Chapter 7
Digital Control Strategies for Active Vibration Control—The Bases

7.1 The Digital Controller

The basic equation for the polynomial digital controller to be used in active vibration control (subsequently called RS controller) is (see Fig. 7.1)

$$S(q^{-1})u(t) = -R(q^{-1})y(t) \tag{7.1}$$

where $u(t)$ is the plant input, $y(t)$ is the measured plant output, and

$$S(q^{-1}) = s_0 + s_1 q^{-1} + \cdots + s_{n_S} q^{-n_S} = s_0 + q^{-1} S^*(q^{-1}), \tag{7.2}$$

$$R(q^{-1}) = r_0 + r_1 q^{-1} + \cdots + r_{n_R} q^{-n_R}, \tag{7.3}$$

are, respectively, the denominator and numerator of the controller

$$K(q^{-1}) = \frac{R(q^{-1})}{S(q^{-1})}. \tag{7.4}$$

Equation (7.1) can also be written as

$$u(t) = \frac{1}{s_0}\left[-S^*(q^{-1})u(t-1) - R(q^{-1})y(t)\right] \tag{7.5}$$

Note that for a number of control algorithms (like pole placement) $s_0 = 1$ in (7.2).

Consider

$$G(q^{-1}) = \frac{q^{-d} B(q^{-1})}{A(q^{-1})} \tag{7.6}$$

as the pulse transfer operator of the cascade DAC + ZOH + continuous-time system + ADC, then the transfer function of the open-loop system is written as

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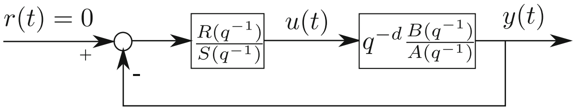

Fig. 7.1 Discrete feedback RS controller

$$H_{OL}(z^{-1}) = K(z^{-1})G(z^{-1}) = \frac{B(z^{-1})R(z^{-1})}{A(z^{-1})S(z^{-1})} \tag{7.7}$$

and the closed-loop transfer function between the reference signal $r(t)$ and the output $y(t)$, using controller (7.4), has the expression

$$S_{yr}(z^{-1}) = \frac{KG}{1+KG} = \frac{B(z^{-1})R(z^{-1})}{A(z^{-1})S(z^{-1}) + B(z^{-1})R(z^{-1})} = \frac{B(z^{-1})R(z^{-1})}{P(z^{-1})}, \tag{7.8}$$

where

$$P(z^{-1}) = A(z^{-1})S(z^{-1}) + z^{-d-1}B^{\star}(z^{-1})R(z^{-1}) \tag{7.9}$$
$$= A(z^{-1})S(z^{-1}) + z^{-d}B(z^{-1})R(z^{-1}) \tag{7.10}$$

is the denominator of the closed-loop transfer function that defines the closed-loop system poles. S_{yr} is known also as the *complementary sensitivity function*.

In the presence of disturbances (see Fig. 7.2), there are other important transfer functions to consider, relating the disturbance to the output and the input of the plant.

The transfer function between the disturbance $p(t)$ and the output $y(t)$ (*output sensitivity function*) is given by

$$S_{yp}(z^{-1}) = \frac{1}{1+KG} = \frac{A(z^{-1})S(z^{-1})}{P(z^{-1})}. \tag{7.11}$$

The transfer function between the disturbance $p(t)$ and the input of the plant $u(t)$ (*input sensitivity function*) is given by

$$S_{up}(z^{-1}) = -\frac{K}{1+KG} = -\frac{A(z^{-1})R(z^{-1})}{P(z^{-1})}. \tag{7.12}$$

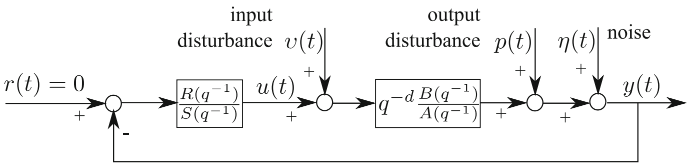

Fig. 7.2 Discrete feedback RS controller with input/output disturbances and measurement noise

Another important transfer function describes the influence on the output of a disturbance $\upsilon(t)$ on the plant input. This sensitivity function (*input disturbance-output sensitivity function*) is given by

$$S_{y\upsilon}(z^{-1}) = \frac{G}{1+KG} = \frac{B(z^{-1})S(z^{-1})}{P(z^{-1})}. \tag{7.13}$$

The feedback system presented in Fig. 7.2 is asymptotically stable if and only if all the four sensitivity functions S_{yr}, S_{yp}, S_{up} and $S_{y\upsilon}$ are asymptotically stable.

As it will be shown soon, the perfect rejection of disturbances with known characteristics or conversely opening of the loop for certain disturbances will require the introduction of some fixed pre-specified polynomials in S and R. The general structure of R and S will be of the form:

$$S(z^{-1}) = S'(z^{-1})H_S(z^{-1}) \tag{7.14}$$

$$R(z^{-1}) = R'(z^{-1})H_R(z^{-1}) \tag{7.15}$$

where $H_S(z^{-1})$ and $H_R(z^{-1})$ are monic fixed polynomials, which are introduced in the controller for achieving certain performances with respect to disturbances. Using this parametrization, the closed-loop poles will be given by

$$P(z^{-1}) = A(z^{-1})H_S(z^{-1})S'(z^{-1}) + z^{-d-1}B^{\star}(z^{-1})H_R(z^{-1})R'(z^{-1}) \tag{7.16}$$

Note that $H_S(z^{-1})$ and $H_R(z^{-1})$ can be interpreted as an "augmentation" of the plant model (for computation purposes).

The design of the RS controller can be done in the frequency domain using transfer functions (operators).

7.2 Pole Placement

The pole placement strategy is applicable to plant models of the form of Eq. (7.6). We will make the following hypothesis upon the plant model of Eq. (7.6):

(H1) No restrictions upon the orders of the polynomials $A(z^{-1})$, $B(z^{-1})$ and the value of the delay d.
(H2) The orders n_A, n_B, the delay d and the coefficients of $A(z^{-1})$ and $B(z^{-1})$ are known.
(H3) The zeros of $B(z^{-1})$ can be inside or outside the unit circle.
(H4) $A(z^{-1})$ and $B(z^{-1})$ (or AH_S and BH_R) do not have any common factors.
(H5) The zeros of $A(z^{-1})$ can be inside or outside the unit circle.

The control law is of the form (7.1) and the polynomials $R(z^{-1})$ and $S(z^{-1})$ have the structure of Eqs. (7.14) and (7.15).

The closed-loop behaviour is defined by

- the desired closed-loop poles;
- the choice of the fixed parts $H_R(z^{-1})$ and $H_S(z^{-1})$.

The desired closed-loop poles are chosen under the form as follows:

$$P(z^{-1}) = P_D(z^{-1}) \cdot P_F(z^{-1}) \tag{7.17}$$

where $P_D(z^{-1})$ defines the *dominant poles* and $P_F(z^{-1})$ defines the *auxiliary poles*.

Often $P_D(z^{-1})$ is chosen to include all the stable poles of the plant in open-loop with the option of eventually modifying the damping of the complex poles.

The role of $P_F(z^{-1})$ is on one hand to introduce a filtering effect at certain frequencies and on the other hand to improve the robustness of the controller.

With the notations:

$$n_A = \deg A \ ; \quad n_B = \deg B$$
$$n_{H_S} = \deg H_S \ ; \quad n_{H_R} = \deg H_R$$

and under the hypotheses H1 to H5, (7.16) has a unique solution for S' and R', of minimal degree for

$$n_P = \deg P(z^{-1}) \leq n_A + n_{H_S} + n_B + n_{H_R} + d - 1 \tag{7.18}$$
$$n_{S'} = \deg S'(z^{-1}) = n_B + n_{H_R} + d - 1 \tag{7.19}$$
$$n_{R'} = \deg R'(z^{-1}) = n_A + n_{H_S} - 1 \tag{7.20}$$

with

$$S'(z^{-1}) = 1 + s'_1 z^{-1} + \cdots + s'_{n_S} z^{-n_S} \tag{7.21}$$
$$R'(z^{-1}) = r'_0 + r'_1 z^{-1} + \cdots + r'_{n_R} z^{-n_R} \tag{7.22}$$

For a proof see [1, 2]. Various methods for solving this equation are available.[1]

7.2.1 Choice of H_R and H_S—Examples

Opening the Loop

In a number of applications, the measured signal may contain specific frequencies which should not be attenuated by the regulator. In such cases the system should be in open-loop at these frequencies.

[1] See functions *bezoutd.m* (MATLAB®) or *bezoutd.sci* (Scilab) on the book website.

