Graded Generalized Hexagon in Fuzzy Natural Logic

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Abstract. In our previous papers, we formally analyzed the generalized Aristotle's square of opposition using tools of fuzzy natural logic. Namely, we introduced general definitions of selected intermediate quantifiers, constructed a generalized square of opposition consisting of them and syntactically analyzed the emerged properties. The main goal of this paper is to extend the generalized square of opposition to graded generalized hexagon.

Keywords: Intermediate quantifiers · Fuzzy natural logic · Evaluative linguistic expressions · Generalized Peterson square · Graded generalized hexagon

1 Introduction

Fuzzy natural logic (FNL) is a formal mathematical theory that consists of three theories: (1) a formal theory of evaluative linguistic expressions (explained in detail in [25]), (2) a formal theory of fuzzy IF-THEN rules and approximate reasoning (presented in [24,27]), and (3) a formal theory of intermediate and generalized fuzzy quantifiers (presented in [16,18,20,26]). This paper is a contribution to (3), namely to extension of the generalized square of opposition to graded generalized hexagon.

Recall that the classical Aristotle's square of opposition [35] consists of the following formulas:

$$^{*}\mathbf{A}: \text{All } B \text{ are } A \qquad (\forall x)(Bx \Rightarrow Ax) \wedge (\exists x)Bx, \tag{1}$$

$$\mathbf{E}: \text{No } B \text{ are } A \qquad (\forall x)(Bx \Rightarrow \neg Ax), \tag{2}$$

$$\mathbf{I}: \text{Some } B \text{ are } A \qquad (\exists x)(Bx \wedge Ax), \tag{3}$$

$$^{*}\mathbf{O}: \text{Some } B \text{ are not } A \qquad (\exists x)(Bx \wedge \neg Ax) \vee \neg(\exists x)Bx. \tag{4}$$

The diagonals correspond to the relation of *contradiction* between the universal affirmative **A** ("All") and the particular negative **O** ("Not all"), as well as between the universal negative **E** ("No") and the particular positive **I** ("Some"). The property of *contrary* holds horizontally at the top between **A**

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("All") and **E** ("No"). The inverse relation of *sub-contraries* holds horizontally at the bottom between **I** ("Some") and **O** "Not all". Finally, the vertical relation between **A** and **I** as well as between **E** and **O** describes the relation of *subaltern* (superaltern).

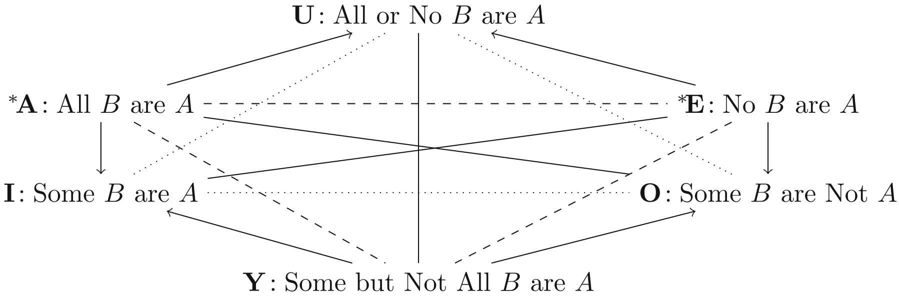
All the relations can be characterized using the definitions formulated in [19, p. 17] (see also below). Note that the Aristotle's square of opposition works fully with *presupposition* only. In [30,34]), the authors draw a crucial distinction between the "classical" Aristotelian square of opposition and the "modern" duality one based on the concepts of inner and outer negation.

Béziau in [2,3] suggested to extend a square of opposition into a hexagon. This technically means to add two new formulas **U** and **Y**[1] that are defined as disjunction of the two top corners of the square and conjunction of the two bottom corners:

$$\mathbf{U} = \mathbf{A} \vee \mathbf{E} : \text{All or No } B \text{ are } A. \tag{5}$$

$$\mathbf{Y} = \mathbf{I} \wedge \mathbf{O} : \text{Some but Not All } B \text{ are } A. \tag{6}$$

Then we obtain the following Aristotelian hexagon.



The diagonal lines represent contradictories (denoted by straight lines), the formulas **A** and **E** are contraries (denoted by dashed lines), **A** and **E** entail **U** (denoted by arrow), while **Y** entails both formulas **I** as well as **O**. The formulas **I** and **O** are sub-contraries (denoted by dotted lines). It is interesting to see that the logical hexagon obtains three Aristotle's squares of opposition, namely, **AEIO**, **AYOU** and **EYUI**.

In [32], we can find differences between the Aristotle hexagon and the Duality hexagon. A logical hexagon with many examples and also the cube of opposition was described in [15]. A more complex 3D generalization of the hexagon was proposed by Moretti [17], Pellissier [29] and Smessaert [33]. Applications of the square of opposition in philosophical and mathematical logic, linguistics and psychology were studied in [4–6].

