau' \le \frac{g(0)+c_2}{\gamma} \log \frac{\tilde{a}(s)}{\tilde{a}_0},$$

the inequality (86) is reduced to

$$\begin{aligned} I_2 &\ge -\frac{g(a_*)+c_1+c_2}{3} C_g \int_0^{\tau_1(\delta)} \left(\frac{\tilde{a}(s)}{\tilde{a}_0} \right)^{\frac{g(0)+c_2}{\gamma}} ds \\ &\ge -\frac{C_g}{3} \frac{g(a_*)+c_1+c_2}{g(0)+c_2} \left\{ 1 - \left(\frac{\tilde{a}(\tau_1(\delta))}{a_*} \right)^{\frac{g(0)+c_2}{\gamma}} \right\} =: L_2^-(\delta). \end{aligned}$$

Moreover, we find

$$\begin{aligned} I_3 &= \frac{-f_1(0)}{3\gamma} \log \frac{\tilde{a}(\tau_1(\delta))}{\tilde{a}_0} + \frac{1}{3\gamma} \int_{\tilde{a}(\tau_1(\delta))}^{\tilde{a}_0} f_1'(\tilde{z})\, dz \\ &\ge \frac{-f_1(0)}{3\gamma} \log \frac{\tilde{a}(\tau_1(\delta))}{a_*} =: L_3^-(\delta), \end{aligned} \tag{87}$$

where $\tilde{z} \in (0, \tilde{a}_0)$. The last inequality is followed from the monotonicity of f_1, and it is strict for any $\delta \in (\,0,\ f_1(\tilde{a}_0) - f_1(0)\,)$. Setting $L^-(\delta) := \sum_{i=1}^3 L_i^-(\delta)$ and recalling (73), we observe that L^- is well-defined on $(\,0,\ f_1(\tilde{a}_0) - f_1(0)\,]$.

Next, we derive L^+. By a similar argument as in the derivation of L_2^-, we obtain

$$\begin{aligned} I_1 &\le \frac{c_1+c_2}{3} \int_0^{\tau_1(\delta)} \overline{w}(s)^2\, ds \le \frac{c_1+c_2}{3} C_g^2 \int_0^{\tau_1(\delta)} \left(\frac{\tilde{a}(s)}{\tilde{a}_0} \right)^{2\frac{g(0)+c_2}{\gamma}} ds \\ &\le \frac{C_g}{6} \frac{c_1+c_2}{g(0)+c_1+2c_2} =: L_1^+(\delta). \end{aligned}$$

Since $I_2 \le 0$, we set $L_2^+(\delta) \equiv 0$. From the first equality in (87), we have

$$I_3 \le \frac{-f_1(0)}{3\gamma} \log \frac{\tilde{a}(\tau_1(\delta))}{f_1^{-1}(0)} + \frac{a_*}{3\gamma} \| f_1' \|_\infty =: L_3^+(\delta).$$

Setting $L^+(\delta) := \sum_{i=1}^3 L_i^+(\delta)$, we see that L^+ is well-defined on $(\,0,\ f_1(\tilde{a}_0) - f_1(0)\,]$.

From the definitions of L^- and L^+, it is clear that (83), (84), and (85) hold true. We have completed the proof. □

We are in the position to prove Theorem 1.1.

Proof of Theorem 1.1. To begin with, we prove the existence of switching solution of (IAS). The key of the proof is how to determine the appropriate thresholds r_0 and r_1. We divide the proof of the existence into 4 steps. Finally we shall prove a boundedness of the switching solution and its regularity.

Step 1: Fix $r_0 \in (\,0, \infty\,)$ arbitrarily. Let r_1 satisfy

$$r_1 \geq r_0 \exp\big[-L^-(-f_1(0))\big], \tag{88}$$

where remark that $L^-(-f_1(0)) < 0$. We claim the following: if $(\tilde{u}_0, \widetilde{R}_0, \tilde{a}_0)$ satisfies

$$\min_{\rho\in[0,1]} \tilde{u}_0(\rho) \geq \underline{\omega} := \max\{K^*(0), 1-\frac{1}{2}C_g\}, \quad \widetilde{R}_0 = r_1, \quad \tilde{a}_0 > f_1^{-1}(0), \tag{89}$$

and (IC), then there exists $\beta_1 \in (0, -f_1(0))$ such that the solution of (P1) satisfies

$$\widetilde{R}(\tau_1(\beta_1)) = r_0, \quad \frac{d\widetilde{R}}{dt}(\tau_1(\beta_1)) < 0. \tag{90}$$

Let $\delta_0 := f_1(\tilde{a}_0) - f_1(0)$, i.e., $\tilde{a}(\tau_1(\delta_0)) = \tilde{a}_0$. Remark that the third inequality in (89) yields $\delta_0 > -f_1(0)$. Since (89) allows us to apply Lemma 4.7, there exists $\beta_1' \in (0, \delta_0)$ such that

$$\widetilde{R}(\tau_1(\beta_1')) = r_0 \quad \text{and} \quad \widetilde{R}(\tau_1(\delta)) > r_0 \quad \text{for any} \quad \delta \in (\beta_1', \delta_0].$$

Moreover, we infer from (89) that Lemma 4.6 implies that

$$\frac{d\widetilde{R}}{dt}(\tau_1(\delta)) < 0 \quad \text{for any} \quad \delta \in (0, -f_1(0)).$$

Therefore it is sufficient to prove that $\beta_1' < -f_1(0)$. Then β_1' is nothing but the required constant β_1. Combining the relation (83) with (88), we have

$$r_0 = \widetilde{R}(\tau_1(\beta_1')) > r_1 \exp L^-(\beta_1') \geq r_0 \exp\big[L^-(\beta_1') - L^-(-f_1(0))\big].$$

Then the monotonicity of L^- yields $\beta_1' < -f_1(0)$.

Step 2: We shall show that, there exists $\varepsilon_1^* \in (0, f_1(a_*) - f_1(0))$ such that for any $r_0 \in (0, \infty)$ and $r_1 \geq r_0 \exp[M^+(\varepsilon_1^*)]$, the following holds: if $(\tilde{u}_0, \widetilde{R}_0, \tilde{a}_0)$ satisfies

$$\min_{\rho\in[0,1]} \tilde{u}_0(\rho) \geq \mathscr{U}_0 = 3\int_0^1 u_0(\rho)\rho^2\,d\rho, \quad \widetilde{R}_0 = r_0, \quad \tilde{a}_0 < f_1^{-1}(0), \tag{91}$$

and (IC), then there exists $\beta_2 \in (0, \varepsilon_1^*)$ such that the solution of (P0) satisfies

$$\widetilde{R}(\tau_0(\beta_2)) = r_1, \quad \frac{d\widetilde{R}}{dt}(\tau_0(\beta_2)) > 0. \tag{92}$$

Let $\varepsilon_0 := f_1(a_*) - f_1(\tilde{a}_0)$, i.e., $\tilde{a}(\tau_0(\varepsilon_0)) = \tilde{a}_0$. Remark that $\varepsilon_0 > f_1(a_*)$ by the third inequality in (91). By Lemma 4.4, there exists a constant $\beta_2' \in (0, \varepsilon_0)$ such that

$$\widetilde{R}(\tau_0(\beta_2')) = r_1 \quad \text{and} \quad \widetilde{R}(\tau_0(\varepsilon)) < r_1 \quad \text{for any} \quad \varepsilon \in (\beta_2', \varepsilon_0]. \tag{93}$$

We define ε_1^* as ε_1 in Lemma 4.2 with $A = \mathscr{U}_0$ and $\kappa = f_1^{-1}(0)$, i.e.,

$$1 - \Gamma_0(\varepsilon_1^*; \mathscr{U}_0, f_1^{-1}(0)) = z_0(f_1^{-1}(f_1(a_*) - \varepsilon_1^*)). \tag{94}$$

Then Lemma 4.2 asserts that $\varepsilon_1^* \in (0, f_1(a_*))$ and

$$\frac{d\widetilde{R}}{dt}(\tau_0(\varepsilon)) > 0 \quad \text{for any} \quad \varepsilon \in (0, \varepsilon_1^*].$$

Thus it is sufficient to prove that $\beta_2' \in (0, \varepsilon_1^*)$. Then β_2' is nothing but the required constant β_2. Letting r_1 satisfy

$$r_0 \exp M^+(\varepsilon_1^*) \leq r_1, \tag{95}$$

we show that $\beta_2' \in (0, \varepsilon_1^*)$. Indeed, since the relation (72) in Lemma 4.4 holds true, we observe from (95) that

$$r_0 \exp M^+(\varepsilon_1^*) \leq r_1 = \widetilde{R}(\tau_0(\beta_2')) < r_0 \exp M^+(\beta_2').$$

Then the monotonicity of M^+ clearly yields $\varepsilon_1^* > \beta_2'$.

Step 3: We shall prove that, there exists $\varepsilon_0^* \in (0, f_1(a_*) - f_1(0))$ such that for any $r_0 \in (0, \infty)$ and $r_1 \geq R_0 \exp M^+(\varepsilon_0^*)$, the following holds: if $(\tilde{u}_0, \widetilde{R}_0, \tilde{a}_0) = (u_0, R_0, a_0)$, then there exists $\beta_0 \in (0, \varepsilon_0^*)$ such that the solution of (P0) satisfies the following:

$$\widetilde{R}(\tau_0(\beta_0)) = r_1, \quad \frac{d\widetilde{R}}{dt}(\tau_0(\beta_0)) > 0; \tag{96}$$

$$\min_{\rho \in [0,1]} \tilde{u}(\rho, \tau_0(\beta_0)) \geq \max\{\underline{\omega}, \mathscr{U}_0\}, \quad \tilde{a}(\tau_0(\beta_0)) > f_1^{-1}(0). \tag{97}$$

Setting $\tilde{\varepsilon}_1$ as ε_1 in Lemma 4.2 with $A = \min_{\rho \in [0,1]} u_0(\rho)$ and $\kappa = a_0$, we have

$$\frac{d\widetilde{R}}{dt}(\tau_0(\varepsilon)) > 0 \quad \text{for any} \quad \varepsilon \in (0, \tilde{\varepsilon}_1] \quad \text{with} \quad \tilde{\varepsilon}_1 \in (0, f_1(a_*)).$$

By way of the function Γ_0^* defined by

$$\Gamma_0^*(\varepsilon) := \Gamma_0(\varepsilon; \min\{\min_{\rho \in [0,1]} u_0(\rho), \underline{\omega}\}, \max\{a_0, f_1^{-1}(0)\}),$$

we define $\tilde{\varepsilon}_2$ as follows:

$$1 - \Gamma_0^*(\tilde{\varepsilon}_2) = \max\{\underline{\omega}, \mathscr{U}_0, 1 - \Gamma_0^*(f_1(a_*) - f_1(0))\}. \tag{98}$$

From now on, we set $\varepsilon_0^* := \min\{\tilde{\varepsilon}_1, \tilde{\varepsilon}_2\}$ and let r_1 satisfy $r_1 \geq R_0 \exp M^+(\varepsilon_0^*)$. Let $\varepsilon_0' := f_1(a_*) - f_1(a_0)$, i.e., $\tilde{a}(\tau_0(\varepsilon_0')) = a_0$. With the aid of Lemma 4.3, we find a constant $\beta_0' \in (0, \varepsilon_0')$ such that (93) holds for $\beta_2' = \beta_0'$. Noting that the latter relation in (97) is equivalent to $\beta_0' < f_1(a_*)$ and recalling $\varepsilon_0^* \leq \tilde{\varepsilon}_1 < f_1(a_*)$, we have $\beta_0' < \varepsilon_0^*$. The same argument as in Step 2 implies that

$$R_0 \exp M^+(\varepsilon_0^*) \leq r_1 = \tilde{R}(\tau_0(\beta_0')) < R_0 \exp M^+(\beta_0'),$$

where the last inequality is followed from Lemma 4.3. Then the monotonicity of M^+ gives us the required relation. Finally we prove the former relation in (97). Thanks to the monotonicity of Γ_0, we observe from Lemma 4.1 that, for any $\varepsilon \in [\beta_0', \tilde{\varepsilon}_2]$,

$$\min_{\rho \in [0,1]} \tilde{u}(\rho, \tau_0(\varepsilon)) \geq 1 - \Gamma_0(\varepsilon; \min_{\rho \in [0,1]} u_0(\rho), a_0) \geq 1 - \Gamma_0^*(\varepsilon) \geq \max\{\underline{\omega}, \mathscr{U}_0\}.$$

Therefore β_0' is nothing but the required constant β_0.

Step 4: We shall prove that, for a suitable pair of theresholds (r_0, r_1), the system (P) has a unique solution with the property (i) in Theorem 1.1. Fix $r_0 \in (0, \infty)$ and let r_1 satisfy

$$r_1 \geq \max\left\{R_0 \exp M^+(\varepsilon_0^*), r_0 \exp M^+(\varepsilon_1^*), r_0 \exp\left[-L^-(-f_1(0))\right]\right\}. \tag{99}$$

We note that (99) yields $\max\{r_0, R_0\} < r_1$, for M^+ is positive in $(0, f_1(a_*) - f_1(0)]$.

With the aid of Step 3, there exist $\beta_0 \in (0, \varepsilon_0^*)$ and a unique solution $(\tilde{u}, \tilde{v}, \widetilde{R}, \tilde{a})$ of (P0) with $(\tilde{u}_0, \widetilde{R}_0, \tilde{a}_0) = (u_0, R_0, a_0)$ such that (96) and (97) hold. Since β_0 is uniquely determined, setting $(u, v, R, a) = (\tilde{u}, \tilde{v}, \widetilde{R}, \tilde{a})$ in $\bar{I} \times [0, t_1]$, we observe from (96) and the proof of Theorem 3.1 that (u, v, R, a, S) is a unique solution of (P) in $\bar{I} \times [0, \tau_0(\beta_0))$ such that $S(t) = 0$ in $[0, t_1)$ and $S(t)$ switches from 0 to 1 at t_1, where $t_1 := \tau_0(\beta_0)$.

Since (96)–(97) asserts that (89) holds for $(\tilde{u}_0, \widetilde{R}_0, \tilde{a}_0) = (u, R, a)|_{t=t_1}$, it follows from Step 1 that there exist $\beta_1 \in (0, -f_1(0))$ and a unique solution $(\tilde{u}_1, \tilde{v}_1, \widetilde{R}_1, \tilde{a}_1)$ of (P1), with $(\tilde{u}_0, \widetilde{R}_0, \tilde{a}_0) = (u, R, a)|_{t=t_1}$, satisfying (90). Since β_1 is uniquely determined, setting $(u, v, R, a) = (\tilde{u}_1, \tilde{v}_1, \widetilde{R}_1, \tilde{a}_1)$ in $\bar{I} \times [t_1, t_2]$ and $S(t) = 1$ in $[t_1, t_2)$, we deduce from (90) and the proof of Theorem 3.1 that (u, v, R, a, S) is a unique solution of (P) in $\bar{I} \times [0, t_2)$ satisfying the following: $S(t) = 1$ in $[t_1, t_2)$; $S(t)$ switches from 1 to 0 at t_2, where t_2 is the time determined by $\tau_1(\beta_1)$.

Here we claim that (91) holds for $(\tilde{u}_0, \widetilde{R}_0, \tilde{a}_0) = (u, R, a)|_{t=t_2}$. Since (96)–(97) implies that $\min_{\rho \in [0,1]} u(\rho, t_1) \geq \max\{\underline{\omega}, \mathscr{U}_0\}$, we infer from Lemma 4.5 that

$$\min_{\rho \in [0,1]} u(\rho, t_2) \geq \max\{\underline{\omega}, \mathscr{U}_0\}.$$

Thus the claim holds true. Then it follows from Step 2 that there exist $\beta_2 \in (0, \varepsilon_1^*)$ and a unique solution $(\tilde{u}_2, \tilde{v}_2, \widetilde{R}_2, \tilde{a}_2)$ of (P0), with $(\tilde{u}_0, \widetilde{R}_0, \tilde{a}_0) = (u, R, a)|_{t=t_2}$, satisfying (92). Thanks to the uniqueness of β_2, setting

$$(u, v, R, a) = (\tilde{u}_2, \tilde{v}_2, \widetilde{R}_2, \tilde{a}_2) \quad \text{in} \quad \bar{I} \times [\, t_2, t_3\,],$$

where t_3 is the time determined by $\tau_0(\beta_2)$, we deduce from the same argument as above that (u, v, R, a, S) is a unique solution of (P) in $\bar{I} \times [\, 0, t_3\,)$ satisfying the following: $S(t) = 1$ in $[\, t_2, t_3\,)$; $S(t)$ switches from 0 to 1 at t_3.

In order to apply Step 1 again, we verify that $u(\cdot, t_3)$ satisfies the first property in (89). Combining Lemma 4.4 with (99), we see that

$$R_0 \exp M^+(\varepsilon_0^*) \le r_1 = \widetilde{R}_2(\tau_0(\beta_2)) < r_0 \exp M^+(\beta_2) \le R_0 \exp M^+(\beta_2), \quad (100)$$

and then the monotonicity of M^+ yields $\varepsilon_0^* > \beta_2$. Recalling the monotonicity of Γ_0 and using Lemma 4.1, we have for any $\varepsilon \in [\, \beta_2, \varepsilon_0^*\,]$

$$\begin{aligned}\min_{\rho \in [\,0,1\,]} u(\rho, \tau_0(\varepsilon)) &\ge 1 - \Gamma_0(\varepsilon; \max\{\underline{\omega}, \mathscr{U}_0\}, f_1^{-1}(0)) \\ &\ge 1 - \Gamma_0^*(\varepsilon) \ge 1 - \Gamma_0^*(\varepsilon_0^*) \ge \max\{\underline{\omega}, \mathscr{U}_0\}.\end{aligned}$$

Thus Step 1 is applicable again. Therefore we can construct inductively a solution of (P) with the property (i) in Theorem 1.1.

Step 5: We prove the property (ii) in Theorem 1.1. Using the sequence $\{t_j\}_{j=0}^{\infty}$ obtained by Step 4, we inductively define sequences $\{\varepsilon_0^{2j}\}_{j=0}^{\infty}$, $\{\delta_0^{2j+1}\}_{j=0}^{\infty}$, and $\{\beta_j\}_{j=0}^{\infty}$. Let $\varepsilon_0^0 := f_1(a_*) - f_1(a_0)$, i.e., $\tau_0(\varepsilon_0^0) = t_0 = 0$. Set

$$\beta_0 := f_1(a_*) - f_1(a(t_1)). \quad (101)$$

By the definition of τ_0, the relation (101) is equivalent to $a(\tau_0(\beta_0)) = a(t_1)$. We set

$$\delta_0^1 := f_1(a_*) - f_1(0) - \beta_0.$$

The definitions of τ_0 and τ_1 yield $a(\tau_1(\delta_0^1)) = a(\tau_0(\beta_0))$. Since $a(\cdot)$ is monotone in $[\, 0, t_1\,]$, it holds that $\tau_0(\beta_0) = t_1 = \tau_1(\delta_0^1)$. Next we set

$$\beta_1 := f_1(a(t_2)) - f_1(0); \quad (102)$$

$$\varepsilon_0^2 := f_1(a_*) - f_1(0) - \beta_1. \quad (103)$$

Then, from (102) and (103), we find $a(\tau_1(\beta_1)) = a(t_2)$ and $a(\tau_0(\varepsilon_0^2)) = a(\tau_1(\beta_1))$. The monotonicity of $a(\cdot)$ in $[\, t_1, t_2\,]$ gives us the relation $\tau_1(\beta_1) = \tau_0(\varepsilon_0^2)$. Along the same manner as above, we define inductively ε_0^{2j}, δ_0^{2j+1}, and β_j for each $j \ge 2$ as follows:

$$\beta_j := \begin{cases} f_1(a_*) - f_1(a(t_{j+1})) & \text{if } j \text{ is even}, \\ f_1(a(t_{j+1})) - f_1(0) & \text{if } j \text{ is odd}, \end{cases}$$

$$\delta_0^{2j-1} := f_1(a_*) - f_1(0) - \beta_{2j-2}, \quad \varepsilon_0^{2j} := f_1(a_*) - f_1(0) - \beta_{2j-1}.$$

We note that the monotonicity of $a(\cdot)$ in $[\, t_j, t_{j+1}\,]$ implies $\tau_0(\beta_{2j}) = \tau_1(\delta_0^{2j+1})$ and $\tau_1(\beta_{2j+1}) = \tau_0(\varepsilon_0^{2j+2})$ for each $j \in \mathbb{N} \cup \{0\}$. Then, it follows from the definitions of the sequences that, for any $j \in \mathbb{N} \cup \{0\}$,

$$R(\tau_0(\beta_{2j})) = r_1, \quad R(\tau_0(\varepsilon)) < r_1 \quad \text{and} \quad S(\tau_0(\varepsilon)) \equiv 0, \quad \text{on} \quad (\,\beta_{2j}, \varepsilon_0^{2j}\,]; \tag{104}$$

$$R(\tau_1(\beta_{2j+1})) = r_0, \quad R(\tau_1(\delta)) > r_0 \quad \text{and} \quad S(\tau_1(\delta)) \equiv 1, \quad \text{on} \quad (\,\beta_{2j+1}, \delta_0^{2j+1}\,]. \tag{105}$$

We give the lower and upper bounds of R when $S \equiv 0$, i.e., for the case of (104). We note that, for the case of $j = 0$, it clearly follows from Lemma 4.3 that

$$R_0 \exp M^-(f_1(a_*) - f_1(a_0)) \le R(\tau_0(\varepsilon)) < r_1 \quad \text{on} \quad (\,\beta_0, \varepsilon_0^0\,], \tag{106}$$

where the first inequality was obtained by the monotonicity of M^-. For any $j \in \mathbb{N}$, we observe from Lemma 4.3 that

$$r_0 \exp M^-(\varepsilon_0^{2j}) \le r_0 \exp M^-(\varepsilon) \le R(\tau_0(\varepsilon)) < r_1 \quad \text{on} \quad (\,\beta_{2j}, \varepsilon_0^{2j}\,]. \tag{107}$$

Here, by (105) and Lemma 4.7, we find $\log(r_0/r_1) \le L^+(\beta_{2j-1})$. Since $L^+(\delta)$ is monotone and diverges to $-\infty$ as $\delta \downarrow 0$, there exists $\hat{\delta} \in (\,0, \beta_{2j-1}\,]$, independent of j, such that $L^+(\hat{\delta}) = \log(r_0/r_1)$. Thus, setting $\hat{\varepsilon} := f_1(a_*) - f_1(0) - \hat{\delta}$, we obtain

$$f_1(a_*) - \hat{\varepsilon} = f_1(0) + \hat{\delta} \le f_1(0) + \beta_{2j-1} = f_1(a_*) - \varepsilon_0^{2j}, \quad \text{i.e.,} \quad \hat{\varepsilon} \ge \varepsilon_0^{2j}. \tag{108}$$

Since $j \in \mathbb{N}$ is arbitral, we observe from (107) and (108) that

$$r_0 \exp M^-(\hat{\varepsilon}) \le R(\tau_0(\varepsilon)) < r_1 \quad \text{on} \quad (\,\beta_{2j}, \varepsilon_0^{2j}\,] \quad \text{for any} \quad j \in \mathbb{N}. \tag{109}$$

In particular, we see that

$$r_0 = R(\tau_0(\varepsilon_0^{2j})) \ge r_0 \exp M^-(\varepsilon_0^{2j}) \ge r_0 \exp M^-(\hat{\varepsilon}). \tag{110}$$

Next, we derive the lower and upper bounds of R when $S \equiv 1$, i.e., for the case of (105). For any $j \in \mathbb{N} \cup \{0\}$, we observe from (105) and Lemma 4.7 that

$$r_0 < R(\tau_1(\delta)) \le r_1 \exp L^+(\delta) \le r_1 \exp L^+(\delta_0^{2j+1}) \quad \text{on} \quad (\,\beta_{2j+1}, \delta_0^{2j+1}\,], \tag{111}$$

where the last inequality was followed from the monotonicity of L^+. Here, it follows from (104) and Lemma 4.3 that $M^-(\beta_{2j}) \le \log(r_1/\min\{R_0, r_0\})$. Since $M^-(\varepsilon)$ is monotone and diverges to ∞ as $\varepsilon \downarrow 0$, there exists $\bar{\varepsilon} \in (\,0, \beta_{2j}\,]$, independent of j, such that $M^-(\bar{\varepsilon}) = \log(r_1/\min\{R_0, r_0\})$. Setting $\bar{\delta} := f_1(a_*) - f_1(0) - \bar{\varepsilon}$, we deduce from a similar argument as in (108) that the relation $\bar{\delta} \ge \delta_0^{2j+1}$ holds. Combining the fact with (111), we have

$$r_0 < R(\tau_1(\delta)) \le r_1 \exp L^+(\bar{\delta}) \quad \text{on} \quad (\beta_{2j+1}, \delta_0^{2j+1}] \quad \text{for any} \quad j \in \mathbb{N} \cup \{0\}. \tag{112}$$

In particular, we see that

$$r_1 = R(\tau_1(\delta_0^{2j+1})) \le r_1 \exp L^+(\delta_0^{2j+1}) \le r_1 \exp L^+(\bar{\delta}). \tag{113}$$

Consequently, by virtue of (106), (109)–(110), and (112)–(113), we conclude that the property (ii) in Theorem 1.1 holds for

$$C_1 = \min\{R_0 \exp M^-(f_1(a_*) - f_1(a_0)), r_0 \exp M^-(\hat{\varepsilon})\}, \quad C_2 = r_1 \exp L^+(\bar{\delta}).$$

Step 6: Finally we prove the regularity of the switching solution constructed by the above arguments. The equation of a implies

$$\left| \frac{da}{dt}(t) \right| = |\gamma(a_* - a(t)) - \gamma a_* S(t)| \le 2\gamma a_* \quad \text{in} \quad [0, \infty) \setminus \{t_j\}_{j=0}^{\infty}.$$

Fix $j \in \mathbb{N}$ arbitrarily. Then, for any t and s with $t_{j-1} \le t < t_j < s < t_{j+1}$, we have

$$\begin{aligned} |a(t) - a(s)| &\le |a(t) - a(t_j)| + |a(t_j) - a(s)| \\ &= \left| \frac{da}{dt}(\tau_1) \right| |t - t_j| + \left| \frac{da}{dt}(\tau_2) \right| |t_j - s| \le 4\gamma a_* |t - s|, \end{aligned} \tag{114}$$

where $\tau_1 \in (t, t_j)$ and $\tau_2 \in (t_j, s)$. Since j is arbitrary, we see that $a \in C^{0,1}(\mathbb{R}_+)$.

