

A Formal Setting for Network Dynamics

Ian Stewart

Abstract This chapter is an introduction to coupled cell networks, a formal setting in which to analyse general features of dynamical systems that are coupled together in a network. Such networks are common in many areas of application. The nodes ('cells') of the network represent system variables, and directed edges ('arrows') represent how variables influence each other. Cells and arrows are assigned types, which determine the form of admissible differential equations—those compatible with the network structure. By analogy with the modern theory of dynamical systems, emphasis is placed on phenomena that are typical of entire classes of model equations with a given network structure, rather than on specific models. Such phenomena include symmetry and synchrony relations among cells, leading to a clustering effect embodied in a quotient network described by a balanced colouring. Rigid patterns of synchrony (those preserved by admissible perturbations) for equilibria and periodic states are classified by the balanced colourings. Bifurcations in which network structure can cause anomalous power-law growth rates are briefly mentioned. The formal concepts are motivated and explained in terms of typical examples.

Keywords Bifurcation • Dynamics • Network • Symmetry • Synchrony

1 Introduction

In recent years it has become increasingly apparent that networks play a highly significant role in many areas of science and technology. Examples include the spread of epidemics, food webs in ecosystems, gene regulation, intercellular signalling, neuroscience, market trading, control, and communications.

I. Stewart (✉)

Mathematics Institute, University of Warwick, Coventry CV4 7AL, UK
e-mail: I.N.Stewart@warwick.ac.uk

The defining features of a network are a set of nodes, which interact through a system of connections. In mathematics, such a structure has traditionally been called a graph, but ‘network’ is more evocative. Nodes are also known as vertices or dots, and connections as edges or lines; these may or may not be directed. We will shortly rename nodes as ‘cells’ and directed edges as ‘arrows’ to emphasise the extra structure that will be brought into play.

Many different aspects of network structure and behaviour have been studied, ranging from statistical features to dynamics. Applications include the rate of spread of an epidemic, stability of the population distribution in an ecosystem, the development of organisms, broken connections in communications networks, ‘small world’ phenomena, stock market crashes, and internet search engines. The literature is vast, with many different viewpoints and philosophies, and we make no attempt to summarise it here. Instead, we focus on one specific area: the nonlinear dynamics of networks of coupled dynamical systems. By a dynamical system we mean a system of ordinary differential equations in one or more variables, which we abbreviate to ‘ODE’.

Just over a decade ago an analogy between symmetric dynamical systems (Golubitsky et al. 1988) and network dynamics began to be explored (Golubitsky and Stewart 2006; Golubitsky et al. 2005; Stewart et al. 2003). The aim was to apply, in a network context, the modern philosophy of nonlinear dynamics. This approach was pioneered by Poincaré (1881, 1882, 1885, 1886) in his work on the qualitative theory of differential equations. Among other things, this viewpoint led him to discover chaotic dynamics in the three-body problem for Newtonian gravitation (Poincaré 1892, 1893, 1899). His qualitative approach to differential equations was developed into a systematic theory by several mathematicians, especially in the Soviet Union, and became firmly established as a new branch of mathematics with the work of Arnold (1963), Smale (1967), and others. The central idea in this approach to dynamics is that significant structural phenomena are invariant under appropriate coordinate changes, and are thus determined purely by the *topology* of the trajectories in phase space—the phase portrait. For example, the presence of a time-periodic state (limit cycle) is a topological feature, but the detailed waveform, the period, and the shape of the cycle in phase space are not.

This approach deliberately ignores many details of the system, which have to be supplied by other means—typically numerical solutions, because few interesting nonlinear ODEs can be solved explicitly. So why do we need topological dynamics when any specific problem can be understood by numerical computation? Often we do not: numerical simulations provide all the answers required. However, numerical solutions sometimes make little sense on their own—they reveal some form of behaviour, but do not explain why it is occurring, or whether it is typical or unusual in the appropriate context. Moreover, most real-world models include numerical parameters that can take on many values, and it is often important to understand how the solutions change as parameters vary. Some numerical schemes exist that can explore such issues, but in general such questions may require infeasibly lengthy calculations. Moreover, it can be difficult to organise the results into a sensible description of the system’s behaviour. Topological dynamics can help here, because

it provides a systematic framework for organising, classifying, and recognising the basic types of behaviour. It relates them to each other, and allows insights to be transferred from one area of application to others. It is in some ways a coarse instrument, but that is a virtue as well as a vice, because it removes inessential information.

The pioneers of the subject realised that the topological approach can be a highly effective approach to a basic, *general* question: ‘what can dynamical systems do?’. The effect of this change of viewpoint was a bit like the zoological move from butterfly-collecting to Linnaean taxonomy. Post-Linnaeus, you still had to collect butterflies to find out what existed in nature, but you began to appreciate how they related to other butterflies—and, more crucially, to other species.

Many special classes of dynamical system have extra structure. For example Hamiltonian systems are defined by a Hamiltonian function, which is conserved along trajectories and induces a symplectic structure, Smale (1967). Symmetric dynamical systems are defined by ‘equivariant’ vector fields with specific symmetry properties. In networks of coupled dynamical systems, the variables that appear in the differential equation, and the form of that equation, respect the network architecture. When the system has special structure, it is sensible to require the permissible coordinate changes to preserve this structure. This restriction can lead to new phenomena, invariant under this more limited type of coordinate change. Examples, in these three contexts, are the topology of energy levels, the symmetry group of a solution, and synchrony of specific nodes of a network.

Our focus here is on the network case. Network dynamics has been widely studied in many specific settings. Often the network structure is treated informally. However, it makes sense to develop a general overview by defining an appropriate formal structure, analogous to that for general dynamical systems. Here we survey some of the basic ideas in one systematic approach to this issue (Golubitsky and Stewart 2006; Golubitsky et al. 2005; Stewart et al. 2003). The main motivation in those papers was to seek analogies with symmetric dynamics (Golubitsky et al. 1988) and to devise alternatives when new issues arose. As they did.

1.1 Outline of Chapter

We begin by describing some examples of networks and their dynamics, to act as motivation. This leads to a formal definition of a coupled cell network and the corresponding class of ‘admissible’ differential equations. A key concept here is the input set of a node, which determines how the rest of the network is coupled to (drives) that cell.

Analogies with other special classes of dynamical systems help to motivate some basic questions and concepts. In particular, we take inspiration from symmetric dynamics, where the ODE respects a group of symmetry transformations.

An immediate obstacle arises, which causes technical difficulties, but cannot easily be avoided. In symmetric dynamics, the composition of two symmetric (that

is, equivariant, see Golubitsky et al. 1988) maps is always symmetric. The analogue for network dynamics is false in general: the composition of two admissible maps need not be admissible. However, there is a partial substitute: strongly admissible maps. The composition of a strongly admissible map and an admissible map, in either order, is always admissible.

We include a brief discussion of symmetries of networks, an important area that combines (often in an uneasy alliance) features of symmetric dynamics and network dynamics.

One important issue in network dynamics is the possibility of synchrony, in which two (or more) cells have identical time series. One way to approach synchrony is through the concept of a balanced colouring of the cells. Suppose that the state of the network exhibits some pattern of synchrony; that is, certain cells are synchronous with others. Assign the same colour to all cells that are synchronous with each other. Intuitively, synchronous cells should receive the same input from the network: if not, the synchrony would be destroyed. The most natural way to ensure this is if the corresponding input sets match up in a manner that preserves colours. That is, cells with the same colour have inputs that are related by a colour-preserving permutation. This is the balance condition.

(An alternative is that some kind of cancellation of inputs takes place, but this would be ‘accidental’ and would disappear after a small admissible perturbation—unless the network equations have extra special features. In such cases, a generalised form of the balance condition must still apply.)