The graded Aristotle square of opposition and also a cube of opposition and its graded version that associates the traditional square of opposition with the dual one were introduced in [13,14]. The structures of opposition in rough set

[1] Blanché in [7] introduced **Y** at first, before completing it with **U** in [8].

theory were analyzed in [10,11]. The gradual hexagon with fuzzy relation and a connection of hexagon and a cube of opposition was described in [12].

In [19], we demonstrated how the generalized[2] Aristotle square of opposition can be represented formally in fuzzy natural logic. In this paper, we first extend this theory and analyze graded Aristotle hexagon as a generalization of the Aristotle classical one. Then we propose a generalization of the Peterson's square of opposition (cf. [31]) to a graded hexagon with intermediate quantifiers (we will call it *graded generalized hexagon*). Let us remark that these results contribute also to development of FNL because the discovered relations among intermediate quantifiers can be applied to formulation of general rules of human reasoning.

The paper is structured as follows: First of all we very briefly remember the basic mathematical formal system. In Sect. 3, we remember the basic definitions of contrary, contradictories, sub-contrary and sub-alterns. Then we prove the main properties of interpretation of three squares of opposition (**AEIO**, **AYOU** and **EYUI**). Finally, we introduce the graded generalized hexagon with five basic intermediate quantifiers.

2 Preliminaries

2.1 The Basic Formal System

The fuzzy natural logic is formulated using tools of Łukasiewicz fuzzy type theory (Ł-FTT) which is a higher-order fuzzy logic. All the details can be found in the papers [19,23,25]. Recall that the basic syntactical objects of Ł-FTT are classical, namely the concepts of type and formula (cf. [1]).

The semantics is defined using the concept of general model in which the type o of truth values is assigned a linearly ordered $\mathrm{MV}_{\boldsymbol{\Delta}}$-algebra which is an MV-algebra extended by the delta operation (see [9,28]). In this paper we will consider only models whose algebra of truth values forms the standard Łukasiewicz MV_{Δ}-algebra

$$\mathcal{L} = \langle [0,1], \vee, \wedge, \otimes, \rightarrow, 0, 1, \Delta\rangle. \tag{7}$$

The following special formulas are important in our theory:

$$\Upsilon_{oo} \equiv \lambda z_o \cdot \neg\boldsymbol{\Delta}(\neg z_o), \qquad \text{(nonzero truth value)}$$

$$\hat{\Upsilon}_{oo} \equiv \lambda z_o \cdot \neg\boldsymbol{\Delta}(z_o \vee \neg z_o). \qquad \text{(general truth value)}$$

Thus, $\mathcal{M}(\Upsilon(A_o)) = 1$ iff $\mathcal{M}(A_o) > 0$, and $\mathcal{M}(\hat{\Upsilon}(A_o)) = 1$ iff $\mathcal{M}(A_o) \in (0,1)$ holds in any model $\mathcal{M}$.

The following completeness theorem will be often used below.

Theorem 1 ([23]).

(a) A theory T is consistent iff it has a general model $\mathcal{M}$.
(b) For every theory T and a formula A_o, $T \vdash A_o$ iff $T \models A_o$.

[2] In some papers, the term "generalized Aristotle square" is replaced by "graded on".

2.2 Theories of Evaluative Expressions and Intermediate Quantifiers

The main constituent of FNL is the theory of evaluative linguistic expressions. These are special natural language expressions such as *small, medium, big, very short, more or less deep, quite roughly strong, extremely high*, etc. A formal theory of their semantics was introduced in [25].

Another constituent of FNL is the theory of intermediate quantifiers. These are natural language expressions such as *most, almost all, a few,* etc. Their semantics is formalized using a special theory of Ł-FTT denoted by $T^{\mathrm{IQ}}[\mathcal{S}]$, which extends the theory of evaluative expressions. The detailed structure of $T^{\mathrm{IQ}}[\mathcal{S}]$ and precise definitions can be found in [18,19,26].

Definition 1. *Let $\mathcal{S} \subset$ Types be a set of selected types. Let $T^{IQ}[\mathcal{S}]$ be a theory of intermediate quantifiers. Let $z \in Form_{o\alpha}$, $x \in Form_{\alpha}$ and $A, B \in Form_{o\alpha}$. Then the following special intermediate quantifiers can be introduced:*

