Chapter 3
Uniform Global Attractors for Nonautonomous Evolution Inclusions

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Abstract In this note, we prove the existence and provide basic structure properties of compact (in the natural phase space) uniform global attractor for all global weak solutions of the general classes of nonautonomous evolution equations and inclusions that satisfy standard sign and polynomial growth conditions. The obtained results allow to reduce the problem of the complete qualitative investigation of various nonlinear systems into the "small" (compact) part of the natural phase space.

3.1 Introduction and Setting of the Problem

For evolution triple $(V_i; H; V_i^*)$[1] and multivalued map $A_i : \mathbb{R}_+ \times V \rightrightarrows V^*$, $i = 1, 2, \ldots, N$, $N = 1, 2, \ldots,$ we consider a problem of longtime behavior (in the natural phase space H) of all globally defined weak solutions for nonautonomous evolution inclusion

$$y'(t) + \sum_{i=1}^{N} A_i(t, y(t)) \ni \bar{0}, \tag{3.1}$$

as $t \to +\infty$. Let $\langle \cdot, \cdot \rangle_{V_i} : V_i^* \times V_i \to \mathbb{R}$ be the pairing in $V_i^* \times V_i$ that coincides on $H \times V_i$ with the inner product $(\cdot, \cdot)$ in the Hilbert space H.

[1] That is, V_i is a real reflexive separable Banach space continuously and densely embedded into a real Hilbert space H, H is identified with its topologically conjugated space H^*, V_i^* is a dual space to V_i. So, there is a chain of continuous and dense embeddings: $V_i \subset H \equiv H^* \subset V_i^*$ (see, e.g., Gajewski, Gröger, and Zacharias [1, Chap. I]).

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To introduce the assumptions on parameters of Problem (3.1) let us introduce additional constructions. A function $\varphi \in L_\gamma^{\mathrm{loc}}(\mathbb{R}_+)$, $\gamma > 1$, is called *translation bounded* in $L_\gamma^{\mathrm{loc}}(\mathbb{R}_+)$, if

$$\sup_{t\geq 0}\int_t^{t+1} |\varphi(s)|^\gamma ds < +\infty;$$

Chepyzhov and Vishik [2, p. 105]. A function $\varphi \in L_1^{\mathrm{loc}}(\mathbb{R}_+)$ is called *translation uniform integrable (t.u.i.)* in $L_1^{\mathrm{loc}}(\mathbb{R}_+)$, if

$$\lim_{K\to+\infty}\sup_{t\geq 0}\int_t^{t+1} |\varphi(s)|\mathbf{I}\{|\varphi(s)|\geq K\}ds = 0.$$

Note that Dunford–Pettis compactness criterion provides that a function $\varphi \in L_1^{\mathrm{loc}}(\mathbb{R}_+)$ is t.u.i. in $L_1^{\mathrm{loc}}(\mathbb{R}_+)$ if and only if for every sequence of elements $\{\tau_n\}_{n\geq 1} \subset \mathbb{R}_+$ the sequence $\{\varphi(\,\cdot\, + \tau_n)\}_{n\geq 1}$ contains a subsequence which converges weakly in $L_1^{\mathrm{loc}}(\mathbb{R}_+)$. Note that for any $\gamma > 1$ every translation bounded in $L_\gamma^{\mathrm{loc}}(\mathbb{R}_+)$ function is t.u.i. in $L_1^{\mathrm{loc}}(\mathbb{R}_+)$; Gorban et al. [3].

Throughout this paper, we suppose that the listed below assumptions hold:

Assumption 1 Let $p_i \geq 2$, $q_i > 1$ are such that $\frac{1}{p_i} + \frac{1}{q_i} = 1$, for each for $i = 1, 2, \ldots, N$, and the embedding $V_i \subset H$ is compact one, for some for $i = 1, 2, \ldots, N$.

Assumption 2 (*Growth Condition*) There exist a t.u.i. in $L_1^{\mathrm{loc}}(\mathbb{R}_+)$ function $c_1 : \mathbb{R}_+ \to \mathbb{R}_+$ and a constant $c_2 > 0$ such that

$$\max_{i=1}^{N} \|d_i\|_{V_i^*}^q \leq c_1(t) + c_2\sum_{i=1}^{N}\|u\|_{V_i}^p$$

for any $u \in V_i$, $d_i \in A_i(t, u)$, $i = 1, 2, \ldots, N$, and a.e. $t > 0$.

