Unbalanced OWA Operators for Atanassov Intuitionistic Fuzzy Sets

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Abstract. In this work we introduce a new class of OWA operators for Atanassov intuitionistic fuzzy sets which distinguishes between the weights for the membership degree and the weights for the nonmembership degree; we call these operators Unbalanced Atanassov Intuitionistic OWA operators. We also study under which conditions these operators are aggregation functions with respect to the Atanassov intuitionistic admissible linear orders. Finally, we apply these aggregation functions in an illustrative example of a decision making problem.

Keywords: Atanassov Intuitionistic Fuzzy Set · OWA operators · Unbalanced OWA operators

1 Introduction

Aggregation functions have shown to be a useful tool in problems where information should be fused. Although a partial order is used in some generalizations of aggregation function on other sets (see, for example [1]), some particular classes of these functions such as OWA operators and Choquet or Sugeno integrals require all the elements being comparable. Consequently a linear order is needed. However, these orders are not trivially generated in the extensions of fuzzy sets where more than one value is used to define the membership degree. This is the case, for instance of Interval-Valued Fuzzy Sets (IVFSs) [2] or Atanassov Intuitionistic Fuzzy Sets (AIFSs) [3].

Although some constructions of linear orders on AIFSs have already been studied [4], more works generalizing different notions using linear orders are indispensable for its use in applications. In particular, we aim to define on AIFSs a new class of OWA operators which may apply different weight vectors for the membership and nonmembership degree. We denote these operators Unbalanced Atanassov Intuitionistic OWA operators (UAIOWAs). Taking into account that OWA operators are a particular class of aggregation functions frequently used

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in applications, our final goal is to study when $UAIOWAs$ satisfy the properties demanded to the aggregation functions. Finally, we introduce an illustrative example on a decision making problem where the Unbalanced Atanassov intuitionistic OWA operators are a suitable option to solve the problem.

The structure of the work is as follows: In Sect. 2 we introduce some well-known concepts which are necessary for the development of this work. The notion of Unbalanced Atanassov intuitionistic OWA operators is introduced in Sect. 3 where we study when these operators are aggregation functions. Section 4 shows an example where Unbalanced Atanassov intuitionistic OWA operators are applied to a decision making problem. We close the study with some conclusions and open problems for future research.

2 Preliminaries

This section is devoted to briefly introduce several well-known basic concepts and to fix the notation used in this work. We first recall the notion of aggregation function on a poset which becomes crucial in the development of this work. For more information see [1,5].

Definition 1. *Given a poset $(P, \preceq)$ with bottom and top, 0_P and 1_P respectively, an aggregation function M on P with respect to the order $\preceq$ is a mapping $M : P^n \to P$ satisfying:*

- $M(0_P, \ldots, 0_P) = 0_P, \quad M(1_P, \ldots, 1_P) = 1_P,$
- $M(x_1, \ldots, x_n) \preceq M(y_1, \ldots, y_n)$ *whenever* $(x_1, \ldots, x_n) \preceq (y_1, \ldots, y_n)$,

where $(x_1, \ldots, x_n) \preceq (y_1, \ldots, y_n)$ if and only if $x_i \preceq y_i$ for all $i \in \{1, \ldots, n\}$.

A particular instance of aggregation functions frequently used in many applications are OWA operators given by Yager [6].

Definition 2. [6] *Let w be a weight vector, i.e., $w = (w_1, \ldots, w_n) \in [0, 1]^n$ with $w_1 + \ldots + w_n = 1$. The Ordered Weighted Averaging operator associated with w, OWA_w, is a mapping $OWA_w : [0, 1]^n \longrightarrow [0, 1]$ defined by*

$$OWA_w(x_1, \ldots, x_n) = \sum_{i=1}^{n} w_i x_{(i)},$$

where $x_{(i)}$, $i = 1, \ldots, n$, denotes the $i-th$ greatest component of the input $(x_1, \ldots, x_n)$.

In this work, we focus on Atanassov intuitionistic fuzzy sets which were presented in 1986 by Atanassov.

