Progress in Probability
71

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High Dimensional Probability VII

The Cargèse Volume

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High Dimensional Probability VII

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Editors
Christian Houdré
Georgia Institute of Technology
Atlanta, GA, USA

David M. Mason
University of Delaware
Department of Applied Economics
and Statistics
Newark, DE, USA

Patricia Reynaud-Bouret
Université Côte d'Azur
Centre national de la recherche scientifique
Laboratoire J.A. Dieudonné
Nice, France

Jan Rosiński
Department of Mathematics
University of Tennessee
Knoxville, TN, USA

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Preface

The *High-Dimensional Probability* proceedings continue a well-established tradition which began with the series of eight *International Conferences on Probability in Banach Spaces,* starting with Oberwolfach in 1975. An earlier conference on Gaussian processes with many of the same participants as the 1975 meeting was held in Strasbourg in 1973. The last Banach space meeting took place in Bowdoin, Maine, in 1991. It was decided in 1994 that, in order to reflect the widening audience and interests, the name of this series should be changed to the *International Conference on High-Dimensional Probability*.

The present volume is an outgrowth of the *Seventh High-Dimensional Probability Conference* (HDP VII) held at the superb *Institut d'Études Scientifiques de Cargèse* (IESC), France, May 26–30, 2014. The scope and the quality of the contributed papers show very well that high-dimensional probability (HDP) remains a vibrant and expanding area of mathematical research. Four of the participants of the first probability on Banach spaces meeting—Dick Dudley, Jim Kuelbs, Jørgen Hoffmann-Jørgensen, and Mike Marcus—have contributed papers to this volume.

HDP deals with a set of ideas and techniques whose origin can largely be traced back to the theory of Gaussian processes and, in particular, the study of their paths properties. The original impetus was to characterize boundedness or continuity via geometric structures associated with random variables in high-dimensional or infinite-dimensional spaces. More precisely, these are geometric characteristics of the parameter space, equipped with the metric induced by the covariance structure of the process, described via metric entropy, majorizing measures and generic chaining.

This set of ideas and techniques turned out to be particularly fruitful in extending the classical limit theorems in probability, such as laws of large numbers, laws of iterated logarithm, and central limit theorems, to the context of Banach spaces and in the study of empirical processes.

Similar developments took place in other mathematical subfields such as convex geometry, asymptotic geometric analysis, additive combinatorics, and random matrices, to name but a few topics. Moreover, the methods of HDP, and especially its offshoot, the concentration of measure phenomenon, were found to have a number of important applications in these areas as well as in statistics, machine learning theory, and computer science. This breadth is very well illustrated by the contributions in the present volume.

Most of the papers in this volume were presented at HDP VII. The participants of this conference are grateful for the support of the Laboratoire Jean Alexandre Dieudonné of the Université de Nice Sophia-Antipolis, of the school of Mathematics at the Georgia Institute of Technology, of the CNRS, of the NSF (DMS Grant # 1441883), of the French Agence Nationale de la Recherche (ANR 2011 BS01 010 01 project Calibration), and of the IESC. The editors also thank Springer-Verlag for agreeing to publish the proceedings of HDP VII.

The papers in this volume aptly display the methods and breadth of HDP. They use a variety of techniques in their analysis that should be of interest to advanced students and researchers. This volume begins with a dedication to the memory of our close colleague and friend, Evarist Giné-Masdeu. It is followed by a collection of contributed papers that are organized into four general areas: inequalities and convexity, limit theorems, stochastic processes, and high-dimensional statistics. To give an idea of their scope, we briefly describe them by subject area in the order they appear in this volume.

Dedication to Evarist Giné-Masdeu

- *Evarist Giné-Masdeu July 31, 1944–March 15, 2015.* This article is made up of reminiscences of Evarist's life and work, from many of the people he touched and influenced.

Inequalities and Convexity

- *Stability of Cramer's Characterization of the Normal Laws in Information Distances,* by S.G. Bobkov, G.P. Chistyakov, and F. Götze. The authors establish the stability of Cramer's theorem, which states that if the convolution of two distributions is normal, both have to be normal. Stability is studied for probability measures that have a Gaussian convolution component with small variance. Quantitative estimates in terms of this variance are derived with respect to the total variation norm and the entropic distance. Part of the arguments used in the proof refine Sapogov-type theorems for random variables with finite second moment.
- *V.N. Sudakov's Work on Expected Suprema of Gaussian Processes,* by Richard M. Dudley. The paper is about two works of V.N. Sudakov on expected suprema of Gaussian processes. The first was a paper in the Japan-USSR Symposium on probability in 1973. In it he defined the expected supremum (without absolute values) of a Gaussian process with mean 0 and showed its usefulness. He gave an upper bound for it as a constant times a metric entropy integral, without proof. In 1976 he published the monograph, "Geometric Problems in the Theory

of Infinite-Dimensional Probability Distributions," in Russian, translated into English in 1979. There he proved his inequality stated in 1973. In 1983. G. Pisier gave another proof. A persistent rumor says that R. Dudley first proved the inequality, but he disclaims this. He defined the metric entropy integral, as an equivalent sum in 1967 and then as an integral in 1973, but the expected supremum does not appear in these papers.

- *Optimal Concentration of Information Content for Log-Concave Densities* by Matthieu Fradelizi, Mokshay Madiman, and Liyao Wang. The authors aim to generalize the fact that a standard Gaussian measure in $\mathbb{R}^n$ is effectively concentrated in a thin shell around a sphere of radius $\sqrt{n}$. While one possible generalization of this—the notorious "thin-shell conjecture"—remains open, the authors demonstrate that another generalization is in fact true: any log-concave measure in high dimension is effectively concentrated in the annulus between two nested convex sets. While this fact was qualitatively demonstrated earlier by Bobkov and Madiman, the current contribution identifies sharp constants in the concentration inequalities and also provides a short and elegant proof.
- *Maximal Inequalities for Dependent Random Variables,* by J. Hoffmann-Jørgensen. Recall that a maximal inequality is an inequality estimating the maximum of partial sum of random variables or vectors in terms of the last sum. In the literature there exist plenty of maximal inequalities for sums of independent random variables. The present paper deals with dependent random variables satisfying some weak independence, for instance, maximal inequalities of the Rademacher-Menchoff type or of the Ottaviani-Levy type or maximal inequalities for negatively or positively correlated random variables or for random variables satisfying a Lipschitz mixing condition.
- *On the Order of the Central Moments of the Length of the Longest Common Subsequences in Random Words,* by Christian Houdré and Jinyong Ma. The authors study the order of the central moments of order r of the length of the longest common subsequences of two independent random words of size n whose letters are identically distributed and independently drawn from a finite alphabet. When all but one of the letters are drawn with small probabilities, which depend on the size of the alphabet, a lower bound of order $n^{r/2}$ is obtained. This complements a generic upper bound also of order $n^{r/2}$.
- *A Weighted Approximation Approach to the Study of the Empirical Wasserstein Distance,* by David M. Mason. The author shows that weighted approximation technology provides an effective set of tools to study the rate of convergence of the Wasserstein distance between the cumulative distribution function [c.d.f] and the empirical c.d.f. A crucial role is played by an exponential inequality for the weighted approximation to the uniform empirical process.
- *On the Product of Random Variables and Moments of Sums Under Dependence,* by Magda Peligrad. This paper establishes upper and lower bounds for the moments of products of dependent random vectors in terms of mixing coefficients. These bounds allow one to compare the maximum term, the characteristic function, the moment-generating function, and moments of sums of a dependent vector with the corresponding ones for an independent vector with the same

marginal distributions. The results show that moments of products and partial sums of a phi-mixing sequence are close in a certain sense to the corresponding ones of an independent sequence.

- *The Expected Norm of a Sum of Independent Random Matrices: An Elementary Approach,* by Joel A. Tropp. Random matrices have become a core tool in modern statistics, signal processing, numerical analysis, machine learning, and related areas. Tools from high-dimensional probability can be used to obtain powerful results that have wide applicability. Tropp's paper explains an important inequality for the spectral norm of a sum of independent random matrices. The result extends the classical inequality of Rosenthal, and the proof is based on elementary principles.
- *Fechner's Distribution and Connections to Skew Brownian Motion,* by Jon A. Wellner. Wellner's paper investigates two aspects of Fechner's two-piece normal distribution: (1) Connections with the mean-median-mode inequality and (strong) log-concavity (2) Connections with skew and oscillating Brownian motion processes.

Limit Theorems

- *Erdös-Rényi-Type Functional Limit Laws for Renewal Processes*, by Paul Deheuvels and Joseph G. Steinebach. The authors discuss functional versions of the celebrated Erdős-Rényi strong law of large numbers, originally stated as a local limit theorem for increments of partial sum processes. We work in the framework of renewal and first-passage-time processes through a duality argument which turns out to be deeply rooted in the theory of Orlicz spaces.
- *Limit Theorems for Quantile and Depth Regions for Stochastic Processes,* by James Kuelbs and Joel Zinn. Contours of multidimensional depth functions often characterize the distribution, so it has become of interest to consider structural properties and limit theorems for the sample contours. Kuelbs and Zinn continue this investigation in the context of Tukey-like depth for functional data. In particular, their results establish convergence of the Hausdorff distance for the empirical depth and quantile regions.
- *In Memory of Wenbo V. Li's Contributions,* by Q.M. Shao. Shao's notes are a tribute to Wenbo Li for his contributions to probability theory and related fields and to the probability community. He also discusses several of Wenbo's open questions.

Stochastic Processes

- *Orlicz Integrability of Additive Functionals of Harris Ergodic Markov Chains,* by Radosław Adamczak and Witold Bednorz. Adamczak and Bednorz consider integrability properties, expressed in terms of Orlicz functions, for "excursions" related to additive functionals of Harris Markov chains. Applying the obtained inequalities together with the regenerative decomposition of the functionals, we obtain limit theorems and exponential inequalities.

- *Bounds for Stochastic Processes on Product Index Spaces,* by Witold Bednorz. In many questions that concern stochastic processes, the index space of a given process has a natural product structure. In this paper, we formulate a general approach to bounding processes of this type. The idea is to use a so-called majorizing measure argument on one of the marginal index spaces and the entropy method on the other. We show that many known consequences of the Bernoulli theorem—complete characterization of sample boundedness for canonical processes of random signs—can be derived in this way. Moreover we establish some new consequences of the Bernoulli theorem, and finally we show the usefulness of our approach by obtaining short solutions to known problems in the theory of empirical processes.
- *Permanental Vectors and Self Decomposability,* by Nathalie Eisenbaum. Exponential variables and more generally gamma variables are self-decomposable. Does this property extend to the class of multivariate gamma distributions? We consider the subclass of the permanental vectors distributions and show that, obvious cases excepted, permanental vectors are never self-decomposable.
- *Permanental Random Variables, M-Matrices, and M-Permanents,* by Michael B. Marcus and Jay Rosen. Marcus and Rosen continue their study of permanental processes. These are stochastic processes that generalize processes that are squares of certain Gaussian processes. Their one-dimensional projections are gamma distributions, and they are determined by matrices, which, when symmetric, are covariance matrices of Gaussian processes. But this class of processes also includes those that are determined by matrices that are not symmetric. In their paper, they relate permanental processes determined by nonsymmetric matrices to those determined by related symmetric matrices.
- *Convergence in Law Implies Convergence in Total Variation for Polynomials in Independent Gaussian, Gamma or Beta Random Variables,* by Ivan Nourdin and Guillaume Poly. Nourdin and Poly consider a sequence of polynomials of bounded degree evaluated in independent Gaussian, gamma, or beta random variables. Whenever this sequence converges in law to a nonconstant distribution, they show that the limit distribution is automatically absolutely continuous (with respect to the Lebesgue measure) and that the convergence actually takes place in the total variation topology.

High-Dimensional Statistics

- *Perturbation of Linear Forms of Singular Vectors Under Gaussian Noise,* by Vladimir Koltchinskii and Dong Xia. The authors deal with the problem of estimation of linear forms of singular vectors of an $m \times n$ matrix A perturbed by a Gaussian noise. Concentration inequalities for linear forms of singular vectors of the perturbed matrix around properly rescaled linear forms of singular vectors of A are obtained. They imply, in particular, tight concentration bounds for the perturbed singular vectors in the ℓ_∞-norm as well as a bias reduction method in the problem of estimation of linear forms.

- *Optimal Kernel Selection for Density Estimation,* by M. Lerasle, N. Magalhães, and P. Reynaud-Bouret. The authors provide new general kernel selection rules for least-squares density estimation thanks to penalized least-squares criteria. They derive optimal oracle inequalities using concentration tools and discuss the general problem of minimal penalty in this framework.

Atlanta, GA, USA — Christian Houdré
Newark, DE, USA — David M. Mason
Nice, France — Patricia Reynaud-Bouret
Knoxville, TN, USA — Jan Rosiński

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List of Participants

Radosław Adamczak	University of Warsaw, Poland
Mélisande Albert	Université de Nice Sophia-Antipolis, France
Sylvain Arlot	ENS-Paris, France
Benjamin Arras	École Centrale de Paris, France
Yannick Baraud	Université Nice Sophia Antipolis, France
Witold Bednorz	University of Warsaw, Poland
Bernard Bercu	Université Bordeaux 1, France
Sergey Bobkov	University of Minnesota, USA
Stéphane Boucheron	Université Paris-Diderot, France
Silouanos Brazitikos	University of Athens, Greece
Sébastien Bubeck	Princeton University, USA
Dariusz Buraczewski	University of Wroclaw, Poland
Djalil Chafaï	Université Paris-Dauphine, France
Julien Chevallier	University of Nice, France
Ewa Damek	University of Wroclaw, Poland
Yohann de Castro	Université Paris Sud, France
Victor de la Peña	Columbia University, USA
Paul Deheuvels	Université Paris VI, France
Dainius Dzindzalieta	Vilnius University, Lithuania
Peter Eichelsbacher	Ruhr-Universität Bochum, Germany
Nathalie Eisenbaum	Université Paris VI, France
Xiequan Fan	Tianjin University, China
José Enrique Figueroa-López	Purdue University, USA
Apostolos Giannopoulos	University of Athens, Greece
Nathael Gozlan	Université Paris-Est-Marne-la Vallée, France
Labrini Hioni	University of Athens, Greece
Jørgen Hoffmann-Jørgensen	Aarhus University, Denmark
Christian Houdré	Georgia Institute of Technology, USA
Vladimir Koltchinskii	Georgia Institute of Technology, USA
Rafal Latała	University of Warsaw, Poland
Mikhail Lifshits	St. Petersburg, Russia

Karim Lounici	Georgia Institute of Technology, USA
Mokshay Madiman	University of Delaware and Yale University, USA
Philippe Marchal	Université Paris 13, France
Eleftherios Markessinis	University of Athens, Greece
David Mason	University of Delaware, USA
Mario Milman	Florida Atlantic University, USA
Nelo Molter Magalhães	Université Pierre et Marie Curie, France
Ivan Nourdin	Institut Élie Cartan, France
Krzysztof Oleszkiewicz	University of Warsaw, Poland
Giovanni Peccati	Luxembourg University, Luxembourg
Magda Peligrad	University of Cincinnati, USA
Ionel Popescu	Georgia Institute of Technology, USA
Patricia Reynaud-Bouret	Université Côte d'Azur, France
Jay Rosen	The City University of New York, USA
Adrien Saumard	University of Washington, USA
Pierre-André Savalle	École Centrale Paris, France
Qi-Man Shao	The Chinese University of Hong Kong, Hong Kong
Maud Thomas	Université Paris-Diderot, France
Joel Tropp	California Institute of Technology, USA
Mark Veraar	Delft University of Technology, The Netherlands
Olivier Wintenberger	Université Pierre et Marie Curie-Paris VI, France
Pawel Wolff	University of Warsaw, Poland

Evarist Giné-Masdeu

This volume is dedicated to the memory of our dear friend and colleague, Evarist Giné-Masdeu, who passed away at age 70 on March 13, 2015. We greatly miss his supportive and engendering influence on our profession. Many of us in the high-dimensional probability group have had the pleasure of collaborating with him on joint publications or were strongly influenced by his ideas and suggestions. Evarist has contributed profound, lasting, and beautiful results to the areas of probability on Banach spaces, the empirical process theory, the asymptotic theory of the bootstrap and of U-statistics and processes, and the large sample properties of nonparametric statistics and function estimators. He has, as well, given important service to our profession as an associate editor for most of the major journals in probability theory such as *Annals of Probability, Journal of Theoretical Probability, Electronic Journal of Probability, Bernoulli Journal*, and *Stochastic Processes and Their Applications*.

Evarist received his Ph.D. from MIT in 1973 under the direction of Richard M. Dudley and subsequently held academic positions at Universitat Autonoma of Barcelona; Universidad de Carabobo, Venezuela; University of California, Berkeley; Louisiana State University; Texas A&M; and CUNY. His last position was at the University of Connecticut, where he was serving as chair of the Mathematics Department, at the time of his death. He guided eight Ph.D. students. One of whom, the late Miguel Arcones, was a fine productive mathematician and a member of our high-dimensional Probability group.

More information about Evarist's distinguished career and accomplishments, including descriptions of his books and some of his major publications, are given in his obituary on page 8 of the June/July 2015 issue of the IMS Bulletin.

Here are remembrances by some of Evarist's many colleagues.

Rudolf Beran

I had the pleasure of meeting Evarist, through his work and sometimes in person, at intervals over many years. Though he was far more mathematical than I am, not to mention more charming, our research interests interacted at least twice. In a 1968 paper, I studied certain rotationally invariant tests for uniformity of a distribution on a sphere. Evarist saw a way, in 1975 work, to develop invariant tests for uniformity on compact Riemannian manifolds, a major technical advance. It might surprise some that Evarist's theoretical work has facilitated the development of statistics as a tested practical discipline no longer limited to analyzing Euclidean data. I am not surprised. He was a remarkable scholar with clear insight as well as a gentleman.

Tasio del Barrio

I first met Evarist Giné as a Ph.D. student through his books and papers in probability on Banach spaces and empirical processes. I had already come to admire his work in these fields when I had the chance to start joint research with him. It turned out to be a very rewarding experience. This was not only for his mathematical talent but also for his kind support in my postdoc years. I feel a great loss of both a mathematician and a friend.

Victor de la Peña

From the first moment I met Evarist, I felt the warmth with which he welcomed others. I met him in College Station during an interview and was fortunate to be able to interact with him. I took a job at Columbia University in New York but frequently visited College Station where he was a professor of mathematics. Eventually, Evarist moved to CUNY, New York. I was fortunate to have him as a role model and in some sense mentor.

He was a great mathematician with unsurpassed insight into problems. On top of this, he was great leader and team player. I had the opportunity to join one of his multiple teams in the nascent area of U-processes. These statistical processes are extensions of the sample average and sample variance. The theory and applications of U-processes have been key tools in the advancement of many important areas. To cite an example, work in this area is important in assessing the speed at which information (like movies) is transmitted through the Internet.

I can say without doubt that the work I did under his mentorship helped launch my career. His advice and support were instrumental in me eventually getting tenure at Columbia University. In 1999 we published a book summarizing the theory and applications of U-processes (mainly developed by Evarist and coauthors). Working on this project, I came to witness his great mathematical power and generosity.

I will always remember Evarist as a dear friend and mentor. The world of mathematics has lost one of its luminaries but his legacy lives for ever.

Friedrich Götze

It was at one of the conferences on probability in Banach spaces in the eighties that I met Evarist for the first time. I was deeply impressed by his mathematical talent and originality, and at the same time, I found him to be a very modest and likeable person. In the summer, he used to spend some weeks with Rosalind in Barcelona and often traveled in Europe, visiting Bielefeld University several times in the nineties. During his visits, we had very stimulating and fruitful collaborations on tough open questions concerning inverse problems for self-normalized statistics. Later David Mason joined our collaboration during his visits in Bielefeld. Sometimes, after intensive discussions in the office, Evarist needed a break, which often meant that they continued in front of the building, while he smoked one of his favorite cigars. We carried on our collaboration in the new millennium, and I warmly remember Evarist's and Rosalind's great hospitality at their home, when I visited them in Storrs.

I also very much enjoyed exchanging views with him on topics other than mathematics, in particular, concerning the history and future of the Catalan nation, a topic in which he engaged himself quite vividly. I learned how deeply he felt about this issue in 2004, when we met at the Bernoulli World Congress in his hometown Barcelona. One evening, we went together with our wives and other participants of the conference for an evening walk in the center to listen to a concert in the famous cathedral Santa Maria del Mar. We enjoyed the concert in this jewel of Catalan Gothic architecture and Evarist felt very much at home. After the concert, we went to a typical Catalan restaurant. But then a waiter spoiled an otherwise perfect evening by insisting on responding in Spanish only to Evarist's menu inquiries in Catalan. Evarist got more upset than I had ever seen him.

It was nice to meet him again at the Cambridge conference in his honor in 2014, and we even discussed plans for his next visit to Bielefeld, to continue with one of our long-term projects. But fate decided against it.

With Evarist we have all lost much too early a dear colleague and friend.

Marjorie Hahn

Together Evarist Giné and I were Ph.D. students of Dick Dudley at MIT, and I have benefited from his friendship and generosity ever since. Let me celebrate his life, accomplishments, and impact with a few remarks on the *legacy by example he leaves for all of us*.

- *Evarist had incredible determination*. On several occasions, Evarist reminded me that his mathematical determination stemmed largely from the following experience: After avoiding Dick's office for weeks because of limited progress on his research problem, Evarist requested a new topic. Dick responded, "If I had worked on a problem for that long, I wouldn't give up." This motivated Evarist to try again with more determination than ever, and as a result, he solved his problem. As Evarist summarized it: "Solving mathematical problems can be really hard, but the determination to succeed can make a huge difference."
- *Evarist was an ideal collaborator*. Having written five papers with Evarist, I can safely say that he always did more than his share, yet always perceived that he didn't do enough. Moreover, he viewed a collaboration as an opportunity for us to learn from each other, and I surely learned a lot from him.
- *Evarist regarded his contributions and his accomplishments with unfailing humility*. Evarist would tell me that he had "a small result that he kind of liked." After explaining the result, I'd invariably tell him that his result either seemed major or should have major implications. Only then would his big well-known smile emerge as he'd admit that deep down he really liked the result.
- *Evarist gave generously of his time to encourage young mathematicians*. Due to Evarist's breadth of knowledge and skill in talking to and motivating graduate students, I invited him to be the outside reader on dissertation committees for at least a half dozen of my Ph.D. students. He took his job seriously, giving the students excellent feedback that included ideas for future work.

We can honor Evarist and his mathematical legacy the most by following his example of quiet leadership.

Christian Houdré
Two things come to my mind when thinking of Evarist. First is his generosity, simple and genuine, which I experienced on many occasions, in particular when he involved me into the HDP organization. Second is his fierce Catalan nationalism to which I was definitively very sympathetic with my Québec background. He occasionally wrote to me in Catalan and I also warmly remember his statistic that one out of three French people in Perpignan spoke Catalan. (He had arrived to that statistic after a short trip to Perpignan where although fluent in French, he refused to speak it since he was in historic Catalonia. If I recall correctly after two failed attempts at trying to be understood in Catalan, the third trial was the good one.) He was quite fond of this statistic.

Vladimir Koltchinskii
I met Evarist for the first time at a conference on probability and mathematical statistics in Vilnius, in 1985. This was one of very few conferences where probabilists from the West and from the East were able to meet each other before the fall of the Berlin Wall. I was interested in probability in Banach spaces and knew some of Evarist's work. A couple of years earlier, Evarist got interested in

empirical processes. I started working on the same problems several years earlier, so this was our main shared interest back then. I remember that around 1983 one of my colleagues, who was, using Soviet jargon of the time, “viezdnoj” (meaning that he was allowed to travel to the West), brought me a preprint of a remarkable paper by Evarist Giné and Joel Zinn that continued some of the work on symmetrization and random entropy conditions in central limit theorems for empirical processes that I started in my own earlier papers. In some sense, Evarist and Joel developed these ideas to perfection. Our conversations with Evarist in 1985 (and also at the First Bernoulli Congress in Tashkent 1 year later) were mostly about these ideas. At the same time, Evarist was trying to convince me to visit him at Texas A&M; I declined the invitation since I was pretty sure that I would not be allowed to leave the country. However, our life is full of surprises: the Soviet Union, designed to stay for ages, all of a sudden started crumbling and then collapsing and then ceased to exist, and in January of 1992, I found myself on a plane heading to New York. Evarist picked me up at JFK airport and drove me up to Storrs, Connecticut. For anybody who moved across the Atlantic Ocean and settled in the USA, America starts with something. For me, the beginning of America was Evarist’s old Mazda. The first meal I had in the USA was a bar of Häagen Dazs ice cream that Evarist highly recommended and bought for us at a gas station on our way to Storrs.

In 1992, I spent one semester at Storrs. I do not recall actively working with Evarist on any special project during these 4 months, but we had numerous conversations (on mathematics and far beyond) in Evarist’s office filled with the smoke of his cigar, and we had numerous dinners together with him and his wife Rosalind in their apartment or in one of the local restaurants (most often, at Wilmington Pizza House). In short, I had not found a collaborator in Evarist during this first visit, but I found a very good friend. It was very easy to become a friend with Evarist. There was something about his personality that we all have as children (when we make friends fast), but we are losing this ability as we grow older. His contagious love of life was seen in his smile and in his genuine interest in many different things ranging from mathematics to music and arts and also to food, wine, and good conversation. It is my impression that on March 13, 2015, many people felt that they lost a friend (even those who met him much later than myself and have not interacted with him as much as myself).

In the years that followed my first visit to Storrs, we met with Evarist very frequently: in Storrs, in Boston, in Albuquerque, in Atlanta, in Paris, in Cambridge, in Oberwolfach, in Seattle, and in his beloved Catalonia. In fact, he stayed in all the houses or apartments where I lived in the USA. The last time we met was in Boston, in October 2014. I was giving a talk at MIT. Evarist could not come for the talk, but he came with Rosalind on Sunday. My wife and I went with them to the Museum of Fine Arts to see Goya’s exhibition and had lunch together. Nothing was telling me that it was the last time I would see him.

We always had lengthy conversations about mathematics (most often, in front of the board) and about almost anything else in life and numerous dinners together, but we had also worked together for a number of years, which resulted in 7 papers we published jointly. I really liked Evarist’s attitude toward mathematics: there was almost Mozartian mix of seriousness and joyfulness about it. He was extremely honest about what he was doing, and, being a brilliant and ambitious mathematician, he never got in a trap of working on something just because it was a "hot topic." He probably had a "daimonion" inside of him (as Socrates called it) that prohibited him from doing this. There have been many things over the past 30 years that were becoming fashionable all of a sudden and were going out of fashion without leaving a trace. I remember Evarist hearing some of the talks on these fashionable subjects and losing his interest after a minute or two. Usually, you would not hear a negative comment from him about the talk. He would only say with his characteristic smile: "I know nothing about it." He actually believed that other people were as honest as he was and would not do rubbish (even if it sounded like rubbish to him and it was, indeed, rubbish) and he just "knew nothing about it." We do not remember many of these things now. But we will remember what Evarist did. A number of his results and the tools he developed in probability in Banach spaces, empirical processes, and U-statistics are now being used and will be used in probability, statistics, and beyond. And those of us, who were lucky to know him and work with him, will always remember his generosity and warmth.

Jim Kuelbs
Evarist was an excellent mathematician, whose work will have a lasting impact on high-dimensional probability. In addition, he was a very pleasant colleague who provided a good deal of wisdom and wit about many things whenever we met. It was my good fortune to interact with him at meetings in Europe and North America on a fairly regular basis for nearly 40 years, but one occasion stands out for me. It was not something of great importance, or even mathematical, but we laughed about it for many years. In fact, the last time was only a few months before his untimely death, so I hope it will also provide a chuckle for you.

The story starts when Evarist was at IVIC, the Venezuelan Institute of Scientific Research, and I was visiting there for several weeks. My wife's mother knew that one could buy emeralds in Caracas, probably from Columbia, so 1 day Evarist and I went to look for them. After visits to several shops, we got a tip on an address that was supposedly a good place for such shopping. When we arrived there, we were quite surprised as the location was an open-air tabac on a street corner. Nevertheless, they displayed a few very imperfect green stones, so we asked about emeralds. We were told these were emeralds, and that could well have been true, but they had no clarity in their structure. We looked at various stones a bit and were about ready to give up on our chase, when Evarist asked for clear cuts of emeralds. Well, the guy reached under the counter and brought out a bunch of newspaper packages, and in these packages, we found something that was much more special. Eventually we bought some of these items, and as we walked back to the car, Evarist summarized the experience exceedingly well by saying: "We bought some very nice emeralds at a reasonable price, or paid a lot for some green glass." The stones proved to be real, and my wife still treasures the things made from them.

Rafał Latała

I spent the fall semester of 2001 at Storrs and was overwhelmed with the hospitality of Evarist and his wife Rosalind. They invited me to their home many times, helped me with my weekly shopping, (I did not have a car then), and took me to Boston several times, where their daughters lived. We had pizza together on Friday evenings at their favorite place near Storrs. It was always a pleasure to talk with them, not only about mathematics, academia, and related issues but also about family, friends, politics, Catalan and Polish history, culture, and cuisine.

Evarist was a bright, knowledgeable, and modest mathematician, dedicated to his profession and family. I enjoyed working with him very much. He was very efficient in writing down the results and stating them in a nice and clean way. I coauthored two papers with him on U-statistics.

Michel Ledoux

In Cambridge, England, June 2014, a beautiful and cordial conference was organized to celebrate Evarist's 70th birthday. At the end of the first day's sessions, I went to a pizzeria with Evarist, Rosalind, Joel, Friedrich Götze, and some others. Evarist ordered pizza (with no tomato!) and ice cream.

For a moment, I felt as though it was 1986 when I visited Texas A&M University as a young assistant professor, welcomed by Evarist and his family at their home, having lunch with him, Mike, and Joel and learning about (nearly measurable!) empirical processes. I was simply learning how to do mathematics and to be a mathematician. Between these two moments, Evarist was a piercing beacon of mathematical vision and a strong and dear friend. He mentioned at the end of the conference banquet that he never expected such an event. But it had to be and couldn't be more deserved. We will all miss him.

Vidyadhar Mandrekar

Prof. Evarist Giné strongly impacted the field of probability on Banach spaces beginning with his thesis work. Unfortunately, at the time he received his Ph.D., it was difficult to get an academic position in the USA, so he moved to Venezuela for his job. In spite of being isolated, he continued his excellent work. I had a good opportunity to showcase him at an AMS special session on limit theorems in Banach spaces (at Columbus). Once researchers saw his ideas, he received job offers in this country and the rest is history. Since he could then easily interact with fellow mathematicians, the area benefited tremendously. I had the good fortune of working with him on two papers. One shows a weakness of general methods in Banach space not being strong to obtain a Donsker theorem. However, Evarist continued to adapt Banach space methods to the study of empirical processes with Joel Zinn, which were very innovative and fundamental with applications to statistics. His death is a great loss to this area in particular and to mathematics in general.

Michael B. Marcus

Evarist and I wrote 5 papers together between 1981 and 1986. On 2 of them, Joel Zinn was a coauthor. But more important to me than our mathematical collaboration was that Evarist and I were friends.

I had visited Barcelona a few times before I met Evarist but only briefly. I was very happy when he invited me to give a talk at Universidad Autonoma de Barcelona in the late spring of 1980. I visited him and Rosalind in their apartment in Barcelona. My visit to Barcelona was a detour on my way to a conference in St. Flour. Evarist was going to the conference also so after a few days in Barcelona we drove off in his car to St. Flour. On the way, we pulled off the highway and drove to a lovely beach town (I think it was Rossas), parked the car by the harbor, and went for a long swim. Back in the car, we crossed into France and stopped at a grocery on the highway near Beziers, for a baguette and some charcuterie. We were having such a good time. Evarist didn't recognize this as France. To him, he was still in Catalonia. He spoke in Catalan to the people who waited on us.

I was somewhat of a romantic revolutionary myself in those days and I thought that Evarist, this gentlest of men, must dream at night of being in the mountains organizing an insurgency to free Catalonia from its Spanish occupiers. I was very moved by a man who was so in love with his country. I learned that he was a farmer's son, whose brilliance was noticed by local priests and who made it from San Cugat to MIT, and he longed to return. He said he would go back when he retired, and I said you will have grandchildren and you will not want to leave them.

In 1981 Joel Zinn and I went to teach at Texas A&M. A year later Evarist joined us. We worked together on various questions in probability in Banach spaces. At this time, Dick Dudley began using the techniques that we had all developed together to study questions in theoretical mathematical statistics. Joel and Evarist were excited by this and began their prolific fine work on this topic. I think that Evarist's work in theoretical statistics was his best work. So did very many other mathematicians. He received a lot of credit which was well deserved.

My own work took a different direction. From 1986 on, we had different mathematical interests but our friendship grew. My wife Jane and I saw Evarist and Rosalind often. We cooked for each other and drank Catalan wine together. I also saw Evarist often at the weeklong specialty conferences that we attended, usually in the spring or summer, usually in a beautiful, exotic location. After a day of talks, we had dinner together and then would talk with colleagues and drink too much wine. I often rested a bit after dinner and then went to the lounge. I walked into the room and looked for Evarist. I would see him. Always with a big smile. Always welcoming. Always glad to see me. Always my dear friend. I miss him very much.

David M. Mason

I thoroughly enjoyed working with Evarist on knotty problems, especially when we were narrowing in on a solution. It was like closing in on the pursuit of an elusive and exotic beast. We published seven joint papers, the most important being our first, in which, with Friedrich Götze, we solved a long-standing conjecture concerning the Student *t*-statistic being asymptotically standard normal. As his other collaborators, I will miss the excitement and intense energy of doing mathematics with him. An extremely talented and dedicated mathematician, as well as a complete gentleman, has left us too soon.

On a personal note, I have fond memories of a beautiful Columbus Day 1998 weekend that I spent as a guest of Evarist and Rosalind at their timeshare near Montpelier, Vermont, during the peak of the fall colors. I especially enjoyed having a fine meal with them at the nearby New England Culinary Institute. On that same visit, Evarist and I met up with Dick Dudley and hiked up to the Owl's Head in Vermont's Groton State Forest. I managed to take a striking photo of Evarist at the rock pausing for a cigar break with the silver blue Kettle Pond in the distance below surrounded by a dense forest displaying its brilliant red and yellow autumn leaf cover.

Richard Nickl

I met Evarist in September 2004, when I was in the 2nd year of my Ph.D., at a summer school in Laredo, Cantabria, Spain, where he was lecturing on empirical processes. From the mathematical literature I had read by myself in Vienna for my thesis, I knew that he was one of the most substantial contributors and co-creators of empirical process theory, and I was excited to be able to meet a great mind like him in person. His lectures (mostly on Talagrand's inequalities) were outstanding. It was unbelievable for me that someone of his distinction would say at some point during his lecture course that "his most important achievement in empirical process theory was that he got Talagrand to work in the area"—at that time, when I thought that mathematics was all about egos and greatness, I could not believe that someone of his stature would say something obviously nonsensical like that! But it was a genuine feature of his humility that I always found excessive but that over the years I learnt was actually at the very heart of his great mathematical talent.

Evarist then was most kind to me as a very junior person, and he supported me from the very beginning, asking me about my Ph.D. work and encouraging me to pursue it further and more importantly getting me an invitation to the "high-dimensional probability" conference in Santa Fe, New Mexico, in 2005, where I met most of the other greats of the field for the first time. More importantly, of course, then Evarist invited me to start a postdoc with him in Connecticut, which I did in 2006–2008. We wrote eight papers and one 700-page monograph, and working with Evarist I can say without doubt was the most impressive period of my life so far as a mathematician. It transformed me completely. Throughout these years, despite his seniority, he was most hard working and passionate, and his mathematical sharpness was as effective as ever (even if, as Evarist said, he was perhaps a bit slower, but the final results didn't show this). It is a great privilege, probably the greatest of my life, that I could work with him over such an intensive period of time and to learn from one of the "masters" of the subject—which he was in the area of mathematics that was relevant for the part of theoretical statistics we were working on. I am very sad that now I cannot really return the favor to equal extent: at least the fact that I could contribute to the organization of a conference in his honor in Cambridge in June 2014 forms a small part of saying thank you for everything he has done for me. This conference, which highlighted his great standing within various fields of mathematics, made him very happy, and I think all of us who were there were very happy to see him earn and finally accept the recognition.

I want to finally mention the many great nonmathematical memories I have with Evarist and his wife Rosalind: From our first dinner out in Storrs with Rosalind at Wilmington Pizza to the many great dinners at their place in Storrs, to the many musical events we have been to together including Mozart's Figaro at the Metropolitan Opera in New York, to hear Pollini play in the musical capitals Storrs and Vienna, to concerts of the Boston Symphony in Boston and Tanglewood, to my visit of "his" St. Cugat near Barcelona, to the hike on Mount Monadnock with Evarist and Dick Dudley in October 2007, and to the last time I saw him in person, having dinner at Legal Seafoods in Cambridge (MA) in September 2014. All these great memories, mathematical or not, will remain as alive as they are now. They make it even more impossible for me to believe that someone as energetic, kind, and passionate as Evarist has left us. He will be so greatly missed.

David Nualart

Evarist Giné was a very kind person and an honest and dedicated professional. His advice was always very helpful to me. We did our undergraduate studies in mathematics at the University of Barcelona. He graduated 5 years before me. After receiving his Ph.D. at the Massachusetts Institute of Technology, he returned to Barcelona to accept a position at the Universitat Autonoma of Barcelona. That is when I met Evarist for the first time.

During his years in Barcelona, Evarist was a mentor and inspiration to me and to the small group of probabilists there. I still remember his series of lectures on the emerging topic of probabilities on Banach spaces. Those lectures represented a source of new ideas at the time, and we all enjoyed them very much.

As years passed, we pursued different areas of research. He was interested in limit theorems with connections to statistics, while I was interested in the analytic aspects of probability theory.

I would meet Evarist occasionally at meetings and conferences and whenever he returned to Barcelona in the summer to visit his family in his hometown of Falset. He used to joke that he considered himself more of a farmer than a city boy.

Mathematics was not Evarist's only passion. He was very passionate about Catalonia. He had unconditional love for his country of origin and never hesitated to express his intense nationalist feelings. He was only slightly less passionate about his small cigars and baking his own bread, even when he was on the road away from home.

Evarist's impact on the field of probability and mathematical statistics was significant. He produced a long list of influential papers and two basic references.

He was a very good friend and an admired and respected colleague. His death has been a great loss for the mathematics community and for me. I still cannot believe that Evarist is no longer among us. He will be missed.

Dragan Radulovic

Evarist once told me, "You are going to make two major decisions in your life: picking your wife and picking your Ph.D. advisor. So choose wisely." And I did. Evarist was a prolific mathematician; he wrote influential books and important papers and contributed to the field in major ways. Curiously, he did not produce many students. I am fortunate to be one of the few. Our student-advisor dynamic was an unusual one. We had frequent but very short interactions. “Prof. Giné, if I have such and such a sequence under these conditions. . . what do you think; does it converge or not?,” I would ask. And, after just a few seconds, he would reply: “No, there is a counterexample. Check Mason’s paper in Annals, 84 or 85 I think.” And that was it. The vast majority of our interactions were conducted in less than 2 min. This suited him well, for he did not need to spend the time lecturing me and I did not like to be lectured. So it worked perfectly. All I needed was the guidance and he was the grandmaster himself.

We would go to the Boston probability seminar, every Tuesday, for 4 years, 2 h by car, each way. That is a lot of hours to be stuck with your advisor. And we seldom talked mathematics. Instead, we had endless discussions about politics, history, philosophy, and life in general. And in the process, we became very good friends. I remember our trip to Montreal, 8 h in the car, without a single dull moment. We jumped from one topic to another and the time flew just like that. We had different approaches to mathematics; I liked the big pictures while he was more concerned with the details. “What technique are you using? What is the trick?,” he would ask. And all I could offer was a general statement like: “You see all these pieces, how they fit together, except in this particular case. There must be something interesting there.” And he would reply: “But what inequality are you going to use?”

Consequently, we never published a paper together. This is rather unusual for a student and his advisor, both publishing in the same field. We tried to keep in touch, but our careers diverged and the time and the distance did their toll. We would meet only occasionally, on our high-dimensional probability retreats, but even there, it was obvious that we drifted apart. I missed those endless car rides. So long Prof. Giné, it is an honor to call myself your student.

Jan Rosiński

I met Evarist for the first time in 1975 at the First International Conference on Probability in Banach Spaces in Oberwolfach, Germany, which was a precursor to the high-dimensional probability conference series. I was a graduate student visiting the West from Soviet-bloc Poland for the first time. Despite plenty of new information to process and meeting many people whom I previously knew only from papers, I remember meeting Evarist clearly for his sincere smile, interest in the well-being of others, ability to listen, and contagious enthusiasm for mathematics.

Several years later, Evarist invited me to visit LSU, Baton Rouge, which eventually evolved into my permanent stay in the USA. Even though we have not had an opportunity for joint work, Evarist’s generosity and care extended into his continuous support of my career, for which I am grateful and deeply indebted. He was also an excellent mentor and friend. He will be deeply missed.

Hailin Sang

Saturday afternoon, March 14, 2015, I was astonished to read Magda Peligrad’s email that Evarist had passed away. I could not believe that he had left us so soon. I had just seen him at the Probability Theory and Statistics in High and Infinite Dimensions Conference, held in honor of his 70th birthday. He looked fine.

He was always full of energy. I thought, because of his love of mathematics and his humor and optimistic attitude toward life, that he would have a long life. I truly believed that he would witness more success from his postdocs and students, including me, on his 80th and 90th birthday. But now we can only see his gentle smile in photographs and recall his lovely Catalan accent in our memory.

Evarist was a very fine mathematician. He published numerous papers in the major journals in probability and statistics and provided important service to mathematics journals and societies. He also received the Alumni Award from the University of Connecticut in 1998.

Evarist was an unconventional instructor. He didn't bore his audience by simply following notes and textbooks. He vigorously presented his lectures with logical arguments. He strived both to provide the simplest possible arguments and to give the big picture. His lectures were an art performance.

I thank Evarist for teaching me how to do research. Although he was an easygoing professor, he was very serious in advising and research. He did not leave holes in any project, even for something intuitively obvious. He did research rigorously with great integrity. Evarist was not only my research advisor, but he was an advisor for my life also. He held no prejudice. He would forgive people with a smile if they did something wrong but not on purpose. I learned a lot from him.

Evarist loved his students as his children. I still remember the sadness and helplessness in his eyes when he told me that Miguel Arcones passed away. Although he devoted his whole life to research and was a very successful academic, he led a simple life. Weather permitting, he rode his bicycle to his office arriving before 8 o'clock. Then he would work through the whole morning with only a 10-min coffee break. He usually had some fruit and nuts for lunch and was at the center of the professors in the math lounge. His colleagues appreciated his humor, as well as his comments on current events.

I can feel the pain of his family. They lost a wonderful husband, an amazing father, and a loving grandfather. We lost an excellent mathematician, a life advisor, and a sincere friend. I have a strong feeling that Evarist will always be with us. May he rest in peace.

Sasha Tsybakov

Evarist was one of the people whom I liked very much and whom I always considered as an example. He was obsessed by the beauty of mathematics. He showed by all his work that statistics is an area of mathematics where difficult problems exist and can be solved by beautiful tools. Overall, he had a highly esthetic feeling for mathematics. He was also very demanding about the quality of his work and was an exceptionally honest scientist. I did not have a joint work with Evarist, but we have met many times at conferences. Our relations were very warm, which I think cannot be otherwise with a person like Evarist. His charisma is still there—it is easy to recall his voice and his smile as if he were alive and to imagine what he would say in this and that situation. It is a sorrow that he left us so early.

Sara van de Geer

Dear Evarist,
If we had talked about this, I think I know what you would say.
You would say: “Don’t worry, it is okay.”
You would smile and look at the ground in the way you do.
You would say: “Just go on and live your lives, it is not important.”
But you are taking such a huge place in so many people’s hearts.
You are taking such a huge place in my heart.
We were just colleagues.
I didn’t even know you that well.
But your being there was enough to give a touch of warmth to everything.
You were not just any colleague.
Having known you is a precious gift.
Sara

Jon Wellner

Evarist Giné was a brilliant and creative mathematician. He had a deep understanding of the interactions between probability theory and analysis, especially in the direction of Banach space theory, and a keen sense of how to formulate sharp (and beautiful) results with conditions both necessary and sufficient. His persistence and acuity in formulating sharp theorems, many in collaboration with others, were remarkable. Evarist’s initial statistical publication concerning tests of uniformity on compact Riemannian manifolds inspired one of my first independent post Ph.D. research projects in the late 1970s. Later, in the 1980s and early 1990s, I had the great pleasure and great fortune of meeting Evarist personally. He became a friend and colleague through mutual research interests and involvement in the research meetings on *probability in Banach spaces* and later *high-dimensional probability*. Evarist was unfailingly generous and open in sharing his knowledge and managed to communicate his excitement and enthusiasm for research to all. I only collaborated with Evarist on two papers, but we jointly edited several proceedings volumes, and I queried him frequently about a wide range of questions and problems. I greatly valued his advice and friendship. I miss him enormously.

Andrei Zaitsev

The news of the death of Evarist Giné came as a shock to me. He died at the height of his scientific career. I first met Evarist at the University of Bielefeld in the 1990s, where we were both guests of Friedrich Götze. I had long been familiar with his remarkable works. After meeting him, I was surprised to see that on his way to becoming a world-renowned mathematician, he had remained a modest and pleasant person. I recall with pleasure going with him mushroom collecting in the woods around Bielefeld.

We have only one joint paper (together with David Mason). Evarist has long been at the forefront of modern probability theory. He had much more to give to mathematics. Sadly, his untimely death prevented this.

Joel Zinn

Evarist and I were friends. I dearly remember the fun we had working together on mathematics. Altogether, beginning around 1977, we wrote 25 joint papers over approximately 25 years. One can imagine that collaborations lasting as long as this can at times give rise to arguments. But I can not recall any. Over the years, each time we met, whether to collaborate or not, we met as friends.

I also remember the many kindnesses that Evarist showed me. One that keeps coming to my mind concerns Evarist's time at Texas A&M. Evarist and I would often arrive early to our offices—often with the intention of working on projects. Evarist liked to smoke a cigar in the morning, but I had allergies which were effected by the smoke. So, Evarist would come to the office especially early, smoke his cigar, and blow the smoke out of the window, so that the smoke would not cause me any problems when I arrived. Sometimes when I arrived at Evarist's office earlier than expected, I would see him almost next to the window blowing out the smoke. This surely must have lessened his pleasure in smoking.

Another concerned the times I visited him at UConn. When I visited, I took a few days to visit my aunt in New York. Evarist always offered to let me use his car for the trip to New York, and whenever I visited him at UConn, I stayed with him and Rosalind. I fondly remember their hospitality and consideration of my peculiarities, especially their attention to my dietary needs.

Photo Credit The photo of Evarist that shows his inimitable good-natured smile was taken by his daughter Núria Giné-Nokes in July 2011, while he was on vacation with his family in his hometown of Falset in his beloved Catalonia.

Part I
Inequalities and Convexity

Stability of Cramer's Characterization of Normal Laws in Information Distances

Sergey Bobkov, Gennadiy Chistyakov, and Friedrich Götze

Abstract Optimal stability estimates in the class of regularized distributions are derived for the characterization of normal laws in Cramer's theorem with respect to relative entropy and Fisher information distance.

Keywords Characterization of normal laws • Cramer's theorem • Stability problems

Mathematics Subject Classification (2010). Primary 60E

1 Introduction

If the sum of two independent random variables has a nearly normal distribution, then both summands have to be nearly normal. This property is called stability, and it depends on distances used to measure "nearness". Quantitative forms of this important theorem by P. Lévy are intensively studied in the literature, and we refer to [7] for historical discussions and references. Most of the results in this direction describe stability of Cramer's characterization of the normal laws for distances which are closely connected to weak convergence. On the other hand, there is no stability for strong distances including the total variation and the relative entropy, even in the case where the summands are equally distributed. (Thus, the answer to a conjecture from the 1960s by McKean [14] is negative, cf. [4, 5].) Nevertheless, the stability with respect to the relative entropy can be established for *regularized* distributions in the model, where a small independent Gaussian noise is added to the

S. Bobkov
School of Mathematics, University of Minnesota, 127 Vincent Hall, 206 Church St. S.E., Minneapolis, MN 55455, USA
e-mail: bobkov@math.umn.edu

G. Chistyakov • F. Götze (✉)
Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, 33501 Bielefeld, Germany
e-mail: chistyak@math.uni-bielefeld.de; goetze@mathematik.uni-bielefeld.de

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summands. Partial results of this kind have been obtained in [7], and in this note we introduce and develop new technical tools in order to reach optimal lower bounds for closeness to the class of the normal laws in the sense of relative entropy. Similar bounds are also obtained for the Fisher information distance.

First let us recall basic definitions and notations. If a random variable (for short—r.v.) X with finite second moment has a density p, the entropic distance from the distribution F of X to the normal is defined to be

$$D(X) = h(Z) - h(X) = \int_{-\infty}^{\infty} p(x) \log \frac{p(x)}{\varphi_{a,b}(x)}\, dx,$$

where

$$\varphi_{a,b}(x) = \frac{1}{b\sqrt{2\pi}}\, e^{-(x-a)^2/2b^2}, \quad x \in \mathbb{R},$$

denotes the density of a Gaussian r.v. $Z \sim N(a, b^2)$ with the same mean $a = \mathbf{E}X = \mathbf{E}Z$ and variance $b^2 = \mathrm{Var}(X) = \mathrm{Var}(Z)$ as for X ($a \in \mathbb{R}$, $b > 0$). Here

$$h(X) = -\int_{-\infty}^{\infty} p(x) \log p(x)\, dx$$

is the Shannon entropy, which is well-defined and is bounded from above by the entropy of Z, so that $D(X) \geq 0$. The quantity $D(X)$ represents the Kullback-Leibler distance from F to the family of all normal laws on the line; it is affine invariant, and so it does not depend on the mean and variance of X.

One of the fundamental properties of the functional h is the entropy power inequality

$$N(X + Y) \geq N(X) + N(Y),$$

which holds for independent random variables X and Y, where $N(X) = e^{2h(X)}$ denotes the entropy power (cf. e.g. [11, 12]). In particular, if $\mathrm{Var}(X + Y) = 1$, it yields an upper bound

$$D(X + Y) \leq \mathrm{Var}(X)D(X) + \mathrm{Var}(Y)D(Y), \tag{1.1}$$

which thus quantifies the closeness to the normal distribution for the sum in terms of closeness to the normal distribution of the summands. The generalized Kac problem addresses (1.1) in the opposite direction: How can one bound the entropic distance $D(X + Y)$ from below in terms of $D(X)$ and $D(Y)$ for sufficiently smooth distributions?

To this aim, for a small parameter $\sigma > 0$, we consider regularized r.v.'s

$$X_\sigma = X + \sigma Z, \qquad Y_\sigma = Y + \sigma Z',$$

where Z, Z' are independent standard normal r.v.'s, independent of X, Y. The distributions of X_σ and Y_σ will be called *regularized* as well. Note that additive white Gaussian noise is a basic statistical model used in information theory to mimic the effect of random processes that occur in nature. In particular, the class of regularized distributions contains a wide class of probability measures on the line which have important applications in statistical theory.

As a main goal, we prove the following reverse of the upper bound (1.1).

Theorem 1.1 *Let X and Y be independent r.v.'s with* $\mathrm{Var}(X+Y)=1$. *Given* $0<\sigma\le 1$, *the regularized r.v.'s X_σ and Y_σ satisfy*

$$D(X_\sigma+Y_\sigma)\ge c_1(\sigma)\left(e^{-c_2(\sigma)/D(X_\sigma)}+e^{-c_2(\sigma)/D(Y_\sigma)}\right), \tag{1.2}$$

where $c_1(\sigma)=e^{c\sigma^{-6}\log\sigma}$ *and* $c_2(\sigma)=c\sigma^{-6}$ *with an absolute constant* $c>0$.

Thus, when $D(X_\sigma+Y_\sigma)$ is small, the entropic distances $D(X_\sigma)$ and $D(Y_\sigma)$ have to be small, as well. In particular, if $X+Y$ is normal, then both X and Y are normal, so we recover Cramer's theorem. Moreover, the dependence with respect to the couple $(D(X_\sigma), D(Y_\sigma))$ on the right-hand side of (1.2) can be shown to be essentially optimal, as stated in Theorem 1.3 below.

Theorem 1.1 remains valid even in extremal cases where $D(X)=D(Y)=\infty$ (for example, when both X and Y have discrete distributions). However, the value of $D(X_\sigma)$ for the regularized r.v.'s X_σ cannot be arbitrary. Indeed, X_σ has always a bounded density $p_\sigma(x)=\frac{1}{\sigma\sqrt{2\pi}}\,\mathbf{E}\,e^{-(x-X)^2/2\sigma^2}\le\frac{1}{\sigma\sqrt{2\pi}}$, so that $h(X_\sigma)\ge -\log\frac{1}{\sigma\sqrt{2\pi}}$. This implies an upper bound

$$D(X_\sigma)\le\frac{1}{2}\log\frac{e\mathrm{Var}(X_\sigma)}{\sigma^2}\le\frac{1}{2}\log\frac{2e}{\sigma^2},$$

describing a general possible degradation of the relative entropy for decreasing σ. If $D_\sigma\equiv D(X_\sigma+Y_\sigma)$ is known to be sufficiently small, say, when $D_\sigma\le c_1^2(\sigma)$, the inequality (1.2) provides an additional constraint in terms of D_σ:

$$D(X_\sigma)\le\frac{c}{\sigma^6\log(1/D_\sigma)}.$$

Let us also note that one may reformulate (1.2) as an upper bound for the entropy power $N(X_\sigma+Y_\sigma)$ in terms of $N(X_\sigma)$ and $N(Y_\sigma)$. Such relations, especially those of the linear form

$$N(X+Y)\le C\,(N(X)+N(Y)), \tag{1.3}$$

are intensively studied in the literature for various classes of probability distributions under the name "reverse entropy power inequalities", cf. e.g. [1–3, 10]. However, (1.3) cannot be used as a quantitative version of Cramér's theorem, since it looses information about $D(X+Y)$, when $D(X)$ and $D(Y)$ approach zero.

A result similar to Theorem 1.1 also holds for the Fisher information distance, which may be more naturally written in the standardized form

$$J_{st}(X) = b^2(I(X) - I(Z)) = b^2 \int_{-\infty}^{\infty} \Big(\frac{p'(x)}{p(x)} - \frac{\varphi'_{a,b}(x)}{\varphi_{a,b}(x)}\Big)^2 p(x)\,dx$$

with parameters a and b as before. Here

$$I(X) = \int_{-\infty}^{\infty} \frac{p'(x)^2}{p(x)}\,dx$$

denotes the Fisher information of X, assuming that the density p of X is (locally) absolutely continuous and has a derivative p' in the sense of Radon-Nikodym. Similarly to D, the standardized Fisher information distance is an affine invariant functional, so that $J_{st}(\alpha + \beta X) = J_{st}(X)$ for all $\alpha, \beta \in \mathbb{R}$, $\beta \neq 0$. In many applications it is used as a strong measure of X being non Gaussian. For example, $J_{st}(X)$ dominates the relative entropy; more precisely, we have

$$\frac{1}{2} J_{st}(X) \geq D(X). \tag{1.4}$$

This relation may be derived from an isoperimetric inequality for entropies due to Stam and is often regarded as an information theoretic variant of the logarithmic Sobolev inequality for the Gaussian measure due to Gross (cf. [6, 9, 16]). Moreover, Stam established in [16] an analog for the entropy power inequality, $\frac{1}{I(X+Y)} \geq \frac{1}{I(X)} + \frac{1}{I(Y)}$, which implies the following counterpart of the inequality (1.1)

$$J_{st}(X+Y) \leq \mathrm{Var}(X) J_{st}(X) + \mathrm{Var}(Y) J_{st}(Y),$$

for any independent r.v.'s X and Y with $\mathrm{Var}(X+Y) = 1$. We will show that this upper bound can be reversed in a full analogy with (1.2).

Theorem 1.2 *Under the assumptions of Theorem 1.1,*

$$J_{st}(X_\sigma + Y_\sigma) \geq c_3(\sigma)\left(e^{-c_4(\sigma)/J_{st}(X_\sigma)} + e^{-c_4(\sigma)/J_{st}(Y_\sigma)}\right), \tag{1.5}$$

where $c_3(\sigma) = e^{c\sigma^{-6}(\log \sigma)^3}$ *and* $c_4(\sigma) = c\sigma^{-6}$ *with an absolute constant* $c > 0$.

Let us also describe in which sense the lower bounds (1.2) and (1.5) may be viewed as optimal.

Theorem 1.3 *For every* $T \geq 1$, *there exist independent identically distributed r.v.'s* $X = X_T$ *and* $Y = Y_T$ *with mean zero and variance one, such that* $J_{st}(X_\sigma) \to 0$ *as*

$T \to \infty$ for $0 < \sigma \le 1$ and

$$D(X_\sigma - Y_\sigma) \le e^{-c(\sigma)/D(X_\sigma)} + e^{-c(\sigma)/D(Y_\sigma)},$$

$$J_{st}(X_\sigma - Y_\sigma) \le e^{-c(\sigma)/J_{st}(X_\sigma)} + e^{-c(\sigma)/J_{st}(Y_\sigma)}$$

with some $c(\sigma) > 0$ depending on σ only.

In this note we prove Theorem 1.1 and omit the proof of Theorem 1.2. The proofs of these theorems are rather similar and differ in technical details only, which can be found in [8]. The paper is organized as follows. In Sect. 2, we describe preliminary steps by introducing truncated r.v.'s X^* and Y^*. Since their characteristic functions represent entire functions, this reduction of Theorem 1.1 to the case of truncated r.v.'s allows to invoke powerful methods of complex analysis. In Sect. 3, $D(X_\sigma)$ is estimated in terms of the entropic distance to the normal for the regularized r.v.'s X_σ^*. In Sect. 4, the product of the characteristic functions of X^* and Y^* is shown to be close to the normal characteristic function in a disk of large radius depending on $1/D(X_\sigma + Y_\sigma)$. In Sect. 5, we deduce by means of saddle-point methods a special representation for the density of the r.v.'s X_σ^*, which is needed in Sect. 6. Finally in Sect. 7, based on the resulting bounds for the density of X_σ^*, we establish the desired upper bound for $D(X_\sigma^*)$. In Sect. 8 we construct an example showing the sharpness of the estimates of Theorems 1.1 and 1.2.

2 Truncated Random Variables

Turning to Theorem 1.1, let us fix several standard notations. By

$$(F * G)(x) = \int_{-\infty}^{\infty} F(x-y)\, dG(y), \qquad x \in \mathbb{R},$$

we denote the convolution of given distribution functions F and G. This operation will only be used when $G = \Phi_b$ is the normal distribution function with mean zero and a standard deviation $b > 0$. We omit the index in case $b = 1$, so that $\Phi_b(x) = \Phi(x/b)$ and $\varphi_b(x) = \frac{1}{b}\,\varphi(x/b)$.

The Kolmogorov (uniform) distance between F and G is denoted by

$$\| F - G\| = \sup_{x \in \mathbb{R}} | F(x) - G(x)|,$$

and $\| F - G\|_{\mathrm{TV}}$ denotes the total variation distance. In general, $\| F - G\| \le \frac{1}{2} \| F - G\|_{\mathrm{TV}}$, while the well-known Pinsker inequality provides an upper bound for the total variation in terms of the relative entropy. Namely,

$$\| F - G\|_{\mathrm{TV}}^2 \le 2 \int_{-\infty}^{\infty} p(x)\ \log \frac{p(x)}{q(x)}\, dx,$$

where F and G are assumed to have densities p and q, respectively.

In the required inequality (1.2) of Theorem 1.1, we may assume that X and Y have mean zero, and that $D(X_\sigma + Y_\sigma)$ is small. Thus, from now on our basic hypothesis may be stated as

$$D(X_\sigma + Y_\sigma) \le 2\varepsilon \qquad (0 < \varepsilon \le \varepsilon_0), \tag{2.1}$$

where ε_0 is a sufficiently small absolute constant. By Pinsker's inequality, this yields bounds for the total variation and Kolmogorov distances

$$|| F_\sigma * G_\sigma - \Phi_{\sqrt{1+2\sigma^2}}|| \le \frac{1}{2}\, || F_\sigma * G_\sigma - \Phi_{\sqrt{1+2\sigma^2}}||_{\mathrm{TV}} \le \sqrt{\varepsilon} < 1, \tag{2.2}$$

where F_σ and G_σ are the distribution functions of X_σ and Y_σ, respectively. Moreover without loss of generality, one may assume that

$$\sigma^2 \ge \tilde{c}(\log\log(1/\varepsilon)/\log(1/\varepsilon))^{1/3} \tag{2.3}$$

with a sufficiently large absolute constant $\tilde{c} > 0$. Indeed if (2.3) does not hold, the statement of the theorem obviously holds.

We shall need some auxiliary assertions about truncated r.v.'s. Let F and G be the distribution functions of independent, mean zero r.v.'s X and Y with second moments $\mathbf{E}X^2 = v_1^2$, $\mathbf{E}Y^2 = v_2^2$, such that $\mathrm{Var}(X + Y) = 1$. Put

$$N = N(\varepsilon) = \sqrt{1 + 2\sigma^2}\,\big(1 + \sqrt{2\log(1/\varepsilon)}\big)$$

with a fixed parameter $0 < \sigma \le 1$.

Introduce truncated r.v.'s at level N. Put $X^* = X$ in case $|X| \le N$, $X^* = 0$ in case $|X| > N$, and similarly Y^* for Y. Note that

$$\mathbf{E}X^* \equiv a_1 = \int_{-N}^{N} x\, dF(x), \qquad \mathrm{Var}(X^*) \equiv \sigma_1^2 = \int_{-N}^{N} x^2\, dF(x) - a_1^2,$$

$$\mathbf{E}Y^* \equiv a_2 = \int_{-N}^{N} x\, dG(x), \qquad \mathrm{Var}(Y^*) \equiv \sigma_2^2 = \int_{-N}^{N} x^2\, dG(x) - a_2^2.$$

By definition, $\sigma_1 \le v_1$ and $\sigma_2 \le v_2$. In particular,

$$\sigma_1^2 + \sigma_2^2 \le v_1^2 + v_2^2 = 1.$$

Denote by F^*, G^* the distribution functions of the truncated r.v.'s X^*, Y^*, and respectively by F^*_σ, G^*_σ the distribution functions of the regularized r.v.'s $X^*_\sigma = X^* + \sigma Z$ and $Y^*_\sigma = Y^* + \sigma Z'$, where Z, Z' are independent standard normal r.v.'s that are independent of (X, Y).

Lemma 2.1 *With some absolute constant C we have*

$$0 \le 1 - (\sigma_1^2 + \sigma_2^2) \le CN^2\sqrt{\varepsilon}.$$

Lemma 2.1 can be deduced from the following observations.

Lemma 2.2 *For any $M > 0$,*

$$\begin{aligned} 1 - F(M) + F(-M) &\le 2\big(1 - F_\sigma(M) + F_\sigma(-M)\big) \\ &\le 4\Phi_{\sqrt{1+2\sigma^2}}(-(M-2)) + 4\sqrt{\varepsilon}. \end{aligned}$$

The same inequalities hold true for G.

Lemma 2.3 *With some positive absolute constant C we have*

$$||F^* - F||_{\mathrm{TV}} \le C\sqrt{\varepsilon}, \qquad ||G^* - G||_{\mathrm{TV}} \le C\sqrt{\varepsilon},$$
$$||F^*_\sigma * G^*_\sigma - \Phi_{\sqrt{1+2\sigma^2}}||_{\mathrm{TV}} \le C\sqrt{\varepsilon}.$$

The proofs of Lemma 2.1 as well as Lemmas 2.2 and 2.3 are similar to those used for Lemma 3.1 in [7]. For details we refer to [8].

Corollary 2.4 *With some absolute constant C, we have*

$$\int_{|x|>N} x^2\, dF(x) \le CN^2\sqrt{\varepsilon}, \qquad \int_{|x|>2N} x^2\, d(F_\sigma(x) + F^*_\sigma(x)) \le CN^2\sqrt{\varepsilon},$$

and similarly for G replacing F.

Proof By the definition of truncated random variables,

$$v_1^2 = \sigma_1^2 + a_1^2 + \int_{|x|>N} x^2\, dF(x), \qquad v_2^2 = \sigma_2^2 + a_2^2 + \int_{|x|>N} x^2\, dG(x),$$

so that, by Lemma 2.1,

$$\int_{|x|>N} x^2\, d(F(x) + G(x)) \le 1 - (\sigma_1^2 + \sigma_2^2) \le CN^2\sqrt{\varepsilon}.$$

As for the second integral of the corollary, we have

$$\int_{|x|>2N} x^2\, dF_\sigma(x) = \int_{|x|>2N} x^2 \Big[\int_{-\infty}^{\infty} \varphi_\sigma(x-s)\, dF(s) \Big]\, dx$$

$$= \int_{-\infty}^{\infty} dF(s) \int_{|x|>2N} x^2 \varphi_\sigma(x-s)\, dx$$

$$\le 2\int_{-N}^{N} s^2\, dF(s) \int_{|u|>N} \varphi_\sigma(u)\, du + 2\int_{|s|>N} s^2\, dF(s) \int_{-\infty}^{\infty} \varphi_\sigma(u)\, du$$

$$+ 2\int_{-N}^{N} dF(s) \int_{|u|>N} u^2 \varphi_\sigma(u)\, du + 2\int_{|s|>N} dF(s) \int_{-\infty}^{\infty} u^2 \varphi_\sigma(u)\, du.$$

It remains to apply the previous step and use the bound $\int_N^\infty u^2 \varphi_\sigma(u) du \le c\sigma N e^{-N^2/(2\sigma^2)}$. The same estimate holds for $\int_{|x|>2N} x^2\, dF_\sigma^*(x)$. □

3 Entropic Distance to Normal Laws for Regularized Random Variables

We keep the same notations as in the previous section and use the relations (2.1) when needed. In this section we obtain some results about the regularized r.v.'s X_σ and X_σ^*, which also hold for Y_σ and Y_σ^*. Denote by p_{X_σ} and $p_{X_\sigma^*}$ the (smooth positive) densities of X_σ and X_σ^*, respectively.

Lemma 3.1 *With some absolute constant C we have, for all $x \in \mathbb{R}$,*

$$|\, p_{X_\sigma}(x) - p_{X_\sigma^*}(x)| \le C\sigma^{-1}\sqrt{\varepsilon}. \tag{3.1}$$

Proof Write

$$p_{X_\sigma}(x) = \int_{-N}^{N} \varphi_\sigma(x-s)\, dF(s) + \int_{|s|>N} \varphi_\sigma(x-s)\, dF(s),$$

$$p_{X_\sigma^*}(x) = \int_{-N}^{N} \varphi_\sigma(x-s)\, dF(s) + (1 - F(N) + F((-N)-)\, \varphi_\sigma(x).$$

Hence

$$|\, p_{X_\sigma}(x) - p_{X_\sigma^*}(x)| \le \frac{1}{\sqrt{2\pi}\sigma} \big(1 - F(N) + F(-N)\big).$$

But, by Lemma 2.2, and recalling the definition of $N = N(\varepsilon)$, we have

$$1 - F(N) + F(-N) \le 2(1 - F_\sigma(N) + F_\sigma(-N)) \le C\sqrt{\varepsilon}$$

with some absolute constant C. Therefore, $|p_{X_\sigma}(x) - p_{X_\sigma^*}| \le C\sigma^{-1}\sqrt{\varepsilon}$, which is the assertion (3.1). The lemma is proved. □

Lemma 3.2 *With some absolute constant $C > 0$ we have*

$$D(X_\sigma) \le D(X_\sigma^*) + C\sigma^{-3}N^3\sqrt{\varepsilon}. \tag{3.2}$$

Proof In general, if a random variable U has density u with finite variance b^2, then, by the very definition,

$$D(U) = \int_{-\infty}^{\infty} u(x)\log u(x)\,dx + \frac{1}{2}\,\log(2\pi e\, b^2).$$

Hence, $D(X_\sigma) - D(X_\sigma^*)$ is represented as

$$\begin{aligned}
&\int_{-\infty}^{\infty} p_{X_\sigma}(x)\log p_{X_\sigma}(x)\,dx - \int_{-\infty}^{\infty} p_{X_\sigma^*}(x)\log p_{X_\sigma^*}(x)\,dx + \frac{1}{2}\log\frac{v_1^2+\sigma^2}{\sigma_1^2+\sigma^2}\\
&= \int_{-\infty}^{\infty} (p_{X_\sigma}(x) - p_{X_\sigma^*}(x))\log p_{X_\sigma}(x)\,dx + \int_{-\infty}^{\infty} p_{X_\sigma^*}(x)\log\frac{p_{X_\sigma}(x)}{p_{X_\sigma^*}(x)}\,dx\\
&\quad + \frac{1}{2}\log\frac{v_1^2+\sigma^2}{\sigma_1^2+\sigma^2}.
\end{aligned} \tag{3.3}$$

Since $\mathbf{E}X^2 \le 1$, necessarily $F(-2) + 1 - F(2) \le \frac{1}{2}$, hence

$$\frac{1}{2\sigma\sqrt{2\pi}}\, e^{-(|x|+2)^2/(2\sigma^2)} \le p_{X_\sigma^*}(x) \le \frac{1}{\sigma\sqrt{2\pi}}, \tag{3.4}$$

and therefore

$$|\log p_{X_\sigma^*}(x)| \le C\sigma^{-2}(x^2+4), \quad x \in \mathbb{R}, \tag{3.5}$$

with some absolute constant C. The same estimate holds for $|\log p_{X_\sigma}(x)|$.