From (7.12) in the absence of the reference, the input to the plant is given by

$$u(t) = S_{up}(q^{-1})p(t) = \frac{A(q^{-1})H_R(q^{-1})R'(q^{-1})}{P(q^{-1})}p(t) \tag{7.23}$$

and therefore in order to make the input sensitivity function zero at a given frequency f, one should introduce a pair of undamped zeros in $H_R(q^{-1})$, i.e.,:

$$H_R(q^{-1}) = (1 + \beta q^{-1} + q^{-2}) \tag{7.24}$$

where

$$\beta = -2\cos(\omega T_S) = -2\cos(2\pi\frac{f}{f_S})$$

In many cases it is desired that the controller does not react to signals of frequencies close to $0.5 f_S$ (where the gain of the system is in general very low). In such cases, one uses

$$H_R(q^{-1}) = (1 + \beta q^{-1}) \tag{7.25}$$

where

$$0 < \beta \leq 1$$

Note that $(1 + \beta q^{-1})^2$ corresponds to a second order with a damped resonance frequency equal to $\omega_S/2$ as follows:

$$\omega_0\sqrt{1-\zeta^2} = \frac{\omega_S}{2}$$

and the corresponding damping ζ is related to β by

$$\beta = e^{-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi}$$

For $\beta = 1$, the system will operate in open-loop at $f_S/2$.

In active vibration control systems, the gain of the secondary path at 0 Hz is zero (double differentiator behaviour). It is therefore not reasonable to send a control signal at this frequency. The system should operate in open-loop at this frequency. To achieve this, one uses

$$H_R(q^{-1}) = (1 - q^{-1}) \tag{7.26}$$

Perfect Rejection of an Harmonic Disturbance

The disturbance $p(t)$ can be represented as the result of a Dirac function $\delta(t)$ passed through a filter $D(q^{-1})$ (called the model of the disturbance)

$$D(q^{-1})p(t) = \delta(t) \tag{7.27}$$

In the case of an harmonic disturbance, the model is

$$(1 + \alpha q^{-1} + q^{-2})p(t) = \delta(t) \tag{7.28}$$

with

$$\alpha = -2\cos(\omega T_S) = -2\cos(2\pi \frac{f}{f_S}) \tag{7.29}$$

From (7.11) in the absence of a reference, one has

$$y(t) = \frac{A(q^{-1})H_S(q^{-1})S'(q^{-1})}{P(q^{-1})}p(t) \tag{7.30}$$

The problem can be viewed as choosing $H_S(q^{-1})$ such that the gain of the transfer function between $p(t)$ and $y(t)$ be zero at this frequency.

To achieve this one should choose

$$H_S(q^{-1}) = (1 + \alpha q^{-1} + q^{-2}) \tag{7.31}$$

In this case the expression of $y(t)$ taking into account (7.28), (7.30) and (7.31) becomes

$$y(t) = \frac{A(q^{-1})S'(q^{-1})}{P(q^{-1})}\delta(t) \tag{7.32}$$

and it results that asymptotically $y(t)$ tends to zero since $P(q^{-1})$ is asymptotically stable. This result is nothing else than the *internal model principle* which will be stated next.

7.2.2 Internal Model Principle (IMP)

Suppose that $p(t)$ is a deterministic disturbance, so it can be written as

$$p(t) = \frac{N_p(q^{-1})}{D_p(q^{-1})} \cdot \delta(t), \tag{7.33}$$

where $\delta(t)$ is a Dirac impulse and $N_p(z^{-1})$, $D_p(z^{-1})$ are coprime polynomials in z^{-1}, of degrees n_{N_p} and n_{D_p}, respectively (see also Fig. 7.1). In the case of stationary disturbances, the roots of $D_p(z^{-1})$ are on the unit circle. The energy of the disturbance is essentially represented by D_p. The contribution of the terms of N_p is weak asymptotically compared to the effect of D_p, so one can neglect the effect of N_p for a steady-state analysis of the effect of the disturbance upon the system.

Internal Model Principle: The effect of the disturbance given in (7.33) upon the output:

$$y(t) = \frac{A(q^{-1})S(q^{-1})}{P(q^{-1})} \cdot \frac{N_p(q^{-1})}{D_p(q^{-1})} \cdot \delta(t), \tag{7.34}$$

where $D_p(z^{-1})$ is a polynomial with roots on the unit circle and $P(z^{-1})$ is an asymptotically stable polynomial, converges asymptotically towards zero if and only if the polynomial $S(z^{-1})$ in the RS controller has the following form:

$$S(z^{-1}) = D_p(z^{-1})S'(z^{-1}). \tag{7.35}$$

In other terms, the pre-specified part of $S(z^{-1})$ should be chosen as $H_S(z^{-1}) = D_p(z^{-1})$ and the controller is computed using (7.16), where P, D_p, A, B, H_R and d are given.[2]

The IMC principle says that *in order to completely reject a disturbance asymptotically (i.e., in steady state), the controller should include the model of the disturbance*.

7.2.3 Youla–Kučera Parametrization

Using the Youla–Kučera parametrization (Q-parametrization) of all stable controllers ([3, 4]), the controller polynomials $R(z^{-1})$ and $S(z^{-1})$ get the following form:

$$R(z^{-1}) = R_0(z^{-1}) + A(z^{-1})Q(z^{-1}) \tag{7.36}$$
$$S(z^{-1}) = S_0(z^{-1}) - z^{-d}B(z^{-1})Q(z^{-1}) \tag{7.37}$$

where (R_0, S_0) is the so-called central controller and Q is the YK or Q filter which can be a FIR or an IIR filter. Figure 7.3 gives a representation of the Youla–Kučera parametrization of the R–S controllers. The *central* controller (R_0, S_0) can be computed by pole placement (but any other design technique can be used). Given the plant model (A, B, d) and the desired closed-loop poles specified by the roots of $P(z^{-1})$ one has to solve

$$P(z^{-1}) = A(z^{-1})S_0(z^{-1}) + z^{-d}B(z^{-1})R_0(z^{-1})\,. \tag{7.38}$$

If $Q(z^{-1})$ is considered to be a polynomial of the form (FIR filter):

$$Q(z^{-1}) = q_0 + q_1 z^{-1} + \cdots + q_{n_Q} z^{-n_Q}\,. \tag{7.39}$$

Equations (7.36) and (7.37) characterize the set of all stabilizable controllers assigning the closed-loop poles as defined by $P(z^{-1})$. It can be easily verified by simple computation, that the poles of the closed-loop remain unchanged; however, the par-

[2] Of course it is assumed that D_p and BH_R do not have common factors.

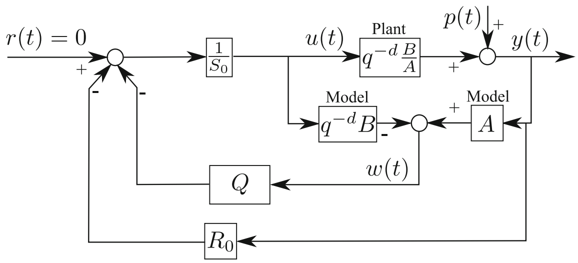

Fig. 7.3 The Youla–Kučera parametrized RS digital controller

ticular interest of the YK parametrization is the fact that the internal model of the disturbance can be incorporated in the controller by an appropriate choice of the filter Q. This filter should be such that the resulting polynomial S has the form $S = S'D_p$, i.e.,:

$$S'(z^{-1})D_p(z^{-1}) = S_0(z^{-1}) - z^{-d}B(z^{-1})Q(z^{-1}) \,, \tag{7.40}$$

To compute $Q(z^{-1})$ in order that the polynomial $S(z^{-1})$ given by (7.37) incorporates the internal model of the disturbance (7.33) one has to solve the following diophantine equation:

$$S'(z^{-1})D_p(z^{-1}) + z^{-d}B(z^{-1})Q(z^{-1}) = S_0(z^{-1}) \,, \tag{7.41}$$

where $D_p(z^{-1})$, d, $B(z^{-1})$ and $S_0(z^{-1})$ are known and $S'(z^{-1})$ and $Q(z^{-1})$ are unknown. Equation (7.41) has a unique solution for $S'(z^{-1})$ et $Q(z^{-1})$ with: $n_{S_0} \leq n_{D_p} + n_B + d - 1, n_{S'} = n_B + d - 1, n_Q = n_{D_p} - 1$. One sees that the order n_Q of the polynomial Q depends upon the structure of the disturbance model.

Consider now the case of a Q filter as ratio of rational polynomials (IIR filter) with an asymptotically stable denominator as follows:

$$Q(z^{-1}) = \frac{B_Q(z^{-1})}{A_Q(z^{-1})} \tag{7.42}$$

The YK controller will have the structure:

$$R(z^{-1}) = A_Q(z^{-1})R_0(z^{-1}) + A(z^{-1})B_Q(z^{-1}) \tag{7.43}$$
$$S(z^{-1}) = A_Q(z^{-1})S_0(z^{-1}) - z^{-d}B(z^{-1})B_Q(z^{-1}) \tag{7.44}$$

but in this case the poles of the closed-loop will be given by

$$P(z^{-1})_{QIIR} = P(z^{-1})A_Q(z^{-1}) \tag{7.45}$$

In the case of IIR Q filters, the poles of the denominator of Q will appear as additional poles of the closed-loop. This parametrization will be discussed in detail in Sects. 7.4 and 12.2 together with the preservation of the pre-specified fixed part of the controller H_R and H_S.

