Chapter 5
State Estimation and EMPC

5.1 Introduction

In the previous chapters, all EMPC schemes are developed under the assumption of state feedback. However, this assumption does not hold in many practical applications. To address this issue, in this chapter, we introduce two output feedback EMPC schemes based on state estimation.

First, working with a class of full-state feedback linearizable nonlinear systems, a high-gain observer-based output feedback EMPC scheme is presented. In this scheme, a high-gain observer is used to estimate the system state using output measurements and the EMPC uses the observer state estimates. Sufficient conditions for the stability of the closed-loop system are derived using singular perturbation arguments. A chemical process example is used to demonstrate the ability of the high-gain observer-based EMPC to achieve time-varying operation that leads to a superior economic performance compared to the performance achieved under steady-state operation.

To improve the robustness of the closed-loop system especially to plant/model mismatch and uncertainties and to reduce the sensitivity of the state observer to measurement noise, a robust moving horizon estimation (RMHE) based output feedback EMPC design is subsequently presented. Bounded process and measurement noise is considered. To achieve fast convergence of the state estimates to the actual state (inducing an effective separation principle between the state observer and controller designs), a deterministic (high-gain) observer is first applied for a small time period with continuous output measurements to drive the estimation error to a small value. After the initial time period, a RMHE designed based on the deterministic observer is used to provide more accurate and smooth state estimates to the EMPC and thus, improves the robustness of the closed-loop system to noise. In the RMHE design, the deterministic observer is used to calculate a reference estimate and a confidence region for the state estimate. The confidence region is subsequently used as a constraint in the RMHE problem. Closed-loop stability is rigorously analyzed, and

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conditions that ensure closed-loop stability are derived. Extensive simulations based on a chemical process example illustrate the effectiveness of the second scheme.

5.1.1 System Description

We consider nonlinear systems described by the following state-space model:

$$\begin{aligned} \dot{x}(t) &= f(x(t)) + g(x(t))u(t) + l(x(t))w(t) \\ y(t) &= h(x(t)) + v(t) \end{aligned} \tag{5.1}$$

where $x(t) \in \mathbb{R}^n$ denotes the state vector, $u(t) \in \mathbb{R}^p$ denotes the control (manipulated) input vector, $w(t) \in \mathbb{R}^m$ denotes the disturbance vector, $y(t) \in \mathbb{R}^q$ denotes the measured output vector and $v(t) \in \mathbb{R}^q$ is the measurement noise vector. The control input vector is restricted to a nonempty convex set $\mathbb{U} \subseteq \mathbb{R}^p$ such that $\mathbb{U} := \{u \in \mathbb{R}^p : |u| \leq u^{\max}\}$ where $u^{\max}$ is the magnitude of the input constraint. It is assumed that the noise vectors are bounded such as $w(t) \in \mathbb{W}$ and $v(t) \in \mathbb{V}$ for all $t \geq 0$ where

$$\begin{aligned} \mathbb{W} &:= \{w \in \mathbb{R}^m : |w| \leq \theta_w,\ \theta_w > 0\} \\ \mathbb{V} &:= \{v \in \mathbb{R}^q : |v| \leq \theta_v,\ \theta_v > 0\} \end{aligned}$$

where θ_w and θ_v are positive real numbers. Moreover, it is assumed that the output measurement vector y of the system is continuously available at all times. It is further assumed that f, g, l and h are sufficiently smooth functions and $f(0) = 0$ and $h(0) = 0$.

5.1.2 Stabilizability Assumption

We assume that there exists a state feedback controller $u = k(x)$, which renders the origin of the nominal closed-loop system asymptotically and locally exponentially stable while satisfying the input constraints for all the states x inside a given stability region. Using converse Lyapunov theorems, this assumption implies that there exist class $\mathcal{K}$ functions $\alpha_i(\cdot)$, $i = 1, 2, 3, 4$ and a continuously differentiable Lyapunov function $V : D \to \mathbb{R}$ for the closed-loop system, that satisfy the following inequalities:

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \tag{5.2a}$$

$$\frac{\partial V(x)}{\partial x}(f(x) + g(x)k(x)) \leq -\alpha_3(|x|) \tag{5.2b}$$

$$\left| \frac{\partial V(x)}{\partial x} \right| \leq \alpha_4(|x|) \tag{5.2c}$$

$$k(x) \in \mathbb{U} \tag{5.2d}$$

for all $x \in D \subseteq \mathbb{R}^n$ where D is an open neighborhood of the origin. We denote the region $\Omega_\rho \subseteq D$ as the stability region of the closed-loop system under the controller $k(x)$. Using the smoothness assumed for f and g, and taking into account that the manipulated input is bounded, there exists a positive constant M such that

$$|f(x) + g(x)u| \leq M \tag{5.3}$$

for all $x \in \Omega_\rho$ and $u \in \mathbb{U}$. In addition, by the continuous differentiable property of the Lyapunov function V and the smoothness of f and g, there exist positive constants L_x, L_u, C_x, $C_{g'}$ and C_g such that

$$\begin{aligned}
\left|\frac{\partial V(x)}{\partial x} f(x) - \frac{\partial V(x')}{\partial x} f(x')\right| &\leq L_x |x - x'| \\
\left|\frac{\partial V(x)}{\partial x} g(x) - \frac{\partial V(x')}{\partial x} g(x')\right| &\leq L_u |x - x'| \\
|f(x) - f(x')| &\leq C_x |x - x'| \\
|g(x) - g(x')| &\leq C_{g'} |x - x'| \\
\left|\frac{\partial V(x)}{\partial x} g(x)\right| &\leq C_g
\end{aligned} \tag{5.4}$$

for all $x, x' \in \Omega_\rho$ and $u \in \mathbb{U}$.

5.2 High-Gain Observer-Based EMPC Scheme

To simplify the presentation but without loss of generality, we restrict our consideration to single-input single-output nonlinear systems in this section. Moreover, we consider systems without process disturbances and measurement noise. The later assumption is relaxed in the subsequent section where robustness is explicitly addressed. The system in Eq. 5.1 reduces to the following system:

$$\begin{aligned}
\dot{x} &= f(x) + g(x)u \\
y &= h(x)
\end{aligned} \tag{5.5}$$

where $u \in \mathbb{R}$ and $y \in \mathbb{R}$. The presented approach may be extended to multi-input multi-output systems in a conceptually straightforward manner.

It is assumed that the system in Eq. 5.5 is full-state feedback linearizable. Thus, the relative degree of the output with respect to the input is n. Assumption 5.1 below states this requirement.

Assumption 5.1 There exists a set of coordinates

$$z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = T(x) = \begin{bmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{n-1} h(x) \end{bmatrix} \tag{5.6}$$

such that the system of Eq. 5.1 may be written as:

$$\begin{aligned} \dot{z}_1 &= z_2 \\ &\vdots \\ \dot{z}_{n-1} &= z_n \\ \dot{z}_n &= L_f^n h(T^{-1}(z)) + L_g L_f^{n-1} h(T^{-1}(z))u \\ y &= z_1 \end{aligned}$$

where $L_g L_f^{n-1} h(x) \neq 0$ for all $x \in \mathbb{R}^n$ ($L_f h(x)$ and $L_g h(x)$ denote Lie derivatives of the function h with respect to f and g, respectively).

Using Assumption 5.1, the system of Eq. 5.5 may be rewritten in the following compact form:

$$\begin{aligned} \dot{z} &= Az + B[L_f^n h(T^{-1}(z)) + L_g L_f^{n-1} h(T^{-1}(z))u] \\ y &= Cz \end{aligned}$$

where

$$A = \begin{bmatrix} 0_{n-1} & I_{n-1} \\ 0 & 0_{n-1}^T \end{bmatrix}, \quad B = \begin{bmatrix} 0_{n-1} \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 \\ 0_{n-1} \end{bmatrix}^T,$$

0_{n-1} denotes a $n-1$ dimensional vector with all elements equal to zero, and I_{n-1} denotes the $n-1$ by $n-1$ identity matrix.

Remark 5.1 Assumption 5.1 imposes certain practical restrictions on the applicability of the method. However, this should be balanced with the nature of the results achieved by the output feedback controller in the sense that for a sufficiently large observer gain, the closed-loop system under the output feedback controller approaches the closed-loop stability region and performance of the state feedback controller. Essentially, a nonlinear separation-principle is achieved because of Assumption 5.1 and the use of a high-gain observer (please see Theorem 5.1). This is an assumption imposed in most previous works that use high-gain observers for state estimation, starting from the early work of Khalil and co-workers [1]. With respect to practical restrictions, our example demonstrates that the method is applicable to a class of chemical reactor models. The requirement of full state linearizability may be relaxed to input/output linearizability where the relative degree r is smaller than the system dimension n. For the input/output linearizability case, an additional observer is required to estimate the state of the inverse dynamics; please see [2] for a detailed development of this case.

5.2.1 State Estimation via High-Gain Observer

The state estimation-based EMPC developed in this section takes advantage of a high-gain observer [1, 3], which obtains estimates of the output derivatives up to order $n-1$ and consequently, computes estimates of the transformed state z. From the estimated transformed state, the system state may be estimated through the inverse transformation $T^{-1}(\cdot)$. The state estimate is denoted by $\hat{x}$. Proposition 5.1 below defines the high-gain observer equations and establishes precise conditions under which the combination of the high-gain observer and of the controller $k(x)$ together with appropriate saturation functions to eliminate wrong estimates enforce asymptotic stability of the origin in the closed-loop system for sufficiently large observer gain. The proof of the proposition follows from the results in [2, 4].

Proposition 5.1 *Consider the nonlinear system of Eq. 5.5 for which Assumption 5.5 holds. Also, assume that there exists a $k(x)$ for which Eq. 5.2 holds and it enforces local exponential stability of the origin in the closed-loop system. Consider the nonlinear system of Eq. 5.5 under the output feedback controller*

$$u = k(\hat{x}) \tag{5.7}$$

where

$$\hat{x} = T^{-1}(sat(\hat{z})) \tag{5.8}$$

and

$$\dot{\hat{z}} = A\hat{z} + L(y - C\hat{z}) \tag{5.9}$$

with

$$L = \left[\frac{a_1}{\varepsilon}\ \frac{a_2}{\varepsilon^2}\ \cdots\ \frac{a_n}{\varepsilon^n}\right]^T,$$

and the parameters a_i are chosen such that the roots of

$$s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n = 0 \tag{5.10}$$

are in the open left-half of the complex plane. Then given δ, there exists ε^* such that if $\varepsilon \in (0, \varepsilon^*]$, $|\hat{z}(0)| \le z_m$, $x(0) \in \Omega_\delta$ with z_m being the maximum of the vector $\hat{z}$ for $|\hat{z}| \le \beta_z(\delta_z, 0)$ where β_z is a class $\mathcal{KL}$ function and $\delta_z = \max\{|T(x)|, x \in \Omega_\delta\}$; the origin of the closed-loop system is asymptotically stable. This stability property implies that for $\varepsilon \in (0, \varepsilon^*]$ and given some positive constant $e_m > 0$ there exists positive real constant $t_b > 0$ such that if $x(0) \in \Omega_\delta$ and $|\hat{z}(0)| \le z_m$, then $|x(t) - \hat{x}(t)| \le e_m$ for all $t \ge t_b$.

Remark 5.2 In Proposition 5.1, the saturation function, $sat(\cdot)$, is used to eliminate the peaking phenomenon associated with the high-gain observer, see for example [1]. Also, the estimated state $\hat{x}$ is considered to have converged to the actual state x when the estimation error $|x - \hat{x}|$ is less than a given bound e_m. The time needed to converge, is given by t_b which is proportional to the observer gain $1/\varepsilon$. During this transient, the value of the Lyapunov function $V(x)$ may increase.

5.2.2 High-Gain Observer-Based EMPC

In this section, we consider the design of an estimation-based LEMPC for nonlinear systems. We assume that the LEMPC is evaluated at synchronous time instants $\{t_{k\geq 0}\}$ with $t_k = t_0 + k\Delta,\ k = 0, 1, \ldots$ where $t_0 = 0$ is the first time that LEMPC is evaluated and $\Delta > 0$ is the LEMPC sampling time.

5.2.2.1 Implementation Strategy

The high-gain observer of Eq. 5.9 continuously receives output measurements and computes estimated system states. At each sampling time t_k, the LEMPC obtains the estimated system state, which is denoted by $\hat{x}(t_k)$, from the observer. Based on $\hat{x}(t_k)$, the LEMPC uses the system model of Eq. 5.5 to predict the future evolution of the system over a finite prediction horizon while minimizing an economic cost function.