$$
\begin{aligned}
&\textbf{A: } \textit{All } B \textit{ are } A := (Q^{\forall}_{Bi\boldsymbol{\Delta}}x)(B, A) \equiv (\forall x)(Bx \Rightarrow Ax),\\
&\textbf{E: } \textit{No } B \textit{ are } A := (Q^{\forall}_{Bi\boldsymbol{\Delta}}x)(B, \neg A) \equiv (\forall x)(Bx \Rightarrow \neg Ax),\\
&\textbf{P: } \textit{Almost all } B \textit{ are } A := (Q^{\forall}_{Bi\,Ex}x)(B, A) \equiv\\
&\qquad (\exists z)((\boldsymbol{\Delta}(z \subseteq B)\,\&(\forall x)(zx \Rightarrow Ax)) \wedge (Bi\,Ex)((\mu B)z)),\\
&\textbf{B: } \textit{Almost all } B \textit{ are not } A := (Q^{\forall}_{Bi\,Ex}x)(B, \neg A) \equiv\\
&\qquad (\exists z)((\boldsymbol{\Delta}(z \subseteq B)\,\&(\forall x)(zx \Rightarrow \neg Ax)) \wedge (Bi\,Ex)((\mu B)z)),\\
&\textbf{T: } \textit{Most } B \textit{ are } A := (Q^{\forall}_{Bi\,Ve}x)(B, A) \equiv\\
&\qquad (\exists z)((\boldsymbol{\Delta}(z \subseteq B)\,\&(\forall x)(zx \Rightarrow Ax)) \wedge (Bi\,Ve)((\mu B)z)),\\
&\textbf{D: } \textit{Most } B \textit{ are not } A := (Q^{\forall}_{Bi\,Ve}x)(B, \neg A) \equiv\\
&\qquad (\exists z)((\boldsymbol{\Delta}(z \subseteq B)\,\&(\forall x)(zx \Rightarrow \neg Ax)) \wedge (Bi\,Ve)((\mu B)z)),\\
&\textbf{K: } \textit{Many } B \textit{ are } A := (Q^{\forall}_{\neg(Sm\,\bar{\boldsymbol{\nu}})}x)(B, A) \equiv\\
&\qquad (\exists z)((\boldsymbol{\Delta}(z \subseteq B)\,\&(\forall x)(zx \Rightarrow Ax)) \wedge \neg(Sm\,\bar{\boldsymbol{\nu}})((\mu B)z)),\\
&\textbf{G: } \textit{Many } B \textit{ are not } A := (Q^{\forall}_{\neg(Sm\,\bar{\boldsymbol{\nu}})}x)(B, \neg A) \equiv\\
&\qquad (\exists z)((\boldsymbol{\Delta}(z \subseteq B)\,\&(\forall x)(zx \Rightarrow \neg Ax)) \wedge \neg(Sm\,\bar{\boldsymbol{\nu}})((\mu B)z)),\\
&\textbf{I: } \textit{Some } B \textit{ are } A := (Q^{\exists}_{Bi\boldsymbol{\Delta}}x)(B, A) \equiv (\exists x)(Bx \wedge Ax),\\
&\textbf{O: } \textit{Some } B \textit{ are not } A := (Q^{\exists}_{Bi\boldsymbol{\Delta}}x)(B, \neg A) \equiv (\exists x)(Bx \wedge \neg Ax).
\end{aligned}
$$

3 Graded Aristotle Hexagon

3.1 From Aristotle Square to Graded Square

In this subsection, the main definitions and results that will be used later are summarized.

Definition 2. *Let T be a consistent theory of Ł-FTT and $P_1, P_2 \in Form_o$ be closed formulas of type o.*

(i) *P_1 and P_2 are* contraries *if $T \vdash \neg(P_1 \,\&\, P_2)$. By completeness, this is equivalent to*

$$\mathcal{M}(P_1) \otimes \mathcal{M}(P_2) = 0$$

for every model $\mathcal{M} \models T$[3]*.*

(ii) *P_1 and P_2 are* weak contraries *if $T \vdash \hat{\Upsilon}(P_1 \,\&\, P_2)$. By completeness, this is equivalent to*

$$0 < \mathcal{M}(P_1) \otimes \mathcal{M}(P_2) < 1$$

for every model $\mathcal{M} \models T$.

(iii) *P_1 and P_2 are* sub-contraries *if $T \vdash (P_1 \,\nabla\, P_2)$. By completeness, this is equivalent to*

$$\mathcal{M}(P_1) \oplus \mathcal{M}(P_2) = 1$$

for every model $\mathcal{M} \models T$.

(iv) *P_1 and P_2 are* weak sub-contraries *if $T \vdash \Upsilon(P_1 \vee P_2)$. By completeness, this is equivalent to*

$$\mathcal{M}(P_1) \vee \mathcal{M}(P_2) > 0$$

for every model $\mathcal{M} \models T$.

(v) *P_1 and P_2 are* contradictories *if both*

$$T \vdash \neg(\boldsymbol{\Delta} P_1 \,\&\, \boldsymbol{\Delta} P_2) \text{ as well as } T \vdash \boldsymbol{\Delta} P_1 \,\nabla\, \boldsymbol{\Delta} P_2.$$

By completeness, this means that both $\mathcal{M}(\boldsymbol{\Delta} P_1) \otimes \mathcal{M}(\boldsymbol{\Delta} P_2) = 0$ as well as $\mathcal{M}(\boldsymbol{\Delta} P_1) \oplus \mathcal{M}(\boldsymbol{\Delta} P_2) = 1$ hold for every model $\mathcal{M} \models T$.

