Chapter 13
Uniform Global Attractor for Nonautonomous Reaction–Diffusion Equations with Carathéodory's Nonlinearity

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Abstract We consider nonautonomous reaction–diffusion system with Carathéodory's nonlinearity. We investigate the long-time dynamics of all globally defined weak solutions under the standard sign and polynomial growth conditions. We obtain new topological properties of solutions, in particular flattening property, prove the existence of uniform global attractor for multivalued semiflow generated by considered problem.

13.1 Introduction and Statement of the Problem

Let $N, M = 1, 2, \ldots$. In a bounded domain $\Omega \subset \mathbf{R}^N$ with sufficiently smooth boundary $\partial\Omega$, we consider the following problem:

$$\begin{cases} u_t = a\Delta u - f(x,t,u), \quad x \in \Omega,\ t > 0, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{13.1}$$

where $u = u(x,t) = (u^{(1)}(x,t), \ldots, u^{(M)}(x,t))$ is unknown vector function, a is real $M \times M$ matrix, $f = f(x,t,u) = (f^{(1)}(x,t,u), \ldots, f^{(M)}(x,t,u))$ is given interaction function.

Note that Problem (13.1) is a nonautonomous reaction–diffusion system. There are a lot of papers on qualitative behavior of solutions for evolution systems of reaction–diffusion type. This is due to theoretical and applied importance of such objects. The partial cases of reaction–diffusion problem are Kolmogorov–Petrovsky–Piskunov equations (the problem on the gene diffusion) [1], models of Belousov–Zhabotinsky reaction [2, 3], Gause–Vitta models [4, 5], and Selkov model for glycolysis [6, 7].

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Reaction–diffusion equations are actively used for modeling various biological and chemical processes.

Remark that existence and properties of global attractors for autonomous reaction–diffusion equations with smooth interaction functions are well-known results (see [8, 9]). The autonomous equations and inclusions without uniqueness are investigated in [10–15]. In [16, 17] for autonomous reaction–diffusion inclusion of subgradient type, the existence of Lyapunov function is obtained, the structure of global attractor is studied, and the application to climatology model is considered. For nonautonomous equations of such type with almost periodic interaction functions, the results on trajectory attractors are obtained in [18]. In [19], the existence of uniform trajectory attractor for nonautonomous Problem (13.1) with Carathéodory's nonlinearity is proved. In this chapter, we prove the existence of uniform global attractor for Problem (13.1).

Remark 13.1 Let $\gamma \geq 1$ and $\mathscr{Y}$ be a real separable Banach space. We consider the Fréchet space $L_\gamma^{\mathrm{loc}}(\mathbb{R}_+;\mathscr{Y})$ of all locally integrable functions with values in $\mathscr{Y}$, i.e., $\varphi \in L_\gamma^{\mathrm{loc}}(\mathbb{R}_+;\mathscr{Y})$ if and only if for any finite interval $[\tau, T] \subset \mathbb{R}_+$ the restriction of φ on $[\tau, T]$ belongs to the space $L_\gamma(\tau, T;\mathscr{Y})$ [19].

Definition 13.1 ([19]) A function $\varphi \in L_1^{\mathrm{loc}}(\mathbb{R}_+; L_1(\Omega)$ is called a translation uniform integrable one in $L_1^{\mathrm{loc}}(\mathbb{R}_+; L_1(\Omega))$, if

$$\lim_{K\to+\infty}\sup_{t\geq 0}\int_t^{t+1}\int_\Omega |\varphi(x,s)|\chi_{\{|\varphi(x,s)|\geq K\}}dxds = 0.$$

Remark 13.2 A function $\varphi \in L_1^{\mathrm{loc}}(\mathbb{R}_+; L_1(\Omega))$ is a translation uniform integrable one in $L_1^{\mathrm{loc}}(\mathbb{R}_+; L_1(\Omega))$ if and only if for every sequence of elements $\{\tau_n\}_{n\geq 1} \subset \mathbb{R}_+$ the sequence $\{\varphi(\cdot + \tau_n)\}_{n\geq 1}$ contains a subsequence which converges weakly in $L_1^{\mathrm{loc}}(\mathbb{R}_+; L_1(\Omega))$.

The following condition

$$\sup_{t\geq 0}\int_t^{t+1} \|\varphi(s)\|_{\mathscr{E}}^{\gamma}ds < +\infty$$

is the sufficient condition for the translation uniform integrability of function φ; see [19].