7.2.4 Robustness Margins

The Nyquist plot of the open-loop transfer function allows one to assess the influence of the modelling errors and to derive appropriate specifications for the controller design in order to assure the *robust stability* of the closed-loop system for certain classes of plant model uncertainties.

The open-loop transfer function corresponding to the use of an RS controller is:

$$H_{OL}(z^{-1}) = \frac{z^{-d}B(z^{-1})R(z^{-1})}{A(z^{-1})S(z^{-1})} \tag{7.46}$$

By making $z = e^{j\omega}$, where ω is the normalized frequency ($\omega = \omega T_s = 2\pi f/f_s$, f_s sampling frequency, T_s sampling period), the Nyquist plot of the open-loop transfer function $H_{OL}(e^{-j\omega})$ can be drawn. In general, one considers for the normalized frequency ω the domain between 0 and π (i.e., between 0 and $0.5f_s$). Note that the Nyquist plot between π and 2π is symmetric with respect to the real axis of the Nyquist plot between 0 and π. An example of a Nyquist plot is given in Fig. 7.4.

The vector connecting a point of the Nyquist plot with the origin corresponds to $H_{OL}(e^{-j\omega})$ for a certain normalized frequency. The point $[-1, j0]$ on the diagram of Fig. 7.4 corresponds to the *critical point*. From Fig. 7.4, it results that the vector connecting the *critical point* with the Nyquist plot of $H_{OL}(e^{-j\omega})$ has the following expression:

$$1 + H_{OL}(z^{-1}) = \frac{A(z^{-1})S(z^{-1}) + z^{-d}B(z^{-1})R(z^{-1})}{A(z^{-1})S(z^{-1})} = S_{yp}^{-1}(z^{-1}) \tag{7.47}$$

This vector corresponds to the inverse of the output sensitivity function $S_{yp}(z^{-1})$ given by Eq. (7.11) and the zeros of S_{yp}^{-1} are the poles of the closed-loop system. In order that the closed-loop system be asymptotically stable, it is necessary that all the zeros of S_{yp}^{-1} lie inside the unit circle.

The necessary and sufficient conditions in the frequency domain for the asymptotic stability of the closed-loop system are given by the Nyquist criterion. For the case of

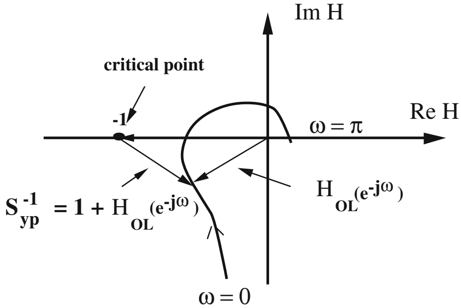

Fig. 7.4 The Nyquist plot of a discrete-time transfer function and the critical point

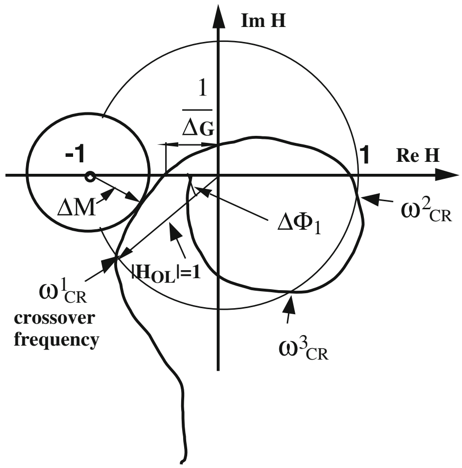

Fig. 7.5 Modulus, gain and phase margins

open-loop stable systems (in our case this corresponds to $A(z^{-1}) = 0$ and $S(z^{-1}) = 0 \implies |z| < 1$), the Nyquist criterion is expressed as:

Stability Criterion (Open-Loop Stable Systems)

The Nyquist plot of $H_{OL}(z^{-1})$ *traversed in the sense of growing frequencies (from* $\omega = 0$ *to* $\omega = \pi$ *leaves the critical point* $[-1, j0]$ *on the left.*

Using *pole placement*, the Nyquist criterion will be satisfied for the *nominal* plant model because $R(z^{-1})$ and $S(z^{-1})$ are computed using Eq. 7.10 for an asymptotically stable polynomial $P(z^{-1})$ defining the desired closed-loop poles ($P(z^{-1}) = 0 \implies |z| < 1$). Of course, we are assuming at this stage that the resulting $S(z^{-1})$ is also stable.[3]

The minimal distance between the Nyquist plot of $H_{OL}(z^{-1})$ and the critical point will define a *stability margin*. This minimal distance according to Eq. (7.47) will depend upon the maximum of the modulus of the output sensitivity function.

This stability margin which we will call subsequently the *modulus margin* could be linked to the uncertainties upon the plant model.

The following indicators serve for characterizing the distance between the Nyquist plot of $H_{OL}(z^{-1})$ and the critical point $[-1, j0]$ (see Fig. 7.5):

- modulus margin (ΔM)
- delay margin ($\Delta \tau$)
- phase margin ($\Delta \phi$)
- gain margin (ΔG)

[3] See [5] for the case of open-loop unstable systems.

Below are the definitions of the modulus margin and delay margin which will be used in the robust control design (for the definition of the gain and phase margin, see any classical control text):

Modulus Margin (ΔM)

The *modulus margin* (ΔM) is defined as the radius of the circle centred in $[-1, j0]$ and tangent to the Nyquist plot of $H_{OL}(z^{-1})$.

From the definition of the vector connecting the critical point $[-1, j0]$ with the Nyquist plot of $H_{OL}(z^{-1})$ (see Eq. (7.47)), it results that

$$\Delta M = |1 + H_{OL}(e^{-j\omega})|_{\min} = (|S_{yp}(e^{-j\omega})|_{\max})^{-1} = (\|S_{yp}\|_{\infty})^{-1} \tag{7.48}$$

As a consequence, the reduction (or minimization) of $|S_{yp}(e^{-j\omega})|_{\max}$ will imply the increase (or maximization) of the modulus margin ΔM.

In other terms the *modulus margin* ΔM is equal to the inverse of the maximum modulus of the output sensitivity function $S_{yp}(z^{-1})$ (i.e., the inverse of the H_{∞} norm of $S_{yp}(z^{-1})$). If the modulus of $S_{yp}(z^{-1})$ is expressed in dB, one has the following relationship:

$$|S_{yp}(e^{-j\omega})|_{\max}(\text{dB}) = (\Delta M)^{-1}(\text{dB}) = -\Delta M(\text{dB}) \tag{7.49}$$

The *modulus margin* is very important because

- It defines the maximum admissible value for the modulus of the output sensitivity function.
- It gives a bound for the characteristics of the nonlinear and time-varying elements tolerated in the closed-loop system (it corresponds to the circle criterion for the stability of nonlinear systems) [6].

Delay Margin ($\Delta \tau$)

For a certain frequency the phase lag introduced by a pure time delay τ is:

$$\angle\phi(\omega) = \omega\tau$$

If the Nyquist plot crosses the unit circle only once, one can therefore convert the phase margin in a delay margin, i.e., to compute the additional delay which will lead to instability. It results that:

$$\Delta\tau = \frac{\Delta\phi}{\omega_{cr}} \tag{7.50}$$

where ω_{cr} is the crossover frequency (where the Nyquist plot intersects the unit circle) and $\Delta\phi$ is the phase margin. If the Nyquist plot intersects the unit circle at several frequencies ω_{cr}^{i} (see Fig. 7.5), characterized by the associated phase margins $\Delta\phi_i$, the phase margin is defined as:

$$\Delta\phi = \min_{i} \Delta\phi_i \tag{7.51}$$

and the delay margin is defined by

$$\Delta\tau = \min_i \frac{\Delta\phi_i}{\omega_{cr}^i} \tag{7.52}$$

Remark This situation appears systematically for systems with pure time delays or with multiple vibration modes.

Typical values of the robustness margins for a *robust* controller design are

- Modulus margin: $\Delta M \geq 0.5(-6\text{ dB})[\min : 0.4(-8\text{ dB})]$
- Delay margin: $\Delta\tau \geq T_s[\min : 0.75T_s]$

where T_s is the sampling period.
Important remarks:

1. A modulus margin $\Delta M \geq 0.5$ implies that $\Delta G \geq 2(6\text{ dB})$ and $\Delta\phi > 29°$. The converse is not generally true. *Systems with satisfactory gain and phase margins may have a very small modulus margin.*
2. Phase margin can be misleading according to Eq. (7.50). *A good phase margin may lead to a very small tolerated additional delay if ω_{cr} is high.*

The modulus margin is an intrinsic measure of the stability margin and will be subsequently used together with the delay margin for the design of robust controllers (instead of the phase and gain margin).