The above statement can be made precise. Balanced colourings define a distinguished class of subspaces of phase space that are invariant under any admissible map. The dynamics on this subspace leads to the pattern of synchrony determined by the colours. In contrast, an unbalanced colouring does not have this invariant subspace property.

If cells with the same colour are identified, the result is a ‘quotient network’ on a smaller number of cells, whose dynamics corresponds to synchronous dynamics in the original network with the corresponding pattern of synchrony.

There are some stronger results which apply to suitable equilibrium and periodic states. Say that a pattern of synchrony is rigid if it persists after any sufficiently small admissible perturbation of the ODE. Then rigid synchrony of equilibria defines a balanced colouring. So does rigid synchrony of periodic states; the current proof uses a mild technical hypothesis but it seems likely that this can be removed. There is also a version of this theorem for patterns of phase-related cells, rather than synchronous ones. It leads to a characterisation of conditions under which clusters of synchronous cells have a ‘rotating wave’ spatio-temporal symmetry.

The final topic is bifurcation theory, where states of the system undergo qualitative changes as some parameter varies. In particular we remark that network architecture can create anomalous bifurcation behaviour—that is, different from the typical bifurcations in general dynamical systems. As an illustration we exhibit an example of anomalous power-law growth of the amplitude of bifurcating branches of periodic states in a three-cell feed-forward network.

2 Network Diagrams and Admissible Maps

2.1 Motivation

The theory of networks goes back to the work of Euler (1741) on the puzzle of the Königsberg bridges. Contrary to common belief, he did not introduce the concept of a graph in its familiar geometric form; instead, he employed a symbolic representation of paths and argued combinatorially, Wilson (1985). However, the standard graphical representation soon followed. The main ingredients for a *graph* are a set of nodes, represented by dots, connected by a set of edges, represented by lines. The edges may be undirected (line segments) or directed (arrows). The main objects of study initially were the combinatorics and topology of graphs. As the subject developed, extra structure was imposed: directed edges were assigned numerical probabilities, connection strengths, flow rates, or durations (for example in critical path analysis).

In applications, especially to neuroscience, ODEs are associated with a given network, and the form of these equations reflects the network architecture. For example in a neuroscience model, nodes might represent neurons and edges axons, coupled via electrical signals passing along the axons. The state of each node i is represented by a variable x_i , which might be a scalar or a vector. Each node typically has an internal dynamic, an ODE that determines how it would behave if it were not coupled to other nodes. Connections from one node to another lead to coupling terms in the equations: if there is an input from node j to node i , then dx_i/dt is a function of both x_i and x_j .

Example 1 The β IG model of diabetes, Topp et al. (2000), takes the form

$$\begin{aligned}\dot{G} &= a - (b + cI)G \\ \dot{I} &= \beta \left(\frac{dG^2}{e + G^2} \right) - fI \\ \dot{\beta} &= (-g + hG - iG^2)/\beta\end{aligned}$$

Here dots are time derivatives. The terms G = glucose level, I = insulin level, and β = beta-cell mass depend on time t . The other terms $a, b, c, d, e, f, g, h, i$ are parameters, whose value is constant during any particular run of the model or in any particular real system.

The network structure arises when we consider which variables depend on which. Here:

- The change in G depends on G, I but not on β .
- The change in I depends on G, I , and β .
- The change in β depends on G, β but not on I .

It is natural to encode these relationships as the network (called a block diagram in some areas of application) shown in Fig. 1. Here each variable is represented by a cell symbol (circle, square, hexagon) and arrows show which variable affects any given cell variable. The different cell symbols indicate different ‘cell types’, meaning that the form of the equation is different for those cells. The different arrow symbols (solid, dotted, and so on) indicate different ‘arrow types’, meaning that the form of the coupling is different for those cells.

In such a representation individual cell or arrow symbols have no further meaning on their own. Their interpretation depends on the entire diagram. For example, in this case the coupling from G and β to I is not a sum of terms in G and β separately, but a combination of both variables. Coupling terms need not be additive; for example in the equation for G the variable I appears as a product cIG .

Example 2 Consider an ODE representing three coupled FitzHugh-Nagumo neurons:

$$\begin{aligned} \dot{v}_1 &= v_1(a - v_1)(v_1 - 1) - w_1 - cv_2 & \dot{w}_1 &= bv_1 - \gamma w_1 \\ \dot{v}_2 &= v_2(a - v_2)(v_2 - 1) - w_2 - cv_3 & \dot{w}_2 &= bv_2 - \gamma w_2 \\ \dot{v}_3 &= v_3(a - v_3)(v_3 - 1) - w_3 - cv_1 & \dot{w}_3 &= bv_3 - \gamma w_3 \end{aligned} \tag{1}$$

Here v_i is the membrane potential of cell i , w_i is a surrogate for an ionic current, and a, b, γ are parameters with $0 < a < 1, b > 0, \gamma > 0$.

In (1) the dynamic equations are the same for each neuron, subject to appropriate permutations of the variables. In other words, the individual neurons are identical, and the couplings are also identical. So in this case the natural diagram is a ring of three identical cells (same cell symbol) with identical unidirectional coupling (same arrow symbol). See Fig. 2.

The state space of cell i is now 2-dimensional, with variables (v_i, w_i) . Because the variables enter the equations in the same manner for each i , subject to the cyclic

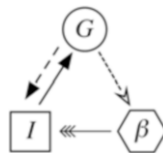


Fig. 1 Network representation of the βIG model

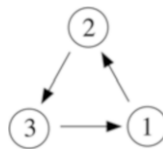


Fig. 2 Network representation of a ring of three identical FitzHugh-Nagumo neurons with identical unidirectional coupling

permutation, the cells have the same type and so do the arrows. In the diagram, we represent this by using circles for all three cells and the same kind of arrow for all three couplings.

2.2 Modelling

A network diagram does not specify an ODE as such. In particular it tells us nothing about the functional form of the equations. Instead, the diagram acts as a schematic representation of which variables affect which, and specifies when the same equation arises for corresponding variables. So each diagram determines a *class* of ODEs that ‘respect the network structure’. Moreover, certain dynamical features may be common to all ODEs in this class, and are thus typical features for that network. These include possible patterns of synchrony, phase relations in periodic states, and a singularity-theoretic interpretation of homeostasis (Golubitsky and Stewart 2016a,b). Other features depend on the precise equations. So the formal theory separates the features that are typical for all networks with a given diagram from those that are special, and depend on the precise terms in the equations.

In a conventional approach to modelling, the equations are set up from the beginning using specific terms that reflect known (or presumed) aspects of the biology or physics of the system being modelled. For example, the term $dG^2/(e + G^2)$ in the β IG model tends to a constant d for large G , modelling a feature of the insulin response to large glucose levels. Other ODEs consistent with the network architecture need not behave in that manner, but would probably not be appropriate to model diabetes. Having set up specific equations that incorporate various assumptions of this kind, they can then be studied analytically or numerically to see how solutions behave. In circumstances when there is strong justification for choosing a particular formula, this type of model is an accurate representation of the real system.

However, especially in biology, there is often a lot of flexibility in the choice of formula, and the literature typically considers many variants. This is where the ‘model-independent’ philosophy presented here differs from this conventional ‘model-dependent’ approach. It offers some advantages by distinguishing between aspects of the solution that are sensitive to the precise formula employed, and those that are relatively robust and depend mainly on the network architecture. Specific models are still important; for example, to work out which parameter values lead to particular types of behaviour. But they can be used in the context of knowledge of what kind of behaviour should be expected on the basis of the network structure. This avoids the danger of attributing predicted behaviour to a specific formula, when it is mainly a result of the network structure and would occur for other formulas.

This viewpoint shifts the emphasis to a two-stage approach. First, understand model-independent features. Second, consider model-dependent features in the context of the model-independent ones to find out what extra information or insight the specific choice of model adds. The first step motivates defining a formal setting

for network dynamics and working out the general principles that apply. The initial aim is to use the network structure to define a natural *class* of differential equations whose structure is compatible with a given network. We say that these ODEs are ‘admissible’ for that network.