The two-mode operation paradigm presented in Chap. 4 is adopted in the design of the LEMPC. From the initial time t_0 up to a specific time t_s the LEMPC operates in the first operation mode to optimize the economic cost function while maintaining the closed-loop system state in the stability region Ω_ρ. Without loss of generality, t_s is assumed to be a multiple of LEMPC sampling time. In the first operation mode, a subset of the stability region, denoted by Ω_{ρ_e} with $\rho_e < \rho$, is defined in order to account for the high-gain observer effect, i.e., there is a discrepancy between the estimated state and the actual state. If the estimated state is in the region Ω_{ρ_e}, the LEMPC minimizes the cost function while constraining the predicted state trajectory to be within the region Ω_{ρ_e} over the prediction horizon. If the estimated state is in the region $\Omega_\rho \backslash \Omega_{\rho_e}$, the LEMPC computes control actions that optimize the economic cost subject to a condition that ensures that the control actions drive the system state to the region Ω_{ρ_e}. After time t_s, the LEMPC operates in the second operation mode and calculates the inputs in a way that the state of the closed-loop system is driven to a neighborhood of the desired steady-state.

The above described implementation strategy of the LEMPC may be summarized as follows:

Algorithm 5.1 High-gain observer-based LEMPC implementation algorithm

1. Based on the output measurements $y(t)$, the high-gain observer continuously estimates the state $\hat{x}(t)$ (for all $t \geq t_0 = 0$). The LEMPC receives the estimated state at a sampling time t_k from the observer.

2. If $t_k < t_s$, go to Step 3. Else, go to Step 4.
3. If $\hat{x}(t_k) \in \Omega_{\rho_e}$, go to Step 3.1. Else, go to Step 3.2.
   3.1. The controller optimizes the economic cost function while constraining the predicted state trajectory to lie within Ω_{ρ_e}. Go to Step 5.
   3.2. The controller optimizes the economic cost function while ensuring the computed control actions drive the state to the region Ω_{ρ_e}. Go to Step 5.
4. The controller computes control actions that drive the state to a small neighborhood of the origin.
5. Go to Step 1 ($k \leftarrow k + 1$).

5.2.2.2 LEMPC Formulation

The LEMPC is evaluated at each sampling time to obtain the future input trajectories based on estimated state $\hat{x}(t_k)$ provided by the high-gain observer. Specifically, the optimization problem of the LEMPC is as follows:

$$\min_{u \in S(\Delta)} \int_{t_k}^{t_{k+N}} l_e(\tilde{x}(\tau), u(\tau))\, d\tau \tag{5.11a}$$

$$\text{s.t.} \quad \dot{\tilde{x}}(t) = f(\tilde{x}(t)) + g(\tilde{x}(t))u(t) \tag{5.11b}$$

$$\tilde{x}(t_k) = \hat{x}(t_k) \tag{5.11c}$$

$$u(t) \in \mathbb{U},\ \forall\, t \in [t_k, t_{k+N}) \tag{5.11d}$$

$$V(\tilde{x}(t)) \leq \rho_e,\ \forall\, t \in [t_k, t_{k+N}), \quad \text{if } V(\hat{x}(t_k)) \leq \rho_e \text{ and } t_k < t_s \tag{5.11e}$$

$$L_g V(\hat{x}(t_k))u(t_k) \leq L_g V(\hat{x}(t_k))k(\hat{x}(t_k)), \quad \text{if } V(\hat{x}(t_k)) > \rho_e \text{ or } t_k \geq t_s \tag{5.11f}$$

where $\tilde{x}$ is the predicted trajectory of the system with control inputs calculated by the LEMPC. The notation used in the LEMPC of Eq. 5.11 is similar to that of the LEMPC of Eq. 4.3. The constraint of Eq. 5.11b is the system model used to predict the future evolution of the system. The model is initialized with the estimated state $\hat{x}(t_k)$ computed by the high-gain observer. The constraint of Eq. 5.11d accounts for the inputs constraints. The constraint of Eq. 5.11e is associated with the mode 1 operation of the LEMPC, which restricts the predicted system state to be in the set Ω_{ρ_e}, while the constraint of Eq. 5.11f is associated with the mode 2 operation of the LEMPC. The latter constraint restricts the control input for the first sampling period of the prediction horizon so that the amount of reduction of the Lyapunov function value is at least at the same level as that achieved by applying the controller $k(x)$. The constraint of Eq. 5.11f is used when $\hat{x}(t_k) \notin \Omega_{\rho_e}$ or when $t_k \geq t_s$.

The optimal solution to the optimization problem is denoted by $u^*(t|t_k)$ for $t \in [t_k, t_{k+N})$. The control actions computed by the LEMPC that are applied to the system are defined as follows:

$$u(t) = u^*(t|t_k),\ \forall\, t \in [t_k, t_{k+1}) \tag{5.12}$$

which is computed at each sampling time.

5.2.3 Closed-Loop Stability Analysis

In this subsection, the closed-loop stability of the output feedback EMPC is analyzed and a set of sufficient conditions is derived. In order to present the results, we need the following proposition, which states the closed-loop stability properties under the LEMPC with full state feedback. The proposition is a slight variation of Theorem 4.1, and therefore, its proof is omitted.

Proposition 5.2 (Theorem 4.1) *Consider the system of Eq. 5.5 in closed-loop under the LEMPC of Eq. 5.11 with state feedback, i.e.,* $\tilde{x}(t_k) = x(t_k)$*, based on a controller* $k(\cdot)$ *that satisfies the conditions of Eq. 5.2. Let* $\varepsilon_w > 0$, $\Delta > 0$ *and* $\rho > \rho_s > 0$ *satisfy the following constraint:*

$$-\,\alpha_3(\alpha_2^{-1}(\rho_s)) + L_x M \Delta \leq -\varepsilon_w/\Delta. \tag{5.13}$$

If $x(0) \in \Omega_\rho$*, then* $x(t) \in \Omega_\rho$ *for all* $t \geq 0$*. Furthermore, there exists a class* $\mathcal{KL}$ *function* β *and a class* $\mathcal{K}$ *function* γ *such that*

$$|x(t)| \leq \beta(|x(t^*)|, t - t^*) + \gamma(\rho^*) \tag{5.14}$$

with $\rho^* = \max\{V(x(t+\Delta)) : V(x(t)) \leq \rho_s\}$*, for all* $x(t^*) \in B_\delta \subset \Omega_\rho$ *and for all* $t \geq t^* > t_s$ *where* t^* *is such that* $x(t^*) \in B_\delta$*.*

Theorem 5.1 below provides sufficient conditions such that the state estimation-based LEMPC of Eq. 5.11 with the high-gain observer of Eq. 5.9 guarantees that the state of the closed-loop system of Eq. 5.5 is always bounded and is ultimately bounded in a small region containing the origin. To this end, let:

$$e_i = \frac{1}{\varepsilon^{n-i}}(y^{(i-1)} - \hat{z}_i),\ i = 1, \ldots, n, \tag{5.15}$$

$$e^T = [e_1\, e_2\, \cdots\, e_n] \tag{5.16}$$

and

$$A^* = \begin{bmatrix} -a_1 & 1\ 0 \cdots 0\ 0 \\ \vdots & \vdots\ \vdots\ \ddots\ \vdots\ \vdots \\ -a_{n-1} & 0\ 0 \cdots 0\ 1 \\ -a_n & 0\ 0 \cdots 0\ 0 \end{bmatrix},\ b = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{5.17}$$

where $y^{(i-1)}$ is the $(i-1)$-th derivative of the output measurement y and $\hat{z}_i$ is the i-th component of $\hat{z}$.

Theorem 5.1 *Consider the closed-loop system of Eq. 5.5 with the state estimation-based LEMPC of Eq. 5.11 based on a feedback controller $k(\cdot)$ that satisfies the conditions of Eq. 5.2. Let Assumption 5.1, Eqs. 5.13, 5.15–5.17 hold and choose the parameters a_i $(i = 1, \ldots, n)$ such that the roots of Eq. 5.10 are in the open left-half of the complex plane. Then there exist a class $\mathscr{KL}$ function β, a class $\mathscr{K}$ function γ, a pair of positive real numbers (δ_x, d_x), $0 < \rho_e < \rho$, $\varepsilon^* > 0$ and $\Delta^* > 0$ such that if* $\max\{|x(0)|, |e(0)|\} \leq \delta_x$, $\varepsilon \in (0, \varepsilon^*]$, $\Delta \in (0, \Delta^*]$,

$$-\alpha_3(\alpha_1^{-1}(\rho_s)) + (M\Delta + e_m)(L_x + L_u u^{\max}) < 0 \tag{5.18}$$

and

$$\rho_e \leq \rho - \alpha_4(\alpha_1^{-1}(\rho))M \max\{t_b(\varepsilon), \Delta\} \tag{5.19}$$

with t_b defined in Proposition 5.1, then $x(t) \in \Omega_\rho$ for all $t \geq 0$. Furthermore, for all $t \geq t^* > t_s$, the following bound holds:

$$|x(t)| \leq \beta(|x(t^*)|, t - t^*) + \gamma(\rho^*) + d_x \, . \tag{5.20}$$

Proof When the control action applied to the closed-loop system of Eq. 5.5 is obtained from the state estimation-based LEMPC of Eq. 5.11, the closed-loop system takes the following singularly perturbed form:

$$\begin{aligned} \dot{x} &= f(x) + g(x)u^*(\hat{x}) \\ \varepsilon\dot{e} &= A^*e + \varepsilon b L_f^n h(T^{-1}(z)) + \varepsilon b L_g L_f^{n-1} h(T^{-1}(z))u^*(\hat{x}) \end{aligned} \tag{5.21}$$

where the notation $u^*(\hat{x})$ is used to emphasize that the control action computed by the state estimation-based LEMPC is a function of the estimated state.

First, we compute the reduced-order slow and fast closed-loop subsystems related to Eq. 5.21 and prove the closed-loop stability of the slow and fast subsystems. Setting $\varepsilon = 0$ in Eq. 5.21, we obtain the corresponding slow subsystem as follows:

$$\dot{x} = f(x) + g(x)u^*(\hat{x}) \tag{5.22a}$$

$$A^*e = 0 \tag{5.22b}$$

Taking into account the fact that A^* is non-singular and $e = [0 \ 0 \ \cdots \ 0]^T$ is the unique solution of Eq. 5.22b, we may obtain $\hat{z}_i = y^{(i-1)}$, $i = 1, \ldots, n$ and $x(t) = \hat{x}(t)$. This means that the closed-loop slow subsystem is reduced to the one studied in Proposition 5.2 under state feedback. According to Proposition 5.2, if $x(0) \in$

$B_\delta \subset \Omega_\rho$, then $x(t) \in \Omega_\rho$ for all $t \geq 0$ and for all $t \geq t^* > t_s$, the following bound holds:

$$|x(t)| \leq \beta(|x(t^*)|, t - t^*) + \gamma(\rho^*) \tag{5.23}$$

where ρ^* and t^* have been defined in Proposition 5.2.

Introducing the fast time scale $\bar{\tau} = t/\varepsilon$ and setting $\varepsilon = 0$, the closed-loop fast subsystem may be represented as follows:

$$\frac{de}{d\bar{\tau}} = A^* e\ . \tag{5.24}$$

Since A^* is Hurwitz, the closed-loop fast subsystem is also stable. Moreover, there exist $k_e \geq 1$ and $a_e > 0$ such that:

$$|e(\bar{\tau})| \leq k_e |e(0)| e^{-a_e \bar{\tau}} \tag{5.25}$$

for all $\bar{\tau} \geq 0$.

Next, we consider $t \in (0, \max\{\Delta, t_b\}]$ and $t \geq \max\{\Delta, t_b\}$ separately and prove that if conditions stated in Theorem 1 are satisfied, boundedness of the state is ensured. Note that t_b decreases as ε decreases. When $x(0) \in B_{\delta_x} \subset \Omega_{\rho_e} \subset \Omega_\rho$, and $\delta_x < \delta$, considering the closed-loop system state trajectory:

$$\dot{x}(t) = f(x(t)) + g(x(t)) u^*(\hat{x}(0))$$

for $t \in (0, \max\{\Delta, t_b\}]$ and using Eqs. 5.2 and 5.3, we obtain that for all $t \in (0, \max\{\Delta, t_b\}]$:

$$\begin{aligned} V(x(t)) &= V(x(0)) + \int_0^t \dot{V}(x(\tau)) d\tau \\ &= V(x(0)) + \int_0^t \frac{\partial V(x(\tau))}{\partial x} \dot{x}(\tau) d\tau \\ &\leq \rho_e + M \max\{\Delta, t_b(\varepsilon)\} \alpha_4(\alpha_1^{-1}(\rho)) \end{aligned} \tag{5.26}$$

Since t_b decreases as ε decreases, there exist Δ_1 and ε_1 such that if $\Delta \in (0, \Delta_1]$ and $\varepsilon \in (0, \varepsilon_1]$, Eq. 5.19 holds and thus,

$$V(x(t)) < \rho \tag{5.27}$$

for all $t \in (0, \max\{\Delta, t_b\}]$.