Splitting the integration in

$$\begin{aligned}
I_1 &= \int_{-\infty}^{\infty} (p_{X_\sigma}(x) - p_{X_\sigma^*}(x))\log p_{X_\sigma}(x)\,dx = I_{1,1} + I_{1,2}\\
&= \Big(\int_{|x|\le 2N} + \int_{|x|>2N}\Big)(p_{X_\sigma}(x) - p_{X_\sigma^*}(x))\log p_{X_\sigma}(x)\,dx,
\end{aligned}$$

we now estimate the integrals $I_{1,1}$ and $I_{1,2}$. By Lemma 3.1 and (3.5), we get

$$|I_{1,1}| \le C'\sigma^{-3}N^3\sqrt{\varepsilon}$$

with some absolute constant C'. Applying (3.5) together with Corollary 2.4, we also have

$$|I_{1,2}| \le 4C\sigma^{-2}\big(1 - F_\sigma(2N) + F_\sigma(-2N) + 1 - F_\sigma^*(2N) + F_\sigma^*(-2N)\big)$$
$$+ \, C\sigma^{-2}\Big(\int_{|x|>2N} x^2\,dF_\sigma(x) + \int_{|x|>2N} x^2\,dF_\sigma^*(x)\Big) \ \le \ C'\sigma^{-2}N^2\sqrt{\varepsilon}.$$

The two bounds yield

$$|I_1| \le C''\sigma^{-3}N^3\sqrt{\varepsilon} \tag{3.6}$$

with some absolute constant C''.

Now consider the integral

$$I_2 = \int_{-\infty}^{\infty} p_{X_\sigma^*}(x)\log\frac{p_{X_\sigma}(x)}{p_{X_\sigma^*}(x)}\,dx = I_{2,1} + I_{2,2}$$
$$= \Big(\int_{|x|\le 2N} + \int_{|x|>2N}\Big)\, p_{X_\sigma^*}(x)\log\frac{p_{X_\sigma}(x)}{p_{X_\sigma^*}(x)}\,dx,$$

which is non-negative, by Jensen's inequality. Using $\log(1+t) \le t$ for $t \ge -1$, and Lemma 3.1, we obtain

$$I_{2,1} = \int_{|x|\le 2N} p_{X_\sigma^*}(x)\log\Big(1 + \frac{p_{X_\sigma}(x) - p_{X_\sigma^*}(x)}{p_{X_\sigma^*}(x)}\Big)\,dx$$
$$\le \int_{|x|\le 2N} |\,p_{X_\sigma}(x) - p_{X_\sigma^*}(x)|\,dx \ \le \ 4C\sigma^{-1}N\sqrt{\varepsilon}.$$

It remains to estimate $I_{2,2}$. We have as before, using (3.5) and Corollary 2.4,

$$|I_{2,2}| \le C\int_{|x|>2N} p_{X_\sigma^*}(x)\,\frac{x^2+4}{\sigma^2}\,dx \le C'\sigma^{-2}N^2\sqrt{\varepsilon}$$

with some absolute constant C'. These bounds yield

$$I_2 \le C''\sigma^{-2}N^2\sqrt{\varepsilon}. \tag{3.7}$$

In addition, by Lemma 2.1,

$$\log \frac{v_1^2 + \sigma^2}{\sigma_1^2 + \sigma^2} \leq \frac{v_1^2 - \sigma_1^2}{\sigma^2} \leq C\sigma^{-2} N^2 \sqrt{\varepsilon}.$$

It remains to combine this bound with (3.6) and (3.7) and apply them in (3.3). □

4 Characteristic Functions of Truncated Random Variables

Denote by $f_{X^*}(t)$ and $f_{Y^*}(t)$ the characteristic functions of the r.v.'s X^* and Y^*, respectively. As integrals over finite intervals they admit analytic continuations as entire functions to the whole complex plane $\mathbb{C}$. These continuations will be denoted by $f_{X^*}(t)$ and $f_{Y^*}(t)$, $(t \in \mathbb{C})$.

Put $T = \frac{N}{64} = \frac{\sigma'}{64}\left(1 + \sqrt{2\log\frac{1}{\varepsilon}}\right)$, where $\sigma' = \sqrt{1 + 2\sigma^2}$. We may assume that $0 < \varepsilon \leq \varepsilon_0$, where ε_0 is a sufficiently small absolute constant.

Lemma 4.1 *For all $t \in \mathbb{C}$, $|t| \leq T$,*

$$\frac{1}{2}\,|e^{-t^2/2}| \leq |f_{X^*}(t)|\,|f_{Y^*}(t)| \leq \frac{3}{2}\,|e^{-t^2/2}|. \tag{4.1}$$

Proof For all complex t,

$$\begin{aligned}\Big|\int_{-\infty}^{\infty} e^{itx}\, d(F_\sigma^* * G_\sigma^*)(x) - \int_{-\infty}^{\infty} e^{itx} d\Phi_{\sigma'}(x)\Big| &\leq \Big|\int_{-4N}^{4N} e^{itx} d(F_\sigma^* * G_\sigma^* - \Phi_{\sigma'})(x)\Big| \\ + \int_{|x|\geq 4N} e^{-x\mathrm{Im}(t)}\, d(F_\sigma^* * G_\sigma^*)(x) &+ \int_{|x|\geq 4N} e^{-x\mathrm{Im}(t)}\, \varphi_{\sigma'}(x)\, dx. \end{aligned}\tag{4.2}$$

Integrating by parts, we have

$$\begin{aligned}\int_{-4N}^{4N} e^{itx}\, d(F_\sigma^* * G_\sigma^* - \Phi_{\sigma'})(x) \;&=\; e^{4itN}(F_\sigma^* * G_\sigma^* - \Phi_{\sigma'})(4N) \\ - e^{-4itN}(F_\sigma^* * G_\sigma^* - \Phi_{\sigma'})(-4N) &- it\int_{-4N}^{4N}(F_\sigma^* * G_\sigma^* - \Phi_{\sigma'})(x)\, e^{itx}\, dx.\end{aligned}$$

In view of the choice of T and N, we obtain, using Lemma 2.3, for all $|t| \leq T$,

$$\begin{aligned}\Big|\int_{-4N}^{4N} e^{itx}\, d(F_\sigma^* * G_\sigma^* - \Phi_{\sigma'})(x)\Big| &\leq 2C\sqrt{\varepsilon}\, e^{4N|\mathrm{Im}(t)|} + 8C|t|\sqrt{\varepsilon}\, e^{4N\,|\mathrm{Im}(t)|} \\ &\leq \frac{1}{6}\, e^{-(1/2+\sigma^2)\,T^2}.\end{aligned}\tag{4.3}$$

The second integral on the right-hand side of (4.2) does not exceed, for $|t| \le T$,

$$\int_{-2N}^{2N} d(F^* * G^*)(s) \int_{|x|\ge 4N} e^{-x\,\mathrm{Im}(t)} \varphi_{\sqrt{2}\sigma}(x-s)\,dx$$
$$\le \int_{-2N}^{2N} e^{-s\,\mathrm{Im}(t)}\, d(F^* * G^*)(s) \cdot \int_{|u|\ge 2N} e^{-u\,\mathrm{Im}(t)} \varphi_{\sqrt{2}\sigma}(u)\,du$$
$$\le e^{2NT} \cdot \frac{1}{\sqrt{\pi}} \int_{2N/\sigma}^{\infty} e^{\sigma T u - u^2/4}\,du \le \frac{1}{6}\, e^{-(1/2+\sigma^2)T^2}. \tag{4.4}$$

The third integral on the right-hand side of (4.2) does not exceed, for $|t| \le T$,

$$\sqrt{\frac{2}{\pi}} \int_{4N}^{\infty} e^{Tu - u^2/6}\,du \le \frac{1}{6}\, e^{-(1/2+\sigma^2)\,T^2}. \tag{4.5}$$

Applying (4.3)–(4.5) in (4.2), we arrive at the upper bound

$$|e^{-\sigma^2 t^2/2} f_{X^*}(t) e^{-\sigma^2 t^2/2} f_{Y^*}(t) - e^{-(1/2+\sigma^2)t^2}|$$
$$\le \frac{1}{2}\, e^{-(1/2+\sigma^2)T^2} \le \frac{1}{2}\, |e^{-(1/2+\sigma^2)t^2}| \tag{4.6}$$

from which (4.1) follows. □

The bounds in (4.1) show that the characteristic function $f_{X^*}(t)$ does not vanish in the circle $|t| \le T$. Hence, using results from ([13], pp. 260–266), we conclude that $f_{X^*}(t)$ has a representation

$$f_{X^*}(t) = \exp\{g_{X^*}(t)\}, \quad g_{X^*}(0) = 0,$$

where $g_{X^*}(t)$ is analytic on the circle $|t| \le T$ and admits the representation

$$g_{X^*}(t) = i a_1 t - \frac{1}{2}\sigma_1^2 t^2 - \frac{1}{2} t^2 \psi_{X^*}(t), \tag{4.7}$$

where

$$\psi_{X^*}(t) = \sum_{k=3}^{\infty} i^k c_k \Big(\frac{t}{T}\Big)^{k-2} \tag{4.8}$$

with real-valued coefficients c_k such that $|c_k| \le C$ for some absolute constant C. In the sequel without loss of generality we assume that $a_1 = 0$. An analogous representation holds for the function $f_{Y^*}(t)$.

5 The Density of the Random Variable X_σ^*

We shall use the following inversion formula

$$p_{X_\sigma^*}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixt} e^{-\sigma^2 t^2/2} f_{X^*}(t)\,dt, \quad x \in \mathbb{R},$$

for the density $p_{X_\sigma^*}(x)$. By Cauchy's theorem, one may change the path of integration in this integral from the real line to any line $z = t + iy,\ t \in \mathbb{R}$, with parameter $y \in \mathbb{R}$. This results in the following representation

$$p_{X_\sigma^*}(x) = e^{yx} e^{\sigma^2 y^2/2} f_{X^*}(iy) \cdot I_0(x,y), \quad x \in \mathbb{R}. \tag{5.1}$$

Here

$$I_0(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R(t,x,y)\,dt, \tag{5.2}$$

where

$$R(t,x,y) = f_{X^*}(t+iy) e^{-it(x+\sigma^2 y) - \sigma^2 t^2/2} / f_{X^*}(iy). \tag{5.3}$$

Let us now describe the choice of the parameter $y \in \mathbb{R}$ in (5.1). It is well-known that the function $\log f_{X^*}(iy), y \in \mathbb{R}$, is convex. Therefore, the function $\frac{d}{dy}\log f_{X^*}(iy) + \sigma^2 y$ is strictly monotone and tends to $-\infty$ as $y \to -\infty$ and tends to ∞ as $y \to \infty$. By (4.7) and (4.8), this function is vanishing at zero. Hence, the equation

$$\frac{d}{dy}\log f_{X^*}(iy) + \sigma^2 y = -x \tag{5.4}$$

has a unique continuous solution $y = y(x)$ such that $y(x) < 0$ for $x > 0$ and $y(x) > 0$ for $x < 0$. Here and in the sequel we use the principal branch of $\log z$.

We shall need one representation of $y(x)$ in the interval $[-(\sigma_1^2 + \sigma^2)T_1, (\sigma_1^2 + \sigma^2)T_1]$, where $T_1 = c'(\sigma_1^2 + \sigma^2)T$ with a sufficiently small absolute constant $c' > 0$. We see that

$$\begin{aligned} q_{X^*}(t) \equiv \frac{d}{dt}\log f_{X^*}(t) - \sigma^2 t &= -(\sigma_1^2 + \sigma^2)t - r_1(t) - r_2(t) \\ &= -(\sigma_1^2 + \sigma^2)t - t\psi_{X^*}(t) - \frac{1}{2}t^2 \psi'_{X^*}(t). \end{aligned} \tag{5.5}$$

The functions $r_1(t)$ and $r_2(t)$ are analytic in the circle $\{|t| \le T/2\}$ and there, by (4.8), they may be bounded as follows

$$|r_1(t)| + |r_2(t)| \le C|t|^2/T \tag{5.6}$$

with some absolute constant C. Using (5.5), (5.6) and Rouché's theorem, we conclude that the function $q_{X^*}(t)$ is univalent in the circle $D = \{|t| \le T_1\}$, and $q_{X^*}(D) \supset \frac{1}{2}(\sigma_1^2 + \sigma^2)D$. By the well-known inverse function theorem (see [15], pp. 159–160), we have

$$q_{X^*}^{(-1)}(w) = b_1 w + ib_2 w^2 - b_3 w^3 + \dots, \qquad w \in \frac{1}{2}(\sigma_1^2 + \sigma^2)D, \tag{5.7}$$

where

$$i^{n-1} b_n = \frac{1}{2\pi i} \int\limits_{|\zeta| = \frac{1}{2}T_1} \frac{\zeta \cdot q_{X^*}'(\zeta)}{q_{X^*}(\zeta)^{n+1}}\, d\zeta, \qquad n = 1, 2, \dots. \tag{5.8}$$

Using this formula and (5.5) and (5.6), we note that

$$b_1 = -\frac{1}{\sigma_1^2 + \sigma^2} \tag{5.9}$$

and that all remaining coefficients $b_2, b_3, \dots$ are real-valued. In addition, by (5.5) and (5.6),

$$-\frac{q_{X^*}(t)}{(\sigma_1^2 + \sigma^2)t} = 1 + q_1(t) \quad \text{and} \quad -\frac{q_{X^*}'(t)}{\sigma_1^2 + \sigma^2} = 1 + q_2(t),$$

where $q_1(t)$ and $q_2(t)$ are analytic functions in D satisfying there $|q_1(t)| + |q_2(t)| \le \frac{1}{2}$. Therefore, for $\zeta \in D$,

$$\frac{q_{X^*}'(\zeta)}{q_{X^*}(\zeta)^{n+1}} = (-1)^n \frac{q_3(\zeta)}{(\sigma_1^2 + \sigma^2)^n \zeta^{n+1}} \equiv (-1)^n \frac{1 + q_2(\zeta)}{(\sigma_1^2 + \sigma^2)^n (1 + q_1(\zeta))^{n+1} \zeta^{n+1}},$$

where $q_3(\zeta)$ is an analytic function in D such that $|q_3(\zeta)| \le 3 \cdot 2^n$. Hence, $q_3(\zeta)$ admits the representation

$$q_3(\zeta) = 1 + \sum_{k=1}^{\infty} d_k \frac{\zeta^k}{T_1^k}$$

with coefficients d_k such that $|d_k| \le 3 \cdot 2^n$. Using this equality, we obtain from (5.8) that

$$b_n = \frac{d_{n-1}}{(\sigma_1^2 + \sigma^2)^n \, T_1^{n-1}} \quad \text{and} \quad |b_n| \le \frac{3 \cdot 2^n}{(\sigma_1^2 + \sigma^2)^n \, T_1^{n-1}}, \quad n = 2, \dots. \tag{5.10}$$

Now we can conclude from (5.7) and (5.10) that, for $|x| \le T_1/(4|b_1|)$,

$$y(x) = -iq_{X^*}^{(-1)}(ix) = b_1 x - b_2 x^2 + R(x), \quad \text{where} \quad |R(x)| \le 48\, |b_1|^3 |x|^3 / T_1^2. \tag{5.11}$$

In the sequel we denote by θ a real-valued quantity such that $|\theta| \le 1$. Using (5.11), let us prove:

Lemma 5.1 *In the interval $|x| \le c'' T_1/|b_1|$ with a sufficiently small positive absolute constant c'',*

$$y(x)x + \frac{1}{2}\,\sigma^2 y(x)^2 + \log f_{X^*}(iy(x)) = \frac{1}{2}\, b_1 x^2 + \frac{c_3 b_1^3}{2T}\, x^3 + \frac{c\theta b_1^5}{T^2}\, x^4, \tag{5.12}$$

where c is an absolute constant.

Proof From (5.10) and (5.11), it follows that

$$\frac{1}{2}|b_1 x| \le |\, y(x)| \le \frac{3}{2}|b_1 x|. \tag{5.13}$$

Therefore,

$$\frac{1}{2} y(x)^2 \sum_{k=4}^{\infty} |c_k| \Big(\frac{|\, y(x)|}{T}\Big)^{k-2} \le C\Big(\frac{3}{2}\Big)^4 \frac{b_1^4 x^4}{T^2}.$$

On the other hand, with the help of (5.10) and (5.11) one can easily deduce the relation

$$y(x)x + \frac{1}{2}\,(\sigma^2 + \sigma_1^2)\, y(x)^2 + \frac{1}{2}\, c_3 \frac{y(x)^3}{T} = \frac{1}{2}\, b_1 x^2 + \frac{1}{2}\, c_3 b_1^3 \frac{x^3}{T} + \frac{c\theta b_1^5}{T^2}\, x^4$$

with some absolute constant c. The assertion of the lemma follows immediately from the two last relations. □

Now, applying Lemma 5.1 to (5.1), we may conclude that in the interval $|x| \le c'' T_1/|b_1|$, the density $p_{X^*}(x)$ admits the representation

$$p_{X_\sigma^*}(x) = \exp\Big\{\frac{1}{2}\, b_1 x^2 + \frac{1}{2}\, c_3 b_1^3 \,\frac{x^3}{T} + \frac{c\theta b_1^5}{T^2}\, x^4\Big\} \cdot I_0(x, y(x)) \tag{5.14}$$

with some absolute constant c.

As for the values $|x| > c''T_1/|b_1|$, in (5.1) we choose $y = y(x) = y(c''T_1/|b_1|)$ for $x > 0$ and $y = y(x) = y(-c''T_1/|b_1|)$ for $x < 0$. In this case, by (5.13), we note that $|y| \le 3c''T_1/2$, and we have

$$\Big|\frac{1}{2}\,\sigma^2 y^2 + \log f_{X^*}(iy)\Big| \le \frac{y^2}{2|b_1|} + \frac{C}{2}\,\frac{|y|^3}{T}\sum_{k=3}^{\infty}\Big(\frac{|y|}{T}\Big)^{k-3}$$

$$\le \frac{|y|}{2}\Big[\frac{3c''T_1}{2|b_1|} + \frac{9}{4}\,C(c'')^2\,\frac{T_1^2}{T}\sum_{k=3}^{\infty}\Big(\frac{3c''T_1}{2T}\Big)^{k-3}\Big] \le \frac{|y|}{2}\Big(\frac{3}{2}\,|x| + \frac{1}{4}\,|x|\Big) \le \frac{7}{8}\,|yx|.$$

As a result, for $|x| > c''T_1/|b_1|$, we obtain from (5.1) an upper bound $|p_{X_\sigma^*}(x)| \le e^{-\frac{1}{8}|y(x)x|}\,|I_0(x, y(x))|$, which with the help of left-hand side of (5.13) yields the estimate

$$|p_{X_\sigma^*}(x)| \le e^{-cT|x|/|b_1|}|I_0(x, y(x))|, \quad |x| > c''T_1/|b_1|, \tag{5.15}$$

with some absolute constant $c > 0$.

6 The Estimate of the Integral $I_0(x, y)$

In order to study the behavior of the integral $I_0(x, y)$, we need some auxiliary results. We use the letter c to denote absolute constants which may vary from place to place.

Lemma 6.1 *For $t, y \in [-T/4, T/4]$ and $x \in \mathbb{R}$, we have the relation*

$$\log|R(t, x, y)| = -\gamma(y)t^2/2 + r_1(t, y), \tag{6.1}$$

where

$$\gamma(y) = |b_1|^{-1} + \psi_{X^*}(iy) + 2iy\psi'_{X^*}(iy) \tag{6.2}$$

and

$$|r_1(t, y)| \le ct^2(t^2 + y^2)T^{-2} \quad \textit{with some absolute constant } c. \tag{6.3}$$

Proof From the definition of the function $R(t, x, y)$ it follows that

$$\log|R(t, x, y)| = \frac{1}{2}\Big(\frac{1}{b_1} - \psi_{X^*}(iy) - 2iy\psi'_{X^*}(iy)\Big)t^2$$
$$-\frac{1}{2}(\Re\psi_{X^*}(t + iy) - \psi_{X^*}(iy))(t^2 - y^2) + (\mathrm{Im}\psi_{X^*}(t + iy) + it\psi'_{X^*}(iy))ty. \tag{6.4}$$

Since, for $t, y \in [-T/4, T/4]$ and $k = 4, \dots,$

$$\begin{aligned}
&\left|\Re(i^k(t+iy)^{k-2} - i^k(iy)^{k-2})\right| \\
&\qquad = \left|\sum_{l=0}^{(k-2)/2} (-1)^{k+1+l}\binom{k-2}{2l} t^{2l} y^{k-2-2l} - (-1)^{k+1} y^{k-2}\right| \\
&\qquad \le t^2 (T/4)^{k-4} \sum_{l=1}^{(k-2)/2} \binom{k-2}{2l} \le 4t^2 (T/2)^{k-4},
\end{aligned}$$

we obtain an upper bound, for the same t and y, namely

$$|\Re\psi_{X^*}(t+iy) - \psi_{X^*}(iy)| \le \sum_{k=4}^{\infty} \frac{|c_k|}{T^{k-2}} |\Re(i^k(t+iy)^{k-2} - i^k(iy)^{k-2})| \le \frac{2^3 C t^2}{T^2}. \tag{6.5}$$

Since, for $t, y \in [-T/4, T/4]$ and $k = 5, \dots,$

$$\begin{aligned}
&\left|\operatorname{Im}(i^k(t+iy)^{k-2} - i^k(k-2)t(iy)^{k-3})\right| \\
&\qquad = \left|\sum_{l=1}^{(k-3)/2} \binom{k-2}{2l+1} (-1)^{k+l} t^{2l+1} y^{k-3-2l}\right| \\
&\qquad \le |t|^3 (T/4)^{k-5} \sum_{l=1}^{(k-3)/2} \binom{k-2}{2l+1} \le 8|t|^3 (T/2)^{k-5},
\end{aligned}$$

we have

$$\begin{aligned}
&|\operatorname{Im}\psi_{X^*}(t+iy) + it\psi'_{X^*}(iy)| \\
&\le \sum_{k=5}^{\infty} \frac{|c_k|}{T^{k-2}} |\operatorname{Im}(i^k(t+iy)^{k-2} - ti^k(k-2)(iy)^{k-3})| \le \frac{2^4 C|t|^3}{T^3}
\end{aligned} \tag{6.6}$$

for the same t and y. Applying (6.5) and (6.6) in (6.4), we obtain the assertion of the lemma. □

Lemma 6.2 *For $|t| \le c''T/\sqrt{|b_1|}$ and $|y| \le c''T/|b_1|$, we have the estimates*

$$\frac{3}{4|b_1|} \le \gamma(y) \le \frac{5}{4|b_1|} \tag{6.7}$$

and

$$|r_1(t,y)| \le t^2/(8|b_1|). \tag{6.8}$$

Proof Recall that the positive absolute constant c'' is chosen to be sufficiently small. Using the following simple bounds

$$|\psi_{X^*}(iy)| \le \sum_{k=3}^{\infty} |c_k| \Big(\frac{|y|}{T}\Big)^{k-2} \le C\frac{|y|}{T} \sum_{k=3}^{\infty} \Big(\frac{c''}{|b_1|}\Big)^{k-3} \le \frac{1}{8|b_1|}, \tag{6.9}$$

$$\begin{aligned} 2|y\psi'_{X^*}(iy)| &\le \frac{2|y|}{T} \sum_{k=3} |c_k|(k-2)\Big(\frac{|y|}{T}\Big)^{k-3} \\ &\le C\frac{2|y|}{T} \sum_{k=3}^{\infty} (k-2)\Big(\frac{c''}{|b_1|}\Big)^{k-3} \le \frac{1}{8|b_1|}, \end{aligned} \tag{6.10}$$

we easily obtain that

$$\begin{aligned} \frac{3}{4|b_1|} \le \frac{1}{|b_1|} - \psi_{X^*}(iy)| - 2|y\psi'_{X^*}(iy)| \le \gamma(y) \\ \le \frac{1}{|b_1|} + |\psi_{X^*}(iy)| + 2|y\psi'_{x^*}(iy)| \le \frac{5}{4|b_1|}, \end{aligned}$$

and thus (6.7) is proved. The bound (6.8) follows immediately from (6.3). □

Lemma 6.3 *For $t \in [-T/4, T/4]$ and $x \in [-c''T_1/|b_1|, c''T_1/|b_1|]$, we have*

$$\operatorname{Im}\log R(t, x, y(x)) = \frac{i}{2}\, t^3\, \psi'_{X^*}(iy(x)) + r_2(t, x), \tag{6.11}$$

where

$$|r_2(t, x)| \le c(|t| + |y(x)|)|t|^3 T^{-2} \quad \textit{with some absolute constant } c. \tag{6.12}$$

Proof Write, for $t, y \in [-T/4, T/4]$ and $x \in \mathbb{R}$,

$$\operatorname{Im}\log R(t, y, x) = -tx + \frac{ty}{b_1} - ty\,\Re\psi_{X^*}(t + iy) - \frac{t^2 - y^2}{2}\operatorname{Im}\psi_{X^*}(t + iy). \tag{6.13}$$

Now we choose in this formula $y = y(x)$, where $y(x)$ is the solution of Eq. (5.4) for $x \in [-c''T_1/|b_1|, c''T_1/|b_1|]$. For such x, in view of (5.13), we know that $|y(x)| \le T/4$. Let us rewrite (5.4) [see as well (5.5)] in the form

$$-\frac{1}{b_1}y(x) + y(x)\psi_{X^*}(iy(x)) + \frac{i}{2}y^2\psi'_{X^*}(iy(x))) = -x.$$

Applying this relation in (6.13), we obtain the formula

$$\begin{aligned}\operatorname{Im}\log R(t,x,y(x)) &= -ty(x)(\Re\psi_{X^*}(t+iy(x)) - \psi_{X^*}(iy(x))) \\ &+ \frac{i}{2}t^3\psi'_{X^*}(iy(x)) - \frac{1}{2}(t^2 - y(x)^2)\big(\operatorname{Im}\psi_{X^*}(t+iy(x)) + it\psi'_{X^*}(iy(x))\big).\end{aligned}$$

In view of (6.5) and (6.6), we can conclude that

$$\operatorname{Im}\log R(t,x,y(x)) = \frac{i}{2}t^3\psi'_{X^*}(iy(x)) + r_2(t,x),$$

where

$$|r_2(t,x)| \le 8\,C\,|t|^3|\,y(x)|T^{-2} + 8C|t|^3(t^2 + y(x)^2)T^{-3} \le 16\,C(|t| + |\,y(x)|)|t|^3T^{-2}$$

for $|t| \le T/4$ and $|\,y(x)| \le T/4$. Thus, the lemma is proved. □

Our next step is to estimate the integral $I_0(x, y(x))$. To this aim, we need the following lemma.

Lemma 6.4 *With some absolute constants c the following formula holds*

$$I_0(x,y(x)) = \frac{1}{\sqrt{2\pi}\,\gamma(\,y(x))^{1/2}} + r_0(x), \quad |x| \le c''T_1/|b_1|,$$

where

$$|r_0(x)| \le c(|b_1|^{7/2} + |b_1|^{3/2}y(x)^2)T^{-2}. \tag{6.14}$$

Proof For short we write y in place of $y(x)$. Put $T_2 = c''T/\sqrt{|b_1|}$ and write

$$\int_{-\infty}^{\infty}\Re R(t,x,y)\,dt = I_{01} + I_{02} = \Big(\int_{-T_2}^{T_2} + \int_{|t|\ge T_2}\Big)\Re R(t,x,y)\,dt.$$

First consider the integral I_{01}. We have

$$\begin{aligned}I_{01} = I_{01,1} - I_{01,2} \equiv \int_{-T_2}^{T_2}|R(t,x,y)|\,dt \\ - 2\int_{-T_2}^{T_2}|R(t,x,y)|\,\sin^2\Big(\frac{1}{2}\operatorname{Im}\log R(t,x,y)\Big)\,dt.\end{aligned}$$

By (6.1), we see that

$$I_{01,1} = \int_{-T_2}^{T_2} e^{-\frac{\gamma(y)}{2}t^2}\,dt + \int_{-T_2}^{T_2} e^{-\frac{\gamma(y)}{2}t^2}\big(e^{r_1(t,y)} - 1\big)\,dt.$$

Using the inequality $|e^z - 1| \le |z| e^{|z|}$, $z \in \mathbb{C}$, and applying Lemma 6.1 together with (6.3), (6.7), (6.8), we have

$$\Big| \int_{-T_2}^{T_2} e^{-\frac{\gamma(y)}{2} t^2} \big(e^{r_1(t,y)} - 1\big)\, dt \Big| \le \int_{-T_2}^{T_2} e^{-\frac{\gamma(y)}{2} t^2} |r_1(t,y)| e^{|r_1(t,y)|}\, dt$$
$$\le \int_{-T_2}^{T_2} e^{-\frac{1}{4|b_1|} t^2} |r_1(t,y)|\, dt \le c \int_{-T_2}^{T_2} t^2\, e^{-\frac{1}{4|b_1|} t^2} \frac{t^2 + y^2}{T^2}\, dt$$
$$\le c|b_1|^{3/2} (|b_1| + y^2) T^{-2}. \tag{6.15}$$

On the other hand

$$\int_{-T_2}^{T_2} e^{-\frac{\gamma(y)}{2} t^2}\, dt = \frac{\sqrt{2\pi}}{\gamma(y)^{1/2}} - \int_{|t| \ge T_2} e^{-\frac{\gamma(y)}{2} t^2}\, dt, \tag{6.16}$$

where, by (6.7) and the assumption (2.3),

$$\int_{|t| \ge T_2} e^{-\frac{\gamma(y)}{2} t^2}\, dt \le \frac{c}{\gamma(y) T_2} e^{-\frac{1}{2} (T_2 \sqrt{\gamma(y)})^2} \le c|b_1|^{3/2} T^{-1} e^{-c''^2 \frac{\gamma(y)}{2|b_1|} T^2} \le c T^{-4}. \tag{6.17}$$

Therefore in view of (6.15)–(6.17), we deduce

$$I_{01,1} = \frac{\sqrt{2\pi}}{\gamma(y)^{1/2}} + c\theta \frac{|b_1|^{3/2} (|b_1| + y^2)}{T^2}. \tag{6.18}$$

Now let us turn to the integral $I_{01,2}$. By (6.11), we have

$$|I_{01,2}| \le \frac{1}{2} \int_{-T_2}^{T_2} |R(t,x,y)|\, (\operatorname{Im} \log R(t,x,y))^2\, dt$$
$$\le 2 \int_{-T_2}^{T_2} |R(t,x,y)|\, \big(t^6 |\psi'_{X^*}(iy)|^2 + |r_2(t,x)|^2\big)\, dt.$$

By Lemmas 6.1–6.3 and by the estimates (2.3), (6.10), we arrive at the upper bound

$$|I_{01,2}| \le \frac{c}{T^2} \int_{-\infty}^{\infty} t^6 \Big(\frac{t^2 + y^2}{T^2} + 1\Big) e^{-\frac{1}{4|b_1|} t^2}\, dt$$
$$\le \frac{c}{T^2} |b_1|^{7/2} \Big(\frac{|b_1| + y^2}{T^2} + 1\Big) \le \frac{c|b_1|^{7/2}}{T^2}. \tag{6.19}$$

It remains to estimate the integral I_{02}. By (2.3),

$$|I_{02}| \le 2\int_{T_2}^{\infty} |R(t,x,y)|\,dt \le 2\int_{T_2}^{\infty} e^{-\frac{\sigma^2}{2}t^2}\,dt$$
$$\le 2\int_{c''\sigma T}^{\infty} e^{-\frac{\sigma^2}{2}t^2}\,dt \le c\sigma^{-3}T^{-1}e^{-(c'')^2\sigma^4T^2} \le cT^{-4}. \tag{6.20}$$

The assertion of the lemma follows from (6.18)–(6.20). □

Since for $|x| > c''T_1/|b_1|$ we choose $y(x) = y(\pm c''T_1/|b_1|)$ and since $|y(x)| \le c''T/|b_1|$ for such x, we obtain, using Lemmas 6.1 and 6.2, and the assumption (2.3), that

$$|I_0(x,y(x))| \le \frac{1}{2\pi}\int_{|t|\le T_2} |R(t,x,y(x))|\,dt + \frac{1}{2\pi}\int_{|t|>T_2} |R(t,x,y(x))|\,dt$$
$$\le \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{t^2}{4|b_1|}}\,dt + \frac{1}{2\pi}\int_{|t|>T_2} e^{-\frac{\sigma^2t^2}{2}}\,dt$$
$$\le c\Big(|b_1|^{\frac12} + T_2^{-1}\sigma^{-2}e^{-\frac{\sigma^2T_2^2}{2}}\Big) \le c|b_1|^{\frac12} \tag{6.21}$$

with some absolute constant c. The bound (6.21) holds for $|x| \le c''T_1/|b_1|$ as well. Thus (6.21) is valid for all real x.

Lemma 6.4 and the upper bound (6.21) allow us to control the behavior of the integral $I_0(x,y(x))$.

7 End of the Proof of Theorem 1.1

Starting from the hypothesis (2.1), we need to derive a good upper bound for $D(X_\sigma)$, which is equivalent to bounding the relative entropy $D(X_\sigma^*)$, according to Lemma 3.2. This will be done with the help of the relations (5.14), (5.15), Lemma 6.4, and (6.21) for the density $p_{X_\sigma^*}(x)$ of the r.v. X_σ^*. First, let us prove the following lemma.

Lemma 7.1 *For* $|x| \le c''T_1/|b_1|$,

$$\log\frac{p_{X_\sigma^*}(x)}{\varphi_{\sqrt{1/|b_1|}}(x)} = \frac{c_3}{2T}\Big((b_1x)^3 + 3b_1y(x)\Big) + \tilde r(x),$$

where with some absolute constant c

$$|\tilde r(x)| \le \frac{c}{T^2}\big(b_1^2y(x)^2 + |b_1|^3 + |b_1|^5x^4\big). \tag{7.1}$$

Proof By (5.14) and Lemma 6.4, we have, for $|x| \le c'' T_1/|b_1|$,

$$\log \frac{p_{X_\sigma^*}(x)}{\varphi_{1/\sqrt{|b_1|}}(x)} = \frac{1}{2}\, c_3 b_1^3 \,\frac{x^3}{T} + \frac{c\theta b_1^5}{T^2}\, x^4$$
$$- \frac{1}{2} \log(|b_1| \gamma(y(x)) + \log\Big(1 + \sqrt{\frac{\gamma(y(x))}{2\pi}} r_0(x)\Big). \tag{7.2}$$

Recalling (6.2) and (4.8), we see that

$$|b_1|\gamma(y(x)) = 1 + |b_1|(\psi_{X^*}(iy(x)) + 2iy(x)\psi'_{X^*}(iy(x)))$$
$$= 1 + 3c_3|b_1|\, y(x) T^{-1} + \rho_1(x), \tag{7.3}$$

where

$$\rho_1(x) \equiv |b_1| \sum_{k=4}^{\infty} i^k (2k-3)\, c_k \Big(\frac{iy(x)}{T}\Big)^{k-2}.$$

It is easy to see that

$$|\rho_1(x)| \le 8C|b_1|\Big(\frac{y(x)}{T}\Big)^2 \le \frac{1}{4}. \tag{7.4}$$

Since $\frac{|3c_3\, b_1 y(x)|}{T} \le \frac{1}{4}$, and using $|\log(1+u) - u| \le u^2$ ($|u| \le 1/2$), we get from (7.3) that

$$\log(|b_1|\, \gamma(y(x))) = \frac{3c_3|b_1|\, y(x)}{T} + c\theta\Big(\frac{b_1 y(x)}{T}\Big)^2 \tag{7.5}$$

with some absolute constant c. Now we conclude from (6.7) and (6.14) that

$$\sqrt{\frac{\gamma(y(x))}{2\pi}} |r_0(x)| \le c\, |b_1| \frac{b_1^2 + y(x)^2}{T^2} \le \frac{1}{4}$$

and arrive as before at the upper bound

$$\Big|\log\Big(1 + \sqrt{\frac{\gamma(y(x))}{2\pi}} r_0(x)\Big)\Big| \le c\, |b_1| \frac{b_1^2 + y(x)^2}{T^2}. \tag{7.6}$$

Applying (7.5) and (7.6) to (7.2), we obtain the assertion of the lemma. □

To estimate the quantity $D(X_\sigma^*)$, we represent it as

$$J_1 + J_2 = \Big(\int_{-c''T_1/|b_1|}^{c''T_1/|b_1|} + \int_{|x| > c''T_1/|b_1|}\Big) p_{X_\sigma^*}(x) \log \frac{p_{X_\sigma^*}(x)}{\varphi_{\sqrt{1/|b_1|}}(x)}\, dx. \tag{7.7}$$

First let us estimate J_1, using the letters c, C' to denote absolute positive constants which may vary from place to place. By Lemma 7.1,

$$J_1 = \frac{c_3}{T} J_{1,1} + J_{1,2} = \frac{c_3}{2T} \int_{-\frac{c''T_1}{|b_1|}}^{\frac{c''T_1}{|b_1|}} p_{X_\sigma^*}(x)\Big((b_1 x)^3 + 3b_1 y(x)\Big)dx + \int_{-\frac{c''T_1}{|b_1|}}^{\frac{c''T_1}{|b_1|}} p_{X_\sigma^*}(x)\tilde{r}(x)\,dx. \tag{7.8}$$

Using (5.14) and Lemma 6.4, we note that

$$\int_{-c''T_1/|b_1|}^{c''T_1/|b_1|} x^3 p_{X_\sigma^*}(x)\,dx = \int_{-c''T_1/|b_1|}^{c''T_1/|b_1|} x^3 (p_{X_\sigma^*}(x) - \varphi_{\sqrt{1/|b_1|}}(x))\,dx$$
$$= J_{1,1,1} + J_{1,1,2} + J_{1,1,3}$$
$$= \int_{-c''T_1/|b_1|}^{c''T_1/|b_1|} x^3 \varphi_{\sqrt{1/|b_1|}}(x)\Big(\frac{1}{\sqrt{\gamma(y(x))|b_1|}} - 1\Big) e^{c_3 b_1^3 x^3/(2T) + c\theta b_1^5 x^4/T^2}\,dx$$
$$+ \int_{-c''T_1/|b_1|}^{c''T_1/|b_1|} x^3 \varphi_{\sqrt{1/|b_1|}}(x)\Big(e^{c_3 b_1^3 x^3/(2T) + c\theta b_1^5 x^4/T^2} - 1\Big)\,dx$$
$$+ \frac{1}{\sqrt{2\pi|b_1|}} \int_{-c'T_1/|b_1|}^{c''T_1/|b_1|} x^3 \varphi_{\sqrt{1/|b_1|}}(x) e^{c_3 b_1^3 x^3/(2T) + c\theta b_1^5 x^4/T^2} r_0(x)\,dx.$$

It is easy to see that

$$\frac{|c_3|\,|b_1|^3|x|^3}{2T} + \frac{c|b_1|^5 x^4}{T^2} \le \frac{|b_1|x^2}{4} \quad \text{for} \quad |x| \le \frac{c''T_1}{|b_1|}. \tag{7.9}$$

Using (7.3) and (7.4) and the bound $|(1+u)^{-1/2} - 1| \le |u|$, $|u| \le \frac{1}{2}$, we get

$$|\,(\gamma(x)|b_1|)^{-1/2} - 1| \le c\,\frac{|b_1|\,|\,y(x)|}{T}.$$

The last estimates and (5.13) lead to

$$|\,J_{1,1,1}| \;\le\; \frac{c|b_1|}{T} \int_{-\frac{c''T_1}{|b_1|}}^{\frac{c''T_1}{|b_1|}} |x|^3 |\,y(x)| \sqrt{|b_1|}\, e^{-|b_1|x^2/4}\,dx$$
$$\le\; \frac{c|b_1|^{5/2}}{T} \int_{-\infty}^{\infty} x^4 e^{-|b_1|x^2/4}\,dx \;\le\; \frac{c}{T}. \tag{7.10}$$

Applying $|e^u - 1| \le |u|e^{|u|}$, we have, for $|x| \le c' T_1/|b_1|$,

$$\left|e^{c_3 b_1^3 x^3/(2T) + c\theta b_1^5 x^4/T^2} - 1\right| \le c|b_1|^3 |x|^3 \Big(\frac{1}{2T} + \frac{b_1^2|x|}{T^2}\Big)\, e^{|b_1|x^2/4}.$$

Therefore, we deduce the estimate

$$|J_{1,1,2}| \le c|b_1|^{7/2} \int_{-\infty}^{\infty} x^6 \Big(\frac{1}{T} + \frac{b_1^2|x|}{T^2}\Big)\, e^{-|b_1|x^2/4}\, dx \le c\Big(\frac{1}{T} + \frac{|b_1|^{3/2}}{T^2}\Big). \tag{7.11}$$

By (5.13) and (6.14), we immediately get

$$|J_{1,1,3}| \le \frac{c}{T^2} \int_{-\frac{c''T_1}{|b_1|}}^{\frac{c''T_1}{|b_1|}} |x|^3 \left(|b_1|^{7/2} + |b_1|^{3/2}\, y(x)^2\right) e^{-|b_1|\,x^2/4}\, dx \le \frac{c|b_1|^{3/2}}{T^2}. \tag{7.12}$$

Hence, by (7.10)–(7.12) and (2.3),

$$\Big|\int_{-c''T_1/|b_1|}^{c''T_1/|b_1|} x^3 p_{X_\sigma^*}(x)\, dx\Big| \le c\left(\frac{1}{T} + \frac{|b_1|^{3/2}}{T^2}\right) \le \frac{c}{T}. \tag{7.13}$$

In the same way,

$$\Big|\int_{-c''T_1/|b_1|}^{c''T_1/|b_1|} x p_{X_\sigma^*}(x)\, dx\Big| \le c\left(\frac{|b_1|}{T} + \frac{|b_1|^{5/2}}{T^2}\right) \le \frac{c|b_1|}{T}. \tag{7.14}$$

Recalling (5.11), we see that $y(x) = b_1 x + c\theta b_1^2 x^2/T_1$. As a result, using (7.13) and (7.14) and the property $\mathrm{Var}(X) \le 1$, we come to the upper bound

$$|J_{1,1}| \le c\,|b_1|^3 T^{-1}. \tag{7.15}$$

In order to estimate $J_{1,2}$, we employ the inequality (7.1). Recalling (5.14), (6.21) and (7.9), we then have

$$|J_{1,2}| \le \frac{c}{T^2} \int_{-c'T_1/|b_1|}^{c'T_1/|b_1|} \left(b_1^2 y(x)^2 + |b_1|^3 + |b_1|^5 x^4\right) \sqrt{|b_1|}\, e^{-|b_1|x^2 4}\, dx \le \frac{c|b_1|^3}{T^2}. \tag{7.16}$$

Combining (7.15) and (7.16), we arrive at

$$|J_1| \le c|b_1|^3 T^{-2}. \tag{7.17}$$

Let us estimate J_2. From (5.15), (6.21), we have, for all $|x| > c''T_1/|b_1|$,

$$p_{X_\sigma^*}(x) \le C' \sqrt{|b_1|}\, e^{-cT|x|/|b_1|} \le C' \sqrt{|b_1|}\, e^{-cc'c''T^2/|b_1|^3} < 1. \tag{7.18}$$

Here we also used (2.3) and the assumption that $0 < \varepsilon \le \varepsilon_0$, where ε_0 is a sufficiently small absolute constant. Using (7.18) and (2.3), we easily obtain

$$\begin{aligned} J_2 &\le -\int_{|x|>c''T_1/|b_1|} p_{X_\sigma^*}(x)\log\varphi_{\sqrt{1/|b_1|}}(x)\,dx \\ &= \frac{1}{2}\log\frac{2\pi}{|b_1|}\int_{|x|>c''T_1/|b_1|} p_{X_\sigma^*}(x)\,dx + \frac{|b_1|}{2}\int_{|x|>c''T_1/|b_1|} x^2 p_{X_\sigma^*}(x)\,dx \\ &\le C'\sqrt{|b_1|}\int_{|x|>c''T_1/|b_1|}\frac{1}{2}\,(\log(4\pi)+|b_1|x^2)e^{-cT|x|/|b_1|}\,dx \\ &\le C'\,(|b_1|^{3/2}T^{-1}+|b_1|^{-3/2}T)\,e^{-cc'c''T^2/|b_1|^3} \le C'T^{-2}. \end{aligned} \tag{7.19}$$

Thus, we derive from (7.17) and (7.19) the inequality $D(X_\sigma^*) \le c|b_1|^3T^{-2}$. Recalling (3.2) and Lemma 2.1, we finally conclude that

$$D(X_\sigma) \le c\frac{|b_1|^3}{T^2} + c\Big(\frac{N}{\sigma}\Big)^3\sqrt{\varepsilon} \le \frac{c}{(v_1^2+\sigma^2)^3T^2} + c\Big(\frac{N}{\sigma}\Big)^3\sqrt{\varepsilon} \le \frac{c}{(v_1^2+\sigma^2)^3T^2}. \tag{7.20}$$

An analogous inequality also holds for the r.v. Y_σ, and thus Theorem 1.1 follows from these estimates.

Remark 7.2 Under the assumptions of Theorem 1.1, a stronger inequality than (1.2) follows from (7.20). Namely, $D(X_\sigma + Y_\sigma)$ may be bounded from below by

$$e^{c\sigma^{-6}\log\sigma}\Big[\exp\Big\{-\frac{c}{(\mathrm{Var}(X_\sigma))^3\,D(X_\sigma)}\Big\} + \exp\Big\{-\frac{c}{(\mathrm{Var}(Y_\sigma))^3\,D(Y_\sigma)}\Big\}\Big].$$

8 Proof of Theorem 1.3

In order to construct r.v.'s X and Y with the desired properties, we need some auxiliary results. We use the letters $c, c', \tilde{c}$ (with indices or without) to denote absolute positive constants which may vary from place to place, and θ may be any number such that $|\theta| \le 1$. First we analyze the function v_σ with Fourier transform

$$f_\sigma(t) = \exp\{-(1+\sigma^2)\,t^2/2 + it^3/T\}, \qquad t \in \mathbb{R}.$$

Lemma 8.1 *If the parameter $T > 1$ is sufficiently large and $0 \le \sigma \le 2$, the function f_σ admits the representation*

$$f_\sigma(t) = \int_{-\infty}^{\infty} e^{itx} v_\sigma(x)\,dx \tag{8.1}$$

with a real-valued infinitely differentiable function $v_\sigma(x)$ which together with its all derivatives is integrable and satisfies

$$v_\sigma(x) > 0, \qquad \textit{for} \quad x \leq (1+\sigma^2)^2\, T/16; \tag{8.2}$$

$$|v_\sigma(x)| \leq e^{-(1+\sigma^2)Tx/32}, \quad \textit{for} \quad x \geq (1+\sigma^2)^2\, T/16. \tag{8.3}$$

In addition, for $|x| \leq (1+\sigma^2)^2\, T/16$,

$$c_1\, e^{-2(5-\sqrt{7})\,|xy(x)|/4} \leq |v_\sigma(x)| \leq c_2\, e^{-4|xy(x)|/9}, \tag{8.4}$$

where

$$y(x) = \frac{1}{6}\, T\left(-(1+\sigma^2) + \sqrt{(1+\sigma^2)^2 - 12x/T}\right). \tag{8.5}$$

The right inequality in (8.4) continues to hold for all $x \leq (1+\sigma^2)^2 T/16$.

Proof Since $f_\sigma(t)$ decays very fast at infinity, the function v_σ is given according to the inversion formula by

$$v_\sigma(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixt} f_\sigma(t)\, dt, \quad x \in \mathbb{R}. \tag{8.6}$$

Clearly, it is infinitely many times differentiable, and all its derivatives are integrable. It remains to prove (8.2)–(8.4). By the Cauchy theorem, one may also write

$$v_\sigma(x) = e^{yx} f_\sigma(iy)\, \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixt} R_\sigma(t,y)\, dt, \quad \text{where } R_\sigma(t,y) = \frac{f_\sigma(t+iy)}{f_\sigma(iy)}, \tag{8.7}$$

for every fixed real y. Here we choose $y = y(x)$ according to the equality in (8.5) for $x \leq (1+\sigma^2)^2\, T/16$. In this case, it is easy to see that

$$\begin{aligned} e^{-ixt} R_\sigma(t, y(x)) &= \exp\left\{-\frac{(1+\sigma^2)t^2}{2}\left(1 + \frac{6y(x)}{(1+\sigma^2)T}\right) + i\frac{t^3}{T}\right\} \\ &\equiv \exp\left\{-\frac{\alpha(x)}{2}\, t^2 + i\frac{t^3}{T}\right\}. \end{aligned}$$

Note that $\alpha(x) \geq (1+\sigma^2)/2$ for x as above.

For a better understanding of the behaviour of the integral in the right-hand side of (8.7), put

$$\tilde{I} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixt} R_\sigma(t,y)\, dt$$

and rewrite it in the form

$$\tilde{I} = \tilde{I}_1 + \tilde{I}_2 = \frac{1}{2\pi}\Big(\int_{|t|\le T^{1/3}} + \int_{|t|>T^{1/3}}\Big)\, e^{-ixt} R_\sigma(t, y(x))\, dt. \tag{8.8}$$

Using $|\cos u - 1 + u^2/2| \le u^4/4!$ ($u \in \mathbb{R}$), we easily obtain the representation

$$\begin{aligned}\tilde{I}_1 &= \frac{1}{2\pi}\int_{|t|\le T^{1/3}} \Big(1 - \frac{t^6}{2T^2}\Big)\, e^{-\alpha(x)t^2/2}\, dt + \frac{\theta}{4!\,T^4}\,\frac{1}{2\pi}\int_{|t|\le T^{1/3}} t^{12} e^{-\alpha(x)t^2/2}\, dt \\ &= \frac{1}{\sqrt{2\pi\alpha(x)}}\Big(1 - \frac{15}{2\alpha(x)^3\, T^2} + \frac{c\theta}{\alpha(x)^6 T^4}\Big) \\ &\quad - \frac{1}{2\pi}\int_{|t|>T^{1/3}} \Big(1 - \frac{t^6}{2T^2}\Big)\, e^{-\alpha(x)t^2/2}\, dt. \end{aligned} \tag{8.9}$$

The absolute value of last integral does not exceed $c(T^{1/3}\alpha(x))^{-1} e^{-\alpha(x)T^{2/3}/2}$. The integral $\tilde{I}_2$ admits the same estimate. Therefore, we obtain from (8.8) the relation

$$\tilde{I} = \frac{1}{\sqrt{2\pi\alpha(x)}}\Big(1 - \frac{15}{2\alpha(x)^3\, T^2} + \frac{c\theta}{\alpha(x)^6\, T^4}\Big). \tag{8.10}$$

Applying (8.10) in (8.7), we deduce for the half-axis $x \le (1+\sigma^2)^2\, T/16$, the formula

$$v_\sigma(x) = \frac{1}{\sqrt{2\pi\alpha(x)}}\Big(1 - \frac{15}{2\alpha(x)^3\, T^2} + \frac{c\theta}{\alpha(x)^6\, T^4}\Big)\, e^{y(x)x} f_\sigma(iy(x)). \tag{8.11}$$

We conclude immediately from (8.11) that (8.2) holds. To prove (8.3), we use (8.7) with $y = y_0 = -(1+\sigma^2)T/16$ and, noting that

$$x + \frac{1+\sigma^2}{2} y_0 \ge \frac{x}{2} \quad \text{for} \quad x \ge \frac{(1+\sigma^2)^2}{16}\, T,$$

we easily deduce the desired estimate

$$|v_\sigma(x)| \le e^{-(1+\sigma^2)Tx/32}\,\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-5(1+\sigma^2)^2 t^2/16}\, dt \le e^{-(1+\sigma^2)Tx/32}.$$

Finally, to prove (8.4), we apply the formula (8.11). Using the explicit form of $y(x)$, write

$$\begin{aligned} e^{y(x)x} f_\sigma(iy(x)) &= \exp\Big\{ y(x)x + \frac{1+\sigma^2}{2}\, y^2(x) + \frac{y(x)^3}{T}\Big\} \\ &= \exp\Big\{\frac{y(x)}{3}\big(2x + \frac{1+\sigma^2}{2} y(x)\big)\Big\} \end{aligned} \tag{8.12}$$

for $x \le \frac{(1+\sigma^2)^2}{16}\,T$. Note that the function $y(x)/x$ is monotonically decreasing from zero to $-\frac{4}{3}\,(1+\sigma^2)^{-1}$ and is equal to $= -\frac{8}{3}\,\big(-1+\sqrt{\frac{7}{4}}\,\big)\,(1+\sigma^2)^{-1}$ at the point $x = -\frac{(1+\sigma^2)^2}{16}\,T$. Using these properties in (8.12), we conclude that in the interval $|x| \le \frac{(1+\sigma^2)^2}{16}\,T$,

$$e^{-2(5-\sqrt{7})|\,y(x)x|/9} \le e^{y(x)x} f_\sigma(iy(x)) \le e^{-4|\,y(x)x|/9}, \tag{8.13}$$

where the right-hand side continues to hold for all $x \le \frac{(1+\sigma^2)^2}{16}\,T$. The inequalities in (8.4) follow immediately from (8.11) and (8.13). □

Now, introduce independent identically distributed r.v.'s U and V with density

$$p(x) = d_0 v_0(x)\, I_{(-\infty,T/16]}(x), \qquad \frac{1}{d_0} = \int_{-\infty}^{T/16} v_0(u)\, du, \tag{8.14}$$

where I_A denotes the indicator function of a set A. The density p depends on T, but for simplicity we omit this parameter. Note that, by Lemma 8.1, $|1 - d_0| \le e^{-cT^2}$.

Consider the regularized r.v. U_σ with density $p_\sigma = p * \varphi_\sigma$, which we represent in the form

$$p_\sigma(x) = d_0 v_\sigma(x) - w_\sigma(x), \quad \text{where} \quad w_\sigma(x) = d_0\,((v_0 I_{(T/16,\infty)}) * \varphi_\sigma)(x).$$

The next lemma is elementary, and we omit its proof.

Lemma 8.2 *We have*

$$\begin{aligned}
|w_\sigma(x)| &\le \varphi_\sigma(|x| + T/16)\, e^{-cT^2}, && x \le 0, \\
|w_\sigma(x)| &\le e^{-cT^2}, && 0 < x \le T/16, \\
|w_\sigma(x)| &\le e^{-cTx}, && x > T/16.
\end{aligned}$$

Lemma 8.3 *For all sufficiently large $T > 1$ and $0 < \sigma \le 2$,*

$$D(U_\sigma) = \frac{3}{(1+\sigma^2)^3\, T^2} + \frac{c\theta}{T^3}.$$

Proof Put $\mathbf{E}\, U_\sigma = a_\sigma$ and $\mathrm{Var}(U_\sigma) = b_\sigma^2$. By Lemma 8.2, $|a_\sigma| + |b_\sigma^2 - 1 - \sigma^2| \le e^{-cT^2}$. Write

$$\begin{aligned}
D(U_\sigma) = \tilde{J}_1 + \tilde{J}_2 + \tilde{J}_3 &= d_0 \int_{|x| \le c'T} v_\sigma(x) \log \frac{p_\sigma(x)}{\varphi_{a_\sigma,b_\sigma}(x)}\, dx \\
&- \int_{|x| \le c'T} w_\sigma(x) \log \frac{p_\sigma(x)}{\varphi_{a_\sigma,b_\sigma}(x)}\, dx + \int_{|x| > c'T} p_\sigma(x) \log \frac{p_\sigma(x)}{\varphi_{a_\sigma,b_\sigma}(x)}\, dx,
\end{aligned} \tag{8.15}$$

where $c' > 0$ is a sufficiently small absolute constant. First we find two-sided bounds on $\tilde{J}_1$, which are based on some additional information about v_σ.

Using a Taylor expansion for the function $\sqrt{1-u}$ about zero in the interval $-\frac{3}{4} \le u \le \frac{3}{4}$, we easily obtain that, for $|x| \le (1+\sigma^2)^2 T/16$,

$$\frac{6y(x)}{(1+\sigma^2)T} = -1 + \sqrt{1 - \frac{12x}{(1+\sigma^2)^2}\,T}$$
$$= -\frac{6x}{(1+\sigma^2)^2\,T} - \frac{18x^2}{(1+\sigma^2)^4\,T^2} - \frac{108x^3}{(1+\sigma^2)^6\,T^3} + \frac{c\theta x^4}{T^4},$$

which leads to the relation

$$y(x)x + \frac{1+\sigma^2}{2}\,y(x)^2 + \frac{y(x)^3}{T} = -\frac{x^2}{2(1+\sigma^2)} - \frac{x^3}{(1+\sigma^2)^3\,T} - \frac{9x^4}{2(1+\sigma^2)^5\,T^2} + \frac{c\theta x^5}{T^3}. \tag{8.16}$$

In addition, it is easy to verify that

$$\alpha(x) = (1+\sigma^2)\Big(1 - \frac{6x}{(1+\sigma^2)^2\,T} - \frac{18x^2}{(1+\sigma^2)^4\,T^2} + \frac{c\theta x^3}{T^3}\Big). \tag{8.17}$$

Finally, using (8.16) and (8.17), we conclude from (8.11) that v_σ is representable as

$$v_\sigma(x) = g(x)\varphi_{\sqrt{1+\sigma^2}}(x)e^{h(x)}$$
$$= \Big(1 + \frac{3x}{(1+\sigma^2)^2\,T} + \frac{15}{2}\,\frac{3x^2-(1+\sigma^2)}{(1+\sigma^2)^4\,T^2} + \frac{c\theta|x|(1+x^2)}{T^3}\Big)$$
$$\cdot\,\varphi_{\sqrt{1+\sigma^2}}(x)\exp\Big\{-\frac{x^3}{(1+\sigma^2)^3\,T} - \frac{9x^4}{2(1+\sigma^2)^5\,T^2} + \frac{c\theta x^5}{T^3}\Big\} \tag{8.18}$$

for $|x| \le (1+\sigma^2)^2 T/16$.

Now, from (8.18) and Lemma 8.2, we obtain a simple bound

$$|w_\sigma(x)/v_\sigma(x)| \le 1/2 \quad \text{for} \quad |x| \le c'T. \tag{8.19}$$

Therefore we have the relation, using again Lemmas 8.1 and 8.2,

$$\tilde{J}_1 = d_0\int_{|x|\le c'T} v_\sigma(x)\log\frac{v_\sigma(x)}{\varphi_{a_\sigma,b_\sigma}(x)}\,dx + 2\theta\int_{|x|\le c'T}|w_\sigma(x)|\,dx + \theta e^{-cT^2}$$
$$= \int_{|x|\le c'T} v_\sigma(x)\log\frac{v_\sigma(x)}{\varphi_{\sqrt{1+\sigma^2}}(x)}\,dx + \theta e^{-cT^2}. \tag{8.20}$$

Let us denote the integral on the right-hand side of (8.20) by $\tilde{J}_{1,1}$. With the help of (8.18) it is not difficult to derive the representation

$$\tilde{J}_{1,1} = \int_{|x|\le c'T} \varphi_{\sqrt{1+\sigma^2}}(x) e^{h(x)} \Big(-\frac{x^3}{(1+\sigma^2)^3\, T} - \frac{15x^4}{2(1+\sigma^2)^5\, T^2} + \frac{3x}{(1+\sigma^2)^2\, T} + \frac{54x^2 - 15(1+\sigma^2)}{2(1+\sigma^2)^4\, T^2} + \frac{c\theta |x|(1+x^4)}{T^3} \Big)\, dx. \tag{8.21}$$

Since $|e^{h(x)} - 1 - h(x)| \le \frac{1}{2} h(x)^2 e^{|h(x)|}$, and $\varphi^2_{\sqrt{1+\sigma^2}}(x) e^{2h(x)} \le \varphi_{\sqrt{1+\sigma^2}}(x)$ for $|x| \le c'T$, we easily deduce from (8.21) that

$$\tilde{J}_{1,1} = \int_{|x|\le c'T} \varphi_{\sqrt{1+\sigma^2}}(x) \Big(\frac{3(1+\sigma^2)x - x^3}{(1+\sigma^2)^3\, T} + \frac{54x^2 - 15(1+\sigma^2)}{2(1+\sigma^2)^4\, T^2} - \frac{21(1+\sigma^2)x^4 - 2x^6}{2(1+\sigma^2)^6\, T^2} \Big)\, dx + \frac{c\theta}{T^3} = \frac{3}{(1+\sigma^2)^3\, T^2} + \frac{c\theta}{T^3}. \tag{8.22}$$

It remains to estimate the integrals $\tilde{J}_2$ and $\tilde{J}_3$. By (8.19) and Lemma 8.2,

$$|\tilde{J}_2| \le \int_{|x|\le c'T} |w_\sigma(x)| (-\log \varphi_{a_\sigma, b_\sigma}(x) + \log \frac{3}{2} + |\log v_\sigma(x)|)\, dx \le \tilde{c} T^3 e^{-cT^2} \le e^{-cT^2}, \tag{8.23}$$

while by Lemmas 8.1 and 8.2,

$$|\tilde{J}_3| \le \int_{|x|>c'T} (|v_\sigma(x)| + |w_\sigma(x)|)(\sqrt{2\pi} b_\sigma + \frac{x^2}{2b_\sigma^2} + |\log(|v_\sigma(x) + |w_\sigma(x)|)\, dx \le \tilde{c} \int_{|x|>c'T} (1+x^2) e^{-cT|x|} dx + \int_{|x|>c'T} (|v_\sigma(x)| + |w_\sigma(x)|)^{1/2}\, dx \le e^{-cT^2}. \tag{8.24}$$

The assertion of the lemma follows from (8.22)–(8.24). □

To complete the proof of Theorem 1.3, we need yet another lemma.

Lemma 8.4 *For all sufficiently large $T > 1$ and $0 < \sigma \le 2$, we have*

$$D(U_\sigma - V_\sigma) \le e^{-cT^2}.$$

Proof Putting $\bar{p}_\sigma(x) = p_\sigma(-x)$, we have

$$D(U_\sigma - V_\sigma) = \int_{-\infty}^{\infty} (p_\sigma * \bar{p}_\sigma)(x) \log \frac{(p_\sigma * \bar{p}_\sigma)(x)}{\varphi_{\sqrt{2(1+\sigma^2)}}(x)}\, dx$$
$$+ \int_{-\infty}^{\infty} (p_\sigma * \bar{p}_\sigma)(x) \log \frac{\varphi_{\sqrt{2(1+\sigma^2)}}(x)}{\varphi_{\sqrt{\mathrm{Var}(X_\sigma - Y_\sigma)}}(x)}\, dx. \tag{8.25}$$

Note that $\bar{p}_\sigma(x) = d_0\bar{v}_\sigma(x) - \bar{w}_\sigma(x)$ with $\bar{v}_\sigma(x) = v_\sigma(-x)$, $\bar{w}_\sigma(x) = w_\sigma(-x)$, and

$$p_\sigma * \bar{p}_\sigma = d_0^2(v_\sigma * \bar{v}_\sigma)(x) - d_0(v_\sigma * \bar{w}_\sigma)(x) - d_0(\bar{v}_\sigma * w_\sigma)(x) + (w_\sigma * \bar{w}_\sigma)(x). \tag{8.26}$$

By the very definition of v_σ, $v_\sigma * \bar{v}_\sigma = \varphi_{\sqrt{2(1+\sigma^2)}}$. Since $|\mathrm{Var}(U_\sigma - V_\sigma) - 2(1 + \sigma^2)| \le e^{-cT^2}$, using Lemma 8.1, we note that the second integral on the right-hand side of (8.25) does not exceed e^{-cT^2}. Using Lemmas 8.1 and 8.2, we get

$$|(v_\sigma * \bar{w}_\sigma)(x)| + |(\bar{v}_\sigma * w_\sigma)(x)| + |w_\sigma * \bar{w}_\sigma(x)| \le e^{-cT^2}, \quad |x| \le \tilde{c}T, \tag{8.27}$$
$$|(v_\sigma * \bar{w}_\sigma(x)| + |(\bar{v}_\sigma * w_\sigma)(x)| + |(w_\sigma * \bar{w}_\sigma)(x)| \le e^{-cT|x|}, \quad |x| > \tilde{c}T. \tag{8.28}$$

It follows from these estimates that

$$\frac{(p_\sigma * \bar{p}_\sigma)(x)}{\varphi_{\sqrt{2(1+\sigma^2)}}(x)} = 1 + c\theta e^{-cT^2} \tag{8.29}$$

for $|x| \le c'T$. Hence, with the help of Lemmas 8.1 and 8.2, we may conclude that

$$\left| \int_{|x| \le c'T} (p_\sigma * \bar{p}_\sigma)(x) \log \frac{(p_\sigma * \bar{p}_\sigma)(x)}{\varphi_{\sqrt{2(1+\sigma^2)}}(x)}\, dx \right| \le e^{-cT^2}. \tag{8.30}$$

A similar integral over the set $|x| > c'T$ can be estimated with the help of (8.27) and (8.28), and here we arrive at the same bound as well. Therefore, the assertion of the lemma follows from (8.25). □

Introduce the r.v.'s $X = (U - a_0)/b_0$ and $Y = (V - a_0)/b_0$. Since $D(X_\sigma) = D(U_{b_0\sigma})$ and $D(X_\sigma - Y_\sigma) = D(U_{b_0\sigma} - V_{b_0\sigma})$, the statement of Theorem 1.3 for the entropic distance D immediately follows from Lemmas 8.3 and 8.4. As for the distance J_{st}, we need to prove corresponding analogs of Lemmas 8.3 and 8.4 for $J_{st}(U_\sigma)$ and $J_{st}(U_\sigma - V_\sigma)$, respectively. By the Stam inequality (1.4) and Lemma 8.3, we see that

$$J_{st}(U_\sigma) \ge c(\sigma)\, T^{-2} \quad \text{for sufficiently large } T > 1, \tag{8.31}$$

where $c(\sigma)$ denote positive constants depending on σ only. We estimate the quantity $J_{st}(U_\sigma - V_\sigma)$, by using the formula

$$\frac{J_{st}(U_\sigma - V_\sigma)}{2\mathrm{Var}(U_\sigma)} = -\int_{-\infty}^{\infty} (p_\sigma * \bar{p}_\sigma)''(x) \log \frac{(p_\sigma * \bar{p}_\sigma)(x)}{\varphi_{\sqrt{\mathrm{Var}(U_\sigma - V_\sigma)}}(x)}\, dx. \tag{8.32}$$

It is not difficult to conclude from (8.26), using our previous arguments, that

$$(p_\sigma * \bar{p}_\sigma)''(x) = d_0^2 \varphi''_{\sqrt{2(1+\sigma^2)}}(x) + R_\sigma(x), \tag{8.33}$$

where $|R_\sigma(x)| \le c(\sigma)e^{-cT^2}$ for $|x| \le \tilde{c}T$ and $|R_\sigma(x)| \le c(\sigma)e^{-cT|x|}$ for $|x| > \tilde{c}T$. Applying (8.33) in the formula (8.32) and repeating the argument that we used in the proof of Lemma 8.4, we obtain the desired result, namely

$$J_{st}(U_\sigma - V_\sigma) \le c(\sigma)\mathrm{Var}(X_\sigma)e^{-cT^2} \quad \text{for sufficiently large } T > 1. \tag{8.34}$$

By Theorem 1.2, $J_{st}(U_\sigma) \le -c(\sigma)/(\log J_{st}(U_\sigma - V_\sigma))$, so $J_{st}(U_\sigma) \to 0$ as $T \to \infty$. Since $J_{st}(X_\sigma) = J_{st}(U_{b_0\sigma})$ and $J_{st}(X_\sigma - Y_\sigma) = J_{st}(U_{b_0\sigma} - V_{b_0\sigma})$, the statement of Theorem 1.3 for J_{st} follows from (8.31) and (8.34).

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V.N. Sudakov's Work on Expected Suprema of Gaussian Processes

Richard M. Dudley

Dedicated to the memory of Evarist Giné.

Abstract It is noted that the late Volodya N. Sudakov (1934–2016) first published a statement in 1973 and proof in 1976 that the expected supremum of a centered Gaussian process is bounded above by a constant times a metric entropy integral. In particular, the present author (R.M. Dudley) defined such an integral but did not state nor prove such a bound.

Keywords Metric entropy

Mathematics Subject Classification (2010). Primary 60G15

1 Introductory Remarks

Vladimir N. Sudakov reached his 80th birthday in 2014. A rather well known fact, which I'll call a majorization inequality, says that the expected supremum of a centered Gaussian process is bounded above by a constant times a metric entropy integral. Who first (a) called attention to the expected supremum, (b) stated such an inequality, and (c) published a proof of it, when? My answer in all three cases is Sudakov (1973, for (a) and (b); 1976, for (c)) [19, 20]. I defined the metric entropy integral, as an equivalent sum in 1967, then explicitly in 1973, and showed that its finiteness implies sample continuity. Sudakov's work on Gaussian processes has perhaps been best known for his minoration; that he was first to state and give a proof for a majorization inequality seems to have passed almost unnoticed, and I hope to rectify that.