Definition 3. [3] *An Atanassov intuitionistic fuzzy set A over the universe $X \neq \emptyset$ is defined as*

$$A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\},$$

where $\mu_A(x), \nu_A(x) \in [0,1]$ *are respectively, the membership and nonmembership degree of the element* x *to* A *and they satisfy* $\mu_A(x) + \nu_A(x) \leq 1$.

We call $(\mu_A(x), \nu_A(x))$ *Atanassov Intuitionistic Fuzzy pair (AIF-pair) and we denote by* $\mathcal{L}([0,1])$ *the set of all possible AIF-pairs, i.e.*

$$\mathcal{L}([0,1]) = \{(\mu, \nu) \mid \mu, \nu \in [0,1] \text{ and } \mu + \nu \leq 1\}.$$

For the sake of simplicity, when the Atanassov intuitionistic fuzzy set and the element of the referential could not be misunderstood we denote the AIF-pair (μ, ν).

In [3] a partial order on AIF-pairs is introduced. This order is certainly enough for defining some aggregation functions but for a suitable definition of OWA operators on these sets a linear order is required. In this way, some recent studies define and construct admissible orders for the different generalizations of fuzzy sets [7,8].

In the following we introduce a construction method of an Atanassov intuitionistic admissible order [4], namely, a linear order which refines the partial order introduced in [3] by Atanassov. That is, a linear order that satisfies that for all $(\mu_1, \nu_1), (\mu_2, \nu_2) \in \mathcal{L}([0,1])$ such that $\mu_1 \leq \mu_2$ and $\nu_1 \geq \nu_2$ then $(\mu_1, \nu_1) \leq (\mu_2, \nu_2)$.

Proposition 1. *Let* M_1, M_2 *be two aggregation functions* $M_1, M_2 : [0,1]^2 \to [0,1]$ *such that for all* $(\mu_1, \nu_1), (\mu_2, \nu_2) \in \mathcal{L}([0,1])$ *the equalities* $M_1(\mu_1, 1-\nu_1) = M_1(\mu_2, 1-\nu_2)$ *and* $M_2(\mu_1, 1-\nu_1) = M_2(\mu_2, 1-\nu_2)$ *hold simultaneously if and only if* $\mu_1 = \mu_2$ *and* $\nu_1 = \nu_2$.

The relation $\leq_{M_1,M_2}$ *on* $\mathcal{L}([0,1])$ *given by* $(\mu_1, \nu_1) \leq_{M_1,M_2} (\mu_2, \nu_2)$ *if and only if*

(i) $M_1(\mu_1, 1-\nu_1) < M_1(\mu_2, 1-\nu_2)$ *or*
(ii) $M_1(\mu_1, 1-\nu_1) = M_1(\mu_2, 1-\nu_2)$ *and* $M_2(\mu_1, 1-\nu_1) \leq M_2(\mu_2, 1-\nu_2)$

is an admissible order on $\mathcal{L}([0,1])$.

Notice that taking

- $M_1(\mu, 1-\nu) = \mu$ and $M_2(\mu, 1-\nu) = 1-\nu$ we recover the intuitionistic lexicographic 1 order on $\mathcal{L}([0,1])$. We denote it by $\leq_{ilex1}$ and it is given by:

$$(\mu_1, \nu_1) \leq_{ilex1} (\mu_2, \nu_2) \text{ if and only if } \mu_1 < \mu_2 \text{ or } (\mu_1 = \mu_2 \text{ and } \nu_1 \geq \mu_2). \quad (1)$$

- $M_1(\mu, 1-\nu) = 1-\nu$ and $M_2(\mu, 1-\nu) = \mu$ we recover the intuitionistic lexicographic 2 order on $\mathcal{L}([0,1])$. We denote it by $\leq_{ilex2}$ and it is given by:

$$(\mu_1, \nu_1) \leq_{ilex2} (\mu_2, \nu_2) \text{ if and only if } \nu_1 > \nu_2 \text{ or } (\nu_1 = \nu_2 \text{ and } \mu_1 \leq \mu_2). \quad (2)$$

3 Unbalanced Atanassov Intuitionistic OWA Operators

In the literature we can find several constructions of OWA operators on the intuitionistic field. In the following, we introduce the first construction of OWA operators on AIFSs considering the partial order introduced by Atanassov on [9]. It is worth mentioning we do not use the original notation on [9] but we rewrite the method following the notation introduced in Sect. 2.