For $t \geq \max\{\Delta, t_b\}$, we have that $|x(t) - \hat{x}(t)| \leq e_m$ (this follows from Proposition 1 and e_m decreases as ε decreases), and we may write the time derivative of the Lyapunov function along the closed-loop system state of Eq. 5.5 under the state estimation-based LEMPC of Eqs. 5.11f, 5.9 and 5.11 for all $t \in [t_k, t_{k+1})$ (assuming without loss of generality that $t_k = \max\{\Delta, t_b\}$) as follows

$$\dot{V}(x(t)) = \frac{\partial V(x(t))}{\partial x}(f(x(t)) + g(x(t))u^*(\hat{x}(t_k))) \,. \tag{5.28}$$

Adding and subtracting the term $\partial V(\hat{x}(t_k))/\partial x(f(\hat{x}(t_k)) + g(\hat{x}(t_k))u^*(\hat{x}(t_k)))$ to/from the above inequality and from Eqs. 5.2 and 5.11f, we obtain

$$\begin{aligned}\dot{V}(x(t)) \leq & -\alpha_3(\alpha_1^{-1}(\rho_s)) \\ & + \frac{\partial V(x)}{\partial x}\left(f(x(t)) - f(\hat{x}(t_k)) + (g(x(t)) - g(\hat{x}(t_k))u^*(\hat{x}(t_k)))\right)\end{aligned} \tag{5.29}$$

Using the smoothness properties of V, f, g and Eq. 5.4, we may obtain

$$\dot{V}(x(t)) \leq -\alpha_3(\alpha_1^{-1}(\rho_s)) + (L_x + L_u u^{\max})|x(t) - \hat{x}(t_k)| \tag{5.30}$$

From the triangle inequality, Eq. 5.3, and the fact that the estimation error is bounded by e_m for $t \geq \max\{\Delta, t_b\}$,

$$|x(t) - \hat{x}(t_k)| \leq |x(t) - x(t_k)| + |x(t_k) - \hat{x}(t_k)| \leq M\Delta + e_m$$

for $x(t), x(t_k), \hat{x}(t_k) \in \Omega_\rho$ where $|x(t_k) - \hat{x}(t_k)| \leq e_m$. Thus,

$$\dot{V}(x(t)) \leq -\alpha_3(\alpha_1^{-1}(\rho_s)) + (L_x + L_u u^{\max})(M\Delta + e_m) \tag{5.31}$$

Picking ε_2 and Δ_2 such that for all $\varepsilon \in (0, \varepsilon_2]$ and for all $\Delta \in (0, \Delta_2]$, Eq. 5.18 is satisfied, the closed-loop system state $x(t)$ is bounded in Ω_ρ, for all $t \geq \max\{\Delta, t_b\}$. Finally, using similar arguments to the proof of Theorem 1 in [5], we have that there exist class $\mathcal{KL}$ function β, positive real numbers (δ_x, d_x) (note that the existence of $\delta_x < \delta$ such that $|x(0)| \leq \delta_x$ follows from the smoothness of V), and $0 < \varepsilon^* < \min\{\varepsilon_1, \varepsilon_2\}$ and $0 < \Delta^* < \min\{\Delta_1, \Delta_2\}$ such that if $\max\{|x(0)|, |e(0)|\} \leq \delta_x$, $\varepsilon \in (0, \varepsilon^*]$ and $\Delta \in (0, \Delta^*]$, then, the bound of Eq. 5.20 holds for all $t \geq 0$.

Remark 5.3 Under the state feedback LEMPC, the closed-loop system state is always bounded in Ω_ρ for both mode 1 and mode 2 operation; however, for mode 2 operation, after time t^* the closed-loop system state enters the ball B_δ, and the closed-loop system state may be bounded by Eq. 5.23. On the other hand, in state estimation-based LEMPC, the closed-loop system state is always bounded in Ω_ρ, if the initial system state belongs in $B_{\delta_x} \subset \Omega_{\rho_e} \subset \Omega_\rho$.

Remark 5.4 The major motivation for taking advantage of the nonlinear controller $k(x)$ arises from the need for formulating an a priori feasible economic MPC problem for a well-defined set of initial conditions. The control action of $k(x)$ is always a feasible candidate for the LEMPC design (even though the LEMPC via optimization is free to choose a different control action) and the LEMPC may take advantage of $k(x)$ to characterize its own corresponding stability region. In addition, the closed-loop system state is always bounded in the invariant stability region of $k(x)$.

5.2.4 Application to a Chemical Process Example

Consider a well-mixed, non-isothermal continuous stirred tank reactor (CSTR) where an irreversible, second-order, endothermic reaction $A \rightarrow B$ takes place, where A is the reactant and B is the desired product. The feedstock of the reactor consists of the reactant A in an inert solvent with flow rate F, temperature T_0 and molar concentration C_{A0}. Due to the non-isothermal nature of the reactor, a jacket is used to provide heat to the reactor. The dynamic equations describing the behavior of the reactor, obtained through material and energy balances under standard modeling assumptions, are given below:

$$\frac{dC_A}{dt} = \frac{F}{V_L}(C_{A0} - C_A) - k_0 e^{-E/RT} C_A^2 \tag{5.32a}$$

$$\frac{dT}{dt} = \frac{F}{V_L}(T_0 - T) + \frac{-\Delta H}{\rho_L C_p} k_0 e^{-E/RT} C_A^2 + \frac{Q_s}{\rho_L C_p V_L} \tag{5.32b}$$

where C_A denotes the concentration of the reactant A, T denotes the temperature of the reactor, Q_s denotes the steady-state rate of heat supply to the reactor, V_L represents the volume of the reactor, ΔH, k_0, and E denote the enthalpy, pre-exponential constant and activation energy of the reaction, respectively, and C_p and ρ_L denote the heat capacity and the density of the fluid in the reactor, respectively. The values of the process parameters used in the simulations are given in Table 5.1. The process model of Eq. 5.32 is numerically simulated using an explicit Euler integration method with integration step $h_c = 1.0 \times 10^{-3}$ h.

The process model has one stable steady-state in the operating range of interest. The control objective is to economically optimize the process in a region around the stable steady-state (C_{As}, T_s) to maximize the average production rate of B through manipulation of the concentration of A in the inlet to the reactor, C_{A0}. The steady-state C_{A0} value associated with the steady-state point is denoted by C_{A0s}. The process model of Eq. 5.32 belongs to the following class of nonlinear systems:

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t)$$

Table 5.1 CSTR model parameter values

$T_0 = 300$	K	$F = 5.0$	$\text{m}^3\ \text{h}^{-1}$
$V_L = 1.0$	m^3	$E = 5.0 \times 10^3$	kJ kmol^{-1}
$k_0 = 13.93$	$\text{m}^3\ \text{kmol}^{-1}\ \text{h}^{-1}$	$\Delta H = 1.15 \times 10^4$	kJ kmol^{-1}
$C_p = 0.231$	$\text{kJ kg}^{-1}\text{K}^{-1}$	$R = 8.314$	$\text{kJ kmol}^{-1}\text{K}^{-1}$
$\rho_L = 1000$	kg m^{-3}	$C_{As} = 2.0$	kmol m^{-3}
$T_s = 350$	K	$C_{A0s} = 4.0$	kmol m^{-3}
$Q_s = 1.73 \times 10^5$	kJ h^{-1}		

where $x^T = [x_1\ x_2] = [C_A - C_{As}\ T - T_s]$ is the state, $u = C_{A0} - C_{A0s}$ is the input, $f = [f_1\ f_2]^T$ and $g_i = [g_{i1}\ g_{i2}]^T$ $(i = 1, 2)$ are vector functions. The input is subject to constraint as follows: $|u| \leq 3.5\,\text{kmol m}^{-3}$. There is an economic measure considered in this example as follows:

$$\frac{1}{t_f}\int_0^{t_f} k_0 e^{-E/RT(\tau)} C_A^2(\tau)\, d\tau \tag{5.33}$$

where t_f is the time duration of the reactor operation. The economic objective function of Eq. 5.33 describes the average production rate over the entire process operation. We also consider that there is a limitation on the amount of reactant material which may be used over a specific period $t_p = 1.0\,\text{h}$. Specifically, $u = C_{A0} - C_{A0s}$ should satisfy the following constraint:

$$\frac{1}{t_p}\int_0^{t_p} u(\tau)\, d\tau = 1.0\,\text{kmol m}^{-3}. \tag{5.34}$$

It should be emphasized that due to the second-order dependence of the reaction rate on the reactant concentration, the production rate may be improved through switching between the upper and lower bounds of the manipulated input, as opposed to steady-state operation via uniform in time distribution of the reactant in the feed; refer to the discussion contained in Sect. 1.3.2 for further explanation of this point.

In this section, we will design a state estimation-based LEMPC to manipulate the C_{A0} subject to the material constraint. In the first set of simulations, we assume that state feedback information is available at synchronous time instants while in the second set of simulations, we take advantage of a high-gain observer to estimate the reactant concentration from temperature measurements.

In terms of the Lyapunov-based controller, we use a proportional controller (P-controller) of the form $u = -\gamma_1 x_1 - \gamma_2 x_2$ subject to input constraints where $\gamma_1 = 1.6$ and $\gamma_2 = 0.01$ and a quadratic Lyapunov function $V(x) = x^T P x$ where

$$P = \text{diag}([110.11, 0.12])\ ,$$

and $\rho = 430$. It should be emphasized that Ω_ρ has been estimated through evaluation of $\dot{V}$ when we apply the proportional controller. We assume that the full system state $x = [x_1\ x_2]^T$ is measured and sent to the LEMPC at synchronous time instants $t_k = k\Delta$, $k = 0, 1, \ldots$, with $\Delta = 0.01\,\text{h} = 36\,\text{s}$ in the first set of simulations. For the output feedback LEMPC (second set of simulations), only temperature (x_2) is measured and a high-gain observer is utilized to estimate the reactant concentration from temperature measurements.

Considering the material constraint which needs to be satisfied through one period of process operation, a decreasing LEMPC horizon sequence $N_0, \ldots, N_{99}$ where $N_i = 100 - i$ and $i = 0, \ldots, 99$ is utilized at the different sampling times. At each sampling time t_k, the LEMPC with horizon N_k takes into account the leftover amount of reactant material and adjusts its horizon to predict future system state up to time

$t_p = 1.0\,\text{h}$ to maximize the average production rate. Since the LEMPC is evaluated at discrete-time instants during the closed-loop simulation, the material constraint is enforced as follows:

$$\sum_{i=0}^{M-1} u(t_i) = \frac{t_p}{\Delta} 1.0\,\text{kmol}\,\text{m}^{-3} \tag{5.35}$$

where $M = 100$. As LEMPC proceeds at different sampling times, the constraint is adjusted according to the optimal manipulated input at previous sampling times.

The state feedback LEMPC formulation for the chemical process example in question has the following form:

$$\max_{u \in S(\Delta)} \frac{1}{N_k \Delta} \int_{t_k}^{t_{k+N_k}} [k_0 e^{-E/R(\tilde{x}_2(\tau)+T_s)} (\tilde{x}_1(\tau) + C_{As})^2]\, d\tau \tag{5.36a}$$

$$\text{s.t.} \quad \dot{\tilde{x}}(t) = f(\tilde{x}(t)) + g(\tilde{x}(t))u(t),\ \forall\, t \in [t_k, t_{k+N_k}) \tag{5.36b}$$

$$\tilde{x}(t_k) = x(t_k) \tag{5.36c}$$

$$u(t) \in \mathbb{U},\ \forall\, t \in [t_k, t_{k+N_k}) \tag{5.36d}$$

$$\sum_{i=k}^{k+N_k-1} u(t_i|t_k) = \zeta_k \tag{5.36e}$$

$$V(\tilde{x}(t)) \le \rho,\ \forall\, t \in [t_k, t_{k+N_k}) \tag{5.36f}$$

where $x(t_k)$ is the process state measurement at sampling time t_k and the predicted system state along the LEMPC horizon is restricted to lie within the invariant set Ω_ρ through enforcement of the constraint of Eq. 5.36f subject to the manipulated input constraint of Eq. 5.36d. The constraint of Eq. 5.36e implies that the optimal values of u along the prediction horizon should be chosen to satisfy the material constraint where the explicit expression of ζ_k may be computed based on Eq. 5.35 and the optimal manipulated input values prior to sampling time t_k. In other words, this constraint indicates the amount of the remaining reactant material at each sampling time. Thus, it ensures that the material constraint is enforced through one period of process operation.