R.M. Dudley (✉)
Professor Emeritus, Mathematics, Massachusetts Institute of Technology, 92 Lewis St. Newton, MA 02458, USA
e-mail: rmd@math.mit.edu

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2 Sudakov and Strassen

At the International Congress of Mathematicians in Moscow in the summer of 1966, Sudakov gave a talk, in Russian, which applied metric entropy $\log(N(C, d, \varepsilon))$ (see Sect. 3), then called ε-entropy, of sets C in a Hilbert space H, to sample continuity and boundedness of the isonormal process L on H, the Gaussian process having mean 0 and covariance equal to the inner product, restricted to C. As far as I know this was the first presentation, oral or written, of such results by anyone, to an international audience. I attended the Moscow 1966 talk and took notes as best I could with my meager Russian. When I looked back at the notes later, I regretted not having absorbed the significance of Sudakov's talk at first. The notion of isonormal process on a Hilbert space originated, as far as I know, with Irving E. Segal, cf. Segal [12]. I did not give any talk at the 1966 Congress.

2.1 Strassen

Volker Strassen did give a talk at the 1966 Congress, on his then-new form of the law of the iterated logarithm. Whether he attended Sudakov's talk I don't recall, but he had been aware of ε-entropy by about 1964. Strassen was born in 1936. Like me, he got his doctorate in mathematics in 1962 and then spent several years in Berkeley, he in the Statistics Department (where probability resided) until 1968, and I in the Mathematics Department until the end of 1966; there was a seminar with probability topics organized by Jacob Feldman, a student of Segal. While we were both in Berkeley, Strassen and I talked about metric entropy. In the late 1960s Strassen began to work on speed of computation, on which he later won several prizes.

Strassen was invited to give a talk at a probability and information theory meeting in Canada which took place in 1968. He declined the invitation but kindly urged the organizers to invite me in his place, as they did; I went and presented the joint paper [16]. The paper gave two results: one, by Strassen, a central limit theorem in $C[0, 1]$ with a metric entropy hypothesis; and a counter-example, by me, showing that for i.i.d. variables X_j with values in $C[0, 1]$ having mean $EX_j = 0$ and being bounded: for some $M < \infty$, $\|X_1(\omega)\| \leq M$ for all ω, the central limit theorem can fail.

3 Early Papers on Metric Entropy and Gaussian Processes

Let (S, d) be a totally bounded metric space and for each $\varepsilon > 0$ let $N(S, d, \varepsilon)$ be the minimum number of points in an ε-net, within ε of each point of S. If d is a Hilbert space (e.g. L^2) metric it may be omitted from the notation. By "metric entropy integral" I mean

$$\int_0^u \sqrt{\log(N(S,d,\varepsilon))}d\varepsilon \tag{3.1}$$

for $u > 0$. The integrand is 0 for ε large enough, as $N(S,d,\varepsilon)$ is nonincreasing in ε and becomes equal to 1. Thus finiteness of (3.1) for some $u > 0$ implies it for all $u > 0$.

Fortunately for me, Irving Segal was one of the founding co-editors of *Journal of Functional Analysis* and solicited my 1967 paper for vol. 1 of the journal. The paper showed that finiteness of (3.1) for $u = 1$ (or an equivalent sum; formulated as an integral in 1973) for a subset S of a Hilbert space is sufficient for sample continuity and boundedness of the isonormal process restricted to S. Dudley [5] showed that if the metric entropy integral is finite for a Gaussian process, its indefinite integral gives a modulus of continuity for the process.

A weaker statement is that it suffices for sample continuity that for some r with $0 < r < 2$, as $\varepsilon \downarrow 0$,

$$\log N(S,d,\varepsilon) = O(\varepsilon^{-r}). \tag{3.2}$$

In my 1967 paper, p. 293, I wrote that "V. Strassen proved (unpublished) in 1963 or 1964" that condition (3.2) implies sample continuity of L on S. Sudakov stated the implication in his 1966 lecture, as I mentioned in Dudley [6, p. 87]. So before 1967, both Sudakov and Strassen had shown the sufficiency of (3.2) although neither had published a statement or proof of it. The abstract Sudakov [15] (in Russian) is quite short; it has two sentences, one about eigen element expansions as in its title, and the second, "For Gaussian distributions, new results are obtained." In Sudakov (1976, pp. 2–3 of the 1979 translation)[20] he reviews previous work, beginning with his 1966 talk.

In MathSciNet (*Mathematical Reviews* online) there is a gap in indexed reviews of Sudakov's publications. There are ten listed as published in the years 1958–1964, none for 1965–1970 (although at least one paper, Sudakov 1969, existed) and 20 for works published in 1971–1980, of which I was reviewer for 7, beginning with Sudakov [17]. Some of my reviews of Sudakov's works were not very perceptive. I had been a reviewer for Math. Reviews since April 1966. (In 1968 and 1971, I had the chance to review an announcement and then a paper by V.N. Vapnik and A.Ya. Chervonenkis.)

Sudakov [16] was as far as I can find his first publication on Gaussian processes. It made the connection with ε-entropy. Sudakov [17, 19] carried the work further. In particular in 1973 he gave an equivalent condition for sample-boundedness of a Gaussian process $\{X_t : t \in T\}$, namely that

$$E \sup_{t\in T} X_t := \sup\{E \sup_{t\in A} X_t : A \subset T,\ A \text{ countable}\} < +\infty. \tag{3.3}$$

Sudakov [20] gave a book-length presentation of his results on Gaussian processes (and also on doubly stochastic operators). An antecedent of the book is his doctoral dissertation Sudakov [18], which has the same title. In reviewing the book for *Math. Reviews* **MR0431359 (55 #4359)** I said I had learned about metric entropy "from V. Strassen, who wanted to give credit to 'someone' whose name we forgot." And so, I said, Sudakov was "too generous" in saying the application of such ideas to Gaussian processes came "independently" to several authors, although sufficiency of (3.2) for sample continuity seems to have been found independently by Strassen and Sudakov.

4 An Inequality: Majorization of E sup

For a Gaussian process $\{X_t,\ t \in T\}$ with mean 0, such an inequality says that

$$E \sup_{t \in T} X_t \leq K \int_0^{+\infty} \sqrt{\log N(\varepsilon, T, d_X)}\, d\varepsilon \tag{4.1}$$

for some $K < \infty$, where $d_X(s,t) := (E((X_s - X_t)^2))^{1/2}$. This has been attributed to me and called "Dudley's Theorem" by Ledoux and Talagrand, 1991, Theorem 11.17, p. 321. But in fact I am only responsible for the integral (3.1) over a finite interval and the fact that its finiteness implies sample continuity. In (4.1), $+\infty$ can clearly be replaced by

$$u = \operatorname{diam}(T) := \sup\{d_X(s,t) :\ s, t \in T\}.$$

(By the way, the left side of (4.1) may be finite while the right side is infinite.)

Sudakov [19] first defined the left-hand side (3.3) of (4.1). I was slow to appreciate it. My short review of Sudakov [19] in *Math. Reviews*, **MR0443059**, makes no explicit mention of the expected supremum; still less did I mention it in the earlier paper Dudley [4]. The bound (4.1) with $K = 24$ given by Ledoux and Talagrand had, as they say on p. 329, been proved by Pisier [11].

Ten years earlier, Sudakov (1973, Eq. (6))[19], had stated the inequality

$$E \sup_{x \in S} L(x) \leq CS_1 := C \sum_{k=-\infty}^{\infty} 2^{-k} \sqrt{\log_2(N(2^{-k}, S))} \tag{4.2}$$

for $C = 22/\sqrt{2\pi}$. Sudakov (1976, transl. 1979, Proposition 33)[20], gives a proof, pp. 54–56 of the translation. (If one is not convinced by Sudakov's proof, then the bound (4.1) might be attributed to Pisier, but in no case to me. Also Lifshits (2012, pp. 73–75)[10] gives further evidence that Sudakov's statement (or better) is correct.)

My review in Math. Revs. of Sudakov [20] also did not mention the quantity (3.3) and so neither (4.2) nor its proof.

We have straightforwardly for every integer k

$$2^{-k}\sqrt{\log_2 N(2^{-k},S)} = 2\int_{2^{-k-1}}^{2^{-k}} \sqrt{(\log 2)\log(N(2^{-k},S))}dx$$

$$\leq 2\sqrt{\log 2}\int_{2^{-k-1}}^{2^{-k}} \sqrt{\log(N(x,S))}dx.$$

It follows that $S_1 \leq 2\sqrt{\log 2}\int_0^{+\infty} \sqrt{\log(N(x,S))}dx$. This implies inequality (4.1) with the constant 24 improved to $K := 44\sqrt{\log 2}/\sqrt{2\pi} < 14.62$. As will be seen later, Lifshits [10] gave a still smaller constant. But I suggest that in view of Sudakov's priority, the inequality (4.1) for any (correct) finite K be called "Sudakov's majorization," by contrast with Sudakov's useful lower bound for $E\sup_{x\in S} L(x)$ based on metric entropy, well known as "Sudakov's minoration" (e.g., Ledoux and Talagrand, [8, pp. 79–84]). Chevet [2] gave, in a rather long paper, the first published proof of a crucial lemma in the Sudakov minoration.

According to Google Scholar, Sudakov [19] had only 15 citers as of May 19, 2015, but they did include Ledoux and Talagrand [8], also its chapter on Gaussian processes, and TalagrandŠs 1987 [26] paper on characterizing sample boundedness of Gaussian processes via majorizing measures. Sudakov (1976, transl. 1979)[20] had 228 citers (roughly half of them relating to optimal transportation and other non-Gaussian topics) as of June 12, 2015; it was from the list of citing works that I found Lifshits [10].

5 Books on Gaussian Processes

There are chapters on Gaussian processes in several books. For entire books, although there are some on applications such as machine learning, I will comment only on Lifshits [9, 10] and Bogachev [1].

5.1 Bogachev [1]

This book's Theorem 7.1.2, p. 334, states the Sudakov minoration and what I have called his majorization. For proof, Bogachev refers to Ledoux and Talagrand [8], Ledoux [7], and Lifshits [9]. Sudakov (1976, transl. 1979)[20] is not mentioned there; it is in the bibliography as ref. no. [733], p. 422, but I could not find a citation of it in the book.

5.2 Lifshits [9]

This book cites three works by Sudakov, [Sud1] = Sudakov [16], [Sud2] = Sudakov [17], and [Sud3] = Sudakov [20]. It gives an inequality (4.1), apparently as Theorem 14.1, although I have not seen the exact statement. Lifshits gives credit to Dmitrovskii [3] for the statement and proof.

5.3 Lifshits [10]

On p. 75 Lifshits gives the constant $K = 4\sqrt{2}$ in (4.1), which is the best I have seen. The proof seems to be self-contained. Lemma 10.1 on p. 73 says (correctly) that if $X_1, \ldots, X_N$ are centered jointly Gaussian variables and $E(X_j^2) \leq \sigma^2$ for each j, then

$$E \max_{1 \leq j \leq N} X_j \leq \sqrt{2 \log N} \sigma.$$

(Ledoux and Talagrand [8], (3.13), p. 71 have such an inequality with a factor of 3 instead of $\sqrt{2}$.) I was unable in a limited time to check Lifshits's proof of his version of (4.1) via the Lemma and induction.

The bibliography of Lifshits [10] lists 183 items, including Sudakov [16, 17, 20], but no works by Dmitrovskii. Sudakov (1976 and 1979) is his second most-cited work with 234 citations, Google Scholar, Nov. 14 (2015).

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Optimal Concentration of Information Content for Log-Concave Densities

Matthieu Fradelizi, Mokshay Madiman, and Liyao Wang

Abstract An elementary proof is provided of sharp bounds for the varentropy of random vectors with log-concave densities, as well as for deviations of the information content from its mean. These bounds significantly improve on the bounds obtained by Bobkov and Madiman (Ann Probab 39(4):1528–1543, 2011).

Keywords Concentration • Information • Log-concave • Varentropy

Mathematics Subject Classification (2010). Primary 52A40; Secondary 60E15, 94A17

1 Introduction

Consider a random vector Z taking values in $\mathbb{R}^n$, drawn from the standard Gaussian distribution γ, whose density is given by

$$\phi(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{|x|^2}{2}}$$

for each $x \in \mathbb{R}^n$, where $|\cdot|$ denotes the Euclidean norm. It is well known that when the dimension n is large, the distribution of Z is highly concentrated around

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M. Fradelizi
Laboratoire d'Analyse et de Mathématiques Appliquées UMR 8050, Université Paris-Est Marne-la-Vallée, 5 Bd Descartes, Champs-sur-Marne 77454, Marne-la-Vallée Cedex 2, France
e-mail: matthieu.fradelizi@univ-mlv.fr

M. Madiman (✉)
Department of Mathematical Sciences, University of Delaware, 501 Ewing Hall, Newark, DE 19716, USA
e-mail: madiman@udel.edu

L. Wang
J.P. Morgan, New York, NY, USA
e-mail: njuwangliyao@gmail.com

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the sphere of radius $\sqrt{n}$; that $\sqrt{n}$ is the appropriate radius follows by the trivial observation that $\mathbf{E}|Z|^2 = \sum_{i=1}^n \mathbf{E}Z_i^2 = n$. One way to express this concentration property is by computing the variance of $|Z|^2$, which is easy to do using the independence of the coordinates of Z:

$$\mathrm{Var}(|Z|^2) = \mathrm{Var}\left(\sum_{i=1}^n Z_i^2\right) = \sum_{i=1}^n \mathrm{Var}(Z_i^2) = 2n.$$

In particular, the standard deviation of $|Z|^2$ is $\sqrt{2n}$, which is much smaller than the mean n of $|Z|^2$ when n is large. Another way to express this concentration property is through a deviation inequality:

$$\mathbf{P}\left\{\frac{|Z|^2}{n} - 1 > t\right\} \leq \exp\left\{-\frac{n}{2}[t - \log(1+t)]\right\} \tag{1.1}$$

for the upper tail, and a corresponding upper bound on the lower tail. These inequalities immediately follow from Chernoff's bound, since $|Z|^2/n$ is just the empirical mean of i.i.d. random variables.

It is natural to wonder if, like so many other facts about Gaussian measures, the above concentration property also has an extension to log-concave measures (or to some subclass of them). There are two ways one may think about extending the above concentration property. One is to ask if there is a universal constant C such that

$$\mathrm{Var}(|X|^2) \leq Cn,$$

for every random vector X that has an isotropic, log-concave distribution on $\mathbb{R}^n$. Here, we say that a distribution on $\mathbb{R}^n$ is isotropic if its covariance matrix is the identity matrix; this assumption ensures that $\mathbf{E}|X|^2 = n$, and provides the normalization needed to make the question meaningful. This question has been well studied in the literature, and is known as the "thin shell conjecture" in convex geometry. It is closely related to other famous conjectures: it implies the hyperplane conjecture of Bourgain [13, 14], is trivially implied by the Kannan-Lovasz-Simonovits conjecture, and also implies the Kannan-Lovasz-Simonovits conjecture up to logarithmic terms [12]. The best bounds known to date are those of Guédon and Milman [18], and assert that

$$\mathrm{Var}(|X|^2) \leq Cn^{4/3}.$$

The second way that one may try to extend the above concentration property from Gaussians to log-concave measures is to first observe that the quantity that concentrates, namely $|Z|^2$, is essentially the logarithm of the Gaussian density function. More precisely, since

$$-\log \phi(x) = \frac{n}{2}\log(2\pi) + \frac{|x|^2}{2},$$

the concentration of $|Z|^2$ about its mean is equivalent to the concentration of $-\log \phi(Z)$ about its mean. Thus one can ask if, for every random vector X that has a log-concave density f on $\mathbb{R}^n$,

$$\mathrm{Var}(-\log f(X)) \leq Cn \tag{1.2}$$

for some absolute constant C. An affirmative answer to this question was provided by Bobkov and Madiman [2]. The approach of [2] can be used to obtain bounds on C, but the bounds so obtained are quite suboptimal (around 1000). Recently V.H. Nguyen [27] (see also [28]) and Wang [32] independently determined, in their respective Ph.D. theses, that the sharp constant C in the bound (1.2) is 1. Soon after this work, simpler proofs of the sharp variance bound were obtained independently by us (presented in the proof of Theorem 2.3 in this paper) and by Bolley et al. [7] (see Remark 4.2 in their paper). An advantage of our proof over the others mentioned is that it is very short and straightforward, and emerges as a consequence of a more basic log-concavity property (namely Theorem 2.9) of L^p-norms of log-concave functions, which may be thought of as an analogue for log-concave functions of a classical inequality of Borell [8] for concave functions.

If we are interested in finer control of the integrability of $-\log f(X)$, we may wish to consider analogues for general log-concave distributions of the inequality (1.1). Our second objective in this note is to provide such an analogue (in Theorem 4.1). A weak version of such a statement was announced in [3] and proved in [2], but the bounds we provide in this note are much stronger. Our approach has two key advantages: first, the proof is transparent and completely avoids the use of the sophisticated Lovasz-Simonovits localization lemma, which is a key ingredient of the approach in [2]; and second, our bounds on the moment generating function are sharp, and are attained for example when the distribution under consideration has i.i.d. exponentially distributed marginals.

While in general exponential deviation inequalities imply variance bounds, the reverse is not true. Nonetheless, our approach in this note is to first prove the variance bound (1.2), and then use a general bootstrapping result (Theorem 3.1) to deduce the exponential deviation inequalities from it. The bootstrapping result is of independent interest; it relies on a technical condition that turns out to be automatically satisfied when the distribution in question is log-concave.

Finally we note that many of the results in this note can be extended to the class of convex measures; partial work in this direction is done by Nguyen [28], and results with sharp constants are obtained in the forthcoming paper [17].

2 Optimal Varentropy Bound for Log-Concave Distributions

Before we proceed, we need to fix some definitions and notation.

Definition 2.1 Let a random vector X taking values in $\mathbb{R}^n$ have probability density function f. The *information content* of X is the random variable $\tilde{h}(X) = -\log f(X)$. The *entropy* of X is defined as $h(X) = \mathbf{E}(\tilde{h}(X))$. The *varentropy* of a random vector X is defined as $V(X) = \mathrm{Var}(\tilde{h}(X))$.

Note that the entropy and varentropy depend not on the realization of X but only on its density f, whereas the information content does indeed depend on the realization of X. For instance, one can write $h(X) = -\int_{\mathbb{R}^n} f \log f$ and

$$V(X) = \mathrm{Var}(\log f(X)) = \int_{\mathbb{R}^n} f(\log f)^2 - \left(\int_{\mathbb{R}^n} f \log f\right)^2.$$

Nonetheless, for reasons of convenience and in keeping with historical convention, we slightly abuse notation as above.

As observed in [2], the distribution of the difference $\tilde{h}(X) - h(X)$ is invariant under any affine transformation of $\mathbb{R}^n$ (i.e., $\tilde{h}(TX) - h(TX) = \tilde{h}(X) - h(X)$ for all invertible affine maps $T : \mathbb{R}^n \to \mathbb{R}^n$); hence the varentropy $V(X)$ is affine-invariant while the entropy $h(X)$ is not.

Another invariance for both $h(X)$ and $V(X)$ follows from the fact that they only depend on the distribution of $\log(f(X))$, so that they are unchanged if f is modified in such a way that its sublevel sets keep the same volume. This implies (see, e.g., [25, Theorem 1.13]) that if $f^\star$ is the spherically symmetric, decreasing rearrangement of f, and $X^\star$ is distributed according to the density $f^\star$, then $h(X) = h(X^\star)$ and $V(X) = V(X^\star)$. The rearrangement-invariance of entropy was a key element in the development of refined entropy power inequalities in [33].

Log-concavity is a natural shape constraint for functions (in particular, probability density functions) because it generalizes the Gaussian distributions. Furthermore, the class of log-concave distributions is infinite-dimensional, and hence, comprises a nonparametric model in statistical terms.

Definition 2.2 A function $f : \mathbb{R}^n \to [0, \infty)$ is *log-concave* if f can be written as

$$f(x) = e^{-U(x)},$$

where $U : \mathbb{R}^n \mapsto (-\infty, +\infty]$ is a convex function, i.e., $U(tx + (1-t)y) \le tU(x) + (1-t)U(y)$ for any x, y and $0 < t < 1$. When f is a probability density function and is log-concave, we say that f is a *log-concave density*.

We can now state the optimal form of the inequality (1.2), first obtained by Nguyen [27] and Wang [32] as discussed in Sect. 1.

Theorem 2.3 ([27, 32]) *Given a random vector X in $\mathbb{R}^n$ with log-concave density f,*

$$V(X) \leq n$$

Remark 2.4 The probability bound *does not depend* on f– it is universal over the class of log-concave densities.

Remark 2.5 The bound is *sharp*. Indeed, let X have density $f = e^{-\varphi}$, with $\varphi : \mathbb{R}^n \to [0, \infty]$ being positively homogeneous of degree 1, i.e., such that $\varphi(tx) = t\varphi(x)$ for all $t > 0$ and all $x \in \mathbb{R}^n$. Then one can check that the random variable $Y = \varphi(X)$ has a gamma distribution with shape parameter n and scale parameter 1, i.e., it is distributed according to the density given by

$$f_Y(t) = \frac{t^{n-1}e^{-t}}{(n-1)!}.$$

Consequently $\mathbf{E}(Y) = n$ and $\mathbf{E}(Y^2) = n(n+1)$, and therefore $V(X) = \mathrm{Var}(Y) = n$. Particular examples of equality include:

1. The case where $\varphi(x) = \sum_{i=1}^{n} x_i$ on the cone of points with non-negative coordinates (which corresponds to X having i.i.d. coordinates with the standard exponential distribution), and
2. The case where $\varphi(x) = \inf\{r > 0 : x \in rK\}$ for some compact convex set K containing the origin (which, by taking K to be a symmetric convex body, includes all norms on $\mathbb{R}^n$ suitably normalized so that $e^{-\varphi}$ is a density).

Remark 2.6 Bolley et al. [7] in fact prove a stronger inequality, namely,

$$\frac{1}{V(X)} - \frac{1}{n} \geq \Big[\mathbf{E}\big\{\nabla U(X) \cdot \mathrm{Hess}(U(X))^{-1} \nabla U(X)\big\}\Big]^{-1}.$$

This gives a strict improvement of Theorem 2.3 when the density $f = e^{-U}$ of X is strictly log-concave, in the sense that $\mathrm{Hess}(U(X))$ is, almost surely, strictly positive definite. As noted by Bolley et al. [7], one may give another alternative proof of Theorem 2.3 by applying a result of Hargé [19, Theorem 2].

In order to present our proof of Theorem 2.3, we will need some lemmata. The first one is a straightforward computation that is a special case of a well known fact about exponential families in statistics, but we write out a proof for completeness.

Lemma 2.7 *Let f be any probability density function on $\mathbb{R}^n$ such that $f \in L^\alpha(\mathbb{R}^n)$ for each $\alpha > 0$, and define*

$$F(\alpha) = \log \int_{\mathbb{R}^n} f^\alpha.$$

Let X_α be a random variable with density f_α on $\mathbb{R}^n$, where

$$f_\alpha := \frac{f^\alpha}{\int_{\mathbb{R}^n} f^\alpha}.$$

Then F is infinitely differentiable on $(0, \infty)$, and moreover, for any $\alpha > 0$,

$$F''(\alpha) = \frac{1}{\alpha^2} V(X_\alpha).$$

Proof Note that the assumption that $f \in L^\alpha(\mathbb{R}^n)$ (or equivalently that $F(\alpha) < \infty$) for all $\alpha > 0$ guarantees that $F(\alpha)$ is infinitely differentiable for $\alpha > 0$ and that we can freely change the order of taking expectations and differentiation.

Now observe that

$$F'(\alpha) = \frac{\int f^\alpha \log f}{\int f^\alpha} = \int f_\alpha \log f;$$

if we wish, we may also massage this to write

$$F'(\alpha) = \frac{1}{\alpha}[F(\alpha) - h(X_\alpha)]. \tag{2.1}$$

Differentiating again, we get

$$\begin{aligned} F''(\alpha) &= \frac{\int f^\alpha (\log f)^2}{\int f^\alpha} - \left(\frac{\int f^\alpha \log f}{\int f^\alpha}\right)^2 \\ &= \int f_\alpha (\log f)^2 - \left(\int f_\alpha \log f\right)^2 \\ &= \mathrm{Var}[\log f(X_\alpha)] = \mathrm{Var}\left[\frac{1}{\alpha}\{\log f_\alpha(X_\alpha) + F(\alpha)\}\right] \\ &= \frac{1}{\alpha^2}\mathrm{Var}[\log f_\alpha(X_\alpha)] = \frac{V(X_\alpha)}{\alpha^2}, \end{aligned}$$

as desired. □

The following lemma is a standard fact about the so-called perspective function in convex analysis. The use of this terminology is due to Hiriart-Urruty and Lemaréchal [20, p. 160] (see [10] for additional discussion), although the notion has been used without a name in convex analysis for a long time (see, e.g., [30, p. 35]). Perspective functions have also seen recent use in convex geometry [6, 11, 17]) and empirical process theory [31]. We give the short proof for completeness.

Lemma 2.8 *If* $U : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *is a convex function, then*

$$w(z, \alpha) := \alpha U(z/\alpha)$$

is a convex function on $\mathbb{R}^n \times (0, +\infty)$.

Proof First note that by definition, $w(az, a\alpha) = aw(z, \alpha)$ for any $a > 0$ and any $(z, \alpha) \in \mathbb{R}^n \times (0, +\infty)$, which implies in particular that

$$\frac{1}{\alpha} w(z, \alpha) = w\left(\frac{z}{\alpha}, 1\right).$$

Hence

$$\begin{aligned} &w(\lambda z_1 + (1-\lambda) z_2, \lambda \alpha_1 + (1-\lambda)\alpha_2) \\ &= [\lambda \alpha_1 + (1-\lambda)\alpha_2]\, U\left(\frac{\lambda \alpha_1 \frac{z_1}{\alpha_1} + (1-\lambda)\alpha_2 \frac{z_2}{\alpha_2}}{\lambda \alpha_1 + (1-\lambda)\alpha_2}\right) \\ &\leq \lambda \alpha_1 U\left(\frac{z_1}{\alpha_1}\right) + (1-\lambda)\alpha_2 U\left(\frac{z_2}{\alpha_2}\right) \\ &= \lambda w(z_1, \alpha_1) + (1-\lambda) w(z_2, \alpha_2), \end{aligned}$$

for any $\lambda \in [0, 1]$, $z_1, z_2 \in \mathbb{R}^n$, and $\alpha_1, \alpha_2 \in (0, \infty)$. □

The key observation is the following theorem.

Theorem 2.9 *If* f *is log-concave on* $\mathbb{R}^n$*, then the function*

$$G(\alpha) := \alpha^n \int f(x)^\alpha dx$$

is log-concave on $(0, +\infty)$.

Proof Write $f = e^{-U}$, with U convex. Make the change of variable $x = z/\alpha$ to get

$$G(\alpha) = \int e^{-\alpha U(z/\alpha)} dz.$$

The function $w(z, \alpha) := \alpha U(z/\alpha)$ is convex on $\mathbb{R}^n \times (0, +\infty)$ by Lemma 2.8, which means that the integrand above is log-concave. The log-concavity of G then follows from Prékopa's theorem [29], which implies that marginals of log-concave functions are log-concave. □

Remark 2.10 An old theorem of Borell [8, Theorem 2] states that if f is concave on $\mathbb{R}^n$, then $G_f(p) := (p+1)\cdots(p+n)\int f^p$is log-concave as a function of $p \in (0, \infty)$. Using this and the fact that a log-concave function is a limit of α-concave functions with $\alpha \to 0$, one can obtain an alternate, indirect proof of Theorem 2.9.

One can also similarly obtain an indirect proof of Theorem 2.9 by considering a limiting version of [4, Theorem VII.2], which expresses a log-concavity property of $(p-1)\dots(p-n)\int\phi^{-p}$ for any convex function ϕ on $\mathbb{R}^n$, for $p > n+1$ (an improvement of this to the optimal range $p > n$ is described in [6, 17], although this is not required for this alternate proof of Theorem 2.9).

Proof of Theorem 2.3 Since f is a log-concave density, it necessarily holds that $f \in L^\alpha(\mathbb{R}^n)$ for every $\alpha > 0$; in particular, $G(\alpha) := \alpha^n \int f^\alpha$ is finite and infinitely differentiable on the domain $(0,\infty)$. By definition,

$$\log G(\alpha) = n\log\alpha + \log\int f^\alpha = n\log\alpha + F(\alpha).$$

Consequently,

$$\frac{d^2}{d\alpha^2}[\log G(\alpha)] = -\frac{n}{\alpha^2} + F''(\alpha).$$

By Theorem 2.9, $\log G(\alpha)$ is concave, and hence we must have that

$$-\frac{n}{\alpha^2} + F''(\alpha) \le 0$$

for each $\alpha > 0$. However, Lemma 2.7 implies that $F''(\alpha) = V(X_\alpha)/\alpha^2$, so that we obtain the inequality

$$\frac{V(X_\alpha) - n}{\alpha^2} \le 0.$$

For $\alpha = 1$, this implies that $V(X) \le n$.

Notice that if $f = e^{-U}$, where $U : \mathbb{R}^n \to [0,\infty]$ is positively homogeneous of degree 1, then the same change of variable as in the proof of Theorem 2.9 shows that

$$G(\alpha) = \int e^{-\alpha U(z/\alpha)}dz = \int e^{-U(z)}dz = \int f(z)dz = 1.$$

Hence the function G is constant. Then the proof above shows that $V(X) = n$, which establishes the equality case stated in Remark 2.5.

3 A General Bootstrapping Strategy

The purpose of this section is to describe a strategy for obtaining exponential deviation inequalities when one has uniform control on variances of a family of random variables. Log-concavity is not an assumption made anywhere in this section.

Theorem 3.1 *Suppose $X \sim f$, where $f \in L^{\alpha}(\mathbb{R}^n)$ for each $\alpha > 0$. Let $X_\alpha \sim f_\alpha$, where*

$$f_\alpha(x) = \frac{f^\alpha(x)}{\int f^\alpha}.$$

If $K = K(f) := \sup_{\alpha>0} V(X_\alpha)$, then

$$\mathbf{E}\big[e^{\beta\{\widetilde{h}(X)-h(X)\}}\big] \leq e^{Kr(-\beta)}, \quad \beta \in \mathbb{R},$$

where

$$r(u) = \begin{cases} u - \log(1+u) & \text{for } u > -1 \\ +\infty & \text{for } u \leq -1 \end{cases}.$$

Proof Suppose X is a random vector drawn from a density f on $\mathbb{R}^n$, and define, for each $\alpha > 0$, $F(\alpha) = \log \int f^\alpha$. Set

$$K = \sup_{\alpha>0} V(X_\alpha) = \sup_{\alpha>0} \alpha^2 F''(\alpha);$$

the second equality follows from Lemma 2.7. Since $f \in L^{\alpha}(\mathbb{R}^n)$ for each $\alpha > 0$, $F(\alpha)$ is finite and moreover, infinitely differentiable for $\alpha > 0$, and we can freely change the order of integration and differentiation when differentiating $F(\alpha)$.

From Taylor-Lagrange formula, for every $\alpha > 0$, one has

$$F(\alpha) = F(1) + (\alpha - 1)F'(1) + \int_1^\alpha (\alpha - u)F''(u)du.$$

Using that $F(1) = 0$, $F''(u) \leq K/u^2$ for every $u > 0$ and the fact that for $0 < \alpha < u < 1$, one has $\alpha - u < 0$, we get

$$\begin{aligned} F(\alpha) &\leq (\alpha - 1)F'(1) + K\int_1^\alpha \frac{\alpha - u}{u^2}du \\ &= (\alpha - 1)F'(1) + K\left[-\frac{\alpha}{u} - \log(u)\right]_1^\alpha. \end{aligned}$$

Thus, for $\alpha > 0$, we have proved that

$$F(\alpha) \leq (\alpha - 1)F'(1) + K(\alpha - 1 - \log\alpha).$$

Setting $\beta = 1 - \alpha$, we have for $\beta < 1$ that

$$e^{F(1-\beta)} \leq e^{-\beta F'(1)}e^{K(-\beta - \log(1-\beta))}. \tag{3.1}$$

Observe that $e^{F(1-\beta)} = \int f^{1-\beta} = \mathbf{E}[f^{-\beta}(X)] = \mathbf{E}[e^{-\beta \log f(X)}] = \mathbf{E}\big[e^{\beta \tilde{h}(X)}\big]$ and $e^{-\beta F'(1)} = e^{\beta h(X)}$; the latter fact follows from the fact that $F'(1) = -h(X)$ as is clear from the identity (2.1). Hence the inequality (3.1) may be rewritten as

$$\mathbf{E}\big[e^{\beta\{\tilde{h}(X)-h(X)\}}\big] \le e^{Kr(-\beta)}, \quad \beta \in \mathbb{R}. \tag{3.2}$$

□

Remark 3.2 We note that the function $r(t) = t - \log(1+t)$ for $t > -1$, (or the related function $h(t) = t\log t - t + 1$ for $t > 0$, which satisfies $sh(t/s) = tr_1(s/t)$ for $r_1(u) = r(u-1)$) appears in many exponential concentration inequalities in the literature, including Bennett's inequality [1] (see also [9]), and empirical process theory [34]. It would be nice to have a clearer understanding of why these functions appear in so many related contexts even though the specific circumstances vary quite a bit.

Remark 3.3 Note that the function r is convex on $\mathbb{R}$ and has a quadratic behavior in the neighborhood of 0 ($r(u) \sim_0 \frac{u^2}{2}$) and a linear behavior at $+\infty$ ($r(u) \sim_\infty u$).

Corollary 3.4 *With the assumptions and notation of Theorem 3.1, we have for any $t > 0$ that*

$$\mathbf{P}\{\tilde{h}(X) - h(X) \ge t\} \le \exp\left\{-Kr\left(\frac{t}{K}\right)\right\}$$

$$\mathbf{P}\{\tilde{h}(X) - h(X) \le -t\} \le \exp\left\{-Kr\left(-\frac{t}{K}\right)\right\}$$

The proof is classical and often called the Cramér-Chernoff method (see for example Sect. 2.2 in [9]). It uses the Legendre transform φ^* of a convex function $\varphi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ defined for $y \in \mathbb{R}$ by

$$\varphi^*(y) = \sup_x xy - \varphi(x).$$

Notice that if $\min \varphi = \varphi(0)$ then for every $y > 0$, the supremum is reached at a positive x, that is $\varphi^*(y) = \sup_{x>0} xy - \varphi(x)$. Similarly, for $y < 0$, the supremum is reached at a negative x.

Proof The idea is simply to use Markov's inequality in conjunction with Theorem 3.1, and optimize the resulting bound.

For the lower tail, we have for $\beta > 0$ and $t > 0$,

$$\begin{aligned}\mathbb{P}[\tilde{h}(X) - h(X) \le -t] &\le \mathbf{E}\left[e^{-\beta\left(\tilde{h}(X)-h(X)\right)}\right]e^{-\beta t}\\ &\le \exp\left\{K\left(r(\beta) - \frac{\beta t}{K}\right)\right\}.\end{aligned}$$

Thus minimizing on $\beta > 0$, and using the remark before the proof, we get

$$\mathbb{P}[\tilde{h}(X) - h(X) \leq -t] \leq \exp\left\{ -K \sup_{\beta>0} \left(\frac{\beta t}{K} - r(\beta) \right) \right\} = e^{-Kr^*\left(\frac{t}{K}\right)}. \tag{3.3}$$

Let us compute the Legendre transform r^* of r. For every t, one has

$$r^*(t) = \sup_u tu - r(u) = \sup_{u>-1} \left(tu - u + \log(1+u) \right).$$

One deduces that $r^*(t) = +\infty$ for $t \geq 1$. For $t < 1$, by differentiating, the supremum is reached at $u = t/(1-t)$ and replacing in the definition we get

$$r^*(t) = -t - \log(1-t) = r(-t).$$

Thus $r^*(t) = r(-t)$ for all $t \in \mathbb{R}$. Replacing, in the inequality (3.3), we get the result for the lower tail.

For the upper tail, we use the same argument: for $\beta > 0$ and $t > 0$,

$$\mathbb{P}[\tilde{h}(X) - h(X) \geq t] \leq \mathbf{E}\left[e^{\beta\left(\tilde{h}(X) - h(X)\right)} \right] e^{-\beta t}$$
$$\leq \exp\left\{ K\left(r(-\beta) - \frac{\beta t}{K} \right) \right\}.$$

Thus minimizing on $\beta > 0$, we get

$$\mathbb{P}[\tilde{h}(X) - h(X) \geq t] \leq \exp\left\{ -K \sup_{\beta>0} \left(\frac{\beta t}{K} - r(-\beta) \right) \right\}. \tag{3.4}$$

Using the remark before the proof, in the right hand side term appears the Legendre transform of the function $\tilde{r}$ defined by $\tilde{r}(u) = r(-u)$. Using that $r^*(t) = r(-t) = \tilde{r}(t)$, we deduce that $(\tilde{r})^* = (r^*)^* = r$. Thus the inequality (3.4) gives the result for the upper tail.

□

4 Conclusion

The purpose of this section is to combine the results of Sects. 2 and 3 to deduce sharp bounds for the moment generating function of the information content of random vectors with log-concave densities. Naturally these yield good bounds on the deviation probability of the information content $\tilde{h}(X)$ from its mean $h(X) = \mathbf{E}\tilde{h}(X)$. We also take the opportunity to record some other easy consequences.

Theorem 4.1 *Let X be a random vector in $\mathbb{R}^n$ with a log-concave density f. For $\beta < 1$,*

$$\mathbf{E}\left[e^{\beta[\tilde{h}(X)-h(X)]}\right] \leq \mathbf{E}\left[e^{\beta[\tilde{h}(X^*)-h(X^*)]}\right],$$

where X^* has density $f^* = e^{-\sum_{i=1}^n x_i}$, restricted to the positive quadrant.

Proof Taking $K = n$ in Theorem 3.1 (which we can do in the log-concave setting because of Theorem 2.3), we obtain:

$$\mathbf{E}\big[e^{\beta\{\tilde{h}(X)-h(X)\}}\big] \leq e^{nr(-\beta)}, \quad \beta \in \mathbb{R}.$$

Some easy computations will show:

$$\mathbf{E}\big[e^{\beta\{\tilde{h}(X^*)-h(X^*)\}}\big] = e^{nr(-\beta)}, \quad \beta \in \mathbb{R}.$$

This concludes the proof.

□

As for the case of equality of Theorem 2.3, discussed in Remark 2.5, notice that there is a broader class of densities for which one has equality in Theorem 4.1, including all those of the form $e^{-\|x\|_K}$, where K is a symmetric convex body.

Remark 4.2 The assumption $\beta < 1$ in Theorem 4.1 is strictly not required; however, for $\beta \geq 1$, the right side is equal to $+\infty$. Indeed, already for $\beta = 1$, one sees that for any random vector X with density f,

$$\begin{aligned}\mathbf{E}\big[e^{\tilde{h}(X)-h(X)}\big] &= e^{-h(X)}\mathbf{E}\left[\frac{1}{f(X)}\right] = e^{-h(X)} \int_{\mathrm{supp(f)}} dx \\ &= e^{-h(X)}\mathrm{Vol}_n(\mathrm{supp(f)}),\end{aligned}$$

where $\mathrm{supp(f)} = \overline{\{x \in \mathbb{R}^n : f(x) > 0\}}$ is the support of the density f and Vol_n denotes Lebesgue measure on $\mathbb{R}^n$. In particular, this quantity for X^*, whose support has infinite Lebesgue measure, is $+\infty$.

Remark 4.3 Since

$$\lim_{\alpha\to 0} \frac{2}{\alpha^2}\mathbf{E}\left[e^{\alpha(\log f(X)-\mathbf{E}[\log f(X)])}\right] = V(X),$$

we can recover Theorem 2.3 from Theorem 4.1.

Taking $K = n$ in Corollary 3.4 (again because of Theorem 2.3), we obtain:

Corollary 4.4 *Let X be a random vector in $\mathbb{R}^n$ with a log-concave density f. For $t > 0$,*

$$\mathbb{P}[\tilde{h}(X) - h(X) \leq -nt] \leq e^{-nr(-t)},$$
$$\mathbb{P}[\tilde{h}(X) - h(X) \geq nt] \leq e^{-nr(t)},$$

where $r(u)$ is defined in Theorem 3.1.

The original concentration of information bounds obtained in [2] were suboptimal not just in terms of constants but also in the exponent; specifically it was proved there that

$$\mathbf{P}\left\{\frac{1}{n}\left|\tilde{h}(X) - h(X)\right| \geq t\right\} \leq 2\, e^{-ct\sqrt{n}} \tag{4.1}$$

for a universal constant $c > 1/16$ (and also that a better bound with ct^2n in the exponent holds on a bounded range, say, for $t \in (0, 2]$). One key advantage of the method presented in this paper, apart from its utter simplicity, is the correct linear dependence of the exponent on dimension. Incidentally, we learnt from a lecture of Klartag [22] that another proof of (4.1) can be given based on the concentration property of the eigenvalues of the Hessian of the Brenier map (corresponding to optimal transportation from one log-concave density to another) that was discovered by Klartag and Kolesnikov [23]; however, the latter proof shares the suboptimal $\sqrt{n}t$ exponent of [2].

The following inequality is an immediate corollary of Corollary 4.4 since it merely expresses a bound on the support of the distribution of the information content.

Corollary 4.5 *Let X have a log-concave probability density function f on $\mathbb{R}^n$. Then:*

$$h(X) \leq -\log \|f\|_\infty + n.$$

Proof By Corollary 4.4, almost surely,

$$\log f(X) \leq \mathbf{E}[\log f(X)] + n,$$

since when $t \geq 1$, $\mathbb{P}[\log f(X) - \mathbf{E}[\log f(X)] \geq nt] = 0$. Taking the supremum over all realizable values of X yields

$$\log \|f\|_\infty \leq \mathbf{E}[\log f(X)] + n,$$

which is equivalent to the desired statement. □

Corollary 4.5 was first explicitly proved in [4], where several applications of it are developed, but it is also implicitly contained in earlier work (see, e.g., the proof of Theorem 7 in [16]).

An immediate consequence of Corollary 4.5, unmentioned in [4], is a result due to [15]:

Corollary 4.6 *Let X be a random vector in $\mathbb{R}^n$ with a log-concave density f. Then*

$$\|f\|_\infty \leq e^n f(\mathbf{E}[X]).$$

Proof By Jensen's inequality,

$$\log f(\mathbf{E}X) \geq \mathbf{E}[\log f(X)].$$

By Corollary 4.5,

$$\mathbf{E}[\log f(X)] \geq \log \|f\|_\infty - n.$$

Hence,

$$\log f(\mathbf{E}X) \geq \log \|f\|_\infty - n.$$

Exponentiating concludes the proof. □

Finally we mention that the main result may also be interpreted as a small ball inequality for the random variable $f(X)$. As an illustration, we record a sharp form of [24, Corollary 2.4] (cf., [21, Corollary 5.1] and [5, Proposition 5.1]).

Corollary 4.7 *Let f be a log-concave density on $\mathbb{R}^n$. Then*

$$\mathbb{P}\{f(X) \geq c^n \|f\|_\infty\} \geq 1 - \left(e \cdot c \cdot \log\left(\frac{1}{c}\right)\right)^n,$$

where $0 < c < \frac{1}{e}$.

Proof Note that

$$\begin{aligned}\mathbb{P}\{f(X) \leq c^n \|f\|_\infty\} &= \mathbb{P}\{\log f(X) \leq \log \|f\|_\infty + n \log c\} \\ &= \mathbb{P}\{\tilde{h}(X) \geq -\log \|f\|_\infty - n \log c\} \\ &\leq \mathbb{P}\{\tilde{h}(X) \geq h(X) - n(1 + \log c)\}.\end{aligned}$$

using Corollary 4.5 for the last inequality. Applying Corollary 4.4 with $t = -\log c - 1$ yields

$$\mathbb{P}\{f(X) \leq c^n \|f\|_\infty\} \leq e^{-nr(-1-\log c)}.$$

Elementary algebra concludes the proof. □

Such “effective support” results are useful in convex geometry as they can allow to reduce certain statements about log-concave functions or measures to statements about convex sets; they thus provide an efficient route to proving functional or probabilistic analogues of known results in the geometry of convex sets. Instances where such a strategy is used include [5, 24]. These and other applications of the concentration of information phenomenon are discussed in [26].

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Maximal Inequalities for Dependent Random Variables

Jørgen Hoffmann-Jørgensen

Abstract Maximal inequalities play a crucial role in many probabilistic limit theorem; for instance, the law of large numbers, the law of the iterated logarithm, the martingale limit theorem and the central limit theorem. Let $X_1, X_2, \ldots$ be random variables with partial sums $S_k = X_1 + \cdots + X_k$. Then a maximal inequality gives conditions ensuring that the maximal partial sum $M_n = \max_{1 \le i \le n} S_i$ is of the same order as the last sum S_n. In the literature there exist large number of maximal inequalities if $X_1, X_2, \ldots$ are independent but much fewer for dependent random variables. In this paper, I shall focus on random variables $X_1, X_2, \ldots$ having some weak dependence properties; such as positive and negative In-correlation, mixing conditions and weak martingale conditions.

Keywords Demi-martingales • Integral orderings • Mixing conditions • Negative and positive correlation

Mathematics Subject Classification (2010). Primary 60E15; Secondary 60F05

1 Introduction

Throughout this paper, we let $(\Omega, \mathcal{F}, P)$ and $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ denote two fixed probability spaces. We let $\mathbb{R} = (-\infty, \infty)$ denote the real line and we let $\mathbb{R}_+ = [0, \infty)$ denote the non-negative real line. We let $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ denote the set of all non-negative integers, we let $\mathbb{N} = \{1, 2, \ldots\}$ denote the set of all positive integers and we define

$$\mathbf{\Delta}_n = \{(i,j) \in \mathbb{N}_0^2 \mid j - i \ge n\}\ ,\ \mathbf{\Delta}^n = \{(i,j) \in \mathbb{N}_0^2 \mid 0 \le j - i \le n\} \quad \forall\, n \in \mathbb{N}_0\,,$$
$$\nabla = \{(i,k,j) \in \mathbb{N}_0^3 \mid i < k \le j\}\,.$$

J. Hoffmann-Jørgensen (✉)
Department of Mathematics, University of Aarhus, Aarhus, Denmark
e-mail: hoff@math.au.dk

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Let $X_1, X_2, \ldots$ be a sequence of random variables and let us consider the partial sums and the maximal partial sums : $S_{i,i} = M_{i,i} = \overline{M}_{i,i} = 0$ and

$$S_{i,j} = \sum_{i<k\leq j} X_k \ , \ M_{i,j} = \max_{i\leq k\leq j} S_{i,k} \ , \ \overline{M}_{i,j} = \max_{i\leq k\leq j} |S_{i,k}| \quad \forall\, (i,j) \in \mathbf{\Delta}_1 \,. \tag{1.1}$$

Then $M_{i,j}$ and $\overline{M}_{i,j}$ are non-negative random variables. Recall that a maximal inequality is an inequality stating the maximal sum $M_{i,j}$ (or $\overline{M}_{i,j}$) is of the same order as the final sum $S_{i,j}^+$ (or $|S_{i,j}|$); where we let $x^+ = \max(x, 0)$ denote the positive part of x for $x \in \mathbb{R}$.

In the literature there exists a variety of maximal inequalities. Let me review a few of these. Let $X_1, X_2, \ldots$ be random variables and let $S_{i,j}$, $M_{i,j}$ and $\overline{M}_{i,j}$ be given by (1.1). Then we have:

The Rademacher-Menchoff inequality (see [21] and [16]): Let $\tau_1\, \tau_2, \ldots$ be non-negative numbers satisfying $ES_{i,j}^2 \leq \sum_{i<k\leq j} \tau_k$ for all $(i,j) \in \mathbf{\Delta}_1$. Then we have

$$EM_{i,j}^2 \leq \left(1 + \lfloor \tfrac{\log(j-i)}{\log 2} \rfloor\right)^2 \sum_{i<k\leq j} \tau_k \quad \forall\, (i,j) \in \mathbf{\Delta}_1 \,. \tag{1.2}$$

The Minkovski-Hölder inequality (see [8, pp. 166–167]): Let us define $\rho(n) = (E|X_n|^q)^{1/r}$ for all $n \geq 1$ and set $\alpha = \left(1 - \frac{r}{q\vee 1}\right)^+$. Then we have

$$E\overline{M}_{i,j}^q \leq \left(\sum_{i<k\leq j} \rho(k)^{\frac{r}{q\vee 1}}\right)^{q\vee 1} \leq (j-i)^\alpha \left(\sum_{i<k\leq j} \rho(k)\right)^r \quad \forall\, (i,j) \in \mathbf{\Delta}_1 \,. \tag{1.3}$$

Lévy's inequality (see [8, p. 473]): If $(X_1, \ldots, X_k\, , -X_{k+1} \ldots, -X_j)$ and $(X_1, \ldots, X_j)$ have the same distribution for all $0 < k < j$, then we have

$$P(M_{i,j} > t) \leq 2\, P(S_{i,j} > t) \quad \forall\, (i,j,t) \in \mathbf{\Delta}_1 \times \mathbb{R}_+ \,, \tag{1.4}$$

$$P(\overline{M}_{i,j} > t) \leq 2\, P(|S_{i,j}| > t) \quad \forall\, (i,j,t) \in \mathbf{\Delta}_1 \times \mathbb{R}_+ \,. \tag{1.5}$$

Khinchine's inequality (see [8, p. 307]): If $(\epsilon_1 X_1, \ldots, \epsilon_j X_j)$ and $(X_1, \ldots, X_j)$ have the same distribution for all $j \geq 1$ and all signs $\epsilon_1, \ldots, \epsilon_j \in \{-1, +1\}$, then we have

$$\tfrac{1}{2}\, E\overline{M}_{i,j}^q \leq E|S_{i,j}|^q \leq K_q\, E \left(\sum_{i<k\leq j} X_k^2\right)^{q/2} \quad \forall\, (i,j) \in \mathbf{\Delta}_1 \,, \tag{1.6}$$

where $K_q = 1$, if $q \leq 2$ and $K_q = \pi^{-1/2}\, 2^{q/2}\, \Gamma(\frac{q+1}{2})$, if $q \geq 2$.

The prophet inequality (see [4]): Suppose that $X_1, X_2, \ldots$ are independent with $EX_n = 0$ for all $n \geq 1$ and let $\varphi : [0, \infty) \to [0, \infty)$ be an increasing, convex function with $\varphi(0) = 0$. Then we have

$$E\varphi((M_{i,j} - a)^+) \le 2\,E\varphi\big((S_{i,j} - a)^+\big) \quad \forall\,(i,j,a) \in \mathbf{\Delta}_1 \times \mathbb{R}\,, \tag{1.7}$$

$$E\varphi(\overline{M}_{i,j}) \le 2\,E\varphi(|S_{i,j}|) \quad \forall\,(i,j) \in \mathbf{\Delta}_1\,. \tag{1.8}$$

Ottaviani's inequality (see [8, p. 472]) Suppose that $X_1, X_2, \ldots$ are independent and let us define $\gamma_{i,j}(s) = \min_{i<k\le j} P(|S_{i,k}| \le s)$ for all $(i,j,s) \in \mathbf{\Delta}_1 \times \mathbb{R}$. Then we have

$$\gamma_{i,j}(s)\,P(\overline{M}_{i,j} > s + t) \le P(|S_{i,j}| > t) \quad \forall\,(i,j,s,t) \in \mathbf{\Delta}_1 \times \mathbb{R}_+ \times \mathbb{R}_+\,. \tag{1.9}$$

The martingale inequality (see [8, p. 472]): Let $\mathcal{F}_{i,j}$ denote the σ-algebra generated by $(X_{i+1}, \ldots, X_j)$ for all $(i,j) \in \mathbf{\Delta}_1$ and suppose that $S_{i,j} \in L(P)$ and $S_{i,k} \le E(S_{i,j} \mid \mathcal{F}_{k,j})$ a.s. for all $i < k < j$. If $\varphi : \mathbb{R} \to [0,\infty)$ is increasing and convex, we have

$$\varphi(t)\,P(M_{i,j} > t) \le E\big(1_{\{M_{i,j}>t\}}\,\varphi(S_{i,j})\big) \quad \forall\,t \in \mathbb{R}\,. \tag{1.10}$$

If $x \in \mathbb{R}$, we let $\lfloor x \rfloor$ denote the largest integer $\le x$ and we let $\lceil x \rceil$ denote the smallest integer $\ge x$. We let $\overline{\mathbb{R}} = [-\infty, \infty]$ denote the extended real line and I shall use the following extension of the arithmetic on the real line $\mathbb{R} = (-\infty, \infty)$:

$$x + \infty := \infty\ \forall -\infty \le x \le \infty\,,\ x + (-\infty) := -\infty\ \forall -\infty \le x < \infty\,,$$
$$0 \cdot (\pm\infty) := 0\,,\ x \cdot (\pm\infty) := \pm\infty\,,\ (-x) \cdot (\pm\infty) := \mp\infty\ \forall\,0 < x \le \infty\,,$$
$$\tfrac{1}{0} = \log\infty = e^{\infty} := \infty\,,\ \tfrac{1}{\pm\infty} = e^{-\infty} := 0\,,\ \tfrac{x}{y} := x \cdot \tfrac{1}{y}\,,\ x^0 = 1\ \forall\,x,y \in \overline{\mathbb{R}}\,,$$

and I shall use the standard conventions $\inf\emptyset = \min\emptyset := \infty$, $\sup\emptyset = \max\emptyset := -\infty$ and $\sum_{k\in\emptyset} a_k := 0$.

If V is a real vector space, we say that $\psi : V \to \overline{\mathbb{R}}$ is *sub-additive* if $\psi(x+y) \le \psi(x) + \psi(y)$ for all $x, y \in V$. Let $k \ge 1$ be an integer. Then we let $\mathcal{B}^k$ denote the Borel σ-algebra on $\mathbb{R}^k$ and we let $\le$ denote the coordinate-wise ordering on $\mathbb{R}^k$; that is $(x_1, \ldots, x_k) \le (y_1, \ldots, y_k)$ if and only if $x_i \le y_i$ for all $i = 1, \ldots, k$. If $D \subseteq \mathbb{R}^k$ and $F : D \to \mathbb{R}^m$ is a function, we say that F is *increasing* if $F(x) \le F(y)$ for all $x, y \in D$ with $x \le y$. If $u = (u_1, \ldots, u_k)$ and $v = (v_1, \ldots, v_k)$ are given vectors in $\mathbb{R}^k$, we define

$$[*, u] = \{x \in \mathbb{R}^k \mid x \le u\}\,,\ [u, *] = \{x \in \mathbb{R}^k \mid u \le x\}\,,$$
$$u \wedge v = (\min(u_1, v_1), \ldots, \min(u_k, v_k))\,,\ u \vee v = (\max(u_1, v_1), \ldots, \max(u_k, v_k))\,.$$

We say that $F : \mathbb{R}_+^k \to \mathbb{R}$ is *homogeneous* if $F(rx) = r\,F(x)$ for all $x \in \mathbb{R}_+^k$ and all $r \in \mathbb{R}_+$, and we say that $f : \mathbb{R}^k \to \mathbb{R}$ is *super-modular* if $f(x) + f(y) \le f(x \vee y) + f(x \wedge y)$ for all $x, y \in \mathbb{R}^k$; see [10]. We let $\mathrm{B}(\mathbb{R}^k)$ denote the set of all bounded Borel functions from $\mathbb{R}^k$ to $\mathbb{R}$, we let $\mathrm{IB}(\mathbb{R}^k)$ denote the set of all bounded, increasing Borel functions from $\mathbb{R}^k$ to $\mathbb{R}$ and we let $\mathrm{B}_+(\mathbb{R}^k)$ and $\mathrm{IB}_+(\mathbb{R}^k)$ denote the sets of all non-negative functions in $\mathrm{B}(\mathbb{R}^k)$ and $\mathrm{IB}(\mathbb{R}^k)$ respectively.

We let $L^0(P)$ denote the set of all real random variables on $(\Omega, \mathcal{F}, P)$ and we let $L^0_+(P)$ denote the set of all non-negative random variables. If $Z : \Omega \to \mathbb{R}$ is an arbitrary function, we let E^*Z and E_*Z denote the upper and lower expectations of Z. We let $L(P)$ denote the set of all random variables $X : \Omega \to \mathbb{R}$ satisfying $E_*X = E^*X$ and we set $EX := E_*X = E^*X$ for all $X \in L(P)$ and $L^1(P) := \{X \in L(P) \mid EX \neq \pm\infty\}$.

2 Rademacher-Menchoff Type Inequalities

In this section we shall study maximal inequalities of the Rademacher-Menchoff type; see [12, 16–18, 21] and [13]. A triangular scheme $(\mathbb{S}_{i,j}, \mathbb{M}_{i,j})_{(i,j)\in\boldsymbol{\Delta}_0}$ will be called a *max-scheme* if $\mathbb{S}_{i,j}$ and $\mathbb{M}_{i,j}$ are non-negative random variables satisfying

$$\mathbb{S}_{i,i} = \mathbb{M}_{i,i} = 0 \text{ a.s. and } \mathbb{M}_{i,j} \leq \mathbb{M}_{i,k-1} \vee (\mathbb{S}_{i,k} + \mathbb{M}_{k,j}) \text{ a.s.} \quad \forall\, (i,k,j) \in \nabla\,. \tag{2.1}$$

Let $(\mathbb{S}_{i,j}, \mathbb{M}_{i,j})$ be a max-scheme. Then we have $\mathbb{M}_{i,j} \leq \mathbb{M}_{i,j-1} \vee \mathbb{S}_{i,j}$ a.s. and $\mathbb{M}_{i,j} \leq \mathbb{S}_{i,i+1} + \mathbb{M}_{i+1,j}$ a.s. [take $k = j$ and $k = i + 1$ in (2.1)]. So by induction we have

$$\mathbb{M}_{i,j} \leq \max_{i<k\leq j} \mathbb{S}_{i,k} \text{ a.s and } \mathbb{M}_{i,j} \leq \sum_{i<k\leq j} \mathbb{S}_{k-1,k} \text{ a.s} \quad \forall\, (i.j) \in \boldsymbol{\Delta}_1\,. \tag{2.2}$$

Let $(\mathbb{S}_{i,j})_{(i,j)\in\boldsymbol{\Delta}_1}$ be non-negative random variables satisfying

$$\mathbb{S}_{i,j} \leq \mathbb{S}_{i,k} + \mathbb{S}_{k,j} \text{ a.s.} \quad \forall (i,j) \in \boldsymbol{\Delta}_1 \tag{2.3}$$

and set $\mathbb{S}_{i,i} = \mathbb{M}_{i,i} = 0$ and $\mathbb{M}_{i,j} = \max_{i<k\leq j} \mathbb{S}_{i,k}$ for all $(i,j) \in \boldsymbol{\Delta}_1$. Then $(\mathbb{S}_{i,j}, \mathbb{M}_{i,j})$ is a max scheme. In particular, we see that $(S^+_{i,j}, M_{i,j})$ and $(|S_{i,j}|, \overline{M}_{i,j})$ are max-schemes, if $S_{i,j}$, $M_{i,j}$ and $\overline{M}_{i,j}$ are given by (1.1) for some sequence (X_n) of real random variables. Let $(\mathbb{S}_{i,j}, \mathbb{M}_{i,j})$ be a max-scheme. In this section we shall search for conditions ensuring that the maximal "sum" $\mathbb{M}_{i,j}$ is of the same order as the "sum" $\mathbb{S}_{i,j}$. More precisely:

Let $\mathcal{K} : L^0_+(P) \to [0,\infty]$ be a functional. Then we say that $\mathcal{K}$ is *P-increasing* if $\mathcal{K}(0) < \infty$ and $\mathcal{K}(X) \leq \mathcal{K}(Y)$ for all $X, Y \in L^0_+(P)$ with $X \leq Y$ a.s. We say that $\mathcal{K}$ is *weakly sub-additive* if $\mathcal{K}$ is P-increasing and $\mathcal{K}(X \vee Y) \leq \mathcal{K}(X) + \mathcal{K}(Y)$ for all $X, Y \in L^0_+(P)$ and we say that $\mathcal{K}$ is *sub-additive* if $\mathcal{K}$ is P-increasing and $\mathcal{K}(X + Y) \leq \mathcal{K}(X) + \mathcal{K}(Y)$ for all $X, Y \in L^0_+(P)$. Note that every sub-additive functional is weakly sub-additive. Let $\mathcal{K} : L^0_+(P) \to [0,\infty]$ be a P-increasing functional and let $(\mathbb{S}_{i,j}, \mathbb{M}_{i,j})$ be a max-scheme. In this section, I shall search for upper bounds of $\mathcal{K}(\mathbb{M}_{i,j})$ in terms of $\mathcal{K}(\mathbb{S}_{i,j})$. More precisely, let $D \subseteq \mathbb{R}_+$ be a non-empty set and let $f : \mathbb{N}_0 \to \mathbb{R}_+$ and $V_{i,j} : D \to \mathbb{R}_+$ be given functions for $(i,j) \in \boldsymbol{\Delta}_0$

satisfying $V_{i,j}(t) \le V_{i,j+1}(t)$ and $\mathcal{K}(t\,\mathbb{S}_{i,j}) \le f(j-i)\,V_{i,j}(t)$ for all $(i,j,t) \in \mathbf{\Delta}_0 \times D$. I shall then search for functions $F : \mathbb{N}_0 \to \mathbb{R}_+$ and $U_{i,j} : D \to \mathbb{R}_+$ satisfying $\mathcal{K}(t\,\mathbb{M}_{i,j}) \le F(j-i)\,U_{i,j}(t)$ for all $(i,j,t) \in \mathbf{\Delta}_0 \times D$. If $\mathcal{K}$ is weakly sub-additive, then (2.2) shows that

$$\mathcal{K}(t\,\mathbb{M}_{i,j}) \le \sum_{i<k\le j} \mathcal{K}(t\,\mathbb{S}_{i,k}) \le F(j-i)\,V_{i,j}(t) \quad \forall\,(i,j,t) \in \mathbf{\Delta}_1 \times D\,, \tag{2.4}$$

where $F(n) = f(1) + \cdots + f(n)$ for all $n \in \mathbb{N}$. This function is in general too large to be really useful. In order to improve (2.4) I shall use the recursive structure of (2.1) together with an inequality of the following type:

$$\mathcal{K}(X \vee (Y+Z)) \le \Gamma(\mathcal{K}(\lambda X) + \mathcal{K}(\mu Y), \mathcal{K}(\nu Z)) \quad \forall\, X, Y, Z \in L^0_+(P)\,, \tag{2.5}$$

where $\lambda, \mu, \nu \in \mathbb{R}_+$ and $\Gamma : \mathbb{R}^2_+ \to \mathbb{R}_+$ is an increasing, homogeneous function and we use the convention $\Gamma(\infty, x) = \Gamma(x, \infty) = \infty$ for all $x \in [0, \infty]$. To construct the improved function, we shall need following notion:

Let $\Gamma : \mathbb{R}^2_+ \to \mathbb{R}_+$ be an increasing homogeneous function. If $f : \mathbb{N}_0 \to \mathbb{R}_+$ and $r \in \mathbb{R}_+$, we define $f_r^\Gamma(n)$ inductively as follows:

$$f_r^\Gamma(0) = f(0)\,,\; f_r^\Gamma(n) = \Gamma(r f_r^\Gamma(n-1), f(n)) \quad \forall\, r \in \mathbb{R}_+ \;\forall\, n \in \mathbb{N}\,. \tag{2.6}$$

If $f : \mathbb{N}_0 \to \mathbb{R}_+$ is increasing, then an easy induction argument (see the proof of Proposition A.2) shows that f_r^Γ is increasing if and only if $f(0) \le \Gamma(rf(0), f(1))$; for instance if $f(0) = 0$ or if $\Gamma(r, 1) \ge 1$. If $f : \mathbb{N}_0 \to \mathbb{R}_+$ is increasing and $c \ge 1$ is a given number satisfying $\Gamma(rc, 1) \le c$, then an easy induction argument shows that $f^\Gamma(n) \le c f(n)$ for all $n \in \mathbb{N}_0$. In the applications I shall consider the following increasing, homogeneous functions:

$$\Sigma_\gamma(x,y) := (x^{1/\gamma} + y^{1/\gamma})^\gamma\,,\; \Theta_\lambda(x,y) := x^\lambda\, y^{1-\lambda} \quad \forall\,(x,y) \in \mathbb{R}^2_+\,, \tag{2.7}$$

$$\Pi_\chi(x,y) := \inf_{0<\alpha<1} \left(\chi(\tfrac{1}{\alpha})\,x + \chi(\tfrac{1}{1-\alpha})\,y\right) \quad \forall\,(x,y) \in \mathbb{R}^2_+\,, \tag{2.8}$$

where $\gamma > 0$, $0 < \lambda < 1$ and $\chi : \mathbb{R}_+ \to \mathbb{R}_+$ is a given function, together with the following weakly sub-additive functionals:

$$\mathcal{L}_\phi(X) = E\phi(X)\,,\; \mathcal{M}_\phi(X) = \sup_{u\in\mathbb{R}_+} \psi(u)\,P(X>u)\,, \tag{2.9}$$

where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing function and $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is an arbitrary function. Note that $f_r^{\Sigma_\gamma}(n) = \left(\sum_{k=0}^n r^{(n-k)/\gamma} f(k)^{1/\gamma}\right)^\gamma$ for all $n \in \mathbb{N}_0$ and all $r \in \mathbb{R}_+$ and that $\Pi_\chi = \Sigma_{\gamma+1}$ if $\gamma \ge 0$ and $\chi(x) = x^\gamma$ for all $x > 1$.

Let $\Gamma : \mathbb{R}^2_+ \to \mathbb{R}_+$ be an increasing homogeneous function and set $\Gamma(\infty, x) = \Gamma(x, \infty) = \infty$ for all $x \in [0, \infty]$. Let $\mu, \nu \ge 0$ be given numbers and let $\mathcal{K} : L^0_+(P) \to [0, \infty]$ be a weakly sub-additive functional. Let us consider the following condition:

$$\mathcal{K}(X+Y) \le \Gamma(\mathcal{K}(\mu X), \mathcal{K}(\nu Y)) \quad \forall X, Y \in L^0_+(P)\,. \tag{2.10}$$

Let $0 < p < 1$ be given. Since $x + y \le \frac{x}{p} \vee \frac{y}{1-p}$ for all $x, y \ge 0$, we see that $(\mathcal{K}, \Sigma_1)$ satisfies (2.10) with $(\mu, \nu) = (\frac{1}{p}, \frac{1}{1-p})$ and since $x \vee (y+z) \le (x \vee y) + z$ for all $x, y, z \ge 0$, we see that (2.10) implies that $(\mathcal{K}, \Gamma)$ satisfies (2.5) with $(\lambda, \mu, \nu) = (\mu, \mu, \nu)$. Since $\mathcal{K}$ is weakly subadditive, we see that $(\mathcal{K}, \Sigma_1)$ satisfies (2.5) with $(\lambda, \mu, \nu) = (1, \frac{1}{p}, \frac{1}{1-p})$. If $\mathcal{K}(\cdot)^{1/\gamma}$ is sub-additive for some $\gamma > 0$, we see that $(\mathcal{K}, \Sigma_\gamma)$ satisfies (2.10) with $(\mu, \nu) = (1, 1)$ and (2.5) with $(\lambda, \mu, \nu) = (1, 1, 1)$. If $\chi : \mathbb{R}_+ \to \mathbb{R}_+$ is a given function satisfying $\mathcal{K}(sX) \le \chi(s)\,\mathcal{K}(X)$ for all $s > 1$ and all $X \in L^0_+(P)$. then $(\mathcal{K}, \Pi_\chi)$ satisfies (2.10) with $(\mu, \nu) = (1, 1)$ and (2.5) with $(\lambda, \mu, \nu) = (1, 1, 1)$.

Let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing function. Then $\mathcal{L}_\phi$ is weakly sub-additive. If $\gamma > 0$ and $\phi(\cdot)^{1/\gamma}$ is subadditive; for instance, if $\phi(st) \le s^\gamma \phi(t)$ for all $s > 1$ and all $t \ge 0$, then Minkovski's inequality shows that $(\mathcal{L}_\phi, \Sigma_\gamma)$ satisfies (2.5) with $(\lambda, \mu, \nu) = (1, 1, 1)$. If $\chi : \mathbb{R}_+ \to \mathbb{R}_+$ is a given function satisfying $\phi(st) \le \chi(s)\,\phi(t)$ for all $s > 1$ and all $t \ge 0$, then $\mathcal{L}_\phi(sX) \le \chi(s)\,\mathcal{L}_\phi(X)$ for all $s > 1$ and all $X \in L^0_+(P)$ and so we see that $(\mathcal{L}_\phi, \Sigma_\gamma)$ satisfies (2.5) with $(\lambda, \mu, \nu) = (1, 1, 1)$. If ϕ is log-convex and $0 < p < 1$, then Hölder's inequality shows that $(\mathcal{L}_\phi, \Theta_p)$ satisfies (2.5) with $(\lambda, \mu, \nu) = (\frac{1}{p}, \frac{1}{p}, \frac{1}{1-p})$.

Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a given function. Then $\mathcal{M}_\psi$ is a weakly sub-additive functional and if $\chi : \mathbb{R}_+ \to \mathbb{R}_+$ is a given function satisfying $\psi(st) \le \chi(s)\,\psi(t)$ for all $s > 1$ and all $t \ge 0$, then the reader easily verifies that $(\mathcal{M}_\psi, \Pi_\chi)$ satisfies (2.5) with $(\lambda, \mu, \nu) = (1, 1, 1)$.

