Chapter 3
Active Vibration Control Systems—Model Representation

3.1 System Description

3.1.1 Continuous-Time Versus Discrete-Time Dynamical Models

Before discussing the system description aspects, one has to take in account that the control law will be implemented on a digital computer. To do this there are two basic options:

- Represent the system in continuous time, compute the control law in continuous time and then discretize the continuous-time control law for implementation.
- Select the sampling frequency, represent the system in discrete time, compute the control law in discrete time and implement it directly.

Since one deals with mechanical systems, differential equations can be written to describe the dynamical behaviour of the various parts of the system allowing to build a "dynamical model" to be used for the design of the active vibration control system [1, 2].[1] There are however several obstacles in using a continuous-time representation of the system.

First of all, since the physical parameters are not precisely known, the model which can be obtained from the fundamental principles will not be very reliable. In addition there are parts of the systems for which it is difficult to give a perfect representation and to associate the corresponding parameters. For a high performance control design one needs an accurate dynamical model of the specific system to be controlled and therefore one has to consider identifying dynamical models from experimental input/output data, obtained by what is called an "identification protocol" (a "black box model" will be obtained).

It turns out that identification techniques are more efficient and much easier to implement if one considers the identification of discrete time dynamic models.

[1]Modern control design techniques use "model based control design".

I.D. Landau et al., *Adaptive and Robust Active Vibration Control*,
Advances in Industrial Control, DOI 10.1007/978-3-319-41450-8_3

It is also important to point out that using a continuous-time representation, the objective of the discretization of the designed control law will be to copy as much as possible the continuous-time control and this will require in general a very high sampling frequency. Further analysis is required in order to be sure that the discretized control law guarantees the robustness and performance of the system (since discretization introduces an approximation).

It turns out that if one considers the alternative situation, i.e., to discretize the input and output of the system at a sampling frequency which is related to its band-pass, one obtains through system identification a discrete-time dynamical model which can be used to design a discrete-time control algorithm.

Using a discrete-time representation of the system will require a lower sampling frequency[2] (directly related to the higher frequencies to be controlled) and the control algorithm to be implemented results directly from the design (no additional analysis is necessary since the control algorithm has been designed at the sampling frequency used).

Therefore because discrete time dynamical models allow:

- using a lower sampling frequency;
- using simpler and more efficient identification algorithms;
- getting directly the control algorithm to be implemented on a digital computer,

they will be used subsequently for representing active vibration control systems. The design of the control algorithm will be based on identified discrete-time dynamical models of the system.

3.1.2 Digital Control Systems

In this section, one reviews the basic requirements for the implementation of a digital control system. For a more detailed discussion of the various aspects see [3–6]. Figure 3.1 represents the structure of a digital control system. In Fig. 3.1, the set[3]: Digital to Analog Converter (D/A), Zero order hold (ZOH),[4] Continuous-time plant, Analog to Digital Converter (A/D) constitutes the discrete-time system to be controlled by a digital controller implemented on the computer used for control.

3.1.2.1 Selection of the Sampling Frequency

A good rule for the selection of the sampling frequency is [3]:

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[2]Numerous examples show that by using this approach, the sampling frequency can be reduced with respect to the previous approach.

[3]Temporarily in this section t designates the continuous time and k the normalized sampling time ($k = \frac{time}{T_s}$). Starting from Sect. 3.1.3 the normalized discrete time will be denoted by t.

[4]ZOH keeps constant the signal delivered by the D/A converter between two sampling instants.

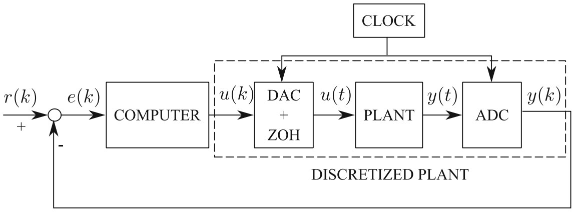

Fig. 3.1 A digital control system

$$f_s = (6 \rightarrow 25) f_B^{CL}, \tag{3.1}$$

where

- f_s = sampling frequency (in Hz);
- f_B^{CL} = desired bandwidth of the closed-loop system (in Hz).

Of course, the desired closed-loop bandwidth is related to the bandwidth of the system to be controlled. The formula (3.1) gives enough freedom for the selection of the sampling frequency.