7.2.5 Model Uncertainties and Robust Stability

Figure 7.6 illustrates the effect of uncertainties or of the variations of the parameters of the nominal model on the Nyquist plots of the open-loop transfer function. In general the Nyquist plot corresponding to the nominal model lies inside a *tube* corresponding to the possible (or accepted) tolerances of parameter variations (or uncertainties) of the plant model.

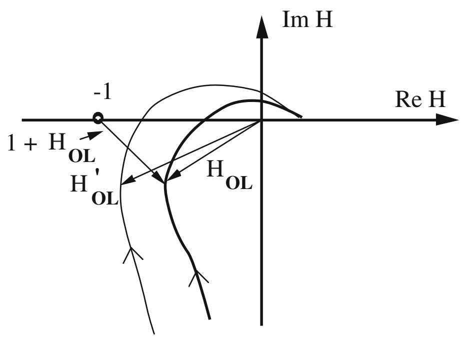

Fig. 7.6 Nyquist plot of the nominal model and perturbed model

We will consider an open-loop transfer function $H'_{OL}(z^{-1})$ which differs from the nominal one. For simplicity one assumes that the nominal transfer function $H_{OL}(z^{-1})$ as well as $H'_{OL}(z^{-1})$ are both stable (the general assumption is that both have the same number of unstable poles, see [7, 8]).

In order to assure the stability of the closed-loop system for an open loop transfer function $H'_{OL}(z^{-1})$ which differs from the nominal one $H_{OL}(z^{-1})$, the Nyquist plot of $H'_{OL}(z^{-1})$ should leave the critical point $[-1, j0]$ on the left side when traversed in the sense of the growing frequencies. Looking at Fig. 7.6 one can see that a sufficient condition for this, is that at each frequency the distance between $H'_{OL}(z^{-1})$ and $H_{OL}(z^{-1})$ be less than the distance between the nominal open-loop transfer function and the critical point. This is expressed by:

$$|H'_{OL}(z^{-1}) - H_{OL}(z^{-1})| < |1 + H_{OL}(z^{-1})| = |S_{yp}^{-1}(z^{-1})| = \left|\frac{P(z^{-1})}{A(z^{-1})S(z^{-1})}\right| \tag{7.53}$$

In other terms, the curve $|S_{yp}(e^{-j\omega})|^{-1}$ in dB (which is obtained by symmetry from $|S_{yp}(e^{-j\omega})|$) will give at each frequency a sufficient condition for the modulus of the tolerated discrepancy between the real open-loop transfer function and the nominal open-loop transfer function in order to guarantee stability.

In general, this tolerance is high in low frequencies and is low at the frequency (or frequencies) where $|S_{yp}(e^{-j\omega})|$ reaches its maximum ($=\Delta M^{-1}$). Therefore, low modulus margin will imply small tolerance to parameter uncertainties in a specified frequency region.

The relationship (7.53) expresses a robustness condition in terms of the variations of the open-loop transfer function (controller + plant). It is interesting to express this in terms of the variations of the plant model. One way to do this, is to observe that (7.53) can be rewritten as:

$$\begin{aligned}\left|\frac{B'(z^{-1})R(z^{-1})}{A'(z^{-1})S(z^{-1})} - \frac{B(z^{-1})R(z^{-1})}{A(z^{-1})S(z^{-1})}\right| &= \left|\frac{R(z^{-1})}{S(z^{-1})}\right| \cdot \left|\frac{B'(z^{-1})}{A'(z^{-1})} - \frac{B(z^{-1})}{A(z^{-1})}\right| \\ &< \left|\frac{P(z^{-1})}{A(z^{-1})S(z^{-1})}\right|\end{aligned} \tag{7.54}$$

Multiplying both sides of Eq. (7.54) by $|\frac{S(z^{-1})}{R(z^{-1})}|$ one gets

$$\left|\frac{B'(z^{-1})}{A'(z^{-1})} - \frac{B(z^{-1})}{A(z^{-1})}\right| \leq \left|\frac{P(z^{-1})}{A(z^{-1})R(z^{-1})}\right| = \left|S_{up}^{-1}(z^{-1})\right| \tag{7.55}$$

The left hand side of Eq. (7.55) expresses in fact an *additive* uncertainty for the nominal plant model. The inverse of the modulus of the input sensitivity function will give a sufficient condition for the tolerated *additive* variations (or uncertainties) of the nominal plant model in order to guarantee stability. Large values of the modulus

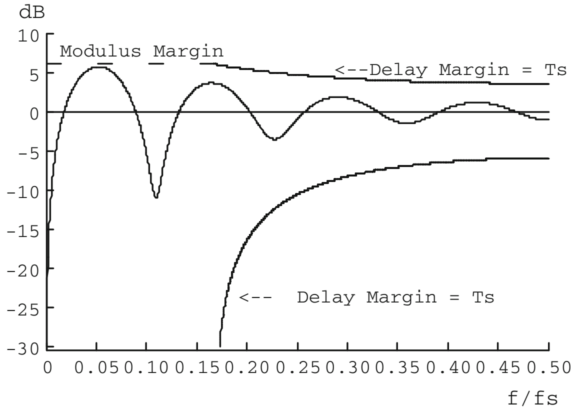

Fig. 7.7 Templates on the output sensitivity function for the modulus margin $\Delta M = 0.5$ and the delay margin $\Delta\tau = T_s$

of the input sensitivity function in certain frequency range will imply low tolerance to uncertainties in this frequency range. It will also mean that at these frequencies high activity of the input will result under the effect of disturbances.

7.2.6 Templates for the Sensitivity Functions

Robustness margins and performance specifications in the frequency domain translates easily in templates for the various sensitivity functions [2, 5]. Figure 7.7 gives the basic template for S_{yp} for assuring the modulus margin constraint ($\Delta M \geqslant 0.5$) and the delay margin ($\Delta\tau \geqslant T_s$). The template on the delay margin is an approximation (for more details see [2]). Violation of the lower or upper template does not necessarily imply violation of the delay margin (which any way can be effectively computed).

To this template, performance specification in terms of imposed attenuation and bound on the "waterbed" effect can be added (see the example in Sect. 7.3).

Templates on the modulus of the input sensitivity function S_{up} are also considered. In particular it is expected that S_{up} is low outside the frequency band of operation of the controller. Low values of the modulus of the input sensitivity functions imply a good robustness with respect to additive model uncertainties. Figure 7.8 gives an example of template on the input sensitivity function. More details can be found on the example given in Sect. 7.3.

7.2.7 Properties of the Sensitivity Functions

7.2.7.1 Output Sensitivity Function

Using an RS controller, the output sensitivity function is given by:

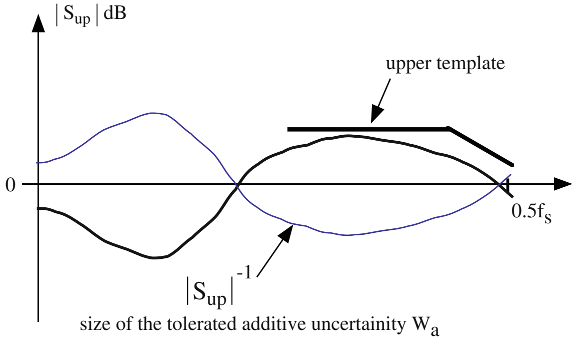

Fig. 7.8 An example of desired template for the input sensitivity function

$$S_{yp}(z^{-1}) = \frac{A(z^{-1})S(z^{-1})}{A(z^{-1})S(z^{-1}) + z^{-d}B(z^{-1})R(z^{-1})} \tag{7.56}$$

where

$$R(z^{-1}) = H_R(z^{-1})R'(z^{-1}) \tag{7.57}$$
$$S(z^{-1}) = H_S(z^{-1})S'(z^{-1}) \tag{7.58}$$

and

$$A(z^{-1})S(z^{-1}) + z^{-d}B(z^{-1})R(z^{-1}) = P_D(z^{-1}) \cdot P_F(z^{-1}) = P(z^{-1}) \tag{7.59}$$

In Eqs. (7.57) and (7.58), $H_R(z^{-1})$ and $H_S(z^{-1})$ correspond to the pre-specified parts of $R(z^{-1})$ and $S(z^{-1})$ respectively. $S'(z^{-1})$ and $R'(z^{-1})$ are the solutions of Eq. (7.16) where $P(z^{-1})$ represents the desired closed-loop poles in pole placement control strategy. The polynomial $P(z^{-1})$ is factorized in order to emphasize the dominant poles defined by $P_D(z^{-1})$ and the auxiliary poles defined by $P_F(z^{-1})$.

Property 1

The modulus of the output sensitivity function at a certain frequency gives the amplification or the attenuation of the disturbance.