(vi) *The formula P_2 is a* subaltern *of P_1 in T if $T \vdash P_1 \Rightarrow P_2$. By completeness, this means that the inequality*

$$\mathcal{M}(P_1) \leq \mathcal{M}(P_2)$$

holds true in every model $\mathcal{M} \models T$. We will call P_1 a superaltern *of P_2.*

Below we recall the main results which were formally proved in [19]. Recall that we fix the set $\mathcal{S}$ and write T^{IQ} instead of $T^{\mathrm{IQ}}[\mathcal{S}]$. Recall that the graded Aristotle's square of opposition in Ł-FTT works with the following four formulas with presupposition:

$$^{*}\mathbf{A}: \text{All } B \text{ are } A \qquad (\forall x)(Bx \Rightarrow Ax) \,\&\, (\exists x)Bx, \tag{8}$$

$$\mathbf{E}: \text{No } B \text{ are } A \qquad (\forall x)(Bx \Rightarrow \neg Ax), \tag{9}$$

$$\mathbf{I}: \text{Some } B \text{ are } A \qquad (\exists x)(Bx \wedge Ax), \tag{10}$$

$$^{*}\mathbf{O}: \text{Some } B \text{ are not } A \qquad (\exists x)(Bx \wedge \neg Ax) \,\nabla\, \neg(\exists x)Bx. \tag{11}$$

[3] Let $\mathcal{M} \models T^{\mathrm{IQ}}$. Then we denote $\mathcal{M}(\top) = 1$ and $\mathcal{M}(\bot) = 0$.

3.2 From Graded Aristotle Square to Graded Aristotle Hexagon

Generalizing the graded Aristotle square to the graded Aristotle hexagon means to define new formulas which will be put to the top and the bottom of the latter:

$$\mathbf{U} := \mathbf{A} \vee \mathbf{E} \quad \text{All or No } B \text{ are } A. \tag{12}$$

$$\mathbf{Y} := \mathbf{I} \,\&\, \mathbf{O} \quad \text{Some but Not All } B \text{ are } A. \tag{13}$$

Lemma 1. *There is no model such that*

(a) $\mathcal{M}(\mathbf{Y}) = 1$ *and* $\mathcal{M}(\mathbf{A}) = 1$.
(b) $\mathcal{M}(\mathbf{Y}) = 1$ *and* $\mathcal{M}(\mathbf{E}) = 1$.

Proof. (a) Let there be a model $\mathcal{M} \models T^{\mathrm{IQ}}$ such that $\mathcal{M}(\mathbf{Y}) = 1$ and $\mathcal{M}(\mathbf{A}) = 1$. Then $\mathcal{M}(\boldsymbol{\Delta}\mathbf{A}) = 1$ and so from the contradictory relation to $\mathbf{O}$ it follows that $\mathcal{M}(\boldsymbol{\Delta}\mathbf{O}) = 0$. Because $\mathbf{A}$ is superaltern of $\mathbf{I}$, $\mathcal{M}(\mathbf{I}) = 1$ and so, $\mathcal{M}(\mathbf{I} \,\&\, \mathbf{O}) = \mathcal{M}(\mathbf{Y}) = 0$ which contradicts the assumption.

(b) Analogously as (a).

Lemma 2. *The following holds in every model* $\mathcal{M} \models T^{IQ}$*:*

(a) $\mathcal{M}(\mathbf{I} \,\boldsymbol{\nabla}\, \mathbf{U}) = 1$,
(b) $\mathcal{M}(\mathbf{O} \,\boldsymbol{\nabla}\, \mathbf{U}) = 1$.

Proof. (a) Let $\mathcal{M} \models T^{\mathrm{IQ}}$. From the contradictory relation of $\mathbf{E}$ to $\mathbf{I}$ it follows that $\mathcal{M}(\boldsymbol{\Delta}\mathbf{E} \,\boldsymbol{\nabla}\, \boldsymbol{\Delta}\mathbf{I}) = 1$. Then

$$1 = \mathcal{M}(\boldsymbol{\Delta}\mathbf{E} \,\boldsymbol{\nabla}\, \boldsymbol{\Delta}\mathbf{I}) \leq \mathcal{M}(\mathbf{E} \,\boldsymbol{\nabla}\, \mathbf{I}) \leq \mathcal{M}(\mathbf{I} \,\boldsymbol{\nabla} (\mathbf{E} \vee \mathbf{A})) = \mathcal{M}(\mathbf{I} \,\boldsymbol{\nabla}\, \mathbf{U}).$$

(b) Analogously as (a).

Corollary 1. *There is no model of* T^{IQ} *such that*

(a) $\mathcal{M}(\mathbf{U}) = 0$ *and* $\mathcal{M}(\mathbf{I}) = 0$.
(b) $\mathcal{M}(\mathbf{U}) = 0$ *and* $\mathcal{M}(\mathbf{O}) = 0$.