The Main Assumptions on Parameters of Problem (13.1)

Assumption (A) There exists a positive constant d such that $\frac{1}{2}(a + a^*) \geq dI$, where I is the identity $M \times M$ matrix, a^* is a transposed matrix for a.

Assumption (B) The interaction function $f = (f^{(1)},\ldots,f^{(M)}) : \Omega \times \mathbb{R}_+ \times \mathbb{R}^M \to \mathbb{R}^M$ satisfies the standard Carathéodory's conditions, i.e., $(x,t,y) \to$

$f(x, t, y)$ is continuous map in $y \in \mathbb{R}^M$ for a.e. $(x, t) \in \Omega \times \mathbb{R}_+$, and it is measurable map in $(x, t) \in \Omega \times \mathbb{R}_+$ for any $y \in \mathbb{R}^M$.
Assumption (C) There exist a translation uniform integrable in $L_1^{\mathrm{loc}}(\mathbb{R}_+; L_1(\Omega))$ function $c_1 : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ and a constant $c_2 > 0$ such that

$$\sum_{i=1}^{M} \left| f^{(i)}(x, t, y) \right|^{q_i} \le c_1(x, t) + c_2 \sum_{i=1}^{M} \left| y^{(i)} \right|^{p_i}$$

for any $y = (y^{(1)}, \dots, y^{(M)}) \in \mathbb{R}^M$ and a.e. $(x, t) \in \Omega \times \mathbb{R}_+$, where $p_i \ge 2$ and $q_i > 1$ are such that $\frac{1}{p_i} + \frac{1}{q_i} = 1$ for any $i = 1, 2, \dots, M$.
Assumption (D) There exist a constant $\alpha > 0$ and a translation uniform integrable in $L_1^{\mathrm{loc}}(\mathbb{R}_+; L_1(\Omega))$ function $\beta : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\sum_{i=1}^{M} f^{(i)}(x, t, y) y^{(i)} \ge \alpha \sum_{i=1}^{M} \left| y^{(i)} \right|^{p_i} - \beta(x, t)$$

for any $y = (y^{(1)}, \dots, y^{(M)}) \in \mathbb{R}^M$ and a.e. $(x, t) \in \Omega \times \mathbb{R}_+$.

Consider the evolution triple (V, H, V^*), where $H = (L_2(\Omega))^M$, $V = (H_0^1(\Omega))^M$, and $V^* = (H^{-1}(\Omega))^M$ with standard respective inner products and norms $(\cdot, \cdot)_H$ and $\| \cdot \|_H$, $(\cdot, \cdot)_V$ and $\| \cdot \|_V$, and $(\cdot, \cdot)_{V^*}$ and $\| \cdot \|_{V^*}$.
Let $0 \le \tau < T < +\infty$. Denote

$$\begin{gathered}
\mathbf{L_p}(\Omega) := L_{p_1}(\Omega) \times \ldots \times L_{p_M}(\Omega), \quad \mathbf{L_q}(\Omega) := L_{q_1}(\Omega) \times \ldots \times L_{q_M}(\Omega), \\
\mathbf{L_p}(\tau, T; \mathbf{L_p}(\Omega)) := L_{p_1}(\tau, T; L_{p_1}(\Omega)) \times \ldots \times L_{p_M}(\tau, T; L_{p_M}(\Omega)), \\
\mathbf{L_q}(\tau, T; \mathbf{L_q}(\Omega)) := L_{q_1}(\tau, T; L_{q_1}(\Omega)) \times \ldots \times L_{q_M}(\tau, T; L_{q_M}(\Omega)),
\end{gathered}$$

where $\mathbf{p} = (p_1, p_2, \dots, p_M)$ and $\mathbf{q} = (q_1, q_2, \dots, q_M)$.

Definition 13.2 A function $u = u(x, t) \in \mathbf{L}_2(\tau, T; V) \cap \mathbf{L_p}(\tau, T; \mathbf{L_p}(\Omega))$ is called a *weak solution* of Problem (13.1) on $[\tau, T]$ if for any function $\varphi = \varphi(x) \in (C_0^\infty(\Omega))^M$ the following equality holds

$$\frac{d}{dt} \int_\Omega u(x, t) \cdot \varphi(x) dx + \int_\Omega \{a \nabla u(x, t) \cdot \nabla \varphi(x) + f(x, t, u(x, t)) \cdot \varphi(x)\} dx = 0$$

in the sense of scalar distributions on (τ, T).