Assumption 3 (*Signed Assumption*) There exists a constant $\alpha > 0$ and a t.u.i. in $L_1^{\mathrm{loc}}(\mathbb{R}_+)$ function $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\sum_{i=1}^{N}\langle d_i, u\rangle_{V_i} \geq \alpha\sum_{i=1}^{N}\|u\|_{V_i}^p - \beta(t)$$

for any $u \in V_i$, $d_i \in A_i(t, u)$, $i = 1, 2, \ldots, N$, and a.e. $t > 0$.

Assumption 4 (*Strong Measurability*) If $C \subseteq V_i^*$ is a closed set, then the set $\{(t, u) \in (0, +\infty) \times V_i \,:\, A_i(t, u) \cap C \neq \emptyset\}$ is a Borel subset in $(0, +\infty) \times V_i$.

Assumption 5 (*Pointwise Pseudomonotonicity*) Let for each $i = 1, 2, \ldots, N$ and a.e. $t > 0$, two assumptions hold:

(a) for every $u \in V_i$ the set $A_i(t, u)$ is nonempty, convex, and weakly compact one in ${V_i}^*$;
(b) if a sequence $\{u_n\}_{n\geq 1}$ converges weakly in V_i toward $u \in V_i$ as $n \to +\infty$, $d_n \in A_i(t, u_n)$ for any $n \geq 1$, and $\limsup_{n\to+\infty} \langle d_n, u_n - u\rangle_{V_i} \leq 0$, then for any $\omega \in V_i$ there exists $d(\omega) \in A_i(t, u)$ such that

$$\liminf_{n\to+\infty} \langle d_n, u_n - \omega\rangle_{V_i} \geq \langle d(\omega), u - \omega\rangle_{V_i}.$$

Let $0 \leq \tau < T < +\infty$. As a *weak solution* of evolution inclusion (3.1) on the interval $[\tau, T]$, we consider an element $u(\cdot)$ of the space $\cap_{i=1}^N L_{p_i}(\tau, T; V_i)$ such that for some $d_i(\cdot) \in L_{q_i}(\tau, T; {V_i}^*)$, $i = 1, 2, \ldots, N$, it is fulfilled:

$$-\int_\tau^T (\xi'(t), y(t))dt + \sum_{i=1}^N \int_\tau^T \langle d_i(t), \xi(t)\rangle_{V_i} dt = 0 \quad \forall \xi \in C_0^\infty([\tau, T]; V_i), \tag{3.2}$$

and $d_i(t) \in A_i(t, y(t))$ for each $i = 1, 2, \ldots, N$ and a.e. $t \in (\tau, T)$.

3.2 Preliminary Properties of Weak Solutions

Zgurovsky and Kasyanov [4, p. 225] provide the existence of a weak solution of Cauchy problem (3.1) with initial data $y(\tau) = y^{(\tau)}$ on the interval $[\tau, T]$, for any $y^{(\tau)} \in H$. For fixed τ and T, such that $0 \leq \tau < T < +\infty$, we denote

$$\mathscr{D}_{\tau,T}(y^{(\tau)}) = \{y(\cdot) \mid y \text{ is a weak solution of (3.1) on } [\tau, T],\ y(\tau) = y^{(\tau)}\}, \quad y^{(\tau)} \in H.$$

We remark that $\mathscr{D}_{\tau,T}(y^{(\tau)}) \neq \emptyset$, if $0 \leq \tau < T < +\infty$ and $y^{(\tau)} \in H$. Moreover, the concatenation of Problem (3.1) weak solutions is a weak solutions too, i.e., if $0 \leq \tau < t < T$, $y^{(\tau)} \in H$, $y(\cdot) \in \mathscr{D}_{\tau,t}(y^{(\tau)})$, and $v(\cdot) \in \mathscr{D}_{t,T}(y(t))$, then

$$z(s) = \begin{cases} y(s), & s \in [\tau, t], \\ v(s), & s \in [t, T], \end{cases}$$

belongs to $\mathscr{D}_{\tau,T}(y^{(\tau)})$; cf. Zgurovsky et al. [5, pp. 55–56].

Gronwall lemma provides that for any finite time interval $[\tau, T] \subset \mathbb{R}_+$ each weak solution y of Problem (3.1) on $[\tau, T]$ satisfies estimates

$$\|y(t)\|_H^2 - 2\int_0^t \beta(\xi)d\xi + 2\alpha\sum_{i=1}^N \int_s^t \|y(\xi)\|_{V_i}^p d\xi \le \|y(s)\|_H^2 - 2\int_0^s \beta(\xi)d\xi, \tag{3.3}$$

$$\|y(t)\|_H^2 \le \|y(s)\|_H^2 e^{-2\alpha\gamma(t-s)} + 2\int_s^t (\beta(\xi)+\alpha\gamma)e^{-2\alpha\gamma(t-\xi)}d\xi, \tag{3.4}$$

where $t, s \in [\tau, T]$, $t \ge s$; γ is a constant that does not depend on y, s, and t; see Zgurovsky and Kasyanov [4, p. 225]. Therefore, any weak solution y of Problem (3.1) on a finite time interval $[\tau, T] \subset \mathbb{R}_+$ can be extended to a global one, defined on $[\tau, +\infty)$.