At the frequencies where $|S_{yp}(\omega)| = 1(0\text{ dB})$, there is no attenuation nor amplification of the disturbance (operation in open-loop). At the frequencies where $|S_{yp}(\omega)| < 1(0\text{ dB})$, the disturbance is attenuated. At the frequencies where $|S_{yp}(\omega)| > 1(0\text{ dB})$, the disturbance is amplified.

Property 2 (The Bode Integral)

The closed-loop being asymptotically stable, the integral of the logarithm of the modulus of the output sensitivity function from 0 *to* $0.5 f_S$ *is equal to* 0 *for the case of stable open-loop systems*[4]:

[4]See [9] for a proof. In the case of unstable open-loop systems but stable in closed-loop, this integral is positive.

$$\int_{0}^{0.5f_S} \log |S_{yp}(e^{-j2\pi f/f_S)}|df = 0$$

In other terms, the sum of the areas between the curve of the modulus of the output sensitivity function and the 0 dB axis taken with their sign is null. **As a consequence, the attenuation of disturbances in a certain frequency region implies necessarily the amplification of disturbances in other frequency regions**.

Property 3

The inverse of the maximum of the modulus of the sensitivity function corresponds to the modulus margin ΔM.

$$\Delta M = (|S_{yp}(e^{-j\omega})|_{\max})^{-1} \tag{7.60}$$

From the Properties 2 and 3, it results that the increase of the attenuation band or of the attenuation in a certain frequency band will in general imply an increase of $|S_{yp}(e^{-j\omega})|_{\max}$ and therefore a decrease of the modulus margin (and therefore less robustness).

Figure 7.9 shows the output sensitivity function for a closed-loop system, corresponding to a plant model $A(z^{-1}) = 1 - 0.7z^{-1}$, $B(z^{-1}) = 0.3z^{-1}$, $d = 2$. The controller has been designed using the pole placement. The desired closed-loop poles correspond to the discretization of a second order system with natural frequency $\omega_0 = 0.1 f_s$ rad/s and damping $\zeta = 0.8$. The system being subject to a tonal disturbance located at $0.15 f_s$ or at $0.151 f_S$, a double internal model corresponding to these frequencies has been introduced in the controller fixed part H_S. In the first case a damping $\zeta = 0.3$ has been considered leading to an attenuation of 8 dB and in

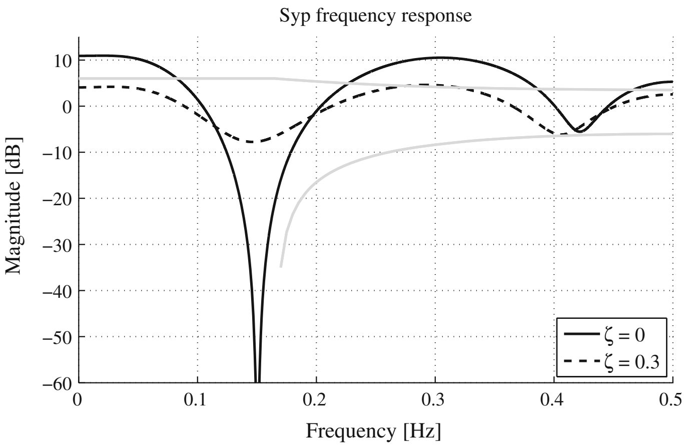

Fig. 7.9 Modulus of the output sensitivity functions for a double internal model with 0 and 0.3 damping

the second case full rejection of the disturbances have been considered using internal models with $\zeta = 0$ leading to an attenuation over 60 dB.[5]

One can clearly see that the increase of attenuation in a certain frequency region implies necessarily a stronger amplification of the disturbances outside the attenuation band. This is a direct consequence of Property 2. A similar phenomenon occurs if for a given attenuation the attenuation band is expanded.

7.2.8 Input Sensitivity Function

The input sensitivity function is extremely important in the design of the linear controller. The modulus of the input sensitivity function should be low at high frequencies in order to assure a good robustness of the system with respect to additive unstructured uncertainties located in the high-frequency region.[6]

The expression of the input sensitivity function using a RS controller with R and S given by (7.57) and (7.58) is

$$S_{up}(z^{-1}) = -\frac{A(z^{-1})H_R(z^{-1})R'(z^{-1})}{A(z^{-1})H_S(z^{-1})S'(z^{-1}) + q^{-d}B(z^{-1})H_R(z^{-1})R'(z^{-1})} \tag{7.61}$$

Property 1

The effect of the output disturbances upon the input is cancelled (i.e., $S_{up} = 0$) at the frequencies where

$$A(e^{-j\omega})H_R(e^{-j\omega})R'(e^{-j\omega}) = 0 \tag{7.62}$$

At these frequencies $S_{yp} = 1$ (open-loop operation). The pre-specified values assuring $S_{up} = 0$ at certain frequencies are of the same form as those used to make $S_{yp} = 1$.

Figure 7.10 illustrates the effect upon S_{up} of a pre-specified $H_R(z^{-1})$ of the form:

$$H_R(z^{-1}) = 1 + \alpha z^{-1}\ ;\ 0 < \alpha \leq 1$$

For $\alpha = 1$, one has $S_{up} = 0$ at $0.5 f_s$. Using $0 < \alpha < 1$ allows to reduce more or less the input sensitivity function around $0.5 f_s$.[7] This structure of $H_R(z^{-1})$ is systematically used for reducing the magnitude of the input sensitivity function in the high-frequency region.

---

[5]The structure of the H_S is $H_s = (1 + \alpha_1 q^{-1} + \alpha_2 q^{-2})(1 + \alpha_1' q^{-1} + \alpha_2' q^{-2})$.

[6]This is indeed true even in adaptive control since the uncertainties in the high-frequency region are not in general handled by the adaptive controller.

[7]The input sensitivity function correspond to the system considered previously which includes in the controller an internal model with zero damping located at $0.15 f_s$.

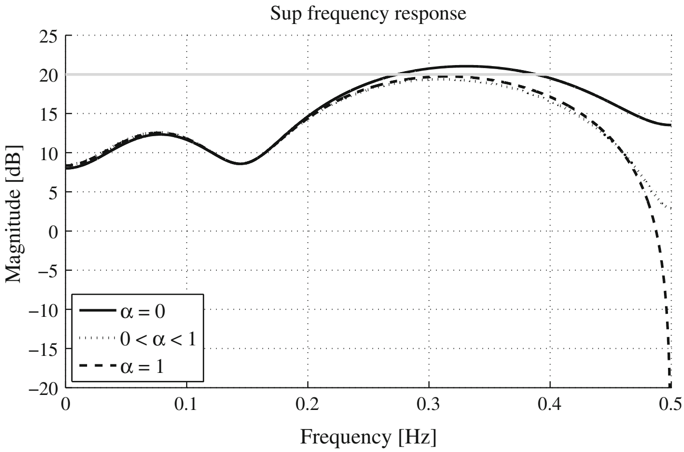

Fig. 7.10 Effect of $H_R(z^{-1}) = 1+\alpha z^{-1}, 0 < \alpha \leq 1$ upon the input sensitivity function for various values of parameter α

Property 2

At the frequencies where:

$$A(e^{-j\omega})H_S(e^{-j\omega})S'(e^{-j\omega}) = 0$$

which corresponds to perfect rejection of the output disturbances ($S_{yp} = 0$ at these frequencies), one has

$$\left|S_{up}(e^{-j\omega})\right| = \left|\frac{A(e^{-j\omega})}{B(e^{-j\omega})}\right| \tag{7.63}$$

i.e., the modulus of the input sensitivity function is equal to the inverse of the gain of the plant at this frequency.

This implies that perfect rejection of disturbances (or more generally attenuation of disturbances) should be done only in the frequency regions, where the gain of the system is large enough. If the gain is too low, $|S_{yp}|$ will be very large at these frequencies. Therefore, the robustness with respect to additive plant model uncertainties will be reduced, and the stress on the actuator will become important [10]. This also indicates that problems will occur if B has complex zeros close to the unit circle (stable or unstable). At these frequencies, rejection of disturbances should be avoided.

7.2.9 Shaping the Sensitivity Functions for Active Vibration Control

Two sensitivity functions are of particular interest in active vibration control:

- Output sensitivity function (the transfer function between the disturbance $p(t)$ and the output of the system $y(t)$):

$$S_{yp}(z^{-1}) = \frac{A(z^{-1})S(z^{-1})}{P(z^{-1})} \tag{7.64}$$

- Input sensitivity function (the transfer function between the disturbance $p(t)$ and the input of the system $u(t)$):

$$S_{up}(z^{-1}) = -\frac{A(z^{-1})R(z^{-1})}{P(z^{-1})} \tag{7.65}$$

In active vibration control they have to be shaped for performance and robustness purposes. The first tool for shaping the sensitivity functions, once the "performance" choices have been done (damping of some complex poles, introduction of the internal model of the disturbance, opening the loop at certain frequencies), is the introduction of the auxiliary poles.