As a general rule, one tries to select the lower sampling frequency compatible with the desired performances.

Except in very particular cases, all the discrete-time models will feature a fractional delay. Fractional delays are reflected as zeros in the transfer function of the discrete-time models (these zeros will be unstable if the fractional delay is larger that half of the sampling period [3]). For continuous-time systems with a relative degree higher or equal to 2, high frequency sampling will induce unstable zeros [5]. The consequence of the presence of unstable zeros in the discrete-time models used for control design is that control strategies based on cancelling the zeros cannot be used.

3.1.2.2 Anti-aliasing Filters

The theory of discrete-time systems [5, 6] indicates that the maximum frequency ($f_{\max}$) of a signal sent to the analog to digital converter should satisfy:

$$f_{\max} < f_s/2, \tag{3.2}$$

where f_s is the sampling frequency and $f_s/2$ is called the Nyquist or Shannon frequency.

Sending frequencies over $f_s/2$ produces distortion of the recovered discrete-time spectrum called *aliasing*. Therefore, *anti-aliasing* filters should always be introduced in order to remove the undesirable components of the signal. Anti-aliasing filters are constituted in general as several second order filters in cascade (Bessel, ITAE, Butterworth type). They should introduce a consequent attenuation of the signal beyond $f_s/2$ but their bandwidth should be larger than the desired closed-loop bandwidth.

Their design will also depend on the level of undesirable signals at frequencies beyond $f_s/2$.

The anti-aliasing filters introduce a high frequency dynamics which can in general be approximated by an additional small time delay. Since one directly estimates a discrete-time model from data, their effect is captured by the estimated model.

3.1.3 Discrete-Time System Models for Control

The discrete-time models will represent the behaviour of the controlled system from the discrete-time control applied to the system through a D/A converter and a ZOH to the output of the A/D converter which will discretize the measured output. Single-input single-output time invariant systems will be considered. They will be described by input–output discrete-time models of the form:

$$y(t) = -\sum_{i=1}^{n_A} a_i y(t-i) + \sum_{i=1}^{n_B} b_i u(t-d-i), \tag{3.3}$$

where t denotes the normalized sampling time (i.e., $t = \frac{time}{T_s}$, T_s = sampling period), $u(t)$ is the input, $y(t)$ is the output, d is the integer number of sampling periods contained in the time delay of the systems, a_i and b_i are the parameters (coefficients) of the models.

As such the output of the system at instant t is a weighted average of the past output over an horizon of n_A samples plus a weighted average of past inputs over an horizon of n_B samples (delayed by d samples).

This input–output model (3.3) can be more conveniently represented using a coding in terms of forward or backward shift operators defined as:

$$qy(t) = y(t+1); \quad q^{-1}y(t) = y(t-1) \tag{3.4}$$

Using the notations

$$1 + \sum_{i=1}^{n_A} a_i q^{-i} = A(q^{-1}) = 1 + q^{-1}A^*(q^{-1}), \tag{3.5}$$

where

$$A(q^{-1}) = 1 + a_1 q^{-1} + \cdots + a_{n_A} q^{-n_A} \tag{3.6}$$

$$A^*(q^{-1}) = a_1 + a_2 q^{-1} + \cdots + a_{n_A} q^{-n_A+1} \tag{3.7}$$

and

$$\sum_{i=1}^{n_B} b_i q^{-i} = B(q^{-1}) = q^{-1}B^*(q^{-1}), \tag{3.8}$$

where

$$B(q^{-1}) = b_1 q^{-1} + b_2 q^{-2} + \cdots + b_{n_B} q^{-n_B} \tag{3.9}$$
$$B^*(q^{-1}) = b_1 + b_2 q^{-1} + \cdots + b_{n_B} q^{-n_B+1}. \tag{3.10}$$