For each $\tau \ge 0$ and $y^{(\tau)} \in H$ let $\mathscr{D}_\tau(y^{(\tau)})$ be the set of all weak solutions (defined on $[\tau, +\infty)$) of Problem (3.1) with initial data $y(\tau) = y^{(\tau)}$. Let us consider the family $\mathscr{K}_\tau^+ = \cup_{y^{(\tau)}\in H}\mathscr{D}_\tau(y^{(\tau)})$ of all weak solutions of Problem (3.1) defined on the semi-infinite time interval $[\tau, +\infty)$.

Consider the Fréchet space $C^{\mathrm{loc}}(\mathbb{R}_+; H)$. We remark that the sequence $\{f_n\}_{n\ge1}$ converges in $C^{\mathrm{loc}}(\mathbb{R}_+; H)$ toward $f \in C^{\mathrm{loc}}(\mathbb{R}_+; H)$ as $n \to +\infty$ iff the sequence $\{\Pi_{t_1,t_2} f_n\}_{n\ge1}$ converges in $C([t_1, t_2]; H)$ toward $\Pi_{t_1,t_2} f$ as $n \to +\infty$ for any finite interval $[t_1, t_2] \subset \mathbb{R}_+$, where Π_{t_1,t_2} is the restriction operator to the interval $[t_1, t_2]$; Chepyzhov and Vishik [6, p. 918]. We denote $T(h)y(\cdot) = y_h(\cdot)$, where $y_h(t) = y(t+h)$ for any $y \in C^{\mathrm{loc}}(\mathbb{R}_+; H)$ and $t, h \ge 0$.

Let us consider *united trajectory space* that includes all globally defined on any $[\tau, +\infty) \subseteq \mathbb{R}_+$ weak solutions of Problem (3.1) shifted to $\tau = 0$:

$$\mathscr{K}^+ = \mathrm{cl}_{C^{\mathrm{loc}}(\mathbb{R}_+;H)}\left[\bigcup_{\tau\ge0}\left\{y(\,\cdot + \tau) \,:\, y \in \mathscr{K}_\tau^+\right\}\right],$$

where $\mathrm{cl}_{C^{\mathrm{loc}}(\mathbb{R}_+;H)}[\,\cdot\,]$ is the closure in $C^{\mathrm{loc}}(\mathbb{R}_+; H)$. Note that $T(h)\{y(\,\cdot+\tau) : y \in \mathscr{K}_\tau^+\} \subseteq \{y(\,\cdot + \tau + h) : y \in \mathscr{K}_{\tau+h}^+\}$ for any $\tau, h \ge 0$. Moreover,

$$T(h)\mathscr{K}^+ \subseteq \mathscr{K}^+ \text{ for any } h \ge 0,$$

because

$$\rho_{C^{\mathrm{loc}}(\mathbb{R}_+;H)}(T(h)u, T(h)v) \le \rho_{C^{\mathrm{loc}}(\mathbb{R}_+;H)}(u, v) \text{ for any } u, v \in C^{\mathrm{loc}}(\mathbb{R}_+; H),$$

where $\rho_{C^{\mathrm{loc}}(\mathbb{R}_+;H)}$ is a standard metric on Fréchet space $C^{\mathrm{loc}}(\mathbb{R}_+; H)$; Zgurovsky and Kasyanov [4, p. 226].

The following Lemma 3.1 and Theorem 3.1 are keynote for the existence of compact (in the natural phase space H) uniform global attractor for all weak solutions of Problem (3.1).

Lemma 3.1 (Zgurovsky and Kasyanov [4]) *Let Assumptions (1)–(5) hold. Then, there exist positive constants c_3 and c_4 such that the following inequalities hold:*

$$\|y(t)\|_H^2 \le \|y(s)\|_H^2 e^{-c_3(t-s)} + c_4,$$

for each $y \in \mathscr{K}^+$, $t \ge s \ge 0$.

Theorem 3.1 (Zgurovsky and Kasyanov [4]) *Let Assumptions (1)–(5) hold. Let* $\{y_n\}_{n\ge1} \subset \mathscr{K}^+$ *be a bounded in* $L_\infty(\mathbb{R}_+; H)$ *sequence. Then, there exist a subsequence* $\{y_{n_k}\}_{k\ge1} \subset \{y_n\}_{n\ge1}$ *and an element* $y \in \mathscr{K}^+$ *such that*

$$\max_{t\in[\tau,T]} \|y_{n_k}(t) - y(t)\|_H \to 0, \quad k \to +\infty,$$

for any finite time interval $[\tau, T] \subset (0, +\infty)$.