There are several general formulations in the literature. For example, Kuramoto (1984) considers nonlinear internal dynamics plus linear coupling:

$$\frac{dx_i}{dt} = f_i(x_i) + \sum_j a_{ij}x_j \quad (2)$$

with nonlinear f_i and constants a_{ij} for some set of input nodes j . The idea is that each cell has a nonlinear internal dynamic f_i , and the couplings are linear, given by the matrix (a_{ij}) . The form (2) can be motivated as a pragmatic low-order approximation to more complicated equations, where linearity corresponds to weak coupling, but this form of coupling is very special. In particular it is not preserved by any obvious type of coordinate change beyond linear maps, contrary to the spirit of topological dynamics. However, it also has some advantages: a specific internal dynamic $f_i(x_i)$, and removal of couplings by setting the relevant a_{ij} to zero.

Another common choice is to assume that the nodes represent ‘phase oscillators’, whose state space is a circle \mathbb{S}^1 , and a state $\theta \in \mathbb{S}^1$ describes the phase of the oscillator. In this model the amplitudes of the oscillations are ignored.

Which formalism is appropriate depends on the questions being asked. The choice described in this chapter avoids restrictive assumptions on the form of the ODEs. It therefore provides a suitable context to study ‘generic’ or ‘typical’ phenomena in network dynamics, offering a useful perspective on more specific models, and it helps to explain some of their features.

3 Coupled Cell Networks and Systems

We now begin to set up a formal structure for network dynamics.

For reasons loosely related to the motivating examples, and to distinguish the topic from standard graph theory, the terms ‘node’ and ‘directed edge’ were replaced by ‘cell’ and ‘arrow’ in early work. For consistency with the literature, we do the same here.

Definition 1 A *coupled cell network* satisfies the following conditions:

- (1) There is a finite set \mathcal{C} of *cells*, usually identified with the standard set $\mathcal{C} = \{1, 2, \dots, n\}$.
- (2) There is a finite set \mathcal{E} of *arrows*.
- (3) Each arrow e has a *head cell* $\mathcal{H}(e) \in \mathcal{C}$ and a *tail cell* $\mathcal{T}(e) \in \mathcal{C}$.
- (4) Cells are classified into *types*. Formally, this is done by defining an equivalence relation \sim_C on \mathcal{C} , called *cell equivalence*. Cells are equivalent if they have the same type.

- (5) Arrows are also classified into *types* by defining an equivalence relation \sim_E on \mathcal{E} , called *arrow equivalence*. Arrows are equivalent if they have the same type.
- (6) Types satisfy two compatibility conditions. If $e_1, e_2 \in \mathcal{E}$ are arrow-equivalent, then $\mathcal{H}(e_1)$ and $\mathcal{H}(e_2)$ are cell-equivalent, and $\mathcal{T}(e_1)$ and $\mathcal{T}(e_2)$ are cell-equivalent.

From now on we often shorten ‘coupled cell network’ to ‘network’. A network can be represented graphically by its *diagram*. Here cells are drawn as dots, circles, squares, hexagons, and so on, with a different symbol for each type. Arrows are drawn as arrows, similarly decorated to distinguish types by using dotted or wavy lines, different shapes of arrowhead, and so on. Each arrow e runs from $\mathcal{T}(e)$ to $\mathcal{H}(e)$. The diagram is a directed labelled graph, where the ‘labels’ are graphical representations of the cell and arrow types.

Warning: An arrow can have the same head and tail, forming a *self-connection* from a cell to itself. Two distinct arrows (of the same or different types) can have the same head and the same tail, giving *multiple connections* between the two cells. Arrows of this kind arise naturally in connection with a basic construction, the ‘quotient network’, which is related to synchrony; the entire formalism works much better if they are permitted from the start. See Sect. 7.

Figure 3 shows a few examples, and we take the opportunity to illustrate some basic types of network *architecture* (that is, topology) at the same time.

Tacit conventions are often used to simplify such diagrams. For example in the ‘all-to-all’ network, pairs of equivalent arrows in opposite directions are shown as

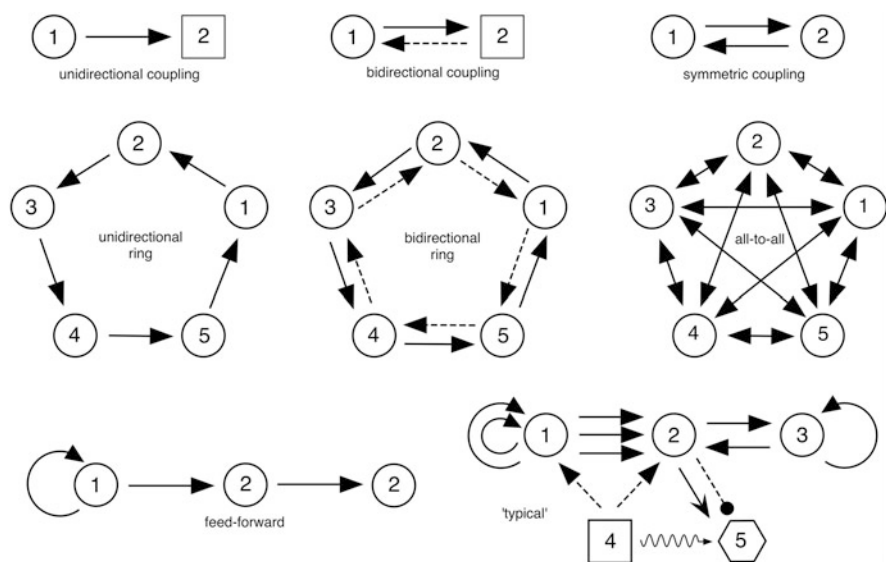


Fig. 3 A sample of coupled cell networks

a single line with two heads. Some examples have a single cell type and a single arrow type; others do not. The final ‘typical’ example illustrates a few possibilities consistent with the formalism, and has no special significance.

A network is *connected* if the underlying graph (ignoring cell and arrow types and arrow directions) is connected; that is, any two nodes are joined by a path of mutually adjacent edges. It is *path-connected* (another term widely used is *transitive*) if any two nodes of the underlying graph are joined by a directed path of mutually adjacent edges. It is *disconnected* if it is not connected, in which case it breaks up into connected components. The examples include some self-connections, multiple arrows (of the same type or different types), and a multiple self-connection.

3.1 Global Symmetries

Symmetries of ODEs have a strong effect on their solutions (Golubitsky et al. 1988; Golubitsky and Stewart 2002a). We therefore make a few remarks about symmetries here and expand on them later.

A (*global*) *symmetry* of a network is a permutation of its cells that preserves the network architecture: how many arrows of each type input to each cell, and how they are connected in the network. Among the examples in Fig. 3, the two-cell network labelled ‘symmetric coupling’ has symmetry group \mathbb{Z}_2 , generated by the transposition (12). The unidirectional ring has cyclic group symmetry \mathbb{Z}_5 generated by the 5-cycle (12345). So does the bidirectional ring, as drawn, because it has two types of arrow. If the dotted arrows were of the same type as the solid ones, the symmetry group would be the dihedral group \mathbb{D}_5 . The all-to-all connected network has symmetry group \mathbb{S}_5 , consisting of all permutations of the cells.

The other networks illustrated have trivial symmetry group.

Networks can also have ‘local’ symmetries, known formally as input isomorphisms, see Definition 2. These have a significant influence, and are central to network dynamics, but their role is less transparent.

4 Admissible Maps

We repeat that the central role of a coupled cell network is to encode a space of ODEs whose couplings model the architecture of the network. We then seek features that are ‘typical’ for all ODEs in this space, as explained below. The formalism does not, and is not intended to, pin down a specific ODE. Instead, it determines a class of ODEs compatible with the network, allowing us to distinguish features that are typical of this class from those that are not.