Hence, we conclude the following:

Theorem 2 (Sub-contraries). *The couples of formulas* $\mathbf{I}$ *and* $\mathbf{U}$*, as well as* $\mathbf{O}$ *and* $\mathbf{U}$ *are sub-contraries in* T^{IQ}.

Lemma 3. *Let* $\mathcal{M} \models T^{IQ}$.

(a) If $\mathcal{M}(\hat{\Upsilon}(\mathbf{A})) = 1$ *then* $\mathcal{M}(\mathbf{O}) = 1$.
(b) If $\mathcal{M}(\hat{\Upsilon}(\mathbf{E})) = 1$ *then* $\mathcal{M}(\mathbf{I}) = 1$.

Proof. (a) Let the assumption hold and $\mathcal{M}(\mathbf{O}) < 1$. From the contradictory relation of $\mathbf{A}$ to $\mathbf{O}$ it follows that $\mathcal{M}(\boldsymbol{\Delta}\mathbf{A}) = 1$ and hence $\mathcal{M}(\mathbf{A}) = 1$ which contradicts the assumption.

(b) is proved analogously.

We see from Lemma 3 that in every model of T^{IQ}, if the truth degree of the formulas **A** and **E** is smaller than 1 then the truth degree of the respective formulas **I** and **O** must be equal to 1. We can also immediately see that if $\mathcal{M}(\hat{\Upsilon}(\mathbf{A})) = 1$ and $\mathcal{M}(\hat{\Upsilon}(\mathbf{E})) = 1$ then also $\mathcal{M}(\hat{\Upsilon}(\mathbf{U})) = 1$.

Lemma 4. *Let $\mathcal{M} \models T^{IQ}$. Let $\mathcal{M}(\hat{\Upsilon}(\mathbf{A})) = 1$ and $\mathcal{M}(\hat{\Upsilon}(\mathbf{E})) = 1$. Then the following is true:*

(a) $\mathcal{M}(\mathbf{I}) \otimes \mathcal{M}(\boldsymbol{\Delta}\mathbf{U}) = 0$
(b) $\mathcal{M}(\mathbf{I}) \oplus \mathcal{M}(\boldsymbol{\Delta}\mathbf{U}) = 1$,
(c) $\mathcal{M}(\mathbf{O}) \otimes \mathcal{M}(\boldsymbol{\Delta}\mathbf{U}) = 0$,
(d) $\mathcal{M}(\mathbf{O}) \oplus \mathcal{M}(\boldsymbol{\Delta}\mathbf{U}) = 1$,
(e) $\mathcal{M}(\mathbf{I}) \otimes \mathcal{M}(\mathbf{U}) = a \in (0, 1)$,
(f) $\mathcal{M}(\mathbf{O}) \otimes \mathcal{M}(\mathbf{U}) = b \in (0, 1)$,
(g) $\mathcal{M}(\mathbf{I}) \oplus \mathcal{M}(\mathbf{U}) = 1$,
(h) $\mathcal{M}(\mathbf{O}) \oplus \mathcal{M}(\mathbf{U}) = 1$.
(i) $\mathcal{M}(\boldsymbol{\Delta}\mathbf{A}) \otimes \mathcal{M}(\boldsymbol{\Delta}\mathbf{Y}) = 0$
(j) $\mathcal{M}(\boldsymbol{\Delta}\mathbf{A}) \oplus \mathcal{M}(\boldsymbol{\Delta}\mathbf{Y}) = 1$,
(k) $\mathcal{M}(\boldsymbol{\Delta}\mathbf{E}) \otimes \mathcal{M}(\boldsymbol{\Delta}\mathbf{Y}) = 0$,
(l) $\mathcal{M}(\boldsymbol{\Delta}\mathbf{E}) \oplus \mathcal{M}(\boldsymbol{\Delta}\mathbf{Y}) = 1$,
(m) $\mathcal{M}(\mathbf{A}) \otimes \mathcal{M}(\mathbf{Y}) = a \in (0, 1)$,
(n) $\mathcal{M}(\mathbf{E}) \otimes \mathcal{M}(\mathbf{Y}) = a \in (0, 1)$,
(o) $\mathcal{M}(\mathbf{A}) \oplus \mathcal{M}(\mathbf{Y}) = 1$,
(p) $\mathcal{M}(\mathbf{E}) \oplus \mathcal{M}(\mathbf{Y}) = 1$.