Definition 4. *The OWA aggregation of intuitionistic fuzzy set associated with* $\tilde{w}$ *a weight vector (*$\tilde{w} = (w_1, \ldots, w_n)$ *in* $[0,1]^n$*) such that* $w_1 + \ldots + w_n = 1$ *is given by*

$$UAIOWA_{[\tilde{w},\tilde{v},\leq]}((\mu_1,\nu_1),\ldots,(\mu_n,\nu_n)) = \left(\sum_{i=1}^{n} w_i\mu_{(i)}, \sum_{i=1}^{n} w_{n-i+1}\nu_{(i)}\right), \quad (3)$$

where $\mu_{(n)} \leq \ldots \leq \mu_{(1)}$ *and* $\nu_{(n)} \leq \ldots \leq \nu_{(1)}$.

More recently, some other works about similar aggregation functions but using linear orders are presented. For instance, in [10] where Xu and Yager order is presented, some geometric operators are defined. These operators are IFWG, IFOWG and IFHG. However, in all these operators the same weight vector is considered for both membership and nonmembership degree. In this way, the novelty of the concept of Unbalanced Atanassov intuitionistic OWA operators lies on the use of two different weight vectors $\tilde{w}$ and $\tilde{v}$.

Definition 5. *An Unbalanced Atanassov Intuitionistic OWA(UAIOWA) operator associated with an admissible order* $\leq$ *on* $\mathcal{L}([0,1])$ *and* $\tilde{w}, \tilde{v}$ *weight vectors (*$\tilde{w} = (w_1, \ldots, w_n), \tilde{v} = (v_1, \ldots, v_n)$ *in* $[0,1]^n$ *such that* $w_1 + \ldots + w_n = 1$ *and* $v_1 + \ldots + v_n = 1$*) is a mapping* $UAIOWA_{[\tilde{w},\tilde{v},\leq]} : (\mathcal{L}([0,1]))^n \longrightarrow [0,1]^2$ *given by*

$$UAIOWA_{[\tilde{w},\tilde{v},\leq]}((\mu_1,\nu_1),\ldots,(\mu_n,\nu_n)) = \left(\sum_{i=1}^{n} w_i\mu_{(i)}, \sum_{i=1}^{n} v_i\nu_{(i)}\right), \quad (4)$$

where $(\mu_{(n)},\nu_{(n)}) \leq \ldots \leq (\mu_{(1)},\nu_{(1)})$.

OWA operators on fuzzy sets are particular instances of aggregation functions. In the following we study under which conditions $UAIOWA$ operators are also particular examples of these functions. Since the boundary conditions $UAIOWA_{[\tilde{w},\tilde{v},\leq]}((1,0),\ldots,(1,0)) = (1,0)$ and $UAIOWA_{[\tilde{w},\tilde{v},\leq]}((0,1),\ldots,(0,1)) = (0,1)$ are trivially satisfied, we only need to study the monotonicity with respect to the considered order $\leq$ and when they are well defined, i.e., the codomain is $\mathcal{L}([0,1])$. That is, we have to study when the image of n AIF-pairs satisfies

$$\sum_{i=1}^{n} w_i\mu(i) + \sum_{i=1}^{n} v_i\nu_{(i)} \leq 1.$$

Proposition 2. *Let* $\leq$ *be the order* $\leq_{ilex1}$ *or* $\leq_{ilex2}$ *on* $\mathcal{L}([0,1])$ *(generated as in Eq. (1) or (2), respectively) and* $\tilde{w}, \tilde{v} \in (0,1]^n$*. Then* $UAIOWA_{[\tilde{w},\tilde{v},\leq]}$ *operator satisfies monotonicity.*

Proof. Straight by the monotonicity of the OWA operators when the space considered is the unit interval.