4.1 Cell Phase Spaces

In order to define an ODE, or a class of them, we need to specify the variables, or *phase space*, and the functions that appear as components of the vector field. For networks, we choose variables that respect the network structure. For each cell $c \in \mathcal{C}$ define a *cell phase space* P_c . In general, this should be a smooth manifold. To avoid too much manifold formalism (tangent bundles on the like) we will assume for most of this chapter that $P_c = \mathbb{R}^{n_c}$ is a real vector space. For local bifurcation theory, this case is all we need. However, for some purposes other choices are necessary; in particular, systems of *phase oscillators* correspond to choosing $P_c = \mathbb{S}^1$, the circle.

The role of cell-equivalence is to identify the phase spaces of equivalent cells. That is, if cells c, d are cell-equivalent then P_c and P_d are required to be equal. The overall phase space of the network is the direct sum

$$P = \bigoplus_{c \in \mathcal{C}} P_c$$

4.2 Input Sets

Networks have a new feature, compared to symmetric systems. Not only can they have global symmetries: they can have ‘partial symmetries’ in which some subnetwork has the same structure as some other subnetwork. This concept is most useful when the subnetworks concerned encode the inputs to cells, because a partial symmetry of this type in effect states that the cells concerned ‘have the same kinds of couplings’.

Definition 2 Let $c, d \in \mathcal{C}$. The *input set* of c is the set $I(c)$ of all arrows e such that $\mathcal{H}(e) = c$.

An *input isomorphism* $\beta : I(c) \rightarrow I(d)$ is a bijection between their input sets that preserves arrow type. That is, e is arrow-equivalent to $\beta(e)$ for all β and all $e \in I(c)$. (It follows that $\beta^{-1}(f)$ is arrow-equivalent to f for all $f \in I(d)$.)

If there exists an input-isomorphism $\beta : I(c) \rightarrow I(d)$ we say that c, d are *input-isomorphic* or *input-equivalent*.

The input set is important because it encodes which cells are connected to which, and by which type of arrow. Input-equivalent cells receive the same couplings from the relevant cells of the network.

Example 3 Let \mathcal{G} be the network of Fig. 4. The input sets of the five cells are shown in Fig. 5.

It is clear from the figure that cells 1 and 2 are input-isomorphic, and so are cells 3 and 5. However, cells 1 and 3 are not input-isomorphic. Although they both receive two inputs, the arrow-types are different.

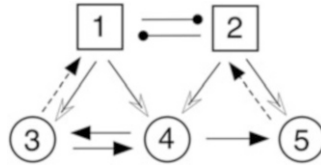


Fig. 4 A 5-cell network

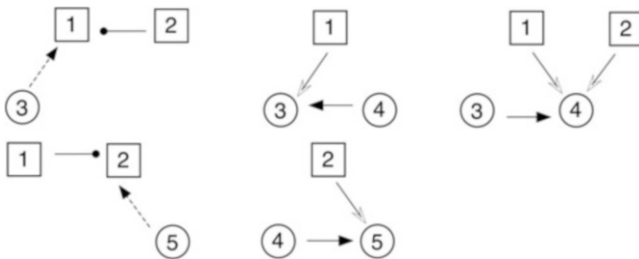


Fig. 5 Input sets of the 5-cell network. From left to right: $I(1), I(3), I(4), I(2), I(5)$. Strictly, the arrows constitute the input set, but is convenient to show the head and tail cells as well

The set of all input-isomorphisms from cell c to cell d is denoted by

$$B(c, d)$$

These maps are closed under composition in the following sense. If a, b, c are input-equivalent cells, and $\alpha \in B(a, b), \beta \in B(b, c)$, then $\beta\alpha \in B(a, c)$. Composition is not always defined, but when it is, it is associative.

It follows that for any $c \in \mathcal{C}$ the set $B(c, c)$ is a finite group, the *vertex group* of c . ‘Cell group’ might be a more consistent choice of terminology, but this choice avoids overusing the word ‘cell’.

The union $\mathcal{B} = \bigcup_{c,d} B(c, d)$ is in general not a group, because its elements may not compose. Technically, it is a *groupoid*, Brandt (1927), Higgins (1971) and Brown (1987). The groupoid structure can be viewed as a side-effect of the formalism rather than a vital ingredient. It does have a few useful implications, but its main influence to date has been through the vertex groups. See Golubitsky and Stewart (2006) for further discussion.

4.3 Admissible Maps

To each network \mathcal{G} , and each specific choice of cell coordinates x_c that preserves cell type, we associate the space of all ODEs that are compatible with the network architecture. Such ODEs are called *coupled cell systems* or *network ODEs*.

To define these ODEs, we associate to \mathcal{G} a space of *admissible vector fields*. When all P_i are real vector spaces, we refer to these as *admissible maps*. (The tangent space of \mathbb{R}^n at any point is \mathbb{R}^n .)

Example 4 Consider once more the network of Fig. 4. Here there are two cell types and four arrow types.

Choose coordinates $(x_1, x_2, x_3, x_4, x_5)$ for cells 1, 2, 3, 4, 5. By cell-equivalence, $P_2 = P_1$ and $P_3 = P_5$. Admissible ODEs take the following form:

$$\begin{aligned} \dot{x}_1 &= f(x_1, x_2, x_3) \\ \dot{x}_2 &= f(x_2, x_1, x_5) \\ \dot{x}_3 &= g(x_3, x_1, x_4) \\ \dot{x}_4 &= h(x_4, \overline{x_1, x_2}, x_3) \\ \dot{x}_5 &= g(x_5, x_2, x_4) \end{aligned} \tag{3}$$

for arbitrary smooth functions

$$\begin{aligned} f &: P_1 \times P_2 \times P_3 \rightarrow P_1 \\ g &: P_3 \times P_1 \times P_4 \rightarrow P_3 \\ h &: P_4 \times P_1 \times P_2 \times P_3 \rightarrow P_4 \end{aligned}$$

(The overline in the fourth equation indicates symmetry, see below.)

First, we explain how this form is obtained from the network. Consider the first equation, for cell 1. The vector field component is $f(x_1, x_2, x_3)$. The first entry x_1 is the cell coordinate, and it represents the internal state of that cell. The other two entries x_2, x_3 are the input coordinates—those of the tail cells of the two input arrows to cell 1, as in Fig. 5. Similarly the equations for cells 2–5 comprise the cell coordinate and the input coordinates, with the cell coordinated being distinguished. We do this because the cell coordinate is not represented by an arrow. (It would be possible to add an explicit self-connection to represent this variable; however, this arrow would naturally be distinguished from any other self-connections in any case.)

A glance at Fig. 5 shows that cells 1 and 2 are input-equivalent; that is, they have the same input sets aside from the numbering of cells. Each cell receives one dashed arrow and one arrow with a dot. Admissibility means that the same function f occurs for cells 1 and 2. The variables are written in an order that respects this equivalence: corresponding variables come from tail cells of arrows of the same type.

The equation for cell 3 has a different function g , because cell 3 is not input-equivalent to cells 1 or 2. Because cell 5 is input-equivalent to cell 3, we use the same g in that equation, with variables again corresponding via the input isomorphism.

In cell 4 we encounter a new feature. Two input arrows are equivalent, those from cells 1 and 2. Therefore there exists an input-isomorphism from $I(4)$ to itself,

which swaps these two arrows. Admissibility requires h to be symmetric in those two variables; that is, $h(x_4, x_1, x_2, x_3) \equiv h(x_4, x_2, x_1, x_3)$. Conventionally the overline on the variables x_1, x_2 in (3) indicates this symmetry.