We consider the following initial boundary problem:

$$\begin{cases} \partial_t \tilde{u}(\rho, t) - \mathscr{L}'(\tilde{v}, \tilde{R})\tilde{u} = P(\tilde{u}(\rho, t), a(t)) & \text{in } I_\infty, \\ \tilde{v}(\rho, t) = \dfrac{1}{\rho^2} \displaystyle\int_0^\rho F(\tilde{u}(r, t), a(t)) r^2 \, dr & \text{in } I_\infty, \\ \dfrac{d\tilde{R}}{dt}(t) = \tilde{v}(1, t)\tilde{R}(t) & \text{in } \mathbb{R}_+, \\ \partial_\rho \tilde{u}(0, t) = \partial_\rho \tilde{u}(1, t) = 0, \quad \dfrac{\tilde{v}}{\rho}\Big|_{\rho=0} = \dfrac{1}{3} F(\tilde{u}(0, t), a(t)) & \text{in } \mathbb{R}_+, \\ \tilde{u}(\rho, 0) = u_0(\rho), \quad \tilde{R}(0) = R_0, & \text{in } I. \end{cases} \tag{$\mathscr{P}$}$$

Since $a \in C^{0,1}(\mathbb{R}_+)$, the proofs of Lemma 2.3 and Theorem 3.1 indicate that ($\mathscr{P}$) has a unique solution $(\tilde{u}, \tilde{v}, \tilde{R})$ in the class

$$C^{2+\alpha, 1+\frac{\alpha}{2}}(Q_\infty) \times (C^{1+\alpha, \frac{\alpha}{2}}([0, 1) \times \mathbb{R}_+) \cap C^1([0, 1) \times \mathbb{R}_+)) \times C^1(\mathbb{R}_+).$$

Recalling that (u, v, R), which is obtained by Step 4, also satisfies ($\mathscr{P}$), we observe from the uniqueness that $(\tilde{u}, \tilde{v}, \tilde{R}) = (u, v, R)$ in Q_∞. We obtain the conclusion. □

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Symmetry Problems on Stationary Isothermic Surfaces in Euclidean Spaces

Shigeru Sakaguchi

Abstract Let S be a smooth hypersurface properly embedded in $\mathbb{R}^N$ with $N \geq 3$ and consider its tubular neighborhood $\mathcal{N}$. We show that, if a heat flow over $\mathcal{N}$ with appropriate initial and boundary conditions has S as a stationary isothermic surface, then S must have some sort of symmetry.

Keywords Heat equation · Cauchy problem · Initial-boundary value problem · Tubular neighborhood · Stationary isothermic surface · Symmetry

1 Introduction

The stationary isothermic surfaces of solutions of the heat equation have been much studied, and it has been shown that the existence of a stationary isothermic surface forces the problems to have some sort of symmetry (see [5, 6, 8, 9, 11–13, 15]). A balance law for stationary zeros of temperature introduced by [7] plays a key role in the proofs. To be more precise, the balance law gives us that for any pair of points x and y in the stationary isothermic surface the heat contents of two balls centered at x and y respectively with an equal radius are equal for every time. The above papers always deal with the cases where each ball touches the boundary only at one point eventually. Then by studying the initial behavior of the heat content of each ball the authors extract some information of the principal curvatures of the boundary at the touching point.

We emphasize that in the present paper we deal with the cases where each ball touches the boundary exactly at two points. Another new point is to give simply a C^2 hypersurface properly embedded in $\mathbb{R}^N$ as a candidate for a stationary isothermic surface from the beginning.

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Let us establish our setting. Let Ω be a C^2 domain in $\mathbb{R}^N$ with $N \geq 3$, whose boundary $\partial\Omega$ is connected and not necessarily bounded. Namely, $\partial\Omega$ is a C^2 hypersurface properly embedded in $\mathbb{R}^N$. Assume that there exists a number $R > 0$ satisfying:

(A-1) : The principal curvatures $\kappa_1(x), \dots, \kappa_{N-1}(x)$ of $\partial\Omega$ at $x \in \partial\Omega$ with respect to the outward normal direction to $\partial\Omega$ satisfy

$$\max_{1 \leq j \leq N-1} |\kappa_j(x)| < \frac{1}{R} \quad \text{for every } x \in \partial\Omega.$$

(A-2) : The tubular neighborhood $\mathcal{N}_R$ of $\partial\Omega$ given by

$$\mathcal{N}_R = \{x \in \mathbb{R}^N : \ \mathrm{dist}(x, \partial\Omega) < R\},$$

is a C^2 domain in $\mathbb{R}^N$ and its boundary $\partial\mathcal{N}_R$ consists of two connected components Γ_+, Γ_- each of which is diffeomorphic to $\partial\Omega$.

Let us introduce two C^2 domains Ω_+, Ω_- in $\mathbb{R}^N$ with $\partial\Omega_+ = \Gamma_+, \partial\Omega_- = \Gamma_-$, respectively, such that the three domains $\Omega_+, \Omega_-, \mathcal{N}_R$ are disjoint, $\Omega_- \subset \Omega$, and $\Omega_+ \cup \Omega_- \cup \overline{\mathcal{N}_R} = \mathbb{R}^N$. Denote by $\mathcal{X}_{\Omega_+}, \mathcal{X}_{\Omega_-}$ the characteristic functions of the sets Ω_+, Ω_-, respectively. Consider the following initial-boundary value problem for the heat equation:

$$u_t = \Delta u \quad \text{in } \mathcal{N}_R \times (0, +\infty), \tag{1}$$
$$u = 1 \ \text{ on } \partial\mathcal{N}_R \times (0, +\infty), \tag{2}$$
$$u = 0 \ \text{ on } \mathcal{N}_R \times \{0\}, \tag{3}$$

and the Cauchy problem for the heat equation:

$$u_t = \Delta u \quad \text{in } \mathbb{R}^N \times (0, +\infty) \ \text{ and } \ u = \mathcal{X}_{\Omega_+} + \mathcal{X}_{\Omega_-} \ \text{ on } \mathbb{R}^N \times \{0\}. \tag{4}$$

We have the following theorem.

Theorem 1 *Let $N = 3$ and let u be the unique bounded solution of either problem (1)–(3) or problem (4). Assume that there exists a function $a(t)$ satisfying*

$$u(x, t) = a(t) \ \text{ for every } (x, t) \in \partial\Omega \times (0, +\infty). \tag{5}$$

Then, $\partial\Omega$ must be either a plane or a sphere, provided at least one of the following conditions is satisfied:

(a) *$\partial\Omega$ has an umbilical point $p \in \partial\Omega$, that is, $\kappa_1(p) = \kappa_2(p)$.*
(b) *There exists a sequence of points $\{p_j\} \subset \partial\Omega$ with $\lim_{j\to\infty} \kappa_1(p_j) = \lim_{j\to\infty} \kappa_2(p_j) \in \mathbb{R}$.*

When $\partial\Omega$ is bounded, the Hopf-Poincaré theorem [4, Theorem II, p. 113] says that the sum of the indices of all the isolated umbilical points equals the Euler number $\chi(\partial\Omega)(= 2 - 2 \times \text{genus})$ of $\partial\Omega$ and hence if the genus of $\partial\Omega$ does not equal 1 then $\partial\Omega$ must have at least one umbilical point. Therefore we have the following direct corollary.

Corollary 1 *Let $N = 3$ and let u be the unique bounded solution of either problem (1)–(3) or problem (4). Assume that (5) holds for some function $a(t)$. Then, if $\partial\Omega$ is bounded and the genus of $\partial\Omega$ does not equal 1, $\partial\Omega$ must be a sphere.*

We next consider the following initial-boundary value problem for the heat equation:

$$u_t = \Delta u \quad \text{in } \mathcal{N}_R \times (0, +\infty), \tag{6}$$

$$u = 1 \ \text{ on } \Gamma_+ \times (0, +\infty), \tag{7}$$

$$u = -1 \ \text{ on } \Gamma_- \times (0, +\infty), \tag{8}$$

$$u = 0 \ \text{ on } \mathcal{N}_R \times \{0\}, \tag{9}$$

and the Cauchy problem for the heat equation:

$$u_t = \Delta u \quad \text{in } \mathbb{R}^N \times (0, +\infty) \ \text{ and } \ u = \mathcal{X}_{\Omega_+} - \mathcal{X}_{\Omega_-} \ \text{ on } \mathbb{R}^N \times \{0\}. \tag{10}$$

Then we have

Theorem 2 *Let $N \geq 3$ and let u be the unique bounded solution of either problem (6)–(9) or problem (10). Assume that (5) holds for some function $a(t)$. Then:*

(a) *If $\partial\Omega$ is bounded, $\partial\Omega$ must be a sphere.*
(b) *If $N = 3$ and $\partial\Omega$ is an entire graph over $\mathbb{R}^2$, $\partial\Omega$ must be a plane.*

By using the asymptotic formula of the heat content $\int_{B_R(x)} u(z, t)\, dz$ of an open ball $B_R(x)$ with radius $R > 0$ centered at $x \in \partial\Omega$ as $t \to +0$ introduced in [10] together with the balance law given in [7], we prove Theorems 1 and 2. Moreover Aleksandrov's sphere theorem and Bernstein's theorem for the minimal surface equation are needed to prove Theorem 2. In Sects. 2 and 3, we prove Theorems 1 and 2, respectively. The final Sect. 4 gives several remarks and problems.

2 Proof of Theorem 1

The proofs of Theorems 1 and 2 have common ingredients. Therefore we begin with general dimensions N for later use, although Theorem 1 assumes that $N = 3$.

Let u be the unique bounded solution of either problem (1)–(3) or problem (4). Denote by $u^{\pm} = u^{\pm}(x, t)$ the unique bounded solutions of the initial-boundary value problems for the heat equation:

$$u_t = \Delta u \quad \text{in} \quad \left(\mathbb{R}^N \setminus \overline{\Omega_\pm}\right) \times (0, +\infty), \tag{11}$$

$$u = 1 \ \text{ on } \Gamma_\pm \times (0, +\infty), \tag{12}$$

$$u = 0 \ \text{ on } \ \left(\mathbb{R}^N \setminus \overline{\Omega_\pm}\right) \times \{0\}, \tag{13}$$

respectively, or of the Cauchy problems for the heat equation:

$$u_t = \Delta u \quad \text{in } \mathbb{R}^N \times (0, +\infty) \ \text{ and } \ u = \mathscr{X}_{\Omega_\pm} \ \text{ on } \mathbb{R}^N \times \{0\}, \tag{14}$$

respectively. Notice that $u = u^+ + u^-$ when u is the solution of problem (4). Then, by a result of Varadhan [16] (see also [13, Theorem A, p. 2024]), we see that

$$-4t \log\left(u^\pm(x,t)\right) \to \ \text{dist}(x, \Gamma_\pm)^2 \ \text{ as } t \to +0 \tag{15}$$

uniformly on every compact sets in $\mathbb{R}^N \setminus \overline{\Omega_\pm}$.

By the assumptions (A-1) and (A-2), every point $x \in \partial\Omega$ determines two points $x_+ \in \Gamma_+$ and $x_- \in \Gamma_-$ satisfying

$$\partial B_R(x) \cap \Gamma_+ = \{x_+\} \ \text{ and } \partial B_R(x) \cap \Gamma_- = \{x_-\},$$

respectively. Moreover, by letting $\kappa_1^\pm(x_\pm), \dots, \kappa_{N-1}^\pm(x_\pm)$ denote the principal curvatures of $\Gamma_\pm$ at $x_\pm$ with respect to the inward normal direction to $\partial\mathscr{N}_R$, respectively, we observe that

$$1 - R\kappa_j^+(x_+) = \frac{1}{1 - R\kappa_j(x)} > 0 \ \text{ and } \ 1 - R\kappa_j^-(x_-) = \frac{1}{1 + R\kappa_j(x)} > 0 \tag{16}$$

for every $x \in \partial\Omega$ and every $j = 1, \dots, N-1$.

On the other hand, it follows from the balance law (see [7, Theorem 4, p. 704] or [8, Theorem 2.1, pp. 934–935]) that (5) gives

$$\int_{B_R(x)} u(z,t)\,dz = \int_{B_R(y)} u(z,t)\,dz \ \text{ for } t > 0 \tag{17}$$

for every $x, y \in \partial\Omega$. Moreover, by virtue of (16), an asymptotic formula given by [10] (see also [13, Theorem B, pp. 2024–2025]) yields that

$$\lim_{t \to +0} t^{-\frac{N+1}{4}} \int_{B_R(x)} u^\pm(z,t)\,dz = c(N) \left\{ \prod_{j=1}^{N-1} \left[\frac{1}{R} - \kappa_j^\pm(x_\pm) \right] \right\}^{-\frac{1}{2}}, \tag{18}$$

respectively. Here, $c(N)$ is a positive constant depending only on N and of course $c(N)$ depends on the problems (11)–(13) or (14). Then we have

Lemma 1 *Let u be the unique bounded solution of either problem* (1)–(3) *or problem* (4)*. Assume that* (5) *holds for some function $a(t)$. Then there exists a constant $c > 0$ satisfying*

$$\left\{\prod_{j=1}^{N-1}(1 - R\kappa_j(x))\right\}^{\frac{1}{2}} + \left\{\prod_{j=1}^{N-1}(1 + R\kappa_j(x))\right\}^{\frac{1}{2}} = c \text{ for every } x \in \partial\Omega, \quad (19)$$

where $\kappa_1(x), \dots, \kappa_{N-1}(x)$ denote the principal curvatures of $\partial\Omega$ given in (A-1).

Proof Let u be the unique bounded solution of problem (4). Then we have that $u = u^+ + u^-$. Hence, combining (17) with (18) yields that there exists a constant $c > 0$ satisfying

$$\left\{\prod_{j=1}^{N-1}\left(1 - R\kappa_j^+(x_+)\right)\right\}^{-\frac{1}{2}} + \left\{\prod_{j=1}^{N-1}\left(1 - R\kappa_j^-(x_-)\right)\right\}^{-\frac{1}{2}} = c \quad (20)$$

for every $x \in \partial\Omega$. Therefore (16) gives the conclusion.

Let u be the solution of problem (1)–(3). It follows from the comparison principle that

$$\max\{u^+, u^-\} \le u \le u^+ + u^- \quad \text{in } \mathcal{N}_R \times (0, \infty).$$

Therefore, in view of (15) and (18), we notice that for every $x \in \partial\Omega$

$$\begin{aligned}
&c(N)\left\{\prod_{j=1}^{N-1}\left[\frac{1}{R} - \kappa_j^+(x_+)\right]\right\}^{-\frac{1}{2}} + c(N)\left\{\prod_{j=1}^{N-1}\left[\frac{1}{R} - \kappa_j^-(x_-)\right]\right\}^{-\frac{1}{2}} \\
&= \lim_{t\to+0} t^{-\frac{N+1}{4}} \int_{B_R(x)} u^+(z,t)\,dz + \lim_{t\to+0} t^{-\frac{N+1}{4}} \int_{B_R(x)} u^-(z,t)\,dz \\
&= \lim_{t\to+0} t^{-\frac{N+1}{4}} \int_{B_R(x)\setminus\Omega} u^+(z,t)\,dz + \lim_{t\to+0} t^{-\frac{N+1}{4}} \int_{B_R(x)\cap\Omega} u^-(z,t)\,dz \\
&= \lim_{t\to+0} t^{-\frac{N+1}{4}} \int_{B_R(x)\setminus\Omega} u(z,t)\,dz + \lim_{t\to+0} t^{-\frac{N+1}{4}} \int_{B_R(x)\cap\Omega} u(z,t)\,dz \\
&= \lim_{t\to+0} t^{-\frac{N+1}{4}} \int_{B_R(x)} u(z,t)\,dz.
\end{aligned}$$

Hence, with the aid of (17), we obtain (20) which yields the conclusion by (16). □

Proof of Theorem 1: Set $N = 3$ in (19). With the aid of the arithmetic-geometric mean inequality, we obtain from (19) that

$$
\begin{aligned}
c &= \sqrt{(1 - R\kappa_1)(1 - R\kappa_2)} + \sqrt{(1 + R\kappa_1)(1 + R\kappa_2)} \\
&\le \frac{2 - R(\kappa_1 + \kappa_2)}{2} + \frac{2 + R(\kappa_1 + \kappa_2)}{2} = 2
\end{aligned}
$$

where $\kappa_j = \kappa_j(x)$ with $j = 1, 2$. By the assumption, $\partial\Omega$ has an umbilical point $p \in \partial\Omega$, that is, $\kappa_1(p) = \kappa_2(p)$, or there exists a sequence of points $\{p_j\} \subset \partial\Omega$ with $\lim_{j\to\infty} \kappa_1(p_j) = \lim_{j\to\infty} \kappa_2(p_j) \in \mathbb{R}$. Then we conclude that $c = 2$ and the equality holds in the above inequality. Hence $\kappa_1 = \kappa_2$ on $\partial\Omega$, that is, $\partial\Omega$ is called totally umbilical. Thus from classical results in differential geometry $\partial\Omega$ must be either a plane or a sphere(see [4, Remark, p. 124] or [14, Theorem 3.30, p. 84] for instance). □

3 Proof of Theorem 2

Let us use the auxiliary functions $u^{\pm} = u^{\pm}(x, t)$ given in Sect. 2. We begin with the following lemma:

Lemma 2 *Let u be the unique bounded solution of either problem* (6)–(9) *or problem* (10). *Assume that* (5) *holds for some function $a(t)$. Then there exists a constant c satisfying*

$$
\left\{\prod_{j=1}^{N-1}(1 - R\kappa_j(x))\right\}^{\frac{1}{2}} - \left\{\prod_{j=1}^{N-1}(1 + R\kappa_j(x))\right\}^{\frac{1}{2}} = c \ \text{ for every } x \in \partial\Omega, \tag{21}
$$

where $\kappa_1(x), \dots, \kappa_{N-1}(x)$ denote the principal curvatures of $\partial\Omega$ given in (A-1).

Proof Let u be the solution of problem (10). Then we have that $u = u^+ - u^-$. Therefore the conclusion follows from the same argument as in the proof of Lemma 1.

Let u be the solution of problem (6)–(9). It follows from the comparison principle that

$$
\max\{-u^-, u^+ - 2u^-\} \le u \le \min\{u^+, 2u^+ - u^-\} \ \text{ in } \mathcal{N}_R \times (0, \infty).
$$

With the aid of these inequalities, in view of (15) and (18), by carrying out calculations similar to those in the proof of Lemma 1 for every $x \in \partial\Omega$, we can reach the conclusion. □

Proof of Theorem 2: Set

$$-\Phi(\kappa_1, \dots, \kappa_{N-1}) = \text{ the left-hand side of (21).}$$

Then we have that $\frac{\partial \Phi}{\partial \kappa_j} > 0$ for $j = 1, \dots, N-1$. Therefore, by introducing local coordinates, the condition $\Phi(\kappa_1, \dots, \kappa_{N-1}) =$ constant on the surface $\partial\Omega$ can be converted into a second order partial differential equation which is of elliptic type. Hence, if $\partial\Omega$ is bounded, then $\partial\Omega$ must be a sphere by Aleksandrov's sphere theorem [1]. Thus proposition (a) is proved.