Notice that in Proposition 2, $w_i, v_i \neq 0$ for all $i = 1, ..., n$ is imposed. This fact is crucial as it can be seen in the following example.

Example 1. Let $\leq_{ilex2}$ be the order generated as in Eq. (2), $w = (0.5, 0.5)$ and $v = (0, 1)$ Then

$UAIOWA_{[w,v,\leq_{ilex2}]}((0.9, 0.1), (0, 1)) = (0.9 \cdot 0.5 + 0 \cdot 0.5, 0.1 \cdot 0 + 1 \cdot 1) = (0.45, 1)$.

Similarly, $UAIOWA_{[\tilde{w},\tilde{v},\leq_{ilex2}]}((0.8, 0), (0, 1)) = (0.8 \cdot 0.5 + 0 \cdot 0.5, 0 \cdot 0 + 1 \cdot 1) = (0.4, 1)$. Since $(0.9, 0.1) \leq_{ilex2} (0.8, 0)$ but $(0.45, 1) \geq_{ilex2} (0.4, 1)$, then $UAIOWA$ is not monotonic.

Finally, we study when the image of the operators are always intuitionistic pairs, namely,

$$\left(\sum_{i=1}^{n} w_i \mu_{(i)} + \sum_{i=1}^{n} v_i \nu_{(i)} \leq 1 \right). \tag{5}$$

It is a simple calculation to see that in the more restrictive case, when $\nu_{(i)} = 1 - \mu_{(i)}$, the equation is reduced to

$$\sum_{i=1}^{n} w_i \mu_{(i)} \leq \sum_{i=1}^{n} v_i \mu_{(i)}. \tag{6}$$

Lemma 1. *Let $\tilde{w}, \tilde{v} \in [0,1]^n$ be two weight vectors. Then the following statements are equivalent.*

(i) $\sum_{j=1}^{i} w_j \leq \sum_{j=1}^{i} v_j$ *for all* $i = 1, \ldots, n$.
(ii) $\sum_{i=1}^{n} w_i t_i \leq \sum_{i=1}^{n} v_i t_i$ *for all* $t_i \in [0, 1]$ *such that* $t_1 \geq t_2 \geq \ldots \geq t_n \geq 0$.

Proof. We first prove (i.) implies (ii.). As

$$\begin{array}{lll}
w_1 \leq v_1 \text{ then } & \text{for all } a_1 \geq 0 & a_1 w_1 \leq a_1 v_1 \\
w_1 + w_2 \leq v_1 + v_2 \text{ then } & \text{for all } a_2 \geq 0 & a_2(w_1 + w_2) \leq a_2(v_1 + v_2) \\
\vdots & & \vdots \\
w_1 + \ldots + w_n \leq v_1 + \ldots + v_n \text{ then } & \text{for all } a_n \geq 0 & a_n(w_1 + \ldots + w_n) \leq a_n(v_1 + \ldots + v_n).
\end{array} \tag{7}$$

If we sum

$(a_1 + \ldots + a_n)w_1 + (a_2 + \ldots + a_n)w_2 + \ldots + a_n w_n \leq (a_1 + \ldots + a_n)v_1 + (a_2 + \ldots + a_n)v_2 + \ldots + a_n v_n$ for all $a_1, \ldots, a_n \geq 0$.

Taking $t_1 = (a_1 + \ldots + a_n)$, $t_2 = (a_2 + \ldots + a_n), \ldots, t_n = a_n$ it satisfies (ii.).

Let us see that (ii.) implies (i.). But this is trivial taking $t_1 = t_2 = \ldots = t_i = 1$ and $t_{i+1} = t_{i+2} = \ldots = t_n = 0$.

Finally, we have the following characterization of $UAIOWA$ operators.

Theorem 1. *Let $\tilde{w}, \tilde{v} \in (0,1]^n$ be weight vectors. Then the following statements are equivalent:*

(i) $UAIOWA$ operator associated with $\tilde{w}, \tilde{v}$ and the order $\leq_{ilex1}$ is an aggregation function.