Equation (3.3) can be rewritten as

$$A(q^{-1})y(t) = q^{-d}B(q^{-1})u(t) = q^{-d-1}B^*(q^{-1})u(t) \tag{3.11}$$

or forward in time

$$A(q^{-1})y(t+d) = B(q^{-1})u(t) \tag{3.12}$$

as well as

$$y(t+1) = -A^*y(t) + q^{-d}B^*u(t) = -A^*y(t) + B^*u(t-d). \tag{3.13}$$

It can be observed that (3.13) can also be expressed as

$$y(t+1) = \theta^T\phi(t), \tag{3.14}$$

where θ defines the vector of parameters

$$\theta^T = [a_1, \dots a_{n_A}, b_1, \dots, b_{n_B}] \tag{3.15}$$

and $\phi(t)$ defines the vector of measurements (or the regressor)

$$\phi^T(t) = [-y(t) \cdots - y(t-n_A+1), u(t-d) \cdots u(t-d-n_B+1)] \tag{3.16}$$

The form of (3.14) will be used in order to estimate the parameters of a system model from input–output data. Filtering both left and right sides of (3.11) through a filter $1/A(q^{-1})$ one gets

$$y(t) = G(q^{-1})u(t), \tag{3.17}$$

where

$$G(q^{-1}) = \frac{q^{-d}B(q^{-1})}{A(q^{-1})} \tag{3.18}$$

is termed the *transfer operator*.[5]

[5]In many cases, the argument q^{-1} will be dropped out, to simplify the notations.

Computing the z-transform of (3.3), one gets the *pulse transfer function* characterizing the input–output model of (3.3)

$$G(z^{-1}) = \frac{z^{-d}B(z^{-1})}{A(z^{-1})} \tag{3.19}$$

It can be observed that the transfer function of the input–output model of (3.3) can be formally obtained from the *transfer operator* by replacing the time operator q by the complex variable z. Nevertheless, one should be careful since the domain of these variables is different. However, in the linear case with constant parameters one can use either one and their appropriate signification will result from the context.

Note also that the transfer operator $G(q^{-1})$ can be defined even if the parameters of the model (3.3) are time-varying, while the concept of pulse transfer function does not simply exist in this case.

The order n of the system model[6] (3.3), is the dimension of the minimal state space representation associated to the input–output model (3.3) and in the case of irreducible transfer function it is equal to

$$n = \max(n_A, n_B + d), \tag{3.20}$$

which corresponds also to the number of the poles of the irreducible transfer function of the system.

The order of the system is immediately obtained by expressing the transfer operator (3.18) or the transfer function (3.19) in the forward operator q and respectively the complex variable z. The passage from $H(z^{-1})$ to $H(z)$ is obtained multiplying by z^n:

$$G(z) = \frac{\bar{B}(z)}{\bar{A}(z)} = \frac{z^{r-d}B(z^{-1})}{z^r A(z^{-1})} \tag{3.21}$$

Example

$$G(z^{-1}) = \frac{z^{-3}(b_1 z^{-1} + b_2 z^{-2})}{1 + a_1 z^{-1}}$$
$$r = \max(1, 5) = 5$$
$$G(z) = \frac{b_1 z + b_2}{z^5 + a_1 z^4}$$

[6]The order of the system will be in general estimated from input/output data.

3.2 Concluding Remarks

- Recursive (differences) equations are used to describe discrete-time dynamic models.
- The delay operator q^{-1} ($q^{-1}y(t) = y(t-1)$) is a simple tool to handle recursive discrete-time equations.
- The input–output relation for a discrete-time model is conveniently described in the time domain by the pulse transfer operator $G(q^{-1})$: $y(t) = G(q^{-1})u(t)$.
- The pulse transfer function of a discrete-time linear system is expressed as function of the complex variable $z = e^{sT_s}$ (T_s = sampling period). The pulse transfer function can be derived from the pulse transfer operator $G(q^{-1})$ by replacing q^{-1} with z^{-1}.
- The asymptotic stability of a discrete-time model is ensured if, and only if, all pulse transfer function poles (in z) lie inside the unit circle.
- The order of a pulse transfer function of the form

$$G(z^{-1}) = \frac{z^{-d}B(z^{-1})}{A(z^{-1})} \tag{3.22}$$

is $n = \max(n_A, n_B + d)$, where n_A and n_B are the orders of the polynomials A and B, respectively, and d is the integer time delay in terms of sampling periods.

3.3 Notes and References

There are many excellent books on digital control systems. The books [3, 5, 6] are probably the most suited for the topics of the present book. The book [7] provides many discrete-time models obtained from the discretization of continuous-time models for various physical systems.

References

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