In terms of the initial guess for solving the optimization problem of Eq. 5.36, at the first sampling time we take advantage of the Lyapunov-based controller while for the subsequent sampling times, a shifted version of the optimal solution of the previous sampling time is utilized. The simulations were carried out using Java programming language in a Pentium 3.20 GHz computer and the optimization problems were solved using the open source interior point optimizer Ipopt [6]. The purpose of the following set of simulations is to demonstrate that: (I) the LEMPC design subject to state and output feedback restricts the system state in an invariant set; (II) the LEMPC design maximizes the economic measure of Eq. 5.36a; and (III) the LEMPC design achieves a higher objective function value compared to steady-state operation with equal distribution in time of the reactant material. We have also performed simulations for the case that the constraint of Eq. 5.36f is not included in the LEMPC design of

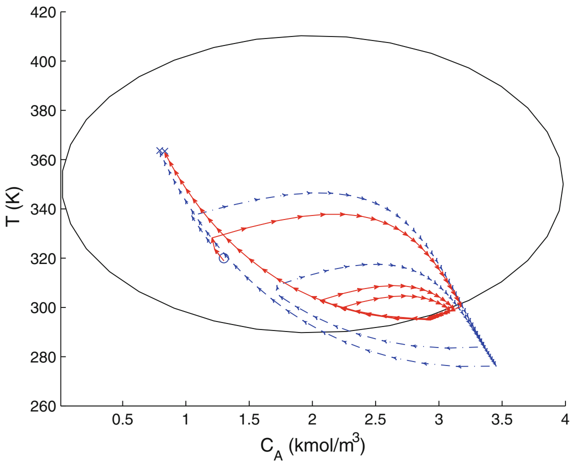

Fig. 5.1 The stability region Ω_ρ and the state trajectories of the process under the LEMPC design of Eq. 5.36 with state feedback and initial state $(C_A(0), T(0)) = (1.3\,\text{kmol m}^{-3}, 320\,\text{K})$ for one period of operation with (*solid line*) and without (*dash-dotted line*) the constraint of Eq. 5.36f. The symbols ∘ and × denote the initial ($t = 0.0$ h) and final ($t = 1.0$ h) state of these closed-loop system trajectories, respectively

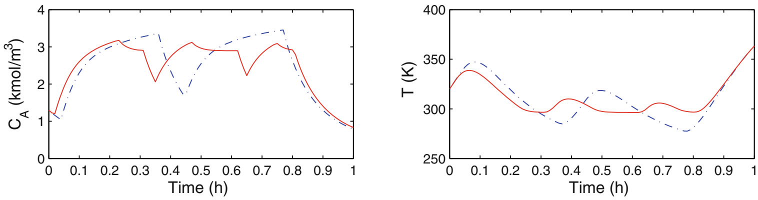

Fig. 5.2 State trajectories of the process under the LEMPC design of Eq. 5.36 with state feedback and initial state $(C_A(0), T(0)) = (1.3\,\text{kmol m}^{-2}, 320\,\text{K})$ for one period of operation with (*solid line*) and without (*dash-dotted line*) the constraint of Eq. 5.36f

Eq. 5.36. In this case, the process state is not constrained to be in a specific invariant set.

In the first set of simulations, we take the CSTR operation time $t_f = t_p = 1.0$ h. Figures 5.1, 5.2 and 5.3 illustrate the process state profile in state space (temperature T versus concentration C_A) considering the stability region Ω_ρ, the time evolution of process state and the manipulated input profile for the LEMPC formulation of Eq. 5.36 with and without the state constraint of Eq. 5.36f, respectively. In both cases, the initial process state is $(1.3\,\text{kmol m}^{-3}, 320\,\text{K})$. For both cases, the material constraint is satisfied while in the unconstrained state case, there is more freedom to compute the optimal input trajectory to maximize the average production rate. It needs to be emphasized that the process state trajectory under the LEMPC design of Eq. 5.36 subject to the constraint of Eq. 5.36f never leaves the invariant level set Ω_ρ when this constraint is enforced.

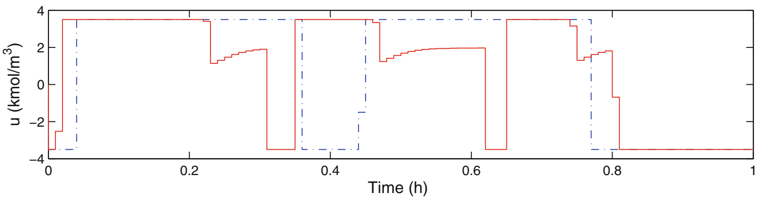

Fig. 5.3 Manipulated input trajectory under the LEMPC design of Eq. 5.36 with state feedback and initial state $(C_A(0), T(0)) = (1.3\,\text{kmol m}^{-3}, 320\,\text{K})$ for one period of operation with (*solid line*) and without (*dash-dotted line*) the constraint of Eq. 5.36f

We have also compared the time-varying operation through LEMPC of Eq. 5.36 to steady-state operation where the reactant material is uniformly distributed in the feed to the reactor over the process operation time of 1 h from a closed-loop performance point of view. To carry out this comparison, we have computed the total cost of each operating scenario based on an index of the following form:

$$J = \frac{1}{t_M} \sum_{i=0}^{M-1} [k_0 e^{-\frac{E}{RT(t_i)}} C_A^2(t_i)]$$

where $t_0 = 0.0\,\text{h}$, $t_M = 1.0\,\text{h}$ and $M = 100$. To be consistent in comparison, both of the simulations have been initialized from the steady-state point $(2.0\,\text{kmol m}^{-3}, 350\,\text{K})$. We find that through time-varying LEMPC operation, there is approximately 7 % improvement with respect to steady-state operation. Specifically, in the case of LEMPC operation with $\rho = 430$ the cost is 13.48, in the case of LEMPC operation with $\rho = \infty$ (LEMPC of Eq. 5.36 without the state constraint of Eq. 5.36f) the cost is 13.55 and in the case of steady-state operation the cost is 12.66.

We have also performed closed-loop simulation with the state estimation-based LEMPC (again, $t_f = t_p = 1.0\,\text{h}$). For this set of simulation, the high-gain observer parameters are $\varepsilon = 0.01$, $a_1 = a_2 = 1$, $\rho_e = 400$ and $z_m = 1685$; the high-gain observer is of the form of Eq. 5.9 with $n = 2$. In this case, the LEMPC formulation at each sampling time is initialized by the estimated system state $\hat{x}(t_k)$ while the output (temperature) measurement is continuously available to the high-gain observer. To ensure that the actual system state is restricted in Ω_ρ, we set $\rho_e = 400$. Figures 5.4, 5.5 and 5.6 illustrate the process state profile in state-space (temperature T versus concentration C_A) considering the stability region Ω_ρ, the time evolution of process states and the manipulated input profile for the LEMPC formulation of Eq. 5.36 using high-gain observer and with the state constraint of Eq. 5.36f, respectively. Similar to the state feedback case, the initial process state is $(1.3\,\text{kmol m}^{-3}, 320\,\text{K})$. Through LEMPC implementation, the material constraint is satisfied while the closed-loop system state is restricted inside the stability region Ω_ρ. The cost is 12.98 which is greater than the one for steady-state operation (12.66).

Also, we performed a set of simulations to compare LEMPC with the Lyapunov-based controller from an economic closed-loop performance point of view for opera-

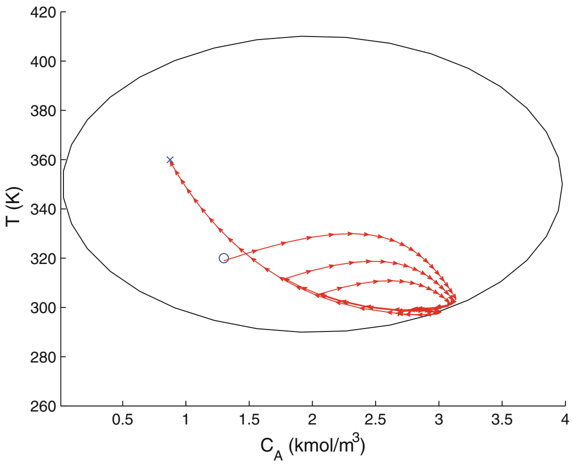

Fig. 5.4 The stability region Ω_ρ and the state trajectories of the process under the state under the state estimation-based LEMPC and initial state $(C_A(0), T(0)) = (1.3\,\text{kmol m}^{-3}, 320\,\text{K})$ for one period of operation subject to the constraint of Eq. 5.36f. The symbols $\circ$ and $\times$ denote the initial ($t = 0.0$ h) and final ($t = 1.0$ h) state of this closed-loop system trajectories, respectively

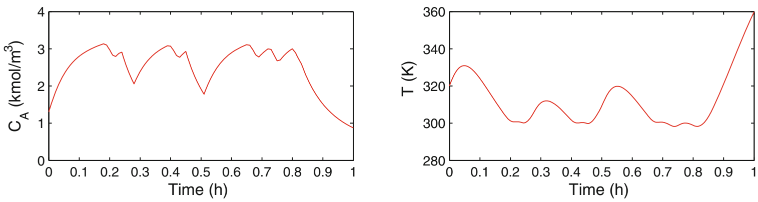

Fig. 5.5 State trajectories of the process under the state estimation-based LEMPC and initial state $(C_A(0), T(0)) = (1.3\,\text{kmol m}^{-3}, 320\,\text{K})$ for one period of operation subject to the constraint of Eq. 5.36f

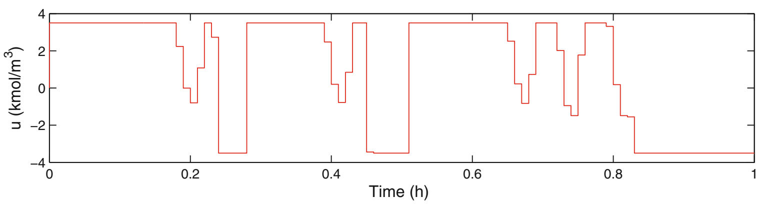

Fig. 5.6 Manipulated input trajectory under the state estimation-based LEMPC and initial state $(C_A(0), T(0)) = (1.3\,\text{kmol m}^{-3}, 320\,\text{K})$ for one period of operation subject to the constraint of Eq. 5.36f

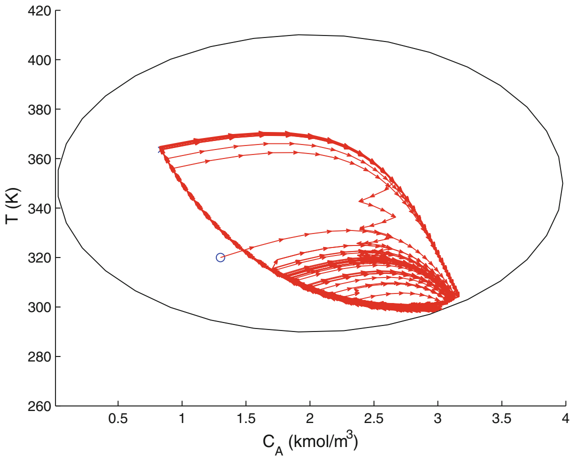

Fig. 5.7 The stability region Ω_ρ and the state trajectories of the process under the state estimation-based LEMPC and initial state $(C_A(0), T(0)) = (1.3\,\text{kmol}\,\text{m}^{-3}, 320\,\text{K})$ for 10 h operation in mode 1, followed by 10 h of operation in mode 2 and finally, 10 h of operation in mode 1. The symbols $\circ$ and $\times$ denote the initial ($t = 0.0$ h) and final ($t = 30.0$ h) state of this closed-loop system trajectories, respectively

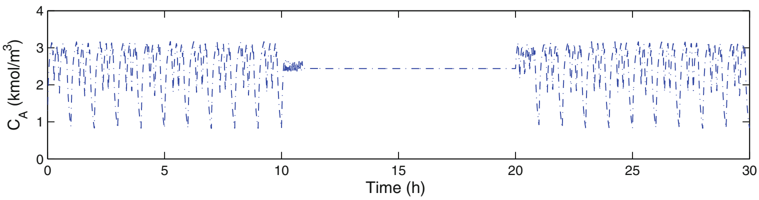

Fig. 5.8 Reactant concentration trajectory of the process under the state estimation-based LEMPC and initial state $(C_A(0), T(0)) = (1.3\,\text{kmol}\,\text{m}^{-3}, 320\,\text{K})$ for 10 h operation in mode 1, followed by 10 h of operation in mode 2 and finally, 10 h of operation in mode 1

tion over two consecutive 1 h periods, i.e., $t_f = 2.0$ h and $t_p = 1.0$ h. To be consistent in this comparison in the sense that both the LEMPC and the Lyapunov-based controller use the same, available amount of reactant material, we start the simulation in both cases from the same initial condition $(2.44\,\text{kmol}\,\text{m}^{-3}, 321.96\,\text{K})$, which corresponds to the steady-state of the process when the available reactant material is uniformly distributed over each period of operation. The objective of the Lyapunov-based controller is to keep the system state at this steady-state, while the output feedback LEMPC leads to time-varying operation that optimizes directly the economic cost. The corresponding economic costs for this 2-h operation are 26.50 for the LEMPC and 25.61 for the Lyapunov-based controller.