(ii) $\sum_{i=1}^{n} w_i t_i \leq \sum_{i=1}^{n} v_i t_i$ *for all* $t_i \in [0,1]$ *such that* $t_1 \geq t_2 \geq \ldots \geq t_n \geq 0$.

Proof. Notice that since the boundary conditions and the monotonicity holds true, $UIOWA$ is an aggregation operator if the codomain of the function is $\mathcal{L}([0,1])$, namely, the image of n AIF-pairs is always an AIF-pair.

Let us show that (i.) implies (ii.). Suppose $UAIOWA$ is an aggregation function. Then it satisfies Eq. (6) for all $\mu_i \in [0,1]$, $i = 1, \ldots, n$. Due to $\leq_{ilex1}$, $\mu_{(1)} \geq ... \geq \mu_{(n)}$, i.e., they are ordered in a decreasing way. Taking $t_i = \mu_{(i)}$ it satisfies (ii.).

Finally, let us show that (ii.) implies (i.). First of all (ii.) can be rewritten as

$$\sum_{i=1}^{n} (w_i - v_i) t_i \leq 0, \text{ for all } t_i \in [0,1] \text{ such that } t_1 \geq t_2 \geq \ldots \geq t_n \geq 0. \qquad (8)$$

Let (μ_i, ν_i) for $i = 1, \ldots, n$, be n intuitionistic pairs. The expression of $UAIOWA$ associated with $\tilde{w}, \tilde{v}$ and the order $\leq_{ilex1}$ is

$$UAIOWA_{[\tilde{w},\tilde{v},\leq_{ilex1}]}((\mu_1,\nu_1),\ldots,(\mu_n,\nu_n)) = \left(\sum_{i=1}^{n} w_i \mu_{(i)}, \sum_{i=1}^{n} v_i \nu_{(i)}\right),$$

where $\mu_{(1)} \geq \mu_{(2)} \geq \ldots \geq \mu_{(n)}$ due to the order $\leq_{ilex1}$ used.

Considering that $\mu_{(i)} + \nu_{(i)} \leq 1$ and $v_1 + v_2 + \ldots + v_n = 1$ then

$$\sum_{i=1}^{n} w_i \mu_{(i)} + \sum_{i=1}^{n} v_i \nu_{(i)} \leq \sum_{i=1}^{n} w_i \mu_{(i)} + \sum_{i=1}^{n} v_i (1 - \mu_{(i)}) = 1 + \sum_{i=1}^{n} (w_i - v_i) \mu_{(i)} \leq 1,$$

where the last inequation is due to Eq. (8).

Corollary 1. *Let* $\tilde{w}, \tilde{v} \in (0,1]^n$ *be weight vectors. Then the following statements are equivalent:*

(i) $UAIOWA$ *operator associated with* $\tilde{w}, \tilde{v}$ *and the order* $\leq_{ilex1}$ *is an aggregation function.*

(ii) $\sum_{j=1}^{i} w_j \leq \sum_{j=1}^{i} v_j$ *for all* $i = 1, \ldots, n$.

Proof. Straight by Lemma 1 and Theorem 1.

Lemma 2. *Let be* $\tilde{w}, \tilde{v} \in [0,1]^n$ *two weight vectors. Then the following statements are equivalent:*

(i) $\sum_{j=i}^{n} w_j \geq \sum_{j=i}^{n} v_j$ *for all* $i = 1, \ldots, n$.

(ii) $\sum_{i=1}^{n} w_i t_i \geq \sum_{i=1}^{n} v_i t_i$ *for all* $t_i \in [0,1]$ *such that* $t_n \geq t_{n-1} \geq \ldots \geq t_1 \geq 0$.

Proof. Similar to Lemma 1.

Theorem 2. *Let $\tilde{w}, \tilde{v} \in (0,1]^n$ be weight vectors. Then the following statements are equivalent:*

(i) UAIOWA operator associated with $\tilde{w}, \tilde{v}$ and the order $\leq_{ilex2}$ is an aggregation function.