3.3 Uniform Global Attractor for all Weak Solutions of Problem (3.1)

Let us define the multivalued semi-flow (*m-semi-flow*) $G : \mathbb{R}_+ \times H \to 2^H$:

$$G(t, y_0) := \{y(t) \ : \ y(\cdot) \in \mathscr{K}^+ \text{ and } y(0) = y_0\}, \quad t \ge 0, \ y_0 \in H. \tag{3.5}$$

For each $t \ge 0$ and $y_0 \in H$, the set $G(t, y_0)$ is nonempty. Moreover, the following two conditions hold:

(i) $G(0, \cdot) = I$ is the identity map;
(ii) $G(t_1 + t_2, y_0) \subseteq G(t_1, G(t_2, y_0))$, $\forall t_1, t_2 \in \mathbb{R}_+$, $\forall y_0 \in H$,

where $G(t, D) = \bigcup_{y\in D} G(t, y)$, $D \subseteq H$.

We denote by $\mathrm{dist}_H(C, D) = \sup_{c\in C} \inf_{d\in D} \rho(c, d)$ the *Hausdorff semi-distance* between nonempty subsets C and D of the Polish space H. Recall that the set $\mathscr{R} \subset H$ is a *global attractor* of the m-semi-flow G if it satisfies the following conditions:

(i) $\mathscr{R}$ attracts each bounded subset $B \subset H$, i.e.,

$$\mathrm{dist}_H(G(t, B), \mathscr{R}) \to 0, \quad t \to +\infty; \tag{3.6}$$

(ii) $\mathscr{R}$ is negatively semi-invariant set, i.e., $\mathscr{R} \subseteq G(t, \mathscr{R})$ for each $t \ge 0$;
(iii) $\mathscr{R}$ is the minimal set among all nonempty closed subsets $C \subseteq H$ that satisfy (3.6).

The main result of this paper has the following form.

Theorem 3.2 *Let Assumptions (1)–(5) hold. Then, the m-semi-flow G, defined in (3.5), has a compact global attractor* $\mathscr{R}$ *in the phase space* H.

3.4 Proof of Theorem 3.2

Lemma 3.1 and Theorem 3.1 imply the following properties for the m-semiflow G, defined in (3.5):

(a) for each $t \geq 0$, the mapping $G(t, \cdot) : H \to 2^H \setminus \{\emptyset\}$ has a closed graph;
(b) for each $t \geq 0$ and $y_0 \in H$, the set $G(t, y_0)$ is compact in H;
(c) the set $G(1, \tilde{C})$, where $\tilde{C} := \{z \in H \ : \ \|z\|_H^2 < c_4 + 1\}$, is precompact and attracts each bounded subset $C \subset H$.

Indeed, property (a) follows from Theorem 3.1; property (b) directly follows from (a) and Theorem 3.1; property (c) holds, because of Lemma 3.1 and since the set $G(1, \tilde{C})$ is precompact in H (Theorem 3.1).

According to properties (a)–(c), Mel'nik and Valero [7, Theorems 1, 2, Remark 2, Proposition 1] yields that the m-semi-flow G has a compact global attractor $\mathscr{R}$ in the phase space H.

3.5 Conclusions

For the class of nonautonomous differential-operator inclusions with pointwise pseudomonotone operators, the dynamics (as $t \to +\infty$) of all global weak solutions defined on $[0, +\infty)$ is examined. The existence of a compact global attractor in the natural phase space H is proved. The results obtained allow one to study the dynamics of solutions for new classes of evolution inclusions related to nonlinear mathematical models of geophysical and socioeconomic processes and for fields with interaction functions of pseudomonotone type satisfying the power growth and sign conditions. For applications, one can consider new classes of problems with degeneracy, feedback control problems, problems on manifolds, problems with delay, stochastic partial differential equations, etc. (see Balibrea et al. [8]; Hu and Papageorgiou [9]; Gasinski and Papageorgiou [10]; Kasyanov [11]; Kasyanov, Toscano, and Zadoianchuk [12]; Mel'nik and Valero [13]; Denkowski, Migórski, and Papageorgiou [14]; Gasinski and Papageorgiou [10]; Zgurovsky et al. [5]; etc., see, also, [16–31]) involving differential operators of pseudomonotone type and the corresponding choice of the phase spaces. This note is a continuation of Zgurovsky and Kasyanov [4, 15].

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