Conditions (A)–(D) guarantee the existence of at least one weak solution on arbitrary interval (τ, T), $0 \le \tau < T < \infty$, with initial condition $u(\tau) = u_\tau$, $u_\tau \in H$ [20, pp. 283–284]. But the uniqueness is not provided.

The main goal of this paper is to investigate the uniform long-time behavior of all globally defined weak solutions for Problem (13.1) with initial data $u_\tau \in H$ under listed above assumptions, in particular to prove the existence of uniform global attractor for all globally defined weak solutions of Problem (13.1).

13.2 Auxiliaries

Let $0 \le \tau < T < \infty, u^{(\tau)} \in H$. Denote by $\mathscr{D}_{\tau,T}(u^{(\tau)})$ the family of all weak solutions on $[\tau, T]$ with initial data $u(\tau) = u^{(\tau)}$; that is,

$$\mathscr{D}_{\tau,T}(u^{(\tau)}) = \{u(\cdot) \,|\, u \text{ is a weak solution of Problem (13.1) on } [\tau, T],\ u(\tau) = u^{(\tau)}\}.$$

Remark that $\mathscr{D}_{\tau,T}(u^{(\tau)}) \neq \emptyset$ and $\mathscr{D}_{\tau,T}(u^{(\tau)}) \subset W_{\tau,T}$ where $u^{(\tau)} \in H$. Moreover, the concatenation of weak solutions of Problem (13.1) is a weak solution too, i.e., if $0 \le \tau < t < T, u^{(\tau)} \in H, u(\cdot) \in \mathscr{D}_{\tau,t}(u^{(\tau)})$, and $v(\cdot) \in \mathscr{D}_{t,T}(u(t))$, then

$$z(s) = \begin{cases} u(s), \ s \in [\tau, t], \\ v(s), \ s \in [t, T] \end{cases}$$

belongs to $\mathscr{D}_{\tau,T}(u^{(\tau)})$ (cf. [21, pp. 55–56]).

Listed above properties of solutions and Grönwall's lemma provide that for any finite time interval $[\tau, T] \subset \mathbb{R}_+$ each weak solution u of Problem (13.1) on $[\tau, T]$ satisfies estimates

$$\begin{aligned} \|u(t)\|_H^2 - 2\int_\tau^t \int_\Omega \beta(x,\xi)dxd\xi + 2\alpha \sum_{i=1}^M \int_s^t \|u^{(i)}(\xi)\|_{L_{p_i}(\Omega)}^{p_i} d\xi \\ +2d\int_s^t \|u(\xi)\|_V^2 d\xi \le \|u(s)\|_H^2 - 2\int_\tau^s \int_\Omega \beta(x,\xi)dxd\xi, \end{aligned} \tag{13.2}$$

$$\|u(t)\|_H^2 \le \|u(s)\|_H^2 e^{-2d\lambda_1(t-s)} + 2\int_s^t \int_\Omega \beta(x,\xi)e^{-2d\lambda_1(t-\xi)}dxd\xi \tag{13.3}$$

for any $t, s \in [\tau, T], t \ge s$, where λ_1 is the first eigenvalue of the scalar operator $-\Delta$ with Dirichlet boundary conditions (cf. [20, p. 285], [21, p. 56], [22] and references therein).

Any weak solution u of Problem (13.1) on a finite time interval $[\tau, T] \subset \mathbb{R}_+$ can be extended to a global one, defined on $[\tau, +\infty)$. For arbitrary $\tau \ge 0$ and $u^{(\tau)} \in H$ denote by $\mathscr{D}_\tau(u^{(\tau)})$ the set of all weak solutions (defined on $[\tau, +\infty)$) of Problem (13.1) with initial data $u(\tau) = u^{(\tau)}$. Consider the family of all weak solutions of Problem (13.1) defined on the semi-infinite time interval $[\tau, +\infty)$:

$$\mathscr{K}_\tau^+ = \cup_{u^{(\tau)} \in H} \mathscr{D}_\tau(u^{(\tau)}).$$

Consider the Fréchet space $C^{\mathrm{loc}}(\mathbf{R}_+; H)$ [23, p. 918]. We denote $T(h)u(\cdot) = u_h(\cdot)$, where $u_h(t) = u(t+h)$ for any $u \in C^{\mathrm{loc}}(\mathbf{R}_+; H)$ and $t, h \ge 0$ [24].