We mention one feature of the formalism that is sometimes misunderstood. When symmetries of this kind are not appropriate in a model, they should be removed by drawing the network using distinct arrow types. Symmetry is an *option*, not a general requirement.

We now describe, informally, a procedure for writing down admissible maps. Formal definitions are given in Golubitsky et al. (2005), Sect. 3.

For each cell $c \in \mathcal{C}$, choose *cell coordinates* x_c on P_c . (In general, x_c may be multidimensional.) Phase space P then comprises all n -tuples

$$x = (x_c)_{c \in \mathcal{C}}$$

A vector field on P , adapted to cell coordinates, comprises components $f_c, c \in \mathcal{C}$ such that

$$f_c : P \rightarrow P_c$$

For admissibility we impose extra conditions on the f_c that reflect network architecture, as follows:

Definition 3 Let \mathcal{G} be a network. A vector field $f : P \rightarrow P$ is \mathcal{G} -admissible if:

- (1) *Domain Condition*: For every cell c , the component f_c depends only on the cell variable x_c and the input variables $x_{\mathcal{I}(e)}$ where $e \in I(c)$.
- (2) *Symmetry Condition*: If c is a cell, f_c is invariant under all permutations of tail cell coordinates for equivalent input arrows.
- (3) *Pullback Condition*: If cells $c \neq d$ are input-equivalent, the components f_c, f_d are identical as functions. The variables to which they are applied correspond under some (hence any, by condition (2)) input-isomorphism.

Formally, conditions (2) and (3) are combined into a single pullback condition applying to any pair c, d of cells, equal or different.

Example 4 exhibits consequences of all three conditions.

Associated with any admissible map f is an *admissible ODE* or *coupled cell system*

$$\frac{dx}{dt} = f(x) \tag{4}$$

If f also depends on a (possibly multidimensional) parameter λ , and is admissible as a function of x for any fixed λ , we have an *admissible family* of maps and ODEs. Such families arise in bifurcation theory.

4.4 Strongly Admissible Maps

A special class of admissible maps plays a key role in the theory, mainly as a technical tool in proofs. Equivariant dynamics has a very useful feature: composing two equivariant maps yields an equivariant map. However, simple examples show that admissible maps often lack this property. It can be regained by considering a more restrictive class of maps:

Definition 4 A *strongly admissible* map is a map g such that:

- (1) g_c depends only on x_c for each cell c .
- (2) If c, d are cell-equivalent then $g_c = g_d$.

It follows that $g(x) = (g_1(x_1), \dots, g_n(x_n))$, where $g_c = g_d$ whenever c, d are cell-equivalent.

Proposition 1 Let $f : P \rightarrow P$ be admissible and let $g : P \rightarrow P$ be strongly admissible. Then

- (1) If g is invertible (that is, a diffeomorphism) then g^{-1} is also strongly admissible.
- (2) Both fg and gf are admissible.

For some networks, other types of map can compose with admissible maps to give admissible maps. See Golubitsky and Stewart (2016b).

5 Global Symmetries

The formalism for networks introduced in Golubitsky et al. (2005) and Stewart et al. (2003) originally emerged from symmetric dynamics, specifically symmetric networks of coupled oscillators, for example Golubitsky and Stewart (1986). We enlarge on our earlier remarks about global symmetry and make them more precise.

Definition 5 Let \mathcal{G} be a network with cells \mathcal{C} and arrows \mathcal{E} . A (*global*) *symmetry* of \mathcal{G} is a permutation π of \mathcal{C} such that the set of arrows from cell c to cell d is isomorphic to the set of arrows from cell $\pi(c)$ to cell $\pi(d)$. That is, the number of arrows of given type is the same in both cases. (It therefore extends naturally to a permutation acting on \mathcal{E} that preserves arrow-type, but it is more convenient to consider the action on cells. The two formulations are equivalent.)

The (*global*) *symmetry group* of \mathcal{G} is the group formed by all such permutations π , and is denoted by $\text{Sym}(\mathcal{G})$.

The action of π on arrows induces one on cells, by requiring $\pi(\mathcal{T}(e)) = \mathcal{T}(\pi(e))$, or $\pi(\mathcal{H}(e)) = \mathcal{H}(\pi(e))$, or both. (These conditions are consistent because equivalent arrows have equivalent heads and tails.)

There is a connection between admissible maps and symmetric (that is, *equivariant*) maps. These satisfy

$$f(\pi(x)) = \pi f(x)$$

where π acts by permuting indices on x_c and f_c .

Theorem 1 Any \mathcal{G} -admissible map is $\text{Sym}(\mathcal{G})$ -equivariant.

Example 5 In general the converse is not true: equivariant maps need not be admissible. The ‘easy’ way for this to occur is when the functions have the wrong domains. But satisfying the domain condition and being equivariant need not imply admissibility. To see why, consider Fig. 6.

This network has dihedral group \mathbb{D}_5 symmetry, determined by all rotations and reflections of the pentagon. There are two types of arrow: short-range (solid) and long-range (dashed).

Consider a global symmetry that fixes cell 1. It is either the identity, or it acts on cells by the reflectional permutation (25)(34).

The vertex group $B(1, 1)$ is larger. Because there are no multiple arrows, we can define its action on arrows by considering the effect on their tail cells. It contains the identity, (25)(34), but also (25) and (34) on their own. Here (25) interchanges the short arrows inputting to cell 1, and (34) interchanges the long arrows inputting to cell 1.

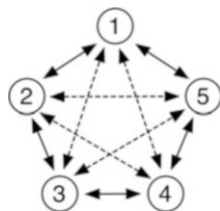
The map

$$f = \begin{bmatrix} x_2x_4 + x_3x_5 \\ x_1x_4 + x_3x_5 \\ x_1x_4 + x_2x_5 \\ x_1x_3 + x_2x_5 \\ x_1x_3 + x_2x_4 \end{bmatrix}$$

is \mathbb{D}_5 -equivariant but not admissible. It is obtained by making f_1 invariant under (25)(34) but not under (25) or (34). Then we use pullback to define the other components.

An analogous admissible map would have $f_1(x) = x_2x_4 + x_2x_3 + x_3x_5 + x_4x_5$, invariant under the whole of $B(1, 1)$.

Fig. 6 Network with dihedral group \mathbb{D}_5 symmetry



Examples like this need to be borne in mind when applying equivariant dynamics and bifurcation theory to symmetric networks. In principle the extra constraints on admissible maps could change the generic behaviour. This effect occurs, for example, in steady-state bifurcation for some regular networks (Stewart 2014; Stewart and Golubitsky 2011), causing higher singularities to be generic. Such networks, however, are very unusual.

5.1 Fixed-Point Subspaces

In equivariant bifurcation theory, it is proved that any symmetric ODE possesses a class of subspaces that are invariant under any equivariant map. These are the *fixed-point subspaces* of subgroups Σ of the overall symmetry group Γ , defined by

$$\text{Fix}(\Sigma) = \{x : \sigma x = x \ \forall \sigma \in \Sigma\}$$

Suppose that the system concerned is an admissible ODE for a symmetric network. Since all admissible maps are equivariant, $\text{Fix}(\Sigma)$ is invariant under all admissible maps. Antoneli and Stewart (2006, 2007, 2008) explore links between symmetry and synchrony in networks, showing in particular that there can be subspaces other than fixed-point subspaces with this invariance property—even when arrows are deemed equivalent if and only if they are related by a symmetry. This again shows that it is necessary to be careful when applying equivariant dynamics to symmetric networks; however, examples of this type are also rare.