The results of this section rely on a purely analytic proposition solving a certain functional inequality (see Proposition A.2 in Appendix). To do this, we need the following notion. If $\xi : \mathbb{N}_0 \to \mathbb{N}_0$ is an increasing function, we define

$$D^\xi_{i,j} = \{k \in \mathbb{N}_0 \mid i < k \le j\,,\ \xi(j{-}k) \vee \xi(k{-}i{-}1) < \xi(j{-}i)\}\ \forall\,(i,j) \in \mathbf{\Delta}_1\,. \tag{2.11}$$

Set $\hat{\xi}(n) = \inf\{k \in \mathbb{N}_0 \mid \xi(k) \ge \xi(n)\}$ for all $n \in \mathbb{N}_0$. Then $\hat{\xi} : \mathbb{N}_0 \to \mathbb{N}_0$ is an increasing function satisfying $\hat{\xi}(n) \le n$ for all $n \in \mathbb{N}_0$ and we have $\hat{\xi}(n) = n$ if and only if either $n = 0$ or $n \ge 1$ and $\xi(n-1) < \xi(n)$. Since ξ is increasing, we have $\xi(k) < \xi(n)$ for all $0 \le k < \hat{\xi}(n)$ and

$$D^\xi_{i,j} = \{k \in \mathbb{N}_0 \mid j - \hat{\xi}(j-i) < k \le i + \hat{\xi}(j-i)\} \quad \forall (i,j) \in \mathbf{\Delta}_1\,. \tag{2.12}$$

Hence, if $(i,j) \in \mathbf{\Delta}_1$, we have that $D^\xi_{i,j} = \{k \in \mathbb{N}_0 \mid i < k \le j\}$ if and only if $\xi(j-i-1) < \xi(j-i)$. Similarly, we have that $D^\xi_{i,j} \ne \emptyset$ if and only if $\xi(\lfloor\frac{j-i}{2}\rfloor) < \xi(j-i)$ and if so we have $\lceil\frac{i+j+1}{2}\rceil \in D^\xi_{i,j}$.

Theorem 2.1 *Let $\mathcal{K} : L^0_+(P) \to [0,\infty]$ be a P-increasing functional, let $\Gamma : \mathbb{R}^2_+ \to \mathbb{R}_+$ be an increasing, homogeneous function and set $\Gamma(\infty, x) = \Gamma(x,\infty) = \infty$ for all $x \in [0,\infty]$. Let $\vartheta : \mathbb{R}_+ \to \mathbb{R}_+$, $f : \mathbb{N}_0 \to \mathbb{R}_+$ and $\xi : \mathbb{N}_0 \to \mathbb{N}_0$ be increasing functions and let $\lambda, \mu, \nu, r \in \mathbb{R}_+$ be given numbers satisfying*

(a) $\mathcal{K}(X \vee (Y+Z)) \le \Gamma(\mathcal{K}(\lambda X) + \mathcal{K}(\mu Y), \mathcal{K}(\nu Z)) \quad \forall\, X,Y,Z \in L^0_+(P)$,
(b) $\xi(0) = 0$ *and* $f(0) \le \Gamma(rf(0), f(1))$.

Let $(\mathbb{M}_{i,j}, \mathbb{S}_{i,j})_{(i,j)\in\mathbf{\Delta}_0}$ be a max-scheme and let $D \subseteq \mathbb{R}_+$ be a non-empty set such that $\lambda t \in D$ and $\mu t \in D$ for all $t \in D$. Let $G_{i,j} : \mathbb{R}_+ \to \mathbb{R}_+$ for $(i,j) \in \mathbf{\Delta}_0$ be given functions satisfying

(c) $\mathcal{K}(t\,\mathbb{S}_{i,j}) \le f(\xi(j-i))\,\vartheta(G_{i,j}(t)) \quad \forall (i,j,t) \in \mathbf{\Delta}_0 \times (\nu D)$,
(d) $G_{i,j}(t) \le G_{i,j+1}(t) \quad \forall (i,j,t) \in \mathbf{\Delta}_0 \times (\nu D)$.

where $\nu D := \{\nu t \mid t \in D\}$. Let us define $V_{i,j}(t) = \vartheta(G_{i,j}(\nu t))$ for all $(i,j,t) \in \mathbf{\Delta}_0 \times \mathbb{R}_+$ and

$$\Upsilon_r = \{(i,j,t) \in \mathbf{\Delta}_1 \times D \mid f^\Gamma_r(\xi(j-i))\, V_{i,j}(t) < \mathcal{K}(t\,\mathbb{M}_{i,j})\}.$$

Then we have $\mathcal{K}(t\,\mathbb{M}_{i,j}) \le f^\Gamma_r(\xi(j-i))\,V_{i,j}(t)$ for all $(i,j,t) \in \mathbf{\Delta}_0 \times D$ if and only if the following condition holds:

(e) $\min_{k\in D^\xi_{i,j}} (V_{i,k-1}(\lambda t) + V_{k,j}(\mu t)) \le r\,V_{i,j}(t) \quad \forall\,(i,j,t) \in \Upsilon_r$.

Suppose that $G_{i,i}(t) = 0$ for all $(i,t) \in \mathbb{N}_0 \times (\nu D)$ and let $\rho, \alpha, \beta, \delta, a, b, q \ge 0$ be given numbers satisfying

(f) $\max\limits_{i<k<j} (G_{i,k}(t) + G_{k,j}(t)) \le \rho\,G_{i,j}(t) \quad \forall\,(i,j,t) \in \mathbf{\Delta}_2 \times (\nu D)$,
(g) $G_{i,j}(\lambda t) \le \alpha\,G_{i,j}(t)$, $G_{i,j}(\mu t) \le \beta\,G_{i,j}(t)\ \forall (i,j,t) \in \Delta_0 \times (\nu D)$,
(h) $\vartheta(bx) \le a\,\vartheta(x)^{1+\delta}$ *and* $\vartheta(x) + \vartheta(y) \le q\,\vartheta(x+y) \quad \forall x,y \ge 0$.

If $\nu \ge 1$, then we have $\mathcal{K}(t\,\mathbb{M}_{i,j}) \le f^\Gamma_r(\xi(j-i))\,V_{i,j}(t)$ for all $(i,j,t) \in \mathbf{\Delta}_0 \times D$ if just one of the following two conditions hold:

(i) $\frac{\alpha\beta\rho}{\alpha+\beta} \le b$, $\xi(n-1) < \xi(n)\ \forall\, n \ge 2$ *and* $r \ge 2a \left(\frac{\mathcal{K}(t\,\mathbb{M}_{i,j})}{f^\Gamma_r(\xi(1))}\right)^\delta\ \forall\,(i,j,t) \in \Upsilon_r$.
(j) $(\alpha \vee \beta)\rho \le b$, $\xi(\lfloor \frac{n}{2} \rfloor) < \xi(n)\ \forall\, n \ge 2$ *and* $r \ge qa \left(\frac{\mathcal{K}(t\,\mathbb{M}_{i,j})}{f^\Gamma_r(\xi(1))}\right)^\delta\ \forall (i,j,t) \in \Upsilon_r$.

Proof We shall apply Proposition A.2 with D, Γ, $V_{i,j}$ as above and

$$A_{i,j}(t) = \mathcal{K}(t\,\mathbb{M}_{i,j})\,, B_{i,j}(t) = \mathcal{K}(\nu t\,\mathbb{S}_{i,j})\,,\ h = f\,,\ (p,q) = (\mu, \lambda)\,.$$

Let $(i,k,j) \in \nabla$ and $t \ge 0$ be given. By (a) and (2.1), we have

$$\mathcal{K}(t\,\mathbb{M}_{i,j}) \le \Gamma(\mathcal{K}(\lambda t\,\mathbb{M}_{i,k-1}) + \mathcal{K}(\mu t\,\mathbb{M}_{k,j}), \mathcal{K}(\nu t\,\mathbb{S}_{i,k}))\,.$$

Hence, we see that condition (a) in Proposition A.2 holds. Since $\mathbb{M}_{i,i} = \mathbb{S}_{i,i} = 0$ a.s., we have $A_{i,i}(t) = B_{i,i}(t) = \mathcal{K}(0)$ and by (c), we have $\mathcal{K}(0) \leq f(0)\, V_{i,i}(t)$. So by (c) and (d) we see that the conditions (b) and (c) in Proposition A.2 hold. Hence, we have that $A_{i,j}$ and $B_{i,j}$ are finite and increasing on D for all $(i,j) \in \mathbf{\Delta}_0$ and by (b), we see that f_r^Γ is increasing. So by Proposition A.2, we see that (e) implies $\mathcal{K}(t\,\mathbb{M}_{i,j}) \leq f_r^\Gamma(\xi(j-i))\, V_{i,j}(t)$ for all $(i,j,t) \in \mathbf{\Delta}_0 \times D$. Conversely, the latter condition implies that $\Upsilon_r = \emptyset$ in which case (e) holds trivially.

Suppose that $G_{i,i}(t) = 0$ for all $(i,t) \in \mathbb{N}_0 \times (\nu D)$ and that (f)–(h) hold. Let $(i,j,t) \in \mathbf{\Delta}_1 \times D$ be given and set $W_{i,j}(t) = \min_{k \in D_{i,j}^\xi} (V_{i,k-1}(\lambda t) + V_{k,j}(\mu t))$. Then I claim that we have

$$\xi(j-i-1) < \xi(j-i) \;\Rightarrow\; W_{i,j}(t) \leq 2\,\vartheta\left(\tfrac{\alpha\beta\rho}{\alpha+\beta}\, G_{i,j}(\nu t)\right), \tag{*}$$

$$\xi(\lfloor \tfrac{j-i}{2} \rfloor) < \xi(j-i) \;\Rightarrow\; W_{i,j}(t) \leq q\,\vartheta(\eta\rho\, G_{i,j}(\nu t)), \tag{**}$$

where $\eta := \alpha \vee \beta$.

Proof of (*) Suppose that $\xi(j-i-1) < \xi(j-i)$. By (2.12), we have $D_{i,j}^\xi = \{k \in \mathbb{N}_0 \mid i < k \leq j\}$. Since $t \in D$, we have $\lambda t \in D$ and $\mu t \in D$ and so we have $G_{m,m}(\nu t) = G_{m,m}(\lambda\nu t) = G_{m,m}(\mu\nu t) = 0$ for all $m \in \mathbb{N}_0$. Hence, if $j = i+1$, we have $W_{i,j}(t) = V_{i,i}(\lambda t) + V_{j,j}(\mu t) = 2\,\vartheta(0) \leq 2\,\vartheta(x)$ for all $x \geq 0$. Suppose that $j \geq i+2$ and set $a = \frac{\beta}{\alpha}$, $g_{m,n} = G_{m,n}(\nu t)$ and $h = \frac{\beta\rho}{\alpha+\beta}\, G_{i,j}(\nu t)$. Then we have $g_{i,i} = g_{j,j} = 0$ and $\rho\, g_{i,j} = (1+\frac{1}{a})\, h$ and so by (f) and Lemma A.1, we have

$$\min_{i<k\leq j} (G_{i,k-1}(\nu t) \vee (\tfrac{\beta}{\alpha}\, G_{k,j}(\nu t)) \leq \tfrac{\beta\rho}{\alpha+\beta}\, G_{i,j}(\nu t)\,.$$

By (g), we have $G_{i,k-1}(\lambda\nu t) \vee G_{k,j}(\mu\nu t) \leq \alpha\,(G_{i,k-1}(\nu t) \vee (\frac{\beta}{\alpha}\, G_{k,j}(\nu t)))$ for all $i < k \leq j$ and since ϑ is increasing, we have

$$\begin{aligned} W_{i,j}(t) &= \min_{i<k\leq j} (V_{i,k-1}(\lambda t) + V_{k,j}(\mu t)) \leq 2\,\vartheta\left(\min_{i<k\leq j} (G_{i,k-1}(\lambda\nu t) \vee G_{k,j}(\mu\nu t))\right) \\ &\leq 2\,\vartheta\left(\tfrac{\alpha\beta\rho}{\alpha+\beta}\, G_{i,j}(\nu t)\right), \end{aligned}$$

which completes the proof of (*).

Proof of (**) Suppose that $\xi(\lfloor \frac{j-i}{2} \rfloor) < \xi(j-i)$ and set $\kappa = \lceil \frac{i+j+1}{2} \rceil$. By (2.12), we have $\kappa \in D_{i,j}^\xi$ and so by (h) we have

$$W_{i,j}(t) \leq V_{i,\kappa-1}(\lambda t) + V_{\kappa,j}(\mu t) \leq q\,\vartheta\big(G_{i,\kappa-1}(\lambda\nu t) + G_{\kappa,j}(\mu\nu t)\big)\,.$$

By (d), we have $G_{i,\kappa-1}(\lambda\nu t) \leq G_{i,\kappa}(\lambda\nu t)$ and so by (f) and (g), we have

$$G_{i,\kappa-1}(\lambda\nu t) + G_{\kappa,j}(\mu\nu t) \leq \alpha\, G_{i,\kappa}(\nu t) + \beta\, G_{\kappa,j}(\nu t) \leq \eta\rho\, G_{i,j}(\nu t)$$

Since ϑ is increasing, we have $W_{i,j}(t) \le q\,\vartheta(\eta\rho\,G_{i,j}(\nu t))$ proves (**).

Suppose that (i) holds and let $(i,j,t) \in \Upsilon_r$ be given. By (2.2), we have $\mathbb{M}_{i,i+1} \le \mathbb{S}_{i,i+1}$ a.s. and by (c), we have $\mathcal{K}(\nu t\,\mathbb{S}_{i,i+1}) \le f(\xi(1))\,V_{i,i+1}(t)$. Since $\nu \ge 1$, we have $\mathcal{K}(t\,\mathbb{M}_{i,i+1}) \le f(\xi(1))\,V_{i,i+1}(t)$. Hence, if $f(\xi(1)) \le f_r^\Gamma(\xi(1))$ we have $(i,i+1,t) \notin \Upsilon_r$ and $j-i \ge 2$, and if $f(\xi(1)) > f_r^\Gamma(\xi(1))$, we have $\xi(1) > 0 = \xi(0)$. Hence, by (i) we have $\xi(j-i-1) < \xi(j-i)$ and since $f_r^\Gamma(\xi(j-i)) \ge f_r^\Gamma(\xi(1))$ and $(i,j,t) \in \Upsilon_r$, we have $V_{i,j}(t) \le \frac{\mathcal{K}(t\,\mathbb{M}_{i,j})}{f_r^\Gamma(\xi(1))}$. So by (i) we have $2a\,V_{i,j}(t)^\delta \le r$ and $\frac{\alpha\beta\rho}{\alpha+\beta} \le b$ and so by (*) and (h) we have

$$W_{i,j}(t) \le 2\,\vartheta(b\,G_{i,j}(\nu t)) \le 2a\,V_{i,j}(t)^{1+\delta} = 2a\,V_{i,j}(t)^\delta\,V_{i,j}(t) \le r\,V_{i,j}(t)\,.$$

Hence, (e) holds and so we have $\mathcal{K}(t\,\mathbb{M}_{i,j}) \le f_r^\Gamma(\xi(j-i))\,V_{i,j}(t)$ for all $(i,j,t) \in \mathbf{\Delta}_0 \times D$.

Suppose that (j) holds and let $(i,j,t) \in \Upsilon_r$ be given. As above, we see that we have either $j-i \ge 2$ or $\xi(1) > 0 = \xi(0)$. Hence, by (j) we have $\xi(\lfloor\frac{j-i}{2}\rfloor) < \xi(j-i)$. Since $(i,j,t) \in \Upsilon_r$ and $f_r^\Gamma(\xi(j-i)) \ge f_r^\Gamma(\xi(1))$, we have $V_{i,j}(t) \le \frac{\mathcal{K}(t\,\mathbb{M}_{i,j})}{f_r^\Gamma(\xi(1))}$. Hence, by (j) we have $qa\,V_{i,j}(t)^\delta \le r$ and $\eta\rho \le b$ and so by (**) and (h) we have

$$W_{i,j}(t) \le q\,\vartheta(b\,G_{i,j}(\nu t)) \le qa\,V_{i,j}(t)^\delta\,V_{i,j}(t) \le r\,V_{i,j}(t)\,.$$

Hence, (e) holds and so we have $\mathcal{K}(t\,\mathbb{M}_{i,j}) \le f_r^\Gamma(\xi(j-i))\,V_{i,j}(t)$ for all $(i,j,t) \in \mathbf{\Delta}_0 \times D$. □

Remark 2.2

(1): Let $\vartheta : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing function and let $b, \tau > 0$ be given. Then ϑ satisfies (h) with $(a,b,\delta,q) = (1,1,0,2)$. If $\vartheta(x) = x^\tau$, then ϑ satisfies (h) with $(a,b,\delta,q) = (b^\tau, b, 0, 2^{(1-\tau)^+})$. If $\vartheta(x) = e^{x^\tau}$ and $b \ge 1$, then ϑ satisfies (h) with $(a,b,\delta,q) = (1,b,b^\tau-1,2)$. If $\vartheta(x) = e^{-x^{-\tau}}$ and $0 < b \le 1$, then ϑ satisfies (h) with $(a,b,\delta,q) = (1,b,b^{-\tau}-1,2)$.

(2): Let $f : \mathbb{N}_0 \to \mathbb{R}_+$ be an increasing function and let $r \ge 0$ and $c \ge 1$ be given numbers such that $\Gamma(rc,1) \le c$. Then we have $f_r^\Gamma(n) \le c f(n)$ for all $n \in \mathbb{N}_0$. Hence, if (a)–(e) hold, we have $\mathcal{K}(t\,\mathbb{M}_{i,j}) \le c f(\xi(j-i))\,\vartheta(G_{i,j}(\nu t))$ for all $(i,j,t) \in \mathbf{\Delta}_0 \times D$. If $\gamma > 0$, we have $\Sigma_\gamma(rc,1) \le c$ if and only if $0 \le r < 1$ and $c \ge (1-r^{1/\gamma})^{-\gamma}$. If $0 < p < 1$, we have $\Theta_p(rc,1) \le c$ if and only if $c \ge r^{p/(1-p)}$. If $\chi : \mathbb{R}_+ \to \mathbb{R}_+$ is a function satisfying $\chi(s) \ge 1$ for all $s > 1$ and $p,q > 0$ are given numbers satisfying $\frac{1}{p} + \frac{1}{q} = 1$ and $r\,\chi(p) < 1$, then we have $\Pi_\chi(rc,1) \le c$ for all $c \ge \chi(q)/(1 - r\,\chi(p))$.

(3): Let me comment on the role of the function ξ. In most applications we use $\xi(n) \equiv n$ and if so we have $D_{i,j}^\xi = \{k \in \mathbb{N}_0 \mid i < k \le j\}$. In order to obtain the logarithmic constant in the classical Rademacher-Menchoff inequality (1.2), we use the function $\ell(n) := \lfloor\frac{\log^+ n}{\log 2}\rfloor$ for $n \in \mathbb{N}_0$; then $\ell : \mathbb{N}_0 \to \mathbb{N}_9$ is increasing with $\ell(0) = \ell(1) = 0$ and we have $\ell(\lfloor\frac{n}{2}\rfloor) < \ell(n)$ for all $n \ge 2$. So by (2.12), we have $\lceil\frac{i+j+1}{2}\rceil \in D^{\ell}_{i,j}$. for all $(i,j) \in \mathbf{\Delta}_2$. Let $\xi : \mathbb{N}_0 \to \mathbb{N}_0$ be an unbounded increasing function and let $\xi^{\sim}(n) := \inf\{k \in \mathbb{N}_0 \mid \xi(k+1) > n\}$ be its "inverse" for $n \in \mathbb{N}_0$. Then we have $\xi(\xi^{\sim}(n)) \le n \le \xi^{\sim}(\xi(n))$ for all $n \in \mathbb{N}_0$. Hence, if $h : \mathbb{N}_0 \to \mathbb{R}_+$ is an increasing function and $f(n) = h(\xi^{\sim}(n))$, then we have $h(n) \le f(\xi(n))$ for all $n \in \mathbb{N}_0$ and observe that $f = h \circ \xi^{\sim}$ is the smallest function with this property. Note that $\ell^{\sim}(n) = 2^{n+1} - 1$ and so we have $h(n) \le f(\ell(n))$ for all $n \in \mathbb{N}_0$ where $f(n) = h(2^{n+1} - 1)$.

Examples In the examples below, we let $(\mathbb{S}_{i,j}, \mathbb{M}_{i,j})_{(i,j)\in\mathbf{\Delta}_0}$ be a max scheme, we let $\mathcal{K} : L^0_+(P) \to [0,\infty]$ be a P-increasing functional and we let $\Gamma : \mathbb{R}^2_+ \to \mathbb{R}_+$ be an increasing homogeneous function.

Example 2.3 Suppose that $(\mathcal{K}, \Gamma)$ satisfies (a) with $\lambda = \mu = \nu = 1$. Let $\rho > 0$ and let $(g_{i,j})_{(i,j)\in\mathbf{\Delta}_0}$ be a triangular scheme of non-negative numbers satisfying $g_{i,i} = 0$ for all $i \in \mathbb{N}_0$ and (cf. (1.a)–(1.c) in [18]):

$$g_{i,j} \le g_{i,j+1} \;\forall\, (i,j) \in \mathbf{\Delta}_1 \text{ and } \max_{i<k<j} (g_{i,k} + g_{k,j}) \le \rho\, g_{i,j} \;\forall (i,j) \in \mathbf{\Delta}_2 \,.$$

Let $f : \mathbb{N}_0 \to \mathbb{R}_+$, $\xi : \mathbb{N}_0 \to \mathbb{N}_0$ and $\vartheta : \mathbb{R}_+ \to \mathbb{R}_+$ be increasing functions such that $\xi(0) = 0$ and let $r, q \ge 0$ be given such that $f(0) \le \Gamma(rf(0), f(1))$ and $\vartheta(x) + \vartheta(y) \le q\,\vartheta(x+y) \;\forall\, x, y \ge 0$. Set $D = \{1\}$ and $G_{i,j}(t) \equiv g_{i,j}$. Then (d), (f) and (g) holds with $\alpha = \beta = 1$. So by Theorem 2.1 we have:

(1.a) If $\vartheta(\rho x) \le \frac{r}{2}\,\vartheta(2x)$ for all $x \ge 0$ and $\mathcal{K}(\mathbb{S}_{i,j}) \le f(j-i)\,\vartheta(g_{i,j})$ for all $(i,j) \in \mathbf{\Delta}_0$, we have $\mathcal{K}(\mathbb{M}_{i,j}) \le f^{\Gamma}_r(j-i)\,\vartheta(g_{i,j})$ for all $(i,j) \in \mathbf{\Delta}_0$ (apply Theorem 2.1 with $\xi(n) \equiv n$).

(1.b): If $\xi(\lfloor\frac{n}{2}\rfloor) < \xi(n) \;\;\forall\, n \ge 2$, $\vartheta(\rho x) \le \frac{r}{q}\,\vartheta(x) \;\forall\, x \ge 0$ and $\mathcal{K}(\mathbb{S}_{i,j}) \le f(\xi(j-i))\,\vartheta(g_{i,j})$ for all $(i,j) \in \mathbf{\Delta}_0$, then we have $\mathcal{K}(\mathbb{M}_{i,j}) \le f^{\Gamma}_r(\xi(j-i))\,\vartheta(g_{i,j})$ for all $(i,j) \in \mathbf{\Delta}_0$.

Let $\gamma \ge 1$ and let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing function such that $\phi(\cdot)^{1/\gamma}$ is sub-additive. Then $(\mathcal{L}_\phi, \Sigma_\gamma)$ satisfies (a) with $\lambda = \mu = \nu = 1$. Hence, we see that (1.a) extends Theorem 3.1 in [18], and that (1.b) with $\xi(n) = \lfloor\frac{\log^+ n}{\log 2}\rfloor$, extends Theorem 3.3 and Corollary 3.1 in [18].

Let $\psi, \chi : \mathbb{R}_+ \to \mathbb{R}_+$ be given functions satisfying $\psi(st) \le \chi(s)\,\psi(t)$ for all $s > 1$ and all t. Then $(\mathcal{M}_\phi, \Pi_\chi)$ satisfies (a) with $\lambda = \mu = \nu = 1$. Hence, we see that (1.a) extends Theorem 3.2 in [18] and gives a general solution to Problem 2 in [18].

Suppose that $\Gamma = \Sigma_\gamma$ for some $\gamma > 0$ and let $(\tau_k)_{k\ge 0}$ be a sequence of non-negative numbers. Applying (1.b) with $\xi(n) = \lfloor\frac{\log^+ n}{\log 2}\rfloor$, $g_{i,j} = \sum_{i<k\le j} \tau_k$, $\rho = 1$ and $f(n) \equiv 1$, we obtain following extension of the classical Rademacher-Menchoff inequality:

(1.c): If $\mathcal{K}(\mathbb{S}_{i,j}) \le \sum_{i<k\le j} \tau_k$ for all $(i,j) \in \mathbf{\Delta}_0$, then we have $\mathcal{K}(\mathbb{M}_{i,j}) \le \Big(1 + \lfloor\frac{\log(j-i)}{\log 2}\rfloor\Big)^{\gamma} \sum_{i<k\le j} \tau_k$ for all $(i,j) \in \mathbf{\Delta}_1$ [see (1.2)].

Example 2.4 Suppose that $(\mathcal{K}, \Gamma)$ satisfies (a) for some $\lambda, \mu, \nu \geq 1$ and let $D \subseteq \mathbb{R}_+$ be a non-empty set such that $\lambda t, \mu t \in D$ for all $t \in D$. Let $G_{i,j} : \mathbb{R}_+ \to \mathbb{R}_+$ be given functions such that $G_{i,i}(t) = 0 \ \forall\, (i,t) \in \mathbb{N}_0 \times \mathbb{R}_+$ and suppose that $(G_{i,j})$ satisfies (d), (f) and (g) for some $\rho, \alpha, \beta \geq 0$. Let $f : \mathbb{N}_0 \to \mathbb{R}_+$ and $\vartheta : \mathbb{R}_+ \to \mathbb{R}_+$ be increasing functions and let $r \geq 0$ be given such that $f(0) \leq \Gamma(rf(0), f(1))$. By Theorem 2.1 with $\xi(n) \equiv n$, we have

(2.a): If $\vartheta(\frac{\alpha\beta\rho}{\alpha+\beta}\, x) \leq \frac{r}{2}\, \vartheta(x)$ for all $x \geq 0$ and $\mathcal{K}(t\, \mathbb{S}_{i,j}) \leq f(j-i)\, \vartheta(G_{i,j}(t))$ for all $(i,j,t) \in \mathbf{\Delta}_0 \times (\nu D)$, then we have $\mathcal{K}(t\, \mathbb{M}_{i,j}) \leq f_r^{\Gamma}(j-i)\, \vartheta(G_{i,j}(\nu t))$ for all $(i,j,t) \in \mathbf{\Delta}_0 \times D$.

Let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be log-convex function and let $\mu, \nu > 0$ be given such that $\frac{1}{\mu} + \frac{1}{\nu} = 1$. Then $(\mathcal{L}_\phi, \Theta_{1/\mu})$ satisfies (a) with $(\lambda, \mu, \nu) = (\mu, \mu, \nu)$. Set $r = 2$ and $c = 2^{1/(1-\mu)}$. Since $\lambda = \mu$, we may take $\alpha = \beta$ and since $\Theta_{1/\mu}(2, 1) \geq 1$ and $\Theta_{1/\mu}(2c, 1) = c$, we have the following extension of Theorem 2.1 in [18]:

(2.b): If $\alpha\rho \leq 2$ and $E\phi(t\, \mathbb{S}_{i,j}) \leq f(j-i)\, \vartheta(G_{i,j}(t))$ for all $(i,j,t) \in \mathbf{\Delta}_0 \times (\nu D)$, then we have $E\phi(t\, \mathbb{M}_{i,j}) \leq 2^{1/(1-\mu)} f(j-i)\, \vartheta(G_{i,j}(\nu t))$ for all $(i,j,t) \in \mathbf{\Delta}_0 \times D$.

Example 2.5 Suppose that $\mathcal{K}$ is weakly subadditive and $\mathcal{K}(X) \leq 1$ for all $X \in L^0_+(P)$. Let $\mu, \nu > 0$ be given such that $\frac{1}{\mu} + \frac{1}{\nu} = 1$ and let $D \subseteq \mathbb{R}_+$ be a non-empty set such that $\mu t \in D$ for all $t \in D$. Let $\beta, \rho \geq 0$ and $G_{i,j} : \mathbb{R}_+ \to \mathbb{R}_+$ be given such that $G_{i,i}(t) = 0 \ \forall\, (i,t) \in \mathbb{N}_0 \times (\nu D)$ and $G_{i,j}(\mu t) \leq \beta\, G_{i,j}(t) \ \forall\, (i,t) \in \mathbf{\Delta}_0 \times (\nu D)$ and $(G_{i,j})$ satisfies (d) and (f). Then $(\mathcal{K}, \Sigma_1)$ satisfies (a) with $(\lambda, \mu\ \nu) = (1, \mu, \nu)$ and since $\lambda = 1$, we see that $(G_{i,j})$ satisfies (g) with $(\alpha, \beta) = (1, \beta)$. Let $f : \mathbb{N}_0 \to \mathbb{R}_+$ and $\vartheta : \mathbb{R}_+ \to \mathbb{R}_+$ be increasing functions. Let $u, \delta > 0$ be given numbers satisfying $u\,(\frac{f(0)}{2} + f(1)) \geq 4^{1/\delta}$. Applying (2.a) with $(r, c) = (\frac{1}{2}, 2)$ and f replaced by uf, we have

(3.a): If $\vartheta(\frac{\beta\rho}{1+\beta}\, x) \leq \vartheta(x)^{1+\delta} \ \forall\, x \geq 0$ and $\mathcal{K}(t\, \mathbb{S}_{i,j}) \leq uf(j-i)\, \vartheta(G_{i,j}(t))$ for all $(i,j,t) \in \mathbf{\Delta}_0 \times D$, then we have $\mathcal{K}(t\, \mathbb{M}_{i,j}) \leq 2u\, \vartheta(G_{i,j}(\nu t))$ for all $(i,j,t) \in \mathbf{\Delta}_0 \times D$.

Note that $\mathcal{T}(X) = P(X > 1)$ is a weakly sub-additive functional such that $\mathcal{T}(X) \leq 1$ and $\mathcal{T}(tX) = P(X > \frac{1}{t})$ for all $X \in L^0_+(P)$ and all $t > 0$. Applying (3.a) on this functional and with $\vartheta(x) = e^{-1/x}$, we obtain Theorem 2.2 in [18].

In the last two results of this section I shall treat the case where $\mathcal{K}$ is a weakly sub-additive functional satisfying $\mathcal{K}(0) = 0$ and $\mathcal{K}(X) \leq 1$ for all $X \in L^0_+(P)$; for instance, if $\mathcal{K} = \mathcal{L}_\phi$ where $\phi : \mathbb{R}_+ \to [0, 1]$ is increasing with $\phi(0) = 0$.

Theorem 2.6 *Let* $(\mathbb{S}_{i,j}, \mathbb{M}_{i,j})$ *be a max-scheme, let* $\mathcal{K} : L^0_+(P) \to [0, 1]$ *be a weakly sub-additive functional such that* $\mathcal{K}(0) = 0$ *and let* $f : \mathbb{N}_0 \to \mathbb{R}_+$ *be an increasing function such that* $f(2) > 0$*. Let* $0 \leq \beta < 1$ *and* $p, q > 0$ *be given numbers such that* $p + q = 1$*, let* $D \subseteq (0, \infty)$ *be a non-empty interval with left endpoint* 0 *and let* $G_0, G_1, \ldots : \mathbb{R}_+ \to \mathbb{R}_+$ *be given functions satisfying:*

(a) $\mathcal{K}\Big(\frac{1}{t}\, \mathbb{S}_{i,j}\Big) \leq f(j-i)\, e^{-G_{j-i}(t)} \quad \forall\, (i,j,t) \in \mathbf{\Delta}_1 \times D\,,$

(b) $G_n(t) \ge G_{n+1}(t)$ *and* $G_n(qt) \ge \beta\, G_n(t) \quad \forall (n,t) \in \mathbb{N}_0 \times D$.

Let us define

$$F(n) = \sum_{k=1}^{n} f(k)\ ,\ Q_n(t) = \max_{1\le m\le n} (G_{m-1}(t) \wedge G_{n-m}(pt))\ ,$$

$$C_n(t) = \{u \in D \mid Q_n(u) < (1+t)\, G_n(u)\}\ ,\ L_n(t) = \inf_{u\in C_{i,j}(t)} G_n(u)\ .$$

for all $(n,t) \in \mathbb{N} \times \mathbb{R}_+$. *If the following condition hold:*

(c) $\exists\, \delta > 0$ *so that* $\liminf_{n\to\infty} \frac{L_n(\delta)}{\log F(n)} > \frac{1}{1-\beta}$,

then there exists $a > 0$ *such that* $\mathcal{K}(\frac{1}{t}\, \mathbb{M}_{i,j}) \le a f(j-i)\, e^{-\beta\, G_{i,j}(t)}$ *for all* $(i,j,t) \in \mathbf{\Delta}_0 \times D$. *More precisely, if* $\delta > 0$ *and* $c \ge 0$ *are given numbers satisfying*

(d) $e^c \ge f(2)^{-1}\, 4^{1/\delta}$ *and* $\log F(n) \le c + \log f(n) + (1-\beta)\, L_n(\delta) \quad \forall\, n \ge 2$,

then we have $\mathcal{K}(\frac{1}{t}\, \mathbb{M}_{i,j}) \le 2e^c f(j-i)\, e^{-\beta\, G_{i,j}(t)}$ *for all* $(i,j,t) \in \mathbf{\Delta}_0 \times D$.

Proof Set $h(0) = 0$ and $h(n) = e^c f(n)$ for $n \ge 1$ and let us define $V_{i,i}(t) = 0$ for $(i,t) \in \mathbb{N}_0 \times D$ and $V_{i,j}(t) = e^{-\beta\, G_{j-i}(t)}$ for $(i,j,t) \in \mathbf{\Delta}_1 \times \mathbb{R}_+$. We shall apply Proposition A.2 with $D, h, V_{i,j}$ as above, $(p,q) = (p,1)$ and

$$A_{i,j}(t) = \mathcal{K}\left(\tfrac{1}{t}\, \mathbb{M}_{i,j}\right)\ ,\ B_{i,j}(t) = \mathcal{K}\left(\tfrac{q}{t}\, \mathbb{S}_{i,j}\right)\ ,\ \Gamma = \Sigma_1\ ,\ \xi(n) \equiv n\ , r = \tfrac{1}{2}\ .$$

Let $(i,k,j) \in \nabla$ and $t > 0$ be given. Since $\mathcal{K}$ is weakly sub-additive and $p+q=1$, we have

$$\mathcal{K}\left(\tfrac{1}{t}\, \mathbb{M}_{i,j}\right) \le \mathcal{K}\left(\tfrac{1}{t}\, \mathbb{M}_{i,k-1}\right) + \mathcal{K}\left(\tfrac{1}{pt}\, \mathbb{M}_{k,j}\right) + \mathcal{K}\left(\tfrac{1}{qt}\, \mathbb{S}_{i,k}\right)\ .$$

Hence we see that condition (a) in Proposition A.2 holds. Since $\mathbb{M}_{i,i} = 0$ a.s. and $\mathcal{K}(0) = 0$, we have $A_{i,i}(t) = 0 = V_{i,i}(t)$ and since $G_n \ge G_{n+1}$, we see that the condition (b) in Proposition A.2 holds. Let $(i,j,t) \in \mathbf{\Delta}_0 \times D$ be given, Since $q \le 1$ and D is an interval with left endpoint 0, we have $qt \in D$. So by (a) and (b) we have $G_{j-i}(qt) \ge \beta\, G_{j-i}(t)$ and $B_{i,j}(t) \le h(j-i)\, V_{i,j}(t)$. Hence, we see that (a)–(c) in Proposition A.2 hold and since $r = \frac{1}{2}$, we have that $h_r^{\Sigma_1}$ is increasing and $h(n) \le h_r^{\Sigma_1}(n) \le 2\, h(n)$ for all $n \in \mathbb{N}_0$. Since $e^c \ge 1$, we have $f(n) \le h(n) \le h_r^{\Sigma_1}(n)$ for all $n \in \mathbb{N}$.

Suppose that (d) holds for some $\delta > 0$ and some $c \ge 0$. Let $(i,j,t) \in \Upsilon_r$ be given, where Υ_r is defined as in Proposition A.2. Since $\mathbb{M}_{i,i+1} \le \mathbb{S}_{i,i+1}$ a.s. and $\beta < 1 \le e^c$, we have by (a)

$$A_{i,i+1}(t) \le \mathcal{K}(\tfrac{1}{t}\, \mathbb{S}_{i,i+1}) \le f(1)\, e^{-G_{i,i+1}(t)} \le h_r^{\Sigma_1}(1)\, V_{i,i+1}(t)$$

and since $(i,j,t) \in \Upsilon_r$, we have $n := j - i \geq 2$. So by (2.4) and weak sub-additivity, we have

$$e^c f(n)\, e^{-\beta\, G_n(t)} \leq h_r^{\Sigma_1}(n)\, V_{i,j}(t) < A_{i,j}(t) \leq F(n)\, e^{-G_n(t)}\,.$$

Taking logarithms and using (c), we have

$$(1-\beta)\, G_n(t) < \log F(n) - \log f(n) - c \leq (1-\beta)\, L_n(\delta)$$

and since $\beta < 1$, we have $G_n(t) < L_n(\rho)$. So we must have $Q_n(t) \geq (1+\delta)\, G_n(t)$. Hence, there exists an integer $1 \leq m \leq n$ such that $G_{m-1}(t) \wedge G_{n-m}(pt) \geq (1+\delta)\, G_n(t)$. Set $k = i + m$. Then we have $i < k \leq j$ and

$$\begin{aligned} V_{i,k-1}(t) + V_{k,j}(pt) &\leq 2\,(V_{i,k-1}(t) \vee V_{k,j}(pt)) = 2\, e^{-\beta\,(G_{m-1}(t) \wedge G_{n-m}(pt))} \\ &\leq 2\, e^{-\beta(1+\delta)\, G_n(t)} = 2\, V_{i,j}(t)^{1+\delta}\,. \end{aligned}$$

Since $\mathcal{K}(X) \leq 1$ and $(i,j,t) \in \Upsilon_r$, we have $A_{i,j}(t) \leq 1$ and $h_r^{\Sigma_1}(n)\, V_{i,j}(t) \leq 1$. Since $e^c f(2) \leq h_r^{\Sigma_1}(n)$ and $\delta > 0$, we have $2\, V_{i,j}(t)^{\delta} \leq 2\, e^{-c\delta} f(2)^{-\delta}$ and since $e^c \geq f(2)^{-1}\, 4^{1/\delta|}$, we have

$$V_{i,k-1}(t) + V_{k,j}(pt) \leq 2\, V_{i,j}(t)^{\delta}\, V_{i,j}(t) \leq \tfrac{1}{2}\, V_{i,j}(t) = r\, V_{i,j}(t)\,.$$

Since ξ is strictly increasing and $i < k \leq j$, we have $k \in D_{i,j}^{\xi}$ and so by Proposition A.2 we have $\mathcal{K}(\frac{1}{t}\, \mathbb{M}_{i,j}) \leq h_r^{\Sigma_1}(j-i)\, V_{i,j}(t)$ for all $(i,j,t) \in \mathbf{\Delta}_0 \times D$. Since $h_r^{\Sigma_1}(n) \leq 2\, h(n) \leq 2e^c f(n)$ and $V_{i,j}(t) = e^{-\beta\, G_{j-i}(t)}$, we see that $\mathcal{K}(\frac{1}{t}\, \mathbb{M}_{i,j}) \leq 2e^c f(j-i)\, e^{-\beta\, G_{j-i}(t)}$ for all $(i,j,t) \in \mathbf{\Delta}_0 \times D$.

Suppose that (c) holds and let $\delta > 0$ be chosen according to (c). Since $F(n) \to \infty$. there exists an integer $n_0 \geq 1$ such that $F(n_0) > e$ and $\log F(n) \leq (1-\beta)\, L_n(\delta)$ for all $n \geq n_0$. But then there exists $c > 0$ such that $e^c \geq f(2)^{-1}\, 4^{1/\delta}$ and $c \geq -\log f(2)$. Since F and f are increasing, we see that (c) implies (d) with this choice of (δ, c). □

Corollary 2.7 (cf. [12] and [13]) *Let* $(\mathbb{S}_{i,j}, \mathbb{M}_{i,j})$ *be a max-scheme, let* $\mathcal{K} : L_+^0(P) \to [0,1]$ *be a weakly sub-additive functional such that* $\mathcal{K}(0) = 0$ *and let* $f : \mathbb{N}_0 \to \mathbb{R}_+$ *be an increasing function such that* $f(2) > 0$. *Let* $0 \leq \alpha, \beta < 1$ *and* $p, q > 0$ *be given numbers such that* $p + q = 1$ *and let* $D \subseteq (0, \infty)$ *be a non-empty interval with left endpoint* 0. *Let* $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ *be a given function and let* $g_1, g_2, \ldots : \mathbb{R}_+ \to \mathbb{R}_+$ *be increasing functions satisfying*

(a) $\mathcal{K}\left(\frac{1}{t}\, \mathbb{S}_{i,j}\right) \leq f(j-i)\, \exp\left(-\frac{\phi(t)}{j-i+g_{j-i}(t)}\right) \quad \forall (i,j,t) \in \mathbf{\Delta}_1 \times D\,,$

(b) $\phi(pt) \geq \alpha\, \phi(1)\,, \phi(qt) \geq \beta\, \phi(t)$ *and* $g_n(t) \leq g_{n+1}(t) \quad \forall\, (n,t) \in \mathbb{N} \times D\,.$

Set $F(n) = f(1) + \cdots + f(n)$ *for* $n \in \mathbb{N}$ *and let us define*

$$C_n^*(\theta) = \{u \in D \mid \phi(u) > 0\,,\ g_n(u) > n\theta\} \ \text{ and } \ L_n^*(\theta) = \inf_{u \in C_n^*(\theta)} \tfrac{\phi(u)}{n+g_n(u)}$$

for all $(\theta, n) \in \mathbb{R}_+ \times \mathbb{N}$. *If the following condition hold:*

(c) $\exists\, 0 \le \theta < \frac{\alpha}{1-\alpha}$ *so that* $\liminf_{n\to\infty} \frac{L_n^*(\theta)}{\log F(n)} > \frac{1}{1-\beta}$,

then there exists $a > 0$ *so that*

$$\mathcal{K}\left(\tfrac{1}{t}\,\mathbb{M}_{i,j}\right) \le a f(j-i)\, \exp\left(-\tfrac{\beta\,\phi(t)}{j-i+g_{j-i}(t)}\right) \quad \forall\,(i,j,t) \in \mathbf{\Delta}_1 \times D\,.$$

More precisely, if $0 \le \theta < \frac{\alpha}{1-\alpha}$ *and* $c \ge 0$ *are given numbers satisfying* $e^c \ge f(2)^{-1}\, 4^{(2\theta+1)/(\alpha-\theta\,(1-\alpha))}$ *and*

(d) $\log F(n) \le c + \log f(n) + (1-\beta)\, L_n^*(\theta) \quad \forall\, n \ge 2$,

then we have $\mathcal{K}(\frac{1}{t}\,\mathbb{M}_{i,j}) \le 2e^c f(j-i)\, \exp\left(-\frac{\beta\,\phi(t)}{j-i+g_{j-i}(t)}\right)$ *for all* $(i,j,t) \in \mathbf{\Delta}_1 \times D$.

Proof Suppose that (d) holds and let us define $G_0(t) = \phi(t)$ and $G_n(t) = \frac{\phi(t)}{n+g_n(t)}$ for $(n,t) \in \mathbb{N} \times \mathbb{R}_+$. Since g_n is increasing, we see that the conditions (a) and (b) in Theorem 2.6 hold and we shall adopt the notation of Theorem 2.6 with this choice of (G_n). Let $0 \le \theta < \frac{\alpha}{1-\alpha}$ and $n \ge 2$ be a given and set $\delta = \frac{\alpha-\theta(1-\alpha)}{2\theta+1}$, $\gamma = \frac{1+\theta(1-\alpha)}{1+\alpha}$ and $m = \lceil n\gamma \rceil$. Then we have $0 < \delta < \alpha$ and $\frac{1}{2} \le \frac{1}{1+\alpha} < \gamma < 1$ and since $n \ge 2$ and $n - m \ge n\,(1-\gamma) > 0$, we have $2 \le m \le n-1$. Let $u \in C_n(\delta)$ be given. Since $\phi(pu) \ge \alpha\,\phi(u)$, we have $G_{m-1}(u) \wedge (\alpha\, G_{n-m}(u)) < (1+\delta)\, G_n(u)$ and since $1 < n\gamma \le m < 1 + n\gamma$, we have

$$\tfrac{n-(1+\delta)(m-1)}{\delta} \ge n\, \tfrac{1-(1+\delta)\gamma}{\delta} = n\theta \ \text{ and } \ \tfrac{n\alpha-(1+\delta)(n-m)}{1+\delta-\alpha} \ge n\, \tfrac{\alpha-(1+\delta)\,(1-\gamma)}{1+\delta-\alpha} = n\theta\,.$$

Let $1 \le j \le n$ be a given integer and let $x \in \mathbb{R}_+$ be a given number. Then an easy computation shows that we have

$$G_j(u) < x\, G_n(u) \ \Rightarrow\ \phi(u) > 0\,,\ x > 1 \ \text{ and } \ g_n(u) > \tfrac{n-xj}{x-1}\,.$$

Applying this with $(j,x) = (m-1, 1+\delta)$ and $(j,x) = (n-m, \frac{1+\delta}{\alpha})$, we have

$$g_n(u) > \tfrac{n-(1+\delta)(m-1)}{\delta} \wedge \tfrac{n\alpha-(1+\delta)(n-m)}{1+\delta-\alpha} \ge n\theta\,.$$

Hence, we have $C_n(\delta) \subseteq C_n^*(\delta)$ and $L_n^*(\theta) \le L_n(\delta)$ and so we see that the corollary follows from Theorem 2.6. □

Remark 2.8 Recall that $\mathcal{T}(X) := P(X > 1)$ is a weakly subadditive functional with $\mathcal{T}(\frac{1}{t}\,X) = P(X > t)$; see Example 2.3 in Remark 2.2. Kevei and Mason (see Theorem 2.1 in [13]) have proved a result similar to Corollary 2.7 in case that $\mathcal{K} =$

$\mathcal{T}$, $\phi(t) = t^2$ and $f(n) \equiv A$ but with condition (c) replaced with the following condition $\lim_{n\to\infty} L_n^\circ(\theta) = \infty \ \forall\, 0 < \theta < 1$, where $L_n^\circ(\theta) := \inf_{u\in C_n^*(\theta)} \frac{\phi(u)}{g_n(u)\,\log u}$.

To compare Theorem 2.1 in [13] with Corollary 2.4, let us consider the setting of Corollary 2.7 and let $\Lambda : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing function such that $\Lambda(0) = 1$ and $\limsup_{n\to\infty} \frac{\log F(n)}{\Lambda(n)} \le 1$. Let us define

$$L_n^\circ(\theta) = \inf_{u\in C_n^*(\theta)} \frac{\phi(u)}{g_n(u)\,\Lambda(\phi(u))} \quad \forall n \in \mathbb{N}\ \forall\, \theta > 0\,.$$

Let $0 < \theta < \frac{\alpha}{1-\alpha}$ and $n \in \mathbb{N}$ be given such that $\theta\, L_n^\circ(\theta) \ge 1$ and let $u \in C_n^*(\theta)$ be given. Since $n\theta < g_n(u)$ and $L_n^\circ(\theta)\, g_n(u)\Lambda(\phi(u)) \le \phi(u)$, we have $n\theta L_n^\circ(\theta)\,\Lambda(\phi(u)) \le \phi(u)$ and since Λ is increasing and ≥ 1, we have $n \le \phi(u)$. Hence, we have $\Lambda(n) \le \Lambda(\phi(u))$ and

$$\frac{\phi(u)}{n+g_n(u)} \ge \frac{\theta}{1+\theta}\,\frac{\phi(u)}{g_n(u)\,\Lambda(\phi(u))}\,\Lambda(\phi(u)) \ge \frac{\theta}{1+\theta}\,L_n^\circ(\theta)\,\Lambda(n) \quad \forall\, u \in C_n^*(\theta)\,.$$

Taking infimum over u, we see that $L_n^\circ(\theta) \le \frac{1+\theta}{\theta}\,\frac{L_n^*(\theta)}{\Lambda(n)}$ for $n \in \mathbb{N}$ and all $0 < \theta < \frac{\alpha}{1-\alpha}$ satisfying $L_n^\circ(\theta) \ge \frac{1}{\theta}$. Hence, if $\liminf_{n\to\infty} L_n^\circ(\theta) > \frac{1+\theta}{\theta\,(1-\beta)}$, then condition (c) holds. In particular Theorem 2.1 in [13] follows from Corollary 2.7.

3 Ottaviani-Lévy Type Inequalities

In this section I shall prove a maximal inequality of the Ottaviani-Lévy type for random vectors with values in a measurable linear space. Recall that $(V, \mathcal{B})$ is a *measurable linear space* if V is a real vector space and $\mathcal{B}$ is a σ-algebra on V such that $(x, y) \curvearrowright x + y$ is measurable from $(V \times V, \mathcal{B} \otimes \mathcal{B})$ to $(V, \mathcal{B})$ and $(s, x) \curvearrowright sx$ is measurable from $(\mathbb{R} \times V, \mathcal{B}(\mathbb{R}) \otimes \mathcal{B})$ to $(V, \mathcal{B})$. If V is a real vector space, we let V^* denote the *algebraic dual* of V; that is the set of all linear functionals from V into $\mathbb{R}$ and if $\Xi \subseteq V^*$ is a non-empty set we let $\mathcal{B}^\Xi(V)$ denote the smallest σ-algebra on V making ξ measurable for all $\xi \in \Xi$. Then $(V, \mathcal{B}^\Xi(V))$ is a measurable linear space and $X : (\Omega, \mathcal{F}) \to (V, \mathcal{B}^\Xi(V))$ is measurable if and only if $\xi(X)$ is a real random variable for all $\xi \in \Xi$. If span(Ξ) denotes the linear span of Ξ, then $X_1, \ldots, X_n : \Omega \to (V, \mathcal{B}^\Xi(V))$ are independent if and only if $\eta(X_1), \ldots, \eta(X_n)$ are independent real random variables for all $\eta \in$ span(Ξ). Let $(V, \|\cdot\|)$ be a Banach space and let $\mathcal{B}(V)$ denote the Borel σ-algebra on V; that is the smallest σ-algebra on V containing all open sets. If $(V, \|\cdot\|)$ is separable, then $(V, \mathcal{B}(V))$ is a measurable linear space but if $(V, \|\cdot\|)$ is non-separable, then $(V, \mathcal{B}(V))$ need not be a measurable linear space; for instance, if ℓ^∞ is the set of all bounded sequences with the sup-norm $\|x\|_\infty = \sup_{n\in\mathbb{N}} |x_n|$, then $(\ell^\infty, \mathcal{B}(\ell^\infty))$ is a measurable linear space if and only if the continuum hypothesis holds.

Let T be a non-empty set and let $V \subseteq \mathbb{R}^T$ be a linear space of real valued functions on T. Let $\pi_t(x) = x(t)$ for $x \in V$ and $t \in T$ denote the evaluation map. Then $\pi_t \in V^*$ for all $t \in T$ and we let $\mathcal{B}^T(V)$ denote the *cylinder σ-algebra*; that is the smallest σ-algebra on V making π_t measurable for all $t \in T$. Note that $\mathcal{B}^T(V) = \mathcal{B}^\Xi(V)$ where $\Xi = \{\pi_t \mid t \in T\}$. Hence, we see that $(V, \mathcal{B}^T(V))$ is a measurable linear space and recall that $X = (X(t) \mid t \in T)$ is a stochastic process with sample paths in V if and only if $X : (\Omega, \mathcal{F}) \to (V, \mathcal{B}^T(V))$ is a measurable function.

If V is a real vector space and $\Xi \subseteq V^*$ is a non-empty set, we let $Q^\Xi(x) := \sup_{\xi \in \Xi} \xi(x)$ for $x \in V$ denote the *support function* of Ξ. Then Q^Ξ is sub-additive and homogeneous with values in $(-\infty, \infty]$ and $Q^\Xi(0) = 0$. Note that Q^Ξ is a semi-norm if and only if $Q^\Xi(x) = Q^\Xi(-x)$ for all $x \in V$. Hence, if $\tilde{\Xi} = \Xi \cup (-\Xi)$, then $Q^{\tilde{\Xi}}$ is a semi-norm and we have $Q^{\tilde{\Xi}}(x) = \sup_{\xi \in \Xi} |\xi(x)|$ for all $x \in V$.

If $G : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing, right continuous function, we let λ_G denote the Lebesgue-Stieltjes measure on $\mathbb{R}_+$ induced by G; that is the unique Borel measure on $\mathbb{R}_+$ satisfying $\lambda_G([0, x]) = G(x)$ for all $x \in \mathbb{R}_+$.

Lemma 3.1 *Let S be a real random variable and let L, M and V be non-negative random variables. Let $\alpha, \beta : \mathbb{R}_+ \to \mathbb{R}_+$ be Borel functions satisfying*

(a) $\alpha(s)\, P(M \geq s) \leq E^*(1_{\{L \geq s\}}\, S) + \beta(s)\, P(V \geq s) \quad \forall\, s \in \mathbb{R}_+.$

Let $G : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing, right continuous function such that $A(x) := \int_{[0,x]} \alpha(s)\, \lambda_G(ds) < \infty$ and $B(x) := \int_{[0,x]} \beta(s)\, \lambda_G(ds) < \infty$ for all $x \geq 0$, Then we have

(b) $EA(M) \leq E^*(S\, G(L)) + EB(V).$

Let $a, b, p, q, u \geq 0$ and $c \geq 1$ be given numbers satisfying

(c) $G(u + cx) \leq a + p\, A(x)\ ,\ B(x) \leq b + q\, G(x) \quad \forall\, x \geq 0.$

If $V \leq u + c\, M$ a.s. and $S \geq 0$ a.s., then we have

(d) $(1 - pq)^+\, EG(V) \leq a + bp + p\, E(S\, G(L)).$

Proof (b): If $E^*(S\, G(L)) = \infty$, then (b) holds trivially. So let us suppose that $E^*(S\, G(L)) < \infty$ and set $H(s, \omega) = 1_{\{L \geq s\}}(\omega)\, S(\omega)$ and $\psi(s) = E^*(S\, 1_{\{L > s\}})$ for all $s \in \mathbb{R}_+$ and all $\omega \in \Omega$. By Tonelli's theorem we have

$$E(S^+\, G(L)) = \int_{\mathbb{R}_+ \times \Omega} H^+(s, \omega)\, (\lambda_G \otimes P)(ds, d\omega) < \infty$$

and so we have $\int_\Omega H^+(s, \omega)\, P(d\omega) < \infty$ for λ_G-a.a. $s \in \mathbb{R}_+$. By (a), we have $\psi(s) \geq -\beta(s)$ for all $s \in \mathbb{R}_+$. Hence, we have $H(s, \cdot) \in L^1(P)$ and $EH(s, \cdot) = \psi(s)$ for λ_G-a.a. $s \in \mathbb{R}_+$. So by the Fubini-Tonelli theorem we have

$$E(S\, G(L)) = \int_{\mathbb{R}_+ \times \Omega} H(s, \omega)\, (\lambda_G \otimes P)(ds, d\omega) = \int_{\mathbb{R}+} \psi(s)\, \lambda_G(ds)$$

and by Tonelli's theorem we have $EB(V) = \int_{\mathbb{R}_+} \beta(s)\, P(V \geq s)\, \lambda_G(ds)$ and

$$EA(M) = \int_{\Omega} P(d\omega) \int_{\mathbb{R}_+} \alpha(s) 1_{\{M \geq s\}}(\omega)\, \lambda_G(ds)$$
$$= \int_{\mathbb{R}_+} \alpha(s)\, P(M \geq s)\, \lambda_G(ds)\,.$$

By (a), we have $\alpha(s)\, P(M \geq s) \leq \psi(s) + \beta(s)\, P(V \geq s)$ for all $s \in \mathbb{R}_+$ and since $G(x) = \lambda_G([0,x])$ for all $x \in \mathbb{R}_+$, we have

$$EA(M) \leq \int_{\mathbb{R}_+} \psi(s)\, \lambda_G(ds) + \int_{\mathbb{R}_+} \beta(s)\, P(V \geq s)\, \lambda_G(ds)$$
$$= E(S\, G(L)) + EB(V)$$

which proves (b).

(d): Suppose that (c) holds and that we have $V \leq u + cM$ a.s. and $S \geq 0$ a.s. Set $M_\rho = M \wedge \rho$ and $V_\rho = V \wedge \rho$ for all $\rho > 0$. Since $S \geq 0$ a.s. we have $\mu := E(S\, G(L)) \geq 0$ and that (a) holds with (M, V) replaced by (M_ρ, V_ρ). So by (b), we have $EA(M_\rho) \leq \mu + EB(V_\rho)$ and since $c \geq 1$ and $V \leq u + c\, M$ a.s., we have $V_\rho \leq u + c\, M_\rho$ a.s. So by (c) we have

$$EG(V_\rho) \leq a + p\, EA(M_\rho) \leq a + p\mu + p\, EB(V_\rho)$$
$$\leq a + p\mu + pb + pqEG(V_\rho)\,.$$

Since $0 \leq G(V_\rho) \leq G(\rho) < \infty$ and $0 \leq a + p\mu + pb$, we have $(1-pq)^+\, EG(M_\rho) \leq a + p\mu + pb$ for all $\rho > 0$ and since $G(M_\rho) \uparrow G(M)$, we see that (d) follows from the monotone convergence theorem. □

Theorem 3.2 *Let* $Q_0, \ldots, Q_n : \Omega \to \overline{\mathbb{R}}$ *and* $R_0, \ldots, R_n : \Omega \to \overline{\mathbb{R}}$ *be (extended) random variables and let us define* $M_0 = Q_0$, $M_i = \max(Q_1, \ldots, Q_i)$ *for* $i = 1, \ldots, n$ *and*

$$\tau_t = \inf\{1 \leq i \leq n \mid Q_i > t\} \quad \forall\, t \in \mathbb{R}\,.$$

Let $\gamma : \mathbb{R}^2 \to [0,1]$ *be a given function and let* $r, s, t \in \mathbb{R}$ *be given numbers satisfying*

(a) $Q_n \leq M_{i-1} \vee (M_{i-1} + R_0 + R_i)$ *a.s.* $\quad \forall\, 1 \leq i \leq n\,.$
(b) $P(R_i > s\,, \tau_t = i) \leq \gamma(s,t)\, P(\tau_t = i) \quad \forall\, 1 \leq i \leq n\,.$
(c) $r + s \geq 0$ *and* $Q_0 \wedge Q_1 \leq t$ *a.s.*

Then we have

(d) $P(Q_n > r + s + t) \leq P(R_0 > r\,,\, M_n > t) + \gamma(s,t)\, P(M_n > t)\,.$

Set $\theta(s,t) = \sup_{u \geq t} \gamma(s,u)$ *and* $\vartheta_k(s,t) = \frac{1-\theta(s,t)^k}{1-\theta(s,t)}$ *for all* $k \in \mathbb{N}_0$ *with the convention that* $\vartheta_k(s,t) = k$ *if* $\theta(s,t) = 1$. *If* $Q_n = M_n$ *a.s., then we have*

(e) $P(M_n > t + k(r+s)) \leq \vartheta_k(s,t)\, P(R_0 > r) + \theta(s,t)^k\, P(M_n > t) \quad \forall\, k \in \mathbb{N}_0$.

Proof Let $N \in \mathcal{F}$ be a P-null set such that $Q_0(\omega) \wedge Q_1(\omega) \leq t$ for all $\omega \in \Omega \setminus N$ and

$$Q_n(\omega) \leq M_{i-1}(\omega) \vee (M_{i-1}(\omega) + R_0(\omega) + R_i(\omega)) \;\forall\, \omega \in \Omega \setminus N \;\forall\, 1 \leq i \leq n\,.$$

Let $1 < i \leq n$ be a given integer and let $\omega \in \{\tau_t = i\,,\ Q_n > r + s + t\} \setminus N$ be a given element. Then we have $M_{i-1}(\omega) \leq t < Q_i(\omega)$ and since $r + s \geq 0$, we have $t \leq r + s + t < Q_n(\omega)$. So we have $M_{i-1}(\omega) \leq t < Q_n(\omega)$ and

$$r + s + t < Q_n(\omega) \leq M_{i-1}(\omega) + R_0(\omega) + R_i(\omega) \leq t + R_0(\omega) + R_i(\omega)\,.$$

Let $\omega \in \{\tau_1 = 1\,,\ Q_n > r + s + t\} \setminus N$ be a given element. Since $Q_1(\omega) > t$ and $Q_0(\omega) \wedge Q_1(\omega) \leq t$, we have $M_0(\omega) = Q_0(\omega) \leq t$ and since $r + s \geq 0$, we have $t \leq r + s + t < Q_n(\omega)$ and so we have

$$r + s + t < Q_n(\omega) \leq M_0(\omega) + R_0(\omega) + R_1(\omega) \leq t + R_0(\omega) + R_1(\omega)\,.$$

Thus, we have $\{\tau_t = i\,,\ Q_n > r + s + t\} \setminus N \subseteq \{R_0 + R_i > r + s\}$ for all $1 \leq i \leq n$ and so by (b) we have

$$\begin{aligned} P(\tau_t = i\,,\ Q_n > r + s + t) &\leq P(\tau_t = i\,,\ R_0 + R_i > r + s) \\ &\leq P(\tau_t = i\,,\ R_0 > r) + \gamma(s,t)\, P(\tau_t = i) \end{aligned}$$

for all $1 \leq i \leq n$. Since $r+s \geq 0$, we have $\{Q_n > r+s+t\} \subseteq \{M_n > t\} = \{\tau_t \leq n\}$. Thus, summing the inequality over $i = 1, \ldots, n$, we obtain (d).

Suppose that $Q_n = M_n$ a.s. and set $c_k = t + k\,(r+s)$ for all $k \in \mathbb{N}_0$. Since $c_0 = t$ and $\vartheta_0(t) = 0$, we see that (e) holds for $k = 0$. Suppose that (e) holds for some integer $k \geq 0$. Since $c_k \geq t$ and $c_{k+1} = r + s + c_k$, we have by (d) and the induction hypothesis:

$$\begin{aligned} P(M_n > c_{k+1}) &= P(Q_n > r + s + c_k) \leq P(R_0 > r) + \gamma(s, c_k)\, P(M_n > c_k) \\ &\leq P(R_0 > r) + \theta(s,t)\, P(M_n > c_k) \\ &\leq (1 + \theta(s,t)\, \vartheta_k(s,t))\, P(R_0 > r) + \theta(s,t)^{k+1}\, P(M_n > t)\,, \end{aligned}$$

and since $\vartheta_{k+1}(s,t) = 1 + \theta(s,t)\, \vartheta_k(s,t)$, we see that (e) follows by induction. □

Theorem 3.3 *Let* $(V, \mathcal{B})$ *be a measurable linear space, let* $n \geq 2$ *be a given integer and let* $X_1, \ldots, X_n : \Omega \to (V, \mathcal{B})$ *be independent random vectors with partial sums* $S_0 = 0$ *and* $S_i = X_1 + \cdots + X_i$ *for* $1 \leq i \leq n$. *Let* $\Xi \subseteq V^*$ *be a non-empty set of*

$\mathcal{B}$-measurable linear functions such that $Q^{\Xi}(x) := \sup_{\xi\in\Xi} \xi(x)$ is $\mathcal{B}$-measurable. Let us define

$$M_n^{\Xi} = \max_{1\le i\le n} Q^{\Xi}(S_i)\ ,\ L_n^{\Xi} = \max_{1\le i\le n} Q^{\Xi}(X_i)\ ,\ M_{i,n}^{\Xi} = \max_{i<\nu\le n} Q^{\Xi}(S_\nu - S_i)\ ,$$

$$\beta_n(s) = \min_{1\le\nu<n} \inf_{\xi\in\Xi} P(\xi(S_\nu - S_n) \le s)\ ,\ \Gamma_n(s) = \max_{1\le i<n} P(M_{i,n}^{\Xi} > s)\ ,$$

$$\gamma_n(s) = \max_{1\le i<n} P(Q^{\Xi}(S_n - S_i) > s)\ ,\ \tilde\gamma_n(s) = \max_{1\le i<n} P(Q^{\Xi}(S_i - S_n) > s)$$

for all $0 \le i < n$ and all $s \in \mathbb{R}$. Set $\Theta_{n,k}(s) = \frac{1-\Gamma_n(s)^k}{1-\Gamma_n(s)}$ for $k \in \mathbb{N}_0$ with the convention that $\Theta_{n,k}(s) = k$ if $\Gamma_n(s) = 1$. Let $f : (-\infty,\infty] \to [0,\infty]$ be an increasing function. Then we have

(a) $\beta_n(u)\, P(M_n^{\Xi} > s + u) \le P(Q^{\Xi}(S_n) > s) \quad \forall\, s \in \mathbb{R}\ \forall\, u \in \mathbb{R}_+$.
(b) $\beta_n(u)\, Ef(M_n^{\Xi} - u) \le Ef(Q^{\Xi}(S_n)) \quad \forall\, u \in \mathbb{R}_+$.
(c) $\beta_n(s) \ge 1 - \tilde\gamma_n(s)$ *and* $\beta_n(u)\,\Gamma_n(s+u) \le \gamma_n(s) \le \Gamma_n(s) \quad \forall\, s \in \mathbb{R}\ \forall\, u \in \mathbb{R}_+$,

and if $r, s, t \in \mathbb{R}_+$ and $k \in \mathbb{N}_0$ are given numbers, then we have

(d) $P(Q^{\Xi}(S_n) > r + s + t) \le P(L_n^{\Xi} > r) + \gamma_n(s)\, P(M_n^{\Xi} > t)$.
(e) $P(M_n^{\Xi} > t + k(r+s)) \le \Theta_{n,k}(s)\, P(L_n^{\Xi} > r) + \Gamma_n(s)^k\, P(M_n^{\Xi} > t)$.

Suppose that $Q^{\Xi}(x) \ge 0$ for all $x \in V$ and let $u \ge 0$ be a given number such that $\Gamma_n(u) < 1$. Let $G : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing, right continuous function and let $K \ge 1$ be number such that $G(2x) \le K\, G(x)$ for all $x \ge 0$ and set

$$\kappa = \frac{\log K}{\log 2}\ ,\ \gamma = \log \frac{1}{\Gamma_n(u)}\ \ \textit{and}\ \ \nu_k = K^2\,(k+1)^{\kappa} \quad \forall\, k \in \mathbb{N}_0 .$$

Then we have

(f) $L_n^{\Xi} \le r$ *a.s.* $\Rightarrow\ Ee^{\lambda M_n^{\Xi}} \le \frac{e^{\lambda(2r+u)}}{1-e^{\lambda(r+u)-\gamma}} \quad \forall\, 0 \le \lambda < \frac{\gamma}{r+u}$.
(g) $(1 - \nu_k\,\Gamma_n(u)^k)^+\, EG(M_n^{\Xi}) \le \nu_k\, G(u) + \nu_k\, EG(L_n^{\Xi}) \quad \forall\, k \in \mathbb{N}_0$.

Proof

(a): Since $(V, \mathcal{B})$ is a measurable linear space and Q^{Ξ} is $\mathcal{B}$-measurable, we see that $Q^{\Xi}(S_1) \ldots, Q^{\Xi}(S_n)$ are (extended) random variables. Since $X_1, \ldots, X_n$ are independent, we see that $(Q^{\Xi}(S_1), \ldots, Q^{\Xi}(S_i))$ and $S_i - S_n$ are independent for all $1 \le i < n$. Hence, we see that (a) follows from Theorem 3.1 in [9].

(b): Let $s, u \ge 0$ be given numbers and set $J_u = f^{-1}((u,\infty])$. Since f is increasing, we see that J_u is an interval of the form $(x,\infty]$ or $[x,\infty]$ for some $x \in [-\infty,\infty]$. So by (a) we have

$$\beta_n(u)\, P(f(M_n^{\Xi} - u) > s) \le \beta_n(u)\, P(M_n^{\Xi} - u \in J_s)$$
$$\le P(Q^{\Xi}(S_n^{\Xi}) \in J_s) = P(f(Q^{\Xi}(S_n)) > s)\, .$$

Hence, we see that (b) follows from (3.30), p. 206 in [8].

(c): Let $s \in \mathbb{R}$ and $u \in \mathbb{R}_+$ be given and set

$$\beta_{i,n}(s) = \min_{i<\nu\leq n} \inf_{\xi\in\Xi} P(\xi(S_\nu - S_n) \leq s) \quad \forall i = 0, 1, \dots n-1 .$$

Then we have $\beta_n(s) \leq \beta_{i,n}(s)$ for all $0 \leq i < n$ and since $\xi(S_\nu - S_n) \leq Q^\Xi(S_\nu - S_n)$ for all $\xi \in \Xi$, we have $\beta_n(s) \geq 1 - \tilde{\gamma}_n(s)$. Applying (a) on $(X_{i+1}, \dots, X_n)$, we see that $\beta_{i,n}(s)\, P(M_{i,n}^\Xi > s + u) \leq P(Q^\Xi(S_n - S_i) > u)$ and since $Q^\Xi(S_n - S_i) \leq M_{i,n}^\Xi$, we have $\beta_n(s)\, \Gamma_n(s + u) \leq \gamma_n(s) \leq \Gamma_n(s)$.