Theorem 3. *Let $\mathcal{M} \models T^{IQ}$. Let $\mathcal{M}(\hat{\Upsilon}(\mathbf{A})) = 1$ and $\mathcal{M}(\hat{\Upsilon}(\mathbf{E})) = 1$. Then the following is true:*

(a) The formulas **I** *and* **U** *can be contradictories, sub-contraries, as well as weak contraries.*
(b) The formulas **O** *and* **U** *can be contradictories, sub-contraries, as well as weak contraries.*
(c) The formulas **A** *and* **Y** *can be contradictories, sub-contraries, as well as weak contraries.*
(d) The formulas **E** *and* **Y** *can be contradictories, sub-contraries, as well as weak contraries.*
(e) The formulas **I** *and* **U** *as well as the formulas* **O** *and* **U** *cannot be contraries.*
(f) The formulas **A** *and* **Y** *as well as the formulas* **E** *and* **Y** *cannot be contraries.*

Lemma 5. *Let* **A**, **E** *and* $\mathbf{I}, \mathbf{O}$ *be intermediate quantifiers introduced above. Then the following holds in every model $\mathcal{M} \models T^{IQ}$:*

(a) $\mathcal{M}(\mathbf{U}) = 0$ *implies* $\mathcal{M}(\mathbf{Y}) = 1$.
(b) $\mathcal{M}(\mathbf{U}) = 1$ *implies* $\mathcal{M}(\mathbf{Y}) = 0$.

Proof. (a) Let there be a model $\mathcal{M} \models T^{\mathrm{IQ}}$ such that $\mathcal{M}(\mathbf{U}) = \mathcal{M}(\mathbf{A} \vee \mathbf{E}) = 0$. Then $\mathcal{M}(\mathbf{A}) = \mathcal{M}((\forall x)(Bx \Rightarrow Ax)) = 0$ and also $\mathcal{M}(\mathbf{E}) = \mathcal{M}((\forall x)(Bx \Rightarrow \neg Ax)) = 0$. Hence,

$$1 = \mathcal{M}((\exists x)(Bx \mathbin{\&} \neg Ax)) \leq \mathcal{M}((\exists x)(Bx \wedge \neg Ax)) = \mathcal{M}(\mathbf{O})$$

and also

$$1 = \mathcal{M}((\exists x)(Bx \mathbin{\&} Ax)) \leq \mathcal{M}((\exists x)(Bx \wedge Ax)) = \mathcal{M}(\mathbf{I}).$$

We conclude that $\mathcal{M}(\mathbf{I} \mathbin{\&} \mathbf{O}) = \mathcal{M}(\mathbf{Y}) = 1$.

(b) immediately follows from (a).

Lemma 6. *Let* $\mathcal{M} \models T^{IQ}$. *Let* $\mathcal{M}(\hat{\Upsilon}(\mathbf{A})) = 1$ *and* $\mathcal{M}(\hat{\Upsilon}(\mathbf{E})) = 1$. *Then the following is true:*

(a) $\mathcal{M}(\boldsymbol{\Delta}\mathbf{U}) \otimes \mathcal{M}(\boldsymbol{\Delta}\mathbf{Y}) = 0$,
(b) $\mathcal{M}(\boldsymbol{\Delta}\mathbf{U}) \oplus \mathcal{M}(\boldsymbol{\Delta}\mathbf{Y}) = 1$.

Proof. (a) Let $\mathcal{M} \models T^{\mathrm{IQ}}$. Let $\mathcal{M}(\hat{\Upsilon}(\mathbf{A})) = 1$ and $\mathcal{M}(\hat{\Upsilon}(\mathbf{E})) = 1$. Then from the definition of $\hat{\Upsilon}$ it follows that $\mathcal{M}(\boldsymbol{\Delta}(\mathbf{A} \vee \mathbf{E})) = 0$. From Lemma 3 we conclude that $\mathcal{M}(\mathbf{I}\,\&\,\mathbf{O}) = 1$ as well as $\mathcal{M}(\boldsymbol{\Delta}(\mathbf{I}\,\&\,\mathbf{O})) = 1$. Finally, $\mathcal{M}(\boldsymbol{\Delta}\mathbf{U}) \otimes \mathcal{M}(\boldsymbol{\Delta}\mathbf{Y}) = 0$ and $\mathcal{M}(\boldsymbol{\Delta}\mathbf{U}) \oplus \mathcal{M}(\boldsymbol{\Delta}\mathbf{Y}) = 1$ are fulfilled.

As a corollary we immediately obtain the following.

Theorem 4 (Contradictories). *The quantifiers* $\mathbf{U}$ *and* $\mathbf{Y}$ *are contradictories in* T^{IQ}.

Finally, we can demonstrate that in the graded Aristotle hexagon the two extra formulas are perfectly united by means of the four arrows of subalterns.

Theorem 5 (Subalterns). *The following holds true in the theory* T^{IQ}:

(a) The formula $\mathbf{U}$ *is subaltern of* $\mathbf{A}$ *and* $\mathbf{E}$, *i.e.,* $T^{IQ} \vdash \mathbf{A} \Rightarrow \mathbf{U}$ *and* $T^{IQ} \vdash \mathbf{E} \Rightarrow \mathbf{U}$.
(a) The formula $\mathbf{Y}$ *is superaltern of* $\mathbf{I}$ *and* $\mathbf{O}$, *i.e.,* $T^{IQ} \vdash \mathbf{Y} \Rightarrow \mathbf{I}$ *and* $T^{IQ} \vdash \mathbf{Y} \Rightarrow \mathbf{O}$.