Let us proceed to proposition (b). Set $N = 3$ in (21). Then

$$\sqrt{(1 - R\kappa_1)(1 - R\kappa_2)} - \sqrt{(1 + R\kappa_1)(1 + R\kappa_2)} = c, \tag{22}$$

where $\kappa_j = \kappa_j(x)$ with $j = 1, 2$, and hence

$$-4RH = c\left(\sqrt{(1 - R\kappa_1)(1 - R\kappa_2)} + \sqrt{(1 + R\kappa_1)(1 + R\kappa_2)}\right), \tag{23}$$

where $H = \frac{1}{2}(\kappa_1 + \kappa_2)$ is the mean curvature of $\partial\Omega$. We distinguish three cases:

$$\text{(i) } c = 0, \;\; \text{(ii) } c > 0, \;\; \text{(iii) } c < 0.$$

In case (i), by (23) we have $H = 0$ on $\partial\Omega$ and hence $\partial\Omega$ is the minimal entire graph of a function over $\mathbb{R}^2$. Therefore, by Bernstein's theorem for the minimal surface equation, $\partial\Omega$ must be a plane. This gives the conclusion desired. (See [2, 3] for Bernstein's theorem.) In case (ii), by (23) we have $H < 0$ on $\partial\Omega$. Suppose that there exists a sequence of points $\{p_n\}$ with $\lim_{n\to\infty} H(p_n) = 0$. Since $R\kappa_1(p_n), R\kappa_2(p_n) \in [-1, 1]$, by the Bolzano-Weierstrass theorem, by taking a subsequence if necessary, we may assume that $\{R\kappa_1(p_n)\}, \{R\kappa_2(p_n)\}$ converge to numbers $\alpha, -\alpha$, respectively, for some $\alpha \in [-1, 1]$. Hence by (22) we get $c = 0$ which is a contradiction. Therefore, there exists a number $\delta > 0$ such that

$$H \le -\delta \;\text{ on } \partial\Omega,$$

which contradicts the fact that $\partial\Omega$ is an entire graph over $\mathbb{R}^2$ with the aid of the divergence theorem as in the proof of [9, Theorem 3.3, pp. 2732–2733]. The remaining case (iii) can be dealt with in a similar manner. Thus proposition (b) is proved. □

Remark 1 In Sect. 2 we did not use the same argument as in Sect. 3, for by introducing local coordinates, the condition (19) on the surface $\partial\Omega$ can not be converted into a second order partial differential equation which is of elliptic type.

4 Concluding Remarks and Problems

In this final section, we mention several remarks and problems.

Concerning Theorem 1, spherical cylinders satisfy the assumption (5). Therefore, as in [5], a theorem including a spherical cylinder as a conclusion is expected. Corollary 1 excludes closed surfaces with genus 1, but this might be technical. Concerning Theorem 2, right helicoids satisfy the assumption (5). Therefore, a theorem including a right helicoid as a conclusion is expected.

Let us set $N = 3$ both in (19) and in (21) and assume that $\partial\Omega$ is a minimal surface properly embedded in $\mathbb{R}^3$. Then (19) yields that the Gauss curvature is constant and hence $\partial\Omega$ must be a plane. On the other hand, (21) holds true for every minimal surface by setting $c = 0$.

Concerning technical points in the theory of partial differential equations, (19) is not of elliptic type but (21) is of elliptic type, as is mentioned in Sect. 3. Therefore, for (21) in general dimensions, Liouville-type theorems characterizing hyperplanes are expected as in [11, 15].

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Improved Rellich Type Inequalities in $\mathbb{R}^N$

Megumi Sano and Futoshi Takahashi

Abstract We consider the second or higher-order Rellich inequalities on the whole space $\mathbb{R}^N$. In spite of the lack of the Poincaré inequality on the whole space, we show that the higher-order Rellich inequalities with optimal constants can be improved, by adding explicit remainder terms to the inequalities.

Keywords Rellich inequality · Hardy inequality · Remainder terms

1 Introduction

Let $N \geq 2$, $1 \leq p < N$, and let Ω be a bounded domain in $\mathbb{R}^N$ with $0 \in \Omega$, or $\Omega = \mathbb{R}^N$. The classical Hardy inequality

$$\int_{\Omega} |\nabla u|^p dx \geq \left(\frac{N-p}{p}\right)^p \int_{\Omega} \frac{|u|^p}{|x|^p} dx \tag{1}$$

holds for all $u \in W_0^{1,p}(\Omega)$, or $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ when $\Omega = \mathbb{R}^N$. Here $W_0^{1,p}(\Omega)$ (resp. $\mathcal{D}^{1,p}(\mathbb{R}^N)$) is the completion of $C_0^\infty(\Omega)$ (resp. $C_0^\infty(\mathbb{R}^N)$) with respect to the norm $\|\nabla \cdot \|_{L^p(\Omega)}$ (resp. $\|\nabla \cdot \|_{L^p(\mathbb{R}^N)}$). It is known that for $1 < p < N$, the best constant $(\frac{N-p}{p})^p$ is never attained in $W_0^{1,p}(\Omega)$, or in $\mathcal{D}^{1,p}(\mathbb{R}^N)$. Therefore, one can expect the existence of remainder terms on the right-hand side of the inequality (1). Indeed, there are many papers that deal with remainder terms for (1) when Ω is a smooth bounded domain (see [1–6], to name a few). For example, Brezis and Vázquez [2] show that the inequality

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$$\int_{\Omega} |\nabla u|^2 dx \geq \left(\frac{N-2}{2} \right)^2 \int_{\Omega} \frac{|u|^2}{|x|^2} dx + z_0^2 \left(\frac{\omega_N}{|\Omega|} \right)^{\frac{2}{N}} \int_{\Omega} |u|^2 dx \tag{2}$$

holds true for all $u \in W_0^{1,2}(\Omega)$ where $z_0 = 2.4048\cdots$ is the first zero of the Bessel function of the first kind.

On the other hand, when $\Omega = \mathbb{R}^N$, the remainder term in (2) becomes trivial and does not provide better inequality than the classical one. More generally, Ghoussoub and Moradifam [7] show that there is no strictly positive $V \in C^1((0, +\infty))$ such that the inequality

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx \geq \left(\frac{N-2}{2} \right)^2 \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} dx + \int_{\mathbb{R}^N} V(|x|)|u|^2 dx$$

holds for all $u \in W^{1,2}(\mathbb{R}^N)$. One of the reasons of it is the lack of the Poincaré inequality: $\|\nabla u\|_{L^2(\Omega)} \geq C\|u\|_{L^2(\Omega)}$ when $\Omega = \mathbb{R}^N$. Although there is a result of refining the Hardy type inequality on the whole space (see Maz'ya's book [8], pp. 139, Corollary 3.), we cannot expect the same type of remainder terms as in (2) on the whole space.

In spite of this fact, the authors of the present paper recently showed the following result [9] : Let $2 \leq p < N$ and $q > 2$. Set $\alpha = \alpha(p, q, N) = \frac{N}{2}(q-2) - \frac{pq}{2} + 2$. Then there exists $D = D(p, q, N) > 0$ such that the inequality

$$\int_{\mathbb{R}^N} |\nabla u|^p dx \geq \left(\frac{N-p}{p} \right)^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} dx + D \left(\frac{\int_{\mathbb{R}^N} |u^\#|^{\frac{pq}{2}} |x|^\alpha dx}{\int_{\mathbb{R}^N} |u^\#|^p |x|^{2-p} dx} \right)^{\frac{2}{q-2}} \tag{3}$$

holds for all $u \in W^{1,p}(\mathbb{R}^N)$, $u \not\equiv 0$. Here $u^\#$ denotes the Schwartz symmetrization of a function u on $\mathbb{R}^N$:

$$u^\#(x) = u^\#(|x|) = \inf \left\{ \lambda > 0 \;\middle|\; \left|\{x \in \mathbb{R}^N \mid |u(x)| > \lambda\}\right| \leq |B_{|x|}(0)| \right\},$$

where $|A|$ denotes the measure of a set $A \subset \mathbb{R}^N$ (see e.g., [10]). Note that the integral $\int_{\mathbb{R}^N} |u^\#|^p |x|^{2-p} dx$ is finite for any $u \in W^{1,p}(\mathbb{R}^N)$.

In this paper, we focus on the higher-order case. A higher-order generalization of (1) was first proved by Rellich [11]: it holds

$$\int_{\Omega} |\Delta u|^2 dx \geq \left(\frac{N(N-4)}{4} \right)^2 \int_{\Omega} \frac{|u|^2}{|x|^4} dx$$

for all $u \in W_0^{2,2}(\Omega)$, where Ω is a domain in $\mathbb{R}^N$, $N \geq 5$. More generally, let $k, m \in \mathbb{N}$ and $k < kp < N$. Define

$$|u|_{k,p}^p = \begin{cases} \int_\Omega |\Delta^m u|^p \, dx & \text{if } k = 2m, \\ \int_\Omega |\nabla(\Delta^m u)|^p \, dx & \text{if } k = 2m+1, \quad \text{and} \end{cases}$$

$$C_{k,p} = \begin{cases} p^{-2m} \prod_{j=1}^m (N - 2jp) \{N(p-1) + 2(j-1)p\} & \text{if } k = 2m, \\ (N-p)p^{-2(m+1)} \prod_{j=1}^m (N - (2j+1)p) \{N(p-1) + (2j-1)p\} & \text{if } k = 2m+1. \end{cases}$$

We put $|u|_{0,p} = \|u\|_{L^p(\mathbb{R}^N)}$ and $C_{0,p} = 1$, $C_{1,p} = \frac{N-p}{p}$ for the convenience of description. Then the inequality

$$|u|_{k,p}^p \geq C_{k,p}^p \int_\Omega \frac{|u|^p}{|x|^{kp}} dx \tag{4}$$

holds for all $u \in W_0^{k,p}(\Omega)$. It is also known that $C_{k,p}^p$ is optimal (see [12, 13], or Proposition 1 in Appendix) and never attained in $W_0^{k,p}(\Omega)$. Furthermore, Gazzola-Grunau-Mitidieri [5] prove the following inequality on a smooth bounded domain: there exist positive constants $A, B > 0$ such that the inequality

$$|u|_{2,2}^2 \geq C_{2,2}^2 \int_\Omega \frac{|u|^2}{|x|^4} dx + A \int_\Omega \frac{|u|^2}{|x|^2} dx + B \int_\Omega |u|^2 dx$$

holds for all $u \in W_0^{2,2}(\Omega)$, where $N \geq 5$. In addition to this, there are many papers that deal with various types of Rellich inequalities with remainder terms on bounded domains (see [14–24] etc.).

A main aim of this paper is to obtain remainder terms for the inequality (4) when $\Omega = \mathbb{R}^N$. Note that the inequalities (1) and (4) have the scale invariance under the scaling

$$u_\lambda(x) = \lambda^{-\frac{N-kp}{p}} u\left(\frac{x}{\lambda}\right) \tag{5}$$

for $\lambda > 0$ when $\Omega = \mathbb{R}^N$. Therefore the possible remainder term to (4) should be invariant under the scaling (5) when $\Omega = \mathbb{R}^N$. In the following, ω_N will denote the area of the unit sphere in $\mathbb{R}^N$, $\|\cdot\|_r = \|\cdot\|_{L^r(\mathbb{R}^N)}$ and $\mathcal{D}^{k,p}(\mathbb{R}^N)$ is the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm $|\cdot|_{k,p}$.

Our main results are as follows:

Theorem 1 (Radial case) *Let $k \geq 2$ be an integer, $k < kp < N$ and $q > 2$. Set $\alpha_k = \frac{N}{2}(q-2) - \frac{kpq}{2} + 2$. Then there exists a constant $C > 0$ such that the inequality*

$$|u|_{k,p}^p \geq C_{k,p}^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{kp}} dx + C \left(\frac{\int_{\mathbb{R}^N} |u|^{\frac{pq}{2}} |x|^{\alpha_k} dx}{\int_{\mathbb{R}^N} |u|^p |x|^{2-kp} dx} \right)^{\frac{2}{q-2}} \tag{6}$$

holds for all radial function $u \in \mathcal{D}^{k,p}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$, $u \neq 0$.

In the non-radial case, we obtain only partial results for $k = 2, 3$.

Theorem 2 (Non-radial case) *For* $k = 2$ *or* $k = 3$, *let* $k < kp < N$ *and* $q > 2$. *Set* $\alpha_k = \frac{N}{2}(q-2) - \frac{kpq}{2} + 2$ *and* $r = \frac{Np}{N+2p}$ *(i.e.* $\frac{1}{p} = \frac{1}{r} - \frac{2}{N}$*). Then there exists a constant* $C > 0$ *such that the inequality*

$$|u|_{k,p}^p \geq C_{k,p}^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{2p}} dx + C \left(\frac{\int_{\mathbb{R}^N} |u^\#|^{\frac{pq}{2}} |x|^{\alpha_k} dx}{|u|_{k,p}^{\frac{kp-2}{k}} \|\Delta u\|_r^{\frac{2}{k}}} \right)^{\frac{2}{q-2}} \tag{7}$$

holds for all $u \in \mathcal{D}^{k,p}(\mathbb{R}^N) \cap \mathcal{D}^{2,p}(\mathbb{R}^N) \cap \mathcal{D}^{2,r}(\mathbb{R}^N)$, $u \neq 0$.

Remark 1 The remainder term of the inequalities (6) and (7) are scale invariant under the scaling (5) on $\mathbb{R}^N$: $u_\lambda(x) = \lambda^{-\frac{N-kp}{p}} u(y)$, $y = \frac{x}{\lambda}$, $x \in \mathbb{R}^N$. Indeed, it holds $|u_\lambda|_{k,p} = |u|_{k.p}$ and for $a, b \in \mathbb{R}$,

$$\int_{\mathbb{R}^N} |u_\lambda(x)|^a |x|^b dx = \lambda^{-\left(\frac{N-kp}{p}\right)a+b+N} \int_{\mathbb{R}^N} |u(y)|^a |y|^b dy. \tag{8}$$

Thus by taking $a = \frac{pq}{2}$ and $b = \alpha_k$, or $a = p$ and $b = 2 - kp$ in (8), we have

$$\int_{\mathbb{R}^N} |u_\lambda(x)|^{\frac{pq}{2}} |x|^{\alpha_k} dx = \lambda^2 \int_{\mathbb{R}^N} |u(y)|^{\frac{pq}{2}} |y|^{\alpha_k} dy,$$
$$\int_{\mathbb{R}^N} |u_\lambda(x)|^p |x|^{2-kp} dx = \lambda^2 \int_{\mathbb{R}^N} |u(y)|^p |y|^{2-kp} dy.$$

Therefore the remainder term in the inequality (6) has the scale invariance. Furthermore from Proposition 2 in Appendix, we obtain

$$\int_{\mathbb{R}^N} |(u_\lambda)^\#|^{\frac{pq}{2}} |x|^{\alpha_k} dx = \int_{\mathbb{R}^N} |(u^\#)_\lambda|^{\frac{pq}{2}} |x|^{\alpha_k} dx = \lambda^2 \int_{\mathbb{R}^N} |u^\#|^{\frac{pq}{2}} |x|^{\alpha_k} dx,$$
$$\|\Delta u_\lambda\|_{L^r(\mathbb{R}^N)}^{\frac{2}{k}} = \lambda^2 \|\Delta u\|_{L^r(\mathbb{R}^N)}^{\frac{2}{k}}.$$

Thus the remainder term in the inequality (7) also has the scale invariance.

Remark 2 If $\alpha_k \leq 0$ in Theorem 2, then $u^\#$ in the RHS of (7) can be replaced by u thanks to the Hardy-Littlewood inequality: $\int_{\mathbb{R}^N} g^\# h^\# \geq \int_{\mathbb{R}^N} gh$ (see e.g., [10]), and the fact $(|x|^{\alpha_k})^\# = |x|^{\alpha_k}$.

2 Proofs of Main Results

In this section, we prove Theorems 1 and 2. The next simple lemma is used in the proof.

Lemma 1 *Let* $p \geq 1$ *and* $a, b \in \mathbb{R}$. *Then it holds*

$$|a-b|^p - |a|^p \geq -p|a|^{p-2}ab.$$

Proof First, we assume $a \geq 0$. We use the mean value theorem for the function $f(t) = (a-t)^p$, which is defined for $t \leq a$. When $b \leq a$, we have

$$f(b) - f(0) = (a-b)^p - a^p = pc^{p-1}(-b) \geq -pa^{p-1}b,$$

where $c \in \mathbb{R}$ satisfies $0 \leq a-b \leq c \leq a$ if $b \geq 0$, or $0 \leq a \leq c \leq a-b$ if $b \leq 0$. When $b \geq a$, then $2a - b \leq a$ and we have

$$f(2a-b) - f(0) = (b-a)^p - a^p = pc^{p-1}(b-2a) \geq -pa^{p-1}b,$$

where $c \in \mathbb{R}$ satisfies $0 \leq a \leq c \leq b-a$ if $b-2a \geq 0$, or $0 \leq b-a \leq c \leq a$ if $b - 2a \leq 0$. This implies the result when $a \geq 0$.

The case $a \leq 0$ follows by considering $a = -\tilde{a}, \tilde{a} \geq 0$ and $b = -\tilde{b}, \tilde{b} \in \mathbb{R}$. □

Proof of Theorem 1

We show the inequality (6) for all radial function $u \in \mathcal{D}^{k,p}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$. By density argument, we may assume $u \in C_0^\infty(\mathbb{R}^N)$ without loss of generality.

First, note that the inequality

$$|u|_{k,p}^p = |\Delta u|_{k-2,p}^p \geq C_{k-2,p}^p \int_{\mathbb{R}^N} \frac{|\Delta u|^p}{|x|^{(k-2)p}}\, dx \tag{9}$$

holds from Rellich's inequality (4). Actually when $k = 2$, this is the equality. Thus, in order to prove Theorem, it is enough to show the RHS of (9) is bounded from below by the RHS of (6).

Since u is radial, u can be written as $u(x) = \tilde{u}(|x|)$ where $\tilde{u} \in C_0^\infty([0, +\infty))$. We define the new function v as follows:

$$\tilde{v}(r) = r^{\frac{N-kp}{p}} \tilde{u}(r), \quad r \in [0, \infty), \quad \text{and} \quad v(y) = \tilde{v}(|y|), \quad y \in \mathbb{R}^2. \tag{10}$$

Note that $\tilde{v}(0) = 0$ and also $\tilde{v}(+\infty) = 0$ since the support of u is compact. We claim that if $u \in \mathcal{D}^{k,p}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$, then $v \in L^p(\mathbb{R}^2)$. Indeed, we have

$$\begin{aligned}\int_{\mathbb{R}^2} |v(y)|^p dy &= \omega_2 \int_0^\infty |\tilde{v}(r)|^p r dr \\ &= \omega_2 \int_0^\infty |\tilde{u}(r)|^p r^{N-kp+1}\, dr = \frac{\omega_2}{\omega_N} \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{kp-2}}\, dx \\ &\le \frac{\omega_2}{\omega_N} \left(\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{kp}}\, dx \right)^{\frac{kp-2}{kp}} \left(\int_{\mathbb{R}^N} |u|^p\, dx \right)^{\frac{2}{kp}} \\ &\le \frac{\omega_2}{\omega_N} C_{k,p}^{\frac{2-kp}{k}} |u|_{k,p}^{\frac{kp-2}{k}} \left(\int_{\mathbb{R}^N} |u|^p\, dx \right)^{\frac{2}{kp}} < \infty, \end{aligned} \tag{11}$$

here we have used Hölder's inequality, Rellich's inequality (4), and the assumption $u \in \mathcal{D}^{k,p}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$. Therefore we have checked $v \in L^p(\mathbb{R}^2)$.

For $k \ge 2$, $k \in \mathbb{N}$ and $k < kp < N$, put

$$\theta_k = \theta(k, N, p) = 2k + \frac{N(p-2)}{p}, \quad \text{and}$$
$$\Delta_{\theta_k} f = f^{''}(r) + \frac{\theta_k - 1}{r} f^{'}(r)$$

for a smooth function $f = f(r)$. Define

$$A_{k,p} = \frac{(N-kp)[(k-2)p + (p-1)N]}{p^2}.$$

Then we see $C_{k-2,p} A_{k,p} = C_{k,p}$ and a direct calculation shows that

$$-\Delta \tilde{u} = r^{k-2-\frac{N}{p}} \left(A_{k,p} \tilde{v}(r) - r^2 \Delta_{\theta_k} \tilde{v}(r) \right).$$

Define

$$J = \int_{\mathbb{R}^N} \frac{|\Delta u|^p}{|x|^{(k-2)p}}\, dx - A_{k,p}^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{kp}}\, dx. \tag{12}$$

Now applying Lemma 1 with the choice

$$a = A_{k,p} \tilde{v}(r) \quad \text{and} \quad b = r^2 \Delta_{\theta_k} \tilde{v}(r),$$

and using the fact $\int_0^\infty |\tilde{v}|^{p-2} \tilde{v} \tilde{v}' dr = 0$ since $\tilde{v}(0) = \tilde{v}(+\infty) = 0$, we have

$$\begin{aligned} J &= \omega_N \int_0^\infty |-\Delta \tilde{u}(r)|^p\, r^{N-1-(k-2)p}\, dr - A_{k,p}^p \omega_N \int_0^\infty |\tilde{u}(r)|^p r^{N-kp-1}\, dr \\ &= \omega_N \int_0^\infty \left(\left| A_{k,p} \tilde{v}(r) - r^2 \Delta_{\theta_k} \tilde{v}(r) \right|^p - |A_{k,p} \tilde{v}(r)|^p \right) r^{-1}\, dr \\ &\ge -p\omega_N A_{k,p}^{p-1} \int_0^\infty |\tilde{v}|^{p-2} \tilde{v} \Delta_{\theta_k} \tilde{v}\, r\, dr \end{aligned}$$

$$
\begin{aligned}
&= -p\omega_N A_{k,p}^{p-1} \int_0^\infty |\tilde{v}|^{p-2}\tilde{v}\left(\tilde{v}'' + \frac{\theta_k - 1}{r}\tilde{v}'\right) r\,dr \\
&= -p\omega_N A_{k,p}^{p-1} \int_0^\infty |\tilde{v}|^{p-2}\tilde{v}\tilde{v}'' r\,dr.
\end{aligned}
$$

Moreover by integration by parts, we observe that

$$
\begin{aligned}
-\int_0^\infty |\tilde{v}|^{p-2}\tilde{v}\tilde{v}'' r\,dr &= (p-1)\int_0^\infty |\tilde{v}|^{p-2}(\tilde{v}')^2 r\,dr + \int_0^\infty |\tilde{v}|^{p-2}\tilde{v}\tilde{v}'\,dr \\
&= \frac{4(p-1)}{p^2}\int_0^\infty |(|\tilde{v}|^{\frac{p-2}{2}}\tilde{v})'|^2 r\,dr \\
&= \frac{4(p-1)}{p^2\omega_2}\int_{\mathbb{R}^2} |\nabla(|v|^{\frac{p-2}{2}}v)|^2\,dy.
\end{aligned}
$$

Combining these, we have

$$
J \geq \frac{4(p-1)\omega_N}{p\omega_2} A_{k,p}^{p-1} \int_{\mathbb{R}^2} |\nabla(|v|^{\frac{p-2}{2}}v)|^2\,dy. \tag{13}
$$

Now, we apply the Gagliardo-Nirenberg inequality to $|v|^{\frac{p-2}{2}}v \in L^2(\mathbb{R}^2)$: for $q > 2$, there exists a constant $C(q) > 0$ such that it holds

$$
\||v|^{\frac{p}{2}}\|_{L^q(\mathbb{R}^2)} \leq C(q)\,\||v|^{\frac{p}{2}}\|_{L^2(\mathbb{R}^2)}^{\frac{2}{q}} \|\nabla(|v|^{\frac{p-2}{2}}v)\|_{L^2(\mathbb{R}^2)}^{\frac{q-2}{q}}. \tag{14}
$$