(d): Let $r, s, t \in \mathbb{R}_+$ be given. We shall apply Theorem 3.2 with $Q_0 = 0$ and $Q_i = Q^\Xi(S_i)$ for $1 \leq i \leq n$ and with $R_0 = L_n^\Xi$, $R_n = 0$ and $R_i = Q^\Xi(S_n - S_i)$ for $1 \leq i < n$. Let $1 < i < n$ be given. Since $S_n = S_{i-1} + X_i + (S_n - S_i)$ and Q^Ξ is subadditive, we have

$$Q_n = Q^\Xi(S_n) \leq Q^\Xi(S_{i-1}) + Q^\Xi(X_i) + Q^\Xi(S_n - S_i) \leq Q_{i-1} + R_0 + R_i$$

and since $S_n = S_{n-1} + X_n$ and $S_n = X_1 + (S_n - S_1)$, we have $Q_n \leq Q_{n-1} + R_0$ and $Q_n \leq R_0 + R_1$. Since $Q_0 = R_n = 0$, we see that condition (a) in Theorem 3.2 holds, and since R_i and $(Q(S_1), \dots, Q(S_i))$ are independent, we see that condition (b) in Theorem 3.2 holds with $\gamma(s, t) = \gamma_n(s)$. Thus, we see that (d) follows from Theorem 3.2.

(e): Let $r, s, t \in \mathbb{R}_+$ be given. We shall apply Theorem 3.2 with $Q_0 = 0$, $Q_i = M_i^\Xi$ for $1 \leq i \leq n$ and with $R_0 = L_n^\Xi$, $R_n = 0$ and $R_i = M_{i,n}^\Xi$ for $1 \leq i < n$. Let $1 \leq i \leq n$ be given. Since $Q^\Xi(S_\nu) \leq Q^\Xi(S_{i-1}) + Q^\Xi(X_i) + Q^\Xi(S_\nu - S_i)$, we have

$$Q^\Xi(S_\nu) \leq Q_{i-1} + R_0 + R_i \quad \forall\, i \leq \nu \leq n ,$$

or equivalently $\max_{i\leq\nu\leq n} Q^\Xi(S_\nu) \leq Q_{i-1} + R_0 + R_i$. Since

$$Q_n = Q_{i-1} \vee \max_{i\leq\nu\leq n} Q^\Xi(S_\nu) ,$$

we see that (a) in Theorem 3.2 holds and since R_i and $(Q^\Xi(S_1), \dots, Q^\Xi(S_i))$ are independent and $Q_0 = 0$, we see that (b) and (c) in Theorem 3.2 holds with $\gamma(s, t) = \Gamma_n(s)$. Thus, we see that (e) follows from Theorem 3.2.

(f)–(g): Suppose that $Q^\Xi(x) \geq 0$ for all $x \in V$ and that $\mu := \Gamma_n(u) < 1$. Let $s \in \mathbb{R}_+$ and $k \in \mathbb{N}_0$ be given. Since $s + ku = \frac{s}{k+1} + k(\frac{s}{k+1} + u)$, we have by (e) with $(r, s, t) := (\frac{s}{k+1}, u, \frac{s}{k+1})$

$$P(M_n^\Xi > s + ku) \leq \tfrac{1}{1-\mu} P((k+1)\, L_n^\Xi > s) + \mu^k P((k+1)\, M_n^\Xi > s) .$$

Suppose that $L_n^\Xi \leq r$ a.s. and let $k \in \mathbb{N}_0$ and $0 \leq \lambda < \frac{\gamma}{v}$ be given where $v = r + u$ and $U = (M_n^\Xi - r)^+$. Taking $s = (k+1)r$, we see that $P(U > kv) \le \mu^k = e^{-k\gamma}$ for all $k \in \mathbb{N}_0$. Since $0 \le \lambda < \frac{\gamma}{v}$ and $P(kv \le U < (k+1)v) \le e^{-k\gamma}$, we have

$$Ee^{\lambda U} = \sum_{k=0}^{\infty} E(e^{\lambda U} 1_{\{kv \le U < (k+1)v\}}) \le e^{\lambda v} \sum_{k=0}^{\infty} e^{(\lambda v - \gamma)k} = \frac{e^{\lambda v}}{1 - e^{\lambda v - \gamma}}$$

and since $M_n^\Xi \le r + U$, we have $Ee^{\lambda M_n^\Xi} \le e^{\lambda r} Ee^{\lambda U}$ which completes the proof of (f). Let $k \in \mathbb{N}_0$ be given and set $M = (\frac{1}{k+1} M_n^\Xi - u)^+$. Applying the inequality above with s replaced by $(k+1)\, s$, we have

$$\begin{aligned} P(M > s) &= P(M_n^\Xi > (k+1)(s+u)) \le P(M_n^\Xi > (k+1)s + ku) \\ &\le \tfrac{1}{1-\mu} P(L_n^\Xi > s) + \mu^k P(M_n^\Xi > s) \end{aligned}$$

for all $s \ge 0$. Hence, we see that condition (a) in Lemma 3.1 holds with $(M, L, S, V) = (M, L_n^\Xi, \frac{1}{1-\mu}, M_n^\Xi)$ and $(\alpha(s), \beta(s)) = (1, \mu^k)$ and note that $M_n^\Xi \le u_k + c_k M$ where $u_k = (k+1)u$ and $c_k = k+1$. Since $G(2x) \le K\, G(x)$, then an easy argument shows that $G(sx) \le K\, s^\kappa\, G(x)$ for all $s \ge 1$ and all $x \ge 0$. In particular, we have

$$G(u_k + c_k\, x) \le K\, G(u_k) + K\, G(c_k x) \le K^2\, (k+1)^\kappa\, (G(u) + G(x))\,.$$

Since $\alpha(s) \equiv 1$ and $\beta(s) \equiv \mu^k$, we see that condition (c) in Lemma 3.1 holds with $a = K^2\,(k+1)^\kappa\, G(u)$, $p = K^2\,(k+1)^\kappa$, $b = 0$ and $q = \mu^k$ and so we see that (g) follows from Lemma 3.1.

□

Remark 3.4 Let $(V, \|\cdot\|)$ be a Banach space and let Ξ be a countable set of continuous linear functionals such that $\|x\| = \sup_{\xi \in \Xi} \xi(x)$ for all $x \in V$. Then the classical Ottaviani inequality (see Lemma 6.2, p. 152 in [14]) states that

$$(1 - \eta_n(s))\, P\left(\max_{1 \le i \le n} \|S_i\| > s + u\right) \le P(\|S_n\| > u) \quad \forall\, s, u > 0$$

where $\eta_n(s) = \max_{1 \le i \le n} P(\|S_n - S_i\| > s)$. Let $\gamma_n(s)$ and $\tilde{\gamma}_n(s)$ be defined as in Theorem 3.3. Since $Q^\Xi(x) = \|x\|$, we have $\gamma_n(s) = \tilde{\gamma}_n(x) = \eta_n(s)$ and so by Theorem 3.3.(c) we have $\beta_n(s) \ge 1 - \eta_n(s)$ and in general we have that $\beta_n(s)$ is much larger than $1 - \eta_n(s)$. Hence, we see that Theorem 3.3.(a) extends and improves the usual Ottaviani inequality. At the same time, we have that (a) extends and improves the usual Lévy inequality. To see this let

$$\chi_Y^\circ(p) = \inf\{x \in \mathbb{R} \mid P(Y \le x) \ge p\}\,, \;\; \chi_Y^*(p) = \inf\{x \in \mathbb{R} \mid P(Y \le x) > p\}$$

denote the smallest and largest p-fractile of the random variable Y for $0 < p < 1$. Then we have

$$\beta_n(s) \geq p \quad \forall\, s \geq \max_{1 \leq i < n} \sup_{\xi \in \Xi} \chi^{\circ}_{\xi(S_i - S_n)}(p)\,.$$

Suppose that $\xi(S_i - S_n)$ has median ≤ 0 for all $0 \leq i < n$ and all $\xi \in \Xi$. Then we have $\beta_n(0) \geq \frac{1}{2}$. So by Theorem 3.3.(a) with $u = 0$ we have $P(M_n^{\Xi} > s) \leq 2\,P(Q^{\Xi}(S_n) > s)$ for all $s \in \mathbb{R}$.

4 Maximal Inequalities for Weakly Dependent Random Variables

In this section I shall establish maximal inequalities under weak dependence assumptions. The weak dependence properties will be stated in terms of an appropriate stochastic ordering. Let $(S, \mathcal{A})$ be a measurable space and let Φ be a non-empty set of functions from S into $\mathbb{R}$. If μ and ν are measures on $(S, \mathcal{A})$, we write $\mu \preceq_{\Phi} \nu$ if $\int^* \phi\, d\mu \leq \int^* \phi\, d\nu$ for all $\phi \in \Phi$, where $\int^* f\, d\mu$ and $\int_* f\, d\mu$ denote the upper and lower μ-integrals of f; see [10]. If $X : (\Omega, \mathcal{F}) \to (S, \mathcal{A})$ is a measurable function, we let $P_X(A) := P(X^{-1}(A))$ for $A \in \mathcal{A}$ denote the *distribution* of X and if $Y : \tilde{\Omega} \to S$ is a measurable function, we write $X \sim Y$ if $P_X = \tilde{P}_Y$ and we write $X \preceq_{\Phi} Y$ if $P_X \preceq_{\Phi} \tilde{P}_Y$.

If $\mathcal{H}$ is a set of subsets of S, we let $W(S, \mathcal{H})$ denote the set of all functions $f : S \to \mathbb{R}$ such that for all $x < y$ there exists a set $H \in \mathcal{H} \cup \{\emptyset, S\}$ satisfying $\{f > y\} \subseteq H \subseteq \{f > x\}$; see [10].

Let $k \in \mathbb{N}$ be an integer. Then we say that $J \subset \mathbb{R}^k$ is an *upper interval* if $[u, *] \subseteq J$ for all $u \in J$ and we define lower intervals similarly. We let $\mathcal{J}(\mathbb{R}^k)$ denote the set of all upper intervals belonging to $\mathcal{B}^k$. Note that $W(\mathbb{R}^k, \mathcal{J}(\mathbb{R}^k))$ is the set of all increasing Borel functions from $\mathbb{R}^k$ into $\mathbb{R}$.

Let $X : \Omega \to \mathbb{R}^n$ and $Y : \Omega \to \mathbb{R}^k$ be random vectors. Then we say that X and Y are *negatively* In-*correlated* if

$$P(X \in J_1\,,\ Y \in J_2) \leq P(X \in J_1)\, P(Y \in J_2)\ \forall\, J_1 \in \mathcal{J}(\mathbb{R}^n)\ \forall\, J_2 \in \mathcal{J}(\mathbb{R}^k) \tag{4.1}$$

and we say that X and Y are *positively* In-*correlated* if

$$P(X \in J_1\,,\ Y \in J_2) \geq P(X \in J_1)\, P(Y \in J_2)\ \forall\, J_1 \in \mathcal{J}(\mathbb{R}^n)\ \forall\, J_2 \in \mathcal{J}(\mathbb{R}^k)\,. \tag{4.2}$$

Recall that a n-dimensional random vector $X = (X_1, \ldots, X_n)$ is *associated* if and only if X and X are positively In-correlated, that X is *negatively associated* if and only if X_α and X_β are negatively In-correlated for all disjoint non-empty sets $\alpha, \beta \subseteq \{1, \ldots, n\}$ and that X is *positively associated* if and only if X_α and X_β are positively In-correlated for all disjoint non-empty sets $\alpha, \beta \subseteq \{1, \ldots, n\}$, where

$X_\alpha = (X_i)_{i\in\alpha}$ is the α'th marginal of X whenever $\alpha \subseteq \{1,\dots,n\}$ is a non-empty set (see [7, 11, 15, 19, 22]). I suggests to give the definition of association, in order to compare a well-known concept with this new definition of dependency.

If μ and ν are measures on $(\mathbb{R}^k, \mathcal{B}^k)$, we shall consider the following integral orderings on $\mathbb{R}^k$, see [10]:

$\mu \preceq_{\text{st}} \nu$ if and only if $\mu(J) \le \nu(J)$ for all $J \in \mathcal{J}(\mathbb{R}^k)$.
$\mu \preceq_{\text{or}} \nu$ if and only if $\mu([x,*]) \le \nu([x,*])$ for all $x \in \mathbb{R}^k$.
$\mu \preceq_{\text{ism}} \nu$ if and only if $\int^* f\,d\mu \le \int^* f\,d\nu$ for all increasing, super-modular functions $f : \mathbb{R}^k \to \mathbb{R}$.
$\mu \preceq_{\text{sm}} \nu$ if and only if $\int^* f\,d\mu \le \int^* f\,d\nu$ for all super-modular Borel functions $f : \mathbb{R}^k \to \mathbb{R}$.
$\mu \preceq_{\text{bsm}} \nu$ if and only if $\int^* f\,d\mu \le \int^* f\,d\nu$ for all bounded, super-modular Borel functions $f : \mathbb{R}^k \to \mathbb{R}$.

Note that the sequence $X_1,\dots,X_n$ is a submartingale if and only if $X_1,\dots,X_n$ are integrable and

$$E((X_{i+1}-X_i)\,\phi(X_1,\dots,X_i)) \ge 0 \quad \forall \phi \in \mathrm{B}_+(\mathbb{R}^i)\ \forall\, 1 \le i < n\,, \tag{4.3}$$

or equivalently if $X_1,\dots,X_n \in L^1(P)$ and $(X_1,\dots,X_i,X_i) \preceq_{\Phi_i} (X_1,\dots,X_i,X_{i+1})$ for all $1 \le i < n$ where Φ_i is the set of all functions of the form $(x_1,\dots x_{i+1}) \curvearrowright x_{i+1}\,\phi(x_1,\dots x_i)$ for some $\phi \in \mathrm{B}_+(\mathbb{R}^i)$. In [20], Newman and Wright have defined a *demi-submartingale* to be a sequence $X_1,\dots X_n \in L^1(P)$ satisfying

$$E((X_{i+1}-X_i)\,\phi(X_1,\dots,X_i)) \ge 0 \quad \forall \phi \in \mathrm{IB}_+(\mathbb{R}^i)\ \forall\, 1 \le i < n\,, \tag{4.4}$$

or equivalently if $X_1,\dots,X_n \in L^1(P)$ and $(X_1,\dots,X_i,X_i) \preceq_{\Psi_i} (X_1,\dots,X_i,X_{i+1})$ for all $1 \le i < n$ where Ψ_i is the set of all functions of the form $(x_1,\dots x_{i+1}) \curvearrowright x_{i+1}\,\psi(x_1,\dots x_i)$ for some $\psi \in \mathrm{IB}_+(\mathbb{R}^i)$. If $X_1,\dots,X_n$ is a demi-submartingale, we have $EX_1 \le \cdots \le EX_n$. If $X_1,\dots,X_n \in L^1(P)$ is a demi-submartingale satisfying $EX_1 = EX_n$, we say that $X_1,\dots,X_n$ is a demi-martingale; see [20].

Proposition 4.1 *Let $\mu,\nu \in \Pr(\mathbb{R}^k)$ be probability measures and let η be a Borel measure on $\mathbb{R}^k$ such that $F_\eta(x) := \eta([*,x]) < \infty$ for all $x \in \mathbb{R}^k$. Let $(\mu_1,\dots,\mu_k)$ and $(\nu_1,\dots,\nu_k)$ denote the 1-dimensional marginals of μ and ν, respectively. Let $\psi_1,\dots,\psi_k : \mathbb{R} \to \mathbb{R}$ be increasing functions and let $\mu_\psi(B) := \mu(\psi^{-1}(B))$ and $\nu_\psi(B) := \nu(\psi^{-1}(B))$ denote the image measures for $B \in \mathcal{B}^k$ where*

$$\psi(x) = (\psi_1(x_1),\dots,\psi_k(x_k))\ ,\ \Sigma(x) = \sum_{i=1}^k \psi_i(x_i) \quad \forall\, x = (x_1,\dots,x_k) \in \mathbb{R}^k\,.$$

Let σ be a Borel measure on $\mathbb{R}$ and let $\alpha > k$ be a number satisfying

$$G_\alpha(t) := \tfrac{1}{\Gamma(\alpha)} \int_{(-\infty,t]} (t-y)^{\alpha-1}\,\sigma(dy) < \infty \quad \forall\, t \in \mathbb{R}\,.$$

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a convex function and let $\varphi_0 : \mathbb{R} \to \mathbb{R}$ be an increasing convex function. Then we have

(a) $\mu \preceq_x \nu \;\Rightarrow\; \mu_\psi \preceq_x \nu_\psi$ *for* $x = \mathrm{st}\,,\ \mathrm{or}\,,\ \mathrm{ism}\,,\ \mathrm{sm}\,,\ \mathrm{bsm}\,.$
(b) $\mu \preceq_{\mathrm{st}} \nu \Leftrightarrow \int^* h\,d\mu \le \int^* h\,d\nu$ *for all increasing functions* $h : \mathbb{R}^k \to \mathbb{R}\,.$
(c) $\mu \preceq_{\mathrm{sm}} \nu \;\Rightarrow\; \mu \preceq_{\mathrm{bsm}} \nu \;\Leftrightarrow\; \mu \preceq_{\mathrm{ism}} \nu$ *and* $\mu_i = \nu_i \quad \forall\, i = 1, \dots, k\,.$
(d) $\mu \preceq_{\mathrm{ism}} \nu \;\Rightarrow\; \int^* (\varphi_0 \circ \Sigma)\,d\mu \le \int^* (\varphi_0 \circ \Sigma)\,d\nu\,.$
(e) $\mu \preceq_{\mathrm{bsm}} \nu \;\Rightarrow\; \int^* (\varphi \circ \Sigma)\,d\mu \le \int^* (\varphi \circ \Sigma)\,d\nu\,.$
(f) If $\mu \preceq_{\mathrm{or}} \nu$*, then we have*

(f.1) $\int_{\mathbb{R}^k} (F_\eta \circ \psi)\,d\mu \le \int_{\mathbb{R}^k} (F_\eta \circ \psi)\,d\nu\,,$
(f.2) $\int_{\mathbb{R}^k} (G_\alpha \circ \Sigma)\,d\mu \le \int_{\mathbb{R}^k} (G_\alpha \circ \Sigma)\,d\nu\,,$
(f.3) $\int_{\mathbb{R}^k} \big(\prod_{i=1}^k \psi_i^+\big)\,d\mu \le \int_{\mathbb{R}^k} \big(\prod_{i=1}^k \psi_i^+\big)\,d\nu\,.$

(g) $\mu \preceq_{\mathrm{ism}} \nu \;\Rightarrow\; \mu \preceq_{\mathrm{or}} \nu$*; and if* $k = 2$*, then the converse implication holds.*

Proof

(a): Since μ, ν are Radon measures, we have $\int^* f\,d\mu_\psi = \int^* (f \circ \psi)\,d\mu$ and $\int^* f\,d\nu_\psi = \int^* (f \circ \psi)\,d\nu$ for all functions $f : \mathbb{R}^k \to \mathbb{R}$. If $J \subseteq \mathbb{R}^k$ is an upper interval, then so is $\psi^{-1}(J)$. Hence, we see that (a) holds for $x = \mathrm{st}$. Let $x = (x_1, \dots, x_k) \in \mathbb{R}^k$ and set $J_i := \psi_i^{-1}([x_i, \infty))$ for $i = 1, \dots, k$. Then $J_1, \dots, J_k$ are upper intervals on $\mathbb{R}$ and since $\psi^{-1}([x, *]) = \prod_{i=1}^k J_i$, we see that (a) holds for $x = \mathrm{or}$. If $f : \mathbb{R}^k \to \mathbb{R}$ is increasing or super-modular, then so is $f \circ \psi$ (see Proposition 4.1 in [10]). Hence, we see that (a) holds for $x = \mathrm{ism}\,,\ \mathrm{sm}\,,\ \mathrm{bsm}$.

(b) follows from Theorem 3.3.(3+4) in [10] and (c) follows from Theorem 4.7 in [10]. By Proposition 4.1 and Theorem 4.4 in [10], we have that $\varphi_0 \circ \Sigma$ is increasing and super-modular. Hence, we see that (d) holds.

(e): Suppose that $\mu \preceq_{\mathrm{bsm}} \nu$ and let $\tilde\mu(B) := \mu(-B)$ and $\tilde\nu(B) := \nu(-B)$ denote the reflected measures for $B \in \mathcal{B}^k$. Since $f(-x)$ is super-modular for every super-modular function $f(x)$, we have $\tilde\mu \preceq_{\mathrm{bsm}} \tilde\nu$. So by (d) we see that $\int^* (\varphi \circ \Sigma)\,d\mu \le \int^* (\varphi \circ \Sigma)\,d\mu$ if φ is either increasing or decreasing. So suppose that φ is neither increasing nor decreasing. Then we must have $m := \inf_{t \in \mathbb{R}} \varphi(t) > -\infty$ and there exist convex functions $\varphi_1, \varphi_2 : \mathbb{R} \to [0, \infty)$ such that φ_1 is increasing, φ_2 is decreasing and $\varphi(t) = m + \varphi_1(t) + \varphi_2(t)$ for all $t \in \mathbb{R}$. By the argument above we have $\int_{\mathbb{R}^k} (\varphi_j \circ \Sigma)\,d\mu \le \int_{\mathbb{R}^k} (\varphi_j \circ \Sigma)\,d\nu$ for $j = 1, 2$ and since $\mu(\mathbb{R}^k) = 1 = \nu(\mathbb{R}^k)$, we have $\int_{\mathbb{R}^k} (\varphi \circ \Sigma)\,d\mu \le \int_{\mathbb{R}^k} (\varphi \circ \Sigma)\,d\nu$ which completes the proof of (e).

(f.1): Suppose that $\mu \preceq_{\mathrm{or}} \nu$. By (a), we have $\mu_\psi \preceq_{\mathrm{or}} \nu_\psi$ and since $F_\eta(x) < \infty$ for all $x \in \mathbb{R}^k$, we have that η is σ-finite. So by Theorem 3.3.(7) in [10], we have

$$\int_{\mathbb{R}^k} (F_\eta \circ \psi)\,d\mu = \int_{\mathbb{R}^k} F_\eta\,d\mu_\psi \le \int_{\mathbb{R}^k} F_\eta\,d\nu_\psi = \int_{\mathbb{R}^k} (F_\eta \circ \psi)\,d\nu$$

which proves (f.1).

(f.2): Let $\beta > 0$ and $x \in \mathbb{R}$ be given. Since $\Gamma(\beta + 1) = \beta\,\Gamma(\beta)$ we have

$$\int_{-\infty}^{x} G_\beta(t)\,dt = \int_{-\infty}^{x} dt \int_{(-\infty,t]} \tfrac{(t-y)^{\beta-1}}{\Gamma(\beta)} \sigma(dy)$$
$$= \int_{(-\infty,x]} \sigma(dy) \int_{y}^{x} \tfrac{(t-y)^{\beta-1}}{\Gamma(\beta)}\,dt = \int_{(-\infty,x]} \tfrac{(x-y)^{\beta}}{\beta\,\Gamma(\beta)} \sigma(dy) = G_{\beta+1}(x)\,.$$

Let $x_1, \ldots, x_k \in \mathbb{R}$ be given. Applying the equality above with $\beta = \alpha - 1$, we find

$$G_\alpha(x_1 + \cdots + x_k) = \int_{-\infty}^{x_1} G_{\alpha-1}(t_1 + x_2 + \cdots + x_k)\,dt_1\,.$$

Since $\alpha > k$, we may iterate this equality k times and if so we obtain the following equality

$$G_\alpha(x_1 + \cdots + x_k) = \int_{-\infty}^{x_1} dt_1 \cdots \int_{-\infty}^{x_k} G_{\alpha-k}(t_1 + \cdots + t_k)\,dt_k\,.$$

Set $U(x) = x_1 + \cdots + x_k$ for all $x = (x_1, \ldots, x_k) \in \mathbb{R}^k$ and let us define $\sigma_\alpha(B) = \int_B G_{\alpha-k}(U(t))\,dt$ for all $B \in \mathcal{B}^k$. Then σ_α is a Borel measure on $\mathbb{R}^k$ satisfying $\sigma_\alpha([*, x]) = G_\alpha(U(x)) < \infty$ for all $x \in \mathbb{R}^k$. By (a), we have $\mu_\psi \preceq_{\mathrm{or}} \nu_\psi$ and so by (f.1) we have

$$\int_{\mathbb{R}^k} (G_\alpha \circ \Sigma)\,d\mu = \int_{\mathbb{R}^k} (G_\alpha \circ U)\,d\mu_\psi \le \int_{\mathbb{R}^k} (G_\alpha \circ U)\,d\nu_\psi = \int_{\mathbb{R}^k} (G_\alpha \circ \Sigma)\,d\nu$$

which proves (f.2).

(f.3): Let λ^k denote the k-dimensional Lebesgue measure on $\mathbb{R}^k$ and set $\lambda_+^k(B) := \lambda^k(B \cap [0, \infty)^k)$ for $B \in \mathcal{B}^k$. Then we have $\lambda_+^k([*, x]) = \prod_{i=1}^k x_i^+$ for all $x = (x_1, \ldots, x_k) \in \mathbb{R}^k$. Hence, we see that (f.3) follows from (f.1).

(g): Suppose that $\mu \preceq_{\mathrm{ism}} \nu$ and let $u \in \mathbb{R}^k$ be given. Since $1_{[u,*]}$ is increasing and super-modular, we see that $\mu \preceq_{\mathrm{or}} \nu$. So suppose that $k = 2$ and $\mu \preceq_{\mathrm{or}} \nu$. Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a bounded, continuous, increasing, super-modular function and let $(a_1, a_2) \le (b_1, b_2)$ be given vectors. Since g is super-modular, $(a_1, a_2) = (a_1, b_2) \wedge (b_1, a_2)$ and $(b_1, b_2) = (a_1, b_2) \vee (b_1, a_2)$, we have

$$g(b_1, b_2) + g(a_1, a_2) - g(a_1, b_2) - g(b_1, a_2) \ge 0$$

and since g is bounded and continuous, we have that the Lebesgue-Stieltjes measure λ_g is a finite measure on $(\mathbb{R}^2, \mathcal{B}^2)$ satisfying

$$\lambda_g(]a_1, b_1] \times]a, , b_2]) = g(b_1, b_2) + g(a_1, a_2) - g(a_1, b_2) - g(b_1, a_2)$$

for all $(a_1, a_2) \le (b_1, b_2)$; see [8, pp. 37–38]. Let $F_g(x) = \lambda_g([*, x])$ be the distribution function of λ_g. Since g is bounded, we have that $m := \inf_{x \in \mathbb{R}^2} g(x)$ is finite. Let us define $g_1(s) := \inf_{t \in \mathbb{R}} (g(s,t) - m)$ and $g_2(s) := \inf_{t \in \mathbb{R}} (g(t,s) - m)$ for all $s \in \mathbb{R}$. Since g is increasing and bounded, we see that g_1 and g_2 are bounded, non-negative and increasing on $\mathbb{R}$ and that we have

$$F_g(x_1, x_2) = g(x_1, x_2) - g_1(x_1) - g_2(x_2) - m \quad \forall\, (x_1, x_2) \in \mathbb{R}^2 .$$

Hence, we have $g(x_1, x_2) = F_g(x_1, x_2) + g_1(x_1) + g_2(x_2) + m$ and by (f.1) we have $\int_{\mathbb{R}^2} F_g \, d\mu \le \int_{\mathbb{R}^2} F_g \, d\nu$ and since $\mu_i \preceq_{\mathrm{st}} \nu_i$ and g_i is increasing for $i = 1, 2$, we have

$$\int_{\mathbb{R}^2} (g_1(s) + g_2(t))\, \mu(ds, dt) \le \int_{\mathbb{R}^2} (g_1(s) + g_2(t))\, \nu(ds, dt) .$$

Since $\mu(\mathbb{R}^2) = 1 = \nu(\mathbb{R}^2)$, we have $\int_{\mathbb{R}^2} g \, d\mu \le \int_{\mathbb{R}^2} g \, d\nu$ for all bounded, continuous, increasing super-modular functions $g : \mathbb{R}^2 \to \mathbb{R}$. Hence, by Theorem 4.7 in [10], we have $\mu \preceq_{\mathrm{ism}} \nu$ which completes the proof of (g).

□

Remark 4.2 Theorem 3.9.11, p. 118 in [19] states that the first implication in Proposition 4.1(c) is an equivalence. This is true in dimension 1 and 2, but fails in dimension 3 or more (see Example C in [10]). It seems, that this mistake has been overlooked in the later literature and I have not found any attempt to correct this mistake.

Proposition 4.3 *Let $(S_1, \mathcal{A}_1)$ and $(S_2, \mathcal{A}_2)$ be measurable spaces. Let $X_i : \Omega \to S_i$ and $\tilde{X}_i : \tilde{\Omega} \to S_i$ be measurable functions for $i = 1, 2$. Let $\mathcal{H}_1 \subseteq \mathcal{A}_1$ and $\mathcal{H}_2 \subseteq \mathcal{A}_2$ be non-empty sets satisfying*

(a) $P(X_1 \in H_1 , X_2 \in H_2) \le \tilde{P}(\tilde{X}_1 \in H_1 , \tilde{X}_2 \in H_2)$,
(b) $P(X_1 \in H_1) = \tilde{P}(\tilde{X}_1 \in H_1)$, $P(X_2 \in H_2) = \tilde{P}(\tilde{X}_2 \in H_2)$

for all $H_1 \in \mathcal{H}_1$ and all $H_2 \in \mathcal{H}_2$. Let $\phi_i \in W(S_i, \mathcal{H}_i)$ be a given function for $i = 1, 2$. Then we have

(c) $E^*\phi_1(X_1) = \tilde{E}^*\phi_1(\tilde{X}_1)$, $E^*\phi_2(X_2) = \tilde{E}^*\phi_2(\tilde{X}_2)$.
(d) $E^*(\phi_1(X_1)\, \phi_2(X_2)) \le \tilde{E}^*(\phi_1(\tilde{X}_1)\, \phi_2(\tilde{X}_2))$.
(e) $E_*(\phi_1(X_1)\, \phi_2(X_2)) \le \tilde{E}_*(\phi_1(\tilde{X}_1)\, \phi_2(\tilde{X}_2))$.

Proof Set $\mathcal{H}_i^* = \{\emptyset, S_i\} \cup \mathcal{H}_i$ and $\mathcal{C}_i = \{S_i \setminus H \mid H \in \mathcal{H}_i^*\}$ for $i = 1, 2$. Then (a) and (b) holds with $(\mathcal{H}_1, \mathcal{H}_2)$ replaced by $(\mathcal{H}_1^*, \mathcal{H}_2^*)$ and (c) follows from (b) and Theorem 3.3 in [10].

Let W_i^+ denote the set of all non-negative functions in $W(S_i, \mathcal{H}_i)$ and let V_i^+ denote the set of all non-negative functions in $W(S_i, \mathcal{C}_i)$. Applying Theorem 3.3 in [10] twice, we see that

$$E(\psi_1(X_1)\, \psi_2(X_2)) \le \tilde{E}(\psi_1(\tilde{X}_1)\, \psi_2(\tilde{X}_2)) \quad \forall\, \psi_1 \in W_1^+ \ \forall\, \psi_2 \in W_2^+ . \qquad \text{(i)}$$

Let $C_1 \in \mathcal{C}_1$ and $H_2 \in \mathcal{H}_2^*$ be given sets. Then $H_1 := S_1 \setminus C_1$ belongs to $\mathcal{H}_1^*$ and by (b) we have $P(X_1 \in C_1) = \tilde{P}(\tilde{X}_1 \in C_1)$. So by (a) we have

$$\tilde{P}(\tilde{X}_1 \in C_1\ ,\ \tilde{X}_2 \in H_2) = P(\tilde{X}_2 \in H_2) - \tilde{P}(\tilde{X}_1 \in H_1\ ,\ \tilde{X}_2 \in H_2)$$
$$\leq P(X_2 \in H_2) - P(X_1 \in H_1\ ,\ X_2 \in H_2) = P(X_1 \in C_1\ ,\ X_2 \in H_2)\,.$$

Hence, as above we see that

$$\tilde{E}(\psi_1(\tilde{X}_1)\,\psi_2(\tilde{X}_2)) \leq E(\psi_1(X_1)\,\psi_2(X_2)) \quad \forall\,\psi_1 \in V_1^+\ \forall\,\psi_2 \in W_2^+\,. \tag{ii}$$

In the same manner, we see that

$$\tilde{E}(\psi_1(\tilde{X}_1)\,\psi_2(\tilde{X}_2)) \leq E(\psi_1(X_1)\,\psi_2(X_2)) \quad \forall\,\psi_1 \in W_1^+\ \forall\,\psi_2 \in V_2^+\,, \tag{iii}$$

and since $\mathcal{H}_i^* = \{S_i \setminus C \mid C \in \mathcal{C}_i\}$, we have

$$E(\psi_1(X_1)\,\psi_2(X_2)) \leq \tilde{E}(\psi_1(\tilde{X}_1)\,\psi_2(\tilde{X}_2)) \quad \forall\,\psi_1 \in V_1^+\ \forall\,\psi_2 \in V_2^+\,. \tag{iv}$$

Set $U_i = \phi_i(X_i)$ and $\tilde{U}_i = \phi_i(\tilde{X}_i)$ for $i = 1, 2$. Since $\phi_i \in W(S_i, \mathcal{H}_i)$, we have $\phi_i^+ \in W_i^+$ and $\phi_i^- \in V_i^+$ and so by (i)–(iv), we have

$$E(U_1^+U_2^+) \leq \tilde{E}(\tilde{U}_1^+\tilde{U}_2^+)\ ,\ E(U_1^-U_2^-) \leq \tilde{E}(\tilde{U}_1^-\tilde{U}_2^-)\,,$$
$$\tilde{E}(\tilde{U}_1^+\tilde{U}_2^-) \leq E(U_1^+U_2^-)\ ,\ \tilde{E}(\tilde{U}_1^-\tilde{U}_2^+) \leq E(U_1^-U_2^+)\,.$$

Since $(xy)^+ = x^+y^+ + x^-y^-$, we see that $E((U_1U_2)^+) \leq \tilde{E}((\tilde{U}_1\tilde{U}_2)^+)$ and since $(xy)^- = x^+y^- + x^-y^+$, we have $\tilde{E}((\tilde{U}_1\tilde{U}_2)^-) \leq E((U_1U_2)^-)$. Recalling the equalities $E^*Y = EY^+ - EY^-$ and $E_*Y = -E^*(-Y)$, we obtain (d) and (e). □

Corollary 4.4 *Let $X : \Omega \to \mathbb{R}^n$ and $Y : \Omega \to \mathbb{R}^k$ be random vectors and let $\Lambda_{n,k}$ be the set of all functions $h : \mathbb{R}^{n+k} \to \mathbb{R}$ of the form $h(x,y) = f(x)\,g(y)$ for some increasing Borel functions $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^k \to \mathbb{R}$. Then the following four statements are equivalent:*

(a) *X and Y are negatively* In-*correlated.*
(b) *$-X$ and Y are positively* In-*correlated.*
(c) $\mathrm{cov}(f(X), h(Y)) \leq 0 \quad \forall f \in \mathrm{IB}(\mathbb{R}^n)\ \forall\, h \in \mathrm{IB}(\mathbb{R}^k)\,.$
(d) $P_{(X,Y)} \preceq_{\Lambda_{n,k}} P_X \otimes P_Y\,,$

and if $n = k = 1$, then (a)–(d) are equivalent to either of the following three statements:

(e) $P_{(X,Y)} \preceq_{\mathrm{or}} P_X \otimes P_Y\,.$
(f) $P_{(X,Y)} \preceq_{\mathrm{ism}} P_X \otimes P_Y\,.$
(g) $P_{(X,Y)} \preceq_{\mathrm{bsm}} P_X \otimes P_Y\,.$

Proof Since $\mathbb{R}^n \setminus (-J) \in \mathcal{J}(\mathbb{R}^n)$ for all $J \in \mathcal{J}(\mathbb{R}^n)$, we see that (a) and (b) are equivalent and so by Proposition 4.3, we see that (a)–(d) are equivalent. Suppose that $n = k = 1$. By Proposition 4.1, we see that (e)–(g) are equivalent and since J is a non-empty upper interval in $\mathbb{R}$ if and only if $J = [a, \infty)$ or $J = (a, \infty)$ for some $a \in \mathbb{R}$, we see that (e) and (a) are equivalent. □

Theorem 4.5 *Let $Y_1, \dots, Y_n : \Omega \to \mathbb{R}$ be real random variables and let $S_1, \dots, S_n \in L^1(P)$ be integrable random variables satisfying*

(a) $S_{i+1} \wedge Y_i \le Y_{i+1} \le S_{i+1} \vee Y_i$ *a.s.* $\quad \forall\, 1 \le i < n$.
(b) $E\big(1_{\{Y_i > t\}}\,(S_{i+1} - S_i)\big) \ge 0 \quad \forall\, 1 \le i < n\ \forall\, t \ge 0$.

Then we have

(c) $t\,P(Y_n > t) \le E\big(1_{\{Y_1 > t\}}\,(t - S_1)\big) + E\big(1_{\{Y_n > t\}}\,S_n\big) \quad \forall\, t \ge 0$,
(d) $Y_1 \le S_1$ *a.s. on* $\{Y_1 > 0\} \Rightarrow t\,P(Y_n > t) \le E\big(1_{\{Y_n > t\}}\,S_n\big) \quad \forall\, t \ge 0$,

and if $Y_1 = S_1$ a.s. and $E(1_{\{Y_i \ge -t\}}\,(S_{i+1} - S_i)) \ge E(S_{i+1} - S_i)$ for all $1 \le i < n$ and all $t \in \mathbb{R}_+$, then we have

(e) $t\,P(Y_n < -t) \le -E(1_{\{Y_n < -t\}}\,S_n) \quad \forall\, t \ge 0$.
(f) $t\,P(|Y_n| > t) \le E(1_{\{|Y_n| > t\}}\,|S_n|) \quad \forall\, t \ge 0$.

Proof Let $t \ge 0$ be given and set $U_i = 1_{\{Y_i > t\}}$ for $i = 1, \dots, n$ and $U_0 = 0$. Let $1 \le i < n$ be given and let me show that $t\,(U_{i+1} - U_i) \le S_{i+1}\,(U_{i+1} - U_i)$ a.s. If $U_i = U_{i+1}$, this holds trivially. If $U_{i+1} = 1$ and $U_i = 0$, we have $Y_i \le t < Y_{i+1}$ and since $Y_{i+1} \le S_{i+1} \vee Y_i$ a.s., we have

$$S_{i+1}\,(U_{i+1} - U_i) = S_{i+1} \ge Y_{i+1} > t = t\,(U_{i+1} - U_i) \ \text{ a.s.}$$

If $U_{i+1} = 0$ and $U_i = 1$, we have $Y_{i+1} \le t < Y_i$ and since $S_{i+1} \wedge Y_i \le Y_{i+1}$ a.s., we have

$$S_{i+1}\,(U_{i+1} - U_i) = -S_{i+1} \ge -Y_{i+1} \ge -t = t\,(U_{i+1} - U_i) \ \text{ a.s.}$$

which proves the claim. So by partial summation, we have

$$\begin{aligned} S_n\,U_n - S_1\,U_1 &= \sum_{i=1}^{n-1} S_{i+1}\,(U_{i+1} - U_i) + \sum_{i=1}^{n-1} U_i\,(S_{i+1} - S_i) \\ &\ge t\,(U_n - U_1) + \sum_{i=1}^{n-1} U_i\,(S_{i+1} - S_i)\,, \end{aligned}$$

and by (b) we have $E(U_i\,(S_{i+1} - S_i)) \ge 0$ for all $1 \le i < n$. Since $S_1, \dots, S_n \in L^1(P)$, we see that (c) holds, and (d) is an easy consequence of (c).

(e)–(f): Suppose that $Y_1 = S_1$ a.s. and that $E(1_{\{Y_i \geq -t\}}(S_{i+1} - S_i)) \geq E(S_{i+1} - S_i)$ for all $1 \leq i < n$ and all $t \in \mathbb{R}_+$. Note that $(-Y_i, -S_i)$ satisfies (a) and since

$$E\big(1_{\{-Y_i > t\}}(S_i - S_{i+1})\big) = E(S_i - S_{i+1}) - E(1_{\{Y_i \geq -t\}}(S_i - S_{i+1})) \geq 0\,,$$

we see that $(-Y_i, -S_i)$ satisfies (b). Since $Y_1 = S_1$ a.s. we see that (e) follows from (d) applied to $(-Y_i, -S_i)$. By (d) and (e) we have

$$\begin{aligned} t\,P(|Y_n| > t) &= t\,P(Y_n > t) + t\,P(Y_n < -t) \\ &\leq E(1_{\{Y_n > t\}} S_n) - E(1_{\{Y_n < -t\}} S_n) \leq E(1_{\{|Y_n| > t\}} |S_n|) \end{aligned}$$

which proves (f). □

Remark 4.6 Let Y and S be random variables such that $S \in L^1(P)$, $Y \geq 0$ a.s. and $t\,P(Y > t) \leq E(1_{\{Y > t\}} S)$ for all $t \geq 0$; see (d) and (f) in Theorem 4.5. Let $G : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing, right continuous function. By Lemma 3.1 with $(S, L, M, V) = (S, Y, Y, 0)$, $\alpha(s) = s$ and $\beta(s) = 0$, we have $EG^\diamond(Y) \leq E^*(S\,G(Y))$ where $G^\diamond(x) = \int_{[0,x]} t\,\lambda_G(dt)$ for all $x \geq 0$. Taking $G(x) = x^{p-1}$ for some $p > 1$, we have $G^\diamond(x) = \frac{p-1}{p}\, x^p$. Hence, we have $EY^p \leq \frac{p}{p-1}\, E(S\,Y^{p-1})$ and so by Hölder's inequality, we have $(EY^p)^{1/p} \leq \frac{p}{p-1}\,(E|S|^p)^{1/p}$. Taking $G(x) = \log(1 + x)$, we have $G^\diamond(x) = x - \log(1 + x)$ and so we have $E(Y - \log(1 + Y)) \leq E(S \log(1 + Y))$. In particular, we see that (d) and (f) give a variety of moment inequalities.

Corollary 4.7 *Let $(S_1, \ldots S_n)$ be a demi-submartingale and let $f_i : \mathbb{R}^i \to \mathbb{R}$ be an increasing Borel function for $i = 1, \ldots n$ satisfying*

$$x_{i+1} \wedge f_i(x_1, \ldots, x_i) \leq f_{i+1}(x_1, \ldots, x_{i+1}) \leq x_{i+1} \vee f_i(x_1, \ldots, x_i)$$

for all $1 \leq i < n$ and all $x_1, \ldots, x_{i+1} \in \mathbb{R}$ and set $Y_i = f_i(S_1, \ldots, S_i)$ for $i = 1, \ldots, n$. Then we have

(a) $t\,P(Y_n > t) \leq E\big(1_{\{Y_1 > t\}}(t - S_1)\big) + E\big(1_{\{Y_n > t\}} S_n\big) \quad \forall\, t \geq 0\,,$
(b) $f_1(x) \leq x\ \forall\, x \in \mathbb{R} \ \Rightarrow\ t\,P(Y_n > t) \leq E\big(1_{\{Y_n > t\}} S_n\big) \quad \forall\, t \geq 0\,,$

and if $f_1(x) = x$ for all $x \in \mathbb{R}$ and $ES_1 = ES_n$, then we have

(c) $t\,P(Y_n < -t) \leq -E(1_{\{Y_n < -t\}} S_n) \quad \forall\, t \in \mathbb{R}_+\,.$
(d) $t\,P(|Y_n| > t) \leq E(1_{\{|Y_n| > t\}} |S_n|) \quad \forall\, t \in \mathbb{R}_+\,.$

Proof By hypothesis, we see that $(Y_i, S_i)_{1 \leq i \leq n}$ satisfies (a) in Theorem 4.5 and since $(S_1, \ldots, S_n)$ is a demi-submartingale, we see that $(Y_i, S_i)_{1 \leq i \leq n}$ satisfies (b) in Theorem 4.5. Hence, we see that (a)–(b) follow from Theorem 4.5. Suppose that $f_1(x) = x$ for all $x \in \mathbb{R}$ and that $ES_1 = ES_n$. Let $1 \leq i < n$ be a given integer and let $t \in \mathbb{R}$. Then we have $Y_1 = S_1$ and since f_i is increasing, we have $1_{\{f_i \geq -t\}} \in \mathrm{IB}_+(\mathbb{R}^i)$. Since $ES_1 \leq ES_{i+1} \leq ES_n = ES_1$, we have $E(S_{i+1} - S_i) = 0 \leq E(1_{\{Y_i \geq -t\}}(S_{i+1} - S_i))$. Hence, we see that (c)–(d) follow from Theorem 4.5. □

Corollary 4.8 (cf. [20]) *Let* $(S_1,\ldots,S_n)$ *be a demi-submartingale and for* $1\le i\le n$*, let* $S_{i,i}\le\cdots\le S_{1,i}$ *denote the order statistics of* $S_1,\ldots,S_i$ *for all* $1\le i\le n$*. Let* $1\le k\le n$ *be a given integer and let us define*

$$U_i^k := S_{k\wedge i,i} \text{ and } V_i^k := S_{1+(i-k)^+,i} \quad \forall\, i=1,\ldots,n\,.$$

Let $t\in\mathbb{R}_+$ *be given. Then we have*

(a) $t\,P(U_n^k>t)\le E(1_{\{U_n^k>t\}}\,S_n)$ *and* $t\,P(V_n^k>t)\le E(1_{\{V_n^k>t\}}\,S_n)$,

and if $ES_1=ES_n$*, then we have*

(b) $t\,P(U_n^k<-t)\le -E(1_{\{U_n^k<-t\}}\,S_n)$ *and* $t\,P(V_n^k<-t)\le -E(1_{\{V_n^k<-t\}}\,S_n)$.
(c) $t\,P(|U_n^k|>t)\le E(1_{\{|U_n^k|>t\}}\,|S_n|)$ *and* $t\,P(|V_n^k|>t)\le E(1_{\{|V_n^k|>t\}}\,|S_n|)$.

Proof If $i\in\mathbb{N}$ and $x=(x_1,\ldots,x_i)\in\mathbb{R}^i$, we let $\pi_{i,i}(x)\le\cdots\le\pi_{1,i}(x)$ denote the order statistics of $x_1,\ldots,x_i$. Then the hypotheses of Corollary 4.8 holds with $f_i=\pi_{k\wedge i,i}$ and $f_i=\pi_{1+(i-k)^+,i}$ and since $\pi_{1,1}(x)=x$ for all $x\in\mathbb{R}$, we see that the corollary follows from Corollary 4.8. □

Theorem 4.9 *Let* $n\ge 2$ *be a given integer and let* $X_1,\ldots,X_n:\Omega\to\mathbb{R}$ *and* $\tilde X_1,\ldots,\tilde X_n:\tilde\Omega\to\mathbf{R}$ *be random variables such that* $X_i\sim\tilde X_i$ *for all* $1\le i\le n$*. Let* $f_1,\ldots,f_{n-1}:\mathbb{R}\to\mathbb{R}$ *be Borel functions and let* $F_j:\mathbb{R}^j\to\mathbb{R}$ *be defined inductively as follows* $F_1(x):=x$ *for* $x\in\mathbb{R}$ *and*

$$F_{j+1}(x_1,\ldots,x_{j+1})=f_j(F_j(x_1,\ldots,x_j))+x_{j+1}\ \forall x_1,\ldots,x_{j+1}\in\mathbb{R}\ \forall 1\le j<n\,.$$

Let $\mathrm{Co}(\mathbb{R})$ *denote the set of all convex functions from* $\mathbb{R}$ *into* $\mathbb{R}$*. Set* $\Phi_1=\mathrm{Co}(\mathbb{R})$ *and*

$$\Phi_{j+1}=\{\phi\in\mathrm{Co}(\mathbb{R})\mid\phi(f_j(\cdot)+a)\in\Phi_j\ \forall\, a\in\mathbb{R}\}\quad\forall\,1\le j<n\,.$$

If $f_i(F_i(X_1,\ldots,X_i))$ *and* X_{i+1} *are negatively* In-*correlated for all* $1\le i<n$ *and* $f_i(F_i(\tilde X_1,\ldots,\tilde X_i))$ *and* $\tilde X_{i+1}$ *are positively* In-*correlated for all* $1\le i<n$*, then we have*

(a) $E^*\phi(F_j(X_1,\ldots,X_j))\le\tilde E^*\phi(F_j(\tilde X_1,\ldots,\tilde X_j))\quad\forall\,\phi\in\Phi_j\ \forall\,1\le j\le n$.

Proof Set $U_i=F_i(X_1,\ldots,X_i)$ and $\tilde U_i=F_i(\tilde X_1,\ldots,\tilde X_i)$ for $1\le i\le n$ and set $V_i=f_i(U_i)$ and $\tilde V_i=f_i(\tilde U_i)$ for $1\le i<n$. Let Φ_j^+ denote the set of all non-negative functions in Φ_j and let me first show that (a) holds for all $1\le j\le n$ and all $\phi\in\Phi_j^+$. Since $X_1\sim\tilde X_1$, we see that this holds for $j=1$. Let $1\le j<n$ be a given integer such that $E\phi(U_j)\le\tilde E\phi(\tilde U_j)$ for all $\phi\in\Phi_j^+$. Let $\phi\in\Phi_{j+1}^+$ be given and let me show that $E\phi(U_{j+1})\le\tilde E\phi(\tilde U_{j+1})$. Since V_j and X_{j+1} are negatively In-correlated, we have $P_{(V_j,X_{j+1})}\preceq_{\mathrm{bsm}}P_{V_j}\otimes P_{X_{j+1}}$ and since $U_{j+1}=V_j+X_{j+1}$ and ϕ is convex

and non-negative, we have by Proposition 4.1.(e)

$$E\phi(U_{j+1}) = E\phi(V_j + X_{j+1}) \le \int_{\mathbb{R}} E\phi(V_j + t)\, P_{X_{j+1}}(dt)\,.$$

Since $\tilde V_j$ and $\tilde X_{j+1}$ are positively In-correlated, we have $\tilde P_{\tilde V_j} \otimes \tilde P_{\tilde X_{j+1}}) \preceq_{\mathrm{bsm}} P_{(\tilde V_j, \tilde X_{j+1})}$ and since $\tilde U_{j+1} = \tilde V_j + \tilde X_{j+1}$ and ϕ is convex and non-negative, we have by Proposition 4.1.(e)

$$\tilde E\phi(\tilde U_{j+1}) = \tilde E\phi(\tilde V_j + \tilde X_{j+1}) \ge \int_{\mathbb{R}} \tilde E\phi(\tilde V_j + t)\, \tilde P_{\tilde X_{j+1}}(dt)\,.$$

Let $t \in \mathbb{R}$ be given and set $\psi_{j,t}(s) = \phi(f_j(s) + t)$ for all $s \in \mathbb{R}$. Since $\phi \in \Phi^+_{j+1}$, we have $\psi_{j,t} \in \Phi^+_j$ and so by induction hypothesis we have $E\psi_{j,t}(U_j) \le \tilde E\psi_{j,t}(\tilde U_j)$ and since $X_{j+1} \sim \tilde X_{j+1}$, we have

$$E\phi(U_{j+1}) \le \int_{\mathbb{R}} E\psi_{j,t}(U_j)\, P_{X_{j+1}}(dt) \le \int_{\mathbb{R}} \tilde E\psi_{j,t}(\tilde U_j) \tilde P_{\tilde X_{j+1}}(dt) \le \tilde E\phi(\tilde U_{j+1})\,.$$

So by induction, we see that $E\phi(U_j) \le \tilde E\phi(\tilde U_j)$ for all $1 \le j \le n$ and all $\phi \in \Phi^+_j$.

Now let me show that $(\phi(\cdot) + a)^+ \in \Phi^+_j$ for all $1 \le j \le n$, all $\phi \in \Phi_j$ and all $a \in \mathbb{R}$. If $j = 1$, this is evident. Let $1 \le j < n$ be a given integer satisfying $(\phi(\cdot) + a)^+ \in \Phi^+_j$ for all $\phi \in \Phi_j$ and all $a \in \mathbb{R}$. Let $\phi \in \Phi_{j+1}$ and $a, b \in \mathbb{R}$ be given and set $\phi_a(t) = (\phi(t) + a)^+$ and $\psi_b(t) = \phi(f_j(t) + b)$ for all $t \in \mathbb{R}$. Since $\phi \in \Phi_{j+1}$, we have $\psi_b \in \Phi_j$ and so by induction hypothesis we have

$$\phi_a(f_j(t) + b) = (\phi(f_j(t) + b) + a)^+ = (\psi_b(t) + a)^+ \in \Phi_j\,.$$

Hence, we have $\phi_a \in \Phi_{j+1}$ for all $a \in \mathbb{R}$. So by induction, we see that $(\phi(\cdot)+a)^+ \in \Phi^+_j$ for all $1 \le j \le n$ all $\phi \in \Phi_j$ and all $a \in \mathbb{R}$. Since $E\phi(U_j) \le \tilde E\phi(\tilde U_j)$ for all $1 \le j \le n$ and all $\phi \in \Phi^+_j$, we see that (a) follows from Theorem 3.3 in [10]. □

Theorem 4.10 *Let $X_1, \ldots, X_n : \Omega \to \mathbb{R}$ and $\tilde X_1, \ldots, \tilde X_n : \tilde\Omega \to \mathbb{R}$ be random variables such that $X_i \sim \tilde X_i$ for all $1 \le i \le n$. Set $S_0 = 0 = \tilde S_0$ and let $S_k = X_1 + \cdots + X_k$ and $\tilde S_k = \tilde X_1 + \cdots + \tilde X_k$ denote the partial sums for $1 \le k \le n$. Let us define $\overline{M}_n = \max(|S_1|, \ldots, |S_n|)$ and*

$$M_{i,j} = \max_{i<\nu\le j} (S_\nu - S_i)\ ,\ \tilde M_{i,j} = \max_{i<\nu\le j} (\tilde S_\nu - \tilde S_i) \quad \forall\, 0 \le i < j \le n$$

$$L_{i,j} = \min_{i<\nu\le j} (S_\nu - S_i)\ ,\ \tilde L_{i,j} = \min_{i<\nu\le j} (\tilde S_\nu - \tilde S_i) \quad \forall\, 0 \le i < j \le n$$

for all $j = 1, \ldots, n$.

(1): Suppose that S_j and X_{j+1} are negatively In*-correlated for all $1 \le j < n$ and that $\tilde{S}_j$ and $\tilde{X}_{j+1}$ are positively* In*-correlated for all $1 \le j < n$. Then we have $E^*\phi(S_n) \le \tilde{E}^*\phi(\tilde{S}_n)$ for every convex function $\phi : \mathbb{R} \to \mathbb{R}$.*

(2): Suppose that $M_{i,n}$ and X_i are negatively In*-correlated for all $1 \le i < n$ and that $\tilde{M}_{i,n}$ and $\tilde{X}_i$ are positively* In*-correlated for all $1 \le i < n$. Then we have $E^*\phi(M_{0,n}) \le \tilde{E}^*\phi(\tilde{M}_{0,n})$ for every convex, increasing function $\phi : \mathbb{R} \to \mathbb{R}$.*

(3): Suppose that $L_{i,n}$ and X_i are negatively In*-correlated for all $1 \le i < n$ and that $\tilde{L}_{i,n}$ and $\tilde{X}_i$ are positively* In*-correlated for all $1 \le i < n$. Then we have $E^*\phi(L_{0,n}) \le \tilde{E}^*\phi(\tilde{L}_{0,n})$ for every convex, decreasing function $\phi : \mathbb{R} \to \mathbb{R}$.*

(4): Suppose that $M_{i,n}$ and X_i are negatively In*-correlated, that $L_{i,n}$ and X_i are negatively* In*-correlated, that $\tilde{M}_{i,n}$ and $\tilde{X}_i$ are positively* In*-correlated and that $\tilde{L}_{i,n}$ and $\tilde{X}_i$ are positively* In*-correlated for all $1 \le i < n$. Then we have*

$$E\phi(\overline{M}_n) \le \tilde{E}\phi(\tilde{M}^+_{0,n}) + \tilde{E}\phi(\tilde{L}^-_{0,n}) \le 2\, E\phi(\overline{M}_{0,n})$$

for every increasing, convex function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$.

Proof

(1): We shall apply Theorem 4.9 with $f_i(x) = x$ for $x \in \mathbb{R}$ and $1 \le i < n$. Let F_j and Φ_j be defined as in Theorem 4.9 and let $\phi : \mathbb{R} \to \mathbb{R}$ be a convex function. Then $F_j(x_1, \ldots, x_j) = x_1 + \cdots + x_j$ and we have $\phi \in \Phi_n$. Hence, we see that (1) follows from Theorem 4.9.

(2): We shall apply Theorem 4.9 on the sequences $(Y_1, \ldots, Y_n) = (X_n, \ldots, X_1)$ and $(\tilde{Y}_1, \ldots, \tilde{Y}_n) = (\tilde{X}_n, \ldots, \tilde{X}_1)$ with $f_i(x) = x \vee 0$. Let F_j and Φ_j be defined as in Theorem 4.9. Let $\phi : \mathbb{R} \to \mathbb{R}$ be an increasing convex function. Then it follows easily that $\phi \in \Phi_n$ and that we have

$$F_j(x_1, \ldots, x_j) = \max_{0 \le i < j} \left(\sum_{i < \nu \le j} x_\nu \right) \quad \forall\, 1 \le j \le n\,. \tag{i}$$

Let $1 \le j < n$ be given. Since $F_j(Y_1, \ldots, Y_j) = M_{n-j,n}$ and $Y_{j+1} = X_{n-j}$, we see that $F_j(Y_1, \ldots, Y_j)$ and Y_{j+1} are negatively In-correlated. Similarly, we see that $F_j(\tilde{Y}_1, \ldots, \tilde{Y}_j)$ and $\tilde{Y}_{j+1}$ are positively In-correlated. Hence, we see that (3) follows from Theorem 4.9.

(3): We shall apply Theorem 4.9 on the sequences $(Z_1, \ldots, Z_n) = -(X_n, \ldots, X_1)$ and $(\tilde{Z}_1, \ldots, \tilde{Z}_n) = -(\tilde{X}_n, \ldots, \tilde{X}_1)$ with $f_i(x) = x \vee 0$. Let F_j and Φ_j be defined as in Theorem 4.9 and let $\phi : \mathbb{R} \to \mathbb{R}$ be a decreasing convex function. Then $\psi(x) := \phi(-x)$ is increasing and convex and belongs to Φ_n and F_j is given by (i). Let $1 \le j < n$ be given. Since $L_{n-j,n}$ and X_{n-j} are negatively In-correlated, we see that $-L_{n-j,n}$ and $-X_{n-j}$ are negatively In-correlated and since $Z_{j+1} = -X_{n-j}$ and $-L_{n-j,n} = F_j(Z_1, \ldots, Z_j)$, we see that Z_{j+1} and $F_j(Z_1, \ldots, Z_j)$ are negatively In-correlated. Similarly, we see that $\tilde{Z}_{j+1}$ and

$F_j(\tilde{Z}_1, \ldots, \tilde{Z}_j)$ are positively In-correlated. So by Theorem 4.9 we have

$$E^* \phi(L_{0,n}) = E^* \psi(-L_{0,n}) \leq \tilde{E}^* \psi(-\tilde{L}_{0,n}) = \tilde{E}^* \phi(\tilde{L}_{0,n})$$

which proves (3).

(4): Let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing convex function and set $\phi_1(x) = \phi(x^+)$ and $\phi_2(x) = \phi(x^-)$ for all $x \in \mathbb{R}$. Then ϕ_1 is increasing and convex and ϕ_2 is decreasing and convex. So by (2) and (3) we have $E\phi_1(M_{0,n}) \leq \tilde{E}\phi_1(\tilde{M}_{0,n})$ and $E\phi_2 L_{0,n}) \leq \tilde{E}\phi_2(\tilde{L}_{0,n})$. Note that $M_{0,n} = \max(S_1, \ldots, S_n)$ and $-L_{0,n} = \max(-S_1, \ldots, -S_n)$. Hence, we have $\overline{M}_n = M_{0,n} \vee (-L_{0,n})$ and since ϕ is non-negative, we have $\phi(\overline{M}_n) \leq \phi_1(M_{0,n}) + \phi_2(L_{0,n})$ and

$$E\phi(\overline{M}_{0,n}) \leq E\phi_1(M_{0,n}) + E\phi_2(L_{0,n}) \leq \tilde{E}\phi_1(\tilde{M}_{0,n}) + \tilde{E}\phi_2(\tilde{L}_{0,n})$$

which proves (4).

□

Remark 4.11

(1): Let $(Y_1, \ldots, Y_n)$ be a positively associated (respectively, negatively associated) random vector; for instance, if $Y_1, \ldots, Y_n$ are independent. Let $1 \leq i < n$ be a given integer and let $\phi : \mathbb{R}^i \to \mathbb{R}$ and $\psi : \mathbb{R}^{n-i} \to \mathbb{R}$ be increasing Borel functions. Then $\phi(Y_1, \ldots, Y_i)$ and Y_{i+1} are positively (negatively) In-correlated and $\psi(Y_{i+1}, \ldots, Y_n)$ and Y_i are positively (negatively) In-correlated.

(2): Suppose that $\tilde{X}_1, \ldots, \tilde{X}_n$ are independent such that $X_i \sim \tilde{X}_i$ and $EX_i = 0$ for all $1 \leq i \leq n$. Let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing convex function with $\phi(0) = 0$. By the prophet inequality of Choi and Klass (see [4]), we have $E\phi(\tilde{M}_{0,n}^+) \leq 2\,E\phi(\tilde{S}_n^+)$ and $E\phi(\tilde{L}_{0,n}^-) \leq 2\,E\phi(\tilde{S}_n^-)$. Hence, we have the following prophet inequalities:

If $M_{i,n}$ and X_i are negatively In-correlated for all $1 \leq i < n$, then we have $E\phi(L_{0,n}^-) \leq 2\,E\phi(\tilde{S}_n^-)$

If $L_{i,n}$ and X_i are negatively In-correlated for all $1 \leq i < n$, then we have $E\phi(M_{0,n}^+) \leq 2\,E\phi(\tilde{S}_n^+)$

If $M_{i,n}$ and X_i are negatively In-correlated and $L_{i,n}$ and X_i are negatively In-correlated for all $1 \leq i < n$, then we have $E\phi(\overline{M}_{0,n}) \leq 2\,E\phi(|\tilde{S}_n|)$

5 The Lipschitz' Mixing Coefficient

In [20], Newman and Wright have proved a central limit theorem for associated stationary sequences and in the literature there exists a variety of central limit theorems under various mixing conditions (see for instance [1–3, 5, 6]). In this section, I shall introduce the Lipschitz' mixing coefficient (see (5.2) below) and show that it is closely related to both negative and positive In-correlation and that the Lipschitz' mixing coefficient can used to establish the central limit theorem for sequences of identically distributed random variables and for stationary sequences.

Let $\mathbb{C}$ denote the set of all complex numbers. If $h : \mathbb{R} \to \mathbb{C}$ is a complex-valued function, we define the *Lipschitz' norm* as usual:

$$\|h\|_{\text{Lip}} = \inf\{c \geq 0 \mid |h(x) - h(y)| \leq c\,|x - y| \;\forall\, x, y \in \mathbb{R}\}\,. \tag{5.1}$$

If $U, V : \Omega \to \mathbb{C}$ are complex random variables , we say that the *covariance* of U and V exists if U, V and UV are P-integrable and if so, we define the covariance as usual; that is, $\text{cov}(U, V) := E(U - EU)(V - EV) = E(UV) - (EU)(EV)$.

We let $\text{Lip}_1(\mathbb{R})$ denote the set of all bounded functions $f : \mathbb{R} \to \mathbb{R}$ satisfying $\|f\|_{\text{Lip}} \leq 1$, and if $X, Y : \Omega \to \mathbb{R}$ and are random variables, we define the *Lipschitz' mixing coefficient* of (X, Y) as follows

$$\ell(X, Y) := \sup\{\,|\text{cov}(f(X), f(Y))| \mid f \in \text{Lip}_1(\mathbb{R})\}\,. \tag{5.2}$$

Since $x \curvearrowright \frac{1}{|\alpha|} f(u + \alpha x)$ belongs to $\text{Lip}_1(\mathbb{R})$ for all $f \in \text{Lip}_1(\mathbb{R})$, we have

$$\ell(u + \alpha\, X, u + \alpha\, Y) = |\alpha|^2\, \ell(X, Y) \quad \forall\, u, \alpha \in \mathbb{R}\,. \tag{5.3}$$

Let $f : \mathbb{R} \to \mathbb{C}$ be a bounded Lipschitz' function and let f_1 and f_2 denote the real and imaginary parts of f. Then f_1 and f_2 are bounded Lipschitz' functions. Let A and B denote the real and imaginary parts of $\text{cov}(f(X), f(Y))$ and set $h = f_1 + f_2$ and $g = f_1 - f_2$. Then we have

$$\begin{aligned} A &= \text{cov}(f_1(X), f_1(Y)) - \text{cov}(f_2(X), f_2(Y)) \\ B &= \text{cov}(f_1(X), f_2(Y)) + \text{cov}(f_2(X), f_1(Y)) \\ 2B &= \text{cov}(h(X), h(Y)) - \text{cov}(g(X), g(Y))\,. \end{aligned}$$

Hence, we have $|A| \leq q\,\ell(X, Y)$ and $|B| \leq \frac{r}{2}\,\ell(X, Y)$ where $q = \|f_1\|_{\text{Lip}}^2 + \|f_2\|_{\text{Lip}}^2$ and $r = \|h\|_{\text{Lip}}^2 + \|g\|_{\text{Lip}}^2$. Since $|a| + |b| \leq \sqrt{2}\,\sqrt{a^2 + b^2}$, we have $\|h\|_{\text{Lip}} \leq \sqrt{2}\,\|f\|_{\text{Lip}}$ and $\|g\|_{\text{Lip}} \leq \sqrt{2}\,\|f\|_{\text{Lip}}$. Hence, we have $r \leq 4\,\|f\|_{\text{Lip}}^2$ and since $q \leq 2\,\|f\|_{\text{Lip}}^2$ we have

$$|\text{cov}(f(X), f(Y))| \leq |A| + |B| \leq 4\,\|f\|_{\text{Lip}}^2\,\ell(X, Y) \tag{5.4}$$

for every bounded Lipschitz' function $f : \mathbb{R} \to \mathbb{C}$.

Let (X^*, Y^*) be a symmetrization of the random vector (X, Y), that is, $(X^*, Y^*) \sim (X' - X, Y' - Y)$ where (X', Y') is an independent copy of (X, Y). Let $f \in \text{Lip}_1(\mathbb{R})$ be a given function and set $\pi_f(x, y) = (f(x) - Ef(X))(f(y) - Ef(Y))$ for all $(x, y) \in \mathbb{R}^2$. Then we have $\text{cov}(f(X), f(Y)) = E\pi_f(X, Y)$ and since $f \in \text{Lip}_1(\mathbb{R})$, we have $|\pi_f(x, y)| \leq E(|x - X| \cdot |y - Y|)$ for all $(x, y) \in \mathbb{R}^2$. Integrating this inequality with respect to $P_{(X,Y)}$, we see that

$$|\text{cov}(f(X),f(Y))| \le E(|X^*Y^*|) \quad \forall f \in \text{Lip}_1(\mathbb{R}), \tag{5.5}$$

$$X, Y \in L^2(P) \Rightarrow \ell(X,Y) \le 2\sqrt{\text{var}(X)} \cdot \sqrt{\text{var}(Y)}. \tag{5.6}$$

If X is a random variable, we let $\varphi_X(t) := Ee^{itX}$ denote the *characteristic function* of X for all $t \in \mathbb{R}$.

Theorem 5.1 *Let $n, k \ge 1$ be given integers and let $X = (X_1, \ldots, X_n)$ and $Y = (Y_1, \ldots, Y_k)$ be random vectors such that the covariances* $\text{cov}(X_i, Y_j)$ *exists for all $1 \le i \le n$ and all $1 \le j \le k$. Let $a_1, \ldots, a_n \ge 0$ and $b_1, \ldots, b_k \ge 0$ be non-negative numbers and let $f : \mathbb{R}^n \to \mathbb{R}$ and $h : \mathbb{R}^k \to \mathbb{R}$ be given functions satisfying the following Lipschitz' conditions:*

(a) $|f(x) - f(y)| \le \sum_{i=1}^{n} a_i |x_i - y_i| \quad \forall\, x = (x_1, \ldots, x_n),\ y = (y_1, \ldots, y_n) \in \mathbb{R}^n,$

(b) $|h(u) - h(v)| \le \sum_{j=1}^{k} b_j |u_j - v_j| \quad \forall\, u = (u_1, \ldots, u_k),\ v = (v_1, \ldots, v_k) \in \mathbb{R}^k.$

Then the covariance $\text{cov}(f(X), h(Y))$ *exists and if X and Y are either positively* In-*correlated or negatively* In-*correlated, then we have we have*

(c) $|\text{cov}(f(X), h(Y))| \le \left| \sum_{i=1}^{n} \sum_{j=1}^{k} a_i\, b_j\, \text{cov}(X_i, Y_j) \right|.$

(d) $\ell(X_i, Y_j) = |\text{cov}(X_i, Y_j)| \quad \forall\, 1 \le i \le n\ \forall\, 1 \le j \le k.$

Proof Since the covariances $\text{cov}(X_i, Y_j)$ exists. we have that X_i, Y_j and X_iY_j are P-integrable for all $1 \le i \le n$ and all $1 \le j \le k$ and by (a) and (b), we have $|f(X)| \le |f(0)| + \sum_{1 \le i \le n} a_i\, |X_i|$ and $|h(Y)| \le |h(0)| + \sum_{1 \le j \le k} b_j\, |Y_j|$. Hence, we see that the covariance $\text{cov}(f(X), h(Y))$ exists.