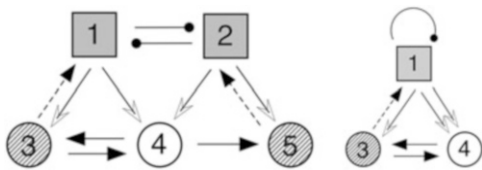
6 Quotient Networks and Synchrony

A basic question in network dynamics is: when are two cells synchronous? We define synchrony by identical time-series: if $x = x(t)$ is a solution of an admissible ODE, we say that cells c, d are *synchronous* on x if $x_c(t) = x_d(t)$ for all times t . This definition is a strong one, and many applications employ a weaker version in which the time series are close together, or are equal most of the time. However, it lets us prove precise theorems that yield useful insights.

A very strong kind of synchrony occurs for *any* admissible ODE, and is associated with a subspace of phase space that is invariant under all admissible maps f . Here cells synchronise in clusters, so that all cells in a given cluster have identical time-series. To introduce this idea we return to Example 4.

Example 6 In Fig. 7 (right) we have assigned ‘colours’ to the cells, shown as grey shading and diagonal hatching. In this example, cells 1 and 2 have the same colour, and cells 3 and 5 have the same colour. So the set of cells \mathcal{C} is partitioned into three subsets, determined by ‘same colour’; namely $\{1, 2\}, \{3, 5\}, \{4\}$. (Technically

Fig. 7 *Left:* Balanced colouring of the 5-cell network. *Right:* Corresponding quotient network



these can be considered as the equivalence classes for the equivalence relation ‘same colour’, but intuitively it seems simpler to think about colours.)

A given network can be coloured in many ways, but this choice has a special feature, which becomes apparent if we look for solutions in which cells of the same colour are synchronous. That is, we set $x_1 = x_2 = u, x_3 = x_5 = v, x_4 = w$, so

$$(x_1, x_2, x_3, x_4, x_5) = (u, u, v, w, v)$$

The admissible ODE (3) now becomes

$$\begin{aligned} \dot{u} &= f(u, u, v) \\ \dot{u} &= f(u, u, v) \\ \dot{v} &= g(v, u, w) \\ \dot{w} &= h(w, u, u, v) \\ \dot{v} &= g(v, u, w) \end{aligned} \tag{5}$$

Although we have five equations in only three unknowns, the system is not overdetermined because the second equation is the same as the first, and the fifth is the same as the third.

If we project (u, u, v, w, v) to (u, v, w) we get a *restricted ODE*

$$\begin{aligned} \dot{u} &= f(u, u, v) \\ \dot{v} &= g(v, u, w) \\ \dot{w} &= h(w, u, u, v) \end{aligned}$$

We recognise this as an admissible ODE for a smaller network, in which cells of the same colour are identified with a single cell, and input sets of arrows remain unchanged (but tail cells with the same colour are identified). This *quotient network* is shown in Fig. 7 (left).

This construction works because the space

$$\Delta = \{(u, u, v, w, v) : u \in P_1, v \in P_3, w \in P_4\}$$

is invariant under all admissible maps, hence under the flow of the corresponding ODEs. It has the pleasant feature that the space of restricted ODEs is *precisely* the space of admissible ODEs for the quotient network, provided the same cell coordinates are used.

Here the quotient network has a double arrow from cell 1 to cell 4, and a self-connection from cell 1 to itself. However, the original network does not have multiple arrows (pointing in the same direction). Multiple arrows and self-connections are natural consequences of the restricted ODE. The equation for \dot{w} involves two entries u , corresponding to the two arrows from cell 1 to cell 4; the equation for \dot{u} has two entries u : one for the cell coordinate and another for the input coordinate from cell 2. The ‘single-arrow’ network formalism in Stewart et al. (2003) failed to take proper account of this effect, leading to complications when characterising restricted ODEs (Dias and Stewart 2004). The modified ‘multi-arrow’ formalism of Golubitsky et al. (2005) relates the space of restricted ODEs to a network in a satisfactory manner by permitting multiple arrows and self-connections.

7 Balanced Colourings

It so happens that in Fig. 7 cells are coloured according to input-equivalence. However, this type of colouring does not always produce a consistent synchrony relation. The next step is to characterise those that do.

Definition 6 A *colouring* of a network \mathcal{G} is a map

$$k : \mathcal{C} \rightarrow K$$

where K is a finite set, whose members are called *colours*.

We say that c, d have the same colour if $k(c) = k(d)$, and write $c \sim_k d$.

A colouring k of a network is *balanced* if whenever cells c, d have the same colour, there exists an input isomorphism $\beta : I(c) \rightarrow I(d)$ such that i and $\beta(i)$ have the same colour for all $i \in \mathcal{I}(I(i))$.

Informally, a colouring is balanced if there exists a *colour-preserving* input isomorphism for any two cells of the same colour. In particular, cells of the same colour must be input-equivalent, so a balanced colouring is a refinement of input equivalence. That is, if $c \sim_k d$ then $c \sim_I d$.

Definition 7 The *polydiagonal* defined by a colouring k of \mathcal{G} is the space

$$\Delta_k = \{x \in P : k(c) = k(d) \implies x_c = x_d\}$$

That is, cells of the same colour are synchronous for $x \in \Delta$.

Theorem 2 A polydiagonal Δ_k is invariant under every admissible map if and only if k is balanced.

One consequence is that when k is balanced, initial conditions that have the pattern of synchrony defined by k (that is, lie in Δ_k) give rise to solutions that remain

inside Δ_k . However, this result does not guarantee that the pattern of synchrony is stable: perturbations that break the synchrony could cause the solution to deviate from Δ_k instead of returning close to it. This kind of stability depends on the admissible vector field; more precisely, on its component transverse to Δ_k .

Definition 8 Let k be a balanced colouring on \mathcal{G} , with colour set K . The associated quotient network \mathcal{G}_k has K as its set of cells (that is, there is one cell per colour).

The cell type of cell $i \in K$ is that of any cell $c \in \mathcal{C}$ with colour i (that is, $k(c) = i$).

The arrows in $I(i)$ in \mathcal{G}_k are obtained from the input set $I(c)$ of any cell c with colour i by copying each arrow e to create an arrow with head $k(\mathcal{H}(e))$ and tail $k(\mathcal{T}(e))$, of the same type as e .

The set of arrows of \mathcal{G}_k is the union of the $I(i)$ as i runs through K .

Deville and Lerman (2015) have reformulated the notion of quotient in a more general manner, in terms of network fibrations. Nijholt et al. (2016) have developed this idea in a very interesting manner to set up a form of semigroup equivariance for some classes for networks, which explains many hitherto puzzling phenomena.

Example 7 We now return to Example 6 in the light of the above definition of a balanced colouring.

First, we check that the colouring in Fig. 7 (left) is balanced.

Cells 1 and 2 have the same colour. So we must check that their input sets are coloured in the same manner.

Cell 1 has two input arrows: one from cell 2 (with a dot for its head) and one from cell 3 (dashed line).

Cell 2 has two input arrows: one from cell 2 (with a dot for its head) and one from cell 3 (dashed line).

The tail cells are (2,3) and (1,5) respectively. Corresponding cells 1 and 2 have the same colour, and corresponding cells 3 and 5 have the same colour.

Similarly, cells 3 and 5 have the same colour and their input sets match up in a way that preserves colours.

Finally, cell 4 has a different colour from all other cells so there is nothing more to check.

Figure 7(right) shows the corresponding quotient network. This has one cell for each colour. For convenience we label these by representatives 1, 3, 4 of those colours. Arrows are drawn to mimic the input sets in the original network, Fig. 7(left).

We emphasise that although in this particular case colours correspond to input-equivalence classes of cells, colouring by input-equivalence need not be balanced. On the other hand, many other balanced colourings may exist, depending on the network.

Theorem 2 is the first and weakest in a series of results that demonstrate the central role played by balanced colourings. Intuitively, the result is straightforward: if two cells remain synchronised as time passes, the inputs to those cells must also be synchronised. however, this does not necessarily imply that the *states* of those input

cells are synchronised. Nonetheless, this ought to be the case for most admissible vector fields, and the proof of Theorem 2 is relatively straightforward: it just requires a sensible choice of admissible vector field.