Combining (13) and (14), we obtain

$$
\begin{aligned}
J &\geq \frac{4(p-1)\omega_N A_{k,p}^{p-1}}{p\omega_2} C(q)^{-\frac{2q}{q-2}} \left(\frac{\int_{\mathbb{R}^2} |v(y)|^{\frac{pq}{2}}\,dy}{\int_{\mathbb{R}^2} |v(y)|^p\,dy}\right)^{\frac{2}{q-2}} \\
&= \frac{4(p-1)\omega_N A_{k,p}^{p-1}}{p\omega_2} C(q)^{-\frac{2q}{q-2}} \left(\frac{\int_{\mathbb{R}^N} |u|^{\frac{pq}{2}}|x|^{\alpha_k}\,dx}{\int_{\mathbb{R}^N} |u|^p|x|^{2-kp}\,dx}\right)^{\frac{2}{q-2}}.
\end{aligned} \tag{15}
$$

Consequently, from (9), (12), (15) and $C_{k-2,p}A_{k,p} = C_{k,p}$, we obtain

$$
\begin{aligned}
|u|_{k,p}^p &\geq C_{k-2,p}^p \int_{\mathbb{R}^N} \frac{|\Delta u|^p}{|x|^{(k-2)p}}\,dx \\
&= C_{k-2,p}^p \left(A_{k,p}^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{kp}}\,dx + J\right) \\
&\geq C_{k,p}^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{kp}}\,dx + C\left(\frac{\int_{\mathbb{R}^N} |u|^{\frac{pq}{2}}|x|^{\alpha_k}\,dx}{\int_{\mathbb{R}^N} |u|^p|x|^{2-kp}\,dx}\right)^{\frac{2}{q-2}}
\end{aligned}
$$

where $C = \frac{4(p-1)\omega_N}{p\omega_2} C_{k-2,p} C_{k,p}^{p-1} C(q)^{-\frac{2q}{q-2}}$. This proves Theorem 1. □

Proof of Theorem 2

First, we treat the case $k = 2$. We show the inequality

$$\int_{\mathbb{R}^N} |\Delta u|^p \, dx \geq C_{2,p}^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{2p}} \, dx + C \left(\frac{\int_{\mathbb{R}^N} |u^\#|^{\frac{pq}{2}} |x|^{\alpha_2} \, dx}{\| \Delta u \|_p^{p-1} \| \Delta u \|_r} \right)^{\frac{2}{q-2}} \tag{16}$$

for all $u \in \mathcal{D}^{2,p}(\mathbb{R}^N) \cap \mathcal{D}^{2,r}(\mathbb{R}^N)$. Set $f = -\Delta u \in L^p(\mathbb{R}^N)$ and $w(x) = \frac{1}{(N-2)\omega_N} \int_{\mathbb{R}^N} \frac{f^\#(y)}{|x-y|^{N-2}} \, dy$. Since $w(Ox) = w(x)$ for any $O \in O(N)$, the group of orthogonal matrices in $\mathbb{R}^N$, we see w is a radial function. Also since $f^\# \in L^p(\mathbb{R}^N)$, the Calderon-Zygmund inequality (see [25] Theorem 9.9.) implies that $w \in \mathcal{D}^{2,p}(\mathbb{R}^N)$ and satisfies $-\Delta w = f^\#$ a.e. in $\mathbb{R}^N$. Therefore we have

$$\| \Delta w \|_p = \| \Delta u \|_p. \tag{17}$$

By Talenti's comparison principle [26], we know $w \geq u^\# \geq 0$. Hence we have

$$\begin{aligned} \int_{\mathbb{R}^N} |w|^\beta |x|^\gamma \, dx &\geq \int_{\mathbb{R}^N} |u^\#|^\beta |x|^\gamma \, dx \quad \text{if } \beta \geq 0, \\ &\geq \int_{\mathbb{R}^N} |u|^\beta |x|^\gamma \, dx \quad \text{if } \beta \geq 0 \text{ and } \gamma \leq 0. \end{aligned} \tag{18}$$

where the second inequality comes from the Hardy-Littlewood inequality. Furthermore there exists a constant $H > 0$ such that the inequality

$$\|w\|_p \leq H \| f^\# \|_r = H \| (-\Delta u)^\# \|_r = H \| (-\Delta u) \|_r \tag{19}$$

holds from the Hardy-Littlewood-Sobolev inequality, where $\frac{1}{p} = \frac{1}{r} - \frac{2}{N}$. From (17), Theorem 1, (18) and (19), we obtain

$$\begin{aligned} |u|_{2,p}^p &= |w|_{2,p}^p \\ &\geq C_{2,p}^p \int_{\mathbb{R}^N} \frac{|w|^p}{|x|^{2p}} \, dx + C \left(\frac{\int_{\mathbb{R}^N} |w|^{\frac{pq}{2}} |x|^{\alpha_2} \, dx}{\int_{\mathbb{R}^N} |w|^p |x|^{2-2p} \, dx} \right)^{\frac{2}{q-2}} \\ &\geq C_{2,p}^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{2p}} \, dx + C \left(\frac{\int_{\mathbb{R}^N} |u^\#|^{\frac{pq}{2}} |x|^{\alpha_2} \, dx}{C_{2,p}^{1-p} \| \Delta w \|_p^{p-1} \| w \|_p} \right)^{\frac{2}{q-2}} \\ &\geq C_{2,p}^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{2p}} \, dx + C \left(\frac{\int_{\mathbb{R}^N} |u^\#|^{\frac{pq}{2}} |x|^{\alpha_2} \, dx}{\| \Delta u \|_p^{p-1} \| \Delta u \|_r} \right)^{\frac{2}{q-2}}, \end{aligned}$$

which concludes (16).

Next, we treat the case $k = 3$. As before, set $f = -\Delta u \in L^p(\mathbb{R}^N) \cap \mathcal{D}^{1,p}(\mathbb{R}^N)$ and $w(x) = \frac{1}{(N-2)\omega_N} \int_{\mathbb{R}^N} \frac{f^\#(y)}{|x-y|^{N-2}}\, dy$. Again we obtain $w \in \mathcal{D}^{2,p}(\mathbb{R}^N)$, w radial, $w \geq u^\# > 0$ and $-\Delta w = f^\#$ a.e. in $\mathbb{R}^N$. By Pólya-Szegö inequality (see e.g., [10]), we have

$$|u|_{3,p}^p = \int_{\mathbb{R}^N} |\nabla \Delta u|^p dx = \int_{\mathbb{R}^N} |\nabla f|^p dx \geq \int_{\mathbb{R}^N} |\nabla f^\#|^p dx = |w|_{3,p}^p.$$

In the same way as $k = 2$ case, we use Theorem 1. Then we obtain

$$\begin{aligned}
|u|_{3,p}^p &\geq |w|_{3,p}^p \\
&\geq C_{3,p}^p \int_{\mathbb{R}^N} \frac{|w|^p}{|x|^{3p}}\, dx + C \left(\frac{\int_{\mathbb{R}^N} |w|^{\frac{pq}{2}} |x|^{\alpha_3}\, dx}{\int_{\mathbb{R}^N} |w|^p |x|^{2-3p}\, dx} \right)^{\frac{2}{q-2}} \\
&\geq C_{3,p}^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{3p}}\, dx + C \left(\frac{\int_{\mathbb{R}^N} |u^\#|^{\frac{pq}{2}} |x|^{\alpha_3}\, dx}{C_{3,p}^{\frac{2-3p}{3}} |w|_{3,p}^{\frac{3p-2}{3}} \|w\|_p^{\frac{2}{3}}} \right)^{\frac{2}{q-2}} \\
&\geq C_{3,p}^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{3p}}\, dx + C \left(\frac{\int_{\mathbb{R}^N} |u^\#|^{\frac{pq}{2}} |x|^{\alpha_3}\, dx}{|u|_{3,p}^{\frac{3p-2}{3}} \|\Delta u\|_r^{\frac{2}{3}}} \right)^{\frac{2}{q-2}},
\end{aligned}$$

which concludes (7). □

Remark 3 Up to now, we do not obtain the result for $k \geq 4$ in Theorem 2. For example, put $f = -\Delta u \in \mathcal{D}^{2,p}(\mathbb{R}^N)$ for $u \in \mathcal{D}^{4,p}(\mathbb{R}^N)$. Since we do not know the validity of the inequality

$$\int_{\mathbb{R}^N} |\Delta f|^p\, dx \geq \int_{\mathbb{R}^N} |\Delta f^\#|^p\, dx,$$

the argument of the proof of Theorem 2 does not work for $k = 4$ case. Instead, if we define $f = (-\Delta)^2 u \in L^p(\mathbb{R}^N)$ and $w(x) = C_N \int_{\mathbb{R}^N} \frac{f^\#(y)}{|x-y|^{N-4}}\, dy$, then we obtain $(-\Delta)^2 w = f^\#$ in $\mathbb{R}^N$ and $|u|_{4,p}^p = |w|_{4,p}^p$. However in this case, we do not know whether the comparison $u^\# \leq w$ hold or not, which invalidates the proof of Theorem 2.

3 Another Improved Rellich Inequality

In this section, we prove another improved Rellich inequality on the whole space. In Theorem 1, we have used the Gagliardo-Nirenberg inequality as a substitute for the Poincaré inequality, which is usually used to improve the Rellich inequality on bounded domains. In the next theorem, we will employ *the logarithmic Sobolev inequality* on the whole space.

Theorem 3 *Let $k \geq 2$ be a integer and $k \leq kp < N$. Then the inequality*

$$\begin{aligned} &|u|_{k,p}^p - C_{k,p}^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{kp}}\,dx \\ &\geq BE(u)\exp\left(1 + E(u)^{-1}\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{kp-2}} \log\left(\frac{\omega_N |x|^{N-kp}|u|^p}{\omega_2 E(u)}\right)dx\right) \end{aligned} \tag{20}$$

holds for all radial function $u \in W^{k,p}(\mathbb{R}^N)$, where $B = \frac{4\pi(p-1)}{p} C_{k-2,p}^p A_{k,p}^{p-1}$ and $E(u) = \int_{\mathbb{R}^N} |u|^p |x|^{2-kp} dx$.

Proof of Theorem 3 We proceed as in the proof of Theorem 1. From the proof of Theorem 1, we observe that

$$|u|_{k,p}^p - C_{k,p}^p \int_{\mathbb{R}^N} \frac{|\Delta u|^p}{|x|^{kp}}\,dx \geq B(k,p,N)\int_{\mathbb{R}^2} \left|\nabla |v|^{\frac{p-2}{2}} v\right|^2 dy, \tag{21}$$

where $B(k,p,N) = \frac{4(p-1)\omega_N}{p\omega_2} C_{k-2,p} C_{k,p}^{p-1}$. Differently from the proof of Theorem 1, here, instead of the Gagliardo-Nirenberg inequality, we apply the logarithmic Sobolev inequality (see [27]) on $\mathbb{R}^2$:

$$\int_{\mathbb{R}^2} f^2(y)\log f^2(y)\,dy \leq \log\left(\frac{1}{\pi e}\int_{\mathbb{R}^2} |\nabla f(y)|^2\,dy\right) \tag{22}$$

for the function $f = \|v\|_{L^p(\mathbb{R}^2)}^{-\frac{p}{2}} |v|^{\frac{p-2}{2}} v$, $\|f\|_{L^2(\mathbb{R}^2)} = 1$, where v is defined in (10). By (21) and (22), we obtain

$$\begin{aligned} &|u|_{k,p}^p - C_{k,p}^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{kp}}\,dx \geq B(k,p,N)\int_{\mathbb{R}^2} |\nabla(|v|^{\frac{p-2}{2}} v)|^2\,dy \\ &\geq \pi B(k,p,N)\|v\|_{L^p(\mathbb{R}^2)}^p \exp\left(1 + \int_{\mathbb{R}^2} \frac{|v(y)|^p}{\|v\|_{L^p(\mathbb{R}^2)}^p} \log\left(\frac{|v(y)|^p}{\|v\|_{L^p(\mathbb{R}^2)}^p}\right)dy\right) \\ &= \pi B(k,p,N)\frac{\omega_2}{\omega_N} E(u)\exp\left(1 + \frac{\omega_N}{E(u)}\int_0^\infty r^{N-kp}|u(r)|^p \log\left(\frac{\omega_N r^{N-kp}|u(r)|^p}{\omega_2 E(u)}\right) r\,dr\right) \\ &= BE(u)\exp\left(1 + E(u)^{-1}\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{kp-2}} \log\left(\frac{\omega_N |x|^{N-kp}|u|^p}{\omega_2 E(u)}\right)dx\right) \end{aligned}$$

where $E(u) = \int_{\mathbb{R}^N} |u|^p |x|^{2-kp} dx = \frac{\omega_N}{\omega_2}\|v\|_{L^p(\mathbb{R}^2)}^p$. Hence the inequality (20) holds. □

Remark 4 The inequality (20) has an invariance under the scaling $u_\lambda(x) = \lambda^{-\frac{N-kp}{p}} u(y)$ where $y = \frac{x}{\lambda}$, $(\lambda > 0, x \in \mathbb{R}^N)$. Indeed, we have $E(u_\lambda) = \lambda^2 E(u)$ and

$$
\begin{aligned}
&E(u_\lambda)\exp\left(1+E(u_\lambda)^{-1}\int_{\mathbb{R}^N}\frac{|u_\lambda(x)|^p}{|x|^{kp-2}}\log\left(\frac{\omega_N|x|^{N-kp}|u_\lambda(x)|^p}{\omega_2 E(u_\lambda)}\right)dx\right)\\
&=\lambda^2 E(u)\exp\left(1+E(u)^{-1}\int_{\mathbb{R}^N}\frac{|u(y)|^p}{|y|^{kp-2}}\left(\log\lambda^{-2}+\log\left(\frac{\omega_N|y|^{N-kp}|u(y)|^p}{\omega_2 E(u)}\right)\right)dy\right)\\
&=E(u)\exp\left(1+E(u)^{-1}\int_{\mathbb{R}^N}\frac{|u(y)|^p}{|y|^{kp-2}}\log\left(\frac{\omega_N|y|^{N-kp}|u(y)|^p}{\omega_2 E(u)}\right)dy\right),
\end{aligned}
$$

so the inequality (20) also enjoys a scale invariance.

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Appendix

Davies-Hinz [12] showed that the constant $C_{k,p}^p$ in the inequality (4) is optimal when $\Omega = \mathbb{R}^N$. In this Appendix, we will show the fact when Ω is a general bounded domain.

Proposition 1 *Let $k \in \mathbb{N}$, $k < kp < N$ and let Ω be a bounded domain with $0 \in \Omega$ in $\mathbb{R}^N$. Then the constant $C_{k,p}^p$ in the inequality (4) is optimal. That is*

$$
\inf_{0\neq u\in W_0^{k,p}(\Omega)}\frac{|u|_{k,p}^p}{\int_\Omega \frac{|u(x)|^p}{|x|^{kp}}\,dx}=C_{k,p}^p.
$$

Proof of Proposition 1

By the scaling (5) and zero extension, we may assume $B_1(0)\subset\subset\Omega$ without loss of generality. First, we show the optimality of $C_{k,p}^p$ in the even case $k=2m$, $m\in\mathbb{N}$. For $0<\varepsilon\ll 1$, we define the function $u_\varepsilon \in W_0^{2m,p}(\Omega)$ as follows:

$$
u_\varepsilon(x)=\begin{cases}\varepsilon^{-\frac{N-2mp}{p}}\log\frac{1}{\varepsilon}, & \text{if } 0\le|x|\le\varepsilon\\ |x|^{-\frac{N-2mp}{p}}\log\frac{1}{|x|}, & \text{if } \varepsilon\le|x|\le 1,\\ 0, & \text{if } x\in\Omega\setminus B_1(0).\end{cases}
$$

Let $\alpha=\frac{N-2mp}{p}$. By using the formula

$$
\begin{aligned}
&\Delta r^{-\alpha}=\alpha(\alpha-N+2)r^{-\alpha-2},\\
&\Delta\left(r^{-\alpha}\log\frac{1}{r}\right)=\alpha(\alpha-N+2)r^{-\alpha-2}\log\frac{1}{r}+(2\alpha-N+2)r^{-\alpha-2},
\end{aligned}
$$

we compute that

$$\Delta^m u_\varepsilon = \begin{cases} 0, & \text{if } 0 \le |x| \le \varepsilon, \\ A_m |x|^{-(\alpha+2m)} \log \frac{1}{|x|} + B_m |x|^{-(\alpha+2m)}, & \text{if } \varepsilon \le |x| \le 1, \\ 0, & \text{if } x \in \Omega \setminus B_1(0), \end{cases}$$

where $A_m(\alpha)$ and $B_m(\alpha)$ are determined by the iterative formula:

$$\begin{aligned} &A_1(\alpha) = \alpha(\alpha - N + 2), \\ &A_{j+1}(\alpha) = (\alpha + 2j)\,(\alpha + 2(j+1) - N)\, A_j, \quad j = 1, 2, \ldots, \\ &B_1(\alpha) = 2\alpha - N + 2, \\ &B_{j+1}(\alpha) = (\alpha + 2j)\,(\alpha + 2(j+1) - N)\, B_j + 2\alpha + 2(2j+1) - N \quad j = 1, 2, \ldots. \end{aligned}$$

Thus we have

$$A_m = A_m(\alpha) = \prod_{j=0}^{m-1} (\alpha + 2j)(\alpha + 2(j+1) - N), \quad |A_m(\alpha)| = C_{2m,p}.$$

We compute

$$\begin{aligned} \int_\Omega |\Delta^m u_\varepsilon(x)|^p \, dx &= \omega_N \int_\varepsilon^1 \left| A_m \log \frac{1}{r} + B_m \right|^p r^{-(\alpha+2m)p+N-1} dr \\ &= \omega_N \left(\frac{1}{A_m} \right) \int_{B_m}^{B_m + A_m \log \frac{1}{\varepsilon}} |t|^p \, dt \\ &= \omega_N \left(\frac{1}{A_m(p+1)} \right) \left(\left| B_m + A_m \log \frac{1}{\varepsilon} \right|^p (B_m + A_m \log \frac{1}{\varepsilon}) - |B_m|^p B_m \right). \end{aligned} \tag{23}$$

On the other hand, we have

$$\begin{aligned} &\int_\Omega \frac{|u_\varepsilon(x)|^p}{|x|^{2mp}} \, dx \\ &= \omega_N \, \varepsilon^{-\alpha p} \left(\log \frac{1}{\varepsilon} \right)^p \int_0^\varepsilon r^{N-2mp-1} \, dr + \omega_N \int_\varepsilon^1 r^{-1} \left(\log \frac{1}{r} \right)^p \, dr \\ &= \omega_N \, \frac{\varepsilon^{N-2mp}}{N-2mp} \left(\log \frac{1}{\varepsilon} \right)^p + \omega_N \int_0^{\log \frac{1}{\varepsilon}} t^p \, dt \\ &= \omega_N \, \frac{\varepsilon^{N-2mp}}{N-2mp} \left(\log \frac{1}{\varepsilon} \right)^p + \omega_N \frac{1}{p+1} \left(\log \frac{1}{\varepsilon} \right)^{p+1}. \end{aligned} \tag{24}$$

By (23), (24) and the fact $|A_m| = C_{2m,p}$, we obtain

$$\frac{\int_{B_1(0)} |\Delta^m u_\varepsilon(x)|^p \, dx}{\int_{B_1(0)} \frac{|u_\varepsilon(x)|^p}{|x|^{2mp}} \, dx} \to |A_m(\alpha)|^p = C^p_{2m,p} \text{ as } \varepsilon \to 0,$$

which implies the optimality of $C^p_{2m,p}$.

Next, in the odd case $k = 2m+1$, $m \in \mathbb{N}$, we consider the function $u_\varepsilon \in W_0^{2m+1,p}(B_1(0))$ as follows:

$$u_\varepsilon(x) = \begin{cases} \varepsilon^{-\frac{N-(2m+1)p}{p}} \log \frac{1}{\varepsilon}, & \text{if } 0 \le |x| \le \varepsilon, \\ |x|^{-\frac{N-(2m+1)p}{p}} \log \frac{1}{|x|}, & \text{if } \varepsilon \le |x| \le 1, \\ 0, & \text{if } x \in \Omega \setminus B_1(0). \end{cases}$$

Let $\beta = \frac{N-(2m+1)p}{p}$. Note that

$$\nabla(\Delta^m u_\varepsilon) = \begin{cases} 0, \quad \text{if } 0 \le |x| \le \varepsilon, \\ |x|^{-(\beta+2m+2)} x \left\{ -A_m(\beta)(\beta+2m) \log \frac{1}{|x|} - (A_m(\beta) + (\beta+2m)B_m(\beta)) \right\}, \\ \qquad \text{if } \varepsilon \le |x| \le 1, \\ 0, \quad \text{if } x \in \Omega \setminus B_1(0). \end{cases}$$

If we make a calculation similar to the even case, we obtain

$$\frac{\int_\Omega |\nabla(\Delta^m u_\varepsilon)(x)|^p \, dx}{\int_\Omega \frac{|u_\varepsilon(x)|^p}{|x|^{(2m+1)p}} \, dx} \to |A_m(\beta)|^p (\beta+2m)^p \text{ as } \varepsilon \to 0,$$

which implies the optimality of $C^p_{2m+1,p}$ by $\beta + 2m = \frac{N-p}{p}$ and $C^p_{2m+1,p} = |A_m(\beta)|^p (\beta+2m)^p$. □

Proposition 2 *Put $r = |x|$, $x \in \mathbb{R}^N$ and let*

$$u^\#(r) = \inf\{\tau > 0 \mid \mu_u(\tau) \le |B_r(0)|\}$$

be the symmetric decreasing rearrangement of a function u, where μ_u is a distribution function of u: $\mu_u(\tau) = \left|\{x \in \mathbb{R}^N \mid |u(x)| > \tau\}\right|$, $\tau \ge 0$. Define $u_\lambda(x) = \lambda^{-\frac{N-kp}{p}} u\left(\frac{x}{\lambda}\right)$ for $\lambda > 0$. Then the equality

$$(u_\lambda)^\#(r) = (u^\#)_\lambda(r) \tag{25}$$

holds for any $r, \lambda > 0$.