Set $f_0(x) = \sum_{i=1}^{n} a_i x_i$ for all $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $h_0(y) = \sum_{j=1}^{k} b_j y_j$ for all $y = (y_1, \ldots, y_k) \in \mathbb{R}^k$. By (a), we see that $f_1(x) := f_0(x) + f(x)$ and $f_2(x) := f_0(x) - f(x)$ are increasing Borel functions on $\mathbb{R}^n$ and by (b), we see that $h_1(y) := h_0(y) + h(y)$ and $h_2(y) := h_0(y) - h(y)$ are increasing Borel functions on $\mathbb{R}^k$. Let us define $U_\nu := f_\nu(X)$ and $V_\nu := h_\nu(Y)$ for $\nu = 0, 1, 2$. Then we have

$$U_1 = U_0 + f(X),\ V_1 = V_0 + h(Y),\ U_2 = U_0 - f(X),\ V_2 = V_0 - h(Y)$$

and so we have

$$\text{cov}(U_1, V_1) + \text{cov}(U_2, V_2) = 2\,\text{cov}(U_0, V_0) + 2\,\text{cov}(f(X), h(Y)), \tag{i}$$

$$\text{cov}(U_2, V_1) + \text{cov}(U_1, V_2) = 2\,\text{cov}(U_0, V_0) - 2\,\text{cov}(f(X), h(Y)). \tag{ii}$$

Suppose that X and Y are negatively In-correlated. Since f_ν and h_ν are increasing for $\nu = 0, 1, 2$, we see that $\text{cov}(U_0, V_0) \le 0$ and that the covariances on the left hand sides of (i) and (ii) are ≤ 0. Hence, we have $\pm\text{cov}(f(X), h(Y)) \leq -\text{cov}(U_0, V_0)$ and so we have

$$|\text{cov}(f(X), h(Y))| \leq |\text{cov}(U_0, V_0)| = \left| \sum_{i=1}^{n} \sum_{j=1}^{k} a_i b_j \,\text{cov}(X_i, Y_j) \right| .$$

Suppose that X and Y are positively In-correlated. Since f_ν and h_ν are increasing for $\nu = 0, 1, 2$, we see that $\text{cov}(U_0, V_0) \geq 0$ and that the covariances on the left hand sides of (i) and (ii) are ≥ 0. Hence, as above we have

$$|\text{cov}(f(X), h(Y))| \leq |\text{cov}(U_0, V_0)| = \left| \sum_{i=1}^{n} \sum_{j=1}^{k} a_i b_j \,\text{cov}(X_i, Y_j) \right|$$

which completes the proof of (c).

(d): Let $1 \leq i \leq n$ and $1 \leq j \leq k$ be given integers. By (c) we have $\ell(X_i, Y_j) \leq |\text{cov}(X_i, Y_j)|$. Let $n \in \mathbb{N}$ be given and let $\eta_n(x) = (-x) \vee (x \wedge n)$ for $x \in \mathbb{R}$ denote the truncation function. Then we have $\eta_n \in \text{Lip}_1(\mathbb{R})$ and $|\eta_n(x)| = n \wedge |x|$ for all $x \in \mathbb{R}$. In particular, we have $|\eta_n(X_i)| \leq |X_i|$, $|\eta_n(Y_j)| \leq |Y_j|$ and $|\eta_n(X_i)\eta_n(Y_j)| \leq |X_i Y_j|$ and since $\eta_n(x) \to x$ for all $x \in \mathbb{R}$ and X_i, Y_j and $X_i Y_j$ belong to $L^1(P)$, we have by Lebesgue dominated convergence theorem that $\text{cov}(\eta_n(X_i), \eta_n(Y_j)) \to \text{cov}(X_i, Y_j)$. Since $\eta_n \in \text{Lip}_1(\mathbb{R})$, we have $|\text{cov}(\eta_n(X_i), \eta_n(Y_j))| \leq \ell(X_i, Y_j)$ and so we see that $|\text{cov}(X_i, Y_j)| \leq \ell(X_i, Y_j)$ which completes the proof of (d). □

Lemma 5.2 *Let $X_1, \ldots, X_n$ be random variables with partial sums $S_0 = 0$ and $S_k = X_1 + \cdots + X_k$ for $1 \leq k \leq n$. Then we have*

(a) $$\left| \varphi_{S_k}(t) - \prod_{\nu=1}^{k} \varphi_{X_\nu}(t) \right| \leq 4t^2 \sum_{\nu=1}^{k} \ell(S_{\nu-1}, X_\nu) \quad \forall\, t \in \mathbb{R}\ \forall\, 1 \leq k \leq n .$$

Proof Let $t \in \mathbb{R}$ be given. If $k = 1$, then (a) holds trivially. So let $1 \leq k < n$ be a given integer such that (a) holds for this k and set $f(x) = e^{itx}$ for all $x \in \mathbb{R}$. Then f is bounded with $\|f\|_{\text{Lip}} \leq |t|$ and so by (5.4) we have

$$|\varphi_{S_{k+1}}(t) - \varphi_{S_k}(t) \cdot \varphi_{X_{k+1}}(t)| = |\text{cov}(f(S_k), f(X_{k+1}))| \leq 4t^2 \ell(S_k, X_{k+1}) .$$

Recall that $|\varphi_{X_{k+1}}(t)| \leq 1$. So by the induction hypothesis we have

$$\left| \varphi_{S_k}(t)\, \varphi_{X_{k+1}}(t) - \prod_{\nu=1}^{k+1} \varphi_{X_\nu}(t) \right| \leq \left| \varphi_{S_k}(t) - \prod_{\nu=1}^{k} \varphi_{X_\nu}(t) \right| \leq 4t^2 \sum_{\nu=1}^{k} \ell(S_{\nu-1}, X_\nu) .$$

Summing the two inequalities, we see that (a) follows by induction. □

Lemma 5.3 *Let X be a random variable with mean* 0 *and finite variance $v = EX^2$ and set $r(x) = 1 \wedge \frac{|x|}{3}$ for $x \in \mathbb{R}$ and $R_X(t) = E[X^2\, r(tX)]$ for $t \in \mathbb{R}$. Then we have*

(a) $|\varphi_X(t) - (1 - \frac{vt^2}{2})| \leq \frac{t^2}{2}\, R_X(t) \quad \forall\, t \in \mathbb{R}\,.$
(b) $|\varphi_X(\frac{t}{\sqrt{n}})^n - e^{-vt^2/2}| \leq t^2 e^{vt^2}\big(\frac{v^2t^2}{n} + \frac{1}{2}\, R_X(\frac{t}{\sqrt{n}})\big) \quad \forall t \in \mathbb{R}\; \forall\, n \in \mathbb{N}\,.$

Proof By Taylor's formula we have $|e^{ix} - (1 + ix - \frac{x^2}{2})| \leq \frac{x^2}{2}\, r(x)$ and since $EX = 0$ and $EX^2 = v$, we see that (a) holds. To prove (b), I shall need the following inequalities:

$$|e^u - e^v| \leq |u - v|\, e^{\Re u \vee \Re v} \quad \forall\, u, v \in \mathbb{C}\,, \tag{i}$$

$$|e^z - (1 + \tfrac{z}{n})^n| \leq \tfrac{1}{n}\, |z|^2\, e^{|z|} \quad \forall\, z \in \mathbb{C}\; \forall\, n \in \mathbb{N}\,. \tag{ii}$$

Proof of (i) Let $z = x + iy$ be a complex number. By the mean value theorem, we have $|1 - e^x| \leq |x|\, e^{x^+}$ and $0 \leq 1 - \cos y \leq \frac{y^2}{2}$ and so we have

$$|e^z - 1|^2 = (e^x - 1)^2 + 2e^x\,(1 - \cos y) \leq |x|^2\, e^{2x^+} + y^2\, e^x \leq (x^2 + y^2)\, e^{2x^+}\,.$$

Hence, we have $|e^z - 1| \leq |z|\, e^{x^+}$. Let $u, v \in \mathbb{C}$ be given and set $a = \Re u$ and $b = \Re v$. Since $a \vee b = b + (a - b)^+$, we have

$$|e^u - e^v| = e^b\, |e^{u-v} - 1| \leq |u - v|\, e^{b + (a-b)^+} = |u - v|\, e^{a \vee b}$$

which proves (i).

Proof of (ii) Let $z \in \mathbb{C}$ and $n \in \mathbb{N}$ be given. If $n = 1$, then (ii) is easy and well-known. Let $2 \leq j \leq n$ be given. Then we have

$$\frac{j!}{n^j}\binom{n}{j} = \prod_{\nu=1}^{j-1}(1 - \tfrac{\nu}{n}) \geq (1 - \tfrac{j-1}{n})^{j-1}\,,$$

and by the mean value theorem, there exists a number θ such that $1 - \frac{j-1}{n} < \theta < 1$ and

$$1 - \frac{j!}{n^j}\binom{n}{j} \leq 1 - (1 - \tfrac{j-1}{n})^{j-1} = \frac{(j-1)^2}{n}\,\theta^{j-2} \leq \frac{(j-1)^2}{n} \quad \forall\, 2 \leq j \leq n\,.$$

Observe that the left hand side is 0 for $j = 0, 1$. Hence, we have

$$|e^z - (1 + \tfrac{z}{n})^n| \le \sum_{j=0}^{n} \tfrac{|z|^j}{j!}\Big(1 - \tfrac{j!}{n^j!}\tbinom{n}{j}\Big) + \sum_{j=n+1}^{\infty} \tfrac{|z|^j}{j!}$$
$$\le \sum_{j=2}^{n} \tfrac{|z|^j}{j!}\,\tfrac{(j-1)^2}{n} + \sum_{j=n+1}^{\infty} \tfrac{|z|^j}{j!}$$
$$\le \tfrac{|z|^2}{n}\sum_{j=2}^{n} \tfrac{|z|^{j-2}}{(j-2)!} + \tfrac{|z|^2}{n}\sum_{j=n+1}^{\infty} \tfrac{|z|^{j-2}}{(j-2)!} = \tfrac{1}{n}|z|^2 e^{|z|}$$

which proves (ii).

Let $t \in \mathbb{R}$ and $n \in \mathbb{N}$ be given and set $t_n = \frac{t}{\sqrt{n}}$ and $z_n = n\,(\varphi_X(t_n) - 1)$. By (a), we have $|z_n + \frac{v}{2}t^2| \le \frac{t^2}{2}R_X(t_n)$ and since $0 \le R_X(u) \le v$, we have $|z_n| \le v\,t^2$. By (ii), we have

$$|e^{z_n} - \varphi_X(t_n)^n| = |e^{z_n} - (1 + \tfrac{z_n}{n})^n| \le \tfrac{1}{n}\,|z_n|^2 e^{|z_n|} \le \tfrac{v^2t^4}{n} e^{vt^2}\,,$$

and by (i), we have

$$|e^{z_n} - e^{-vt^2/2}| \le |z_n + \tfrac{v}{2}t^2|\, e^{(-vt^2/2)\vee \Re z_n} \le \tfrac{t^2}{2}e^{vt^2}\, R_X(t_n)\,.$$

Hence, we have

$$|\varphi_X(t_n)^n - e^{-vt^2/2}| \le t^2 e^{vt^2}\Big(\tfrac{v^2t^2}{n} + \tfrac{1}{2}\,R_X(t_n)\Big)$$

which proves (b). □

Theorem 5.4 *Let $X_1, X_2, \ldots \in L^2(P)$ be identically distributed, random variables with mean 0 and variance v and let $S_0 = 0$ and $S_n = X_1 + \cdots + X_n$ denote the partial sums for $n \ge 1$. Then we have*

(a) $\lim_{n\to\infty}\Big(\frac{1}{n}\sum_{j=1}^{n} \ell(S_{j-1}, X_j)\Big) = 0 \;\Rightarrow\; \frac{S_n}{\sqrt{n}} \overset{\sim}{\to} N(0, v)\,.$

Proof Let $\phi(t) = \varphi_{X_1}(t)$ and $\phi_n(t) = \varphi_{S_n}(t)$ denote the characteristic functions of the X_1 and S_n. By the classical central limit theorem we have $\phi(\frac{t}{\sqrt{n}})^n \to e^{-vt^2/2}$ for all $t \in \mathbb{R}$ and by Lemma 5.2 we have

$$|\phi_n(\tfrac{t}{\sqrt{n}}) - \phi(\tfrac{t}{\sqrt{n}})^n| \le 4t^2\tfrac{1}{n}\sum_{j=1}^{n} \ell(S_{j-1}, X_j)\,,$$

which proves the theorem. □

Remark 5.5 Recall that the sequence (a_n) tends to a in *Cesàro mean* if and only if $\frac{1}{n}\sum_{k=1}^{n} a_k \to a$. Let $X_1, X_2, \ldots \in L^2(P)$ be identically distributed (not necessarily independent) random variables with mean 0 and variance v. Then Theorem 5.4 shows that the central limits theorem holds if the Lipschitz' mixing coefficients $\ell(S_n, X_{n+1})$ tends to 0 in Cesàro mean and recall that we have $\ell(S_n, X_{n+1}) = |\mathrm{cov}(S_n, X_{n+1})|$ if S_n and X_{n+1} are either positively In-correlated or negatively In-correlated.

Theorem 5.6 *Let $(X_n)_{n\ge 1} \subseteq L^2(P)$ be a strictly stationary sequence, let $S_0 = 0$ and $S_n = X_1 + \cdots + X_n$ for $n \ge 1$ denote the partial sums and set $v(0) = 0$ and*

$$\rho(n) = \mathrm{cov}(X_1, X_{n+1}) \ , \ v(n) = \mathrm{var}\, S_n \ , \ C_{n,k} = \sum_{j=1}^{n} \ell(S_{(j-1)k}\, , S_{jk} - S_{(j-1)k})$$

for all $n, k \ge 1$. Then we have

(a) $\frac{v(n)}{n} = v(1) + 2\sum_{k=1}^{n} (1 - \frac{k}{n})\, \rho(k) \quad \forall\, n, k \ge 1$,

(b) $\sum_{j=1}^{n} \mathrm{cov}(S_{(j-1)k}\, , S_{jk} - S_{(j-1)k}) = \frac{1}{2}\,(v(nk) - n\, v(k)) \quad \forall\, n, k \ge 1$,

(c) $C_{n,k} \le 2\sqrt{v(k)} \sum_{j=0}^{n-1} \sqrt{v(jk)} \quad \forall\, n, k \ge 1$.

Suppose that $EX_1 = 0$ and let $\sigma^2 \ge 0$ be a non-negative number satisfying

(d) $\liminf_{k\to\infty} \left(\limsup_{n\to\infty} \frac{C_{n,k}}{nk} \right) = 0$ *and* $\lim_{k\to\infty} \frac{v(k)}{k} = \sigma^2$,

Then we have $\frac{S_n}{\sqrt{n}} \stackrel{\sim}{\to} N(0, \sigma^2)$ and if $\sigma^2 > 0$, we have $\frac{S_n}{\sqrt{v(n)}} \stackrel{\sim}{\to} N(0, 1)$.

Proof (a) and (b) are easy consequences of (weak) stationarity and since $v(k) = \mathrm{var}(S_{n+k} - S_k)$ for all $n, k \ge 0$, we see that (c) follows from (5.6).

So suppose that $EX_1 = 0$ and that (d) holds. Set $U_n = n^{-1/2}\, S_n$ for $n \in \mathbb{N}$ and let $\phi_n = \varphi_{S_n}$ and $\psi_n = \varphi_{U_n}$ denote the characteristic functions of S_n and U_n for all $n \ge 1$. Then we have $\psi_n(t) = \phi_n(\frac{t}{\sqrt{n}})$ for all $t \in \mathbb{R}$ and all $n \in \mathbb{N}$.

Let $t \in \mathbb{R}$ and $0 < \delta < 1$ be given numbers. By (d), there exists an integer $k \ge 1$ (which will be fixed for the rest of the proof) such that

$$|\sigma^2 - \tfrac{v(k)}{k}| \le \tfrac{\delta}{1+t^2} \ \text{ and } \ \limsup_{n\to\infty} \tfrac{C_{n,k}}{nk} < \tfrac{\delta}{1+16\,t^2} \, . \tag{i}$$

Set $X_j^k = S_{jk} - S_{(j-1)k}$ for $j \ge 1$. Since (X_i) is strictly stationary, we see that $X_1^k, X_2^k, \ldots$ are identically distributed with common characteristic function ϕ_k and partial sums

$X_1^k + \cdots + X_j^k = S_{jk}$. So by Lemma 5.2 we have

$$|\psi_{mk}(t) - \psi_k(\tfrac{t}{\sqrt{m}})^m| = \left|\phi_{mk}(\tfrac{t}{\sqrt{mk}}) - \phi_k(\tfrac{t}{\sqrt{mk}})^m\right|$$
$$\leq 4t^2 \tfrac{1}{mk} \sum_{j=1}^{m} \ell(S_{k(j-1)}, S_{kj} - S_{k(j-1)}) = 4t^2 \tfrac{C_{m,k}}{mk}$$

for all $m \geq 1$. Let $m \geq 1$ be given and let $R_{U_k}(s)$ be defined as in Lemma 5.3. Since $EU_k = 0$ and $EU_k^2 = \frac{v(k)}{k}$, we have by Lemma 5.3

$$\left|\psi_k(\tfrac{t}{\sqrt{m}})^m - e^{-\frac{v(k)}{2k}t^2}\right| \leq t^2 e^{\frac{v(k)}{k}t^2} \left(\tfrac{v(k)^2}{m k^2} t^2 + \tfrac{1}{2} R_{U_k}(\tfrac{t}{\sqrt{m}})\right).$$

Note that $R_{U_k}(s)$ is continuous with $R_{U_k}(0) = 0$. So by (i) there exists an integer $m_k \geq 1$ such that

$$|\psi_{mk}(t) - \psi_k(\tfrac{t}{\sqrt{m}})^m| < \tfrac{\delta}{4} \quad \forall\, m \geq m_k$$
$$\left|\psi_k(\tfrac{t}{\sqrt{m}})^m - e^{-\frac{v(k)}{2k}t^2}\right| \leq \tfrac{\delta}{4} \quad \forall\, m \geq m_k$$

Since $|\sigma^2 - \frac{v(k)}{k}| \leq \frac{\delta}{1+t^2}$, we have $|e^{-\frac{v(k)}{2k}t^2} - e^{-\sigma^2 t^2/2}| \leq \frac{t^2}{2}|\sigma^2 - \frac{v(k)}{k}| \leq \frac{\delta}{2}$ and so we have $|\psi_{mk}(t) - e^{-\sigma^2 t^2/2}| \leq \delta$ for all $m \geq m_k$.

Set $q = 1 + E|tX_1|$ and $r = 4(qk)^2\delta^{-2}$. Let $n > r + k\,m_k$ be given and set $m = \lfloor \frac{n}{k} \rfloor$. Then we have $0 \leq n - mk < k$ and by the mean value theorem, we have

$$0 \leq \sqrt{n} - \sqrt{mk} \leq (n - mk)\,\tfrac{1}{2\sqrt{mk}} \leq \tfrac{k}{2\sqrt{mk}}.$$

Since $E|tX_i| = E|tX_1| \leq q$, we have $E|t\,S_{mk}| \leq qmk$ and $E|t(S_n - S_{mk})| \leq q\,(n - mk) \leq qk$ and since $|e^{ix} - e^{iy}| \leq |x - y|$ and $n \geq r$, we have

$$|\psi_n(t) - \psi_{mk}(t)| \leq |t|\,E|U_n - U_{mk}| \leq \tfrac{t}{\sqrt{n}}E|t(S_n - S_{mk})| + \tfrac{\sqrt{n}-\sqrt{mk}}{\sqrt{nmk}}E|tS_{mk}|$$
$$\leq q\,\tfrac{k}{\sqrt{n}} + q\,\tfrac{k}{2\sqrt{n}} \leq \tfrac{2qk}{\sqrt{n}} \leq \delta,$$

Since $n > k\,m_k$, we have $m \geq m_k$ and so we have $|\psi_{mk}(t) - e^{-\sigma^2 t^2/2}| < \delta$ and

$$|\psi_n(t) - e^{-\sigma^2 t^2/2}| \leq |\psi_n(t) - \psi_{mk}(t)| + |\psi_{mk}(t) - e^{-\sigma^2 t^2/2}| < 2\delta$$

for all $n > r + k\,m_k$. Hence, we have $U_n \overset{\sim}{\to} N(0, \sigma^2)$ and since $\frac{v(n)}{n} \to \sigma^2$, we have $\frac{S_n}{\sqrt{v(n)}} \overset{\sim}{\to} N(0, 1)$ if $\sigma^2 > 0$. □

Remark 5.7

(1): Recall that the series $\sum_{n=1}^{\infty} a_n$ is *Cesàro summable* with *Cesàro sum A* if and only if the partial sums $a_1 + \cdots + a_n$ tends to A in Cesáro mean or equivalent if $\sum_{k=1}^{n} (1 - \frac{k}{n})\, a_k \to A$. By (a), we see that $\lim_{n\to\infty} \frac{v(n)}{n} = \sigma^2$ if and only if $\sum_{n=1}^{\infty} \rho(n)$ is Cesàro summable with Cesàro sum $\frac{\sigma^2 - v(1)}{2}$.

(2): Suppose that $\rho(n) \geq 0$ for all $n \geq 1$. By (a), we see that $(\frac{v(n)}{n})_{n\geq 1}$ is increasing and $\lim_{n\to\infty} \frac{v(n)}{n} = v(1) + 2 \sum_{k=1}^{\infty} \rho(k)$.

(3): Suppose that $\rho(n) \leq 0$ for all $n \geq 1$. By (a), we see that $(\frac{v(n)}{n})_{n\geq 1}$ is decreasing and we have $\sum_{k=1}^{\infty} |\rho(k)| \leq \frac{v(1)}{2}$ and $\lim_{n\to\infty} \frac{v(n)}{n} = v(1) + 2 \sum_{k=1}^{\infty} \rho(k)$.

(4): Suppose that S_n and $S_{n+k} - S_n$ are positively In-correlated or negatively In-correlated for every $(n,k) \in \mathbb{N}^2$. Then Theorem 5.1 shows $\frac{C_{n,k}}{nk} = \frac{1}{2}\,(\frac{v(nk)}{nk} - \frac{v(k)}{k})$ for all $n, k \geq 1$ and so we see that condition (d) holds if and only if $\lim_{k\to\infty} \frac{v(k)}{k} = \sigma^2$.

Appendix

In this appendix, I shall give a purely analytic solution to a certain recursive, functional inequality which is closely linked to the Rademacher-Menchoff inequalities of Sect. 2. But first let me prove the following simple lemma.

Lemma A.1 *Let $(g_{i,j})_{(i,j)\in\mathbf{\Delta}_0}$ be a triangular schemes of non-negative numbers. Let $(i,j) \in \mathbf{\Delta}_2$ be a given pair and let $a > 0$ and $h \geq 0$ be given numbers satisfying*

$$g_{i,i} \vee (a\, g_{j,j}) \leq h \ \ \textit{and} \ \ \max_{i<k<k} (g_{i,k} + g_{k,j}) \leq (1 + \tfrac{1}{a})\, h\,.$$

Then we have $\min_{i<k\leq j} (g_{i,k-1} \vee (a\, g_{k,j})) \leq h$.

Proof I shall split the proof in three cases:

Case 1 : $g_{i,j-1} \leq h$. Since $a\, g_{j,j} \leq h$, we have $g_{i,j-1} \vee (a\, g_{j,j}) \leq h$.

Case 2 : $g_{i,i+1} \leq h < g_{i,j-1}$. Then there exists an integer $i < k < j$ such that $g_{i,k-1} \leq h \leq g_{i,k}$ and since $h + g_{k,j} \leq g_{i,k} + g_{k,j} \leq (1 + \frac{1}{a})\, h$, we have $g_{i,k-1} \vee (a\, g_{k,j}) \leq h$.

Case 3 : $g_{i,i+1} > h$. Since $j \geq 2$, we have $i < i+1 < j$ and so we have $h + g_{i+1,j} \leq g_{i,i+1} + g_{i+1,j} \leq (1 + \frac{1}{a})\, h$ and since $g_{i,i} \leq h$, we have $g_{i,i} \vee (a\, g_{i+1,j}) \leq h$.

Since the three cases exhaust all possibilities, we see that there exists an integer $i < k \leq j$ such that $g_{i,k-1} \vee (a\, g_{k,j}) \leq h$. □

Proposition A.2 *Let $p, q \in \mathbb{R}_+$ be given numbers and let $D \subseteq \mathbb{R}_+$ be a non-empty set such that $pt \in D$ and $qt \in D$ for all $t \in D$. Let $\Gamma : \mathbb{R}_+^2 \to \mathbb{R}_+$, be an increasing homogeneous function and set $\Gamma(x, \infty) = \Gamma(\infty, x) = \infty$ for all $x \in [0, \infty]$. Let $A_{i,j}, B_{i,j}, V_{i,j} : D \to [0, \infty]$ be given functions for $(i,j) \in \mathbf{\Delta}_0$ and let $h : \mathbb{N}_0 \to \mathbb{R}_+$ and $\xi : \mathbb{N}_0 \to \mathbb{N}_0$ be increasing functions such that $\xi(0) = 0$ and*

(a) $A_{i,j}(t) \le \Gamma(A_{k,j}(pt) + A_{i,k-1}(qt), B_{i,k}(t)) \quad \forall (i,k,j,t) \in \nabla \times D\,,$
(b) $A_{i,i}(t) \le h(0)\, V_{i,i}(t)$ *and* $V_{i,j}(t) \le V_{i,j+1}(t) < \infty \quad \forall\, (i,j,t) \in \mathbf{\Delta}_0 \times D\,,$
(c) $B_{i,j}(t) \le h(\xi(j-i))\, V_{i,j}(t) \quad \forall\, (i,j,t) \in \mathbf{\Delta}_0 \times D\,.$

Then we have $A_{i,j}(t) < \infty$ and $B_{i,j}(t) < \infty$ for all $(i,j,t) \in \mathbf{\Delta}_0 \times D$. Let $s \ge 0$ be a given number satisfying $h(0) \le \Gamma(s\, h(0), h(1))$ and let us define

$$\Upsilon_s = \{(i,j,t) \in \mathbf{\Delta}_1 \times D \mid h_s^\Gamma(\xi(j-i))\, V_{i,j}(t) < A_{i,j}(t)\}\,.$$

Then h_s^Γ is increasing and if

(d) $\min\limits_{k \in D_{i,j}^\xi} (V_{i,k-1}(qt) + V_{k,j}(pt)) \le s\, V_{i,j}(t) \quad \forall\, (i,j,t) \in \Upsilon_r\,,$

then we have $A_{i,j}(t) \le h_s^\Gamma(\xi(j-i))\, V_{i,j}(t)$ for all $(i,j,t) \in \mathbf{\Delta}_0 \times D$.

Proof By (b) and (c), we have $A_{i,i}(t) < \infty$ for all $(i,t) \in \mathbb{N}_0 \times D$ and $B_{i,j}(t) < \infty$ for all $(i,j,t) \in \mathbf{\Delta}_0 \times D$. Let $n \ge 0$ be a given integer such that $A_{i,i+n}(t) < \infty$ for all $(i,t) \in \mathbb{N}_0 \times D$ and let $(i,t) \in \mathbb{N}_0 \times D$ be given. Since $pt \in D$ and $qt \in D$, we have $A_{i,i+n}(qt) + A_{i+n+1,i+n+1}(pt) < \infty$. Hence, by (a), we see that $A_{i,i+n+1}(t) < \infty$ and so by induction, we have $A_{i,j}(t) < \infty$ for all $(i,j,t) \in \mathbf{\Delta}_0 \times D$.

Suppose that $h(0) \le \Gamma(s\, h(0), h(1))$. By (2.6), we have $h_s^\Gamma(0) = h(0) \le \Gamma(s\, h(0), h(1)) = h_r^\Gamma(1)$. Let $n \ge 1$ be a given integer such that $h_s^\Gamma(n-1) \le h_s^\Gamma(n)$. By (2.6), we have $h_s^\Gamma(n+1) = \Gamma(s\, h_s^\Gamma(n), h(n+1))$ and since Γ and h are increasing we have $h_s^\Gamma(n+1) \ge \Gamma(s\, h_s^\Gamma(n-1)\,, h(n)) = h_s^\Gamma(n)$. So by induction, we see that h_s^Γ is increasing.

Suppose in addition that (d) holds. Since $\xi(0) = 0$, we have $h(0) = h_s^\Gamma(\xi(0))$ and so by (b) we have $A_{i,j}(t) \le h_s^\Gamma(\xi(j-i))\, V_{i,j}(t)$ for all $(i,j,t) \in \mathbf{\Delta}^0 \times D$. Let $n \ge 0$ be a given integer such that $A_{i,j}(t) \le f_s^\Gamma(\xi(j-i))\, V_{i,j}(t)$ for all $(i,j,t) \in \mathbf{\Delta}^n \times D$. Let $(i,j,t) \in \mathbf{\Delta}^{n+1} \times D$ be given and let me show that $A_{i,j}(t) \le h_s^\Gamma(\xi(j-i))\, V_{i,j}(t)$.

If $j - i \le n$, this follows from the induction hypothesis and if $A_{i,j}(t) \le h_s^\Gamma(\xi(j-i))\, V_{i,j}(t)$, this holds trivially. So suppose that $j - i = n+1$ and $h_s^\Gamma(\xi(j-i))\, V_{i,j}(t) < A_{i,j}(t)$ and set $\nu = \xi(n+1)$. Since $j - i = n + 1 \ge 1$, we have $(i,j,t) \in \Upsilon_s$. Recall that $\min_{k \in \emptyset} a_k = \infty$ and $V_{i,j}(t) < \infty$. So by (d) there exists $k \in D_{i,j}^\xi$ such that $V_{i,k-1}(qt) + V_{k,j}(pt) \le s\, V_{i,j}(t)$.

Since $k \in D_{i,j}^\xi$, we have $\xi(j-k) \vee \xi(k-i-1) \le \nu - 1$ and since h_s^Γ is increasing, we have $h_s^\Gamma(\xi(j-k)) \le h_s^\Gamma(\nu-1)$ and $h_s^\Gamma(\xi(k-i-1)) \le h_s^\Gamma(\nu-1)$. Since h and ξ are increasing and $k - i \le n+1$, we have $h(\xi(k-i)) \le h(\nu)$ and since $i < k \le j$ and $n + 1 = j - i$, we have $(i, k-1) \in \mathbf{\Delta}^n$ and $(k,j) \in \mathbf{\Delta}^n$. Since $t \in D$, we have $pt, qt \in D$ and so by induction hypothesis, we have $A_{i,k-1}(qt) \le h_s^\Gamma(\nu-1))\, V_{i,k-1}(qt)$

and $A_{k,j}(pt) \leq h_s^{\Gamma}(\nu-1)\, V_{k,j}(pt)$. Since $V_{i,k-1}(qt) + V_{k,j}(pt)) \leq s\, V_{i,j}(t)$, we have

$$A_{i,k-1}(qt) + A_{k,j}(pt) \leq s\, h_s^{\Gamma}(\nu-1)\, V_{i,j}(t)\,.$$

By (c), we have $B_{i,k}(t) \leq h(\xi(k-i))\, V_{i,k}(t)$ and since ξ and h are increasing and $k-i \leq j-i = n+1$, we have $h(\xi(k-i)) \leq h(\nu)$. So we have $B_{i,k}(t) \leq h(\nu)\, V_{i,k}(t)$ and by (a) and homogeneity and monotonicity of Γ we have

$$\begin{aligned} A_{i,j}(t) &\leq \Gamma(A_{i,k-1}(qt) + A_{k,j}(pt), B_{i,k}(t)) \\ &\leq \Gamma(s\, h_s^{\Gamma}(\nu-1),\, h(\nu))\, V_{i,j}(t) = h_s^{\Gamma}(\nu)\, V_{i,j}(t)\,. \end{aligned}$$

Hence, by induction we see that $A_{i,j}(t) \leq h_s^{\Gamma}(\xi(j-i))\, V_{i,j}(t)$ for all $(i,j,t) \in \mathbf{\Delta}_0 \times D$. □

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On the Order of the Central Moments of the Length of the Longest Common Subsequences in Random Words

Christian Houdré and Jinyong Ma

Abstract We investigate the order of the r-th, $1 \le r < +\infty$, central moment of the length of the longest common subsequences of two independent random words of size n whose letters are identically distributed and independently drawn from a finite alphabet. When all but one of the letters are drawn with small probabilities, which depend on the size of the alphabet, a lower bound is shown to be of order $n^{r/2}$. This result complements a generic upper bound also of order $n^{r/2}$.

Keywords Burkholder inequality • Efron-Stein inequality • Last passage percolation • Longest common subsequence • r-th central moment

Mathematics Subject Classification (2010). 60K35; 60C05; 05A05

1 Introduction and Statements of Results

Let $X = (X_i)_{i \ge 1}$ and $Y = (Y_i)_{i \ge 1}$ be two independent sequences of iid random variables taking their values in a finite alphabet $\mathcal{A}_m = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$, $m \ge 2$, with $\mathbb{P}(X_1 = \alpha_k) = \mathbb{P}(Y_1 = \alpha_k) = p_k$, $k = 1, 2, \ldots, m$. Let also LC_n be the length of the longest common subsequence of the random words $X_1 \cdots X_n$ and $Y_1 \cdots Y_n$, i.e., $LC_n := LC_n(X_1 \cdots X_n; Y_1 \cdots Y_n)$ is the largest k such that there exist $1 \le i_1 < i_2 < \cdots < i_k \le n$ and $1 \le j_1 < j_2 < \cdots < j_k \le n$, with $X_{i_s} = Y_{j_s}$, $s = 1, \ldots, k$.

The study of the asymptotic behavior of LC_n has a long history starting with the well known result of Chvátal and Sankoff [5] asserting that

$$\lim_{n \to \infty} \frac{\mathbb{E} LC_n}{n} = \gamma_m^*. \tag{1.1}$$

C. Houdré (✉) • J. Ma
School of Mathematics, Georgia Institute of Technology, Atlanta, GA, 30332-0160, USA
e-mail: houdre@math.gatech.edu; jinyma@gmail.com

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However, to this day, the exact value of γ_m^* (which depends on the distribution of X_1 and on the size of the alphabet) is still unknown even in "simple cases" such as for uniform Bernoulli random variables. This first asymptotic result was sharpened by Alexander [1] who showed that

$$\gamma_m^* n - K_A \sqrt{n \log n} \leq \mathbb{E} LC_n \leq \gamma_m^* n, \tag{1.2}$$

where $K_A > 0$ is a constant depending neither on n nor on the distribution of X_1. Next, Steele [13] was the first to investigate the order of the variance proving, in particular, that $VarLC_n \leq n$. However, finding the order of the lower bound is more illusive. For Bernoulli random variables and in various instances where there is a strong "bias" such as high asymmetry or mixed common and increasing subsequence problems, the lower bound is also shown to be of order n [6, 8, 9]. The uniform case is still unresolved and tight lower variance estimates seem to be lacking (however, see [2, 3], where a situation "as close as we want" to uniformity is treated).

Below, starting with a generic upper bound, we investigate the order of the r-th, $r \geq 1$, central moment of LC_n in case of finite alphabets (of course, as far as the order is concerned only the case $1 \leq r \leq 2$ is really of interest for this lower bound).

The upper bound obtained in [13] relies on an asymmetric version of the Efron-Stein inequality which can be viewed as a tensorization property of the variance. The symmetric Efron-Stein inequality has seen a generalization, due to Rhee and Talagrand [12], to the r-th moment where it is, in turn, viewed as a consequence of Burkholder's square function inequality. As described next, in the asymmetric case, a similar extension also holds thus providing a generic upper bound on the r-th central moment of LC_n. First, let $S : \mathbb{R}^n \to \mathbb{R}$ be a Borel function and let $(Z_i)_{1 \leq i \leq n}$ and $(\hat{Z}_i)_{1 \leq i \leq n}$ be two independent families of iid random variables having the same law. Now, and with suboptimal notation, let $S = S(Z_1, Z_2, \ldots, Z_n)$, and let $S_i = S(Z_1, Z_2, \ldots, Z_{i-1}, \hat{Z}_i, Z_{i+1}, \ldots, Z_n)$, $1 \leq i \leq n$. Then, as shown next, for any $r \geq 2$,

$$\|S - \mathbb{E}S\|_r := (\mathbb{E}|S - \mathbb{E}S|^r)^{1/r} \leq \frac{r-1}{2^{1/r}} \left(\sum_{i=1}^{n} \|S - S_i\|_r^2 \right)^{1/2}. \tag{1.3}$$

Indeed, for $i = 1, \ldots, n$, let $\mathcal{F}_i = \sigma(Z_1, \ldots, Z_i)$ be the σ-field generated by $Z_1, \ldots, Z_i$, let $\mathcal{F}_0 = \{\Omega, \emptyset\}$ be trivial, and let $d_i := \mathbb{E}(S|\mathcal{F}_i) - \mathbb{E}(S|\mathcal{F}_{i-1})$. Thus, $(d_i, \mathcal{F}_i)_{1 \leq i \leq n}$ is a martingale differences sequence and from Burkholder's square function inequality, with optimal constant, e.g., see [11], for $r \geq 2$,

$$\|S - \mathbb{E}S\|_r = \left\| \sum_{i=1}^{n} d_i \right\|_r \leq (r-1) \left\| \left(\sum_{i=1}^{n} d_i^2 \right)^{1/2} \right\|_r \leq (r-1) \left(\sum_{i=1}^{n} \|d_i^2\|_{r/2} \right)^{1/2}. \tag{1.4}$$

Moreover, and as in [12], letting $\mathcal{G}_i = \sigma(Z_1, Z_2, \ldots, Z_i, \hat{Z}_i)$, $1 \le i \le n$,

$$\begin{aligned}\mathbb{E}|S - S_i|^r &= \mathbb{E}(\mathbb{E}(|S - S_i|^r|\mathcal{G}_i)) \\ &\ge \mathbb{E}(|\mathbb{E}(S|\mathcal{G}_i) - \mathbb{E}(S|\mathcal{F}_{i-1}) + \mathbb{E}(S_i|\mathcal{F}_{i-1}) - \mathbb{E}(S_i|\mathcal{G}_i)|^r) \\ &:= \mathbb{E}|U + V|^r, \end{aligned} \tag{1.5}$$

where $U = \mathbb{E}(S|\mathcal{G}_i) - \mathbb{E}(S|\mathcal{F}_{i-1})$ and $V = \mathbb{E}(S_i|\mathcal{F}_{i-1}) - \mathbb{E}(S_i|\mathcal{G}_i)$. But, given $\mathcal{F}_{i-1}$, U and V are independent, with moreover $\mathbb{E}(U|\mathcal{F}_{i-1}) = \mathbb{E}(V|\mathcal{F}_{i-1}) = 0$ and $\mathbb{E}|U|^r = \mathbb{E}|V|^r = \mathbb{E}|d_i|^r$, thus,

$$\mathbb{E}|U + V|^r = \mathbb{E}(\mathbb{E}(|U + V|^r|\mathcal{F}_{i-1})) \ge \mathbb{E}|U|^r + \mathbb{E}|V|^r = 2\mathbb{E}|d_i|^r, \tag{1.6}$$

using the calculus inequality, valid for any $r \ge 2$, $u \in \mathbb{R}$ and $v \in \mathbb{R}$, $|u+v|^r \ge |u|^r + r sign(u)|u|^{r-1}v + |v|^r$, and taking conditional expectations. Combining (1.4), (1.5) and (1.6) gives (1.3).

Next, apply (1.3) to LC_n viewed as a function of the $2n$ random variables $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ and note, at first, that replacing X_i (resp. Y_i) by an independent copy $\hat{X}_i$ (resp. $\hat{Y}_i$), changes $|LC_n - LC_n(X_1 \cdots \hat{X}_i \cdots X_n; Y_1 \cdots Y_n)|$ (resp. $|LC_n - LC_n(X_1 \cdots X_n; Y_1 \cdots \hat{Y}_i \cdots Y_n)|$) by at most 1. Thus, following Steele [13] and for each $i = 1, \ldots, n$,

$$\begin{aligned}&||LC_n - LC_n(X_1 \cdots \hat{X}_i \cdots X_n; Y_1 \cdots Y_n)||_r^2 \\ &= \Big(\mathbb{E}(|LC_n - LC_n(X_1 \cdots \hat{X}_i \cdots X_n; Y_1 \cdots Y_n)|^r \mathbf{1}_{X_i \ne \hat{X}_i})\Big)^{2/r} \\ &\le \Big(\mathbb{P}(X_i \ne \hat{X}_i)\Big)^{2/r} = \left(1 - \sum_{k=1}^m p_k^2\right)^{2/r}. \end{aligned} \tag{1.7}$$

Combining (1.7), and its version for $(Y_i)_{1 \le i \le n}$, with (1.3) yields, for any $r \ge 2$,

$$\mathbb{E}|LC_n - \mathbb{E}LC_n|^r \le \frac{(r-1)^r}{2}\left(1 - \sum_{k=1}^m p_k^2\right)(2n)^{r/2}, \tag{1.8}$$

which further yields,

$$\mathbb{E}|LC_n - \mathbb{E}LC_n|^r \le \left(\left(1 - \sum_{k=1}^m p_k^2\right) n\right)^{r/2},$$

for any $0 < r \le 2$, by the Cauchy-Schwarz inequality.

Therefore, (1.8) provides an upper bound whose order could also be obtained, in a simpler way, by integrating out the tail inequality given via Hoeffding's exponential martingale inequality. Let us now state the main result of the paper which provides a lower bound on the r-th central moment of LC_n, when all but one of the symbols are drawn with very small probabilities.

Theorem 1.1 *Let $1 \leq r < +\infty$, and let $(X_i)_{i\geq 1}$ and $(Y_i)_{i\geq 1}$ be two independent sequences of iid random variables with values in $\mathcal{A}_m = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$, $m \geq 2$, such that $\mathbb{P}(X_1 = \alpha_k) = p_k$, $k = 1, 2, \ldots, m$. Further, let $j_0 \in \{1, \ldots, m\}$ be such that $\max_{j\neq j_0} p_j \leq \min(2^{-2}e^{-5}K_m/m, K_m/2m^2)$, where $K_m = \min(K, 1/800m)$ and $K = 2^{-4}10^{-2}e^{-67}$. Then, there exists a constant $C > 0$ depending on r, m, p_{j_0} and $\max_{j\neq j_0} p_j$, such that, for all $n \geq 1$,*

$$\mathbb{M}_r(LC_n) := \mathbb{E}\,|LC_n - \mathbb{E}LC_n|^r \geq Cn^{\frac{r}{2}}. \tag{1.9}$$

An estimate on the constant C present in (1.9) is given in Remark 2.1.

In contrast to [6, 8] or [9] which deal only with binary words, our results are proved for alphabets of arbitrary, but fixed size m, and are thus novel in that context as well even for the variance, i.e., $r = 2$. Moreover, our results are no longer existential, but provide precise constants depending on the alphabet size. As well known, e.g., see [2, 3], the LCS problem is a last passage percolation (LPP) problem with strictly increasing paths and dependent weights and, therefore, in our context, the order of the variance is linear. For the LPP problem with independent weights the variance is conjectured to be sublinear. In view of (1.8) and (1.9), it is tempting to conjecture, and we do so, that when properly centered (by $\gamma_m^* n$) and normalized (by $\sqrt{n}$), asymptotically, LC_n has a normal component. (The limiting law is in fact normal, see [7].) This conjecture might appear surprising since in LPP with independent weights different limiting laws are conjectured and have been proved to be such in the closely related Bernoulli matching model [10]. It should finally also be noted that, as seen in [4], with another closely related model, the order $n^{r/2}$ on the central moments does not guarantee normal convergence, but nevertheless a normal component is present.

As for the content of the rest of paper, Sect. 2 presents a proof of Theorem 1.1 which relies on a key preliminary result, Theorem 2.1, whose proof is given in Sect. 3.

2 Proof of Theorem 1.1

The strategy of proof to obtain the lower bound is to first represent LC_n as a random function of the number of most probable letters α_{j_0}. In turn, this random function locally satisfies a reversed Lipschitz condition which ultimately gives the lower bound in Theorem 1.1. This methodology extends, modifies and simplifies (and at times corrects) the binary strategy of proof of [6] or [9] providing also a more quantitative result.

To start, and as in [6], pick a letter equiprobably at random from all the non-α_{j_0} letters in either one of the two finite sequences, of length n, X or Y (*Throughout the* *paper, by finite sequences X and Y, of length n, it is meant that* $X = (X_i)_{1\le i\le n}$ *and* $Y = (Y_i)_{1\le i\le n}$). Next, change it to the most probable letter α_{j_0} and call the two new finite sequences $\tilde{X}$ and $\tilde{Y}$. Then the length of the longest common subsequence of $\tilde{X}$ and $\tilde{Y}$, denoted by $\widetilde{LC_n}$, tends, on an event of high probability, to be larger than LC_n. This is the content of the following theorem which is proved in the next section.

Theorem 2.1 *Let the hypothesis of Theorem 1.1 hold. Then, for all* $n \ge 1$, *there exists a set* $\mathcal{B}_n \subset \mathcal{A}_m^n \times \mathcal{A}_m^n$, *such that,*

$$\mathbb{P}\left((X,Y) \in \mathcal{B}_n\right) \ge 1 - 125 \exp\left(-\frac{n(\max_{j\neq j_0} p_j)^6}{5}\right), \tag{2.1}$$

and such that for all $(x, y) \in \mathcal{B}_n$,

$$\mathbb{P}(\widetilde{LC_n} - LC_n = 1|X = x, Y = y) \ge \frac{K}{m}, \tag{2.2}$$

$$\mathbb{P}(\widetilde{LC_n} - LC_n = -1|X = x, Y = y) \le \frac{K}{2m}, \tag{2.3}$$

where $K = 2^{-4}10^{-2}e^{-67}$.

As already mentioned, the proof of Theorem 2.1 is given in the next section, let us nevertheless indicate how it leads to the lower bound on $\mathbb{M}_r(LC_n)$ given in Theorem 1.1. In fact, the arguments leading to the conclusion of Theorem 1.1 remain valid under any hypotheses for which the conclusions of Theorem 2.1 remain valid.

From now on, assume without loss of generality that $p_1 > 1/2$ *and that* $p_2 = \max_{2\le j\le m} p_j$, *so that* α_1 *is the most probable letter and* α_2 *the second most probable one.*

To begin with, let us present a few definitions. For the two finite random sequences $X = (X_i)_{1\le i\le n}$ and $Y = (Y_i)_{1\le i\le n}$, let N_1 be the total number of letters α_1 present in both sequences, i.e., N_1 is a binomial random variable with parameters $2n$ and p_1. Next, by induction, define a finite collection of pairs of finite random sequences $(X^k, Y^k)_{0\le k\le 2n}$, which are independent of X and Y, and therefore independent of N_1, as follows: First, let $X^0 = (X_i^0)_{1\le i\le n}$ and $Y^0 = (Y_i^0)_{1\le i\le n}$ be independent, with X_i^0 and Y_i^0, $i = 1, \ldots, n$, iid random variables with values in $\{\alpha_2, \ldots, \alpha_m\}$ and such that $\mathbb{P}(X_1^0 = \alpha_k) = \mathbb{P}(Y_1^0 = \alpha_k) = p_k/(1 - p_1)$, $2 \le k \le m$. In other words, X^0 and Y^0 are two independent finite sequences of iid random variables whose joint law is the law of $((X, Y)|N_1 = 0)$. Once (X^k, Y^k) is defined, let (X^{k+1}, Y^{k+1}) be the pair of finite random sequences obtained by taking (pathwise) with equal probability, one letter from all the letters $\alpha_2, \alpha_3, \ldots, \alpha_m$ in the pair (X^k, Y^k) and replacing it with α_1, and for this path iterating the process till $k = 2n$. Clearly, for $1 \le k \le 2n - 1$, X^k and Y^k are not independent, while $(X_i^{2n}, Y_i^{2n})_{1\le i\le n}$ is a deterministic sequence made up only of the letter α_1.

Rigorously, the random variables can be defined as follows: let Ω be our underlying space, and let Ω^{2n+1} be its $(2n + 1)$-fold Cartesian product. For each

$\omega = (\omega_0, \omega_1, \dots, \omega_{2n}) \in \Omega^{2n+1}$ and $0 \le k \le 2n$, $(X^k(\omega), Y^k(\omega))$ only depends on $\omega_0, \omega_1, \dots, \omega_k$. Then, $(X^{k+1}(\omega), Y^{k+1}(\omega))$ is obtained from $(X^k(\omega), Y^k(\omega))$ by replacing with equal probability any non-α_1 letter by α_1, while the choice of the non-α_1 letter to be replaced in $(X^k(\omega), Y^k(\omega))$ is determined by ω_{k+1}.

Next, let $LC_n(k)$ denote the length of the longest common subsequence of X^k and Y^k (with a slight abuse of notation and terminology with the identification of finite sequences and words). The lemma below shows that (X^k, Y^k) has the same law as (X, Y) conditional on $N_1 = k$, and therefore the law of $LC_n(k)$ is the same as the conditional law of LC_n given $N_1 = k$.

Lemma 2.1 *For any* $k = 0, 1, \dots, 2n$,

$$(X^k, Y^k) \stackrel{d}{=} ((X, Y)|N_1 = k), \tag{2.4}$$

and moreover,

$$(X^{N_1}, Y^{N_1}) \stackrel{d}{=} (X, Y), \tag{2.5}$$

where $\stackrel{d}{=}$ *denotes equality in distribution.*

Proof The proof is by induction on k. By definition, (X^0, Y^0) has the same law as (X, Y) conditional on $N_1 = 0$. For any $(\alpha_{j_1}, \dots, \alpha_{j_{2n}}) \in \mathcal{A}_m^n \times \mathcal{A}_m^n$, let

$$q_\ell = \left|\left\{1 \le i \le 2n : \ \alpha_{j_i} = \alpha_\ell\right\}\right|,$$

$1 \le \ell \le m$. Now assume that (2.4) is true for k, i.e., assume that for any $(\alpha_{j_1}, \dots, \alpha_{j_{2n}}) \in \mathcal{A}_m^n \times \mathcal{A}_m^n$, with $q_1 = k$,

$$\mathbb{P}\left((X_1^k, \dots, X_n^k, Y_1^k, \dots, Y_n^k) = (\alpha_{j_1}, \dots, \alpha_{j_{2n}})\right) = \binom{2n}{k}^{-1} \prod_{\ell=2}^{m} \left(\frac{p_\ell}{1-p_1}\right)^{q_\ell}. \tag{2.6}$$

Then, for any $(\alpha_{j_1}, \dots, \alpha_{j_{2n}}) \in \mathcal{A}_m^n \times \mathcal{A}_m^n$, with $q_1 = k+1$,

$$\mathbb{P}\left((X_1^{k+1}, \dots, X_n^{k+1}, Y_1^{k+1}, \dots, Y_n^{k+1}) = (\alpha_{j_1}, \dots, \alpha_{j_{2n}})\right) =$$
$$\sum_{i=1}^{k+1} \mathbb{P}\big((X_1^{k+1}, \dots, X_n^{k+1}, Y_1^{k+1}, \dots, Y_n^{k+1}) = (\alpha_{j_1}, \dots, \alpha_{j_{2n}})|B_i^{k+1}\big)\mathbb{P}(B_i^{k+1}), \tag{2.7}$$

where B_i^{k+1}, $1 \le i \le k+1$, is the event that the i-th α_1 in $(\alpha_{j_1}, \dots, \alpha_{j_{2n}})$ is changed from a non-α_1 letter when passing from (X^k, Y^k) to (X^{k+1}, Y^{k+1}). (Conditional on B_i^{k+1}, the i-th α_1 in $(\alpha_{j_1}, \dots, \alpha_{j_{2n}})$ could have been changed from any letter in $\{\alpha_2, \alpha_3, \dots, \alpha_m\}$.) Assuming this α_1 has been changed, say, from α_s, $2 \le s \le m$, the corresponding probability is given by:

$$\mathbb{P}\left((X^k, Y^k) = (\alpha_{j_1}, \ldots, \alpha_s, \ldots, \alpha_{j_{2n}})\right) = \binom{2n}{k}^{-1} \prod_{\ell=2}^{m} \left(\frac{p_\ell}{1-p_1}\right)^{q_\ell} \left(\frac{p_s}{1-p_1}\right),$$

where, above, α_s takes the place of the i-th α_1 in the sequence $(\alpha_{j_1}, \ldots, \alpha_{j_{2n}})$. Thus,

$$\mathbb{P}\left((X_1^{k+1}, \ldots, X_n^{k+1}, Y_1^{k+1}, \ldots, Y_n^{k+1}) = (\alpha_{j_1}, \ldots, \alpha_{j_{2n}}) | B_i^{k+1}\right) \mathbb{P}(B_i^{k+1})$$
$$= \binom{2n}{k}^{-1} \prod_{\ell=2}^{m} \left(\frac{p_\ell}{1-p_1}\right)^{q_\ell} \left(\sum_{s=2}^{m} \frac{p_s}{1-p_1}\right) \frac{1}{2n-k},$$

which when incorporated into (2.7), gives

$$\mathbb{P}\left((X_1^{k+1}, \ldots, X_n^{k+1}, Y_1^{k+1}, \ldots, Y_n^{k+1}) = (\alpha_{j_1}, \ldots, \alpha_{j_{2n}})\right)$$
$$= \binom{2n}{k+1}^{-1} \prod_{\ell=2}^{m} \left(\frac{p_\ell}{1-p_1}\right)^{q_\ell}, \qquad (2.8)$$

finishing the proof of the first part of the lemma.

Next, from (2.4) and the independence of N_1 and $\{(X^k, Y^k)\}_{0 \le k \le 2n}$, for any $(u, v) \in \mathbb{R}^n \times \mathbb{R}^n$,

$$\mathbb{E}\left(e^{i<u,X>+i<v,Y>}\right) = \sum_{k=0}^{2n} \mathbb{E}\left(e^{i<u,X>+i<v,Y>} | N_1 = k\right) \mathbb{P}(N_1 = k)$$
$$= \sum_{k=0}^{2n} \mathbb{E}\left(e^{i<u,X^k>+i<v,Y^k>}\right) \mathbb{P}(N_1 = k)$$
$$= \sum_{k=0}^{2n} \mathbb{E}\left(e^{i<u,X^k>+i<v,Y^k>} | N_1 = k\right) \mathbb{P}(N_1 = k)$$
$$= \sum_{k=0}^{2n} \mathbb{E}\left(e^{i<u,X^{N_1}>+i<v,Y^{N_1}>} | N_1 = k\right) \mathbb{P}(N_1 = k)$$
$$= \mathbb{E}\left(e^{i<u,X^{N_1}>+i<v,Y^{N_1}>}\right),$$

finishing the proof of the lemma. ■

Let now $LC_n(N_1)$ be the length of the longest common subsequence of X^{N_1} and Y^{N_1}. The above lemma implies that LC_n and $LC_n(N_1)$ have the same law and, therefore,

$$\mathbb{M}_r(LC_n(N_1)) = \mathbb{M}_r(LC_n). \tag{2.9}$$

To lower bound the right hand side of (2.9) (and to prove Theorem 1.1) the following simple inequality will prove useful.

Lemma 2.2 *Let $f : Dom \to \mathbb{Z}$ satisfy a local reversed Lipschitz condition, i.e., let $\ell \geq 0$ and let f be such that for any $i, j \in D$ with $j \geq i + \ell$,*

$$f(j) - f(i) \geq c(j - i),$$

for some $c > 0$. Let T be a Dom-valued random variable with $\mathbb{E}|f(T)|^r < +\infty$, $r \geq 1$, then

$$\mathbb{M}_r(f(T)) \geq \left(\frac{c}{2}\right)^r (\mathbb{M}_r(T) - \ell^r). \tag{2.10}$$

Proof Let $r \geq 1$, and let $\widehat{T}$ be an independent copy of T. First, and clearly, $\mathbb{M}_r(T) \leq \mathbb{E}(|T - \widehat{T}|^r) \leq 2^r \mathbb{M}_r(T)$. Hence,

$$\begin{aligned}
\mathbb{M}_r(f(T)) &\geq \frac{1}{2^r}\mathbb{E}(|f(T) - f(\widehat{T})|^r) \\
&\geq \left(\frac{c}{2}\right)^r \left(\mathbb{E}(T - \widehat{T})^r \mathbf{1}_{T - \widehat{T} \geq \ell} + \mathbb{E}(\widehat{T} - T)^r \mathbf{1}_{\widehat{T} - T \geq \ell}\right) \\
&\geq \left(\frac{c}{2}\right)^r \left(\mathbb{E}|T - \widehat{T}|^r - \ell^r\right) \\
&\geq \left(\frac{c}{2}\right)^r (\mathbb{M}_r(T) - \ell^r).
\end{aligned}$$

■

The above lemma will prove useful in providing a lower bound on $\mathbb{M}_r(LC_n(N_1))$ by showing that, after removing the randomness of $LC_n(\cdot)$, $LC_n(\cdot)$ satisfies a local reversed Lipschitz condition. To do so, for a random variable U with finite r-th moment and for a random vector V, let $\mathbb{M}_r(U|V) := \mathbb{E}\left(|U - \mathbb{E}(U|V)|^r \middle| V\right)$. Clearly, by convexity and the conditional Jensen's inequality,

$$\begin{aligned}
\mathbb{M}_r(U|V) &\leq 2^r \left(\mathbb{E}\left(|U - \mathbb{E}U|^r \middle| V\right)/2 + \mathbb{E}\left(|\mathbb{E}(U|V) - \mathbb{E}U|^r \middle| V\right)/2\right) \\
&\leq 2^r \mathbb{E}\left(|U - \mathbb{E}U|^r \middle| V\right)
\end{aligned} \tag{2.11}$$

and so, for any $n \geq 1$,

$$\begin{aligned}\mathbb{M}_r(LC_n(N_1)) &\geq \frac{1}{2^r}\mathbb{E}(\mathbb{M}_r(LC_n(N_1)|(LC_n(k))_{0\leq k\leq 2n})) \\ &= \frac{1}{2^r}\int_\Omega \mathbb{M}_r(LC_n(N_1)|(LC_n(k))_{0\leq k\leq 2n}(\omega))\mathbb{P}(d\omega) \\ &\geq \frac{1}{2^r}\int_{O_n} \mathbb{M}_r(LC_n(N_1)|(LC_n(k))_{0\leq k\leq 2n}(\omega))\mathbb{P}(d\omega), \qquad (2.12)\end{aligned}$$

where for each $n \geq 1$,

$$O_n := \bigcap_{\substack{i,j\in I \\ j\geq i+\ell(n)}} \left\{LC_n(j) - LC_n(i) \geq \frac{K}{4m}(j-i)\right\}, \qquad (2.13)$$

where K is given in Theorem 2.1 and where $\ell(n) \geq 0$ is to be chosen later. (Of course, above and everywhere, intersections, unions and sums are taken over countable sets of integers.) In words, on the event O_n the random function LC_n has a slope of at least $K/4m$, when restricted to the interval I and when i and j are at least $\ell(n)$ apart from each other.

Since N_1 is independent of $(LC_n(k))_{0\leq k\leq 2n}$, and from (2.11), for each $\omega \in \Omega$,

$$\begin{aligned}&\mathbb{M}_r(LC_n(N_1)|(LC_n(k))_{0\leq k\leq 2n}(\omega)) \\ &\geq \frac{1}{2^r}\mathbb{M}_r(LC_n(N_1)|(LC_n(k))_{0\leq k\leq 2n}(\omega), \mathbf{1}_{N_1\in I} = 1)\mathbb{P}(N_1 \in I|(LC_n(k))_{0\leq k\leq 2n}(\omega)) \\ &= \frac{1}{2^r}\mathbb{M}_r(LC_n(N_1)|(LC_n(k))_{0\leq k\leq 2n}(\omega), \mathbf{1}_{N_1\in I} = 1)\mathbb{P}(N_1 \in I), \qquad (2.14)\end{aligned}$$

where

$$I = \left[2np_1 - \sqrt{2n(1-p_1)p_1}, 2np_1 + \sqrt{2n(1-p_1)p_1}\right]. \qquad (2.15)$$

Again, for each $\omega \in O_n$, from Lemma 2.2, and since N_1 is independent of $(LC_n(k))_{0\leq k\leq 2n}$,

$$\begin{aligned}&\mathbb{M}_r(LC_n(N_1)|(LC_n(k))_{0\leq k\leq 2n}(\omega), \mathbf{1}_{N_1\in I} = 1) \\ &\qquad \geq \left(\frac{K}{8m}\right)^r (\mathbb{M}_r(N_1|\mathbf{1}_{N_1\in I} = 1) - \ell(n)^r). \qquad (2.16)\end{aligned}$$

Now, (2.12), (2.14) and (2.16) lead to

$$\mathbb{M}_r(LC_n(N_1)) \geq \frac{1}{4^r}\left(\frac{K}{8m}\right)^r (\mathbb{M}_r(N_1|\mathbf{1}_{N_1\in I}=1) - \ell(n)^r)\,\mathbb{P}(N_1 \in I)\mathbb{P}(O_n), \tag{2.17}$$

and it remains to estimate each one of the three terms on the right hand side of (2.17). By the Berry-Esséen inequality, and all $n \geq 1$,

$$\left|\mathbb{P}(N_1 \in I) - \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} e^{-\frac{x^2}{2}}dx\right| \leq \frac{1}{\sqrt{2np_1(1-p_1)}}. \tag{2.18}$$

Moreover,

$$\begin{aligned} &\mathbb{M}_r(N_1|\mathbf{1}_{N_1\in I}=1) \\ &= \mathbb{E}(|N_1 - 2np_1 + 2np_1 - \mathbb{E}(N_1|\mathbf{1}_{N_1\in I}=1)|^r|\mathbf{1}_{N_1\in I}=1) \\ &\geq \left|\mathbb{E}(|N_1 - 2np_1|^r|\mathbf{1}_{N_1\in I}=1)^{1/r} - |2np_1 - \mathbb{E}(N_1|\mathbf{1}_{N_1\in I}=1)|\right|^r, \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} &|\mathbb{E}(N_1|\mathbf{1}_{N_1\in I}=1) - 2np_1| \\ &= \sqrt{2np_1(1-p_1)}\left|\mathbb{E}\left(\frac{N_1-2np_1}{\sqrt{2np_1(1-p_1)}}\middle|\mathbf{1}_{N_1\in I}=1\right)\right| \\ &= \sqrt{2np_1(1-p_1)}\frac{\left|F_n(1) - \Phi(1) + F_n(-1) - \Phi(-1) - \int_{-1}^{1}(F_n(x)-\Phi(x))dx\right|}{\mathbb{P}(N_1\in I)} \\ &\leq \sqrt{2np_1(1-p_1)}\frac{4\max_{x\in[-1,1]}|F_n(x)-\Phi(x)|}{\mathbb{P}(N_1\in I)} \\ &\leq \frac{2}{\int_{-1}^{1} e^{-\frac{x^2}{2}}dx/\sqrt{2\pi} - 1/\sqrt{2np_1(1-p_1)}}, \end{aligned} \tag{2.20}$$

where F_n is the distribution functions of $(N_1 - 2np_1)/\sqrt{2np_1(1-p_1)}$, while Φ is the standard normal one. Likewise,

$$\begin{aligned} &\mathbb{E}(|N_1 - 2np_1|^r|\mathbf{1}_{N_1\in I}=1) \\ &\geq (2np_1(1-p_1))^{r/2}\frac{\int_{-1}^{1}|x|^r d\Phi(x) - 4\max_{x\in[-1,1]}|F_n(x)-\Phi(x)|}{\mathbb{P}(N_1\in I)} \\ &\geq (2np_1(1-p_1))^{r/2}\frac{\int_{-1}^{1}|x|^r e^{-\frac{x^2}{2}}dx - 2\sqrt{\pi}/\sqrt{np_1(1-p_1)}}{\int_{-1}^{1} e^{-\frac{x^2}{2}}dx + \sqrt{\pi}/\sqrt{np_1(1-p_1)}}. \end{aligned} \tag{2.21}$$

Next, (2.19)–(2.21) lead to:

$$\mathbb{M}_r(N_1|\mathbf{1}_{N_1\in I}=1)$$
$$\geq \left|(2np_1(1-p_1))^{\frac{1}{2}}\left(\frac{\int_{-1}^{1}|x|^r e^{-\frac{x^2}{2}}dx-2\sqrt{\pi}/\sqrt{np_1(1-p_1)}}{\int_{-1}^{1}e^{-\frac{x^2}{2}}dx+\sqrt{\pi}/\sqrt{np_1(1-p_1)}}\right)^{\frac{1}{r}}\right.$$
$$\left.-\frac{2}{\int_{-1}^{1}e^{-\frac{x^2}{2}}dx/\sqrt{2\pi}-1/\sqrt{2np_1(1-p_1)}}\right|^r. \tag{2.22}$$

Finally, assuming Theorem 2.1, the estimates (2.17)–(2.22) combined with the estimate on $\mathbb{P}(O_n)$ obtained in the next lemma give the lower bound (1.9), whenever $33m^2\log n/K^2 \leq \ell(n) \leq K_1\sqrt{n}$ (where K_1 is given and estimated in Remark 2.1).