Corollary 2. *The graded Aristotle hexagon forms three graded squares of opposition, namely* **AEIO**, **AYOU** *and* **EYIU**.

3.3 From Graded Generalized Peterson's Square to Graded Generalized Hexagon

In the papers [19,21], we syntactically analyzed and semantically verified the generalized Peterson square (**5**-square) of opposition in FNL. In this section, we will introduce basic concepts using which the graded generalized hexagon can be formed. We start with the following definitions of new generalized intermediate quantifiers:

$$\mathbf{U}_{Ex\,Bi} := \mathbf{P} \vee \mathbf{B} \quad \text{Almost all } B \text{ are } A \text{ or Almost all } B \text{ are not } A \tag{14}$$

$$\mathbf{U}_{Ve\,Bi} := \mathbf{T} \vee \mathbf{D} \quad \text{Most } B \text{ are } A \text{ or Most } B \text{ are not } A \tag{15}$$

$$\mathbf{Y}_{\neg\,Sm} := \mathbf{K}\,\&\,\mathbf{G} \quad \text{Many } B \text{ are } A \text{ and Many } B \text{ are not } A. \tag{16}$$

We will suppose that the basic fuzzy set used in the definition of the intermediate quantifier is a normal fuzzy set. This is specified by the following definition.

Definition 3. *Let* $B \in Form_{o\alpha}$. *By* $T[B]$ *we denote an extension of the theory* T^{IQ} *such that*

$$T[B] \vdash (\exists x_\alpha)\boldsymbol{\Delta} Bx.$$

Theorem 6 (Contraries, [19]). *The following couples of formulas are contraries in* $T[B]$*:* $\mathbf{B}$ *and* $\mathbf{P}$*, and also* $\mathbf{D}$ *and* $\mathbf{T}$*.*

Other properties of five basic generalized intermediate quantifiers were proved in [19]. In [22], we analyzed the intermediate quantifier "Many" and its position inside of 5-square of opposition.

Recall that the formula $\mathbf{A}$ is a superaltern of all the formulas $\mathbf{P}, \mathbf{T}, \mathbf{K}, \mathbf{I}$. At the same time, $\mathbf{E}$ is a superaltern of $\mathbf{B}, \mathbf{D}, \mathbf{G}, \mathbf{O}$.

Lemma 7. *The following is provable:*

(a) $T[B] \vdash \mathbf{I} \nabla \mathbf{U}_{Ex\,Bi}$,
(b) $T[B] \vdash \mathbf{I} \nabla \mathbf{U}_{Ve\,Bi}$.

Proof. (a) Because $\mathbf{E}$ is a superaltern of $\mathbf{B}$ then by the properties of the delta operation we obtain $T[B] \vdash \Delta\mathbf{E} \Rightarrow \Delta\mathbf{B}$. Then

$$T[B] \vdash \Delta\mathbf{E} \nabla \Delta\mathbf{I} \Rightarrow \Delta\mathbf{B} \nabla \Delta\mathbf{I}. \tag{17}$$

Furthermore, by the properties of delta we obtain $T[B] \vdash \Delta\mathbf{B} \Rightarrow \mathbf{B}$ as well as, $T[B] \vdash \Delta\mathbf{I} \Rightarrow \mathbf{I}$. Then by properties of Ł-FTT we get

$$T[B] \vdash \Delta\mathbf{B} \nabla \Delta\mathbf{I} \Rightarrow \mathbf{B} \nabla \mathbf{I}. \tag{18}$$

Joining (17) and (18) we obtain

$$T[B] \vdash \Delta\mathbf{E} \nabla \Delta\mathbf{I} \Rightarrow \mathbf{B} \nabla \mathbf{I}. \tag{19}$$

But we know that The quantifiers $\mathbf{E}$ and $\mathbf{I}$ are contradictories and so $T[B] \vdash \Delta\mathbf{E} \nabla \Delta\mathbf{I}$, which gives $T[B] \vdash \mathbf{B} \nabla \mathbf{I}$ and also $T[B] \vdash (\mathbf{B} \vee \mathbf{P}) \nabla \mathbf{I}$.

(b) Analogously as (a).

Theorem 7. *The following couples of formulas are subcontraries in T[B]:* $\mathbf{U}_{Ex\,Bi}$ *and* $\mathbf{I}$*, and also* $\mathbf{U}_{Ve\,Bi}$ *and* $\mathbf{I}$*.*