Proof of Proposition 2 The distribution function of u_λ can be written as

$$\begin{aligned}
\mu_{u_\lambda}(\tau) &= \left|\{x \in \mathbb{R}^N \mid |u_\lambda(x)| > \tau\}\right| \\
&= \left|\{x \in \mathbb{R}^N \;\middle|\; \lambda^{-\frac{N-kp}{p}} \left|u\left(\frac{x}{\lambda}\right)\right| > \tau\}\right| \\
&= |\{\lambda y \in \mathbb{R}^N \mid |u(y)| > \lambda^{\frac{N-kp}{p}} \tau\}| \\
&= \lambda^N \, |\{y \in \mathbb{R}^N \mid |u(y)| > \lambda^{\frac{N-kp}{p}} \tau\}| \\
&= \lambda^N \mu_u(\lambda^{\frac{N-kp}{p}} \tau).
\end{aligned} \tag{26}$$

Hence by the definition of $(u_\lambda)^\#$ and (26), we obtain

$$\begin{aligned}
(u_\lambda)^\#(r) &= \inf\{\tau > 0 \mid \mu_{u_\lambda}(\tau) \le |B_r|\} \\
&= \inf\{\tau > 0 \mid \lambda^N \mu_u(\lambda^{\frac{N-kp}{p}} \tau) \le |B_r|\} \\
&= \inf\{\lambda^{-\frac{N-kp}{p}} \tilde{\tau} > 0 \mid \mu_u(\tilde{\tau}) \le \lambda^{-N} |B_r|\} \\
&= \lambda^{-\frac{N-kp}{p}} \inf\{\tilde{\tau} > 0 \mid \mu_u(\tilde{\tau}) \le |B_{\frac{r}{\lambda}}|\} \\
&= \lambda^{-\frac{N-kp}{p}} u^\# \left(\frac{r}{\lambda}\right) = (u^\#)_\lambda(r).
\end{aligned}$$

The proof of Proposition 2 is now complete. □

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Solvability of a Semilinear Parabolic Equation with Measures as Initial Data

Jin Takahashi

Abstract We study a sharp condition for the solvability of the Cauchy problem $u_t - \Delta u = u^p, u(\cdot, 0) = \mu$, where $N \geq 1, p \geq (N+2)/N$ and μ is a Radon measure on $\mathbf{R}^N$. Our results show that the problem does not admit any local nonnegative solutions for some μ satisfying $\mu(\{y \in \mathbf{R}^N; |x-y| < \rho\}) \leq C\rho^{N-2/(p-1)}(\log(e + 1/\rho))^{-1/(p-1)}$ $(x \in \mathbf{R}^N,\ \rho > 0)$ with a constant $C > 0$. On the other hand, the problem admits a local solution if $\mu(\{y \in \mathbf{R}^N; |x-y| < \rho\}) \leq C\rho^{N-2/(p-1)}(\log(e + 1/\rho))^{-1/(p-1)-\varepsilon}$ $(x \in \mathbf{R}^N, \rho > 0)$ with a constant $\varepsilon \in (0, 1/(p-1))$.

Keywords Solvability · Semilinear parabolic equations · Radon measures

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1 Introduction

This paper concerns the Cauchy problem

$$\begin{cases} u_t - \Delta u = u^p & \text{in } \mathbf{R}^N \times (0, T), \\ u(\cdot, 0) = \mu & \text{in } \mathbf{R}^N, \end{cases} \tag{1}$$

where $N \geq 1$, $T > 0$, $p > 1$ and μ is a nonnegative Radon measure on $\mathbf{R}^N$. We say that u is a solution of (1) if u is a nonnegative function satisfying the equation in the classical sense and $u(\cdot, t) \to \mu$ as $t \downarrow 0$ weakly as measures on each fixed open ball. In this paper, we study a sharp condition on μ for the solvability of (1) under the condition that

$$p \geq p_F := (N+2)/N.$$

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A necessary and sufficient condition on μ for the existence of solutions of (1) was indirectly characterized by Baras and Pierre [4, THÉORÈME 3.2]. They also gave an explicit necessary condition for existence in [4, PROPOSITION 3.2] (see also Andreucci and DiBenedetto [3, Part I, Proposition 4.3, Remark 4.9]). More precisely, they proved that if $u \geq 0$ satisfies $u_t - \Delta u = u^p$ in $\mathbf{R}^N \times (0, T)$, then there exists a unique Radon measure μ on $\mathbf{R}^N$ such that $u(\cdot, t) \to \mu$ as $t \downarrow 0$ weakly as measures on each fixed open ball. Furthermore, μ must satisfy

$$\sup_{x \in \mathbf{R}^N} \mu(B_1(x)) < +\infty \quad \text{if } p < p_F, \tag{2}$$

and for any compact subset K of $\mathbf{R}^N$ there exists a constant $C > 0$ such that

$$\mu(B_\rho(x)) \leq \begin{cases} C(\log(1/\rho))^{-\frac{N}{2}} & \text{if } p = p_F, \\ C\rho^{N - \frac{2}{p-1}} & \text{if } p > p_F, \end{cases} \tag{3}$$

for any $x \in K$ and $\rho > 0$ small. Here $B_\rho(x)$ is the N-dimensional open ball of radius $\rho > 0$ centered at $x \in \mathbf{R}^N$. We remark that (2) is also a sufficient condition for the existence of local solutions of (1) (see [4, COROLLAIRE 3.4.i]). For $p > p_F$, by the result of Robinson and Sierżęga [17, Theorem 3], we can check that the problem (1) admits a solution for the initial data $\mu_1(A) := \int_A |f_1(x)|dx$ if $|f_1(x)| \leq c|x|^{-2/(p-1)}$ with $c > 0$ small (see Proposition 2 in the last part of Sect. 3 for more details). Therefore it is expected that (3) is also a sufficient condition.

In this paper, by taking an approach similar to that of Kan and the author [11], we first show that (3) is not a sufficient condition. Indeed, Theorem 1 below says that the problem (1) does not admit any local solutions for some initial data μ_2 satisfying

$$\mu_2(B_\rho(x)) \leq C\rho^{N - \frac{2}{p-1}}(\log(e + 1/\rho))^{-\frac{1}{p-1}} \quad \text{if } p \geq p_F \tag{4}$$

for any $x \in \mathbf{R}^N$ and $\rho > 0$ with a constant $C > 0$.

We next consider sufficient conditions on μ for the existence of solutions of (1). The case where μ is an L^q function was first investigated by Weissler [20, 21]. For subsequent developments, see [5, 7, 8, 14, 17], [16, Sect. 15] and the references therein. In the case where the initial data is a Radon measure, sufficient conditions have been studied by many papers, see for instance [1–4, 12, 13, 15, 19]. Among others, in Niwa [15, Subsect. 1.5, Theorem] (see also Andreucci [2, Theorem 1.1, Remark 1.2]), it was shown that (1) admits a local solution if there exist constants $C > 0$ and $\varepsilon > 0$ such that

$$\mu(B_\rho(x)) \leq C\rho^{N - \frac{2}{p-1} + \varepsilon} \quad \text{if } p \geq p_F$$

for any $x \in \mathbf{R}^N$ and $\rho > 0$ small. Our second result improves this condition. Roughly speaking, Theorem 2 below says that the above condition can be replaced by

$$\mu(B_\rho(x)) \le C\rho^{N-\frac{2}{p-1}}(\log(e+1/\rho))^{-\frac{1}{p-1}-\varepsilon} \quad \text{if } p \ge p_F.$$

Before stating main results precisely, we introduce a Morrey-type space. Let $p \ge p_F$ and let ϕ be a positive function defined on $(0, \infty)$. Set

$$X = X^p_\phi(\mathbf{R}^N) := \left\{\mu;\ \mu \text{ is a Radon measure on } \mathbf{R}^N \text{ satisfying } \|\mu\|_X < +\infty\right\},$$
$$\|\mu\|_X = \|\mu\|_{X^p_\phi(\mathbf{R}^N)} := \sup_{x\in\mathbf{R}^N}\sup_{\rho>0}\left(\phi(\rho)^{-1}\rho^{-(N-\frac{2}{p-1})}\mu(B_\rho(x))\right).$$

For nonexistence, we impose the following conditions on ϕ:

$$\int_0^1 \frac{\phi(\eta)^{p-1}}{\eta}d\eta = +\infty, \tag{5}$$

$$\text{there exists } \alpha \in (0, \frac{2}{p-1}) \text{ such that } \rho^{-\alpha}\phi(\rho) \text{ is nonincreasing,} \tag{6}$$

$$\text{there exists } \beta \in (0, N-\frac{2}{p-1}) \text{ such that } \rho^{\beta}\phi(\rho) \text{ is nondecreasing,} \tag{7}$$

$$\phi(\rho) \text{ is nondecreasing and } \lim_{\rho\downarrow 0}\phi(\rho) = 0. \tag{8}$$

Theorem 1 *Let $p \ge p_F$ and let ϕ be a positive and continuous function defined on $(0, \infty)$ satisfying (5) and (6). Suppose that ϕ satisfies (7) if $p > p_F$ or ϕ satisfies (8) if $p = p_F$. Then there exists $\mu \in X$ such that for all $T > 0$ the problem (1) does not admit any solutions.*

We note that, for every constant $c_1 > 0$, the positive function $\phi_1(\rho) := c_1(\log(e + 1/\rho))^{-1/(p-1)}$ is defined on $(0, \infty)$ and satisfies (5), (6) and (8). If $p > p_F$, (7) is also satisfied. Hence the conditions stated in Theorem 1 hold for the choice of ϕ in (4).

For existence, we introduce the following conditions:

$$\int_0^1 \frac{\phi(\eta)^{p-1}}{\eta}d\eta < +\infty, \tag{9}$$

or the stronger condition

$$\int_0^\infty \frac{\phi(\eta)^{p-1}}{\eta}d\eta < +\infty. \tag{10}$$

Theorem 2 *Let ϕ be a positive function defined on $(0, \infty)$.*

(i) *Let $p \ge p_F$. Suppose that (6) and (9) hold. Then for any $\mu \in X$ with $\mu \ne 0$, there exists a constant $T_0 = T_0(N, p, \phi, \|\mu\|_X) > 0$ such that the problem (1) for $T = T_0$ admits a positive solution.*

(ii) *Let $p > p_F$. Suppose that (6) and (10) hold. Then there exists a constant $c_0 = c_0(N, p, \phi) > 0$ such that for any $\mu \in X$ with $\mu \ne 0$ and $\|\mu\|_X < c_0$, the problem (1) for $T = +\infty$ admits a positive solution u satisfying*

$$\|u(\cdot,t)\|_{L^\infty(\mathbf{R}^N)} \le C\|\mu\|_X \phi(t^{\frac{1}{2}})t^{-\frac{1}{p-1}} \quad \text{for all } t>0$$

with a constant $C = C(N, p, \alpha) > 0$*.*

We remark that, for every $\varepsilon \in (0, 1/(p-1))$ and positive constants c_2 and c_3, the functions $\phi_2(\rho) := c_2(\log(e+1/\rho))^{-1/(p-1)-\varepsilon}$ and $\phi_3(\rho) := c_3 \min\{(\log(e+1/\rho))^{-1/(p-1)-\varepsilon}, (\log(e+\rho))^{-1/(p-1)-\varepsilon}\}$ satisfy the conditions stated in Theorem 2 (i) and (ii), respectively. We also remark that the equation in (1) with $T = +\infty$ has no positive solution for $p \le p_F$ (see for example [16, Theorem 18.1]).

This paper is organized as follows. In Sect. 2, Theorem 1 is proved. The proof is based on showing the unboundedness of some fractional maximal operator by utilizing a fractal-type set. In Sect. 3, we show Theorem 2 by the construction of a monotone iteration scheme.

2 Nonexistence of Solutions

In this section, we use the following notation. Let $Q_\rho(x)$ be the N-dimensional open cube of side 2ρ (not "ρ") centered at $x \in \mathbf{R}^N$. We write $Q_\rho = Q_{\rho/2}(\rho/2, \ldots, \rho/2)$, that is, $Q_\rho = (0,\rho) \times \cdots \times (0,\rho)$. Let $p \ge p_F$ and let ϕ be a positive and continuous function defined on $(0, \infty)$. Define

$$Y := \left\{f ; f \text{ is a Lebesgue measurable function on } \mathbf{R}^N \text{ with } \|f\|_Y < +\infty\right\},$$
$$\|f\|_Y := \sup_{x\in\mathbf{R}^N} \sup_{\rho>0} \left(\phi(\rho)^{-1}\rho^{-(N-\frac{2}{p-1})} \int_{Q_\rho(x)} |f(y)|dy\right).$$

We always regard a function $f \in Y$ as a Radon measure on $\mathbf{R}^N$ defined by $\mu_f(A) := \int_A |f(x)|dx$. We remark that $Y \subset X$, since $\|\mu_f\|_X \le \|f\|_Y$ for $f \in Y$. For a Radon measure ν, we define

$$U_\nu(x,t) := \int_{\mathbf{R}^N} G(x-y,t)d\nu(y), \tag{11}$$

where $G(x,t) := (4\pi t)^{-N/2} e^{-|x|^2/(4t)}$.

We assume that (6) and (7) hold for $p > p_F$ and that (6) and (8) hold for $p = p_F$. Theorem 1 will be proved once we prove the next proposition.

Proposition 1 *If (5) holds, then there exists a nonnegative function* $g_* \in Y$ *such that* $g_* = 0$ *a.e. in* $\mathbf{R}^N \setminus \overline{Q_1}$ *and* $\|U_{g_*}\|_{L^p(Q_{\sqrt{T}}\times(0,T))} = +\infty$ *for any* $T \in (0,1)$*, where* U_{g_*} *is defined by (11) with* $d\nu(y) = g_*(y)dy$*.*

Before proving the proposition, we show Theorem 1 by modifying the argument of Brezis and Cazenave [5, Proof of Theorem 11].

Proof (*Proof of Theorem* 1) Let us consider the solvability of the problem (1) under the condition that $\mu = g_*$, where $g_* \in Y$ is given by Proposition 1. Contrary to the conclusion, suppose that there exists $T_0 > 0$ such that the problem (1) for $T = T_0$ admits a solution u. Without loss of generality, we may assume that $T_0 < 1$.

Let $\varepsilon \in (0, T_0/2)$ and let $\psi \in C_0^\infty(\mathbf{R}^N)$ satisfy $0 \le \psi \le 1$, $\psi = 1$ in Q_1 and $\psi = 0$ in $\mathbf{R}^N \setminus Q_2(0)$. Integration by parts shows that

$$\begin{aligned}\|u\|^p_{L^p(Q_1\times(\varepsilon,T_0/2))} &\le \int_\varepsilon^{T_0/2}\int_{\mathbf{R}^N} u^p\psi\,dxdt\\ &= \int_{\mathbf{R}^N}(u(x,T_0/2)-u(x,\varepsilon))\psi(x)dx - \int_\varepsilon^{T_0/2}\int_{\mathbf{R}^N} u\Delta\psi\,dxdt\\ &\le \int_{\mathbf{R}^N} u(x,T_0/2)\psi(x)dx + \int_\varepsilon^{T_0/2}\int_{\mathbf{R}^N} u|\Delta\psi|dxdt.\end{aligned}$$

Since $|\Delta\psi| \in C_0(\mathbf{R}^N)$ and $u(\cdot,t) \to g_*$ as $t \downarrow 0$ weakly as measures on each fixed open ball, we have $u|\Delta\psi| \in L^1(\mathbf{R}^N \times (0, T_0/2))$. Thus letting $\varepsilon \to 0$ yields $\|u\|_{L^p(Q_1\times(0,T_0/2))} < +\infty$.

On the other hand, for $x \in \mathbf{R}^N$ and $s, t > 0$ with $t + s < T_0/2$, the maximum principle for classical solutions shows that

$$u(x,t+s) \ge \int_{\mathbf{R}^N} G(x-y,t)u(y,s)dy \ge \int_{\mathbf{R}^N} \psi(y)G(x-y,t)u(y,s)dy.$$

Since $\psi(\cdot)G(x-\cdot,t) \in C_0(\mathbf{R}^N)$ and $g_* = 0$ a.e. in $\mathbf{R}^N \setminus \overline{Q_1}$, letting $s \downarrow 0$ gives

$$u(x,t) \ge \int_{\mathbf{R}^N} \psi(y)G(x-y,t)g_*(y)dy = U_{g_*}(x,t)$$

for $(x,t) \in \mathbf{R}^N \times (0, T_0/2)$. By the relation $T_0/2 < 1$ and Proposition 1, we obtain

$$+\infty > \|u\|_{L^p(Q_1\times(0,T_0/2))} \ge \|U_{g_*}\|_{L^p(Q_{\sqrt{T_0/2}}\times(0,T_0/2))} = +\infty,$$

a contradiction. □

In the rest of this section, we prove Proposition 1 by means of several lemmas. The proof is based on showing the unboundedness of some fractional maximal operator by using

$$f_n(x) := \textstyle\prod_{i=1}^N \chi_{I_n}(x_i), \quad x = (x_1, \ldots, x_N) \in \mathbf{R}^N,$$

where $\{I_n\}_{n=0}^\infty$ is a sequence of sets defined later. We remark that $\cap_{n=0}^\infty I_n$ becomes a fractal-type set and that fractal sets are utilized for showing the optimality of maximal operators, see for instance Carro, Pérez, Soria and Soria [6, Theorem 2.1] and Sawano, Sugano and Tanaka [18, Proposition 4.1]. Recently, Kan and the author [11, Sect. 4] modified their method and proved the nonexistence of solutions with a time-dependent singularity by using $\chi_{E_n}(t)$, $t \in \mathbf{R}$, where $\{E_n\}_{n=0}^{\infty}$ is a suitable sequence of sets. The following is a multidimensional generalization of the argument of [11].

To define $\{I_n\}_{n=0}^{\infty}$, we construct a sequence $\{R_n\}_{n=0}^{\infty}$ as follows. Let $\theta \in (0, 1/2^N]$. We consider the equation

$$F(R) := R^{N-\frac{2}{p-1}}\phi(R) - \phi(1/2)\theta = 0.$$

The assumptions (7) and (8) imply that

$$\lim_{R\downarrow 0} F(R) = 0 - \phi(1/2)\theta < 0,$$
$$F(1/2) = \left\{(1/2)^{N-\frac{2}{p-1}} - \theta\right\}\phi(1/2) > 0.$$

We deduce from the continuity of F that there exists a positive solution of $F(R) = 0$ and that the smallest positive solution, denoted by $R(\theta)$, can be determined for each $\theta \in (0, 1/2^N]$. Set $R_0 := 1$ and $R_n := R(1/2^{Nn})$ for $n \geq 1$. One can observe that $1/2 > R_1 > R_2 > \cdots > R_n \to 0$ as $n \to \infty$ and

$$2^{Nn}R_n^{N-\frac{2}{p-1}}\phi(R_n) = \phi(1/2) \quad \text{for } n \geq 1. \tag{12}$$

By the following lemma, we see that $R_{n+1} < R_n/2$ for all $n \geq 0$.

Lemma 1 *Let $r_n := R_n/R_{n-1}$ for $n \geq 1$. Then there exist constants $\underline{r}$ and $\overline{r}$ with $0 < \underline{r} < \overline{r} < 1/2$ such that $\underline{r} \leq r_n \leq \overline{r}$ for $p > p_F$ and $0 < r_n \leq \overline{r}$ for $p = p_F$.*

Proof Let $n \geq 1$. By (12) and the monotonicity conditions (6) and (7), we have

$$2^{Nn}R_n^{N-\frac{2}{p-1}}\phi(R_n) = 2^{N(n-1)}R_{n-1}^{N-\frac{2}{p-1}}\phi(R_{n-1})$$
$$\begin{cases} \leq 2^{N(n-1)}R_{n-1}^{N-\frac{2}{p-1}+\alpha}R_n^{-\alpha}\phi(R_n) & \text{for } p \geq p_F, \\ \geq 2^{N(n-1)}R_{n-1}^{N-\frac{2}{p-1}-\beta}R_n^{\beta}\phi(R_n) & \text{for } p > p_F. \end{cases}$$

Simple calculations show that

$$0 < (1/2)^{N/(N-\frac{2}{p-1}-\beta)} \leq r_n \leq (1/2)^{N/(N-\frac{2}{p-1}+\alpha)} < 1/2 \qquad \text{for } p > p_F,$$
$$0 < r_n \leq (1/2)^{N/(N-\frac{2}{p-1}+\alpha)} < 1/2 \qquad \text{for } p \geq p_F,$$

which proves the lemma. □

Define $\{I_n\}_{n=0}^{\infty}$ by $I_n = \bigcup_{l=1}^{2^n}(a_{n,l}, b_{n,l})$, where each of the elements of $\{a_{n,l}\}$ and $\{b_{n,l}\}$ is inductively determined in the following way:

$$a_{0,1} := 0, \qquad b_{0,1} := 1,$$
$$a_{n+1,2l-1} = a_{n,l}, \qquad b_{n+1,2l-1} = a_{n,l} + R_{n+1},$$
$$a_{n+1,2l} = b_{n,l} - R_{n+1}, \qquad b_{n+1,2l} = b_{n,l}.$$

From definition and Lemma 1, it follows that

$$0 = a_{n,1} < b_{n,1} < a_{n,2} < b_{n,2} < \cdots < a_{n,2^n} < b_{n,2^n} = 1,$$

$b_{n,l} - a_{n,l} = R_n$ and $I_n \supset I_{n+1} \supset I_{n+2} \supset \cdots$.

Set $\gamma := 2/(Np)$. For $f \in Y$, we define a fractional maximal function M_f by

$$M_f(x) := \sup_{P=(\rho_1,\ldots,\rho_N)\in Q_1} \left(|P|^{-N(1-\gamma)} \int_{D_P(x)} |f(y)|dy \right), \tag{13}$$

$$D_P(x) := (x_1 + (1-2\overline{r})\rho_1, x_1+\rho_1) \times \cdots \times (x_N + (1-2\overline{r})\rho_N, x_N+\rho_N). \tag{14}$$

We will show the unboundedness of a fractional maximal operator $M : f \longmapsto M_f$ in the sense that $\|M_{f_n}\|_{L^p(Q_1)}/\|f_n\|_Y \to +\infty$ as $n \to \infty$, where $f_n(x) = \prod_{i=1}^N \chi_{I_n}(x_i)$. Before starting estimation, we note that the monotonicity conditions (6)–(8) imply that

$$\eta^{-\frac{2}{p-1}}\phi(\eta) \text{ is decreasing} \tag{15}$$

and

$$\eta^{N-\frac{2}{p-1}}\phi(\eta) \text{ is nondecreasing.} \tag{16}$$

Lemma 2 $\|f_n\|_Y \leq C(2^nR_n)^N$, *where* $C > 0$ *is a constant independent of* n.

Proof From (15) and the definition of $Q_\rho(x)$, we see that

$$\begin{aligned}\|f_n\|_Y &\leq 2^{\frac{2}{p-1}} \sup_{x\in\mathbf{R}^N}\sup_{\rho>0}\left(\phi(2\rho)^{-1}\rho^{-(N-\frac{2}{p-1})}\int_{Q_\rho(x)} f_n(y)dy\right)\\ &\leq 2^{\frac{2}{p-1}}\prod_{i=1}^N \sup_{x_i\in\mathbf{R}}\sup_{\rho_i>0}\left(\phi(2\rho_i)^{-\frac{1}{N}}\rho_i^{-\frac{1}{N}(N-\frac{2}{p-1})}\int_{x_i-\rho_i}^{x_i+\rho_i}\chi_{I_n}(\eta)d\eta\right).\end{aligned}$$

Taking $a_i = x_i - \rho_i$ and $b_i = x_i + \rho_i$, we have

$$\begin{aligned}\|f_n\|_Y &\leq 2^{\frac{2}{p-1}}\prod_{i=1}^N \sup_{a_i<b_i}\left(\phi(b_i-a_i)^{-\frac{1}{N}}\{(b_i-a_i)/2\}^{-\frac{1}{N}(N-\frac{2}{p-1})}\int_{a_i}^{b_i}\chi_{I_n}(\eta)d\eta\right)\\ &= 2^N\left\{\sup_{a<b}\left(\phi(b-a)^{-\frac{1}{N}}(b-a)^{-\frac{1}{N}(N-\frac{2}{p-1})}\int_a^b\chi_{I_n}(\eta)d\eta\right)\right\}^N\\ &=: 2^N\left(\sup_{a<b}A_n(a,b)\right)^N.\end{aligned}$$

We put $\tilde{A}_n = \sup_{a<b} A_n(a,b)$.