Theorem 3 *Let k be a balanced colouring of \mathcal{G} . Then*

- (1) *The restriction of any \mathcal{G} -admissible map to Δ_k is \mathcal{G}_k -admissible.*
- (2) *Every \mathcal{G}_k -admissible map is a restriction to Δ_k of a \mathcal{G} -admissible map.*

Another way to say (2) is that every \mathcal{G}_k -admissible map on Δ_k lifts to a \mathcal{G} -admissible map on P .

If f is \mathcal{G} -admissible, the restricted map $f|_{\Delta_k}$ determines the dynamics under f of the synchronous clusters determined by the colouring k .

8 Rigid Synchrony for Equilibria

In dynamical systems theory an equilibrium x^0 of an ODE $\dot{x} = f(x)$ is said to be *hyperbolic* if no eigenvalues of the derivative (or Jacobian) $D_x f|_{x^0}$ lie on the imaginary axis. It can then be proved that if g is a small perturbation of f there exists a unique equilibrium y^0 of the ODE $\dot{x} = g(x)$ with y^0 near x^0 . See Hirsch and Smale (1974) and Guckenheimer and Holmes (1983).

Definition 9 A hyperbolic equilibrium x^0 of a network ODE $\dot{x} = f(x)$ is *rigid* if its pattern of synchrony is preserved by any sufficiently small admissible perturbation. That is, suppose that $g = f + \varepsilon p$ is any admissible perturbation of f and ε is sufficiently small. Let y^0 be the unique perturbed equilibrium near x^0 . Then whenever $x_c^0 = x_d^0$, we have $y_c^0 = y_d^0$.

Golubitsky et al. (2005) prove the Rigid Equilibrium Theorem:

Theorem 4 *Let x^0 be a hyperbolic rigid equilibrium of a network ODE. Define the relation \sim by $c \sim d \iff y_c^0 = y_d^0$ for the perturbed equilibrium y^0 of any sufficiently small admissible perturbation of f . Then \sim is balanced.*

Briefly: rigid synchrony patterns of equilibria are balanced. Another very different proof can be found in Aldis (2010).

9 Rigid Synchrony and Phase Relations for Periodic States

The Rigid Equilibrium Theorem 4 has an analogue for periodic states. We introduce this idea with an example, the coupled FitzHugh-Nagumo equations (1) represented by a ring of three identical cells with unidirectional coupling as in Fig. 2. This network has \mathbb{Z}_3 symmetry, which has implications for periodic states.

When $a = b = \gamma = 0.5$ and $c = 0.8$, the origin is a stable equilibrium for the full six-dimensional system, and the cells undergo a *synchronous* oscillation. That is, their time-series are identical. However, when $a = b = \gamma = 0.5$ and $c = 2$, the system has a stable periodic state in which successive cells are one third of a period out of phase. Figure 8, which shows the pattern for the v_j ; the same pattern occurs for the w_j . This state is a *discrete rotating wave*. It has *spatio-temporal* symmetry:

$$x_2(t) = x_1(t - T/3) \quad x_3(t) = x_1(t - 2T/3)$$

That is, $x(t)$ is invariant if we permute the labels using the 3-cycle $\rho = (123)$ and shift phase by $T/3$. So

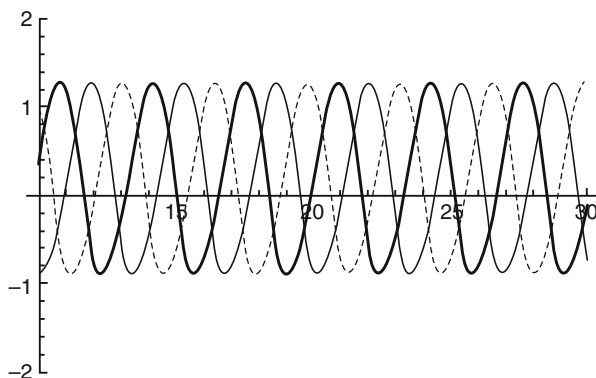
$$\rho x(t + T/3) = x(t)$$

where $x_i = (v_i, w_i)$. Thus $x(t)$ is fixed by $(\rho, T/3) \in \Gamma \times \mathbb{S}^1$, where \mathbb{S}^1 is the circle group of phase shifts modulo the period.

The Equivariant Hopf Theorem (Golubitsky and Schaeffer 1985; Golubitsky and Stewart 2002b; Golubitsky et al. 1988) provides conditions under which phase-related states of this type occur; the H/K Theorem (Buono and Golubitsky 2001) classifies the possible spatio-temporal symmetries. This theorem has been applied to analyse central pattern generators for quadruped locomotion. Different gait patterns exhibit different phase relations between various legs, and these can be read off from the network structure of the central pattern generator by considering symmetries. See Buono (2001), Buono and Golubitsky (2001), Collins and Stewart (1993a,b), Golubitsky and Stewart (2002a).

Example 8 Figure 9 shows a chain of 7 identical cells with identical couplings, driven by a ring of three cells 1, 2, 3. (There is nothing special about the numbers here, and both 3 and 7 can be replaced by arbitrary positive integers for appropriate chains.)

Fig. 8 Periodic oscillations of the 3-cell ring exhibiting a $\frac{1}{3}$ -period out of phase periodic solution. Time series of v_1 (thick solid), v_2 (thin solid), v_3 (dashed)



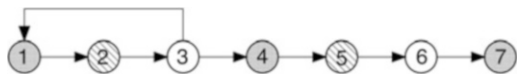


Fig. 9 Balanced colouring of a feed-forward chain leading to travelling wave

The colouring shown is balanced, and the corresponding quotient network is the \mathbb{Z}_3 -symmetric ring of Fig. 2. With suitable admissible equations, this ring supports a rotating wave with $1/3$ period phase shifts as above. Therefore, lifting, the original chain supports a state with three synchronous clusters, formed by cells $\{3k + 1\}, \{3k + 2\}, \{3k\}$, with x_i, x_{i+1} being synchronous except for a phase shift of one third of a period. The effect is similar to a travelling wave in which cells 1, 2, 3, 4, . . . ‘fire’ in turn, and cells $i, i - 3, i - 6, \dots$ are synchronous.

The 7-cell chain has no global symmetry, but its symmetric 3-cell quotient implies that certain synchronised states in the chain can behave in a manner that is typical of symmetric rings of cells.

This example motivates (and illustrates the answer to) an interesting converse question: if certain cells have identical time-series apart from a phase shift, does this imply some kind of global symmetry of the network? Remarkably, the answer, subject to reasonable conditions, is ‘yes’. But, as Example 8 shows, we must first pass to a quotient.

The main condition required is rigidity: the phase relation must stay unchanged (as a proportion of the period) after any sufficiently small admissible perturbation of the underlying ODE. To state this precisely, we need the following concept from dynamical systems theory. A periodic state $x(t)$ is *hyperbolic* if it has no Floquet exponent on the imaginary axis. Hyperbolicity implies that after a small perturbation of the vector field there exists a unique periodic orbit near $x(t)$ in the C^1 topology, Katok and Hasselblatt (1995), and its period is near that of $x(t)$. Thus we may talk of ‘the’ perturbed periodic state.

Definition 10 Suppose that $x(t)$ is a hyperbolic periodic state of period T of a \mathcal{G} -admissible ODE. A phase relation

$$x_c(t) = x_d(t - \theta) \quad c, d \in \mathcal{C}, \quad \theta \in \mathbb{R}/T\mathbb{Z} \tag{6}$$

is *rigid* if for all sufficiently small admissible perturbations the perturbed periodic state $\tilde{x}(t)$ satisfies

$$\tilde{x}_c(t) = \tilde{x}_d(t - \theta) \quad c, d \in \mathcal{C}, \quad \theta \in \mathbb{R}/\tilde{T}\mathbb{Z}$$

where \tilde{T} is the period of $\tilde{x}(t)$.