(ii) $\sum_{i=1}^{n} w_i t_i \geq \sum_{i=1}^{n} v_i t_i$ *for all* $t_i \in [0,1]$ *such that* $t_n \geq t_{n-1} \geq \ldots \geq t_1 \geq 0.$

Proof. Similar to Theorem 1.

Corollary 2. *Let $\tilde{w}, \tilde{v} \in (0,1]^n$ be weight vectors. Then the following statements are equivalent*

(i) UAIOWA operator associated with $\tilde{w}, \tilde{v}$ and the order $\leq_{ilex2}$ is an aggregation function.

(ii) $\sum_{j=1}^{i} w_j \leq \sum_{j=1}^{i} v_j$ *for all* $i = 1, \ldots, n$

Proof. Straight by Lemma 2 and Theorem 2.

Remark 1.
It can be seen that:

$$\sum_{j=1}^{i} w_j \leq \sum_{j=1}^{i} v_j \qquad \text{for } i = 1, ..., n-1 \text{ (the condition for } i = n \text{ is trivial)}$$

if and only if

$$1 + \sum_{j=1}^{i} w_j \leq 1 + \sum_{j=1}^{i} v_j \qquad \text{for } i = 1, ..., n-1$$

if and only if

$$1 - \sum_{j=1}^{i} v_j \leq 1 - \sum_{j=1}^{i} w_j \qquad \text{for } i = 1, ..., n-1$$

if and only if

$$\sum_{j=i+1}^{n} v_j \leq \sum_{j=i+1}^{n} w_j \qquad \text{for } i = 1, ..., n-1.$$

Consequently, taking into account that $\tilde{w}$ and $\tilde{v}$ are weight vectors, i.e., $\sum_{i=1}^{n} w_i = 1$ and $\sum_{i=1}^{n} v_i = 1$, the condition that the weight vector must satisfy to be aggregation functions is the same for the orders $\leq_{ilex1}, \leq_{ilex2}$.

Notice that the weight vectors $w_i = v_i$ for all $i = 1, \ldots, n$ satisfy the condition required. In this way, the UAIOWA operators obviously increase the expressiveness and add more flexibility to the aggregation result than other operators. In fact, we believe a deep study on the optimization of the weight vectors could be useful in the improvement of the applications.

Example 2. Take $\tilde{w} = (0.2, 0.3, 0.5)$, $\tilde{v} = (0.4, 0.25, 0.35)$ and $\leq_{ilex2}$. Given the AIF-pairs $(0.3, 0.7)$, $(0.4, 0.2)$ and $(0.1, 0.8)$

$$UAIOWA_{[\tilde{w},\tilde{v},\leq_{ilex2}]}((0.3, 0.7), (0.4, 0.2), (0.1, 0.8)) =$$
$$(0.2 \cdot 0.4 + 0.3 \cdot 0.3 + 0.5 \cdot 0.1, 0.4 \cdot 0.2 + 0.25 \cdot 0.7 + 0.35 \cdot 0.8) = (0.22, 0.535)$$

which satisfies $0.22 + 0.535 \leq 1$.

4 Illustrative Example: Application to a Decision Making Problem

In this section we make use of $UAIOWA$ operators in a decision making problem where information represented by Atanassov intuitionistic fuzzy set needs to be fused. However, we never intended to introduce a real application, but rather to show how it can be used.

We recall that a decision making problem consists on finding which is the best alternative in a set of n elements, $\mathcal{X} = \{x_1, \ldots, x_n\}$. In particular, in this problem we consider a set of four companies where some money can be invested. We ask a set of 50 experts who give their opinion in the following way:

- If they believe investing in the company is a good option, they vote *in favour* of the company.
- If they believe investing in the company is not a good option, they vote *against* the company.
- If they are not sure they vote *abstain*.

In this way, after all the votes we have the results given in Table 1.

We can construct the Atanassov intuitionistic fuzzy set, considering the universe $X = \{\text{Company } 1, \text{Company } 2, \text{Company } 3, \text{Company } 4\}$ and generating the membership and nonmembership degrees dividing the values in favour and against the company by the number of experts.