Remark that in the autonomous case the set $\mathscr{K}^+ := \mathscr{K}_0^+$ is *translation semi-invariant*, i.e., $T(h)\mathscr{K}^+ \subseteq \mathscr{K}^+$ for any $h \ge 0$. Such autonomous problems were investigated in [20, Chap. XIII], [25–28], [21, Chap. 2] and references therein; see also [29]. In the nonautonomous case, we have that $T(h)\mathscr{K}_0^+ \nsubseteq \mathscr{K}_0^+$. So, we consider *a united trajectory space* [19] of the following form:

$$\mathcal{K}_{\cup}^{+} := \bigcup_{\tau \geq 0} \left\{ u(\cdot + \tau) \in C^{\mathrm{loc}}(\mathbb{R}_+; H) \ : \ u(\cdot) \in \mathcal{K}_{\tau}^{+} \right\}.$$

Then $T(h)\{u(\cdot + \tau) : u \in \mathcal{K}_{\tau}^{+}\} \subseteq \{u(\cdot + \tau + h) : u \in \mathcal{K}_{\tau+h}^{+}\}$ for any $\tau, h \geq 0$. So, $T(h)\mathcal{K}_{\cup}^{+} \subseteq \mathcal{K}_{\cup}^{+}$ for any $h \geq 0$. Then, we consider *an extended united trajectory space* for Problem (13.1):

$$\mathcal{K}_{C^{\mathrm{loc}}(\mathbf{R}_+;H)}^{+} = \mathrm{cl}_{C^{\mathrm{loc}}(\mathbf{R}_+;H)} \left[\mathcal{K}_{\cup}^{+} \right], \tag{13.4}$$

where $\mathrm{cl}_{C^{\mathrm{loc}}(\mathbf{R}_+;H)}[\,\cdot\,]$ is the closure in $C^{\mathrm{loc}}(\mathbf{R}_+; H)$. Note that

$$T(h)\mathcal{K}_{C^{\mathrm{loc}}(\mathbf{R}_+;H)}^{+} \subseteq \mathcal{K}_{C^{\mathrm{loc}}(\mathbf{R}_+)}^{+} \text{ for any } h \geq 0 \tag{13.5}$$

(cf. [19, 23, 25]).

The following theorem characterizes the compactness properties of shifted solutions for Problem (13.1) in the induced topology from $C^{\mathrm{loc}}(\mathbb{R}_+; H)$.

Theorem 13.1 ([19, Theorem 4.1]) *Let Assumptions (A)–(D) hold. If $\{u_n\}_{n \geq 1} \subset \mathcal{K}_{C^{\mathrm{loc}}(\mathbb{R}_+;H)}^{+}$ is an arbitrary sequence, which is bounded in $L_\infty(\mathbf{R}_+; H)$, then there exist a subsequence $\{u_{n_k}\}_{k \geq 1} \subseteq \{u_n\}_{n \geq 1}$ and an element $u \in \mathcal{K}_{C^{\mathrm{loc}}(\mathbb{R}_+;H)}^{+}$ such that*

$$\|\Pi_{\tau,T} u_{n_k} - \Pi_{\tau,T} u\|_{C([\tau,T];H)} \to 0, \quad k \to +\infty,$$

for any finite time interval $[\tau, T] \subset (0, +\infty)$. Moreover, for any $u \in \mathcal{K}_{C^{\mathrm{loc}}(\mathbb{R}_+;H)}^{+}$ the following estimate holds:

$$\|u(t)\|_H^2 \leq \|u(0)\|_H^2 e^{-c_3 t} + c_4$$

for any $t \geq 0$, where positive constants c_3 and c_4 do not depend on $u \in \mathcal{K}_{C^{\mathrm{loc}}(\mathbb{R}_+;H)}^{+}$ and $t \geq 0$.

Let us define the multivalued map $G : \mathbb{R}_+ \times H \to 2^H \setminus \{\emptyset\}$ as

$$G(t, u_0) = \{u(t) \in H \mid u(\cdot) \in \mathcal{K}_{C^{\mathrm{loc}}(\mathbf{R}_+;H)}^{+} : \ u(0) = u_0\}. \tag{13.6}$$

Then the multivalued map G is a multivalued semiflow (see [21], (13.4) and (13.5)).