The introduction of auxiliary asymptotically stable real poles $P_F(z^{-1})$ will cause in general a decrease of the modulus of the sensitivity function in the domain of attenuation of $1/P_F(z^{-1})$.

From Eqs. (7.56) and (7.59), one can see that the term $1/P_D(z^{-1})P_F(z^{-1})$ will introduce a stronger attenuation in the frequency domain than the term $1/P_D(z^{-1})$ if the auxiliary poles $P_F(z^{-1})$ are real (aperiodic) and asymptotically stable; however, since $S'(z^{-1})$ depends upon the poles through Eq. (7.16), one cannot guarantee this property for all the values of $P_F(z^{-1})$.

The auxiliary poles are generally chosen as high-frequency real poles under the form:

$$P_F(z^{-1}) = (1 - p_1 z^{-1})^{n_F} \; ; \; 0.05 \leq p_1 \leq 0.5$$

where:

$$n_F \leq n_p - n_D \; ; \; n_p = (\deg P)_{\max} \; ; \; n_D = \deg P_D$$

The effect of the introduction of the auxiliary poles is illustrated in Fig. 7.11, for the same system considered previously with a controller including an internal model with 0 damping at $0.05 f_S$. One observes that the introduction of 5 auxiliary real poles located at 0.5 "squeezes" the modulus of the output sensitivity function around 0 dB axis in the high-frequency range.

Note that in many applications the introduction of high-frequency auxiliary poles allows to satisfy the requirements for robustness margins.

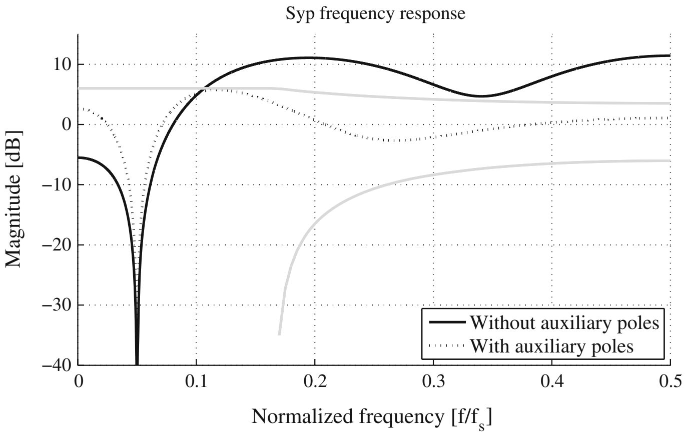

Fig. 7.11 Effect of auxiliary poles on the output sensitivity function

Simultaneous introduction of a fixed part H_{S_i} *and of a pair of auxiliary poles* P_{F_i} *in the form*

$$\frac{H_{S_i}(z^{-1})}{P_{F_i}(z^{-1})} = \frac{1 + \beta_1 z^{-1} + \beta_2 z^{-2}}{1 + \alpha_1 z^{-1} + \alpha_2 z^{-2}} \tag{7.66}$$

resulting from the discretization of the continuous-time band-stop filter (BSF):

$$F(s) = \frac{s^2 + 2\zeta_{num}\omega_0 s + \omega_0^2}{s^2 + 2\zeta_{den}\omega_0 s + \omega_0^2} \tag{7.67}$$

using the bilinear transformation[8]

$$s = \frac{2}{T_s}\frac{1 - z^{-1}}{1 + z^{-1}} \tag{7.68}$$

introduces an attenuation (a "hole") at the normalized discretized frequency

$$\omega_{disc} = 2\arctan(\frac{\omega_0 T_s}{2}) \tag{7.69}$$

as a function of the ratio $\zeta_{num}/\zeta_{den} < 1$. *The attenuation at* ω_{disc} *is given by*

$$M_t = 20\log(\frac{\zeta_{num}}{\zeta_{den}})\ ;\ (\zeta_{num} < \zeta_{den}) \tag{7.70}$$

[8]The bilinear transformation assures a better approximation of a continuous-time model by a discrete-time model in the frequency domain than the replacement of differentiation by a difference, i.e., $s = (1 - z^{-1})T_s$ (see [6]).

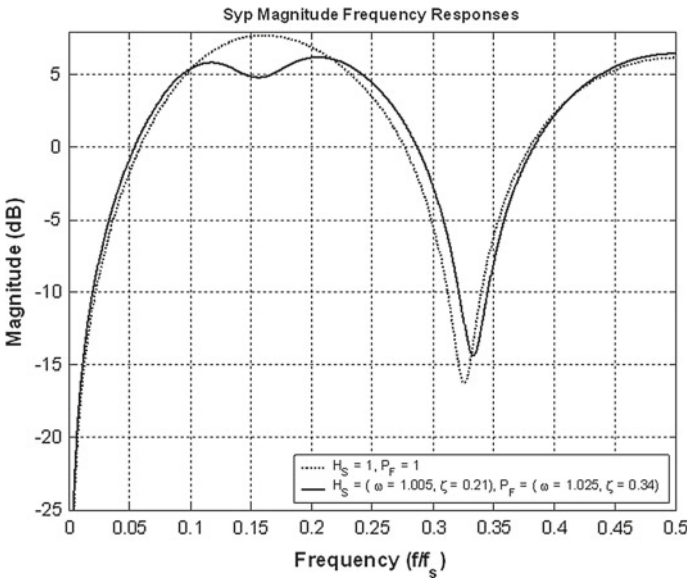

Fig. 7.12 Effects of a resonant filter H_{S_i}/P_{F_i} on the output sensitivity functions

The effect upon the frequency characteristics of S_{yp} at frequencies $f << f_{disc}$ and $f >> f_{disc}$ is negligible.

Figure 7.12 illustrates the effect of the simultaneous introduction of a fixed part H_S and a pair of poles in P, corresponding to the discretization of a resonant filter of the form of (7.67). One observes its weak effect on the frequency characteristics of S_{yp}, far from the resonance frequency of the filter.

This pole-zero filter (band-stop filter) is essential for an accurate shaping of the modulus of the sensitivity functions in the various frequency regions in order to satisfy the constraints. It allows to reduce the interaction between the tuning in different regions.

Design of the Band-Stop Filter H_{S_i}/P_{F_i}

The computation of the coefficients of H_{S_i} and P_{F_i} is done in the following way:

Specifications:

- Central normalized frequency f_{disc} ($\omega_{disc} = 2\pi f_{disc}$)
- Desired attenuation at frequency f_{disc} : M_t (dB)
- Minimum accepted damping for auxiliary poles

$$P_{F_i} : (\zeta_{den})_{min} \geq 0.3$$

Step I: Design of the continuous-time filter

$$\omega_0 = \frac{2}{T_s}\tan(\frac{\omega_{disc}}{2}) \quad 0 \leq \omega_{disc} \leq \pi \quad \zeta_{num} = 10^{M_t/20}\zeta_{den}$$

Step II: Design of the discrete-time filter using the bilinear transformation (7.68). Using (7.68) one gets

$$F(z^{-1}) = \frac{a_{z0} + a_{z1}z^{-1} + a_{z2}z^{-2}}{a_{z0} + a_{z1}z^{-1} + a_{z2}z^{-2}} = \gamma \frac{1 + \beta_1 z^{-1} + \beta_2 z^{-2}}{1 + \alpha_1 z^{-1} + \alpha_2 z^{-2}} \tag{7.71}$$

which will be effectively implemented as[9]

$$F(z^{-1}) = \frac{H_S(z^{-1})}{P_i(z^{-1})} = \frac{1 + \beta_1 z^{-1} + \beta_2 z^{-2}}{1 + \alpha_1 z^{-1} + \alpha_2 z^{-2}}$$

where the coefficients are given by[10]

$$\begin{aligned} b_{z0} &= \frac{4}{T_s^2} + 4\frac{\zeta_{num}\omega_0}{T_s} + \omega_0^2 \; ; \; b_{z1} = 2\omega_0^2 - \frac{8}{T_s^2} \\ b_{z2} &= \frac{4}{T_s^2} - 4\frac{\zeta_{num}\omega_0}{T_s} + \omega_0^2 \\ a_{z0} &= \frac{4}{T_s^2} + 4\frac{\zeta_{den}\omega_0}{T_s} + \omega_0^2 \; ; \; a_{z1} = 2\omega_0^2 - \frac{8}{T_s^2} \\ a_{z2} &= \frac{4}{T_s^2} - 4\frac{\zeta_{den}\omega_0}{T_s} + \omega_0^2 \end{aligned} \tag{7.72}$$

$$\begin{aligned} \gamma &= \frac{b_{z0}}{a_{z0}} \\ \beta_1 &= \frac{b_{z1}}{b_{z0}} \; ; \; \beta_2 = \frac{b_{z2}}{b_{z0}} \\ \alpha_1 &= \frac{a_{z1}}{a_{z0}} \; ; \; \alpha_2 = \frac{a_{z2}}{a_{z0}} \end{aligned} \tag{7.73}$$

Remark For frequencies below 0.17 f_s the design can be done with a very good precision directly in discrete-time. In this case, $\omega_0 = \omega_{0den} = \omega_{0num}$ and the damping of the discrete-time filters H_{S_i} and P_{F_i} is computed as a function of the attenuation directly using Eq. (7.70).