Furthermore, to demonstrate long-term reactor operation, i.e., $t_f = 30.0$ h and $t_p = 1.0$ h, we operate the process in a time-varying fashion to optimize the economic cost in mode 1 for the first 10 h, then switch to mode 2 to drive the closed-loop state

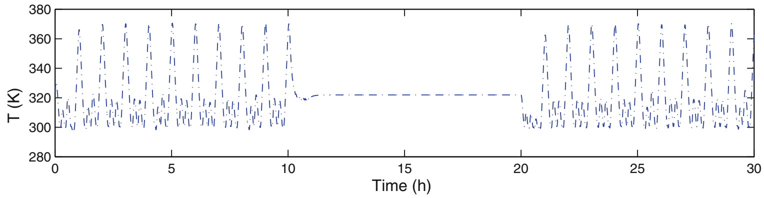

Fig. 5.9 Temperature trajectory of the process under the state estimation-based LEMPC and initial state $(C_A(0), T(0)) = (1.3\,\text{kmol m}^{-3}, 320\,\text{K})$ for 10 h operation in mode 1, followed by 10 h of operation in mode 2 and finally, 10 h of operation in mode 1

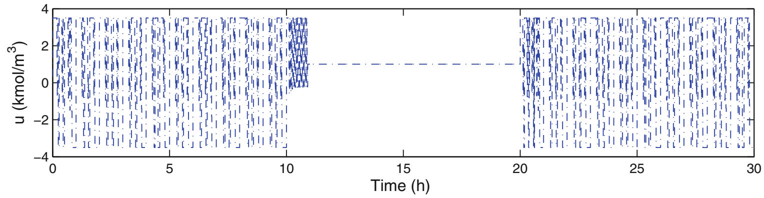

Fig. 5.10 Manipulated input trajectory under the state estimation-based LEMPC and initial state $(C_A(0), T(0)) = (1.3\,\text{kmol m}^{-3}, 320\,\text{K})$ for 10 h operation in mode 1, followed by 10 h of operation in mode 2 and finally, 10 h of operation in mode 1

to the steady-state corresponding to $u = 1.0$, i.e., equal distribution with time of the reactant material, for the next 10 h, and finally, operate the process in mode 1 for the last 10 h. Figures 5.7, 5.8, 5.9 and 5.10 display the results for this case, where the closed-loop system successfully alternates between the two different types (time-varying versus steady-state) of operation.

Finally, we performed a set of simulations to evaluate the effect of bounded measurement noise. Figures 5.11, 5.12 and 5.13 display the closed-loop system state and manipulated input of the state-estimation-based LEMPC subject to bounded output (temperature) measurement noise whose absolute value is bounded by 1.0 K. As it may be seen in Figs. 5.11, 5.12 and 5.13, the controller may tolerate the effect of measurement noise; in this case, Ω_{ρ_e} was reduced to 370 to improve the robustness margin of the controller to measurement noise. Economic closed-loop performance in this case is 12.95.

5.3 RMHE-Based EMPC Scheme

In the previous section, a high-gain observer is used in the design of the output feedback EMPC without explicitly considering process and measurement noise. In order to improve the robustness of the observer to plant-model mismatch and uncertainties while reducing its sensitivity to measurement noise significantly, a robust moving horizon estimation (RMHE) based output feedback LEMPC design is presented in this section. We consider systems that may be described by Eq. 5.1.

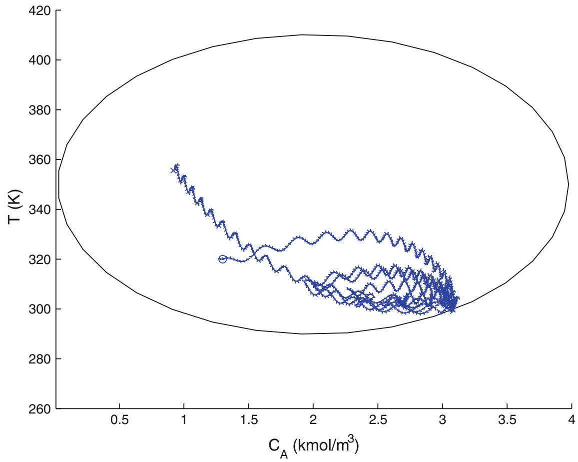

Fig. 5.11 The stability region Ω_ρ and the state trajectories of the process under the state estimation-based LEMPC and initial state $(C_A(0), T(0)) = (1.3\,\text{kmol m}^{-3}, 320\,\text{K})$ for one period of operation subject to the constraint of Eq. 5.36f and bounded measurement noise. The symbols $\circ$ and $\times$ denote the initial ($t = 0.0$ h) and final ($t = 1.0$ h) state of this closed-loop system trajectories, respectively

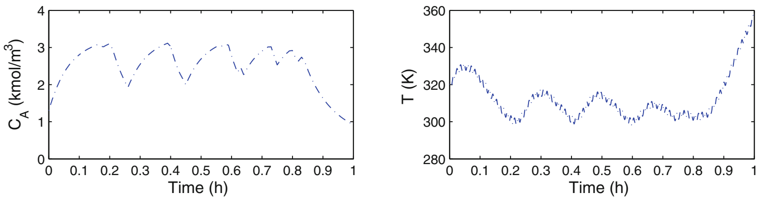

Fig. 5.12 State trajectories of the process under the state estimation-based LEMPC and initial state $(C_A(0), T(0)) = (1.3\,\text{kmol m}^{-3}, 320\,\text{K})$ for one period of operation subject to the constraint of Eq. 5.36f and bounded measurement noise

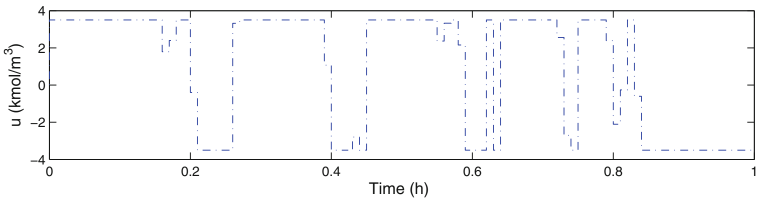

Fig. 5.13 Manipulated input trajectory under the state estimation-based LEMPC and initial state $(C_A(0), T(0)) = (1.3\,\text{kmol m}^{-3}, 320\,\text{K})$ for one period of operation subject to the constraint of Eq. 5.36f and bounded measurement noise

5.3.1 Observability Assumptions

It is assumed that there exists a deterministic observer that takes the following general form:

$$\dot{z} = F(\varepsilon, z, y) \tag{5.37}$$

where z is the observer state, y is the output measurement vector and ε is a positive parameter. This observer together with the state feedback controller $u = k(x)$ of Sect. 5.1.2 form an output feedback controller: $\dot{z} = F(\varepsilon, z, y)$, $u = k(z)$ which satisfies the following assumptions:

(1) there exist positive constants θ_w^*, θ_v^* such that for each pair $\{\theta_w, \theta_v\}$ with $\theta_w \leq \theta_w^*$, $\theta_v \leq \theta_v^*$, there exist $0 < \rho_1 < \rho$, $e_{m0} > 0$, $\varepsilon_L^* > 0$, $\varepsilon_U^* > 0$ such that if $x(t_0) \in \Omega_{\rho_1}$, $|z(t_0) - x(t_0)| \leq e_{m0}$ and $\varepsilon \in (\varepsilon_L^*, \varepsilon_U^*)$, the trajectories of the closed-loop system are bounded in Ω_ρ for all $t \geq t_0$;
(2) and there exists $e_m^* > 0$ such that for each $e_m \geq e_m^*$, there exists t_b such that $|z(t) - x(t)| \leq e_m$ for all $t \geq t_b(\varepsilon)$.

Note that a type of observer that satisfies the above assumptions is a high-gain observer like that presented in Sect. 5.2; see, also, [7] for results on high-gain observers subject to measurement noise. From an estimate error convergence speed point of view, it is desirable to pick the observer parameter ε as small as possible; however, when the parameter ε is too small, i.e., the observer gain is too large, it will make the observer very sensitive to measurement noise. In the observer assumptions, a key idea is to pick the gain ε in a way that balances the estimate error convergence speed to zero and the effect of the noise. In the remainder of this section, the estimate given by the observer F will be denoted as z.

Remark 5.5 It is important to point out the difference between the positive constants θ_w^* and θ_v^* and the bounds θ_w and θ_v. Specifically, the positive constants θ_w^* and θ_v^* correspond to theoretical bounds on the noise such that the closed-loop system under the output feedback controller: $\dot{z} = F(\varepsilon, z, y)$, $u = k(z)$ is maintained in Ω_ρ. The constants θ_w^* and θ_v^* depend on the stability properties of a given system under the output feedback controller. On the other hand, the bounds θ_w and θ_v correspond to the actual bound on the process and measurement noise for a given (open-loop) system.

5.3.2 Robust MHE

The idea of RMHE was initially developed in [8] integrating deterministic observer techniques and optimization-based estimation techniques in a unified framework. In the RMHE, an auxiliary deterministic nonlinear observer that is able to asymptotically track the nominal system state is used to calculate a confidence region. In the calculation of the confidence region, bounded process and measurement noise are taken into account. The RMHE problem is constrained to ensure that it computes a state estimate that is within the confidence region. By this approach, the RMHE gives bounded estimation error in the case of bounded process noise. Moreover, the RMHE could be used together with different arrival cost approximation techniques and was shown to compensate for the error in the arrival cost approximation [8]. The RMHE has been applied to the design of a robust output feedback Lyapunov-based MPC [9] and has also been extended to estimate the state of large-scale systems in a distributed manner [10]. The RMHE scheme in [8] will be adopted in this section to take advantage of the tunable convergence speed of the observer presented in the previous subsection while significantly reducing its sensitivity to measurement noise.

The RMHE is evaluated at discrete time instants denoted by the time sequence $\{t_k\}_{k\geq 0}$ with $t_k = t_0 + k\Delta$, $k = 0, 1, \ldots$ where t_0 is the initial time. In the RMHE scheme, the deterministic observer, which is denoted by F, will be used to calculate a reference state estimate at each sampling time from continuous output measurements. Based on the reference state estimate, the RMHE determines a confidence region for the actual system state. The RMHE computes a state estimate within the confidence region based on a sequence of previous output measurements, a system model, and bounds information of the process and measurement noise. The RMHE scheme at time instant t_k is formulated as follows:

$$\min_{\tilde{x}(t_{k-N_e}),\ldots,\tilde{x}(t_k)} \sum_{i=k-N_e}^{k-1} |w(t_i)|^2_{Q_m^{-1}} + \sum_{i=k-N_e}^{k} |v(t_i)|^2_{R_m^{-1}} + \hat{V}_T(t_{k-N_e}) \tag{5.38a}$$

$$\text{s.t.} \quad \dot{\tilde{x}}(t) = f(\tilde{x}(t)) + g(\tilde{x}(t))u(t) + l(\tilde{x}(t))w(t_i),\ t \in [t_i, t_{i+1}], \tag{5.38b}$$

$$v(t_i) = y(t_i) - h(\tilde{x}(t_i)),\ i = k - N_e, k - N_e + 1, \ldots, k \tag{5.38c}$$

$$w(t_i) \in \mathbb{W},\ i = k - N_e, k - N_e + 1, \ldots, k - 1 \tag{5.38d}$$

$$v(t_i) \in \mathbb{V},\ i = k - N_e, k - N_e + 1, \ldots, k \tag{5.38e}$$

$$\tilde{x}(t) \in \Omega_\rho,\ \forall\, t \in [t_{k-N_e}, t_k] \tag{5.38f}$$

$$|\tilde{x}(t_k) - z(t_k)| \leq \kappa |y(t_k) - h(z(t_k))| \tag{5.38g}$$

where N_e is the estimation horizon, Q_m and R_m are the estimated covariance matrices of w and v respectively, $\hat{V}_T(t_{k-N_e})$ denotes the arrival cost which summarizes past information up to t_{k-N_e}, $\tilde{x}$ is the predicted state x in the above optimization problem, $y(t_i)$ is the output measurement at t_i, $z(t_k)$ is an estimate given by the observer F based on continuous measurements of y, and κ is a positive constant which is a design parameter.