Let us examine the function $A_n(a, b)$. We observe the following 4 cases: (i) $a \in I_n$ and $b \in I_n$, (ii) $a \notin I_n$ and $b \in I_n$, (iii) $a \in I_n$ and $b \notin I_n$, (iv) $a \notin I_n$ and $b \notin I_n$. Then by (15) and (16), we see that $A_n(a, b)$ is decreasing in $a \in I_n$, increasing in $a \notin I_n$, increasing in $b \in I_n$ and decreasing in $b \notin I_n$. This observation shows that

$$\begin{aligned}\tilde{A}_n &\le \max_{1\le l_1\le l_2\le 2^n}\left(\phi(b_{n,l_2}-a_{n,l_1})^{-\frac{1}{N}}(b_{n,l_2}-a_{n,l_1})^{-\frac{1}{N}(N-\frac{2}{p-1})}\int_{a_{n,l_1}}^{b_{n,l_2}}\chi_{I_n}(\eta)d\eta\right)\\ &= R_n \max_{1\le l_1\le l_2\le 2^n}\left((l_2-l_1+1)\left\{(b_{n,l_2}-a_{n,l_1})^{N-\frac{2}{p-1}}\phi(b_{n,l_2}-a_{n,l_1})\right\}^{-\frac{1}{N}}\right).\end{aligned}$$

Let $1 \le l_1 \le l_2 \le 2^n$. Then $2^m \le l_2 - l_1 + 1 \le 2^{m+1} - 1$ for some $0 \le m \le n$. We observe that $b_{n,l_2} \ge b_{n,l_1+2^m-1}$ and that

$$\begin{aligned}&b_{n,l_1} - a_{n,l_1} = R_n, && b_{n,l_1+2^1-1} - a_{n,l_1} \ge R_{n-1} - 2R_n,\\ &b_{n,l_1+2^2-1} - a_{n,l_1} \ge R_{n-2} - 2R_{n-1}, \quad \dots, && b_{n,l_1+2^m-1} - a_{n,l_1} \ge R_{n-m} - 2R_{n-m+1}.\end{aligned}$$

By Lemma 1, we have

$$b_{n,l_2} - a_{n,l_1} \ge R_{n-m} - 2R_{n-m+1} = R_{n-m}(1 - 2r_{n-m+1}) \ge R_{n-m}(1 - 2\overline{r}).$$

Hence (16) yields

$$\tilde{A}_n \le R_n \max_{0\le m\le n}\left(2^{m+1}\left[\{R_{n-m}(1-2\overline{r})\}^{N-\frac{2}{p-1}}\phi(R_{n-m}(1-2\overline{r}))\right]^{-\frac{1}{N}}\right).$$

By (15), we have

$$\left[\{R_{n-m}(1-2\overline{r})\}^{-\frac{2}{p-1}}\phi(R_{n-m}(1-2\overline{r}))\right]^{-\frac{1}{N}} \le \left(R_{n-m}^{-\frac{2}{p-1}}\phi(R_{n-m})\right)^{-\frac{1}{N}}.$$

This together with (12) gives

$$\begin{aligned}\tilde{A}_n &\le 2R_n(1-2\overline{r})^{-1}\max_{0\le m\le n}\left(2^m\left(R_{n-m}^{N-\frac{2}{p-1}}\phi(R_{n-m})\right)^{-\frac{1}{N}}\right)\\ &= 2R_n(1-2\overline{r})^{-1}\max_{0\le m\le n}\left(2^m\left\{2^{-N(n-m)}\phi(1/2)\right\}^{-\frac{1}{N}}\right)\\ &= 2(1-2\overline{r})^{-1}\phi(1/2)^{-\frac{1}{N}}(2^nR_n),\end{aligned}$$

and the lemma follows. □

Lemma 3 *If $p > p_F$, then there exists a constant $C > 0$ independent of n such that*

$$\|M_{f_n}\|^p_{L^p(Q_1)} \geq \frac{1}{C}(2^n R_n)^{Np} \sum_{j=0}^{n-1} \phi(R_j)^{p-1}.$$

Proof Let $n \geq 1$. For $0 \leq j \leq n-1$, $1 \leq i \leq N$ and $1 \leq l_i \leq 2^j$, taking into account that $b_{j,l_i} - a_{j,l_i} = R_j > R_{j+1} + R_{j+1}/(2\overline{r})$ by Lemma 1, we define

$$J_{j,l_i} := (a_{j,l_i} + R_{j+1}, b_{j,l_i} - R_{j+1}/(2\overline{r})), \quad J_{j,l_1,\dots,l_N} := J_{j,l_1} \times \cdots \times J_{j,l_N}.$$

We see that $(\bigcup_{1\leq l_1,\dots,l_N\leq 2^j} J_{j,l_1,\dots,l_N}) \cap (\bigcup_{1\leq l'_1,\dots,l'_N\leq 2^{j'}} J_{j',l'_1,\dots,l'_N}) = \emptyset$ for $j \neq j'$ and $\bigcup_{j=0}^{n-1} \bigcup_{1\leq l_1,\dots,l_N\leq 2^j} J_{j,l_1,\dots,l_N} \subset Q_1$.

Let $x \in J_{j,l_1,\dots,l_N}$. Then $R_{j+1}/(2\overline{r}) \leq b_{j,l_i} - x_i \leq R_j - R_{j+1} < 1/2$. Substituting $P = (\rho_1, \dots, \rho_N) = (b_{j,l_1} - x_1, \dots, b_{j,l_N} - x_N)$ into (14) gives

$$\begin{aligned} D_P(x) &= (b_{j,l_1} - 2\overline{r}(b_{j,l_1} - x_1), b_{j,l_1}) \times \cdots \times (b_{j,l_N} - 2\overline{r}(b_{j,l_N} - x_N), b_{j,l_N}) \\ &\supset (b_{j,l_1} - R_{j+1}, b_{j,l_1}) \times \cdots \times (b_{j,l_N} - R_{j+1}, b_{j,l_N}). \end{aligned}$$

Hence by (13), we obtain

$$M_{f_n}(x) \geq \left\{ \sum_{i=1}^N (b_{j,l_i} - x_i)^2 \right\}^{-\frac{N}{2}(1-\gamma)} \prod_{i=1}^N \int_{b_{j,l_i}-R_{j+1}}^{b_{j,l_i}} \chi_{I_n}(\eta) d\eta.$$

By counting intervals in I_n, we see that $\int_{b_{j,l_i}-R_{j+1}}^{b_{j,l_i}} \chi_{I_n}(\eta) d\eta = 2^{n-j-1} R_n$. Thus,

$$M_{f_n}(x) \geq (2^{n-j-1} R_n)^N \left\{ \sum_{i=1}^N (b_{j,l_i} - x_i)^2 \right\}^{-\frac{N}{2}(1-\gamma)}$$

for $x \in J_{j,l_1,\dots,l_N}$, so that

$$\begin{aligned} \|M_{f_n}\|^p_{L^p(Q_1)} &\geq \sum_{j=0}^{n-1} \left(\sum_{1\leq l_1,\cdots,l_N\leq 2^j} \int_{J_{j,l_1,\dots,l_N}} M_{f_n}(y)^p dy \right) \\ &\geq \sum_{j=0}^{n-1} (2^{n-j-1} R_n)^{Np} \sum_{1\leq l_1,\cdots,l_N\leq 2^j} \int_{J_{j,l_1,\dots,l_N}} \left\{ \sum_{i=1}^N (b_{j,l_i} - y_i)^2 \right\}^{-\frac{Np}{2}(1-\gamma)} dy. \end{aligned}$$

The change of variables $\tilde{y}_i = b_{j,l_i} - y_i$ and $\tilde{y} := (\tilde{y}_1, \dots, \tilde{y}_N)$ gives

$$\|M_{f_n}\|^p_{L^p(Q_1)} \geq \frac{1}{2^{Np}} (2^n R_n)^{Np} \sum_{j=0}^{n-1} 2^{-Nj(p-1)} \int_{D_j} |\tilde{y}|^{-Np(1-\gamma)} d\tilde{y}$$

for $p > p_F$, where $D_j := (R_{j+1}/(2\overline{r}), R_j - R_{j+1}) \times \cdots \times (R_{j+1}/(2\overline{r}), R_j - R_{j+1})$. Note that even if $p = p_F$ the above inequalities hold, and so

$$\|M_{f_n}\|^p_{L^p(Q_1)} \geq \frac{1}{C}(2^n R_n)^{Np} \sum_{j=0}^{n-1} 2^{-Nj(p-1)} \int_{D_j} |\tilde{y}|^{-N} d\tilde{y}. \tag{17}$$

We continue to examine the case $p > p_F$. Since $R_{j+1} = R_j r_{j+1}$, we have

$$\begin{aligned}\int_{D_j} |\tilde{y}|^{-Np(1-\gamma)} d\tilde{y} &\geq \left\{N(R_j - R_{j+1})^2\right\}^{-\frac{Np}{2}(1-\gamma)} \left\{R_j - R_{j+1} - R_{j+1}/(2\overline{r})\right\}^N \\ &\geq N^{-\frac{Np}{2}(1-\gamma)} \left\{1 - r_{j+1} - r_{j+1}/(2\overline{r})\right\}^N R_j^{N-Np(1-\gamma)}.\end{aligned}$$

From Lemma 1 and the relation $N - Np(1-\gamma) = -(p-1)\{N - 2/(p-1)\}$, it follows that

$$\|M_{f_n}\|^p_{L^p(Q_1)} \geq \frac{(1/2 - \overline{r})^N}{2^{Np} N^{Np(1-\gamma)/2}} (2^n R_n)^{Np} \sum_{j=0}^{n-1} \left(2^{Nj} R_j^{N-\frac{2}{p-1}}\right)^{-(p-1)}.$$

Then (12) shows the lemma. □

To estimate the norm of M_{f_n} for the case $p = p_F$, we need the following lemma.

Lemma 4 *Let $b > a > 0$. Then for any $\varepsilon \in (0, (b/a) - 1)$, there exists a constant $C > 0$ depending only on N and ε such that*

$$\int_a^b \cdots \int_a^b |y|^{-N} dy_1 \cdots dy_N \geq \frac{1}{C}(\log b - \log a - \log(1+\varepsilon)).$$

Proof Since the case $N = 1$ is easy, we only consider the case $N \geq 2$. We prove this lemma by geometric observation. Define points $P^a_1, \ldots P^a_N, P^b_1, \ldots P^b_N$ in $\mathbf{R}^N$ by

$$\begin{aligned}&P^a_1 := ((1+\varepsilon)a, a, \ldots, a), \quad P^a_2 := (a, (1+\varepsilon)a, a, \ldots, a), \\ &\quad \ldots, \quad P^a_N := (a, a, \ldots, a, (1+\varepsilon)a), \\ &P^b_1 := \left(b, \frac{b}{1+\varepsilon}, \frac{b}{1+\varepsilon}, \ldots, \frac{b}{1+\varepsilon}\right), \quad P^b_2 := \left(\frac{b}{1+\varepsilon}, b, \frac{b}{1+\varepsilon}, \ldots, \frac{b}{1+\varepsilon}\right), \\ &\quad \ldots, \quad P^b_N := \left(\frac{b}{1+\varepsilon}, \ldots, \frac{b}{1+\varepsilon}, b\right).\end{aligned}$$

Note that, for each i, the three points P^a_i, P^b_i and 0 lie on the same line.

Let $A^a \subset \mathbf{R}^N$ be the regular $(N-1)$-simplex with vertices $P_1^a, \ldots, P_N^a$ and let B_{in}^a be the $(N-1)$-dimensional ball inscribed in A^a with radius r_{in}^a and centered at P_{in}^a. Let $K^a \subset \mathbf{R}^N$ be the cone of vertex $0 \in \mathbf{R}^N$ and base B_{in}^a. The height of K^a is $|P_{\text{in}}^a|$. Let r_s^a denote the slant height of K^a. In the same manner, we define A^b, B_{in}^b, r_{in}^b, P_{in}^b, K^b and r_s^b as associated with $P_1^b, \ldots, P_N^b$. Note that

$$r_{\text{in}}^a = \frac{\varepsilon}{\sqrt{(N-1)N}}a, \quad P_{\text{in}}^a = \left(\frac{N+\varepsilon}{N}a, \ldots, \frac{N+\varepsilon}{N}a\right),$$
$$(r_s^a)^2 = (r_{\text{in}}^a)^2 + |P_{\text{in}}^a|^2 = \left\{\frac{\varepsilon^2}{(N-1)N} + \frac{(N+\varepsilon)^2}{N}\right\} a^2 > \left(\sqrt{N}a\right)^2$$

and that

$$r_{\text{in}}^b = \frac{\varepsilon}{(1+\varepsilon)\sqrt{(N-1)N}}b, \quad P_{\text{in}}^b = \left(\frac{N+\varepsilon}{(1+\varepsilon)N}b, \ldots, \frac{N+\varepsilon}{(1+\varepsilon)N}b\right),$$
$$(r_s^b)^2 = (r_{\text{in}}^b)^2 + |P_{\text{in}}^b|^2 = \left\{\frac{\varepsilon^2}{(N-1)N} + \frac{(N+\varepsilon)^2}{N}\right\} \frac{1}{(1+\varepsilon)^2}b^2 < \left(\sqrt{N}b\right)^2.$$

Since $\varepsilon < (b/a) - 1$, we have $|P_{\text{in}}^b|^2 - |P_{\text{in}}^a|^2 > 0$. Then we see that $K^b \setminus K^a \subset [a,b]^N$.

Define $\tilde{C} := |P_{\text{in}}^a|/r_s^a$. Simple calculations show that

$$\left(\frac{|P_{\text{in}}^a|}{r_s^a}\right)^2 = \left(\frac{|P_{\text{in}}^b|}{r_s^b}\right)^2 = \frac{(N+\varepsilon)^2}{\varepsilon^2(N-1)^{-1} + (N+\varepsilon)^2},$$

so that $0 < \tilde{C} < 1$ and $\tilde{C}$ depends only on N and ε. Let S^a be the spherical cap of $B_{r_s^a}(0)$ associated with the cone K^a, that is, $S^a \subset B_{r_s^a}(0)$ is a spherical cap of height $r_s^a - |P_{\text{in}}^a|$ and radius r_s^a. Let S^b be the spherical cap of $B_{r_s^b}(0)$ determined by the same way. Then $S^b \setminus S^a \subset [a,b]^N$ and the area of S^a, denoted by S_{area}^a, can be written as

$$S_{\text{area}}^a = \frac{1}{2}N\omega_N (r_s^a)^{N-1} \frac{\int_0^{1-\tilde{C}^2} \eta^{\frac{N-1}{2}-1}(1-\eta)^{\frac{1}{2}-1}d\eta}{\int_0^1 \eta^{\frac{N-1}{2}-1}(1-\eta)^{\frac{1}{2}-1}d\eta},$$

where ω_N is the volume of the unit ball in $\mathbf{R}^N$. Hence we can estimate that

$$\int_a^b \cdots \int_a^b |y|^{-N} dy_1 \cdots dy_N \ge \frac{S_{\text{area}}^a}{N\omega_N (r_s^a)^{N-1}} \int_{B_{r_s^b}(0)\setminus B_{r_s^a}(0)} |y|^{-N} dy$$
$$= \frac{1}{C}(\log b - \log a - \log(1+\varepsilon))$$

for some constant $C > 0$ depending only on N and ε. Thus the lemma follows. □

Lemma 4 yields the following estimate.

Lemma 5 *If $p = p_F$, then there exists a constant $C > 0$ independent of n such that*

$$\|M_{f_n}\|^p_{L^p(Q_1)} \geq (2^n R_n)^{Np} \left\{ \frac{1}{C} \sum_{j=0}^{n-1} \left(\phi(R_j)^{p-1} \log \frac{R_j}{R_{j+1}} \right) - C \right\}.$$

Proof Let us examine the integral in the right-hand side of (17). Since

$$\frac{R_j - R_{j+1}}{R_{j+1}/(2\overline{r})} = \frac{1 - r_{j+1}}{r_{j+1}/(2\overline{r})} \geq 2(1 - \overline{r}) > 1,$$

we can apply Lemma 4 with $a = R_{j+1}/(2\overline{r})$, $b = R_j - R_{j+1}$ and $\varepsilon = 2(1 - \overline{r}) - 1$. Therefore we can estimate that

$$\begin{aligned}
\int_{D_j} |\tilde{y}|^{-N} d\tilde{y} &\geq \frac{1}{C} \left\{ \log(R_j - R_{j+1}) - \log(R_{j+1}/(2\overline{r})) - \log(2(1 - \overline{r})) \right\} \\
&= \frac{1}{C} \left\{ \log \frac{R_j}{R_{j+1}} - \log \left(\frac{R_j}{R_{j+1}} \times \frac{R_{j+1}}{2\overline{r}} \times \frac{2(1 - \overline{r})}{R_j(1 - r_{j+1})} \right) \right\} \\
&\geq \frac{1}{C} \left(\log \frac{R_j}{R_{j+1}} - \log \frac{1}{\overline{r}} \right).
\end{aligned}$$

Hence by (17) and (12), we obtain

$$\begin{aligned}
\|M_{f_n}\|^p_{L^p(Q_1)} &\geq (2^n R_n)^{Np} \left\{ \frac{1}{C} \sum_{j=0}^{n-1} 2^{-Nj(p-1)} \log \frac{R_j}{R_{j+1}} - C \sum_{j=0}^{\infty} 2^{-Nj(p-1)} \right\} \\
&\geq (2^n R_n)^{Np} \left\{ \frac{1}{C} \sum_{j=0}^{n-1} \left(\frac{\phi(1/2)}{\phi(R_j)} \right)^{-(p-1)} \log \frac{R_j}{R_{j+1}} - C \right\},
\end{aligned}$$

which proves the lemma. □

Lemmas 1–3 and 5 show the unboundedness of $M : f \longmapsto M_f$.

Lemma 6 *If (5) holds, then $\|M_{f_n}\|_{L^p(Q_1)}/\|f_n\|_Y \to +\infty$ as $n \to \infty$.*

Proof First, we consider the case $p > p_F$. By Lemmas 2 and 3, we have

$$\frac{\|M_{f_n}\|^p_{L^p(Q_1)}}{\|f_n\|^p_Y} \geq \frac{1}{C} \sum_{j=1}^{n-1} \phi(R_j)^{p-1} = \frac{1}{C} \sum_{j=1}^{n-1} \frac{\phi(R_j)^{p-1}}{R_j^2} (R_{j-1} - R_j) \times \frac{R_j^2}{R_{j-1} - R_j}.$$

Since $\eta^{-2}\phi(\eta)^{p-1}$ is decreasing, Lemma 1 and (5) give

$$\begin{aligned}
\frac{\|M_{f_n}\|^p_{L^p(Q_1)}}{\|f_n\|^p_Y} &\geq \frac{1}{C}\sum_{j=1}^{n-1}\int_{R_j}^{R_{j-1}} \frac{\phi(\eta)^{p-1}}{\eta^2}d\eta \times \frac{R_j^2}{R_{j-1}-R_j} \\
&\geq \frac{1}{C}\sum_{j=1}^{n-1}\int_{R_j}^{R_{j-1}} \frac{\phi(\eta)^{p-1}}{\eta}d\eta \times \frac{R_j^2}{R_{j-1}(R_{j-1}-R_j)} \\
&\geq \frac{r^2}{C}\int_{R_{n-1}}^{1} \frac{\phi(\eta)^{p-1}}{\eta}d\eta \to +\infty \quad \text{as } n\to\infty.
\end{aligned}$$

We next examine the case $p = p_F$. Lemmas 2 and 5 yield

$$\frac{\|M_{f_n}\|^p_{L^p(Q_1)}}{\|f_n\|^p_Y} \geq \frac{1}{C}\sum_{j=0}^{n-1}\phi(R_j)^{p-1}\log\frac{R_j}{R_{j+1}} - C = \frac{1}{C}\sum_{j=0}^{n-1}\int_{R_{j+1}}^{R_j}\frac{\phi(R_j)^{p-1}}{\eta}d\eta - C.$$

By (16) and (5), we obtain

$$\begin{aligned}
\frac{\|M_{f_n}\|^p_{L^p(Q_1)}}{\|f_n\|^p_Y} &\geq \frac{1}{C}\sum_{j=0}^{n-1}\int_{R_{j+1}}^{R_j}\frac{\phi(\eta)^{p-1}}{\eta}d\eta - C \\
&= \frac{1}{C}\int_{R_n}^{1}\frac{\phi(\eta)^{p-1}}{\eta}d\eta - C \to +\infty \quad \text{as } n\to\infty.
\end{aligned}$$

Thus the lemma follows. □

By using Lemma 6 and the closed graph theorem, we prove the following lemma. We remark that the idea of using the theorem is due to Brezis and Cazenave [5, Proof of Theorem 11].

Lemma 7 *Let $T > 0$. If (5) holds, then there exists a nonnegative function $g_T \in Y$ with $g_T = 0$ a.e. in $\mathbf{R}^N \setminus \overline{Q_{\sqrt{T}}}$ such that $\|U_{g_T}\|_{L^p(Q_{\sqrt{T}}\times(0,T))} = +\infty$, where U_{g_T} is defined by (11) with $d\nu(y) = g_T(y)dy$.*

Proof First, we show the lemma for the case $T = 1$. Define

$$\tilde{Y} := \{g \in Y; g = 0 \text{ a.e. in } \mathbf{R}^N \setminus \overline{Q_1}\}.$$

We recall $I_0 = (0, 1)$. To derive a contradiction, suppose that $U_g \in L^p(Q_1 \times I_0)$ for any nonnegative function $g \in \tilde{Y}$. Then $U_g \in L^p(Q_1 \times I_0)$ for any $g \in \tilde{Y}$, so that a linear operator $L : \tilde{Y} \ni g \mapsto U_g \in L^p(Q_1 \times I_0)$ is well defined.

We will prove that L is closed. Let $\{g_n\}_{n=1}^\infty \subset \tilde{Y}$. Suppose that there exist functions $g_\infty \in \tilde{Y}$ and $\tilde{g}_\infty \in L^p(Q_1 \times I_0)$ such that $g_n \to g_\infty$ in Y and $Lg_n \to \tilde{g}_\infty$ in $L^p(Q_1 \times I_0)$. By the definition of U_g, we have

$$\int_{I_0}\int_{Q_1} |Lg_n(x,t) - Lg_\infty(x,t)|dxdt \le \int_{I_0}\int_{\mathbf{R}^N}\int_{\mathbf{R}^N} G(x-y,t)|g_n(y) - g_\infty(y)|dydxdt.$$

The Fubini theorem and the definition of $\tilde{Y}$ show that

$$\begin{aligned}\|Lg_n - Lg_\infty\|_{L^1(Q_1\times I_0)} &\le \int_{Q_{1/2}((1/2,\ldots,1/2))} |g_n(y) - g_\infty(y)|dy \\ &\le \phi(1/2)(1/2)^{N-\frac{2}{p-1}}\|g_n - g_\infty\|_Y \to 0 \quad \text{as } n\to\infty,\end{aligned}$$

and so $Lg_n \to Lg_\infty$ in $L^1(Q_1 \times I_0)$. Since $Lg_n \to \tilde{g}_\infty$ in $L^1(Q_1 \times I_0)$, we see that $Lg_\infty = \tilde{g}_\infty$ and L is closed. Hence the closed graph theorem shows that L is bounded.