The $T/3$ and $2T/3$ phase shifts in the rotating wave state for the coupled FitzHugh-Nagumo system is rigid. This can be proved for any rotating wave state arising by Hopf bifurcation in a symmetric system, indeed for any such state consistent with the H/K Theorem.

When $\theta = 0$ in (6) we say that cells c and d are *rigidly synchronous*.

Intuitively, whenever (6) holds, we expect the states $x_{I(c)}(t)$ and $x_{I(d)}(t)$ of the input sets of cells c and d to be phase-related by the same θ , up to some input isomorphism. Taken literally, this statement is false: the inputs states could differ in a way that does not affect the coupling to cells c, d . But we expect such a relationship to be destroyed by most small perturbations. For several years this belief was conjectural (Stewart and Parker 2007); the main difficulty in proving it was to keep track of how the periodic state perturbed.

We now introduce a mild technical condition, which some authors include in the definition of a coupled cell network:

Definition 11 A network is *cell-homogeneous* if all cell-equivalent cells are input-equivalent.

Assuming this condition, Golubitsky et al. (2010) proved the Rigid Synchrony Theorem:

Theorem 5 *Suppose that \mathcal{G} is a cell-homogeneous path-connected network and two cells c, d are rigidly synchronous. Then there exists an input isomorphism $\beta : I(c) \rightarrow I(d)$ such that for all $j \in \mathcal{T}(d)$ cells j and $\beta^*(j)$ are rigidly synchronous.*

Corollary 1 *Suppose that \mathcal{G} is a cell-homogeneous path-connected network. Then the colouring K in which cells have the same colour if and only if they are rigidly synchronous is balanced.*

Their method is inspired by singularity theory, and requires studying a space of perturbations large enough to destroy any spurious synchrony but small enough to control. Shortly afterwards, Golubitsky et al. (2012) extended their methods to handle nonzero phase shifts, obtaining the Rigid Phase Theorem:

Theorem 6 *Suppose that \mathcal{G} is a cell-homogeneous path-connected network and two cells c, d are rigidly phase related by a phase shift that is a proportion θ of the period of the perturbed periodic state. Then there exists an input isomorphism $\beta : I(c) \rightarrow I(d)$ such that for all $j \in \mathcal{T}(d)$ cells j and $\beta^*(j)$ are phase related by a phase shift that is the same proportion of the period of the perturbed periodic state.*

It is conjectured that the condition of cell-homogeneity can be removed, and it seems likely that the methods of Golubitsky et al. (2010, 2012) can be modified to prove this, but this issue is currently unresolved.

A key consequence had already been observed in Stewart and Parker (2008):

Theorem 7 *Suppose that \mathcal{G} is a cell-homogeneous path-connected network and two cells c, d are rigidly phase related by a phase shift that is a proportion θ of the period of the perturbed periodic state. Let \mathcal{G}_1 be the quotient of \mathcal{G} by the balanced coloring corresponding to rigid synchrony of cells. Then there exist integers m, k such that $\theta = m/k$, \mathcal{G}_1 has a global group of symmetries that is the cyclic group \mathbb{Z}_k , and all rigid phase relations between cells are determined by a discrete rotating wave consistent with these symmetries.*

Informally: whenever a rigid phase shift is observed in a periodic state for a path-connected network, it is a consequence of a *global* cyclic-group symmetry of the quotient network in which rigidly synchronous cells are identified.

10 Bifurcations

Informally, a bifurcation occurs in a family of ODEs

$$\frac{dx}{dt} = f(x, \lambda)$$

with a parameter λ when the qualitative description of states changes near some parameter value λ_0 . For example the number of steady states may change as λ passes through λ_0 , or a stable steady state may become unstable and throw off a periodic cycle.

Local bifurcation, where the states branch along different curves in (x, λ) -space, is governed by the eigenvalues of the Jacobian matrix $J = D_x f|_{(x, \lambda)}$. If an eigenvalue of J is zero at some point (x_0, λ_0) then typically a new branch of steady states appears. If a complex conjugate pair of eigenvalues are purely imaginary, equal to $\pm i\omega$, then typically there is a Hopf bifurcation to a branch of time-periodic states with frequency close to $2\pi/\omega$, Hassard et al. (1981). Such eigenvalues are said to be critical.

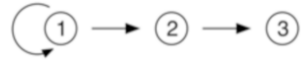
For standard dynamical systems, ‘typically’ here requires the critical eigenvalues to be simple. Moreover, they should pass through the imaginary axis with nonzero speed as λ passes through λ_0 . In equivariant dynamics, symmetry constraints can force eigenvalues to be multiple, and new phenomena occur. A notable one is spontaneous symmetry-breaking, where solutions have less symmetry than the equations (Golubitsky and Stewart 2002a; Golubitsky et al. 1988).

In networks, local bifurcation is more complicated. The network architecture can have a strong effect not only on the eigenvalues, but also on the nonlinearities along the bifurcating branch. For example, there exist networks for which ‘typical’ steady-state bifurcation is more degenerate, in a singularity-theoretic sense, than the usual transcritical or pitchfork bifurcations. This affects the typical growth rate of the bifurcating branch (Stewart 2014; Stewart and Golubitsky 2011).

Instead of symmetry-breaking bifurcations, networks can exhibit synchrony-breaking bifurcations. Here a state with some pattern of synchrony loses stability and the pattern of synchrony changes: some cells that were synchronous cease to be synchronous. The interplay between network architecture and eigenvalues (and eigenvectors) of the Jacobian plays a central role in the theory of synchrony-breaking bifurcations. Rink and Sanders (2012, 2013a, 2014) explain this relationship in terms of a modified type of equivariance, using semigroups rather than groups.

A very surprising synchrony-breaking bifurcation occurs at Hopf bifurcation in a 3-cell feed-forward network, Fig. 10. Generic Hopf bifurcation in a general

Fig. 10 A 3-cell
feed-forward network



dynamical system creates a bifurcating branch of equilibria whose amplitude grows like $\lambda^{1/2}$. However, Elmhirst and Golubitsky (2006) proved that typically there is a bifurcating branch of periodic states in which cell 1 is steady, the amplitude of cell 2 grows like $\lambda^{1/2}$, and the amplitude of cell 3 has the anomalous growth rate $\lambda^{1/6}$.

There is an analogous result for a feed-forward chain of m nodes. Hopf bifurcation can then lead to states that grow like $\lambda^{1/18}$ in the fourth node, $\lambda^{1/54}$ in the fifth node, and so on. This has been proved by Rink and Sanders (2013b) using a far-reaching generalisation of the notion of symmetry for networks. See also Rink and Sanders (2012, 2013a, 2014).

There is also a potential application to a nonlinear filter that selects and amplifies periodic oscillations close to a specific frequency (Golubitsky et al. 2009; McCullen et al. 2007).

11 Conclusions

The main message of this chapter is very simple. Networks are becoming increasingly important as models of many real systems, across the whole range of sciences. Moreover, the dynamics of networks has its own special flavour and differs considerably from the standard theory of dynamical systems. There is now a growing understanding of network dynamics, which in particular makes it possible to distinguish typical phenomena common to many networks with a given architecture from special phenomena that depend on the modelling equations. Among them are patterns of synchrony and phase relations, but the approach is not limited to these types of behaviour.

Placing network dynamics in a formal, abstract setting makes the above distinction clear, and offers several benefits, which are already sufficiently interesting to justify setting up such a formalism. Many types of behaviour become comprehensible and natural within this setting. On the other hand, it is important to recognise that the general abstract results must be augmented by special considerations, either for classes of networks with extra structure, or for specific models. The area of network dynamics is developing rapidly with many new results and open questions.

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