Once the optimization problem of Eq. 5.38 is solved, an optimal trajectory of the system state, $\tilde{x}^*(t_{k-N_e}), \ldots, \tilde{x}^*(t_k)$, is obtained. The optimal estimate of the current system state is denoted:

$$\hat{x}^*(t_k) = \tilde{x}^*(t_k). \tag{5.39}$$

Note that in the optimization problem of Eq. 5.38, w and v are assumed to be piecewise constant variables with sampling time Δ to ensure that Eq. 5.38 is a finite dimensional optimization problem.

In the optimization problem of Eq. 5.38, $z(t_k)$ is a reference estimate calculated by the observer F. Based on the reference estimate and the current output measurement, $y(t_k)$, a confidence region that contains the actual system state is constructed, i.e., $\kappa|y(t_k) - h(z(t_k))|$. The estimate of the current state provided by the RMHE is only allowed to be optimized within the confidence region. This approach ensures that the RMHE inherits the robustness of the observer F and gives estimates with bounded errors.

Remark 5.6 In order to account for the effect of historical data outside the estimation window, an arrival cost which summarizes the information of those data is included in the cost function of an MHE optimization problem. The arrival cost plays an important role in the performance and stability of an MHE scheme. Different methods have been developed to approximate the arrival cost including Kalman filtering and smoothing techniques for linear systems [11], extended Kalman filtering for nonlinear systems [12], and particle filters for constrained systems [13].

5.3.3 RMHE-Based EMPC

Without loss of generality, it is assumed that the LEMPC is evaluated at time instants $\{t_k\}_{k\geq 0}$ with sampling time Δ as used in the RMHE. In the LEMPC design, we will take advantage of both the fast convergence rate of the observer F and the robustness of the RMHE to measurement noise.

5.3.3.1 Implementation Strategy

In the approach, the observer F is initially applied for a short period to drive the state estimate from the observer to a small neighborhood of the actual system state. Once the estimate has converged to a small neighborhood of the actual system state, the RMHE takes over the estimation task and provides smoother and optimal state estimates to the LEMPC. Without loss of generality, we assume that t_b is a multiple integer of the sampling time Δ in the sense that $t_b = b\Delta$ where b is a strictly positive integer. In the first b sampling periods, the observer F is applied with continuously output measurements, i.e., the observer is continuously evaluated and provides state estimates to the LEMPC at every sampling time. Starting from t_b, the RMHE is activated and provides an optimal estimates of the system state to the LEMPC at every subsequent sampling time. The LEMPC evaluates its optimal input trajectory based on either the estimates provided by the observer F or the estimates from the RMHE.

The two-mode operation scheme is adopted in the LEMPC design. From the initial time t_0 up to a time t_s, the LEMPC operates in the first operation mode to minimize the economic cost function while maintaining the closed-loop system state in the stability region Ω_ρ. In this operation mode, in order to account for the uncertainties in state estimates and process noise, a region Ω_{ρ_e} with $\rho_e < \rho$ is used. If the estimated state is in the region Ω_{ρ_e}, the LEMPC optimizes the cost function while constraining the predicted state trajectory be within the region Ω_{ρ_e}; if the estimated state is in the region $\Omega_\rho \backslash \Omega_{\rho_e}$, the LEMPC computes control actions such that the state is forced to the region Ω_{ρ_e}. After time t_s, the LEMPC operates in the second operation mode and calculates the inputs in a way that the state of the closed-loop system is driven to a neighborhood of the desired steady-state. The implementation strategy of the output feedback LEMPC described above may be summarized as follows:

Algorithm 5.2 RMHE-based LEMPC implementation algorithm

1. Initialize the observer F with $z(t_0)$ and continuously execute the observer F based on the output measurements y.
2. At a sampling time t_k, if $t_k < t_b$, go to Step 2.1; otherwise, go to Step 2.2.
   - 2.1. The LEMPC gets a sample of the estimated system state $z(t_k)$ at t_k from the observer F, and go to Step 3.
   - 2.2. Based on the estimate $z(t_k)$ provided by the observer F and output measurements at the current and previous N_e sampling instants, i.e., $y(t_i)$ with $i = k - N_e, \dots, k$, the RMHE calculates the optimal state estimate $\hat{x}^*(t_k)$. The estimate $\hat{x}^*(t_k)$ is sent to the LEMPC. Go to Step 3.
3. If $t_k < t_s$ and if $z(t_k) \in \Omega_{\rho_e}$ (or if $\hat{x}^*(t_k) \in \Omega_{\rho_e}$), go to Step 3.1. Otherwise, go to Step 3.2.
   - 3.1. Based on $z(t_k)$ or $\hat{x}^*(t_k)$, the LEMPC calculates its input trajectory to minimize the economic cost function while ensuring that the predicted state trajectory over the prediction horizon lies within Ω_{ρ_e}. The first value of the optimal input trajectory is applied to the system. Go to Step 4.
   - 3.2. Based on $z(t_k)$ or $\hat{x}^*(t_k)$, the LEMPC calculates its input trajectory to drive the system state towards the origin. The first value of the input trajectory is applied to the system. Go to Step 4.
4. Go to Step 2 ($k \leftarrow k + 1$).

In the remainder, we will use $\hat{x}$ to denote the state estimate used in the LEMPC. Specifically, $\hat{x}$ at time t_k is defined as follows:

$$\hat{x}(t_k) = \begin{cases} z(t_k), & \text{if } t_k < t_b \\ \hat{x}^*(t_k), & \text{if } t_k \geq t_b \end{cases} \tag{5.40}$$

Remark 5.7 In the implementation Algorithm 5.2 as well as in the RMHE design of Eq. 5.38, the observer F provides state estimate to the RMHE at every sampling time and is independently evaluated from the RMHE. To improve the quality of the estimates provided by the observer F, the state of the observer F may be set to the estimate of the RMHE at every sampling time since the estimates obtained from the RMHE are expected to be more accurate. That is, at Step 2.2, the estimate $\hat{x}^*(t_k)$ is also sent to the observer F and the observer F resets its state to $z(t_k) = \hat{x}^*(t_k)$. The state estimate $z(t_{k+1})$ of the observer F at the next sampling time is computed with continuous output measurements received over the sampling period ($t \in [t_k, t_{k+1}]$) initialized with $z(t_k) = \hat{x}^*(t_k)$.

5.3.3.2 LEMPC Design

The LEMPC is evaluated every sampling time to obtain the optimal input trajectory based on estimated state $\hat{x}(t_k)$ provided by the observer F or the RMHE. Specifically, the optimization problem of the LEMPC is formulated as follows:

$$\min_{u \in S(\Delta)} \int_{t_k}^{t_{k+N}} l_e(\tilde{x}(\tau), u(\tau))\, d\tau \tag{5.41a}$$

$$\text{s.t.} \quad \dot{\tilde{x}}(t) = f(\tilde{x}(t)) + g(\tilde{x}(t))u(t), \ t \in [t_k, t_{k+N}) \tag{5.41b}$$

$$\tilde{x}(t_k) = \hat{x}(t_k) \tag{5.41c}$$

$$u(\tau) \in \mathbb{U}, \ \forall\, t \in [t_k, t_{k+N}) \tag{5.41d}$$

$$V(\tilde{x}(t)) \leq \rho_e, \ \forall\, t \in [t_k, t_{k+N}), \quad \text{if } t_k < t_s \ \text{ and } V(\hat{x}(t_k)) \leq \rho_e \tag{5.41e}$$

$$L_g V(\hat{x}(t_k))u(t_k) \leq L_g V(\hat{x}(t_k))k(\hat{x}(t_k)), \quad \text{if } t_k \geq t_s \ \text{ or } V(\hat{x}(t_k)) > \rho_e \tag{5.41f}$$

where N is the control prediction horizon and $\tilde{x}$ is the predicted trajectory of the system with control inputs calculated by this LEMPC. The constraint of Eq. 5.41b is the nominal system model used to predict the future evolution of the system initialized with the estimated state at t_k (Eq. 5.41c). The constraint of Eq. 5.41d accounts for the input constraint. The constraint of Eq. 5.41e is active only for mode 1 operation of the LEMPC which requires that the predicted state trajectory be within the region defined by Ω_{ρ_e}. The constraint of Eq. 5.41f is active for mode 2 operation of the LEMPC as well as mode 1 operation when the estimated system state is out of Ω_{ρ_e}. This constraint forces the LEMPC to generate control actions that drive the closed-loop system state towards the origin.

The optimal solution to this optimization problem is denoted by $u^*(t|t_k)$, which is defined for $t \in [t_k, t_{k+N})$. The manipulated input of the LEMPC is defined as follows:

$$u(t) = u^*(t|t_k), \ \forall t \in [t_k, t_{k+1}). \tag{5.42}$$

The control input applied to the closed-loop system from t_k to t_{k+1} is $u^*(t_k|t_k)$.

5.3.4 Stability Analysis

The stability of LEMPC of Eq. 5.41 based on state estimates obtained following Eq. 5.40 is analyzed in this subsection. A set of sufficient conditions is derived under which the closed-loop system state trajectory is ensured to be maintained in the region Ω_ρ and ultimately bounded in an invariant set.

In the remainder of this subsection, we first present two propositions and then summarize the main results in a theorem. Proposition 5.3 characterizes the continuity property of the Lyapunov function V. Proposition 5.4 characterizes the effects of bounded state estimation error and process noise.

Proposition 5.3 (Proposition 4.2) *Consider the Lyapunov function V of system of Eq. 5.1. There exists a quadratic function f_V such that*

$$V(x) \leq V(\hat{x}) + f_V(|x - \hat{x}|) \tag{5.43}$$

for all $x, \hat{x} \in \Omega_\rho$ with

$$f_V(s) = \alpha_4(\alpha_1^{-1}(\rho))s + M_v s^2 \tag{5.44}$$

where M_v is a positive constant.

Proposition 5.4 *Consider the systems*

$$\begin{aligned} \dot{x}_a(t) &= f(x_a) + g(x_a)u(t) + l(x_a)w(t) \\ \dot{x}_b(t) &= f(x_b) + g(x_b)u(t) \end{aligned} \tag{5.45}$$

with initial states $|x_a(t_0) - x_b(t_0)| \leq \delta_x$. If $x_a(t) \in \Omega_\rho$ and $x_b(t) \in \Omega_\rho$ for all $t \in [t_0, t']$, there exists a function $f_W(\cdot, \cdot)$ such that

$$|x_a(t) - x_b(t)| \leq f_W(\delta_x, t - t_0) \tag{5.46}$$

for all $x_a(t), x_b(t) \in \Omega_\rho$ and $u(t) \in \mathbb{U}$, $w(t) \in \mathbb{W}$ for all $t \in [t_0, t']$ with:

$$f_W(s, \tau) = \left(s + \frac{M_l \theta_w}{L_f + L_g u^{\max}}\right) e^{(L_f + L_g u^{\max})\tau} - \frac{M_l \theta_w}{L_f + L_g u^{\max}} \tag{5.47}$$

where L_f, L_g, M_l are positive constants associated with functions f, g, l.

Proof Define $e_x = x_a - x_b$. The time derivative of e_x is given by:

$$\dot{e}_x(t) = f(x_a) + g(x_a)u(t) + l(x_a)w(t) - f(x_b) - g(x_b)u(t). \tag{5.48}$$

By continuity and the smoothness property assumed for f, g, there exist positive constants L_f, L_g such that:

$$|\dot{e}_x(t)| \leq L_f |e_x(t)| + L_g u |e_x(t)| + |l(x_a)w(t)|, \tag{5.49}$$

for all $t \in [t_0, t']$ provided $x_a(t) \in \Omega_\rho$ and $x_b(t) \in \Omega_\rho$ for all $t \in [t_0, t']$. By the boundedness of x_a and the smoothness property assumed for l as well as the boundedness of u and w, there exist positive constants M_l such that:

$$|\dot{e}_x(t)| \leq (L_f + L_g u^{\max})|e_x(t)| + M_l \theta_w. \tag{5.50}$$

for all $t \in [t_0, t']$. Integrating the above inequality and taking into account that $|e_x(t_0)| \leq \delta_x$, the following inequality is obtained:

$$|e_x(t)| \leq \left(\delta_x + \frac{M_l \theta_w}{L_f + L_g u^{\max}}\right) e^{(L_f + L_g u^{\max})(t-t_0)} - \frac{M_l \theta_w}{L_f + L_g u^{\max}}. \tag{5.51}$$

This proves Proposition 5.4.

The following Theorem 5.2 summarizes the stability properties of the output feedback LEMPC. The stability of the closed-loop system is based on the observer F and controller k pair with F implemented continuously and k implemented in a sample-and-hold fashion.