Remark While H_S is effectively implemented in the controller, P_F is only used indirectly. P_F will be introduced in (7.17) and its effect will be reflected in the coefficients of R and S obtained as solutions of Eq. (7.59).

If the S polynomial contains the internal model of a sinusoidal disturbance, i.e., $S = S'D_p$ and D_p is a second-order polynomial with zero damping and a resonance frequency ω, the modulus of the output sensitivity function will be zero at this

[9]The factor γ has no effect on the final result (coefficients of R and S). It is possible, however, to implement the filter without normalizing the numerator coefficients.

[10]These filters can be computed using the functions *filter22.sci* (Scilab) and *filter22.m* (MATLAB®) to be downloaded from the book website.

frequency, which means total rejection of a sinusoidal disturbance. Without any shaping of the sensitivity function, there will be a "waterbed effect" in the vicinity of this frequency; however, if the objective is to introduce just a certain amount of attenuation, we should consider introduction of the "band-stop" filters which have zeros and poles. The numerator will be implemented in the "S" polynomial while the poles will be added to the desired closed-loop poles. In this case the waterbed effect will be less important.

For n narrow-band disturbances, n band-stop filters will be used

$$\frac{S_{BSF}(z^{-1})}{P_{BSF}(z^{-1})} = \frac{\prod_{i=1}^{n} S_{BSF_i}(z^{-1})}{\prod_{i=1}^{n} P_{BSF_i}(z^{-1})}. \tag{7.74}$$

A similar procedure can be used for shaping the input sensitivity function (H_S in Eq. (7.66) is replaced H_R).

7.3 Real-Time Example: Narrow-Band Disturbance Attenuation on the Active Vibration Control System Using an Inertial Actuator

This section illustrates the methodology used for the attenuation of narrow-band disturbances through an example. The active vibration control system with inertial actuator described in Sect. 2.2 will be used as a test bench. An open-loop identification for this system has been done in Sect. 6.2. The sampling frequency is $f_s = 800$ Hz.

One sinusoidal disturbance at 70 Hz is applied to the system. The disturbance is filtered by the primary path and its effects are measured by the residual force transducer. The objective is to strongly attenuate the effect of this disturbance on the residual force. The internal model principle together with the shaping of the sensitivity functions will be used for the design of a linear robust controller.

The specifications are as follows:

- the controller should eliminate the disturbance at 70 Hz (at least 40 dB attenuation).
- the maximum allowed amplification of the output sensitivity function is 6 dB (i.e., the modulus margin will be $\Delta M \geqslant 0.5$).
- a delay margin of at least one sampling period should be achieved.
- the gain of the controller has to be zero at 0 Hz (since the system has a double differentiator behaviour).
- the gain of the controller should be zero at $0.5 f_s$ where the system has low gain and uncertainties exist.
- the effect of disturbances on the control input should be attenuated above 100 Hz in order to improve robustness with respect to unmodelled dynamics ($S_{up}(e^{j\omega}) < -40$ dB, $\forall \omega \in [100, 400$ Hz$]$).

The steps for the design of the linear controller are

1. include all (stable) secondary path poles in the closed-loop characteristic polynomial;
2. design the fixed part of the controller denominator in order to cancel the 70 Hz disturbance (IMP)
$$H_S(q^{-1}) = 1 + a_1 q^{-1} + q^{-2}, \tag{7.75}$$
where $a_1 = -2\cos(2\pi f/f_S)$, $f = 70$ Hz. The modulus of the resulting output sensitivity function is shown in Fig. 7.13 (curve IMP). As one can see the maximum of the modulus of the output sensitivity function is larger than 6 dB;
3. open the loop at 0 Hz and at 400 Hz by setting the fixed part of the controller numerator as
$$H_R = (1 + q^{-1}) \cdot (1 - q^{-1}) = 1 - q^{-2}. \tag{7.76}$$
The resulting output sensitivity function is shown also in Fig. 7.13 (curve IMP + Hr). As it can be seen, it has an unacceptable value around 250 Hz (violation of the delay margin constraint);
4. to improve robustness two complex conjugate poles have been added to the characteristic polynomial, one at 65 Hz and the second at 75 Hz, both of them with 0.2 damping factor. The resulting output sensitivity function (curve IMP + Hr + aux. poles) has the desired characteristics; however, as one can see in Fig. 7.14 (curve IMP + Hr + aux. poles), the modulus of the input sensitivity function is higher than −40 dB between 100 and 400 Hz;

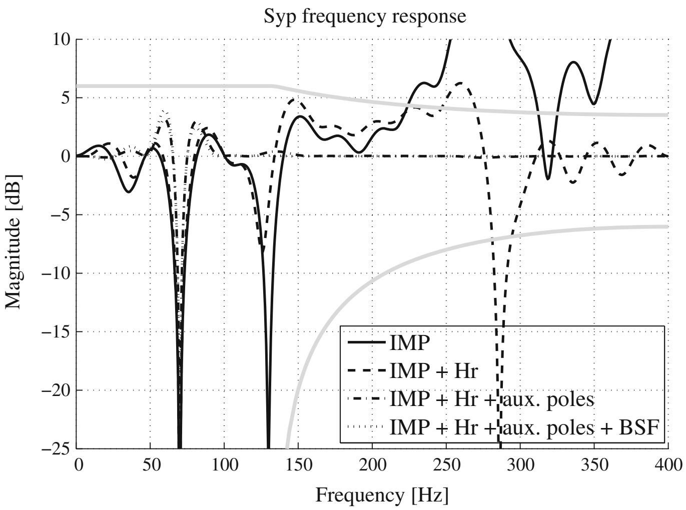

Fig. 7.13 Output sensitivity functions for the various controllers (*grey lines* represent the templates for modulus and delay margins)

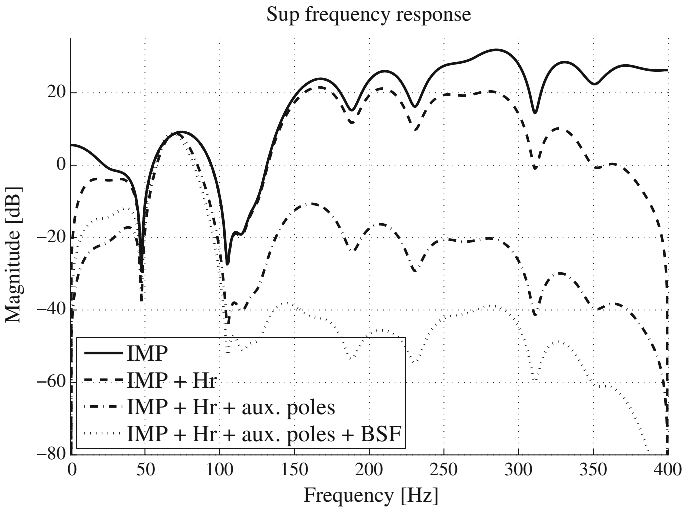

Fig. 7.14 Input sensitivity functions for the various controllers

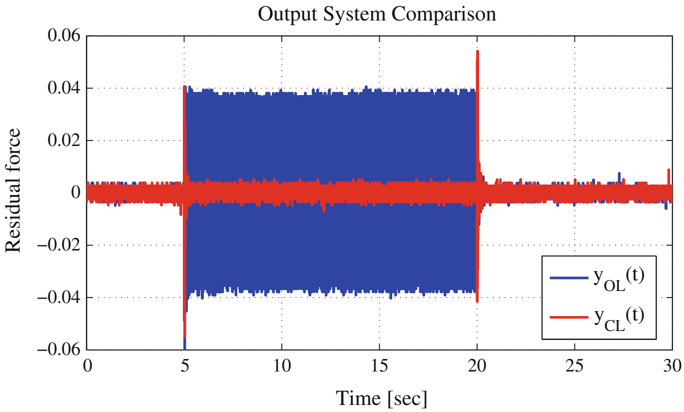

Fig. 7.15 Time response results for a 70 Hz disturbance in open-loop and in closed-loop

5. add band-stop filters (BSF) on the S_{up} sensitivity function: one at 160 Hz, the other at 210 Hz, with -20 and -15 dB attenuation respectively. Both have 0.9 damping factor for the denominator. One can see that this has the desired effect on the input sensitivity functions and no effects on the output sensitivity function.

The resulting modulus margin is 0.637 and the resulting delay margin is $2.012 \cdot T_s$. The final controller satisfies the desired specifications both in terms of performance and robustness.