Let $g \in \tilde{Y}$ be a nonnegative function and let $P = (\rho_1, \ldots, \rho_N) \in Q_1$. For any $x \in Q_1$ and $t \in (|P|^2/(2N), |P|^2/N) \subset I_0$, simple calculations show that

$$\begin{aligned}Lg(x,t) &\ge (4\pi t)^{-\frac{N}{2}}\int_{D_P(x)} e^{-\frac{(y_1-x_1)^2}{4t}}\cdots e^{-\frac{(y_N-x_N)^2}{4t}} g(y)dy \\ &\ge (4\pi/N)^{-\frac{N}{2}} e^{-\rho_1^2/(2|P|^2/N)}\cdots e^{-\rho_N^2/(2|P|^2/N)}|P|^{-N}\int_{D_P(x)} g(y)dy \\ &\ge (4\pi/N)^{-\frac{N}{2}} e^{-\frac{N}{2}}\cdots e^{-\frac{N}{2}}|P|^{-N}\int_{D_P(x)} g(y)dy,\end{aligned}$$

where $D_P(x)$ is defined by (14), so that

$$\begin{aligned}\|Lg(x,\cdot)\|_{L^p(I_0)} &\ge \|Lg(x,\cdot)\|_{L^p((|P|^2/(2N),|P|^2/N))} \\ &\ge \frac{1}{C}|P|^{-N(1-\frac{2}{Np})}\int_{D_P(x)} g(y)dy.\end{aligned}$$

By (13), we have $\|Lg(x,\cdot)\|_{L^p(I_0)} \ge (1/C)M_g(x)$ for any $x \in Q_1$. Hence by the boundedness of L, we obtain

$$\|M_g\|_{L^p(Q_1)} \le C\|Lg\|_{L^p(Q_1\times I_0)} \le C\|g\|_Y.$$

This contradicts Lemma 6, and the lemma is proved for $T = 1$.

We next examine the case where $T > 0$ is chosen arbitrarily. Let $g_1 \in \tilde{Y}$ be a nonnegative function such that $\|U_{g_1}\|_{L^p(Q_1\times I_0)} = +\infty$, and define $g_T(x) := g_1(x/\sqrt{T})$. Then $g_T \in Y$ and $g_T = 0$ a.e. in $\mathbf{R}^N \setminus \overline{Q_{\sqrt{T}}}$. Moreover, by the change of variables $\tilde{x} = T^{-1/2}x$, $\tilde{y} = T^{-1/2}y$ and $\tilde{t} = T^{-1}t$, we can calculate that $\|U_{g_T}\|_{L^p(Q_{\sqrt{T}}\times(0,T))} = T^{(N/2+1)/p}\|U_{g_1}\|_{L^p(Q_1\times I_0)} = +\infty$. Thus the proof is complete. □

We are now in a position to prove Proposition 1.

Proof (*Proof of Proposition* 1) Let $\{q_j\}_{j=1}^\infty$ be the set of all rational numbers in $(0, 1)$. Lemma 7 guarantees that, for each j, there exists a nonnegative function $g_j \in Y$ such that $g_j = 0$ a.e. in $\mathbf{R}^N \setminus \overline{Q_{\sqrt{q_j}}}$ and $\|U_{g_j}\|_{L^p(Q_{\sqrt{q_j}}\times(0,q_j))} = +\infty$. Define a function on

$\mathbf{R}^N$ independent of j by

$$g_*(x) := \sum_{j=1}^{\infty} 2^{-j} \frac{g_j(x)}{\|g_j\|_Y}.$$

We see that $g_* = 0$ a.e. in $\mathbf{R}^N \setminus \overline{Q_1}$ and that $g_* \in Y$, since $\|g_*\|_Y \le 1$. Let $T \in (0, 1)$. Then by choosing j_0 such that $Q_{\sqrt{q_{j_0}}} \times (0, q_{j_0}) \subset Q_{\sqrt{T}} \times (0, T)$, we have

$$\|U_{g_*}\|_{L^p(Q_{\sqrt{T}} \times (0,T))} \ge 2^{-j_0} \|g_{j_0}\|_Y^{-1} \|U_{g_{j_0}}\|_{L^p(Q_{\sqrt{q_{j_0}}} \times (0, q_{j_0}))} = +\infty.$$

Hence g_* has the desired properties. □

As proved before, Theorem 1 follows from Proposition 1. Hence the proof of Theorem 1 is complete.

3 Existence of Solutions

Let $p \ge p_F$ and let $\mu \in X = X^p_\phi(\mathbf{R}^N)$ with a positive function ϕ on $(0, \infty)$ satisfying (6). To obtain solutions of (1), by the variation of constants formula, we consider the following integral equation

$$u = \Phi[u], \quad \Phi[u] := U_\mu + S[u], \tag{18}$$

where U_μ is defined by (11) with $\nu = \mu$ and $S[u]$ is defined by

$$S[u](x, t) := \int_0^t \int_{\mathbf{R}^N} G(x - y, t - s) u(y, s)^p dyds.$$

We will prove Theorem 2 by solving (18).

Let us first observe the properties of U_μ. By Giga and Miyakawa [9, Proposition 3.2.iii], $U_\mu(\cdot, t) \to \mu$ as $t \downarrow 0$ weakly as measures on each fixed open ball. Moreover, the Fubini theorem and integration by parts show that $(U_\mu)_t - \Delta U_\mu = 0$ in $\mathbf{R}^N \times (0, \infty)$ in the sense of distribution. Hence by the standard regularity theory, U_μ is smooth on $\mathbf{R}^N \times (0, \infty)$. If we suppose in addition that $\mu \ne 0$, then $\mu(B_{\rho_0}(x_0)) > 0$ for some $x_0 \in \mathbf{R}^N$ and $\rho_0 > 0$. Hence U_μ is positive if $\mu \ne 0$.

We next give the pointwise estimate of U_μ.

Lemma 8 *There exists a constant $C = C(N, p, \alpha) > 0$ such that*

$$\|U_\mu(\cdot, t)\|_{L^\infty(\mathbf{R}^N)} \le C \|\mu\|_X \phi(t^{\frac{1}{2}}) t^{-\frac{1}{p-1}} \quad \text{for all } t > 0.$$

Proof Since $G(x, t) \le C'|x|^{-N}$ and $G(x, t) \le C' t^{-N/2}$ for some constant $C' = C'(N) > 0$, the change of variables $\rho = (C'/\lambda)^{1/N}$ gives

$$
\begin{aligned}
U_\mu(x,t) &= \int_0^\infty \mu(\{y \in \mathbf{R}^N; G(x-y,t) > \lambda\})d\lambda \\
&\le \int_0^{C' t^{-N/2}} \mu(\{y \in \mathbf{R}^N; C'|x-y|^{-N} > \lambda\})d\lambda \\
&= C'N \int_{t^{1/2}}^\infty \rho^{-N-1}\mu(B_\rho(x))d\rho.
\end{aligned}
$$

By the assumptions $\mu \in X$ and (6), we obtain

$$
\begin{aligned}
U_\mu(x,t) &\le C\|\mu\|_X \int_{t^{1/2}}^\infty \rho^{-1-\frac{2}{p-1}}\phi(\rho)d\rho \\
&= C\|\mu\|_X \int_{t^{1/2}}^\infty \rho^{-1-(\frac{2}{p-1}-\alpha)}\rho^{-\alpha}\phi(\rho)d\rho \\
&\le C\|\mu\|_X t^{-\frac{\alpha}{2}}\phi(t^{\frac{1}{2}}) \int_{t^{1/2}}^\infty \rho^{-1-(\frac{2}{p-1}-\alpha)}d\rho \\
&= C\|\mu\|_X \phi(t^{\frac{1}{2}})t^{-\frac{1}{p-1}},
\end{aligned}
$$

which shows the Lemma. □

We next give the pointwise estimate of $S[U_\mu]$.

Lemma 9 *There exists a constant $C = C(N,p,\alpha) > 0$ such that*

$$
S[U_\mu](x,t) \le C\|\mu\|_X^{p-1}K(t)U_\mu(x,t) \quad \textit{for all } (x,t) \in \mathbf{R}^N \times (0,\infty),
$$

where $K(t) := \int_0^{\sqrt{t}} \eta^{-1}\phi(\eta)^{p-1}d\eta$.

Proof One can check that

$$
G(x-y,t-s)G(y-z,s) = G(x-z,t)G(\xi,\tau),
$$
$$
\xi = \xi(x,y,z,s,t) := y - \frac{s}{t}x - \frac{t-s}{t}z, \quad \tau = \tau(s,t) := \frac{s(t-s)}{t}.
$$

By the Fubini theorem, this relation and Lemma 8, we have

$$
\begin{aligned}
S[U_\mu](x,t) &= \int_{\mathbf{R}^N}\left(\int_0^t \int_{\mathbf{R}^N} G(x-y,t-s)G(y-z,s)U_\mu(y,s)^{p-1}dyds\right)d\mu(z) \\
&= \int_{\mathbf{R}^N} G(x-z,t)\left(\int_0^t \int_{\mathbf{R}^N} G(\xi,\tau)U_\mu(y,s)^{p-1}dyds\right)d\mu(z) \\
&\le C\|\mu\|_X^{p-1}\int_{\mathbf{R}^N} G(x-z,t)\left(\int_0^t \int_{\mathbf{R}^N} G(\xi,\tau)\frac{\phi(s^{1/2})^{p-1}}{s}dyds\right)d\mu(z).
\end{aligned}
$$

From the relation $\int_{\mathbf{R}^N} G(\xi,\tau)dy = 1$ and the change of variables $\eta = s^{1/2}$, we obtain

$$S[U_\mu](x,t) \le C\|\mu\|_X^{p-1}K(t)\int_{\mathbf{R}^N} G(x-z,t)d\mu(z) = C\|\mu\|_X^{p-1}K(t)U_\mu(x,t).$$

Thus the Lemma follows. □

We are now in a position to prove the existence of solutions.

Proof (*Proof of Theorem* 2) Lemma 9 gives

$$\Phi[2U_\mu] = U_\mu + 2^p S[U_\mu] \le \left(1 + C\|\mu\|_X^{p-1}K(t)\right) U_\mu \quad \text{in } \mathbf{R}^N \times (0,\infty). \tag{19}$$

First, we prove the assertion (i). Since (9) yields $\lim_{t\downarrow 0} K(t) = 0$, there exists a positive constant $T_0 = T_0(N, p, \phi, \|\mu\|_X)$ such that $\Phi[2U_\mu] \le 2U_\mu$ in $\mathbf{R}^N \times (0, T_0)$. Setting $u_n := \Phi^n[2U_\mu]$ for $n = 1, 2, \ldots$, we have

$$2U_\mu(x,t) \ge u_1(x,t) \ge u_2(x,t) \ge \cdots \ge 0 \quad \text{for } (x,t) \in \mathbf{R}^N \times (0,T_0).$$

This guarantees that the nonnegative function $u_\infty(x,t) := \lim_{n\to\infty} u_n(x,t)$ is well defined. The monotonicity of $\{u_n\}_{n=1}^\infty$ gives $\lim_{n\to\infty} \Phi[u_n](x,t) = \Phi[u_\infty](x,t)$. Hence u_∞ satisfies (18) in $\mathbf{R}^N \times (0,T_0)$. From the assumption $\mu \ne 0$ and (18), we deduce that U_μ is positive and u_∞ is also positive. Since $u_\infty \le 2U_\mu$ and U_μ is locally bounded in $\mathbf{R}^N \times (0,\infty)$, the function u_∞ satisfies the equation in (1) in the classical sense.

On the other hand, Lemma 9 yields $S[u_\infty] \le 2^p S[U_\mu] \le CK(t)U_\mu$. This together with the fact $\lim_{t\downarrow 0} K(t) = 0$ proves $S[u_\infty](\cdot,t) \to 0$ as $t \downarrow 0$ weakly as measures on each fixed open ball. By (18), the function u_∞ satisfies the initial condition in (1). Thus u_∞ is a positive solution of the problem (1) for $T = T_0$.

Next, we show the assertion (ii). From (10) and (19), it follows that $\Phi[2U_\mu] \le 2U_\mu$ in $\mathbf{R}^N \times (0,\infty)$ if $\|\mu\|_X > 0$ is small enough. Then, similarly to the proof of (i), the problem (1) for $T = +\infty$ admits a positive solution u satisfying $u \le 2U_\mu$ in $\mathbf{R}^N \times (0,\infty)$. Hence by Lemma 8, the decay estimate is also proved, and the proof is complete. □

We note that the above argument concerning global existence cannot be applied to the case $p = p_F$, since (10) gives $X_\phi^{p_F} = \{0\}$. Indeed, suppose that ϕ satisfies (10) and there exists $\mu \in X_\phi^{p_F}$ with $\mu \ne 0$. Then $\phi(\rho) \ge \|\mu\|_X^{-1}\mu(B_\rho(x))$ for all $x \in \mathbf{R}^N$ and $\rho > 0$, so that $\int_0^\infty \eta^{-1}\phi(\eta)^{p-1}d\eta \ge \|\mu\|_X^{-(p-1)}\mu(B_1(x_0))^{p-1}\int_1^\infty \eta^{-1}d\eta = +\infty$ for some $x_0 \in \mathbf{R}^N$, which contradicts (10).

Finally, we see that the problem (1) for the initial data $\mu_1(A) = \int_A |f_1(x)|dx$ and $T = +\infty$ admits a solution if $p > p_F$ and $f_1 \in L^1_{\mathrm{loc}}(\mathbf{R}^N)$ satisfies the condition

$$|f_1(x)| \le c|x|^{-\frac{2}{p-1}} \quad \text{a.e. } x \in \mathbf{R}^N \tag{20}$$

with a small constant $c > 0$. For $p > p_F$, this condition is one of a critical case of (3). Furthermore, since $|\cdot|^{-2/(p-1)} \notin L^{N(p-1)/2}_{\mathrm{loc}}(\mathbf{R}^N)$, the condition (20) is also critical in the sense of Weissler [21, Theorem 1]. We remark that the following proposition is a consequence of Robinson and Sierżęga [17, Theorem 3] and that our proof below is due to the argument of theirs and Ishige, Kawakami and Sierżęga [10, Lemma 2.4]. We repeat it for the convenience of the reader.

Proposition 2 *Let $p > p_F$ and $f_1 \in L^1_{loc}(\mathbf{R}^N)$. Then there exists a constant $c_0 = c_0(N, p) > 0$ such that if $f_1 \neq 0$ and f_1 satisfies (20) with $c = c_0$, the problem (1) for $\mu = \mu_1$ and $T = +\infty$ admits a positive solution u, where μ_1 is defined by $\mu_1(A) := \int_A |f_1(x)|dx$ for a Borel set A in $\mathbf{R}^N$. Moreover, the solution u satisfies*

$$\|u(\cdot, t)\|_{L^\infty(\mathbf{R}^N)} \le Ct^{-\frac{1}{p-1}} \quad \textit{for all } t > 0$$

with a constant $C = C(N, p) > 0$.

Proof Let $1 < \sigma < \min\{N(p-1)/2, p\}$ and let $f_1 \in L^1_{\mathrm{loc}}(\mathbf{R}^N)$ satisfy (20), where $c > 0$ is a constant yet to be determined. Taking into account that $|f_1|^\sigma \in L^1_{\mathrm{loc}}(\mathbf{R}^N)$, we define $\mu_1^\sigma(A) := \int_A |f_1(x)|^\sigma dx$. Then by the Hölder inequality, we have

$$\begin{aligned} U_{\mu_1}(x, t) &\le \left(\int_{\mathbf{R}^N} G(x-y, t)|f_1(y)|^\sigma dy\right)^{\frac{1}{\sigma}} \left(\int_{\mathbf{R}^N} G(x-y, t)dy\right)^{1-\frac{1}{\sigma}} \\ &= U_{\mu_1^\sigma}(x, t)^{\frac{1}{\sigma}} \end{aligned} \tag{21}$$

for $(x, t) \in \mathbf{R}^N \times (0, \infty)$. By (20) and the same calculation as in the proof of Lemma 8, there exists a positive constant C depending on N and p such that

$$U_{\mu_1^\sigma}(x, t) \le Cc^\sigma t^{-\frac{\sigma}{p-1}} \quad \text{for all } (x, t) \in \mathbf{R}^N \times (0, \infty). \tag{22}$$

We will solve (18) with $\mu = \mu_1$. By (21), we have $\Phi[2U_{\mu_1^\sigma}^{1/\sigma}] \le U_{\mu_1^\sigma}^{1/\sigma} + 2^p S[U_{\mu_1^\sigma}^{1/\sigma}]$. From (22) and the relation $p/\sigma - 1 > 0$, it follows that

$$\begin{aligned} S[U_{\mu_1^\sigma}^{1/\sigma}](x, t) &= \int_0^t \int_{\mathbf{R}^N} G(x-y, t-s) U_{\mu_1^\sigma}(y, s)^{\frac{p}{\sigma}-1} U_{\mu_1^\sigma}(y, s) dyds \\ &\le Cc^{p-\sigma} \int_0^t s^{-\frac{p-\sigma}{p-1}} \left(\int_{\mathbf{R}^N} G(x-y, t-s) U_{\mu_1^\sigma}(y, s)dy\right) ds. \end{aligned}$$

Since $\int_{\mathbf{R}^N} G(x-y, t-s) U_{\mu_1^\sigma}(y, s) dy = U_{\mu_1^\sigma}(x, t)$, we have

$$S[U_{\mu_1^\sigma}^{1/\sigma}](x, t) \le Cc^{p-\sigma} t^{\frac{\sigma-1}{p-1}} U_{\mu_1^\sigma}(x, t). \tag{23}$$

This together with (22) shows that

$$S[U_{\mu_1^\sigma}^{1/\sigma}](x,t) \le Cc^{p-\sigma} t^{\frac{\sigma-1}{p-1}} U_{\mu_1^\sigma}(x,t)^{\frac{\sigma-1}{\sigma}} U_{\mu_1^\sigma}(x,t)^{\frac{1}{\sigma}} \le Cc^{p-1} U_{\mu_1^\sigma}(x,t)^{\frac{1}{\sigma}}.$$

From the above calculations, we see that $\Phi[2U_{\mu_1^\sigma}^{1/\sigma}] \le (1 + Cc^{p-1})U_{\mu_1^\sigma}^{1/\sigma}$. Fix $c > 0$ such that $Cc^{p-1} < 1$. Then $\Phi[2U_{\mu_1^\sigma}^{1/\sigma}] \le 2U_{\mu_1^\sigma}^{1/\sigma}$ in $\mathbf{R}^N \times (0,\infty)$. Similarly to the proof of Theorem 2, if $f_1 \ne 0$, there exists a positive function u satisfying (18) and $u \le 2U_{\mu_1^\sigma}^{1/\sigma}$ in $\mathbf{R}^N \times (0,\infty)$. We deduce from (22) that u satisfies the desired decay estimate and the equation in (1) in the classical sense.

We check that u satisfies the initial condition of the problem (1). By (23), we have $S[u] \le CS[U_{\mu_1^\sigma}^{1/\sigma}] \le Ct^{(\sigma-1)/(p-1)}U_{\mu_1^\sigma}$. Since $U_{\mu_1^\sigma}(\cdot,t) \to |f_1|^\sigma$ as $t \downarrow 0$ weakly as measures on each fixed open ball, $S[u](\cdot,t) \to 0$ in the same sense as $t \downarrow 0$. From the integral equation (18), it follows that u satisfies the initial condition in (1) and that u is a positive solution of the problem (1) for $T = +\infty$. Thus the proof is complete. □

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Singular Solutions of the Scalar Field Equation with a Critical Exponent

Jann-Long Chern and Eiji Yanagida

Abstract We consider radially symmetric singular solutions of the scalar field equation with the Sobolev critical exponent. It is shown that there exists a unique special singular solution, and other infinitely many singular solutions are oscillatory around the special singular solution.

Keywords Scalar field equation · Singular solution · Critical exponent

1 Introduction and Main Results

In this paper we consider the elliptic equation

$$\Delta u - u + u^p = 0 \quad \text{in } \mathbf{R}^n \tag{1}$$

with $n > 2$ and $p > 1$. This equation is called the scalar field equation, and has been studied extensively in the past few decades. Our aim is to study the existence and structure of radially symmetric singular solutions $u = u(r)$, $r = |x| > 0$, of (1), where $u(r)$ satisfies

$$u_{rr}(r) + \frac{n-1}{r} u_r(r) - u(r) + u(r)^p = 0, \quad r > 0, \tag{2}$$

and $u(r) \to \infty$ as $r \to 0$.

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It is well known [1–8] that the structure of solutions of (2) changes drastically when the parameter p crosses the Sobolev critical exponent

$$p_S := \frac{n+2}{n-2}.$$

In the subcritical case $1 < p < p_S$, the existence of a positive ground state solution, which is regular at zero and $u(r) \to 0$ as $r \to \infty$, was proved by Berestycki-Lions [9] and the uniqueness was proved by Kwong [5]. Johnson-Pan-Yi [4] proved in the subcritical case that there are infinitely many positive singular solutions that satisfy $u(r) \to 0$ as $r \to \infty$, and are infinitely many singular solutions that vanish at some $r \in (0, \infty)$. For $p \geq p_S$, the nonexistence of regular and singular positive ground state solutions follows from a result of Ni-Serrin [6]. Recently, for $p > p_S$, Chern-Chen-Chen-Tang [2] proved that (2) possesses at most one positive singular solution $u(r)$ which oscillates around 1 as $r \to \infty$. Furthermore, in [2], if $n > 10$ and

$$p > p^* := \frac{(n-2)^2 - 4n + 8\sqrt{n-1}}{(n-2)(n-10)} \quad (> p_S),$$

the unique singular solution can be obtained as the limit of a sequence of regular solutions of (2). They also clarified the entire structure of radial solutions of various types according to their behavior at the origin and infinity.

When $p = p_S$, for singular solutions of (1) that are not necessarily radially symmetric, it was shown in [10–12] that any singular solution of (1) is asymptotically symmetric around the singular point. Moreover, near the singular point, the solution resembles an entire singular solution w of

$$\Delta w + w^{p_S} = 0 \quad \text{in } \mathbf{R}^n \setminus \{0\},$$

which is explicitly expressed as

$$w = L|x|^{-\frac{n-2}{2}}, \qquad L := \left(\frac{n-2}{2}\right)^{\frac{n-2}{2}}.$$

Singular solutions are also important from the viewpoint of the theory of nonlinear parabolic equations. Indeed, singular steady states of superlinear parabolic equations have been studied in various contexts (see, e.g., [13–15] and the references cited therein), and dynamic properties of singular steady states are closely related to the structure of singular solutions for associated elliptic equations.