Table 1. Opinions of the experts with respect of the 4 companies

	Favour	Against	Abstain
Company 1	15	10	25
Company 2	28	14	8
Company 3	30	10	10
Company 4	8	13	29

For instance, with the information of Table 1 the following AIFS is generated:

$$A_1 = \{(C_1, (0.3, 0.2)), (C_2, (0.56, 0.28)), (C_3, (0.6, 0.2)), (C_4, (0.16, 0.26))\}.$$

Due to the international nature of the companies, the cultural differences may have a negative effect on the result. To avoid this situation, we repeat this process in three different countries: Spain, China and Brasil.

The results of the three countries Spain, China and Brasil are summarized in the AIFSs A_1, A_2, A_3, respectively.

$$A_2 = \{(C_1, (0.46, 0.42)), (C_2, (0.4, 0.6)), (C_3, (0.2, 0.5)), (C_4, (0.75, 0.2))\},$$

$$A_3 = \{(C_1, (0.12, 0.34)), (C_2, (0.26, 0.58)), (C_3, (0.7, 0.3)), (C_4, (0.44, 0.26))\}.$$

In the process of choice, the first step to be taken is the fusion of the three AIFSs. In this example, such fusion is carried out using an UAIOWA operator. Since the aim of this section is purely illustrative and for the sake of simplicity and clarity, the order and the weights vectors in this example are set arbitrarily. Nevertheless, in real applications some kind of optimization algorithm should be used to fine-tune them. In the present example the considered order is $\leq_{ilex1}$, and the weight vectors are set to $\tilde{w} = (0.2, 0.25, 0.55)$ and $\tilde{v} = (0.25, 0.35, 0.4)$ (which satisfy the condition of Corollary 1).

The results are

$$UAIOWA_{[\tilde{w},\tilde{v}\leq_{ilex1}]}((0.3, 0.2), (0.46, 0.42), (0.12, 0.34)) = (0.233, 0.311),$$

$$UAIOWA_{[\tilde{w},\tilde{v}\leq_{ilex1}]}((0.56, 0.28), (0.4, 0.6), (0.26, 0.58)) = (0.355, 0.512),$$

$$UAIOWA_{[\tilde{w},\tilde{v}\leq_{ilex1}]}((0.6, 0.2), (0.2, 0.5), (0.7, 0.3)) = (0.4, 0.345),$$

$$UAIOWA_{[\tilde{w},\tilde{v}\leq_{ilex1}]}((0.16, 0.26), (0.75, 0.2), (0.44, 0.26)) = (0.348, 0.245),$$

which generate the AIFS $\tilde{A}$, given by:

$$\tilde{A} = \{(C_1, (0.233, 0.311)), (C_2, (0.355, 0.512)), (C_3, (0.4, 0.345)), (C_4, (0.348, 0.245))\}.$$

Notice that $\tilde{A}$ summarizes the information of the experts of the three countries about the companies. Moreover, since the result is an Atanassov intuitionistic fuzzy set, a linear order is required to take a decision. In this example, we take $\leq_{ilex1}$ since it is the order used for the $UAIOWA$. The ranking of the alternatives is:

Company 3 *better than* Company 2 *better than* Company 4 *better than* Company 1.

Consequently, the best alternative in this illustrative example is to invest in the third company.

5 Conclusion

In the last years there has been an increasing interest in the study of linear orders for the different extensions of Fuzzy Sets. These orders let us define some theoretical notions which could not be trivially generalized. In particular, in this work, we have defined a new class of aggregation functions slightly different from OWA operators which makes use of different weight vectors for the membership and nonmembership degrees. However, the number of linear orders for Atanassov intuitionistic fuzzy sets in which these operators satisfy the monotonicity are really scarce. We let for future work the study of the linear orders in these sets generated by aggregation functions which satisfy the monotonicity.

In the context of the applications, we have only introduced an illustrative example where the parameters are set arbitrarily. Thereby, a deep study of algorithms which fine-tunes the parameters (order and weight vectors) in real applications is left to future researchs.

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