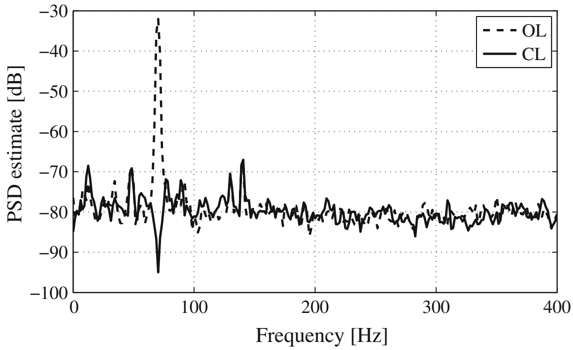

Fig. 7.16 PSD of the open-loop disturbance (*dashed line*) and effective attenuation (*solid line*) for 70 Hz in closed-loop

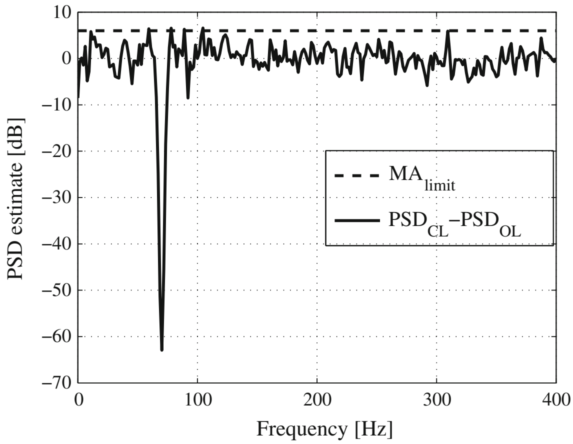

Fig. 7.17 Effective residual attenuation/amplification PSD estimates computed as the difference between the open-loop PSD and the closed-loop PSD

Real-Time Results

Time domain results in open-loop ($y_{OL}(t)$) and in closed-loop ($y_{CL}(t)$) are shown in Fig. 7.15. A frequency domain analysis has been done and is shown in Figs. 7.16 and 7.17. It can be seen that the controller achieves all the desired specifications. Under the effect of the controller the residual force is almost at the level of the system's noise.

7.4 Pole Placement with Sensitivity Function Shaping by Convex Optimisation

In [11] it was shown that the problem of shaping the sensitivity functions in the context of pole placement can be formulated as a convex optimisation problem, and routines for convex optimisation can be used (available in the toolbox OPTREG[11]).

[11] To be downloaded from the book website.

We will present this method which will be used in the context of active damping. This method takes first advantage of the Youla–Kučera parametrization. It is assumed that

- the fixed parts of the controller H_R and H_S have been defined (in order to achieve certain performances);
- a "central" stabilizing controller is already designed;
- the templates for the output and input sensitivity functions have been defined (in order to obtain the required robustness margins and performance specifications).

One considers the Youla–Kučera parmetrization for the controller as follows:

$$R = H_R(R_0 + AH_S Q) \tag{7.77}$$

$$S = H_S(S_0 - z^{-d} B H_R Q) \tag{7.78}$$

where the fixed parts of the controller (H_R, H_S), and (A, B) are polynomials in z^{-1} ($z^{-d}B/A$ is the nominal model) and Q is a rational transfer function proper and asymptotically stable.[12]

The central controller R_0/S_0 ($Q = 0$) can be obtained by solving the following Bezout equation for R_0 and S_0:

$$AH_S S_0 + z^{-d} B H_R R_0 = P_D, \tag{7.79}$$

where P_D is an asymptotically stable polynomial defined by the designer and which contains the desired dominant poles for the closed-loop system. Expressing Q as a ratio of proper transfer functions in z^{-1} such as

$$Q(z^{-1}) = \frac{B_Q(z^{-1})}{A_Q(z^{-1})} \tag{7.80}$$

one gets

$$\frac{R}{S} = \frac{H_R(R_0 A_Q + AH_S B_Q)}{H_S(S_0 A_Q - z^{-d} B H_R B_Q)}. \tag{7.81}$$

The poles of the closed-loop system will be given by

$$P = AS + z^{-d} BR = P_D A_Q,$$

where the zeros of P_D are the fixed poles of the closed-loop (defined by the central controller) and the zeros of A_Q are the additional poles which will be introduced by the optimization procedure. The output and input sensitivity functions can be written as:

$$S_{yp} = \frac{AS}{AS + z^{-d} BR} = \frac{AH_S}{P_D}\left(S_0 - B_{nom} H_R \frac{B_Q}{A_Q}\right) ; \tag{7.82}$$

[12]This particular YK parametrization allows to preserve the fixed parts H_R and H_S in the resulting controller given in Eq. (7.4).

$$S_{up} = \frac{AH_R}{P_D}\left(R_0 + A_{nom}H_S\frac{B_Q}{A_Q}\right). \tag{7.83}$$

As shown in the above Eqs. (7.82) and (7.83), the sensitivity functions can obviously be expressed in the form $T_1 + T_2\frac{\beta}{\alpha}$.

Imposing a certain frequency-dependent limit W (template) on the modulus of the sensitivity functions (attenuation band, modulus margin, delay margin, restrictions on the input sensitivity function) leads then to a condition of the form

$$\left|T_1 \arg z + T_2 \arg z\frac{\beta' \arg z}{\alpha' \arg z}\right| \leq |W \arg z| \quad \forall |z| = 1 \tag{7.84}$$

Condition (7.84) is equivalent to the condition

$$\left\|\bar{T}_1 + \bar{T}_2\frac{\beta'}{\alpha'}\right\|_\infty < 1 \tag{7.85}$$

Thus, Eq. (7.84) implies the existence of α and β such that by setting $\bar{T}_1 = W^{-1}T_1$ and $\bar{T}_2 = W^{-1}T_2$ one obtains

$$\left|W^{-1}T_1\alpha + W^{-1}T_2\beta\right| \leq Re\{\alpha\} \tag{7.86}$$

and this is obviously a convex condition on α and β. Details can be found in [11, 12].

For point-wise testing of the conditions a frequency gridding is carried out (e.g. 32 points between $f = 0$ and $f = 0.5f_s$).

For the optimization procedures the polynomials A_Q et B_Q will take the form (Ritz method):

$$A_Q(x_a) = 1 + \sum_{k=1}^{N} x_{ak}\alpha_k ; \tag{7.87}$$

$$B_Q(x_b) = x_{b_0} + \sum_{k=1}^{N} x_{bk}\beta_k, \tag{7.88}$$

where α_k, β_k are stable polynomials (affine in x_{ak} et x_{bk}) and N is the order of the parametrization (i.e., the number of points on the sensitivity functions where the constraints have to be verified). The parameters to be optimized are x_{ak} et x_{bk}.

For the discrete-time cases α_k and β_k can be chosen as

$$\alpha_k = \beta_k = \left(\frac{z_0 - z^{-1}}{1 - z_0 z^{-1}}\right)^k,$$

where z_0 is the time constant of the parametrization (which can be adjusted).

Using the parametrization and the constraints indicated above an (RS) controller with desired properties can be obtained by convex optimization. For more details on the optimization procedure see [13, 14].

The MATLAB® toolbox Optreg provides the appropriate routines for specifying the constraints and finding the optimal controller. The method will be used in Chap. 10 for active damping.

7.5 Concluding Remarks

- The design of polynomial RS controllers for active vibration control systems has been discussed in this chapter.
- The design of the controller requires the knowledge of the plant model (the secondary path in active vibration control).
- Asymptotic rejection of tonal disturbances can be achieved using the Internal Model Principle (it requires the knowledge of the frequency of the disturbance).
- The Youla–Kučera parametrization of the controller provides a separation between disturbance compensation and feedback stabilization.
- Robustness is not an intrinsic property of a control strategy. It results from an appropriate choice of some control objectives related to the sensitivity functions.
- Two sensitivity functions are of major interest: the output sensitivity function and the input sensitivity function.
- *Modulus margin* and *delay margin* are basic robustness indicators.
- Shaping of the sensitivity functions is a key issue in active vibration control in order to achieve desired performance and robustness objectives.
- Performance and robustness specifications translate in desired templates for the sensitivity functions.
- Pole placement combined with tools for shaping the sensitivity functions is an efficient approach for designing active vibration control systems.
- Shaping of the sensitivity functions can be conveniently achieved by the selection of the auxiliary poles and the use of band-stop filters.
- Pole placement combined with convex optimization can provide almost an automatic solution to the design problem, once the desired templates for the sensitivity functions are defined.

7.6 Notes and References

The first issue in the design of AVC systems (assuming that the plant model is known) is the translation of the performance and robustness specifications in desired templates for the sensitivity functions. Then any design method which allows to achieve the desired sensitivity functions can be used, such as Pole placement

[1, 6, 7, 15], Linear Quadratic Control [6, 7, 15], H_∞ control [8, 16], CRONE control [17–19], Generalized Predictive Control [2, 20].

The shaping of the sensitivity function can be converted in a convex optimization problem [12] and the use of this approach is detailed in [11, 13, 14].

The Bode integral constraint in the context of AVC is discussed in [21, 22].

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