Lemma 2.3 *For $m \geq 2$, let $K_m = \min(K, 1/800m)$ where $K = 2^{-4}10^{-2}e^{-67}$, and let $p_2 \leq \min(2^{-2}e^{-5}K_m/m, K_m/2m^2)$. Then, for all $n \geq 1$,*

$$\mathbb{P}(O_n) \geq 1-\left(500\sqrt{\pi}e^2 n\exp\left(-\frac{np_2^6}{5}\right)+2n\exp\left(-\frac{K^2\ell(n)}{32m^2}\right)\right). \tag{2.23}$$

Proof Let $A_n := \{(X,Y)\in\mathcal{B}_n\}$ and let $A_n^k := \{(X^k,Y^k)\in\mathcal{B}_n\}$. Then,

$$\mathbb{P}\left(\left(\bigcap_{k\in I}A_n^k\right)^c\right) \leq \sum_{k\in I}\mathbb{P}\left(\left(A_n^k\right)^c\right) = \sum_{k\in I}\mathbb{P}\left(A_n^c|N_1=k\right) \leq \sum_{k\in I}\frac{\mathbb{P}(A_n^c)}{\mathbb{P}(N_1=k)}, \tag{2.24}$$

by Lemma 2.1. Next, by Stirling's formula in the form,

$$\sqrt{2\pi}n^{n+\frac{1}{2}}e^{-n+\frac{1}{12n+1}} < n! < \sqrt{2\pi}n^{n+\frac{1}{2}}e^{-n+\frac{1}{12n}},$$

for all $k \in I$ and $n \geq 1$,

$$\begin{aligned}\mathbb{P}(N_1=k) &= \binom{2n}{k}p_1^k(1-p_1)^{2n-k}\\ &\geq \frac{1}{\sqrt{2\pi}e^2}\frac{(2n)^{2n+1/2}}{k^{k+1/2}(2n-k)^{2n-k+1/2}}p_1^k(1-p_1)^{2n-k}\\ &:= \gamma(k,n,p_1).\end{aligned}$$

Hence, for all $k \in I$ and $p_1 \geq 3/4$ (which holds true since $p_2 \leq K/m$), from the property of the probability mass function of the binomial distribution,

$$\begin{aligned}
&\mathbb{P}(N_1 = k) \\
&\geq \min\left(\mathbb{P}(N_1 = 2np_1 - \lfloor\sqrt{2n(1-p_1)p_1}\rfloor), \mathbb{P}(N_1 = 2np_1 + \lfloor\sqrt{2n(1-p_1)p_1}\rfloor)\right) \\
&\geq \min\left(\gamma\left(2np_1 - \lfloor\sqrt{2n(1-p_1)p_1}\rfloor, n, p_1\right), \gamma\left(2np_1 + \lfloor\sqrt{2n(1-p_1)p_1}\rfloor, n, p_1\right)\right) \\
&\geq \frac{1}{2\sqrt{2\pi}e^2\sqrt{n}}. \qquad (2.25)
\end{aligned}$$

This last inequality in conjunction with (2.24) and Theorem 2.1, gives

$$\mathbb{P}\left(\left(\bigcap_{k\in I} A_n^k\right)^c\right) \leq 4\sqrt{\pi}e^2 n\mathbb{P}(A_n^c) \leq 500\sqrt{\pi}e^2 n \exp\left(-\frac{np_2^6}{5}\right). \qquad (2.26)$$

Next, for each $n \geq 1$, letting

$$\Delta_{k+1} = \begin{cases} LC_n(k+1) - LC_n(k), & \text{when } A_n^k \text{ holds,} \\ 1, & \text{otherwise,} \end{cases} \qquad (2.27)$$

it follows from Theorem 2.1 that,

$$\mathbb{E}(\Delta_{k+1}|X^k, Y^k) \geq \frac{K}{2m}. \qquad (2.28)$$

Now, for each $k = 0, 1, \ldots, 2n$, let $\mathcal{F}_k := \sigma(X^0, Y^0, \ldots, X^k, Y^k)$, be the σ-field generated by $X^0, Y^0, \ldots, X^k, Y^k$. Clearly, $(\Delta_k - \mathbb{E}(\Delta_k|\mathcal{F}_{k-1}), \mathcal{F}_k)_{1\leq k\leq 2n}$ forms a martingale differences sequence and since $-1 \leq \Delta_k \leq 1$, Hoeffding's martingale inequality gives, for any $i < j$,

$$\mathbb{P}\left(\sum_{k=i+1}^{j} (\Delta_k - \mathbb{E}(\Delta_k|\mathcal{F}_{k-1})) < -\frac{K}{4m}(j-i)\right) \leq \exp\left(-\frac{K^2(j-i)}{32m^2}\right). \qquad (2.29)$$

Moreover, from (2.28), $\sum_{k=i+1}^{j} \mathbb{E}(\Delta_k|X^{k-1}, Y^{k-1}) \geq K(j-i)/2m$, and therefore

$$\begin{aligned}
\mathbb{P}\left(\sum_{k=i+1}^{j} \Delta_k \leq \frac{K}{4m}(j-i)\right) &\leq \mathbb{P}\left(\sum_{k=i+1}^{j} (\Delta_k - \mathbb{E}(\Delta_k|\mathcal{F}_{k-1})) < -\frac{K}{4m}(j-i)\right) \\
&\leq \exp\left(-\frac{K^2(j-i)}{32m^2}\right). \qquad (2.30)
\end{aligned}$$

For each $n \geq 1$, let now

$$O_n^{\Delta} = \bigcap_{\substack{i,j \in I \\ j \geq i+\ell(n)}} \left\{ \sum_{i+1}^{j} \Delta_k \geq \frac{K}{4m}(j-i) \right\} ,$$

then, from (2.30)

$$\mathbb{P}\left(\left(O_n^{\Delta}\right)^c\right) \leq \sum_{\substack{i,j \in I \\ j \geq i+\ell(n)}} \mathbb{P}\left(\sum_{i+1}^{j} \Delta_k < \frac{K}{4m}(j-i)\right) \leq 2n \exp\left(-\frac{K^2 \ell(n)}{32m^2}\right). \tag{2.31}$$

From the very definition of Δ_k in (2.27), $\bigcap_{k \in I} A_n^k \cap O_n^{\Delta} \subset O_n$, and therefore

$$\begin{aligned}\mathbb{P}\left((O_n)^c\right) &\leq \mathbb{P}\left(\left(\bigcap_{k \in I} A_n^k\right)^c\right) + \mathbb{P}\left(\left(O_n^{\Delta}\right)^c\right) \\ &\leq 500\sqrt{\pi} e^2 n \exp\left(-\frac{np_2^6}{5}\right) + 2n \exp\left(-\frac{K^2 \ell(n)}{32m^2}\right).\end{aligned} \tag{2.32}$$

■

Remark 2.1 The reader might wonder how to estimate the constant C in Theorem 1.1. In view of (2.9), the right hand side of (2.17) needs to be lower bounded. Letting $n \geq p_2^{-12} + m^8$, together with (2.18), (2.22) and (2.23) yield to:

$$\mathbb{P}(N_1 \in I) \geq \frac{1}{2}, \quad \mathbb{P}(O_n) \geq \frac{1}{2},$$

and

$$\mathbb{M}_r(N_1 | \mathbf{1}_{N_1 \in I} = 1) \geq e^{-\frac{1}{2}} 2^{-(1+r)} (1+r)^{-1} (n(1-p_1))^{\frac{r}{2}}.$$

Moreover, choosing

$$\ell(n) = 2^{(-1-r-\frac{1}{r})} e^{-\frac{1}{2r}} (n(1-p_1))^{\frac{1}{2}} \left(\frac{1}{1+r}\right)^{\frac{1}{r}} := K_1 \sqrt{n},$$

in (2.17), gives:

$$\mathbb{M}_r(LC_n) \geq 2^{-4-6r}(1+r)^{-1} e^{-1/2} K^r m^{-r} (1-p_1)^{r/2} n^{r/2}.$$

Letting $C_1 = 2^{-4-6r}(1+r)^{-1}e^{-1/2}K^r m^{-r}(1-p_1)^{r/2}$, and

$$C_2 = \min_{n \le p_2^{-12}+m^8} \frac{\mathbb{M}_r(LC_n)}{n^{r/2}} \le \frac{(r-1)^r}{2} 2^{r/2} \left(1 - \sum_{k=1}^{m} p_k^2\right),$$

by (1.8), then one can choose $C = \min(C_1, C_2)$ in Theorem 1.1.

3 Proof of Theorem 2.1

3.1 Description of Alignments

Let us begin with an example. Let $\mathcal{A}_3 = \{\alpha_1, \alpha_2, \alpha_3\}$, with $\alpha_i = i$, $i = 1, 2, 3$, and, say that

$$X = 121313111211, \quad Y = 111311112112. \tag{3.1}$$

An optimal alignment of X and Y, i.e., an alignment corresponding to a LCS, is

1	2		1	3	1	3	1	1	1	2	1	1	
1		1	1	3	1		1	1	1	2	1	1	2

(3.2)

and another possible optimal alignment is

1	2	1		3	1	3	1	1	1	2	1	1	
1		1	1	3	1		1	1	1	2	1	1	2

(3.3)

both corresponding to the LCS 1131111211.

Comparing these two optimal alignments, it is clear that the way the letters α_1 are aligned, between the aligned non-α_1 letters, is not important as long as a maximal number of such letters α_1 are aligned. Therefore, in general, it is enough to describe which non-α_1 letters are aligned and to assume that between pairs of aligned non-α_1 letters a maximal number of letters α_1 are aligned. In other words, we can identify the two optimal alignments (3.2) and (3.3) as the same.

Next, let a *cell*, be either the beginning of an alignment till and including, if any, its first pair of aligned non-α_1 letter, or be a part of an alignment between pairs of aligned non-α_1 letters.

For example, the alignment (3.2) can be decomposed into two cells $C(1)$ and $C(2)$ as

$$\overbrace{\begin{array}{ccccc}1&2&&1&3\\1&&1&1&3\end{array}}^{C(1),\ v_1=-1}\;\overbrace{\begin{array}{cccccc}1&3&1&1&1&2\\1&&1&1&1&2\end{array}}^{C(2),\ v_2=0}\;\begin{array}{ccc}1&1&\\1&1&2\end{array} \tag{3.4}$$

where, moreover, each v_i denotes the difference between the number of letters α_1 in the X-strand and the Y-strand of the cell $C(i)$, $i = 1, 2$. For the alignment (3.2), this gives the representation $v = (v_1, v_2) = (-1, 0)$. Another optimal alignment is via $v = (v_1, v_2) = (0, -1)$ corresponding to another LCS, namely 1113111211:

$$\overbrace{\begin{array}{cccccc}1&2&1&3&1&3\\1&&1&&1&3\end{array}}^{C(1),\ v_1=0}\;\overbrace{\begin{array}{ccccc}1&1&1&&2\\1&1&1&1&2\end{array}}^{C(2),\ v_2=-1}\;\begin{array}{ccc}1&1&\\1&1&2\end{array} \tag{3.5}$$

Note that any alignment has a cell-decomposition with a corresponding finite vector of differences. (With the convention that when no non-α_1 letters are aligned, then the alignment has no cell.)

Let $X = X_1X_2\cdots X_n$ and $Y = Y_1Y_2\cdots Y_n$ be given. As just conveyed, any alignment has a cell-decomposition with an associated vector representation $v := (v_1, \ldots, v_k)$ indicating the number of cells (k, here) and the differences between the number of letters α_1 in the X-strand and the corresponding number in the Y-strand of each cell. Conversely, any $v \in \mathbb{Z}^k$ corresponds to a, possibly empty, family of cell-decompositions.

Let us now turn to *optimality*. First, clearly any optimal alignment is made of, say, k cells (recall also our convention above), where within each cell a maximum number of letters α_1 are aligned and, if any, the optimal alignment also has a tail part (the part after the last cell, i.e., the part after the last aligned non-α_1 letters) where as many letter α_1 as possible are aligned. Therefore, such an optimal alignment is given via a unique $v \in \mathbb{Z}^k$. On the other hand, every $v = (v_1, \ldots, v_k) \in \mathbb{Z}^k$ also corresponds to a (possibly empty) family of optimal alignments. All of these optimal alignments have the same number of pairs of aligned non-α_1 letters where within each cell a maximal number of letters α_1 are aligned, and where moreover as many letters α_1 as possible are aligned after the pair of aligned non-α_1-letters. These optimal alignments corresponding to the same v can differ in the way the letters α_1 are aligned within each cell and in the tail part. It can also happen, and in contrast to the binary case, that one can align different pairs of non-α_1 letters, which can only happen when no letters α_1 are present between these different pairs of non-α_1 letters. (Take, for example, $X = 1321$ and $Y = 2311$, then the optimal alignments corresponding to $v \in \mathbb{Z}$ can align either the letter 2 or the letter 3.) But in both cases such optimal alignments based on the same v give the same length for the corresponding longest common subsequences. Therefore, we can identify all the optimal alignments in the family associated with v as a single one. In other words, we identify each vector v with an optimal alignment, provided one exists, and vice-versa.

Writing $|v|$ for the number of coordinates of v, i.e., $|v| = k$, if $v \in \mathbb{Z}^k$, the cell-decomposition $\pi - \nu$ associated with $v = (v_1, \ldots, v_k) \in \mathbb{Z}^k$ can now precisely be defined:

Definition 3.1 Let $k \in \mathbb{N}, k \geq 1$ and let $v = (v_1, \ldots, v_k) \in \mathbb{Z}^k$. Let $\pi_v(0) = \nu_v(0) = 0$, and for each $i = 1, \ldots, k$, let $(\pi_v(i), \nu_v(i))$ be any one of the smallest pair of integers (s, t) (where $(s_1, t_1) \leq (s_2, t_2)$ indicates that $s_1 \leq s_2$ and $t_1 \leq t_2$) satisfying the following three conditions:

1. $\pi_v(i-1) < s$ and $\nu_v(i-1) < t$;
2. $X_s = Y_t \in \{\alpha_2, \ldots, \alpha_m\}$;
3. the difference between the number of letters α_1 in the integer intervals $[\pi_v(i-1), s]$ and $[\nu_v(i-1), t]$ is equal to v_i.

If for some $i = 1, \ldots, k$, no such (s, t) exists, then set $\pi_v(i) = \cdots = \pi_v(k) = \infty$ and $\nu_v(i) = \cdots = \nu_v(k) = \infty$.

In other words, above, $\pi_v(i), \nu_v(i), i = 1, \ldots, k$, are the indices corresponding to the i-th aligned non-α_1 pair in v. For $i = 1, \ldots, k$, the i-th cell, $C_v(i)$ is the pair

$$C_v(i) := \left(X_{\pi_v(i-1)+1} \ldots X_{\pi_v(i)}; Y_{\nu_v(i-1)+1} \cdots Y_{\nu_v(i)}\right),$$

and the cell $C_v(i)$ *is called a* v_i*-cell.*

Let us further comment on the above definition, we actually defined a greedy algorithm for each cell (each cell must be *minimal* meaning that the cell ends as soon as all three conditions in Definition 3.1 are met). For any optimal alignment, let us compare its cells with our minimal cells alignment. If any, respectively denote the first two different cells by c_i^{opt} and c_i^{min}, $1 \leq i \leq k$, since these cells correspond to the same $v_i \in \mathbb{Z}$, they only differ in the number of pairs of aligned letters α_1. From the definition of minimality, c_i^{opt} contains more pairs of aligned letters α_1 than c_i^{min}. These pairs of letters α_1, being of same number on the X-strand and Y-strand, can thus be pushed to next cell. By iterating this push-procedure till the tail, then any optimal alignment can be transformed into a minimal (optimal) alignment without reducing the length of the common subsequence. Thus an optimal alignment can always be transformed into a minimal (optimal) alignment.

With the above definition, we can let the alignment associated to v be any alignment (provided one exists) satisfying the following three conditions:

1. $X_{\pi_v(i)}$ is aligned with $Y_{\nu_v(i)}$, for every $i = 1, 2, \ldots, k$;
2. the number of aligned letters α_1 in the cell $C_v(i)$, denoted by $S_v(i)$, is the minimum number of letters α_1 present in either $X_{\pi_v(i-1)+1} \cdots X_{\pi_v(i)}$ or $Y_{\nu_v(i-1)+1} \cdots Y_{\nu_v(i)}$;
3. after having aligned $X_{\pi_v(k)}$ with $Y_{\nu_v(k)}$, then align as many letters α_1 as possible and denote that number by r_v.

From these definitions, for any $v \in \mathbb{Z}^k$, and if there exists a minimal cell-decomposition corresponding to v exists, then $\pi_v(k) \leq n$ and $\nu_v(k) \leq n$. Such a v is then said to be *admissible*. Let V denote the set of all admissible cell-decompositions, that is,

$$V := \left\{ v \in \bigcup_{k=1}^{\infty} \mathbb{Z}^k : \pi_v(|v|) \leq n, \nu_v(|v|) \leq n \right\} . \tag{3.6}$$

Then, for every $v \in V$, and further for $|v| = 0$ in case of no cell, the length of the common subsequence corresponding to this alignment is:

$$\Lambda C_v = |v| + \sum_{i=1}^{|v|} S_v(i) + r_v . \tag{3.7}$$

Therefore the length of the longest common subsequence of X and Y can be expressed as:

$$LC_n = \max_{v \in V} \Lambda C_v , \tag{3.8}$$

and, moreover, an alignment associated to an admissible v is optimal if and only if $\Lambda C_v = LC_n$.

3.2 The Effect of Changing a Non-α_1 Letter into α_1

Again, the main idea behind Theorem 2.1 is that, by changing a randomly picked non-α_1 letter into α_1, the length of the longest common subsequence is more likely to increase by one than to decrease by one. More precisely, conditional on the event $A_n = \{(X, Y) \in \mathcal{B}_n\}$, the probability of an increase of LC_n is at least K/m while the probability of a decrease is at most $K/2m$. Let us illustrate this fact with another example. Let X and Y be given by,

$$X = 112113112131, \; Y = 131111111131, \tag{3.9}$$

with optimal alignment:

$$\begin{array}{c|c} \overbrace{\begin{array}{cccccccccccccc} 1 & & 1 & 2 & 1 & 1 & 3 & 1 & 1 & 2 & 1 & & & 3 \\ 1 & 3 & 1 & & 1 & 1 & & 1 & 1 & & 1 & 1 & 1 & 3 \end{array}}^{C(1),\ v_1=-2} & \begin{array}{c} 1 \\ 1 \end{array} \end{array} \tag{3.10}$$

Above, there are 6 non-α_1 letters, $X_3, X_6, X_9, X_{11}, Y_2, Y_{11}$, and each one has probability $1/6$ to be picked and replaced by α_1. Next, X_3, X_6, X_9 and Y_2 are not aligned with other letters but rather with gaps. Moreover, since X_3, X_6, X_9 are on the top strand which contains a lesser number of letters α_1, picking one of them and replacing it leads to an increase of one in the length of the LCS. On the other hand, since X_{11} and Y_{11} are aligned in this optimal alignment, picking one of them and replacing it could potentially (but not necessarily) decrease the length of the LCS by one. Finally, picking Y_2 may only potentially increase the length of the LCS by modifying the alignment. In conclusion, in this example, by switching a randomly chosen non-α_1 letter into α_1, the probability of an increase of the length of the LCS is at least $1/2$, while the probability of a decrease is at most $1/3$.

To prove Theorem 2.1, we just need to prove that typically there exists an optimal alignment such that:

1. Among all the non-α_1 letters in X and Y, the proportion which are on the cell-strand with the smaller number of letters α_1 is at least K/m.
2. Among all the non-α_1 letters in X and Y, the proportion which is aligned is at most $K/2m$.

Formally, let $v = (v_1, \dots, v_k) \in \mathbb{Z}^k$ be admissible. For each $1 \le i \le k$, if $v_i \neq 0$, let $N_v^-(i)$ be the number of non-α_1 letters on the cell-strand of $C_v(i)$ with the lesser number of letters α_1, i.e., let

$$N_v^-(i) = \begin{cases} \sum_{j=\pi_v(i-1)+1}^{\pi_v(i)-1} \mathbf{1}_{X_j \in \{\alpha_2,\dots,\alpha_m\}}, & \text{if } v_i < 0, \\ \sum_{j=\nu_v(i-1)+1}^{\nu_v(i)-1} \mathbf{1}_{Y_j \in \{\alpha_2,\dots,\alpha_m\}}, & \text{if } v_i > 0, \end{cases} \tag{3.11}$$

while if $v_i = 0$, let $N_v^-(i) = 0$. Then, the total number of non-α_1 letters present on the cell-strands with the smaller number of letters α_1 is equal to

$$N_v^- := \sum_{i=1}^{|v|} N_v^-(i). \tag{3.12}$$

Let N_i be the number of letters α_i in the two finite sequences X and Y, and let

$$N_{>1} = \sum_{i=2}^{m} N_i. \tag{3.13}$$

Next, let

$$\mathcal{B}_n := \left\{(x, y) \in \mathcal{A}_m^n \times \mathcal{A}_m^n : \text{ there exists an optimal alignment of } (x, y) \right.$$
$$\left. \text{with } |v| \ge 1, n_v^- \ge K n_{>1}/m \text{ and } 2|v| \le K n_{>1}/2m \right\},$$

where, above, n_v^- is the value of N_v^- corresponding to v and similarly for $n_{>1}$. Clearly, $\mathcal{B}_n$ depends on K and m. Letting $A_n = \{(X, Y) \in \mathcal{B}_n\}$, our goal is now to prove that for some $\tilde{K} > 0$, independent of n, $\mathbb{P}(A_n) \ge 1 - e^{-\tilde{K} n}$.

To continue, we need an optimal alignment having enough non-α_1 letters in the cell-strands with the smaller number of letters α_1. However, for many optimal alignments, most cells are zero-cells, i.e., cells with the same number of letters α_1 on both strands. To bypass this hurdle, on an optimal alignment where most cells are zero-cells, some of the zero-cells are broken up in order to create enough nonzero-cells while at the same time, maintaining the optimality of the alignment after this breaking procedure. Let us present this breaking operation on an example. Take the two sequences

$$X = 112113113, \quad \text{and } Y = 112131113.$$

One of their optimal alignments is

$$\overbrace{\begin{array}{ccc}1&1&2\\1&1&2\end{array}}^{C(1),\ v_1=0}\ \overbrace{\begin{array}{ccccccc}1&&1&3&1&1&3\\1&3&1&&1&1&3\end{array}}^{C(2),\ v_2=0} \tag{3.14}$$

where both cells $C(1)$ and $C(2)$ are zero-cells. Now in the cell $C(2)$, X_6 and Y_5 are only one position away from being aligned. Thus aligning them, instead of the pair X_5 and Y_6, breaks the cell $C(2)$ into two new cells $\tilde{C}(2)$ and $\tilde{C}(3)$, with $\tilde{v}_2 = 1$ and $\tilde{v}_3 = -1$. The new optimal alignment is then:

$$\overbrace{\begin{array}{ccc}1&1&2\\1&1&2\end{array}}^{\tilde{C}(1),\ \tilde{v}_1=0}\ \overbrace{\begin{array}{ccc}1&1&3\\1&&3\end{array}}^{\tilde{C}(2),\ \tilde{v}_2=1}\ \overbrace{\begin{array}{cccc}1&1&&3\\1&1&1&3\end{array}}^{\tilde{C}(3),\ \tilde{v}_3=-1} \tag{3.15}$$

The advantage of breaking up a zero-cell is that the resulting newly formed cells have different numbers of letters α_1 on each strand, thus N_v^- tends to increase in this process while the length of the common subsequence remains the same. After applying this procedure and getting enough cells with different numbers of letters α_1 on the two strands, there is a high probability of finding enough non-α_1 letters on the strand with the smaller number of letters α_1.

The previous example leads to our next definition.

Definition 3.2 Let $k \in \mathbb{N}, k \geq 1$, let $v \in \mathbb{Z}^k \cap V$, and for $i = 1, \ldots, k$, let $C_v(i)$ be any cell with $v_i = 0$. Then, $C_v(i)$ is said to be breakable if there exist j and j' such that:

1. $X_j = Y_{j'} \in \{\alpha_2, \ldots, \alpha_m\}$;
2. $\pi_v(i-1) < j < \pi_v(i)$ and $\nu_v(i-1) < j' < \nu_v(i)$;
3. the difference between the number of letters α_1 in

$$X_{\pi_v(i-1)+1} X_{\pi_v(i-1)+2} \cdots X_{j-1} \text{ and } Y_{\nu_v(i-1)+1} Y_{\nu_v(i-1)+2} \cdots Y_{j'-1}$$

is plus or minus one.

3.3 Probabilistic Developments

After the combinatorial developments of the previous sections, let us now bring forward some probabilistic tools. We start by introducing a useful way of constructing alignments corresponding to a given vector $v = (v_1, \ldots, v_k) \in \mathbb{R}^k$.

For $1 \leq i \leq n$ and $2 \leq j \leq m$, let R_i^j (resp. S_i^j) be the number of letters α_j between the $(i-1)$-th and i-th α_1 in the infinite sequence $(X_i)_{i \geq 1}$ (resp. $(Y_i)_{i \geq 1}$), with, of course, R_1^j (resp. S_1^j) being the number of letters α_j before the first α_1.

Recall also, from Definition 3.1, that in order to construct a zero-cell, we use the random time T_0, given by

$$T_0 = \min_{2 \leq j \leq m} T_0^j, \tag{3.16}$$

where $T_0^j := \min\{i = 1, 2, \ldots : \quad R_i^j \neq 0, S_i^j \neq 0\}$. For a $-u$-cell ($u > 0$), the random time is

$$T_{-u} = \min_{2 \leq j \leq m} T_{-u}^j, \tag{3.17}$$

where $T_{-u}^j := \min\{i = 1, 2, \ldots : R_i^j \neq 0, S_{i+u}^j \neq 0\}$, and for a u-cell ($u > 0$),

$$T_u = \min_{2 \leq j \leq m} T_u^j, \tag{3.18}$$

where $T_u^j := \min\{i = 1, 2, \ldots : R_{i+u}^j \neq 0, S_i^j \neq 0\}$. In other words, a cell with $v_i = u$ can be constructed in the following way: Begin by keeping the first u letters α_1 in the X-strand, then align consecutive pairs of letter α_1 until meeting the first pair of the same non-α_1 letter. (As previously argued, here different choices of pairs of the same non-α_1 letter are possible, i.e., if there are no letters α_1 between different minimal pairs, but any pair will do if there is more than one choice.)

Let us find the law of R_i^j and, to do so, let $R_i^{>1} = \sum_{j=2}^m R_i^j$ be the total number of non-α_1 letters between the $(i-1)$-th and the i-th α_1. Then, $R_i^{>1} + 1$ is a geometric random variable with parameter p_1, i.e., $\mathbb{P}(R_i^{>1} = k) = (1-p_1)^k p_1$, $k = 0, 1, 2, \ldots$. Moreover, conditionally on $R_i^{>1}$, $(R_i^j)_{j=2}^m$ has a multinomial distribution and therefore

$$\begin{aligned}
\mathbb{P}(R_i^j = k) &= \sum_{\ell=k}^{\infty} \mathbb{P}(R_i^j = k | R_i^{>1} = \ell)\mathbb{P}(R_i^{>1} = \ell) \\
&= \sum_{\ell=k}^{\infty} \binom{\ell}{k} \left(\frac{p_j}{1-p_1}\right)^k \left(\frac{1-p_1-p_j}{1-p_1}\right)^{\ell-k} (1-p_1)^{\ell} p_1 \\
&= \left(\frac{p_1}{p_1+p_j}\right)\left(\frac{p_j}{p_1+p_j}\right)^k,
\end{aligned} \tag{3.19}$$

for $k = 0, 1, 2, \ldots$. Thus, $R_i^j + 1$ has a geometric distribution with parameter $p_1/(p_1 + p_j)$, $2 \le j \le m$.

To continue our probabilistic analysis, let us provide a rough lower bound on the length of the LCS. First, aligning as many letters α_1 as possible in X and Y, would get approximately a common subsequence of length np_1, then aligning as many letters α_2 as possible without disturbing the already aligned α_1, would give an additional $\sum_{i=1}^{np_1} \min\{R_i^2, S_i^2\}$ aligned α_2. Moreover, since R_i^2 and S_i^2 are independent geometric random variables, $\min\{R_i^2, S_i^2\} + 1$ is a geometric random variable with parameter $1 - (p_2/(p_1 + p_2))^2$. So, on average, the aligned letters α_2 contribute to the length of the LCS by an amount of:

$$np_1 \frac{p_2^2}{p_1(p_1 + 2p_2)} = \frac{1}{p_1 + 2p_2} np_2^2 \ge (1 - p_2)np_2^2.$$

This heuristic argument leads to the following lemma:

Lemma 3.1 *Let $p_1 > 1/2$ and let $D_1 := \{LC_n \ge np_1 + \left((1 - p_2)^2 - p_2\right) np_2^2\}$. Then, $\mathbb{P}(D_1) \ge 1 - 4\exp(-2np_2^6) - \exp\left(n(p_2^3 + \log(1 - p_2^3))(p_1 - p_2^3)\right)$.*

Proof For $p_1 > \delta > 0$, let $D_2^x(\delta) := \left\{\left|\sum_{i=1}^n \mathbf{1}_{\{X_i = \alpha_1\}} - np_1\right| \le \delta n\right\}$, let $D_2^y(\delta) := \left\{\left|\sum_{i=1}^n \mathbf{1}_{\{Y_i = \alpha_1\}} - np_1\right| \le \delta n\right\}$, and let $D_2(\delta) := D_2^x(\delta) \cap D_2^y(\delta)$, so that on $D_2(\delta)$, at least $n_1(\delta) := n(p_1 - \delta)$ letters α_1 can be aligned. Clearly, $1 + \min(R_i^2, S_i^2)$ has a geometric distribution with parameter $1 - (p_2/(p_1 + p_2))^2$. Also, if $\mathcal{G}_1, \ldots, \mathcal{G}_r$ are iid geometric random variables with parameter p, then for any $\beta < 1$,

$$\mathbb{P}\left(\sum_{i=1}^r \mathcal{G}_i \le \frac{\beta}{p} r\right) \le \exp\left(-(\beta - 1 - \log\beta)r\right). \tag{3.20}$$

By taking $p = 1 - (p_2/(p_1 + p_2))^2$ and $r = n_1(\delta)$, and since the sequences have same length n, the following equality in law holds true:

$$\sum_{i=1}^{n_1(\delta)} \min(R_i^2, S_i^2) + n_1(\delta) \stackrel{d}{=} \sum_{i=1}^{n_1(\delta)} (\mathcal{G}_i \wedge n).$$

For any $\beta < 1$, let us estimate

$$\mathbb{P}\left(\sum_{i=1}^{n_1(\delta)} \min(R_i^2, S_i^2) < \frac{\beta n_1(\delta)}{1 - \left(\frac{p_2}{p_1 + p_2}\right)^2} - n_1(\delta)\right).$$

First,

$$\frac{\beta n_1(\delta)}{1-\left(\frac{p_2}{p_1+p_2}\right)^2} - n_1(\delta) \le n,$$

and therefore,

$$\mathbb{P}\left(\sum_{i=1}^{n_1(\delta)} \min(R_i^2, S_i^2) < \frac{\beta n_1(\delta)}{1-\left(\frac{p_2}{p_1+p_2}\right)^2} - n_1(\delta)\right) \le e^{-(\beta-1-\log\beta)n_1(\delta)}. \tag{3.21}$$

Next, let

$$D_3(\beta,\delta) := \left\{\sum_{i=1}^{n_1(\delta)} \min(R_i^2, S_i^2) \ge \frac{\beta n_1(\delta)}{1-\left(\frac{p_2}{p_1+p_2}\right)^2} - n_1(\delta)\right\}.$$

Letting $\delta = p_2^3$ and $\beta = 1-p_2^3$, and when $D_2(\delta)$ and $D_3(\beta,\delta)$ both hold, then

$$\begin{aligned}
LC_n &\ge \frac{\beta n_1(\delta)}{1-\left(\frac{p_2}{p_1+p_2}\right)^2} - n_1(\delta) + n_1(\delta) \\
&= np_2^2 \frac{p_1-p_2^3}{(p_1+p_2)^2-p_2^2} + n(p_1-p_2^3) - np_2^2 \frac{p_2(p_1-p_2^3)}{1-\left(\frac{p_2}{p_1+p_2}\right)^2} \\
&= np_1 + \left(\frac{(p_1-p_2^3)(1-p_2(p_1+p_2)^2)}{p_1(p_1+2p_2)} - p_2\right) np_2^2 \\
&\ge np_1 + \left(\frac{(p_1-p_2^3)(1-p_2)}{p_1(1+p_2)} - p_2\right) np_2^2 \\
&\ge np_1 + \left((1-p_2)^2 - p_2\right) np_2^2.
\end{aligned}$$

Since $D_2(p_2^3) \cap D_3(1-p_2^3, p_2^3) \subset D_1$, it follows from Hoeffding's inequality and (3.21) that

$$\mathbb{P}(D_1) \ge 1 - 4\exp(-2np_2^6) - \exp\left(n(p_2^3 + \log(1-p_2^3))(p_1-p_2^3)\right).$$

■

To state our next lemma, let us introduce some more notation. First, let

$$V(k) := \left\{(v_1, v_2, \dots, v_k) \in \mathbb{Z}^k : |v_1| + \cdots + |v_k| \le 2k\right\}, \tag{3.22}$$

and then let

$$P := \bigcup_{2k \geq np_2^2} V(k). \tag{3.23}$$

With these definitions, the previous lemma further yields:

Lemma 3.2 *Let O be the set of all the optimal alignments of $X = (X_i)_{1\leq i\leq n}$ and $Y = (Y_i)_{1\leq i\leq n}$, let $D = \{O \subset P\}$, let $p_1 > 1/2$ and let $p_2 < 1/10$. Then,* $\mathbb{P}(D) \geq 1 - 5\exp\left(-np_2^6/5\right)$.

Proof Let N_1^X be the number of letters α_1 in X, and N_1^Y be the corresponding number in Y, and so $N_1 = N_1^X + N_1^Y$. From the proof of the previous lemma, with its notation, it is clear that:

$$D_1 \cap D_2(p_2^3) \subset \left\{LC_n \geq \frac{N_1}{2} - np_2^3 + \left((1-p_2)^2 - p_2\right) np_2^2\right\} \tag{3.24}$$

$$\subset \left\{LC_n \geq \frac{N_1}{2} + \frac{1}{2}np_2^2\right\} := \tilde{D}_1(p_2^2), \tag{3.25}$$

since $p_2 < 1/10$. But, $D_2(p_2^3) \cap D_3(1-p_2^3, p_2^3) \subset D_1$, so as in the previous lemma,

$$\mathbb{P}\left(LC_n \geq \frac{N_1}{2} + \frac{1}{2}np_2^2\right) \geq 1 - 4\exp(-2np_2^6) - \exp\left(n(p_2^3 + \log(1-p_2^3))(p_1 - p_2^3)\right)$$
$$\geq 1 - 5\exp\left(-np_2^6/5\right),$$

since again $p_2 < 1/10$. It remains to show that $\tilde{D}_1(p_2^2) \subset D$. But, for any alignment with $|v| = k \geq 0$,

$$LC_n \leq \frac{N_1}{2} - \frac{1}{2}\sum_{i=1}^{k} |v_i| + k, \tag{3.26}$$

while on $\tilde{D}_1(p_2^2)$,

$$LC_n \geq \frac{N_1}{2} + \frac{1}{2}np_2^2. \tag{3.27}$$

In case $|v| = 0$, no optimal alignment do satisfy both (3.26) and (3.27), while for $|v| \geq 1$, they both combine to yield $\sum_{i=1}^{k} |v_i| \leq 2k$ and $np_2^2 \leq 2k$, and this finishes the proof. ■

The previous lemma asserts that, with high probability, any optimal alignment belongs to the set P. Hence, in order to prove that the optimal alignments satisfy a property, one needs, essentially, to only prove it for the alignments in P.

3.4 High Probability Events

Recall, from Definition 3.1, that any $v \in \mathbb{Z}^k, k \geq 1$ is associated with an alignment having $k = |v|$ cells $C_v(1), \ldots, C_v(|v|)$, and that a cell is called a nonzero-cell if it contains a different number of letters α_1 on the X-strand and on the Y-strand. For any $\theta > 0$, let W^θ be the subset of P, consisting of the alignments having a proportion of nonzero-cells at least equal to θ, i.e.,

$$W^\theta := \{v \in P : |\{i \in [1,k] : v_i \neq 0\}| \geq \theta |v|\},$$

and let $(W^\theta)^c := P \backslash W^\theta$.

To complete the proof of the theorem, some further relevant events need to be defined.

- For any $v \in P$, let E_v^θ be the event that the proportion of zero-cells in $C_v(1), \ldots, C_v(|v|)$, is at least equal to θ. Then, let

$$E^\theta := \bigcap_{v \in (W^\theta)^c} E_v^\theta := \bigcap_{v \in (W^\theta)^c} \{I_b \geq \theta J_0\},$$

  where J_0 is the number of zero-cells while I_b is the number of breakable zero-cells for v, i.e., E^θ is the event that every $v \in (W^\theta)^c$ has a proportion of breakable zero-cells at least equal to θ.
- Recall also from (3.12) and (3.13), that N_v^- is the total number of non-α_1 letters in the cell strands with the lesser number of α_1, and that $N_{>1}$ is the total number of non-α_1 letters in X and Y. Then, let

$$F^\theta := \bigcap_{v \in W^\theta} F_v := \bigcap_{v \in W^\theta} \left\{N_v^- \geq \frac{K}{m} N_{>1}\right\},$$

  i.e., F^θ is the event that for every $v \in W^\theta$, the proportion of non-α_1 letters which are on the cell-strand with the smaller number of letters α_1, is at least equal to K/m.
- Let

$$G^\theta := \bigcap_{v \in W^\theta} G_v := \bigcap_{v \in W^\theta} \left\{2|v| \leq \frac{K}{2m} N_{>1}\right\},$$

  i.e., G^θ are the alignments $v \in W^\theta$ having a proportion of aligned non-α_1 letters at most equal to $K/2m$.

Finally recall from Sect. 3.2 that $A_n = \{(X,Y) \in \mathcal{B}_n\}$ is the event that there exists an optimal alignment, with $|v| \geq 1$, such that $N_v^- \geq KN_{>1}/m$ and $2|v| \leq KN_{>1}/2m$, and therefore

$$D \cap E^\theta \cap F^\theta \cap G^\theta \subset A_n. \tag{3.28}$$

Our next task is to prove that each one of the events $E^\theta, F^\theta, G^\theta$ hold with high probability. Let us start with E^θ.

Lemma 3.3 *Let* $0 < \theta \leq p_1^2/(1+p_1^2)$*, then*

$$\mathbb{P}(E^\theta) \geq 1 - \sum_{2k \geq np_2^2} \exp\left(-\left(2(1-\theta)\left(\frac{p_1^2}{1+p_1^2} - \theta\right)^2 - \log f(\theta)\right)k\right), \tag{3.29}$$

where $f(\theta) = \left((4+2\theta)/\theta^2\right)^\theta ((2+\theta)/2)^2 (1/(1-\theta))^{1-\theta}$.

Proof For any $v \in P \backslash W^\theta$, let us compute the probability that a zero-cell in the alignment associated with v is breakable. Recalling the definition of T_0 in (3.16), for $2 \leq j \leq m$, let M_j be the event that this cell ends with a pair of letters α_j. So, when M_j holds, then $T_0 = T_0^j$. For $2 \leq j \leq m$, let also

$$U_1^j := \min\{i = 2, 3, \ldots : \quad R_{i-1}^j \neq 0, \quad S_{i-1}^j = 0, \quad R_i^j = 0, \quad S_i^j \neq 0\},$$
$$U_2^j := \min\{i = 2, 3, \ldots : \quad R_{i-1}^j = 0, \quad S_{i-1}^j \neq 0, \quad R_i^j \neq 0, \quad S_i^j = 0\},$$

and

$$U^j := \min\{U_1^j, U_2^j\}.$$

With the above constructions, conditional on the event M_j, if $U^j < T_0^j$ then this zero-cell is breakable and thus, to lower bound the probability that it is breakable, it is enough to lower bound $\mathbb{P}(U^j < T_0^j)$. To do so, let first $(Z_i^j)_{i \geq 1}$ be the independent random vectors given by:

$$Z_i^j = (R_{2i-1}^j, S_{2i-1}^j, R_{2i}^j, S_{2i}^j).$$

Then, let

$$\tilde{U}^j = \min\{i = 1, 2, \ldots : \quad Z_i^j \in B_1 \cup B_2\},$$
$$\tilde{T}_0^j = \min\{i = 1, 2, \ldots : \quad Z_i^j \in B_3 \cup B_4\},$$

where

$$B_1 := \mathbb{N}^* \times \{0\} \times \{0\} \times \mathbb{N}^*, B_2 := \{0\} \times \mathbb{N}^* \times \mathbb{N}^* \times \{0\},$$
$$B_3 := \mathbb{N}^* \times \mathbb{N}^* \times \mathbb{N} \times \mathbb{N}, \quad B_4 := \mathbb{N} \times \mathbb{N} \times \mathbb{N}^* \times \mathbb{N}^*,$$

and where as usual $\mathbb{N}$ is the set of non-negative integers, while $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$. Clearly, $2\tilde{U}^j \geq U^j$ and $2\tilde{T}_0^j - 1 \leq T_0^j$, thus $\mathbb{P}(U^j < T_0^j) \geq \mathbb{P}(2\tilde{U}^j < 2\tilde{T}_0^j - 1) = \mathbb{P}(\tilde{U}^j < \tilde{T}_0^j)$. Now, since the random vectors $(Z_i^j)_{i \geq 1}$ are iid, and since $B_1 \cup B_2$ and $B_3 \cup B_4$ are pairwise disjoint,

$$\begin{aligned}\mathbb{P}(\tilde{U}^j < \tilde{T}_0^j) &= \frac{\mathbb{P}(Z_i^j \in B_1 \cup B_2)}{\mathbb{P}(Z_i^j \in B_1 \cup B_2) + \mathbb{P}(Z_i^j \in B_3 \cup B_4)} \\ &= \frac{2p_1^2}{2p_1^2 + 2(p_1 + p_j)^2 - p_j^2} \geq \frac{p_1^2}{1 + p_1^2}.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbb{P}(\text{a zero-cell is breakable}) &= \sum_{j=2}^{m} \mathbb{P}(\text{a zero-cell is breakable}|M_j)\mathbb{P}(M_j) \\ &= \sum_{j=2}^{m} \mathbb{P}(U^j < T_0^j)\mathbb{P}(M_j) \geq \frac{p_1^2}{1 + p_1^2}.\end{aligned}$$

Let J be the index set of all the zero-cells in the alignment associated with $v \in (W^\theta)^c$, and so $|J| \geq (1 - \theta)|v|$. For each $i \in J$, let I_i be the Bernoulli random variable which is one if the cell $C_v(i)$ is breakable and 0 otherwise. Recall that E_v^θ is the event that the proportion of breakable cells in v is at least equal to θ. Then, since $\theta \leq p_1^2/(1 + p_1^2)$, from Hoeffding's inequality, and after subtracting the mean,

$$\mathbb{P}((E_v^\theta)^c) = \mathbb{P}\left(\sum_{i \in J} I_i < \theta |J|\right) \leq \exp\left(-2(1 - \theta)|v| \left(\frac{p_1^2}{1 + p_1^2} - \theta\right)^2\right).$$

Recall now the definition of $V(k)$ in (3.22) and let $(W^\theta(k))^c := (W^\theta)^c \cap V(k)$. For any two integers, ℓ and $q\ell$, with $0 < q < 1$, Stirling's formula in the form $1 \leq \ell! e^\ell / (\sqrt{2\pi\ell}\ell^\ell) \leq e/\sqrt{2\pi}$, gives

$$\binom{\ell}{q\ell} \leq q^{-q\ell}(1 - q)^{-(\ell - q\ell)}, \tag{3.30}$$

which, when combined with simple estimates yields,

$$|(W^{\theta}(k))^c| \le 2^{\theta k}\binom{2k+\theta k}{\theta k}\binom{k}{\theta k}$$
$$\le (f(\theta))^k := \left(\left(\frac{4+2\theta}{\theta^2}\right)^{\theta}\left(\frac{2+\theta}{2}\right)^{2}\left(\frac{1}{1-\theta}\right)^{1-\theta}\right)^{k}. \tag{3.31}$$

Next, let $E^{\theta}(k) = \bigcap_{v\in(W^{\theta}(k))^c} E^{\theta}_v$, then

$$\mathbb{P}((E^{\theta}(k))^c) \le \sum_{v\in(W^{\theta}(k))^c} \mathbb{P}((E^{\theta}_v)^c)$$
$$\le \exp\left(-\left(2(1-\theta)\left(\frac{p_1^2}{1+p_1^2}-\theta\right)^2 - \log f(\theta)\right)k\right),$$

and therefore,

$$\mathbb{P}((E^{\theta})^c) \le \sum_{2k\ge np_2^2} \mathbb{P}((E^{\theta}(k))^c)$$
$$\le \sum_{2k\ge np_2^2} \exp\left(-\left(2(1-\theta)\left(\frac{p_1^2}{1+p_1^2}-\theta\right)^2 - \log f(\theta)\right)k\right). \tag{3.32}$$

■

Of course, in (3.32), one wants

$$2(1-\theta)\left(\frac{p_1^2}{1+p_1^2}-\theta\right)^2 - \log f(\theta) > 0, \tag{3.33}$$

and choices of θ for which this is indeed the case are given later.

Let u be a non-negative integer. For any $-u$-cell ending with an aligned pair of letters α_j (the event M_j holds for this cell), let $\tau^j_X(\ell)$ be the index of the ℓ-th R^j_i such that $R^j_i \ne 0$, i.e.,

$$\tau^j_X(1) = \min\{i \ge 1 :\ R^j_i \ne 0\},$$

and for any $\ell \ge 1$, $\tau^j_X(l+1) = \min\{i > \tau^j_X(\ell) :\ R^j_i \ne 0\}$. Let

$$\rho^{j,-} := \min\{\ell = 1, 2, \ldots :\ S^j_{u+\tau^j_X(\ell)} \ne 0\}.$$

In words, $\rho^{j,-}$ is the number of nonzero values taken by $R^j = (R_i^j)_{1\le i\le s}$ (where s is the number of letters α_1 in the X-strand of the cell). Since X and Y are independent,

$$\begin{aligned}\mathbb{P}(\rho^{j,-}=k) &= \mathbb{P}(S^j_{u+\tau_X^j(1)}=0,\ldots,S^j_{u+\tau_X^j(k-1)}=0, S^j_{u+\tau_X^j(k)}\neq 0)\\ &= \left(\frac{p_1}{p_1+p_j}\right)^{k-1}\frac{p_j}{p_1+p_j},\end{aligned} \tag{3.34}$$

for $k = 1,2,\ldots$. Thus, $\rho^{j,-}$ has a geometric distribution with parameter $\tilde{p}_j = p_j/(p_1+p_j)$, $2 \le j \le m$. (By just replacing τ_X by τ_Y the random variables $\rho^{j,-}$ can then be defined for u-cells. Hence, since X and Y have the same law, the corresponding law of $\rho^{j,-}$ remains unchanged, therefore taking care of all the cases.) When $-u < 0$, the number of letters α_j in the X-strand (which is the strand with the smaller number of letters α_1) is at least $\rho^{j,-}-1$ and, as shown in the next lemma, this provides a lower bound for N_v^- (the number of non-α_1 letters on the cell-strand with the lesser number of letters α_1) in this $-u$-cell.

Recalling now that $F^\theta = \bigcap_{v\in W^\theta}\{N_v^- \ge KN_{>1}/m\}$, we have:

Lemma 3.4 *Let* $0<\theta<1$*, let* $K = 2^{-4}10^{-2}e^{-67}$*, and let* $p_1 \ge 1-e^{-67}/4$*. Then,* $\mathbb{P}(F^\theta) \ge 1-38\exp(-3np_2^2/200)$.

Proof For any $v \in W^\theta$, let J be the index set of all the nonzero-cells of the alignment corresponding to v, hence, $|J| \ge \theta|v|$. Then,

$$N_v^- = \sum_{i=1}^{|v|} N_v^-(i) = \sum_{i\in J} N_v^-(i) \ge \sum_{i\in J}\left(\rho_i^{j(i),-}-1\right),$$

where $j(i)$ is the index of the last aligned pair of letters α_j in the cell $C_v(i)$, and where $\rho_i^{j(i),-}$ is the number of nonzero $R^{j(i)} = (R_\ell^{j(i)})_{1\le\ell\le s}$ (assuming this is a $-u$-cell, and that s is the number of letters α_1 in the X-strand of $C_v(i)$. In case of a u-cell, by symmetry, the same argument is valid on the Y-strand). From (3.34), $\rho_i^{j(i),-}$ is a geometric random variable with parameter $\tilde{p}_{j(i)}$. Now, let $\varepsilon > 0$, let again $\tilde{p}_2 = p_2/(p_1+p_2)$, and let $F_{1,v} := \{N_v^- \ge \varepsilon|v|/\tilde{p}_2\}$. Then,

$$\begin{aligned}\mathbb{P}(F_{1,v}^c) &\le \mathbb{P}\left(\sum_{i\in J}\left(\rho_i^{j(i),-}-1\right) \le \frac{\varepsilon}{\tilde{p}_2}|v|\right)\\ &\le \mathbb{P}\left(\sum_{i\in J}\rho_i^{j(i),-} \le \frac{\varepsilon/\theta+\tilde{p}_2}{\tilde{p}_2}|J|\right)\\ &\le \mathbb{P}\left(\sum_{i\in J}\rho_i^{j(i),-} \le \frac{\varepsilon/\theta+2p_2}{\tilde{p}_2}|J|\right).\end{aligned} \tag{3.35}$$

The geometric random variables $\rho_i^{j(i),-}$, $i \in J$, are independent each with parameter $\tilde{p}_{j(i)} \leq \tilde{p}_2$, and moreover the sequences have finite length n, therefore,

$$\mathbb{P}\left(\sum_{i\in J}\rho_i^{j(i),-} \leq \frac{\varepsilon/\theta + 2p_2}{\tilde{p}_2}|J|\right) \leq \mathbb{P}\left(\sum_{i\in J}(\mathcal{G}_i \wedge n) \leq \frac{\varepsilon/\theta + 2p_2}{\tilde{p}_2}|J|\right),$$

where the $\mathcal{G}_i$ are iid geometric random variables with parameter $\tilde{p}_2$. As proved later, and using (3.20), when

$$\frac{\varepsilon/\theta + 2p_2}{\tilde{p}_2}|J| < n, \tag{3.36}$$

it follows that

$$\mathbb{P}\left(\sum_{i\in J}\rho_i^{j(i),-} \leq \frac{\varepsilon/\theta + 2p_2}{\tilde{p}_2}|J|\right) \leq \mathbb{P}\left(\sum_{i\in J}\mathcal{G}_i \leq \frac{\varepsilon/\theta + 2p_2}{\tilde{p}_2}|J|\right)$$
$$\leq \exp\left((1 + \log(\varepsilon/\theta + 2p_2))\,\theta|v|\right). \tag{3.37}$$

Let $F_1^\theta(k) := \bigcap_{v\in W^\theta\cap V(k)} F_{1,v} = \bigcap_{v\in W^\theta\cap V(k)} \{N_v^- \geq \varepsilon|v|/\tilde{p}_2\}$, and let $F_1^\theta := \bigcap_{2k\geq np_2^2} F_1^\theta(k)$. From the very definition of $V(k)$ in (3.22), and using (3.30),

$$|V(k)| \leq 2^k\binom{3k}{k} \leq 2^k 3^k \left(\frac{3}{2}\right)^{2k} = \left(\frac{27}{2}\right)^k,$$

which when combined with (3.37) leads to

$$\mathbb{P}(F_1^\theta(k)) \geq 1 - \exp\left(k\log(27/2) + k\left(1 + \log(\varepsilon/\theta + 2p_2)\right)\theta\right). \tag{3.38}$$

Of course, one wants $\log(27/2) + (1 + \log(\varepsilon/\theta + 2p_2))\,\theta < 0$. Choosing $\theta = 1/25$ and $\varepsilon = 10^{-2}e^{-67}$, then $\mathbb{P}((F_1^\theta(k))^c) \leq e^{-3k/100}$, for any $p_1 \geq 1 - 2^{-2}e^{-67}$, and so

$$\mathbb{P}((F_1^\theta)^c) \leq \sum_{2k\geq np_2^2}\mathbb{P}((F_1^\theta(k))^c) \leq \frac{\exp(-3np_2^2/200)}{1-\exp(-3/100)} \leq 34\exp(-3np_2^2/200).$$

Note also that for these choices of θ and p_1, (3.33) is satisfied and so E^θ also holds with high probability.

From the proof of Lemma 3.1, when $D_2((1-p_1))$ holds, the total number of non-α_1 letters in X and Y is at most $4n(1-p_1)$. Thus $N_{>1} \leq 4n(1-p_1)$, and so when

$F_1^\theta \cap D_2((1-p_1))$ holds, for every $v \in W^\theta$,

$$\frac{N_v^-}{N_{>1}} \ge \frac{\varepsilon|v|}{\tilde{p}_2 4n(1-p_1)} \ge \frac{\varepsilon}{\tilde{p}_2 4n(1-p_1)} \frac{np_2^2}{2} \ge \frac{\varepsilon p_2}{16(1-p_1)} \ge \frac{\varepsilon}{16m} \ge \frac{K}{m}.$$

Also note that by properly choosing these constants and under the further condition $400mK < 1$, it follows that (3.36) holds true. Therefore,

$$\begin{aligned}\mathbb{P}((F^\theta)^c) &\le \mathbb{P}((F_1^\theta)^c) + \mathbb{P}((D_2(1-p_1))^c) \\ &\le 34\exp(-3np_2^2/200) + 4\exp(-2n(1-p_1)^2) \\ &\le 38\exp(-3np_2^2/200).\end{aligned}$$

■

Recalling that $G^\theta = \bigcap_{v\in W^\theta}\{2|v| \le KN_{>1}/2m\}$, we finally have:

Lemma 3.5 *Let* $0 < \theta < 1$, *let* $K = 2^{-4}10^{-2}e^{-67}$, *and moreover let* $p_2 \le \min\{2^{-2}e^{-5}K/m, K/2m^2\}$. *Then,* $\mathbb{P}(G^\theta) \ge 1 - 8\exp(-np_2^2/2)$.

Proof For any $v \in W^\theta$, let $C_v(1), \ldots, C_v(|v|)$ be the corresponding cells. If the cell $C_v(i)$ ends with a pair of aligned α_j, $2 \le j \le m$, then let $\rho_i^{j(i)}$ be the number of nonzero values taken by $R^{j(i)}$ in $C_v(i)$. If $v_i \le 0$, by the same arguments as in getting (3.34), $\rho_i^{j(i)}$ has a geometric distribution with parameter $\tilde{p}_{j(i)} = p_{j(i)}/(p_1 + p_{j(i)})$. If $v_i > 0$, then there exists a geometric random variable $\rho_i^{j(i),-}$ with parameter $\tilde{p}_{j(i)}$ such that $\rho_i^{j(i),-} \le \rho_i^{j(i)} \le \rho_i^{j(i),-} + v_i$. Let $N_{>1}^X$ (resp. $N_{>1}^Y$) be the number of non-α_1 letters in X (resp. Y), so that $N_{>1} = N_{>1}^X + N_{>1}^Y$, and let

$$G_v^X := \left\{|v| \le \frac{K}{2m}N_{>1}^x\right\} \text{ and } G_v^Y := \left\{|v| \le \frac{K}{2m}N_{>1}^y\right\},$$

and so $G_v^X \cap G_v^Y \subset G_v$. Since $N_{>1}^X \ge \sum_{i=1}^{|v|}\rho_i^{j(i)}$,

$$\begin{aligned}\mathbb{P}\left((G_v^X)^c\right) &\le \mathbb{P}\left(|v| > \frac{K}{2m}\sum_{i=1}^{|v|}\rho_i^{j(i)}\right) \\ &\le \mathbb{P}\left(|v| > \frac{K}{2m}\left(\sum_{1\le i\le|v|,v_i\le 0}\rho_i^{j(i)} + \sum_{1\le i\le|v|,v_i>0}\rho_i^{j(i),-}\right)\right) \\ &\le \mathbb{P}\left(\sum_{i=1}^{|v|}(\mathcal{G}_i \wedge n) < \frac{2m|v|}{K}\right),\end{aligned}$$

where the $\mathcal{G}_i$ are iid geometric random variables with parameter $\tilde{p}_2$ and the truncation is at n since the sequences have such a length. From the proof of Lemma 3.1, when $D_2((1-p_1))$ holds, $N_{>1} \le 4n(1-p_1)$, then $|v| \le 2n(1-p_1)$. Thus $2m|v| \le 2mn(1-p_1) < 2m^2p_2n$, and so if $2m^2p_2 < K$, then for any $p_2 \le 2^{-2}e^{-5}K/m$,

$$\mathbb{P}\left((G_v^X)^c \cap D_2((1-p_1))\right) \le \mathbb{P}\left(\sum_{i=1}^{|v|}\mathcal{G}_i < \frac{2m|v|}{K}\right) \le \mathbb{P}\left(\sum_{i=1}^{|v|}\mathcal{G}_i < \frac{e^{-5}|v|}{\tilde{p}_2}\right) \le \exp(-4|v|). \tag{3.39}$$

Likewise, $\mathbb{P}\left((G_v^Y)^c \cap D_2((1-p_1))\right) \le \exp(-4|v|)$, and thus

$$\mathbb{P}\left((G_v)^c \cap D_2((1-p_1))\right) \le 2\exp(-4|v|).$$

As before, let $G^\theta(k) := \bigcap_{v\in W^\theta \cap V(k)} G_v$ and $G^\theta = \bigcap_{2k\ge np_2^2} G^\theta(k)$, then

$$\mathbb{P}((G^\theta(k))^c \cap D_2((1-p_1))) \le |V(k)|2\exp(-4k) \le 2\exp(-k),$$

and

$$\begin{aligned}\mathbb{P}((G^\theta)^c) &\le \mathbb{P}((G^\theta)^c \cap D_2((1-p_1))) + \mathbb{P}(D_2((1-p_1))^c)\\ &\le \sum_{2k\ge np_2^2}\mathbb{P}((G^\theta(k))^c \cap D_2((1-p_1))) + 4\exp(-2n(1-p_1)^2)\\ &\le \frac{2}{1-1/e}\exp(-np_2^2/2) + 4\exp(-2n(1-p_1)^2)\\ &\le 8\exp(-np_2^2/2).\end{aligned} \tag{3.40}$$

■

From Lemma 3.2–3.5, using (3.28), letting $\theta = 1/25$, $K = 2^{-4}10^{-2}e^{-67}$ and $K_m := \min(K, 1/800m)$, and for $p_2 \le \min\{2^{-2}e^{-5}K_m/m, K_m/2m^2\}$, it follows that:

$$\begin{aligned}\mathbb{P}(A_n^c) &\le \mathbb{P}(D^c) + \mathbb{P}((E^\theta)^c) + \mathbb{P}((F^\theta)^c) + \mathbb{P}((G^\theta)^c)\\ &\le 5\exp\left(-\frac{np_2^6}{5}\right) + 74\exp\left(-\frac{np_2^2}{10^3}\right) + 38\exp\left(-\frac{3np_2^2}{200}\right) + 8\exp\left(-\frac{np_2^2}{2}\right)\\ &\le 125\exp\left(-\frac{np_2^6}{5}\right).\end{aligned} \tag{3.41}$$

This finishes the proof of Theorem 2.1. ■

Remark 3.1

(i) Our results on the central r-th absolute moments of the LCS continue to be valid for three or more sequences of random words. First, the upper bound methods are very easily adapted to provide the same order $n^{r/2}$. Next, for the lower bound, the alignments can still be represented with a series of cells, each of the cells ending with the same non-α_1 letter from every strand. Then, with exponential bounds techniques, a similar high probability event can be exhibited, also leading to a lower bound of order $n^{r/2}$.

(ii) With the methodology developed here, the results of [2, 6] can also be generalized, beyond the variance or the Bernoulli case, to centered absolute moments, m-letters alphabets and even to a general scoring function framework with scoring functions satisfying bounded differences conditions.

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A Weighted Approximation Approach to the Study of the Empirical Wasserstein Distance

David M. Mason

Dedicated to the memory of Evarist Giné.

Abstract We shall demonstrate that weighted approximation technology provides an effective set of tools to study the rate of convergence of the Wasserstein distance between the cumulative distribution function [c.d.f] and the empirical c.d.f.

Keywords Empirical process • Wasserstein distance • Weighted approximation

Mathematics Subject Classification (2010). Primary 60F17, 62E17; Secondary 62E20

1 Introduction

Let $X, X_1, X_2, \ldots,$ be a sequence of independent [i.i.d.] nondegenerate random variables with common cumulative distribution function F [c.d.f.] and left-continuous inverse or quantile function Q, defined for $s \in (0, 1)$ to be

$$Q(s) = \inf\{x : F(x) \geq s\}. \tag{1.1}$$

For each integer $n \geq 1$ let F_n denote the *empirical distribution function*

$$F_n(x) = \frac{1}{n} \sum_{i=1}^{n} 1\{X_i \leq x\}, \quad -\infty < x < \infty. \tag{1.2}$$

D.M. Mason (✉)
Department of Applied Economics and Statistics, University of Delaware, Newark, DE 19716, USA
e-mail: davidm@udel.edu

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Recall that the Wasserstein distance between any two cdfs F and G with finite means is

$$d_1(G,F) = \int_{-\infty}^{\infty} |G(x) - F(x)|\,dx.$$

(This is called the Kantorovich transport distance. For more about terminology see the footnote on page 4 of Bobkov and Ledoux [1].) In particular, the (empirical) Wasserstein distance between F_n, based on $X_1, , \ldots, X_n$ i.i.d. F, and F is

$$d_1(F_n,F) = \int_{-\infty}^{\infty} |F_n(x) - F(x)|\,dx,$$

where Q is defined as in (1.1). Note that $d_1(F_n,F)$ is finite as long as $E|X_1| < \infty$. The empirical Wasserstein distance $d_1(F_n,F)$ also has the representation

$$d_1(F_n,F) = \int_0^1 |Q_n(t) - Q(t)|\,dt,$$

where Q_n is the empirical quantile function defined for $t \in (0,1)$,

$$Q_n(t) = \inf\{x : F_n(x) \geq t\}.$$

(See for instance, Exercise 3 on page 64 of Shorack and Wellner [14].)

Let $U, U_1, U_2, \ldots,$ be independent Uniform $(0,1)$ random variables. For each integer $n \geq 1$ the *empirical distribution function* based on $U_1, \ldots, U_n$, is defined to be

$$G_n(t) = \frac{1}{n}\sum_{i=1}^{n} 1\{U_i \leq t\},\ t \in [0,1]. \tag{1.3}$$

Note that if $X_1, \ldots, X_n$, $n \geq 1$, are i.i.d. F and $U_1, \ldots, U_n$ are i.i.d. Uniform $(0,1)$ random variables, then by the probability integral transformation

$$(X_1, \ldots, X_n) =_d (Q(U_1), \ldots, Q(U_n)). \tag{1.4}$$

This implies that

$$d_1(F_n,F) =_d \int_0^1 |G_n(t) - t|\,dQ(t). \tag{1.5}$$

We shall show how to use weighted approximation technology to obtain rates of convergence of $Ed_1(F_n,F)$ to zero. This will lead to refinements and complements to Theorem 6.7 of Bobkov and Ledoux [1], which in our notation says that for a universal constant $c > 0$,

$$2c\int_{\left\{t:t(1-t)\le\frac{1}{4n}\right\}} t(1-t)dQ(t) + \frac{2c}{\sqrt{n}}\int_{\left\{t:t(1-t)>\frac{1}{4n}\right\}} \sqrt{t(1-t)}dQ(t)$$
$$\le Ed_1(F_n,F) \le \int_{\left\{t:t(1-t)\le\frac{1}{4n}\right\}} t(1-t)dQ(t)$$
$$+\frac{1}{\sqrt{n}}\int_{\left\{t:t(1-t)>\frac{1}{4n}\right\}} \sqrt{t(1-t)}dQ(t), \tag{1.6}$$

where c may chosen to be $\frac{1}{2}5^{-4}$. They base their proof on their Lemma 3.8 in [1], a version of which is stated in (3.23) below.

In Sect. 2 we shall describe the weighted approximation tools that we shall be using. Next, in Sect. 3 we shall apply them to obtain rates at which $Ed_1(F_n,F)$ converges to zero. Then in Sect. 4 we shall discuss the original motivation to develop the exponential inequality for the weighted approximation to the uniform empirical process stated in Sect. 2.2.

2 The Mason and van Zwet Refinement of KMT (1975)

Much of our analysis will be based on weighted approximations to the *uniform empirical process,* which is defined by

$$\alpha_n(t) = \sqrt{n}\{G_n(t) - t\},\ t \in [0,1]. \tag{2.1}$$

Mason and van Zwet [11] obtained the following refinement of the Komlós, Major and Tusnády [KMT] [9] Brownian bridge approximation to the uniform empirical process.

Theorem 2.1 *There exists a probability space* (Ω, A, P) *with independent Uniform* $(0,1)$ *random variables* $U_1,\ U_2,\ \ldots,$ *and a sequence of Brownian bridges* $B_1, B_2,\ \ldots,$ *such that for all* $n \ge 1$, $1 \le d \le n$, *and* $-\infty < x < \infty$,

$$P\left\{\sup_{0\le t\le d/n}|\alpha_n(t) - B_n(t)| \ge n^{-1/2}(a\log d + x)\right\} \le b\exp(-cx) \tag{2.2}$$

and

$$P\left\{\sup_{1-d/n\le t\le 1}|\alpha_n(t) - B_n(t)| \ge n^{-1/2}(a\log d + x)\right\} \le b\exp(-cx), \tag{2.3}$$

where a, b *and* c *are suitable positive constants independent of* n, d *and* x

Setting $d = n$ into these inequalities yields the original KMT [9] inequality given in their Theorem 3.

Remark Actually, KMT [9] construct for each $n \geq 1$ a probability space on which sit i.i.d. Uniform (0, 1) random variables $U_1, \ldots, U_n$ and a Brownian bridge B_n such that inequality (2.2) holds with $d = 1$. Lemma 3.1.2 of Csörgő [3] details how to use their result holding for each n to construct on the same probability space a sequence of i.i.d. $U_1, U_2 \ldots,$ Uniform (0, 1) random variables and a sequence of Brownian bridges $B_1, B_2 \ldots,$ such that for all $n \geq 1$ inequality (2.2) holds with $d = 1$ on this space. Mason and van Zwet [11] show in their proof that inequalities (2.2) and (2.3) hold for each $n \geq 1$ and $1 \leq d \leq n$ for the original KMT [9] construction and then, in turn, apply Lemma 3.1.2 of [3] to construct the extended space of Theorem 2.1

2.1 Mason and van Zwet Weighted Approximations

Mason and van Zwet [11] pointed out that their inequality leads to the following useful weighted approximations. For any $0 \leq \nu < 1/2$, $n \geq 1$, and $1 \leq d \leq n$ let

$$\Delta_{n,\nu}^{(1)}(d) := \sup_{d/n \leq t \leq 1} \frac{n^{\nu} |\alpha_n(t) - B_n(t)|}{t^{1/2-\nu}}, \tag{2.4}$$

$$\Delta_{n,\nu}^{(2)}(d) := \sup_{0 \leq t \leq 1-d/n} \frac{n^{\nu} |\alpha_n(t) - B_n(t)|}{(1-t)^{1/2-\nu}}, \tag{2.5}$$

and for $1 \leq d \leq n/2$, set

$$\Delta_{n,\nu}(d) := \sup_{d/n \leq t \leq 1-d/n} \frac{n^{\nu} |\alpha_n(t) - B_n(t)|}{(t(1-t))^{1/2-\nu}}. \tag{2.6}$$

By arguing exactly as in the proof of Theorem 2.1 in Csörgő et al. [4] with α_n replacing the uniform quantile process β_n, one easily verifies that on the probability space of Theorem 2.1, one has for any $0 \leq \nu < 1/2$

$$\Delta_{n,\nu}(1) = O_p(1), \tag{2.7}$$

with the same holding with $\Delta_{n,\nu}(1)$ replaced by $\Delta_{n,\nu}^{(1)}(1)$ and $\Delta_{n,\nu}^{(2)}(1)$.

2.2 An Exponential Inequality for the Weighted Approximation to the Uniform Empirical Process

Mason [10] derived the following exponential inequality for the Mason and van Zwet weighted approximations, which was motivated by a question of Evarist Giné. We shall discuss this in Sect. 4.

Theorem 2.2 *On the probability space of Theorem 2.1 for every* $0 \le \nu < 1/2$ *there exist positive constants* A_ν *and* C_ν *such that for all* $n \ge 2$, $1 \le d \le n/2$ *and* $0 \le x < \infty$,

$$P\{\Delta_{n,\nu}(d) \ge x\} \le 2A_\nu \exp(d^{1/2-\nu}C_\nu) \exp\left(-\frac{d^{1/2-\nu}cx}{4}\right),$$

with similar inequalities for $\Delta^{(1)}_{n,\nu}(d)$ *and* $\Delta^{(2)}_{n,\nu}(d)$ *for* $1 \le d \le n$.

2.3 A Moment Bound for the Weighted Approximation

Theorem 2.2 readily yields the following uniform moment bounds for (2.4), (2.5) and (2.6).

Proposition 2.3 *On the probability space of Theorem 2.1, for all* $0 \le \nu < 1/2$ *there exists a* $\gamma > 0$ *such that*

$$\sup_{n\ge 2} E\exp\left(\gamma\Delta_{n,\nu}(1)\right) < \infty,$$

with the same statement holding with $\Delta_{n,\nu}(1)$ *replaced by* $\Delta^{(1)}_{n,\nu}(1)$ *or* $\Delta^{(2)}_{n,\nu}(1)$. *In particular, we have for all* $r > 0$, $\sup_{n\ge 2} E\Delta^{r}_{n,\nu}(1) < \infty$.

There is also a functional version of Proposition 2.3. For each integer $n \ge 2$ let R_n denote a class of nondecreasing left-continuous functions r on $[1/n, 1-1/n]$. Assume there exists a sequence of positive constants D_n such that for some $0 \le \nu < 1/2$

$$\sup_{n\ge 2}\sup_{r\in\mathcal{R}_n} D_n^{-1}\int_{1/n}^{1-1/n}(s(1-s))^{1/2-\nu}dr(s) =: M < \infty. \tag{2.8}$$

From Proposition 2.3 we obtain the following functional form of it.

Proposition 2.4 *Let* $\{R_n, n \ge 2\}$ *denote a sequence of classes of nondecreasing left-continuous functions on* $[1/n, 1-1/n]$ *satisfying (2.8) for some* $0 \le \nu < 1/2$. *On the probability space of Theorem 2.1 there exists a* $\gamma > 0$ *such that*

$$\sup_{n\ge 2} E\exp(\gamma n^{\nu} I_n) < \infty, \tag{2.9}$$

where

$$I_n := \sup_{r\in\mathcal{R}_n} D_n^{-1}\int_{1/n}^{1-1/n}|\alpha_n(s) - B_n(s)|dr(s).$$

Proposition 2.4 follows trivially from Proposition 2.3 by observing that $n^{\nu} I_n \leq \Delta_{n,\nu}(1)M$.

All of the results stated in Sects. 2.2 and 2.3 are found in Mason [10].

3 Use of Theorem 2.2 to Study the Empirical Wasserstein Distance

In order to state our results we first need a definition and some background material. We begin with the definition of the domain of attraction to a normal law.

Let $X, X_1, X_2, \ldots,$ be a sequence of independent nondegenerate random variables with common c.d.f. F and left-continuous inverse or quantile function Q. We say that F is in the domain of attraction of a normal law, written $F \in DN$, if there exist norming and centering constants b_n and c_n such that

$$\frac{\sum_{i=1}^{n} X_i - c_n}{b_n} \to_d Z, \tag{3.1}$$

where here and elsewhere Z denotes a standard normal random variable. Csörgő et al. [5] show that whenever $F \in DN$ one can always choose for $n \geq 2$, $c_n = nEX$ and $b_n = \sqrt{n}\sigma\left(1/n\right)$, where for any $0 < u < 1/2$

$$\sigma^2\left(u\right) := \int_u^{1-u} \int_u^{1-u} \left(s \wedge t - st\right) dQ\left(s\right) dQ\left(t\right). \tag{3.2}$$

For future reference we shall write for any $0 < u < 1/2$

$$\tau^2\left(u\right) = \left(\int_u^{1-u} \sqrt{s\left(1-s\right)} dQ\left(s\right)\right)^2, \tag{3.3}$$

and note that

$$\tau^2\left(u\right) \geq \sigma^2\left(u\right). \tag{3.4}$$

Observe that

$$\sigma^2\left(u\right) = Var\left(\int_u^{1-u} \left(1\left\{U \leq t\right\} - t\right) dQ\left(t\right)\right),$$

where U is Uniform $(0, 1)$. Furthermore $\sigma^2(0) := \sigma^2(0+) < \infty$ if and only if $\sigma^2 := VarX$ is finite, in which case $\sigma^2(0) = \sigma^2$.