Definition 13.3 (*see* [15, 21]) The set $\Theta \subset X$ is called a uniform global attractor for multivalued semiflow G from (13.6) if the following conditions hold:

- Θ is an attracting set for G, that is for arbitrary bounded nonempty set $B \subset H$

$$dist_H(G(t, B), \Theta) \to 0 \text{ as } t \to +\infty,$$

where $dist_H(A, B) = \sup_{x \in A} \inf_{y \in B} ||x - y||_H$ for any non-empty sets $A, B \subset H$.

- Θ is the minimal attracting set, that is $\Theta \subset cl_H Y$ for arbitrary attracting set $Y \subset H$;
- $\Theta \subset G(t, \Theta)$ for all $t \geq 0$.

Uniform global attractor is invariant if $\Theta = G(t, \Theta)$ for all $t \geq 0$.

Definition 13.4 ([30, Definition 2.7]) Multivalued semiflow $\mathscr{G} : \mathbf{R}_+ \times H \to 2^H \setminus \emptyset$ satisfies the flattening property if for arbitrary bounded set $B \subset H$ and $\varepsilon > 0$ there exist $t_0(B, \varepsilon)$ and finite-dimensional subspace E of H such that for bounded projector $P : H \to E$ the set $P(\bigcup_{t>t_0} \mathscr{G}(t, B))$ is bounded in H, and

$$(I - P)(\bigcup_{t>t_0} \mathscr{G}(t, B)) \subset B(0, \varepsilon).$$

The following lemma provides the sufficient condition for justice of flattening property for multivalued semiflow G.

Lemma 13.1 ([30, Lemmas 2.4, 2.6], [31, p. 35]) *Let $\mathscr{G}$ be an asymptotically compact multivalued semiflow in H, that is for arbitrary sequence $\{\varphi_n\}_{n\geq 1} \subset \mathscr{G}$ with $\{\varphi_n(0)\}_{n\geq 1}$ bounded, and for any sequence $\{t_n\}_{n\geq 1}$: $t_n \to +\infty$ as $n \to \infty$, the sequence $\{\varphi_n(t_n)\}_{n\geq 1}$ has a convergent subsequence. Then for $\mathscr{G}$ the flattening property holds.*

13.3 Main Results

The main result of this note has the following formulation:

Theorem 13.2 *Let Assumptions (A)–(D) hold. Then the multivalued semiflow G, defined in (13.6), has a compact uniform global attractor Θ in the phase space H.*

Proof From [21], we have that the following conditions are sufficient for the existence of a compact uniform global attractor for the multivalued semiflow G: for each $t \geq 0$, the mapping $H \ni u \mapsto G(t, u)$ has a closed graph; G is asymptotically compact multivalued semiflow; there exists $R_0 > 0$ such that $\forall R > 0\ \exists T \geq 0$ (depended on R) such that $\forall t \geq T$

$$G(t, \{u \in H \mid \rho(u, 0) \leq R\}) \subset B_0 = \{u \in H \mid \rho(u, 0) \leq R_0\}. \tag{13.7}$$

The first condition follows from (13.6). Theorem 13.1 and assumptions (C), (D) and estimates (13.2), (13.3) provide the asymptotically compactness of multivalued semiflow G and the fulfillment of (13.7).

The following theorem implies that dynamics of all weak solutions of studied problem is finite-dimensional within a small parameter.

Theorem 13.3 *Let Assumptions (A)–(D) on the parameters of Problem (13.1) hold. Then the multivalued semiflow G satisfies the flattening condition.*

Proof The statement of the theorem directly follows from Lemma 13.1 and proof of Theorem 13.2.

Remark 13.3 All statements of Theorems 13.2 and 13.3 hold for function $f(x, t, u)$ equals to the sum of interaction function $f_1(x, t, u)$, satisfying Assumptions (A)–(D), and an external force $g \in L_2^{\text{loc}}(\mathbb{R}_+; V^*)$, which satisfies

$$\sup_{t \geq 0} \int_t^{t+1} \|g(s)\|_{V^*}^2 ds < +\infty.$$

The proofs are similar with some standard technical modifications.

As applications we may consider Fitz–Hugh–Nagumo system (signal transmission across axons), complex Ginzburg–Landau equation (theory of superconductivity), Lotka–Volterra system with diffusion (ecology models), Belousov–Zhabotinsky system (chemical dynamics) and many other reaction–diffusion-type systems [32], whose dynamics are well studied in autonomous case [9, 20], and in nonautonomous case, when all coefficients are uniformly continuous on time variable (see [20, 21] and references therein).

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