Theorem 5.2 *Consider system of Eq. 5.1 in closed loop under LEMPC of Eq. 5.41 with state estimates determined following Eq. 5.40 based on an observer and controller pair satisfying the assumptions in Sect. 5.3.1. Let* $\theta_w \leq \theta_w^*$, $\theta_v \leq \theta_v^*$, $\varepsilon \in (\varepsilon_L^*, \varepsilon_U^*)$ *and* $|z(t_0) - x(t_0)| \leq e_{m0}$. *Also, let* $\varepsilon_w > 0$, $\Delta > 0$ *and* $\rho > \rho_1 > \rho_e > \rho^* > \rho_s > 0$ *and* $\kappa \geq 0$ *satisfy the following conditions:*

$$\rho_e \leq \rho - \max\{f_V(f_W(\delta_x, \Delta)) + f_V(\delta_x), M \max\{\Delta, t_b\}\alpha_4(\alpha_1^{-1}(\rho))\}, \tag{5.52}$$

$$-\alpha_3(\alpha_2^{-1}(\rho_s)) + \left(L_V^f + L_V^g u^{\max}\right)(M\Delta + \delta_x) + M_V^l \theta_w \leq -\varepsilon_w/\Delta \tag{5.53}$$

where $\delta_x = (\kappa L_h + 1)e_m + \kappa\theta_v$, L_V^f, L_V^g *are Lipschitz constants associated with the Lie derivatives* $L_f V$ *and* $L_g V$, *respectively,* M *is a constant that bounds the time derivative of* x, *i.e.,* $|\dot{x}| \leq M$, *and* M_V^l *is a constant that bounds* $|L_l V|$ *for* $x \in \Omega_\rho$. *If* $x(t_0) \in \Omega_{\rho_e}$, *then* $x(t) \in \Omega_\rho$ *for all* $t \geq t_0$ *and is ultimately bounded in an invariant set.*

Proof In this proof, we consider $t \in [t_0, \max\{\Delta, t_b\})$ and $t \geq \max\{\Delta, t_b\}$ separately and prove that if the conditions stated in Proposition 5.2 are satisfied, the boundedness of the closed-loop state is ensured. The proof consists of three parts. In *Part I*, we prove that the closed-loop state trajectory is contained in Ω_ρ for $t \in [t_0, \max\{\Delta, t_b\})$; in *Part II*, we prove that the boundedness of the closed-loop state trajectory under the first operation mode of the LEMPC for $t \geq \max\{\Delta, t_b\}$ when the initial state is within Ω_{ρ_e}; and in *Part III*, we prove that the closed-loop state trajectory is bounded for the first operation mode when the initial state is within $\Omega_\rho \backslash \Omega_{\rho_e}$ and is ultimately bounded in an invariant set for the second operation mode for $t \geq \max\{\Delta, t_b\}$.

Part I: First, we consider the case that $t \in [t_0, \max\{\Delta, t_b\})$. The closed-loop system state may be described as follows:

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t) + l(x(t))w(t) \tag{5.54}$$

with $u(t)$ determined by the LEMPC with $\hat{x} = z$. The Lyapunov function of the state trajectory may be evaluated as follows:

$$V(x(t)) = V(x(t_0)) + \int_{t_0}^{t} \dot{V}(x(t))d\tau = V(x(t_0)) + \int_{t_0}^{t} \frac{\partial V(x(\tau))}{\partial x}\dot{x}(\tau)d\tau \tag{5.55}$$

Using condition of Eq. 5.2 and the boundedness of $\dot{x}$ in the region of interest, if $x(t_0) \in \Omega_{\rho_e} \subset \Omega_{\rho_1} \subset \Omega_{\rho}$, it may be written for all $t \in [t_0, \max\{\Delta, t_b\})$ that:

$$V(x(t)) \leq \rho_e + M \max\{\Delta, t_b\}\alpha_4(\alpha_1^{-1}(\rho)) \tag{5.56}$$

with M a positive constant which bounds $\dot{x}$ in Ω_ρ, i.e., $|\dot{x}| \leq M$. If ρ_e is defined as in Proposition 5.2, then

$$V(x(t)) < \rho, \ \forall\, t \in [t_0, \max\{\Delta, t_b\}). \tag{5.57}$$

Part II: In this part, we consider the case that $t \geq \max\{\Delta, t_b\}$. In this case, we have that $|x(t) - z(t)| \leq e_m$. We consider that the LEMPC is operated in the first operation mode and focus on the evolution of the state trajectory from t_k to t_{k+1}. Moreover, we consider $\tilde{x}(t_k) = \hat{x}(t_k) \in \Omega_{\rho_e}$. In this case, the LEMPC will optimize the economic cost while keeping $\tilde{x}(t)$ within Ω_{ρ_e}. We prove that if $\tilde{x}(t_k) \in \Omega_{\rho_e}$, then $x(t_{k+1}) \in \Omega_\rho$ and $\hat{x}(t_{k+1}) \in \Omega_\rho$.

From t_k to t_{k+1}, the worst case scenario is as shown in Fig. 5.14. At time t_k, the estimate of the state $\hat{x}(t_k) = \tilde{x}(t_k)$ is on the boundary of Ω_{ρ_e} while the actual system state is outside of Ω_{ρ_e} and on the boundary of another set Ω_{ρ_2} due to uncertainty in

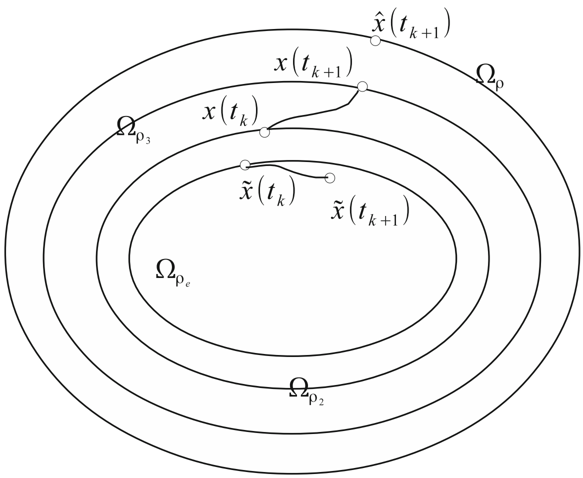

Fig. 5.14 Worst case scenario of the evolution of $\tilde{x}$ and x from t_k to t_{k+1} in the first operation mode

$\hat{x}$. The LEMPC will keep $\tilde{x}(t)$ inside Ω_{ρ_e} from t_k to t_{k+1}. However, due to the error in $\tilde{x}(t_k)$ and the presence of process noise, the actual system state $x(t_{k+1})$ may evolve to a point on the boundary of Ω_{ρ_3} in Fig. 5.14 that is further away of Ω_{ρ_e}. The distance between $\tilde{x}(t_{k+1})$ and $x(t_{k+1})$, is bounded. From Proposition 5.4, it may be obtained that:

$$|\tilde{x}(t_{k+1}) - x(t_{k+1})| \leq f_W(|\hat{x}(t_k) - x(t_k)|, \Delta). \tag{5.58}$$

Recall that when $t \geq t_b$, all the estimates are provided by the RMHE. From the design of the RMHE, it may be written that:

$$|\hat{x}(t_k) - z(t_k)| \leq \kappa |y(t_k) - h(z(t_k))| . \tag{5.59}$$

Using the relation that $|\hat{x} - x| \leq |\hat{x} - z| + |z - x|$, it may be obtained that:

$$|\hat{x}(t_k) - x(t_k)| \leq \kappa |y(t_k) - h(z(t_k))| + |z(t_k) - x(t_k)| . \tag{5.60}$$

Noticing that $|z(t_k) - x(t_k)| \leq e_m$ and $|y(t_k) - h(z(t_k))| = |h(x(t_k)) + v(t_k) - h(z(t_k))|$, and using the Lipschitz property of h, the boundedness of v, the following inequality may be written:

$$|\hat{x}(t_k) - x(t_k)| \leq (\kappa L_h + 1)e_m + \kappa\theta_v. \tag{5.61}$$

From Eqs. 5.58 and 5.61, it may be obtained that:

$$|\tilde{x}(t_{k+1}) - x(t_{k+1})| \leq f_W((\kappa L_h + 1)e_m + \kappa\theta_v, \Delta). \tag{5.62}$$

This implies that if $\tilde{x}$ is maintained in Ω_{ρ_e}, the actual system state x is ensured to be within the set Ω_{ρ_2} with $\rho_2 = \rho_e + f_V(f_W((\kappa L_h + 1)e_m + \kappa\theta_v, \Delta))$ which may be obtained from Proposition 5.3.

Taking into account Eq. 5.61 for $t = t_{k+1}$, the state estimate obtained at t_{k+1} could be outside the region Ω_{ρ_2} but the distance is bounded as follows:

$$|\hat{x}(t_{k+1}) - x(t_{k+1})| \leq (\kappa L_h + 1)e_m + \kappa\theta_v. \tag{5.63}$$

In order to ensure that $\hat{x}(t_{k+1})$ is within Ω_ρ which is required for the feasibility of LEMPC of Eq. 5.41, the following inequality should be satisfied:

$$\rho \geq \rho_e + f_V(f_W((\kappa L_h + 1)e_m + \kappa\theta_v, \Delta)) + f_V((\kappa L_h + 1)e_m + \kappa\theta_v) \tag{5.64}$$

which implies that ρ_e should be picked to satisfy the following condition:

$$\rho_e \leq \rho - f_V(f_W((\kappa L_h + 1)e_m + \kappa\theta_v, \Delta)) - f_V((\kappa L_h + 1)e_m + \kappa\theta_v). \tag{5.65}$$

If ρ_e is defined as in Proposition 5.2, the above condition is satisfied.

Part III: Next, we consider the case that $\hat{x}(t_k) = \tilde{x}(t_k) \in \Omega_\rho \backslash \Omega_{\rho_e}$ in the first operation mode or $t_k \geq t_s$ for the second operation mode. In either case, constraint of Eq. 5.41f will be active. The time derivative of the Lyapunov function may be evaluated as follows:

$$\dot{V}(x(t)) = \frac{\partial V(x(t))}{\partial x}(f(x(t) + g(x(t))u(t_k) + l(x(t))w(t)) \tag{5.66}$$

for $t \in [t_k, t_{k+1})$. Adding and subtracting the term:

$$\frac{\partial V(\hat{x}(t_k))}{\partial x}(f(\hat{x}(t_k) + g(\hat{x}(t_k))u(t_k))$$

to/from the above equation and considering constraint Eq. 5.41f as well as condition of Eq. 5.2, it is obtained that:

$$\begin{aligned}\dot{V}(x(t)) \leq -\alpha_3(|\hat{x}(t_k)|) + \frac{\partial V(x(t))}{\partial x}(f(x(t)) + g(x(t))u(t_k) + l(x(t))w(t)) \\ - \frac{\partial V(\hat{x}(t_k))}{\partial x}(f(\hat{x}(t_k) + g(\hat{x}(t_k))u(t_k))\end{aligned} \tag{5.67}$$

for all $t \in [t_k, t_{k+1})$. By the smooth properties of V, f, g and l, the boundedness of x, u and w, there exist positive constants L_V^f, L_V^g, M_V^l such that:

$$\dot{V}(x(t)) \leq -\alpha_3(|x(t_k)|) + \left(L_V^f + L_V^g u^{\max}\right)|x(t) - \hat{x}(t_k)| + M_V^l \theta_w \tag{5.68}$$

for all $x \in \Omega_\rho$. Noticing that $|x(t) - \hat{x}(t_k)| \leq |x(t) - x(t_k)| + |x(t_k) - \hat{x}(t_k)|$, it is obtained that:

$$|x(t) - \hat{x}(t_k)| \leq |x(t) - x(t_k)| + (\kappa L_h + 1)e_m + \kappa\theta_v. \tag{5.69}$$

By the continuity and smoothness properties of f, g, l and the boundedness of x, u and w, there exists a positive constant M such that $|\dot{x}| \leq M$. From the above inequalities, it may be obtained that:

$$\begin{aligned}\dot{V}(x(t)) \leq -\alpha_3(\alpha_2^{-1}(\rho_s)) + \left(L_V^f + L_V^g u^{\max}\right)(M\Delta + (\kappa L_h + 1)e_m + \kappa\theta_v) \\ +M_V^l \theta_w\end{aligned} \tag{5.70}$$

for all $x \in \Omega_\rho \backslash \Omega_{\rho_s}$. If condition of Eq. 5.53 is satisfied, it may be obtained from Eq. 5.70 that:

$$V(x(t_{k+1})) \leq V(x(t_k)) - \varepsilon_w. \tag{5.71}$$

This means that the function value $V(x)$ is decreasing in the first operation mode if $\tilde{x}(t_k) = \hat{x}(t_k)$ is outside of Ω_{ρ_e}. This implies that $\hat{x}(t_k)$ will eventually enter Ω_{ρ_e}.