In this paper, we study the existence and structure of singular solutions of (2) in the critical case $p = p_S$. The following theorem is our main result.

Theorem 1 *Assume $n > 2$ and $p = p_S$. Then the following properties hold for (2):*

(a) *There exists a unique singular solution $u = u^*(r)$ such that*

$$\lim_{r \to 0} r^{(n-2)/2} u^*(r) = L. \tag{3}$$

(b) *There exist infinitely many singular solutions such that*

$$0 < \liminf_{r \to 0} r^{(n-2)/2} u(r) < L < \limsup_{r \to 0} r^{(n-2)/2} u(r) < M, \tag{4}$$

where

$$M := \left\{ \frac{n(n-2)}{4} \right\}^{\frac{n-2}{4}}.$$

This theorem implies that if $n > 2$ the special singular solution $u^*(r)$ and other singular solution intersect infinitely many times as $r \to 0$. In fact, we shall show that if u is a singular solution satisfying (4), then

$$w(s) := r^{(n-2)/2} u(r), \qquad r = e^{-s}, \tag{5}$$

approaches a periodic function as $s \to \infty$ that oscillates around the constant L.

Here we should note that the existence of radial singular solutions of (2) with $p = p_S$ can be shown by using the method of Han-Li-Teixeira [16]. In their studies on singularities for a Yamabe-type problem, they obtained a general asymptotic result (see Theorem 3 of [16]) for certain ODEs, by which we can show the existence of infinitely many singular solutions of (2). See also [17–19] for related results for analogous equations. In this paper, we adopt a different approach from these papers to derive more precise properties of singular solutions such as the uniqueness of the special singular solution and the oscillatory behavior of other singular solutions.

In order to investigate the structure of the set of singular solutions of (1), we consider the initial value problem (2) with

$$u(c) = \alpha, \qquad u_r(c) = \beta, \tag{6}$$

where $c \in (0, \infty)$ is arbitrarily fixed and $\alpha > 0$, $\beta \in \mathbf{R}$ are given initial data. We denote by $u(r; \alpha, \beta)$ the unique solution of (2) with (6).

The next theorem shows that the set of singular solutions of (2) satisfying (4) is open, and the unique solution satisfying (3) is surrounded by solutions satisfying (4).

Theorem 2 *Let $n > 2$ and $p = p_S$. Assume that $u(r; \alpha_0, \beta_0)$ is a singular solution of (2) with (6) satisfying (3) or (4). Then there exists $\delta > 0$ such that for any (α, β) with $0 < |\alpha - \alpha_0| + |\beta - \beta_0| < \delta$, $u(r; \alpha, \beta)$ is a singular solution of (2) satisfying (4).*

This paper is organized as follows. In Sect. 2, we study the existence of infinitely many singular solutions that are oscillatory around a special singular solutions as r decreases. In Sect. 3, we show the existence and uniqueness of the special singular solution. Section 4 is devoted to a proof of Theorem 2. In the following sections we always assume $n > 2$ and $p = p_S$.

2 Existence of Singular Solutions

By the transformation (5), we can rewrite (2) as

$$w_{ss} + f(w) = e^{-2s}w, \qquad s \in \mathbf{R}. \tag{7}$$

where

$$f(w) := -\frac{(n-2)^2}{4}w + w^{p_S}.$$

By taking a limit as $s \to \infty$, the equation formally reduces to a limiting equation

$$W_{ss} + f(W) = 0, \quad s \in \mathbf{R}. \tag{8}$$

Since L satisfies $f(L) = 0$, $W \equiv L$ is a constant solution of (8).

Let us define an energy functional

$$E[w] := \frac{1}{2}w_s^2 + F(w),$$

where

$$F(w) := \int_0^w f(t)dt = -\frac{(n-2)^2}{8}w^2 + \frac{1}{p_S+1}w^{p_S+1}.$$

We note that F satisfies

$$F(w) \begin{cases} = 0 & \text{if } w = 0, \\ < 0 & \text{if } 0 < w < M, \\ = 0 & \text{if } w = M, \\ > 0 & \text{if } w > M, \end{cases}$$

and takes a minimum value at $w = L$. By (7), we have an identity

$$\frac{d}{ds}E[w] \equiv e^{-2s}ww_s. \tag{9}$$

On the other hand, by (8), we have

$$\frac{d}{ds}E[W] \equiv 0,$$

so that $E[W]$ is constant. Moreover, it is easily shown by the phase plane analysis that the solution $W(s) > 0$ of (8) satisfies $W(s) \equiv L$ if $E[W] = F(L)$, and is periodic in s and satisfies $0 < W(s) < M$ if $F(L) < E[W] < 0$.

Let $s_0 > 0$ be sufficiently large and fixed, and denote by $w = w(s; \alpha, \beta)$ the unique solution of (7) subject to the initial condition

$$w(s_0) = \alpha > 0, \qquad w_s(s_0) = \beta \in \mathbf{R}. \tag{10}$$

We classify positive solutions of (7) with (10) as follows:

Type C: $w(s) = 0$ at some $s \in (s_0, \infty)$.
Type R: $w(s) > 0$ for $s > s_0$ and $w(s) \to 0$ as $s \to \infty$.
Type L: $w(s) > 0$ for $s > s_0$, and $w(s) \to L$ as $s \to \infty$.
Type O: $w(s) > 0$ for $s > s_0$ and $E[w] \to E_\infty$ as $s \to \infty$ for some constant $E_\infty \in (F(L), 0)$.

It is easy to see that if w is of Type C, then the corresponding solution $u(r) = r^{-(n-2)/2} w(-\log r)$ of (2) vanishes at some $r \in (0, e^{-s_0})$. If w is of Type R, then there exists a constant $\alpha > 0$ such that

$$w(s) = \alpha \exp(-\frac{n-2}{2}s) + h.o.t. \quad \text{as } s \to \infty,$$

which implies that $u(r) \to \alpha$ as $r \to 0$. If w is of Type L, then $u^*(r) = r^{-(n-2)/2} w(-\log r)$ is a singular solution of (2) satisfying (3). If w is of Type O, then w approaches a periodic solution of (8) as $s \to \infty$ that oscillates around L, which implies that $u(r)$ is a singular solution of (2) satisfying (4).

Theorem 1 (b) is a direct consequence of the following lemma.

Lemma 1 *Fix* $\alpha \in (0, L)$. *If* $s_0 > 0$ *is sufficiently large, then* $w(s; \alpha, 0)$ *is of Type O.*

Proof For large $s_0 > 0$, $w(s; \alpha, 0)$ is uniformly approximated on any bounded interval of s by the solution $W(s)$ of (8) with $W(s_0) = \alpha$ and $W_s(s_0) = 0$. Since $W(s)$ is periodic in s, there exist s_1 and s_2 ($s_0 < s_1 < s_2 < \infty$) such that

$$\begin{aligned} &w_s(s) > 0 \ \text{ for } s \in (s_0, s_1), \\ &w_s(s) < 0 \ \text{ for } s \in (s_1, s_2), \\ &w_s(s_1) = w_s(s_2) = 0. \end{aligned}$$

Then by the identity (9), we have $E[w(s_0)] < E[w(s_1)]$ and $E[w(s_1)] > E[w(s_2)]$. Moreover, since w is almost symmetric with respect to $s = s_1$ by the even symmetry of W with respect to its critical point, it follows from the non-increase of e^{-2s} that

$$\begin{aligned} E[w(s_1)] - E[w(s_0)] &= \int_{s_0}^{s_1} e^{-2s} w(s) w_s(s) ds \\ &> -\int_{s_1}^{s_2} e^{-2s} w(s) w_s(s) ds \\ &= E[w(s_1)] - E[w(s_2)]. \end{aligned}$$

Hence we obtain $E[w(s_0)] < E[w(s_2)]$. Thus if $s_0 > 0$ is sufficiently large, we have

$$F(L) < E[w(s_0)] < E[w(s_2)] < E[w(s_1)] < 0.$$

Repeating this argument, we can show that there is a sequence $\{s_k\}$ such that

$$\begin{aligned} &w_s(s) > 0 \ \text{ for } s \in (s_{2k}, s_{2k+1}), \\ &w_s(s) < 0 \ \text{ for } s \in (s_{2k+1}, s_{2k+2}), \\ &w_s(s_{2_k}) = w_s(s_{2k+1}) = 0, \end{aligned}$$

and

$$F(L) < E[w(s_0)] < E[w(s_2)] < \cdots < E[w(s_3)] < E[w(s_1)] < 0.$$

Since $E[w(s_{k+1})] - E[w(s_k)] \to 0$ as $k \to \infty$, we have $E[w(s_k)] = F(w(s_k)) \to E_\infty$ as $k \to \infty$ for some constant $E_\infty \in (F(L), 0)$. This shows that $w(s; \alpha, 0)$ is of Type O. □

3 Existence and Uniqueness of a Special Singular Solution

The following lemma shows the existence of a special singular solution of (2) satisfying (3).

Lemma 2 *Let $n > 2$. For any $\gamma \in (-\frac{n}{2} + 3, -\frac{n}{2} + 5)$, there exists a solution of* (2) *such that*

$$u(r) = Lr^{-\frac{n}{2}+1} + L_1 r^{-\frac{n}{2}+3} + O(r^\gamma) \quad \text{as } r \to 0, \tag{11}$$

where $L_1 := L/(4n + 8)$.

Proof of Lemma 2. In the following we let $p = p_S$. We write a solution of (2) as

$$u(r) = Lr^{-\frac{n}{2}+1} + L_1 r^{-\frac{n}{2}+3} + r^\gamma y(r), \tag{12}$$

Note that $\varphi(r) := Lr^{-\frac{n}{2}+1}$ and $\psi(r) := L_1 r^{-\frac{n}{2}+3}$ satisfy

$$\begin{cases} \varphi_{rr} + \dfrac{n-1}{r}\varphi_r + \varphi^p = 0, \\ \psi_{rr} + \dfrac{n-1}{r}\psi_r + \dfrac{pL^{p-1}}{r^2}\psi = \varphi, \end{cases} \qquad r > 0.$$

respectively. Then from (2), we have the equation for y

$$y_{rr} + \frac{A}{r} y_r + \frac{B}{r^2} y + g(r, y) = 0, \qquad r > 0, \tag{13}$$

where

$$
\begin{aligned}
A &:= n-1+2\gamma > n-1+2\Big(-\frac{n}{2}+3\Big) = 5, \\
B &:= pL^{p-1}+\gamma(\gamma+n-2) = \frac{n^2}{4}-1+\gamma(\gamma+n-2) > 0
\end{aligned} \tag{14}
$$

by assumptions on γ and n, and

$$
g(r,y) := L^p r^{-\frac{n}{2}-1-\gamma}\left\{\Big(1+\frac{L_1}{L}r^2+\frac{1}{L}r^{\frac{n}{2}-1+\gamma}y\Big)^p - 1 - \tfrac{pL_1}{L}r^2 - \tfrac{p}{L}r^{\frac{n}{2}-1+\gamma}y\right\}
-L_1 r^{-\frac{n}{2}+3-\gamma} - y.
$$

We note that if y is bounded, then

$$
g(r,y) = O(r^{-\frac{n}{2}+3-\gamma}) \quad \text{as } r\to 0. \tag{15}
$$

First we consider the Euler equation

$$
Y_{\rho\rho}+\frac{A}{\rho}Y_\rho+\frac{B}{\rho^2}Y = 0, \qquad \rho>0. \tag{16}
$$

We define the energy

$$
J[Y] := BY^2+\rho^2 Y_\rho^2,
$$

which satisfies

$$
\begin{aligned}
\frac{d}{d\rho}J[Y(\rho)] &= 2BYY_\rho+2\rho Y_\rho^2+2\rho^2 Y_\rho Y_{\rho\rho} \\
&= 2\rho Y_\rho^2+2\rho^2 Y_\rho\Big(Y_{\rho\rho}+\frac{B}{\rho^2}Y\Big) \\
&= -2(A-1)\rho Y_\rho^2 \\
&\le 0
\end{aligned}
$$

by (14). Here, the last equality holds if and only if $Y_\rho = 0$. This implies that $J[Y(\rho)]$ is strictly decreasing in $\rho>0$ if Y is a non-trivial solution. Let $Y(\rho;\alpha,\beta)$ be the solution of (16) with

$$
Y(1)=\alpha, \qquad Y_\rho(1)=\beta.
$$

Since $J[Y(\rho;\alpha,\beta)]$ is strictly decreasing in $\rho>0$, there exists $\delta>0$ such that if $B\alpha^2+\beta^2\le 1$, then

$$
\begin{cases}
BY(\rho;\alpha,\beta)^2+\rho^2 Y_\rho(\rho;\alpha,\beta)^2 < 1 & \text{for } \rho\in(1,2), \\
BY(2;\alpha,\beta)^2+4Y_\rho(2;\alpha,\beta)^2 < 1-\delta.
\end{cases} \tag{17}
$$

Next, for $d>0$ fixed, let $y(r;\alpha,\beta)$ be the solution of (13) with

$$
y(d)=\alpha, \qquad y_r(d)=d^{-1}\beta.
$$

We set $\rho := d^{-1}r$ and $\tilde{y}(\rho; \alpha, \beta) := y(d\rho; \alpha, \beta)$. Then by (13), we have

$$\tilde{y}_{\rho\rho} + \frac{A}{\rho}\tilde{y}_\rho + \frac{B}{\rho^2}\tilde{y} + d^2 g(d\rho, \tilde{y}) = 0, \qquad \rho > 0,$$

and

$$\tilde{y}(1) = \alpha, \qquad \tilde{y}_\rho(1) = \beta.$$

Here, by (15),

$$d^2 g(d\rho; \tilde{y}) = O(d^{-\frac{n}{2}+5-\gamma}) \to 0 \quad \text{as } d \to 0$$

uniformly in $(\rho, \tilde{y}) \in [1, 2] \times [-2, 2]$. Hence by continuity, we obtain

$$(\tilde{y}(\rho; \alpha, \beta), \tilde{y}_\rho(\rho; \alpha, \beta)) \to (Y(\rho; \alpha, \beta), Y_\rho(\rho; \alpha, \beta)) \quad \text{as } d \to 0 \qquad (18)$$

uniformly in $\rho \in [1, 2]$. Therefore, by (17), there exists $d_0 > 0$ such that if $0 < d < d_0$ and $B\alpha^2 + d^2\beta^2 \leq 1$, then

$$\begin{aligned} &By(r; \alpha, \beta)^2 + r^2 y_r(r; \alpha, \beta)^2 < 2 \quad \text{for } r \in [d, 2d], \\ &By(2d; \alpha, \beta)^2 + (2d)^2 y_r(2d; \alpha, \beta)) < 1. \end{aligned}$$

Now we take a positive decreasing sequence $\{d_k\}$ given by

$$d_k := 2^{-k} d_0, \qquad k = 1, 2, \ldots,$$

and set $r = d_k\rho$. We also define

$$D_k := \{(\alpha, \beta) \in \mathbf{R}^2 : B\alpha^2 + d_k^2\beta^2 \leq 1\}, \qquad k = 0, 1, 2, \ldots.$$

Let $y_k(r; \alpha, \beta)$ be the solution of (13) subject to the initial condition

$$y(d_k) = \alpha, \quad y_r(d_k) = d_k^{-1}\beta,$$

and T_k be a mapping from $\mathbf{R}^2$ to $\mathbf{R}^2$ defined by

$$T_k[\alpha, \beta] := (y_k(d_{k-1}; \alpha, \beta), 2(y_k)_r(d_{k-1}; \alpha, \beta)).$$

Then by (17) and (18), we have $T_k[D_k] \subset D_{k-1}$ for all $k = 1, 2, \ldots$. Moreover, if we define a sequence of sets $\{D_k^0\}$ in $\mathbf{R}^2$ by

$$D_k^0 := T_1 \circ T_2 \circ \cdots \circ T_k[D_k], \qquad k = 1, 2, \ldots,$$

then we have

$$D_0 \supset D_1^0 \supset \cdots \supset D_{k-1}^0 \supset D_k^0 \supset \cdots.$$

Since $D_k^0, k = 1, 2, \ldots$, are nonempty and compact, there exists an element such that

$$(\alpha^*, \beta^*) \in \bigcap_{k=1}^{\infty} D_k^0 \subset D_0.$$

Then from the above construction, it follows that the solution $y(r; \alpha^*, \beta^*)$ of (13) is bounded for $r \in (0, d_0)$. This completes the proof. □

Next, we show the uniqueness of the special singular solution.

Lemma 3 *The singular solution of* (2) *satisfying* (3) *is unique.*

Proof We shall derive a contradiction by assuming that (2) possesses two distinct solutions satisfying (3).

Let $u^*(r)$ be the singular solution obtained by Lemma 2, and $u(r)$ be another singular solution satisfying (3). By (2), $z(r) := u(r)/u^*(r)$ satisfies

$$z_{rr}(r) + \Big(\frac{n-1}{r} + \frac{2u_r^*}{u^*}\Big) z_r(r) + \big(u^{p-1} - (u^*)^{p-1}\big) z(r) = 0, \tag{19}$$

where $p = p_S$. Then $\tilde{z} := z - 1$ satisfies

$$\tilde{z}_{rr}(r) + \Big(\frac{n-1}{r} + \frac{2u_r^*}{u^*}\Big) \tilde{z}_r(r) + \frac{\big(u^{p-1} - (u^*)^{p-1}\big) u}{u - u^*} \tilde{z}(r) = 0.$$

Transforming this equation by

$$Z(s) = \tilde{z}(r), \qquad r = e^{-s},$$

we obtain

$$Z_{ss}(s) + P(r) Z_s(s) + Q(r) Z(s) = 0, \tag{20}$$

where

$$P(r) = -n + 2 - \frac{2ru_r^*}{u^*}$$

and

$$Q(r) = \frac{r^2 \big(u^{p-1} - (u^*)^{p-1}\big) u}{u - u^*}.$$

Here we easily have

$$\lim_{r \to 0} r^{(n-2)/2+1} u_r^*(r) = -\Big(\frac{n-2}{2}\Big) \lim_{r \to 0} r^{(n-2)/2} u^*(r) = \Big(\frac{n-2}{2}\Big)^{\frac{n}{2}},$$

so that

$$\lim_{r \to 0} P(r) = 0, \qquad \lim_{r \to 0} Q(r) = n - 2 > 0.$$

This implies that the solution $Z \not\equiv 0$ of (20) is oscillatory for large s. Hence there exists a sequence $\{s_j\}$ such that $Z(s_j) = 0, Z_s(s_j) > 0$, and $s_{j+1} - s_j \to 2\pi/\sqrt{n-2}$ as $j \to \infty$. Moreover,

$$Z(s)/C_j \to \sin(\omega(s - s_j)), \qquad Z_s(s)/C_j \to \omega \cos(\omega(s - s_j)) \tag{21}$$

as $j \to \infty$ uniformly in $s \in (s_j, s_{j+1})$, where

$$C_j = Z_s(s_j)/\omega > 0, \qquad \omega = \sqrt{n-2}.$$

Now we define an energy function J for (20) by

$$J(s) := \frac{1}{2} Z_s(s)^2 + \frac{n-2}{2} Z(s)^2.$$

Then by (20) we have

$$\begin{aligned} \frac{d}{ds} J(s) &= Z_s(s) Z_{ss}(s) + (n-2) Z(s) Z_s(s) \\ &= -Z_s(s)\big\{P(r) Z_s(s) + Q(r) Z(s)\big\} + (n-2) Z(s) Z_s(s) \\ &= -P(r) Z_s(s)^2 + \{n - 2 - Q(r)\} Z(s) Z_s(s). \end{aligned}$$

Integrating this on (s_j, s_{j+1}), we obtain

$$J(s_{j+1}) - J(s_j) = -\int_{s_j}^{s_{j+1}} P(r(s)) Z_s(s)^2 ds + \int_{s_j}^{s_{j+1}} \{n - 2 - Q(r)\} Z(s) Z_s(s) ds.$$

Here, by Lemma 2, we easily obtain

$$\begin{aligned} r u_r^*(r) + \frac{n-2}{2} u^*(r) &= \Big(-\frac{n}{2} + 3 + \tfrac{n-2}{2}\Big) L_1 r^{-\frac{n}{2}+3} + o(r^{-\frac{n}{2}+3}) \\ &= 2 L_1 r^{-\frac{n}{2}+3} + o(r^{-\frac{n}{2}+3}) \quad \text{as } r \to 0. \end{aligned}$$

This implies $P(r) < 0$ for small $r > 0$ so that the first term in the right-hand side is positive. On the other hand, since

$$n - 2 - Q(r) = \frac{4(n-2)}{n+2} e^{-2s} + o(e^{-2s}) \ \text{ as } \ s \to \infty,$$

we obtain

$$\begin{aligned} &\frac{1}{C_j^2} \int_{s_j}^{s_{j+1}} \{n - 2 - Q(r)\} Z(s) Z_s(s) ds \\ &\quad \to \frac{4(n-2)\omega}{n+2} \int_{s_j}^{s_j + 2\pi} e^{-2s} \sin(\omega(s - s_j)) \cos(\omega(s - s_j)) ds > 0 \ \text{ as } \ s \to \infty. \end{aligned}$$

Thus it is shown that $J(s_{j+1}) > J(s_j)$ for large j. This shows that C_j does not converge to 0 as $j \to \infty$, but then (21) contradicts the fact that

$$Z(s) = \tilde{z}(r) = z(r) - 1 = \frac{u(r) - u^*(r)}{u^*(r)} \to 0 \quad \text{as } r = e^{-s} \to 0.$$

This proves the uniqueness. □

Thus the proof of Theorem 1 (a) is completed.

4 Structure of the Set of Singular Solutions

Finally, we consider the structure of solutions of (7) with (10). The following lemma shows that the set of solutions of Type O is open, and the solution of Type L is isolated and surrounded by solutions of Type O.

Lemma 4 *Assume that $w(s; \alpha_0, \beta_0)$ is of Type O or Type L. Then there exists $\delta > 0$ such that $w(s; \alpha, \beta)$ is of Type O for any (α, β) with $0 < |\alpha - \alpha_0| + |\beta - \beta_0| < \delta$.*

Proof First assume that $w(s; \alpha_0, \beta_0)$ is of Type O. By continuity with respect to initial data, if (α, β) is sufficiently close to (α_0, β_0), then we can take a sufficiently large $s_1 > 0$ such that $w_s(s_1; \alpha, \beta) = 0$ and $w(s_1; \alpha, \beta) \in (0, L)$. Then by the same argument as in the proof of Lemma 1, it is shown that the solution is of Type O.

Next, assume that $w(s; \alpha_0, \beta_0)$ is of Type L. Again by continuity with respect to initial data, for any $\varepsilon > 0$, we can take a sufficiently large $s_1 > 0$ such that $w_s(s_1; \alpha, \beta) = 0$ and $w(s_1; \alpha, \beta) \in (L - \varepsilon, L + \varepsilon)$. Since the solution of Type L is unique, $w(s; \alpha, \beta)$ is not of Type L. This implies that $w(s; \alpha, \beta)$ is of Type O. □

Theorem 2 is an immediate consequence of this lemma.

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