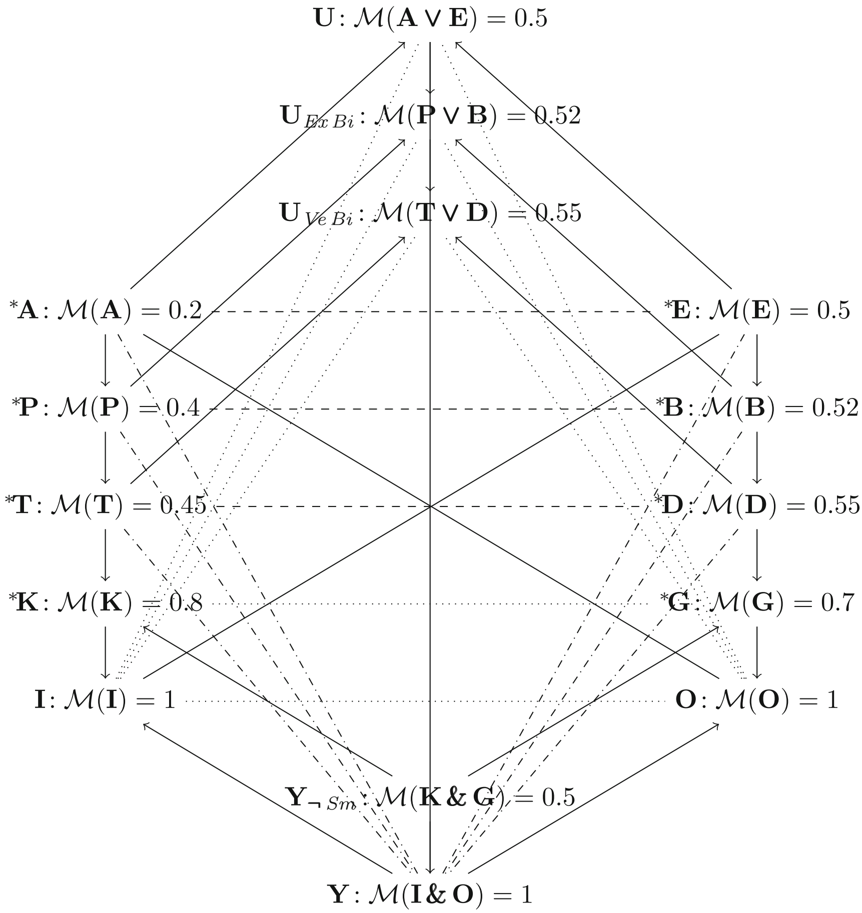
Theorem 8. *The following is provable in* $T[B]$*:*

(a) The quantifier $\mathbf{U}$ *is a superaltern of* $\mathbf{U}_{Ex\,Bi}$*.*
(b) The quantifier $\mathbf{U}_{Ve\,Bi}$ *is a subaltern of* $\mathbf{U}_{Ex\,Bi}$*.*
(c) The quantifier $\mathbf{Y}_{\neg\,Sm}$ *is a superaltern of* $\mathbf{Y}$*.*
(d) The quantifiers $\mathbf{P}$ *and* $\mathbf{B}$ *are a superalterns of* $\mathbf{U}_{Ex\,Bi}$*.*
(e) The quantifiers $\mathbf{T}$ *and* $\mathbf{D}$ *are a superalterns of* $\mathbf{U}_{Ve\,Bi}$*.*

3.4 Example of the Graded Generalized Hexagon

Let us consider a model $\mathcal{M} \models T[B]$ such that $T^{\mathrm{IQ}} \vdash (\exists x)Bx$ and let $\mathcal{M}(\mathbf{A}) = a > 0$ (e.g., $a = 0.2$). The degrees inside of the generalized Peterson's square follow from the definitions of contraries, contradictories, sub-contraries and subalterns.

The following can be proved: The formulas **I** and **O** are sub-contraries with each of $\mathbf{U}, \mathbf{U}_{Ex\,Bi}, \mathbf{U}_{Ve\,Bi}$. The quantifier **U** is superaltern and the formula **Y** is subaltern of all of the other quantifiers. Furthermore, the quantifiers **A** and **E** are weak contraries with **Y**. The motivation and explanation that the quantifier **A** is not a negation of the formula **O** is explained in [19]. Similarly the quantifiers **P**, **T** and **B**, **D** are weak contraries with **Y**.



In every classical Aristotle hexagon, three squares **AEIO**, **AYOU** and **EYIU** are formed on the basis of the properties of contraries, contradictories, sub-contraries and sub-alterns. The graded Aristotle square (**AEIO**) contains the same properties as its classical version. It can be seen that the new graded squares (**AYOU** and **EYIU**) prove that the formulas **A** and **Y** as well as the formulas **E** and **Y** are weak contraries.

4 Conclusion

In this paper we extended the theory of graded classical Aristotle square of opposition to the graded Aristotle hexagon. Then we suggested generalization of the Peterson's square of opposition to a graded generalized hexagon, i.e., the hexagon whose vertices contain intermediate quantifiers.

The future work will focus on more detailed analysis of the properties of the graded generalized hexagon, possibly extended by more intermediate quantifiers. Furthermore, we will also study graded cube of opposition. This may open interesting area of study of relations among important classes of properties. We expect that these results will contribute to the development of fuzzy natural logic, namely to formulation of various kinds of general rules used in human reasoning.

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