This also implies that in the second operation mode, $V(x)$ decreases every sampling time and x will eventually enter Ω_{ρ_s}. Once $x \in \Omega_{\rho_s} \subset \Omega_{\rho^*}$, it will remain in Ω_{ρ^*} because of the definition of ρ^*. This proves Proposition 5.2.

Remark 5.8 Part I of Theorem 5.2 essentially treats the input as a perturbation to the system. Given that the input and the noise are bounded, a bound is derived for how large the Lyapunov function may increase over time t_b (which is small). This follows from the fact that the initial estimation error of the deterministic observer and actual closed-loop state are both bounded in a region containing the origin.

Remark 5.9 Parts II and III prove that if the current state $x(t_k) \in \Omega_\rho$ and if the current estimate $\hat{x}(t_k) \in \Omega_\rho$, the actual closed-loop state and the estimated state at the next sampling period are also within Ω_ρ. Since Part II considers mode 1 operation of the LEMPC, the worst case scenario is considered (Fig. 5.14). Part III considers mode 2 operation of the LEMPC. While the theoretical developments and corresponding bounding inequalities contained in this section are conservative, they do provide valuable insight and guidelines for selecting the parameters of the state feedback controller $k(x)$, the deterministic observer, the RMHE, and the output feedback LEMPC such that the closed-loop system of Eq. 5.1 with bounded process and measurement noise under the output feedback LEMPC of Eq. 5.41 is stable.

Remark 5.10 One could potentially apply the RMHE for t_0 to t_b instead of using the deterministic observer. However, it is difficult to prove closed-loop stability for this case owing to the fact that the estimation error may not have decayed to a small value over this time period with the RMHE, i.e., it is difficult to show that the RMHE satisfies the observability assumptions of Sect. 5.3.1.

5.3.5 Application to a Chemical Process Example

Consider the CSTR described in Sect. 5.2.4 with the same control objective and the same limitation on the available reactant material. To estimate the state from noisy temperature measurements, the RMHE scheme is used. The weighting matrices of the RMHE are given by $Q_e = \text{diag}([\sigma_{w_1}^2\ \sigma_{w_2}^2])$ and $R_e = \sigma_v^2$ where σ denotes the standard deviation of the process or measurement noise. The design parameter of the RMHE is $\kappa = 0.4$, the sampling period is the same as the LEMPC, i.e., $\Delta_e = 0.01$ h), and the estimation horizon of the RMHE is $N_e = 15$. The robust constraint of the RMHE is based on a high-gain observer as in Sect. 5.2.4. For the first 15 sampling periods, the high-gain observer is used to provide the LEMPC with a state estimate. At each subsequent sampling periods, the LEMPC is initialized using the state estimate from the RMHE. To solve the optimization problems of the LEMPC and the RMHE at each sampling period, the open-source software Ipopt [6] is used. The process model is numerically simulated using an explicit Euler integration method with integration step $h_c = 1.0 \times 10^{-3}$ h. To simulate the process and measurement noise, new random numbers are generated and applied over each integration step. The process noise is

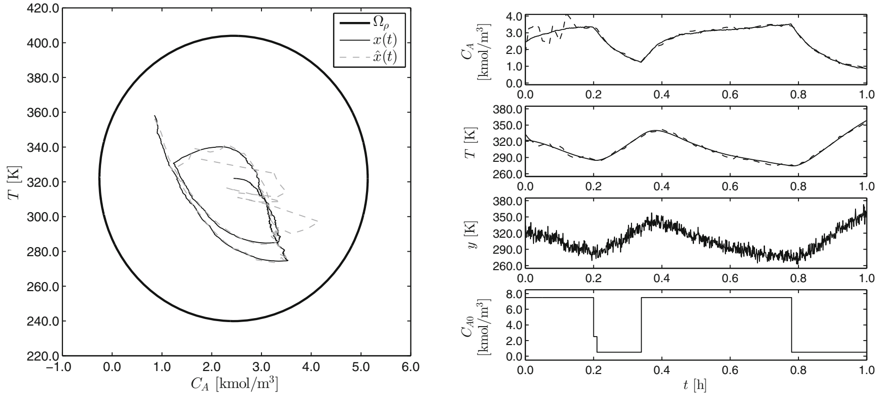

Fig. 5.15 The evolution of the closed-loop CSTR under the RMHE-based LEMPC scheme shown in state-space (*left*) and as a function of time (*right*). The *solid line* is the actual closed-loop state trajectory $x(t)$, while, the *dashed line* is the estimated state $\hat{x}(t)$

assumed to enter the system additively to the right-hand side of the process model ODEs. The random numbers are generated from a zero-mean, bounded Gaussian distribution.

Square bounds of $w_{\max} = [20.0\ 50.0]$ and $v_{\max} = 20.0$ are used to bound the process and measurement noise, respectively, and the standard deviation of the noise terms are $\sigma_w = [7.0\ 20.0]$ and $\sigma_v = 7.0$, respectively. The CSTR is initialized at $x_0^T = [2.44\,\text{kmol}\,\text{m}^{-3}\ 320.0\,\text{K}]$, which corresponds to the steady-state. The evolution of the closed-loop CSTR under the RMHE-based LEMPC is shown in Fig. 5.15. Initially, the estimated reactant concentration is significantly affected by the measurement noise which is expected since the state estimate is computed by the high-gain observer over this initial time period. After the RMHE is activated, the estimated state trajectories are nearly overlapping with the actual closed-loop state trajectories. Furthermore, the LEMPC computes a periodic-like input profile to optimize the process economics over the 1 h period of operation.

The average reaction rate over 1 h period of operation is 13.59 kmol m^{-3}. If, instead, the CSTR was maintained at the steady-state (x_0) without process and measurement noise (nominal operation), the average reaction rate over this 1 h operation would be 12.80 kmol m^{-3}. This is a 6.2 % improvement in the economic cost of the closed-loop system under the RMHE-based LEMPC with process and measurement noise over nominal steady-state operation. We note that the economic performance of the closed-loop system under LEMPC with full state feedback and nominal operation over 1 h operation is 13.60 kmol m^{-3} which is a 6.3 % economic performance improvement.

To assess the estimation performance of the RMHE, another simulation is performed with the same realization of the process and measurement noise and with the high-gain observer presented in Sect. 5.2.4. The evolution of the closed-loop CSTR under the high-gain observer state estimation-based LEMPC is shown in Fig. 5.16.

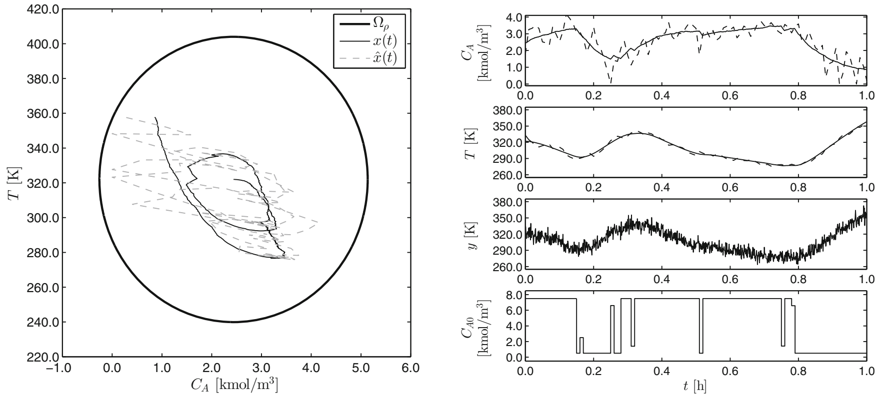

Fig. 5.16 The evolution of the closed-loop CSTR under the state estimation-based LEMPC scheme with the high-gain observer of Sect. 5.2.4 shown in state-space (*left*) and as a function of time (*right*). The *solid line* is the actual closed-loop state trajectory $x(t)$, while, the *dashed line* is the estimated state $\hat{x}(t)$

Not only does the noise impact the estimates provided by the high-gain observer in this case, but also, it impacts the computed input profile (Fig. 5.16). Comparing Figs. 5.15 and 5.16, the RMHE is able to provide estimates of the state within a small neighborhood of the actual process states, while the high-gain observer is not able to estimate the concentration as well as the RMHE. Furthermore, since the RMHE provides better (smoother) estimates of the states, the operation of the closed-loop system under the RMHE-based LEMPC is smoother which may be observed in the input trajectories.

Several additional closed-loop simulations with various bounds and standard deviations on the process and measurement noise and initial conditions are performed to further assess the estimation performance of RMHE compared to the one of the high-gain observer of Sect. 5.2.4. An estimation performance index which is defined as

$$J = \sum_{k=0}^{99} \left| \hat{x}(t_k) - x(t_k) \right|_S^2 \tag{5.72}$$

is used to assess the estimation performance where the matrix S is a positive definite weighting matrix given by $S = \text{diag}([50\ 1])$. The matrix S has been chosen to account for the different numerical ranges of the concentration and temperature. In addition to the assessment on the estimation performance, the total economic performance index over the length of the simulation is defined as

$$J_e = \frac{1}{100} \sum_{k=0}^{99} k_0 e^{-E/RT(t_k)} C_A^2(t_k) \tag{5.73}$$

Table 5.2 Estimation performance comparison of the closed-loop CSTR with various bounds and standard deviation of the disturbances and noise and initial conditions under the high-gain observer state estimation-based LEMPC and under the RMHE-based LEMPC (ordered below by increasing bounds and standard deviation)

	High-gain observer			RMHE		
	J	SSE of C_A	J_e	J	SSE of C_A	J_e
1	310.5	4.450	13.04	104.0	1.277	13.04
2	528.5	7.781	14.19	310.1	4.169	14.19
3	271.6	3.669	13.47	88.1	0.440	13.47
4	506.4	7.066	13.06	181.9	1.476	13.07
5	583.2	8.097	14.20	354.4	3.888	14.20
6	482.1	6.397	13.48	137.7	0.372	13.48
7	592.4	7.821	13.09	257.1	1.734	13.09
8	572.8	8.519	14.23	252.6	3.425	14.23
9	616.4	8.579	13.51	168.6	1.126	13.52
10	992.0	13.700	13.00	429.9	4.355	13.08
11	1079.8	14.871	14.14	888.7	12.076	14.21
12	1012.5	14.304	13.42	552.0	5.817	13.43
13	1643.6	22.606	13.02	665.3	3.523	12.99
14	1758.5	23.396	14.24	771.2	5.492	14.27
15	1591.0	21.740	13.51	561.5	1.717	13.55

The J column refers to the performance index of Eq. 5.72, the "SSE of C_A" column denotes the sum of squared errors of the concentration C_A estimation, and the J_e column refers to the economic performance index of Eq. 5.73

which is the time-averaged reaction rate over the simulation. From the results displayed in Table 5.2, the RMHE consistently provides a significantly better estimates of the state than the high-gain observer which demonstrates the robustness of the RMHE to process and measurement noise. However, the estimation performance does not translate into a significant closed-loop average economic performance improvement of the closed-loop system with the RMHE-based LEMPC over the closed-loop system with the high-gain observer and LEMPC. This relationship is due to the fact that the closed-loop average economic performance over one operation period is not strongly dependent on the initial condition of the LEMPC optimization problem, i.e., $\hat{x}(t_k)$, for this particular example. In other words, providing the LEMPC with an estimate of the actual state anywhere in a neighborhood around the actual state results in a computed input trajectory that leads to nearly the same economic cost for the closed-loop systems. For systems that are more sensitive to the estimate of the current state, it is expected that there would also be improved closed-loop economic performance with the RMHE-based LEMPC in addition to improved estimation performance.

5.4 Conclusions

In this chapter, two output feedback EMPC schemes were presented. In the first scheme, a high-gain observer-based EMPC for the class of full-state feedback linearizable nonlinear systems was introduced. A high-gain observer is used to estimate the nonlinear system state using output measurements and a Lyapunov-based approach is adopted to design the EMPC that uses the observer state estimates. It was proved, using singular perturbation arguments, that the closed-loop system is practically stable provided the observer gain is sufficiently large.

To achieve fast convergence of the state estimate to the actual system state as well as to improve the robustness of the estimator to measurement and process noise, a high-gain observer and a RMHE scheme were used to estimate the system states. In particular, the high-gain observer was first applied for a small time period with continuous output measurements to drive the estimation error to a small value. Once the estimation error had converged to a small neighborhood of the origin, the RMHE was activated to provide more accurate and smoother state estimates. In the design of the RMHE, the high-gain observer was used to provide reference estimates, which were subsequently used to calculate confidence regions. The RMHE was restricted to compute state estimates that are within these confidence regions. The output feedback EMPC was designed via Lyapunov techniques based on state estimates provided by the high-gain observer and the RMHE.

The application of the two schemes to a chemical reactor demonstrated the applicability and effectiveness of the schemes and the ability to deal with measurement noise.

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