Now with F_n as in (1.2), we have by (1.4)

$$\frac{\sum_{i=1}^{n} X_i - nEX}{b_n} = \frac{\int_{-\infty}^{\infty} \sqrt{n}\{F(x) - F_n(x)\}\,dx}{\sigma(1/n)} =_d \frac{-\int_0^1 \alpha_n(s)dQ(s)}{\sigma(1/n)}.$$

In fact one can use weighted approximation technology to show whenever $F \in DN$ that on the probability space on which (2.7) holds

$$\frac{\int_0^1 \alpha_n(s)dQ(s)}{\sigma(1/n)} = \frac{\int_{1/n}^{1-1/n} B_n(s)dQ(s)}{\sigma(1/n)} + o_p(1) =_d Z + o_p(1),$$

Crucial to the proof is the fact proved in Corollary 1 of Csörgő et al. [5] that $F \in DN$ if and only if

$$\lim_{u\searrow 0} u\left(Q^2(\lambda u) + Q^2(1-\lambda u)\right)/\sigma^2(u) = 0, \text{ for all } \lambda > 0 \tag{3.5}$$

if and only if σ is slowly varying at zero, i.e.

$$\lim_{u\searrow 0} \sigma^2(\lambda u)/\sigma^2(u) = 1, \text{ for all } \lambda > 0. \tag{3.6}$$

In our proofs that follow, whenever we apply Proposition 2.3 we assume that we are on the probability space of Theorem 2.1. Our first result related to $Ed_1(F_n, F)$ is the following estimate of a trimmed version of this expectation.

Proposition 3.1 *For any quantile function Q and $p > 1$,*

$$\int_{\frac{1}{n}}^{1-\frac{1}{n}} E|\alpha_n(t)|\,dQ(t) = \int_{\frac{1}{n}}^{1-\frac{1}{n}} E|B(t)|\,dQ(t)(1 + O(1)) \tag{3.7}$$

$$= \sqrt{\frac{2}{\pi}} \int_{\frac{1}{n}}^{1-\frac{1}{n}} \sqrt{t(1-t)}dQ(t)(1 + O(1)), \tag{3.8}$$

where B is a Brownian bridge on $[0, 1]$ *and the big Oh term in (3.8) is bounded in absolute value by $c_p(r(1/n))^{1-1/p}$ for some constant c_p depending on p and*

$$r(1/n) = \frac{\left|Q\left(\frac{1}{n}\right)\right| + \left|Q\left(1-\frac{1}{n}\right)\right|}{n^{1/2}\int_{\frac{1}{n}}^{1-\frac{1}{n}} \sqrt{t(1-t)}dQ(t)}. \tag{3.9}$$

Furthermore, if

$$r(1/n) \to 0, \text{ as } n \to \infty, \tag{3.10}$$

then

$$\int_{\frac{1}{n}}^{1-\frac{1}{n}} E\left|\alpha_n(t)\right| dQ(t) = \int_{\frac{1}{n}}^{1-\frac{1}{n}} E\left|B(t)\right| dQ(t)\left(1+o(1)\right). \tag{3.11}$$

Proof Note that for any finite measure μ on $\left[\frac{1}{n}, 1-\frac{1}{n}\right]$ and random functions f and g in such that Ef and Eg are in $L_1\left(\left[\frac{1}{n}, 1-\frac{1}{n}\right], \mu\right)$,

$$\left|\int_{\frac{1}{n}}^{1-\frac{1}{n}} E\left|f(t)\right| d\mu(t) - \int_{\frac{1}{n}}^{1-\frac{1}{n}} E\left|g(t)\right| d\mu(t)\right| \le \int_{\frac{1}{n}}^{1-\frac{1}{n}} E\left|f(t)-g(t)\right| d\mu(t).$$

Applying this fact we get with obvious choices of f, g and μ

$$\left|\int_{\frac{1}{n}}^{1-\frac{1}{n}} E\left|\alpha_n(t)\right| dQ(t) - \int_{\frac{1}{n}}^{1-\frac{1}{n}} E\left|B(t)\right| dQ(t)\right|$$
$$\le \int_{\frac{1}{n}}^{1-\frac{1}{n}} E\left|\alpha_n(t) - B_n(t)\right| dQ(t).$$

This last bound is in turn with $v = 1/2 - 1/(2p)$

$$\le E\Delta_{n,\nu}(1) \int_{\frac{1}{n}}^{1-\frac{1}{n}} \left(t(1-t)\right)^{1/2-v} dQ(t)\, n^{-v}$$
$$= E\Delta_{n,\nu}(1) \int_{\frac{1}{n}}^{1-\frac{1}{n}} \left(t(1-t)\right)^{1/(2p)} dQ(t)\, n^{-1/2+1/(2p)},$$

which by an application of Proposition 2.3 is for some positive constant C_p,

$$\le C_p \int_{\frac{1}{n}}^{1-\frac{1}{n}} \left(t(1-t)\right)^{1/(2p)} dQ(t)\, n^{-1/2+1/(2p)} \tag{3.12}$$

and by Hölder's inequality is

$$\le C_p \left(\int_{\frac{1}{n}}^{1-\frac{1}{n}} \sqrt{t(1-t)} dQ(t)\right)^{\frac{1}{p}} \left(\left|Q\left(\frac{1}{n}\right)\right| + \left|Q\left(1-\frac{1}{n}\right)\right|\right)^{1-\frac{1}{p}} n^{-\frac{1}{2}+\frac{1}{2p}}$$
$$= C_p \int_{\frac{1}{n}}^{1-\frac{1}{n}} \sqrt{t(1-t)} dQ(t) \left(\frac{\left|Q\left(\frac{1}{n}\right)\right| + \left|Q\left(1-\frac{1}{n}\right)\right|}{n^{1/2}\int_{\frac{1}{n}}^{1-\frac{1}{n}} \sqrt{t(1-t)} dQ(t)}\right)^{1-\frac{1}{p}}.$$

Noting that for each $t \in (0, 1)$,

$$E|B(t)| = E|Z|\sqrt{t(1-t)} = \sqrt{\frac{2}{\pi}}\sqrt{t(1-t)}, \tag{3.13}$$

we see that for $c_p = C_p\sqrt{\frac{\pi}{2}}$ the last bound

$$= \int_{\frac{1}{n}}^{1-\frac{1}{n}} E|B(t)|\, dQ(t)\, c_p\,(r(1/n))^{1-1/p}.$$

Notice that by (3.4)

$$r^2(1/n) \le 2n^{-1}\left(Q^2\left(\frac{1}{n}\right) + Q^2\left(1-\frac{1}{n}\right)\right)/\sigma^2(1/n), \tag{3.14}$$

where $\sigma^2(1/n)$ is defined in (3.2). It is shown in the proof of Lemma 2.1 of Csörgő et al. [6] that

$$\limsup_{n\to\infty} n^{-1}\left(Q^2\left(\frac{1}{n}\right) + Q^2\left(1-\frac{1}{n}\right)\right)/\sigma^2(1/n) \le 1. \tag{3.15}$$

Therefore under absolutely no conditions on Q we have (3.7). Furthermore if (3.10) holds, we have (3.11). □

Corollary 3.2 *If $F \in DN$, then*

$$\int_0^1 E|\alpha_n(t)|\, dQ(t) = \int_{\frac{1}{n}}^{1-\frac{1}{n}} E|B(t)|\, dQ(t)\,(1+o(1)). \tag{3.16}$$

Proof If $F \in DN$, by (3.14) and (3.5), (3.10) holds. Thus

$$\int_{\frac{1}{n}}^{1-\frac{1}{n}} E|\alpha_n(t)|\, dQ(t) = \int_{\frac{1}{n}}^{1-\frac{1}{n}} E|B(t)|\, dQ(t)\,(1+o(1)). \tag{3.17}$$

Observing that

$$\int_{\frac{1}{n}}^{1-\frac{1}{n}} E|B(t)|\, dQ(t) = \sqrt{\frac{2}{\pi}}\tau(1/n),$$

we see that to finish the proof it suffices to prove that

$$\left(\int_0^{\frac{1}{n}} E|\alpha_n(t)|\, dQ(t) + \int_{1-\frac{1}{n}}^1 E|\alpha_n(t)|\, dQ(t)\right)/\tau(1/n) \to 0. \tag{3.18}$$

Since $\tau(1/n) \geq \sigma(1/n)$, to show this it is enough to verify that

$$\left(\int_0^{\frac{1}{n}} E|\alpha_n(t)|\,dQ(t) + \int_{1-\frac{1}{n}}^1 E|\alpha_n(t)|\,dQ(t)\right)/\sigma(1/n) \to 0. \tag{3.19}$$

Notice that since $E|\alpha_n(t)| \leq 2\sqrt{n}t$, we have

$$\int_0^{\frac{1}{n}} E|\alpha_n(t)|\,dQ(t)/\sigma(1/n) \leq 2\sqrt{n}\int_0^{\frac{1}{n}} t dQ(t)/\sigma(1/n)$$

$$\leq \left(\frac{2}{\sqrt{n}}\left|Q\left(\frac{1}{n}\right)\right| + 2\sqrt{n}\int_0^{\frac{1}{n}} |Q(t)|\,dt\right)/\sigma(1/n)$$

$$\leq \left(\frac{2}{\sqrt{n}}\left|Q\left(\frac{1}{n}\right)\right| + 2\sqrt{n}\int_0^{\frac{1}{n}} t^{-1/2}\sigma(t)\,dt \sup_{0<t\leq 1/n}\frac{\sqrt{t}\,|Q(t)|}{\sigma(t)}\right)/\sigma(1/n),$$

$$= \frac{2}{\sqrt{n}}\frac{\left|Q\left(\frac{1}{n}\right)\right|}{\sigma(1/n)} + o(1) = o(1),$$

where in the last step we use the facts that $F \in DN$ is equivalent to σ being slowly varying at zero and that $F \in DN$ implies $\sup_{0<t\leq 1/n}\frac{\sqrt{t}|Q(t)|}{\sigma(t)} = o(1)$. (We pointed out these two facts in (3.5) and (3.6) above.) This proves the first part of (3.19). The second part of (3.19) is proved in the same way. □

Remark Notice that in the special case when F is symmetric about zero and $F(x) = 1 - \frac{1}{2}(1+x)^{-2}$ for $x \geq 0$, we have $F \in DN$ and

$$\int_0^1 E|\alpha_n(t)|\,dQ(t) \sim \frac{\log n}{\sqrt{\pi}}, \text{ as } n \to \infty.$$

Remark Clearly a sufficient condition for (3.16) to hold is that, as $n \to \infty$, $r(1/n) \to 0$, and

$$\left(\int_0^{1/n} E|\alpha_n(t)|\,dQ(t) + \int_{1-1/n}^1 E|\alpha_n(t)|\,dQ(t)\right)/\tau(1/n) \to 0. \tag{3.20}$$

The proof of Corollary 3.2 shows that whenever $F \in DN$, both $r(1/n) \to 0$, as $n \to \infty$, and (3.20) hold.

Assuming that $E|X| < \infty$, write for $0 < u \leq 1/2$ and $n \geq 2$,

$$\overline{\beta}_n(u) = \sqrt{n}\int_0^u t dQ(t) + \sqrt{n}\int_{1-u}^1 (1-t)\,dQ(t) =: \beta_{n,(-)}(u) + \beta_{n,(+)}(u). \tag{3.21}$$

Observation Whenever $E|X| < \infty$, (3.20) is satisfied if and only if

$$\overline{\beta}_n(1/n)/\tau(1/n) \to 0, \text{ as } n \to \infty. \tag{3.22}$$

Proof Lemma 3.8 of Bobkov and Ledoux [1] says that for an absolute constant $c > 0$ for all $0 \le t \le 1$,

$$c \min\left\{2\sqrt{n}t(1-t), \sqrt{t(1-t)}\right\} \le E|\alpha_n(t)|$$
$$\le \min\left\{\sqrt{n}2t(1-t), \sqrt{t(1-t)}\right\}, \tag{3.23}$$

where c may chosen to be $\frac{1}{2}5^{-4}$. This implies that for all $0 < t \le 1/n$ with $1/n \le 1/2$

$$\frac{c\sqrt{n}t}{2} = c\min\left\{\frac{\sqrt{n}t}{2}, \frac{\sqrt{t}}{2}\right\} \le c\min\left\{2\sqrt{n}t(1-t), \sqrt{t(1-t)}\right\}$$
$$\le E|\alpha_n(t)| \le 2\sqrt{n}t(1-t). \tag{3.24}$$

Using this inequality, we get for $n \ge 2$,

$$\frac{c\sqrt{n}}{2}\int_0^{1/n} t dQ(t) \le \int_0^{1/n} E|\alpha_n(t)| dQ(t) \le 2\sqrt{n}\int_0^{1/n} t dQ(t).$$

Obviously this implies that $\beta_{n,(-)}(1/n)/\tau(1/n) \to 0$, as $n \to \infty$, if and only if

$$\left(\int_0^{1/n} E|\alpha_n(t)| dQ(t)\right)/\tau(1/n) \to 0, \text{ as } n \to \infty.$$

In the same way using the version of inequality (3.24) with t replaced by $1-t$, we get $\frac{\beta_{n,(+)}(1/n)}{\tau(1/n)} \to 0$, as $n \to \infty$, if and only if

$$\left(\int_{1-1/n}^{1} E|\alpha_n(t)| dQ(t)\right)/\tau(1/n) \to 0, \text{ as } n \to \infty.$$

□

Remark Whenever

$$0 < \int_{-\infty}^{\infty} \sqrt{F(x)(1-F(x))}dx = \int_0^1 \sqrt{s(1-s)}dQ(s) < \infty, \tag{3.25}$$

we have

$$VarX = \sigma^2(0) = \int_0^1 \int_0^1 (s \wedge t - st)\, dQ(s)\, dQ(t)$$
$$\leq \left(\int_0^1 \sqrt{s(1-s)} dQ(s) \right)^2 < \infty,$$

which implies $0 < VarX < \infty$, and thus $F \in DN$. Hence we can infer from (3.20) that

$$\int_0^1 E|\alpha_n(t)|\, dQ(t) - \int_{\frac{1}{n}}^{1-\frac{1}{n}} E|\alpha_n(t)|\, dQ(t) \to 0$$

and from (3.25) that

$$\int_0^1 E|B(t)|\, dQ(t) - \int_{\frac{1}{n}}^{1-\frac{1}{n}} E|B(t)|\, dQ(t) \to 0,$$

and thus since $r^2(1/n) \to 0$ we can conclude by (3.11) that

$$\int_0^1 E|\alpha_n(t)|\, dQ(t) \to \int_0^1 E|B(t)|\, dQ(t) < \infty.$$

For our next result we shall use the fact (e.g. Inequality 2.1 of Shorack [13]) that for any $0 < \nu < 1/2$ and $0 < c < 1 - d < 1$

$$\int_c^{1-d} (s(1-s))^{1/2-\nu} dQ(s)/\sigma(c,d) \leq (3/\sqrt{\nu})(c \wedge d)^{-\nu}, \tag{3.26}$$

where

$$\sigma^2(c,d) = \int_c^{1-d} \int_c^{1-d} (s \wedge t - st)\, dQ(s)\, dQ(t).$$

Proposition 3.3 *For any quantile function Q, any $p > 1$ and any sequences of positive numbers $0 < c_n < 1 - d_n < 1$, $n \geq 1$,*

$$\left| \int_{c_n}^{1-d_n} E|\alpha_n(t)|\, dQ(t) - \int_{c_n}^{1-d_n} E|B(t)|\, dQ(t) \right|$$
$$\leq \sqrt{\frac{\pi}{2}} C_p (3/\sqrt{\nu})(n(c_n \wedge d_n))^{-\nu}) \int_{c_n}^{1-d_n} E|B(t)|\, dQ(t), \tag{3.27}$$

where $v = 1/2 - 1/(2p)$. *In particular, if* $n(c_n \wedge d_n) \to \infty$, *as* $n \to \infty$,

$$\int_{c_n}^{1-d_n} E\left|\alpha_n(t)\right| dQ(t) = \int_{c_n}^{1-d_n} E\left|B(t)\right| dQ(t)\,(1 + o(1)). \tag{3.28}$$

Proof Notice that for $0 < c < 1 - d < 1$

$$\sigma^2(c,d) \le \left(\int_c^{1-d} \sqrt{s(1-s)}dQ(s)\right)^2,$$

and we get by the Shorack [13] fact (3.26) that for any $0 < c < 1 - d < 1$ and $0 < v < 1/2$,

$$\int_c^{1-d} (s(1-s))^{1/2-\nu} dQ(s) / \int_c^{1-d} \sqrt{s(1-s)}dQ(s) \le (3/\sqrt{\nu})(c \wedge d)^{-\nu}.$$

We see then, as in the proof of Proposition 3.1, that for any $p > 1$ with $v = 1/2 - 1/(2p)$,

$$\left|\int_{c_n}^{1-d_n} E\left|\alpha_n(t)\right| dQ(t) - \int_{c_n}^{1-d_n} E\left|B(t)\right| dQ(t)\right|$$

$$\le C_p \int_{c_n}^{1-d_n} (t(1-t))^{1/(2p)} dQ(t)\, n^{-1/2+1/(2p)}$$

$$= C_p \frac{\int_{c_n}^{1-d_n} (t(1-t))^{1/(2p)} dQ(t)}{\int_{c_n}^{1-d_n} \sqrt{s(1-s)}dQ(s)} n^{-1/2+1/(2p)} \int_{c_n}^{1-d_n} \sqrt{s(1-s)}dQ(s)$$

$$\le \sqrt{\frac{\pi}{2}} C_p (3/\sqrt{\nu})(n(c_n \wedge d_n))^{-\nu}) \sqrt{\frac{2}{\pi}} \int_{c_n}^{1-d_n} \sqrt{s(1-s)}dQ(s)$$

$$= \sqrt{\frac{\pi}{2}} C_p (3/\sqrt{\nu})(n(c_n \wedge d_n))^{-\nu}) \int_{c_n}^{1-d_n} E\left|B(t)\right| dQ(t).$$

□

We immediately get the following corollary.

Corollary 3.4 *If* $E|X| < \infty$, *then for all* $0 < \varepsilon < 1$ *there exists a* $k > 0$ *such that for* $k/n \le 1/2$

$$\int_{(0,1)-[k/n,1-k/n]} E\left|G_n(t) - t\right| dQ(t) + \frac{1}{\sqrt{n}} \int_{k/n}^{1-k/n} E\left|B(t)\right| dQ(t)\,(1-\varepsilon)$$

$$\le Ed_1(F_n, F) \le \int_{(0,1)-[k/n,1-k/n]} E\left|G_n(t) - t\right| dQ(t)$$

$$+ \frac{1}{\sqrt{n}} \int_{k/n}^{1-k/n} E\left|B(t)\right| dQ(t)\,(1+\varepsilon). \tag{3.29}$$

Note that if $E|X| < \infty$, by applying inequality (3.23), we get for $n/2 \geq k$,

$$\int_{(0,1)-[k/n,1-k/n]} E\left|G_n(t) - t\right| dQ(t) \leq 2 \int_{(0,1)-[k/n,1-k/n]} t(1-t)\, dQ(t)\,. \qquad (3.30)$$

Now $E|X| < \infty$ implies that for any $0 < \varepsilon < 1, k > 0$ and all large enough $n/2 \geq k$, the right side of (3.30) is less than ε. Thus from (3.27) and (3.30) we can say that for any $0 < \varepsilon < 1$ there exists a $k > 0$ such that for all large enough $n/2 \geq k$

$$\frac{1}{\sqrt{n}} \int_{k/n}^{1-k/n} E\left|B(t)\right| dQ(t)\,(1-\varepsilon) \leq Ed_1(F_n, F)$$
$$\leq \varepsilon + \frac{1}{\sqrt{n}} \int_{k/n}^{1-k/n} E\left|B(t)\right| dQ(t)\,(1+\varepsilon)\,.$$

4 A Result of del Barrio et al. [7]

Set

$$W_n = n \int_{-\infty}^{\infty} \left|F_n(x) - F(x)\right| dx.$$

del Barrio et al. [7] using a version of the weighted approximation of (2.7) , derived the asymptotic distribution of W_n whenever $F \in DN$ and satisfies some additional conditions. For instance, if (3.25) is satisfied then

$$\sqrt{n} \int_{-\infty}^{\infty} \left|F_n(x) - F(x)\right| dx =_d \int_0^1 \left|\alpha_n(s)\right| dQ(s) \to_d \int_0^1 \left|B(s)\right| dQ(s)\,. \qquad (4.1)$$

Condition (3.25) is a bit stronger than $0 < VarX = \sigma^2 < \infty$ and it is necessary for the limit integral to exist. Notice that if we remove the absolute values signs in (4.1) we get the usual central limit theorem, namely, in the $0 < \sigma^2 < \infty$ case

$$\sqrt{n} \int_{-\infty}^{\infty} \{F_n(x) - F(x)\}\, dx \to_d \sigma Z =_d \int_0^1 B(s) dQ(s)\,.$$

Along the way, in their study, del Barrio et al. [7] proved that whenever $F \in DN$, for all $0 < r < 2$,

$$\sup_{n \geq 1} E\left|\frac{W_n - EW_n}{b_n}\right|^r < \infty, \qquad (4.2)$$

where b_n is as in (3.1). We shall demonstrate how Theorem 2.2 leads to a quick proof of this result.

4.1 An Equivalent Version of the del Barrio, Giné and Matrán result (4.2)

Observing that by (1.4),

$$W_n =_d n \int_0^1 |G_n(t) - t| \, dQ(t),$$

and since, as pointed out above, we can always choose $b_n = \sqrt{n}\sigma\,(1/n)$, we see that their result (4.2) is equivalent to, for all $0 < r < 2$,

$$\sup_{n \geq 2} E \left| \frac{\int_0^1 \{|\alpha_n(t)| - E\,|\alpha_n(t)|\} \, dQ(t)}{\sigma\,(1/n)} \right|^r < \infty. \tag{4.3}$$

In a separate technical lemma they showed that whenever $F \in DN$, for all $0 < r < 2$,

$$\sup_{n \geq 2} E \left| \frac{\int_{(0,1)-[1/n,1-1/n]} \{|\alpha_n(t)| - E\,|\alpha_n(t)|\} \, dQ(t)}{\sigma\,(1/n)} \right|^r < \infty \tag{4.4}$$

and they used the Talagrand [15] exponential inequality to prove that for all $r > 0$,

$$\sup_{n \geq 2} E \left| \frac{\int_{1/n}^{1-1/n} \{|\alpha_n(t)| - E\,|\alpha_n(t)|\} \, dQ(t)}{\sigma\,(1/n)} \right|^r < \infty. \tag{4.5}$$

Clearly (4.4) and (4.5) imply (4.2).

4.2 A Weighted Approximation Approach to (4.5)

Evarist Giné asked the author whether it is true that on the space of Theorem 2.1, for all $r > 0$,

$$\sup_{n \geq 2} E \left[\sup_{1/n \leq t \leq 1-1/n} \frac{n^{\nu} |\alpha_n(t) - B_n(t)|}{(t(1-t))^{1/2-\nu}} \right]^r < \infty? \tag{4.6}$$

In which case, a weighted approximation approach could be used to show that for all $r > 0$, (4.5) holds.

This was the motivation for the author to establish Theorem 2.2, which we have shown in Proposition 2.3 implies (4.6). We shall use Proposition 2.4 and some pieces from del Barrio et al. [7] to prove that (4.5) holds for all $r > 0$, under no assumptions on F. Their proof of (4.5), based on Talagrand [15], assumes $F \in DN$.

Our aim will be to transfer our study of the moment behavior of

$$\frac{\int_{1/n}^{1-1/n} \{|\alpha_n(t)| - E\,|\alpha_n(t)|\}\, dQ(t)}{\sigma\,(1/n)}$$

to that of

$$\frac{\int_{1/n}^{1-1/n} \{|B_n(t)| - E\,|B_n(t)|\}\, dQ(t)}{\sigma\,(1/n)}.$$

What follows is somewhat technical, however, it demonstrates nicely the power of Theorem 2.2.

Step 1.

For any quantile function Q, one has for any $0 < \nu < 1/2$ (see the Shorack [13] fact (3.26))

$$\sup_{n \geq 2} \frac{\int_{1/n}^{1-1/n} (s(1-s))^{1/2-\nu}\, dQ(s)}{n^{\nu} \sigma\,(1/n)} \leq \frac{3}{\sqrt{\nu}}.$$

Thus from Proposition 2.4, (with $M = \frac{3}{\sqrt{\nu}}$, $D_n = n^{\nu} \sigma\,(1/n)$ and obvious choices of $\{R_n, n \geq 2\}$), we get for any $0 < \nu < 1/2$, on the probability space of the KMT [9] approximation there exists a $\gamma > 0$ such that (2.9) holds, where

$$I_n := \frac{\int_{1/n}^{1-1/n} |\alpha_n(s) - B_n(s)| dQ(s)}{n^{\nu} \sigma\,(1/n)}.$$

Step 2.

Noting that

$$n^{\nu} I_n = \frac{\int_{1/n}^{1-1/n} |\alpha_n(s) - B_n(s)| dQ(s)}{\sigma\,(1/n)},$$

we see that (2.9) implies that for any $r > 0$

$$\sup_{n \geq 2} E \left| \frac{\int_{1/n}^{1-1/n} \{|\alpha_n(s)| - |B_n(s)|\}\, dQ(s)}{\sigma\,(1/n)} \right|^r < \infty. \tag{4.7}$$

Step 3.

By recopying steps from the proof of Theorem 5.1 of del Barrio et al. [7], (also see their Proposition 6.2), based on Borell's inequality [2] one gets the exponential inequality, valid for all $t > 0$

$$P\left\{\frac{\left|\int_{1/n}^{1-1/n}\{|B(t)| - E\,|B(t)|\}\,dQ\,(t)\right|}{\sigma\,(1/n)} > t\right\} \leq 2\exp\left(-\frac{2t^2}{\pi^2}\right),$$

which, of course, implies that for all $r > 0$,

$$\sup_{n\geq 2} E\left|\frac{\int_{1/n}^{1-1/n}\{|B(t)| - E\,|B(t)|\}\,dQ(t)}{\sigma\,(1/n)}\right|^r < \infty.$$

This in combination with (4.7) establishes (4.5), which we have just shown holds under absolutely no assumptions on the underlying c.d.f F. As pointed out above, (4.4) and (4.5) imply the del Barrio et al. [7] result (4.3), which on account of (4.4) requires $F \in DN$. In the end, del Barrio et al. [7] decided to use their own proof of (4.5) based on the Talagrand [15] inequality.

4.3 One Can Say More

Piecing all of our inequalities together we obtain the following inequality.

Proposition 4.1 *Under absolutely no conditions on F, for all $n \geq 2$ and $t > 0$,*

$$P\left\{\frac{\left|\int_{1/n}^{1-1/n}\{|\alpha_n(t)| - E\,|\alpha_n(t)|\}\,dQ(t)\right|}{\sigma\,(1/n)} > t\right\} \leq A\exp\left(-Ct\right), \tag{4.8}$$

for suitable constants $A > 0$ and $C > 0$ independent of F.

For additional investigations along this line consult Haeusler and Mason [8], who study the asymptotic distribution of the appropriately centered and normed moderately trimmed Wasserstein distance

$$\int_{Q(a_n/n)}^{Q(1-a_n/n)} |F_n(x) - F(x)|\,dx =_d \int_{a_n/n}^{1-a_n/n} |G_n(t) - t|\,dQ(t),$$

where a_n is a sequence of positive constants satisfying $a_n \to 0$ and $na_n \to \infty$. See Haeusler and Mason [8] for motivation. As part of a general investigation of the trimmed pth Mallows distance, Munk and Czado [12] had previously looked at a somewhat different version of the trimmed Wasserstein distance when $0 < a_n = \alpha < 1/2$. Check their paper for details.

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On the Product of Random Variables and Moments of Sums Under Dependence

Magda Peligrad

Dedicated to the memory of Evarist Giné

Abstract In this paper we compare the moments of products of dependent random vectors with the corresponding ones of independent vectors with the same marginal distributions. Various applications of this result are pointed out, including inequalities for the maximum of dependent random variables and moments of partial sums. The inequalities involve the generalized phi-mixing coefficient.

Keywords Inequalities • Mixing coefficients • Moments for partial sums • Product of dependent random variables

Mathematics Subject Classification (2010). Primary 60E15; Secondary 60F05

1 Introduction

This paper is motivated by the study of moments of products and moments of sums for a dependent vector. In the independent case, the bounds for moments of products or sums is well understood. Various aspects of these bounds can be found in books by Ledoux and Talagrand [14], and by de la Peña and Giné [6], among others. Bounds for moments of sums are also available for various classes of dependent sequences. For martingales, the book by de la Peña and Giné [6] is again an excellent source of information. Various other classes of dependent sequences have been considered in the literature. For positively associated sequences see Birkel [3], negatively associated sequences [25], and classes based on projective conditions [7, 16, 21, 22, 24, 27]. Many of these bounds can be expressed in terms of mixing coefficients surveyed in the book by Bradley [4].

M. Peligrad (✉)
Department of Mathematical Sciences, University of Cincinnati, PO Box 210025, Cincinnati, OH 45221-0025, USA
e-mail: peligrm@ucmail.uc.edu

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By using the expectations of product of moments, we shall compare the maximum term, the characteristics function, the moment generating function and moments of sums of a dependent vector to the corresponding ones for an independent vector with the same marginal distributions.

Relevant to our study is the following coefficient of dependence. Let $(\Omega, \mathcal{K}, \mathbb{P})$ be a probability space and consider $\mathcal{A}, \mathcal{B}$ two sub-sigma algebras $\mathcal{K}$. Denote by $||\cdot||_p$ the norm in $\mathbb{L}_p(\Omega, \mathcal{K}, \mathbb{P})$. Define

$$\bar{\varphi}(\mathcal{A}, \mathcal{B}) = \sup \frac{|\mathrm{cov}(X, Y)|}{||X||_\infty ||Y||_1} ,$$

where supremum is taken over all real-valued functions $X \in \mathbb{L}_\infty(\Omega, \mathcal{A}, P)$ and $Y \in \mathbb{L}_1(\Omega, \mathcal{B}, \mathbb{P})$, where, as usual, $0/0$ is interpreted to be 0. Note that $\bar{\varphi}(\mathcal{A}, \mathcal{B}) \leq 2$.

This coefficient is, up to a constant, comparable with the φ-mixing coefficient introduced by Ibragimov [11] defined in the following way:

$$\begin{aligned}\varphi(\mathcal{A}, \mathcal{B}) &= \sup_{A \in \mathcal{A}, B \in \mathcal{B}, \mathbb{P}(A) \neq 0} |\mathbb{P}(B|A) - \mathbb{P}(B)| \\ &= \sup_{B \in \mathcal{B}} (\text{ess sup}\, |\mathbb{P}(B|\mathcal{A}) - \mathbb{P}(B)|).\end{aligned}$$

Note that by Theorem 3.1 in [4], we have

$$|\mathrm{cov}(X, Y)| \leq 2\varphi(\sigma(X), \sigma(Y))||X||_\infty ||Y||_1, \tag{1.1}$$

and by item (c.1) of Theorem 4.4 in [4], $\bar{\varphi} = 2\varphi$.

Let $(X_k)_{1 \leq k \leq n}$ be a random vector and define the sigma algebras $\mathcal{P}_k = \sigma(X_k)$ and $\mathcal{F}_{k+1}^n = \sigma(X_{k+1}, \ldots X_n)$. Define the $\bar{\varphi}$-mixing coefficient

$$\bar{\varphi} = \bar{\varphi}((X_k)_{1 \leq k \leq n}) = \max_{1 \leq k \leq n-1} \bar{\varphi}(\mathcal{P}_k, \mathcal{F}_{k+1}^n). \tag{1.2}$$

Similarly define

$$\varphi = \varphi((X_k)_{1 \leq k \leq n}) = \max_{1 \leq k \leq n-1} \varphi(\mathcal{P}_k, \mathcal{F}_{k+1}^n).$$

One of our results compares the product of a vector of real positive random variables uniformly bounded by 1 with the product of independent random variables. We shall show that

$$(1 - \bar{\varphi})(1 - \prod_{k=1}^n \mathbb{E}Y_k) \leq (1 - \mathbb{E}(\prod_{k=1}^n Y_k)) \leq (1 + \bar{\varphi})(1 - \prod_{k=1}^n \mathbb{E}Y_k).$$

Actually, we shall derive related results for certain complex-valued random variables, which will allow us to obtain upper and lower bounds for moments of sums of positive random variables in terms of independent variables.

Also, for an arbitrary random vector of positive random variables (Y_k), we compare the moments of sums to the corresponding moments of an independent vector with the same marginal distributions (Y_k^*). For any $0 < p < 1$, we shall show that

$$(1-2\bar{\varphi})K_p^{-1}\mathbb{E}(\sum\nolimits_{k=1}^{n} Y_k^*)^p \le \mathbb{E}(\sum\nolimits_{k=1}^{n} Y_k)^p \le (1+2\bar{\varphi})K_p\mathbb{E}(\sum\nolimits_{k=1}^{n} Y_k^*)^p,$$

where the constant K_p depends only on p via the optimal Khinchin lower constant A_{2p} from [9].

More precisely K_p can be taken $2^{1+(1-2p)\wedge 1}A_{2p}^{-1}$. The inequality in the right hand side provides an alternative approach to the proof of Proposition 1 in [5] (see also relation (1.4.27) of Theorem 1.4.12 in [6]). We also exhibit a lower bound, interesting when $2\bar{\varphi} < 1$.

At the end of the paper we discuss the computability of the φ-mixing coefficient used in the results and we comment about its relation to Doeblin recurrence and the Dobrushin coefficient of ergodicity for Markov chains. We also discuss some aspects of the Ibragimov conjecture on the central limit theorem for a φ-mixing sequence, which motivated this paper.

2 Results and Applications

Our first result points out an identity for any product of random variables.

Lemma 2.1 *Let $(Y_k)_{1\le k\le n}$, $n \ge 2$, be a vector of complex-valued random variables with $\mathbb{E}Y_k \ne 1$ for all k. Define*

$$R_j = \frac{1}{1-\mathbb{E}Y_j}\text{cov}(\prod_{k=j+1}^{n} Y_k, Y_j) \text{ for } 1 \le j \le n-1,\ R_n = 0. \tag{2.1}$$

Then

$$(1-\mathbb{E}(\prod_{k=1}^{n} Y_k)) = \sum_{k=1}^{n}(\prod_{j=1}^{k-1}\mathbb{E}Y_j)(1-\mathbb{E}Y_k)(1-R_k), \tag{2.2}$$

where we understand that $\prod_{j=1}^{0}\mathbb{E}Y_j = 1$.

Proof We start by a remark: For $n \geq 2$,

$$\mathrm{cov}(\prod_{k=j+1}^{n} Y_k, 1 - Y_j) = -\mathrm{cov}(\prod_{k=j+1}^{n} Y_k, Y_j) = -R_j(1 - \mathbb{E}Y_j).$$

By simple algebra,

$$\begin{aligned}(1 - \mathbb{E}(\prod_{k=1}^{n} Y_k)) &= 1 - \mathbb{E}(\prod_{k=2}^{n} Y_k)\mathbb{E}Y_1 + \mathrm{cov}(\prod_{k=2}^{n} Y_k, 1 - Y_1) \\ &= 1 - \mathbb{E}(\prod_{k=2}^{n} Y_k)\mathbb{E}Y_1 + cov(\prod_{k=2}^{n} Y_k, 1 - Y_1).\end{aligned}$$

By using notation (2.1), we write

$$\begin{aligned}(1 - \mathbb{E}(\prod_{k=1}^{n} Y_k)) &= 1 - \mathbb{E}(\prod_{k=2}^{n} Y_k)\mathbb{E}Y_1 - R_1(1 - \mathbb{E}Y_1) \\ &= (1 - \mathbb{E}(\prod_{k=2}^{n} Y_k))\mathbb{E}Y_1 + (1 - R_1)(1 - \mathbb{E}Y_1).\end{aligned}$$

We notice now that we can apply recurrence and obtain the identity (2.2). □

Lemma 2.1 allows us to compare the moments of products of random variables bounded by 1 with the corresponding ones of independent variables. It is convenient to consider complex-valued random variables.

Lemma 2.2 *Let $(Y_k)_{1\leq k\leq n}$, $n \geq 2$, be a vector of complex random variables with $||Y_k||_\infty \leq 1$, $\mathbb{E}Y_k$ real and positive for all k, $1 \leq k \leq n$. Denote*

$$\omega = \omega((Y_k)_{1\leq k\leq n}) = \max_{1\leq k\leq n-1} |\operatorname{Re} R_k|. \tag{2.3}$$

Then

$$(1 - \omega)(1 - \prod_{k=1}^{n} \mathbb{E}Y_k) \leq \operatorname{Re}(1 - \mathbb{E}(\prod_{k=1}^{n} Y_k)) \leq (1 + \omega)(1 - \prod_{k=1}^{n} \mathbb{E}Y_k).$$

Proof Note that our conditions imply that $0 \le \mathbb{E}Y_k \le 1$. By Lemma 2.1 and the definition of ω we obtain

$$\begin{aligned}\mathrm{Re}(1-\mathbb{E}(\prod_{k=1}^{n}Y_k)) &= \sum_{k=1}^{n}(\prod_{j=1}^{k-1}\mathbb{E}Y_j)(1-\mathbb{E}Y_k)(1-\mathrm{Re}\,R_k)\\ &\le (1+\omega)\sum_{k=1}^{n}(\prod_{j=1}^{k-1}\mathbb{E}Y_j)(1-\mathbb{E}Y_k),\end{aligned}$$

and

$$\mathrm{Re}(1-\mathbb{E}(\prod_{k=1}^{n}Y_k)) \ge (1-\omega)\sum_{k=1}^{n}(\prod_{j=1}^{k-1}\mathbb{E}Y_j)(1-\mathbb{E}Y_k).$$

The result follows since

$$\sum_{k=1}^{n}(\prod_{j=1}^{k-1}\mathbb{E}Y_j)(1-\mathbb{E}Y_k) = 1-\prod_{k=1}^{n}\mathbb{E}Y_k.$$

□

If the variables are real-valued we obtain

Lemma 2.3 *Let* $(Y_k)_{1\le k\le n}$, $n \ge 2$, *be real-valued random variables with* $0 \le Y_k \le 1$ *a.s. Then*

$$(1-\omega)(1-\prod_{k=1}^{n}\mathbb{E}Y_k) \le (1-\mathbb{E}(\prod_{k=1}^{n}Y_k)) \le (1+\omega)(1-\prod_{k=1}^{n}\mathbb{E}Y_k),$$

where $\omega = \omega(n)$ *is defined by*

$$\omega = \sup_{1\le j\le n-1}\frac{1}{\mathbb{E}(1-Y_j)}\left|\mathrm{cov}(\prod_{k=j+1}^{n}Y_k, 1-Y_j)\right|.$$

Also $\omega \le \bar{\varphi}$ *where* $\bar{\varphi}$ *is defined by (1.2).*

Proof In this case all the conditions of Lemma 2.2 are satisfied. In addition, for $1 \le j \le n-1$, we have

$$|R_j| = \frac{1}{1-\mathbb{E}Y_j}|\mathrm{cov}(\prod_{k=j+1}^{n}Y_k, Y_j)| = \frac{1}{\mathbb{E}|1-Y_j|}|cov(\prod_{k=j+1}^{n}Y_k, 1-Y_j)| \le \bar{\varphi},$$

and the result follows. Note that, in this lemma, we do not have to assume $\mathbb{E}Y_k \neq 1$ since we can interpret $0/0 = 0$. □

2.1 Application to Indicator Functions

Let $(A_k)_{1\leq k\leq n}$ be events in $\mathcal{K}$. We apply Lemma 2.3 to the indicator functions $Y_k = I_{A_k}$. For these functions

$$\omega = \max_{1\leq j\leq n-1} |\mathbb{P}((\cap_{k=j+1}^n A_k) \cap A_j') - \mathbb{P}(\cap_{k=j+1}^n A_k)\mathbb{P}(A_j')|/\mathbb{P}(A_j'),$$

where A_k' is the complement of A_k and we interpret $0/0 = 0$. Note that for this case $\omega \leq \varphi((I_{A_k})_k)$.

Then, by Lemma 2.3,

$$(1-\omega)(1-\prod_{k=1}^n \mathbb{P}(A_k)) \leq 1 - \mathbb{P}(\cap_{k=1}^n A_k) \leq (1+\omega)(1-\prod_{k=1}^n \mathbb{P}(A_k)). \tag{2.4}$$

We can represent relation (2.4) in the following equivalent way:

$$(1-\omega)\mathbb{P}(\cup_{k=1}^n (A_k^*)^{'}) \leq \mathbb{P}(\cup_{k=1}^n A_k') \leq (1+\omega)\mathbb{P}(\cup_{k=1}^n (A_k^*)^{'}), \tag{2.5}$$

where (A_k^*) are independent with $\mathbb{P}(A_k)= \mathbb{P}(A_k^*)$. Also it can be represented as

$$|\mathbb{P}(\cap_{k=1}^n A_k) - \prod_{k=1}^n \mathbb{P}(A_k)| \leq \omega\mathbb{P}(\cup_{k=1}^n (A_k^*)^{'}).$$

From this last expression we can see that the inequality (2.4) is tighter than Lemma 3 in [17], which has in the right hand side only ω.

2.2 Maximum of Random Variables

Consider now a vector $(X_k^*)_{1\leq k\leq n}$ of independent random variables where each X_k^* is distributed as X_k. Then for all real x,

$$(1-\varphi)\mathbb{P}(\max_{1\leq k\leq n} X_k^* \geq x) \leq \mathbb{P}(\max_{1\leq k\leq n} X_k \geq x) \leq (1+\varphi)\mathbb{P}(\max_{1\leq k\leq n} X_k^* \geq x). \tag{2.6}$$

To prove it, we consider the events $A_k = I(X_k < x)$ and notice that

$$(\max_{1\le k\le n} X_k \ge x) = \cup_{k=1}^n A_k'. \tag{2.7}$$

We then apply relation (2.5) along with the definition of φ-mixing coefficients to conclude that $\omega \le \varphi$.

By using the sequence of inequalities in relation (2.6) and the integration by parts formula (see Theorem 18.4 in [2]), we can obtain various moment inequalities for functions of the maximum of variables in terms of the maximum of independent variables. For instance, for any positive continuous and nondecreasing function g, we have

$$(1-\varphi)\mathbb{E}g(\max_{1\le k\le n}|X_k^*|) \le \mathbb{E}g(\max_{1\le k\le n}|X_k|) \le (1+\varphi)\mathbb{E}g(\max_{1\le k\le n}|X_k^*|).$$

Inequality (2.6) was obtained by a direct approach in [19] and exploited to derive a central limit theorem for φ-mixing sequences.

2.3 Application to Laplace Transform of Positive Functions

Here we shall apply Lemma 2.3 to the Laplace transform of positive functions. For any positive random variable X and positive number t, denote $M_X(t) = \mathbb{E}\exp(-tX)$. Given $(X_k)_{1\le k\le n}$ a vector of positive random variables, for every $t > 0$, define

$$Y_k(t) = \exp(-tX_k).$$

Note that $Y_k(t)$ satisfies the conditions of Lemma 2.3. Also note that $\mathbb{E}(\prod_{k=1}^n Y_k) = \mathbb{E}(\exp(-t\sum_{k=1}^n X_k))$ and $\prod_{k=1}^n \mathbb{E}Y_k = \mathbb{E}(\exp(-t\sum_{k=1}^n X_k^*))$, where $(X_k^*)_{1\le k\le n}$ are independent variables, with each variable X_k^* distributed as X_k. Denote $S_n = \sum_{k=1}^n X_k$ and $S_n^* = \sum_{k=1}^n X_k^*$. We also have

$$\omega(t) = \max_{1\le j\le n-1} \frac{1}{\mathbb{E}(1-\exp(-tX_j))}|\text{cov}(\exp(-t\sum_{k=j+1}^n X_k), 1-\exp(-tX_j))| \le \bar{\varphi}.$$

Therefore

$$(1-\bar{\varphi})(1-M_{S_n^*}(-t)) \le 1-M_{S_n}(-t) \le (1+\bar{\varphi})(1-M_{S_n^*}(-t)). \tag{2.8}$$

2.4 Application to Characteristic Functions

We consider here a vector of random variables $(X_k)_{1\leq k\leq n}$. We introduce the random function of t, $Y_k = \exp(itX_k)$. Denote the characteristic function $f_X(t) = \mathbb{E}\exp(itX)$. We assume that for a t fixed $f_{X_k}(t)$ is positive and different from 1, for all k. For this case

$$\omega(t) = \max_{1\leq j\leq n-1} |\operatorname{Re} R_j(t)|, \tag{2.9}$$

where

$$\operatorname{Re} R_j(t) = \frac{-1}{1-\mathbb{E}\cos(tX_j)}[\operatorname{cov}(\cos(\sum_{k=j+1}^{n} tX_k), 1-\cos(tX_j))$$
$$+ \operatorname{cov}(\sin(\sum_{k=j+1}^{n} tX_k), \sin(tX_j))].$$

By Lemma 2.2, it follows that for such a value of t,

$$(1-\omega(t))(1-f_{S_n^*}(t)) \leq \operatorname{Re}(1-f_{S_n}(t)) \leq (1+\omega(t))(1-f_{S_n^*}(t)), \tag{2.10}$$

where, as before, $(X_k^*)_{1\leq k\leq n}$ are independent, each X_k^* distributed as X_k and S_n, S_n^* are their sums.

If in addition we assume that for such t,

$$\operatorname{cov}(\sin(\sum_{k=j+1}^{n} tX_k), \sin(tX_j)) = 0 \tag{2.11}$$

for all natural numbers j, $0\leq j\leq n$, then, by the definition of $\bar{\varphi}$ we see that $\omega(t)\leq\bar{\varphi}$ and we have

$$(1-\bar{\varphi})(1-f_{S_n^*}(t)) \leq \operatorname{Re}(1-f_{S_n}(t)) \leq (1+\bar{\varphi})(1-f_{S_n^*}(t)). \tag{2.12}$$

2.5 Bounds for Moments of Partial Sums

To compare the moments of partial sums of a dependent sequence with an independent one we shall use the following well-known lemma (see, for instance, relation (4.1) in [8]).

Lemma 2.4 *Let Y be a random variable with characteristic function $f_Y(t)$ and for a real number $r \in (0,2)$ assume that $\mathbb{E}|Y|^r < \infty$. Then,*

$$\mathbb{E}|Y|^r = C_r \int_{-\infty}^{\infty} \frac{(1 - \operatorname{Re} f_Y(u))}{|u|^{1+r}} du,$$

with $C_r = \pi^{-1}\Gamma(1+r)\sin\frac{\pi r}{2}$.

In order to derive moment inequalities for sums of random variables with a positive characteristic function we combine inequality (2.12) with Lemma 2.4. Therefore, by taking also into account the continuity of norms, we obtain:

Lemma 2.5 *Assume that for all natural numbers k, $1 \le k \le n$, the characteristic function $f_{X_k}(t)$ is positive for all t and in addition we have for all natural numbers j, $1 \le j \le n-1$,*

$$\operatorname{cov}(\sin(\sum_{k=j+1}^{n} tX_k), \sin(tX_j)) = 0 \textit{ for almost all } t. \tag{2.13}$$

Then, for every $r \in (0,2]$,

$$(1-\bar{\varphi})\mathbb{E}|S_n^*|^r \le \mathbb{E}|S_n|^r \le (1+\bar{\varphi})\mathbb{E}|S_n^*|^r. \tag{2.14}$$

The condition that the random variables have positive characteristic function can be easily removed at the cost of the constants by symmetrization and de-symmetrization procedures. These procedures will also have the effect of removing condition (2.13). In order to point out some intermediate results we shall proceed in two steps. First we symmetrize with a Rademacher sequence, and then we shall combine two kinds of symmetrization.

2.5.1 Symmetrization with a Rademacher sequence

Assume now that for all k, X_k is such that $\operatorname{Re} f_{X_k}(t) \ge 0$ for all t. For $1 \le k \le n$ we consider now a Rademacher vector of independent random variables (i.e. ε_k are i.i.d. $\mathbb{P}(\varepsilon_k = \pm 1) = 1/2$) which is independent on the vector $(X_k)_{1\le k\le n}$ and introduce the vector $(\varepsilon_k X_k)_{1\le k\le n}$.

By Fubini's theorem, we note that $\mathbb{E}\sin t\varepsilon_k X_k = 0$ and

$$f_{\varepsilon_k X_k}(t) = \mathbb{E}\cos tX_k = \operatorname{Re} f_{X_k}(t) \ge 0.$$

In addition, also by Fubini's theorem, by integrating first with ε_j we can easily get, for all k,

$$\mathrm{cov}(\sin(\sum_{k=j+1}^{n} t\varepsilon_k X_k), \sin(t\varepsilon_j X_j)) = \mathbb{E}(\sin(\sum_{k=j+1}^{n} t\varepsilon_k X_k)\sin(t\varepsilon_j X_j)) = 0,$$

showing that condition (2.13) is satisfied. Therefore the vector $(\varepsilon_k X_k)_{1\le k\le n}$ satisfies the conditions of Lemma 2.5. Furthermore, it is easy to see that for all natural numbers j, $1 \le j \le n-1$, we have

$$\begin{aligned} R_j(t) &= \frac{-1}{1-\mathbb{E}\cos t\varepsilon_j X_j}\mathrm{cov}(\cos(\sum_{k=j+1}^{n} t\varepsilon_k X_k), 1-\cos(t\varepsilon_j X_j)) \\ &= \frac{-1}{1-\mathbb{E}\cos tX_j}\mathrm{cov}(\cos(\sum_{k=j+1}^{n} t\varepsilon_k X_k), 1-\cos(tX_j)). \end{aligned}$$

It follows that for all natural numbers j, $1 \le j \le n-1$,

$$|R_j(t)| \le \bar{\varphi}((X_k)_{1\le k\le n}) = \bar{\varphi}.$$

By Lemma 2.5, we obtain for $0 < r \le 2$,

$$(1-\bar{\varphi})\mathbb{E}|\sum_{k=1}^{n}\varepsilon_k X_k^*|^r \le \mathbb{E}|\sum_{k=1}^{n}\varepsilon_k X_k|^r \le (1+\bar{\varphi})\mathbb{E}|\sum_{k=1}^{n}\varepsilon_k X_k^*|^r. \tag{2.15}$$

It is well-known that, by Khinchin inequalities (see page 21 in [6]), for $0 < r \le 2$ there is a positive constant A_r depending only on r, such that

$$A_r(\sum\nolimits_{k=1}^{n} X_k^2)^{r/2} \le \mathbb{E}_\varepsilon|\sum\nolimits_{k=1}^{n}\varepsilon_k X_k|^r \le (\sum\nolimits_{k=1}^{n} X_k^2)^{r/2}, \tag{2.16}$$

where $\mathbb{E}_\varepsilon$ denotes the integration with respect to variables $(\varepsilon_k)_{1\le k\le n}$. The best constant A_r can be found in [9]. Therefore, by combining (2.15) and (2.16) we obtain

$$\begin{aligned} (1-\bar{\varphi})A_r\mathbb{E}(\sum\nolimits_{k=1}^{n}(X_k^*)^2)^{r/2} &\le (1-\bar{\varphi})\mathbb{E}|\sum\nolimits_{k=1}^{n}\varepsilon_k X_k^*|^r \\ &\le \mathbb{E}|\sum\nolimits_{k=1}^{n}\varepsilon_k X_k|^r \le \mathbb{E}(\sum\nolimits_{k=1}^{n} X_k^2)^{r/2}. \end{aligned}$$

and

$$A_r\mathbb{E}(\sum_{k=1}^{n} X_k^2)^{r/2} \le \mathbb{E}|\sum_{k=1}^{n} \varepsilon_k X_k|^r \le (1+\bar{\varphi})\mathbb{E}|\sum_{k=1}^{n} \varepsilon_k X_k^*|^r$$
$$\le (1+\bar{\varphi})\mathbb{E}(\sum_{k=1}^{n} (X_k^*)^2)^{r/2}.$$

Therefore, we have established the following result:

Proposition 2.6 *Assume that for all natural numbers k,* $1 \le k \le n$, $\mathrm{Re} f_{X_j}(t) \ge 0$ *for all t. Then for* $0 < r \le 2$ *we have*

$$(1-\bar{\varphi})A_r\mathbb{E}(\sum_{k=1}^{n} (X_k^*)^2)^{r/2} \le \mathbb{E}(\sum_{k=1}^{n} X_k^2)^{r/2} \le (1+\bar{\varphi})A_r^{-1}\mathbb{E}(\sum_{k=1}^{n} (X_k^*)^2)^{r/2}. \tag{2.17}$$

where (X_k^*) *are independent with each* X_k^* *distributed as* X_k *and* A_r *is the lower Khinchin constant.*

2.5.2 Second Symmetrization and Desymmetrization

In case where $\mathrm{Re} f_{X_j}(t)$ is not positive for all t, we can remove this restriction by using a combination of symmetrization techniques, at the cost of constants. We shall use the following lemma. It contains two simple inequalities which we formulate only in the setting we apply them. They can be formulated for more general variables. Parts of this lemma are well-known.

Lemma 2.7 *Let X and Y be two i.i.d. symmetric random variables. Choose* $0 < r \le 2$. *Then,*

$$\mathbb{E}|X+Y|^r \le 2\mathbb{E}|X|^r. \tag{2.18}$$

and

$$\mathbb{E}|X|^r \le b_r\mathbb{E}|X+Y|^r, \tag{2.19}$$

where b_r *can be taken* $b_r = 2^{(1-r)\vee 0}$.

Proof For proving the inequality (2.18), we apply Lemma 2.4. By using the fact that by symmetry $f_{X+Y}(u) = f_X(u)f_Y(u) = |f_Y(u)|^2$, and that $\mathrm{Re} f_Y(u) \le |f_Y(u)|$ we obtain, for $0 < r < 2$,

$$\mathbb{E}|X+Y|^r = C_r \int_{-\infty}^{\infty} \frac{(1-|f_Y(u)|^2)}{|u|^{1+r}} du \le 2C_r \int_{-\infty}^{\infty} \frac{(1-|f_Y(u)|)}{|u|^{1+r}} du \le$$
$$2C_r \int_{-\infty}^{\infty} \frac{(1-\mathrm{Re} f_Y(u))}{|u|^{1+r}} du = 2\mathbb{E}|X|^r.$$

We turn now to prove (2.19). Since $2X = (X+Y)+(X-Y)$ and (X,Y) and $(X,-Y)$ have the same distribution, by convexity, it follows that for $0 < r \le 1$ we have

$$2^r \mathbb{E}|X|^r \le \mathbb{E}|X|^r \le \mathbb{E}|X+Y|^r + \mathbb{E}|X-Y|^r = 2\mathbb{E}|X+Y|^r.$$

If $r > 1$, by the triangle inequality we obtain the well-known inequality

$$2||X||_r \le ||X+Y||_r + ||X-Y||_r = 2||X+Y||_r.$$

□

We shall obtain now an analogue of Proposition 2.6 for a general vector $(X_k)_{1\le k\le n}$. With this aim we consider a vector of variables, $(X'_k)_{1\le k\le n}$ which is an independent copy of $(X_k)_{1\le k\le n}$ and two independent Rademacher vectors $(\varepsilon_k)_{1\le k\le n}$, $(\varepsilon'_k)_{1\le k\le n}$ which are independent of both $(X_k)_{1\le k\le n}$ and $(X'_k)_{1\le k\le n}$. Define the vector $\tilde{X}_k = \varepsilon_k X_k + \varepsilon'_k X'_k$, $1 \le k \le n$. Now, by Theorem 6.6 in [4] we know that

$$\bar{\varphi}((\tilde{X}_k)_{1\le k\le n}) \le 2\bar{\varphi}((X_k)_{1\le k\le n}).$$

Furthermore, since $\varepsilon'_k X'_k$ is symmetric and independent of $\varepsilon_k X_k$, we obtain $f_{\tilde{X}_k}(t) = |f_{\varepsilon_k X_k}(t)|^2$. Also by Fubini's theorem condition (2.13) is satisfied.

Therefore we can apply Lemma 2.5 which gives

$$(1-2\bar{\varphi})\mathbb{E}|\sum_{k=1}^n \tilde{X}_k^*|^r \le \mathbb{E}|\sum_{k=1}^n \tilde{X}_k|^r \le (1+2\bar{\varphi})\mathbb{E}|\sum_{k=1}^n \tilde{X}_k^*|^r, \tag{2.20}$$

where $(\tilde{X}_k^*)_{1\le k\le n}$ are independent and each $\tilde{X}_k^*$ is distributed as $\tilde{X}_k$. Without restricting the generality we can take $\tilde{X}_k^*$ of the form $\tilde{X}_k^* = \varepsilon_k X_k^* + \varepsilon'_k (X'_k)^*$ with both vectors $(X_k^*)_{1\le k\le n}$ and $\left((X'_k)^*\right)_{1\le k\le n}$, i.i.d., with the same marginal distributions as $(X_k)_{1\le k\le n}$. We shall denote

$$V_n = \sum_{k=1}^n \varepsilon_k X_k \, , \, W_n = \sum_{k=1}^n \varepsilon'_k X'_k$$

and

$$V_n^* = \sum_{k=1}^{n} \varepsilon_k X_k^* \ , \ W_n^* = \sum_{k=1}^{n} \varepsilon_k'(X_k')^*.$$

Note that V_n, W_n are i.i.d., symmetric random variables and also V_n^* and W_n^* are i.i.d. and symmetric random variables. With these notations we can write

$$\sum_{k=1}^{n} \tilde{X}_k = V_n + W_n, \ \sum_{k=1}^{n} \tilde{X}_k^* = V_n^* + W_n^*,$$

and relation (2.20) as

$$(1-2\bar{\varphi})\mathbb{E}|V_n^* + W_n^*|^r \le \mathbb{E}|V_n + W_n|^r \le (1+2\bar{\varphi})\mathbb{E}|V_n^* + W_n^*|^r. \tag{2.21}$$

So, for $r \in (0, 2]$, by (2.19) and (2.18) of Lemma 2.7, applied together with (2.21) we obtain

$$(1-2\bar{\varphi})\mathbb{E}|V_n^*|^r \le (1-2\bar{\varphi})b_r\mathbb{E}|V_n^* + W_n^*|^r \le b_r\mathbb{E}|V_n + W_n|^r \le 2b_r\mathbb{E}|V_n|^r.$$

By the same arguments, we also have

$$\mathbb{E}|V_n|^r \le b_r\mathbb{E}|V_n + W_n|^r \le (1+2\bar{\varphi})b_r\mathbb{E}|V_n^* + W_n^*|^r \le (1+2\bar{\varphi})2b_r\mathbb{E}|V_n^*|^r.$$

Overall

$$(2b_r)^{-1}(1-2\bar{\varphi})\mathbb{E}|V_n^*|^r \le \mathbb{E}|V_n|^r \le 2b_r(1+2\bar{\varphi})\mathbb{E}|V_n^*|^r. \tag{2.22}$$

Combining this latter inequality with Khinchin inequalities, with the notation $C_r = 2b_rA_r^{-1} = 2^{1+(1-r)\wedge 1}A_r^{-1}$, we obtain for $0 < r \le 2$ and an arbitrary vector $(X_k)_{1\le k\le n}$,

$$\begin{aligned}(1-2\bar{\varphi})C_r^{-1}\mathbb{E}(\textstyle\sum_{k=1}^{n}(X_k^*)^2)^{r/2} &\le \mathbb{E}(\textstyle\sum_{k=1}^{n}X_k^2)^{r/2} \\ &\le (1+2\bar{\varphi})C_r\mathbb{E}(\textstyle\sum_{k=1}^{n}(X_k^*)^2)^{r/2}.\end{aligned} \tag{2.23}$$

Now, giving a positive vector of random variables $(Y_k)_{1\le k\le n}$, we define the sequence $X_k = \sqrt{Y_k}$. By applying inequality (2.23) to $(X_k)_{1\le k\le n}$ we obtain:

Theorem 2.8 *Assume* $(Y_k)_{1\le k\le n}$, $n \ge 2$, *are arbitrary positive random variables and* $0 < p \le 1$. *If* $\mathbb{E}(Y_k^p) < \infty$, $1 \le k \le n$, *then*

$$\begin{aligned}(1-2\bar{\varphi})K_p^{-1}\mathbb{E}(\textstyle\sum_{k=1}^{n}Y_k^*)^p &\le \mathbb{E}(\textstyle\sum_{k=1}^{n}Y_k)^p \\ &\le (1+2\bar{\varphi})K_p\mathbb{E}(\textstyle\sum_{k=1}^{n}Y_k^*)^p,\end{aligned}$$

where $(Y_k^*)_{1\le k\le n}$ *are independent random variables with* Y_k^* *distributed as* Y_k *and* K_p *can be taken* $2^{1+(1-2p)\wedge 1}A_{2p}^{-1}$ *with* A_{2p} *the lower Khinchin constant.*

Since always $\bar{\varphi}_1 \le 2$, we obtain for any vector of positive random variables $(Y_k)_{1\le k\le n}$, that

$$\mathbb{E}(\sum\nolimits_{k=1}^{n} Y_k)^p \le 5K_p\mathbb{E}(\sum\nolimits_{k=1}^{n} Y_k^*)^p, \tag{2.24}$$

where $(Y_k^*)_{1\le k\le n}$ are independent and each Y_k^* is distributed as Y_k. The constant is depending only on p, and is expressed as a function of the lower Khinchin constant. Therefore, our proof provides for power functions an alternative approach to the result given in Proposition 1 in [5] (see also relation (1.4.27) on page 33 in [6]), whose proof is based on a truncation argument. The paper by de la Peña [5] also provides examples showing that, in general, inequality (2.24) cannot be reversed. However, our results in Proposition 2.6 and Theorem 2.8 provide a class of random vectors for which this is possible. In order for the inequality in the left hand side to be meaningful we have to assume that $2\bar{\varphi} < 1$. For this class of random vectors we also obtain a lower bound for the moments of sums of positive random variables, in terms of moments of sum of independent ones. Of course, if we have $\mathrm{Re} f_{X_j}(t) > 0$ for almost all t and all j, by Proposition 2.6, we obtain better constants and in this case, in order to use the lower bound, we have only to assume $\bar{\varphi} < 1$.

2.6 Discussion of the φ-Mixing Coefficient

In general, the computation of the Ibragimov coefficient φ is not an easy task except in the Markovian case. If $(X_k)_{1\le k\le n}$ is a Markov random vector the definition simplifies as follows:

$$\varphi = \varphi((X_k)_{1\le k\le n}) = \max_{1\le k\le n-1} \varphi(\sigma(X_k), \sigma(X_{k+1})).$$

Furthermore, by relation (1.1.2) in [13],

$$\varphi(\sigma(X_k), \sigma(X_{k+1})) \le \sup_{B\in\sigma(X_{k+1})} [\operatorname{ess\,sup} \mathbb{P}(B|\sigma(X_k)) - \operatorname{ess\,inf} \mathbb{P}(B|\sigma(X_k))].$$

We mention that

$$\delta(\sigma(X_k), \sigma(X_{k+1})) = 1 - \sup_{B\in\sigma(X_{k+1})} [\operatorname{ess\,sup} \mathbb{P}(B|\sigma(X_k)) - \operatorname{ess\,inf} \mathbb{P}(B|\sigma(X_k))]$$

is the famous Dobrushin coefficient of ergodicity.

If $(X_k)_{1\le k\le n}$ are discrete random variables, we denote by

$$q_{ij}^{(k)} = \mathbb{P}(X_{k+1} = j | X_k = i).$$

Then by Proposition 1.2.3 in [13],

$$\varphi(\sigma(X_k), \sigma(X_{k+1})) \le \frac{1}{2} \sup_{i,j} \sum_{\ell} |q_{i\ell}^{(k)} - q_{j\ell}^{(k)}|.$$

Now, assume we have a strictly stationary Markov chain $(X_k)_{k\in\mathbb{Z}}$ satisfying the following form of Doeblin condition: There is a Borel set A with $\mathbb{P}(X_0 \in A) = 1$ and there is an $\varepsilon \in (0, 1)$ such that

$$\mathbb{P}(X_1 \in B | X_0 = x) - \mathbb{P}(X_0 \in B) \le 1 - \varepsilon,$$

as soon as $\mathbb{P}(X_0 \in B) \le \varepsilon$. In this case we have $\varphi \le 1 - \varepsilon$ (see for instance Sect. 21.23 in [4, Vol. 2]). Here, because of stationarity φ is defined as $\varphi(\sigma(X_0), \sigma(X_1))$.

In many situations, the following two quantities are relevant to the computation of the coefficient φ $(0/0 = 0)$:

$$\psi^* = \sup_{A,B} \frac{\mathbb{P}(X_0 \in A, X_1 \in B)}{\mathbb{P}(X_0 \in A)(X_1 \in B)} \text{ and } \psi' = \inf_{A,B} \frac{\mathbb{P}(X_0 \in A, X_1 \in B)}{\mathbb{P}(X_0 \in A)(X_1 \in B)}.$$

Then, by Proposition 5.2 in [4, Vol. 1], we have $1 \le \psi^* \le \infty$ and $0 \le \psi' \le 1$. Moreover

$$\varphi \le 1 - \frac{1}{\psi^*} \text{ and } \varphi \le 1 - \psi'.$$

These inequalities are practical in many situations. For instance, it is convenient to specify a stationary Markov chain $(X_k)_{k\in\mathbb{Z}}$ with marginal distribution function F by a copula $C(x_0, x_1)$:

$$P(X_0 \le x_0, X_1 \le x_1) = C(F(x_0), F(x_1)).$$

If for some $0 < \delta \le 1$ the absolutely continuous part of the copula $C(x_0, x_1)$ has a density $c(x_0, x_1) \ge \delta$, then we have $\psi' \ge \delta$ and $\varphi \le 1 - \delta$. For instance, for the Marshall-Olkin copula

$$C_\alpha(x_0, x_1) = \min(x_0 x_1^{1-\alpha}, x_1 x_0^{1-\alpha}),\ 0 \le \alpha < 1,$$

we have $c(x_0, x_1) \ge 1 - \alpha$ and therefore $\varphi \le \alpha$. For a detailed formulation of the φ-mixing coefficients in terms of copula and further examples see for instance [15].

We now give an estimate of the coefficient φ for a specific example related to number theory. For every irrational number x in $(0,1)$ there is a unique sequence of positive integers $x_1, x_2, x_3, \ldots$ such that the following continued fraction expansion holds:

$$x = \frac{1}{x_1 + \frac{1}{x_2 + \frac{1}{x_3 + \cdots}}}.$$

If we introduce on $[0,1]$ the Gauss probability measure with the density $f(x) = (\ln 2)^{-1}(1+x)^{-1}$, then the sequence $(x_1, x_2, x_3, \ldots)$ is strictly stationary. We know from Lemma 2.1 in [23] that for this sequence $\psi^* < 1.8$ and then $\varphi \le 1 - (\psi^*)^{-1} \le 0.45$. For this case, our inequality (2.6) gives

$$(0.55)\left(1 - \left(\frac{\ln(1+x)}{\ln 2}\right)^n\right) \le \mathbb{P}(\max_{1\le k\le n} X_k \ge x) \le (1.55)\left(1 - \left(\frac{\ln(1+x)}{\ln 2}\right)^n\right).$$

Also, the left hand side of (2.8) holds with $1 - \bar{\varphi} = 1 - 2\varphi \ge 0.1$.

As a matter of fact, all the left hand side inequalities obtained in this paper are usable if the coefficient φ is small enough. One way to reduce the size of φ is to use various blocking procedures. A useful blocking procedure is to fix a natural number $p > 1$ and to leave a gap of p between the variables. For instance, when we treat the maximum of random variables, we can look at $(X_p, X_{2p}, \ldots, X_{kp})$ with k being the integer part of n/p. Let us denote by φ_p the mixing coefficient for this sequence. The left hand side of (2.6) gives

$$(1-\varphi_p)\mathbb{P}(\max_{1\le \ell\le k} X^*_{\ell p} \ge x) \le \mathbb{P}(\max_{1\le \ell\le k} X_{\ell p} \ge x) \le \mathbb{P}(\max_{1\le k\le n} X_k \ge x).$$

which is meaningful provided $\varphi_p < 1$. In the Markov setting, by Theorem 7.4 in [4], we know that

$$\bar{\varphi}(\sigma(X_p), \sigma(X_{2p})) = \prod_{k=p}^{2p-1} (\bar{\varphi}(\sigma(X_k), \sigma(X_{k+1})),$$

which gives in the stationary setting $\bar{\varphi}_p \le (\bar{\varphi})^p$ (i.e $2\varphi_p \le (2\varphi)^p$).

When dealing with sums of random variables, an extremely useful procedure is the Bernstein big and small block argument: The variables are divided in large blocks intertwined with small blocks. The partial sum in big blocks are vectors distant enough to have a small mixing coefficient while the sum of variables in small blocks is negligible.

2.7 Discussion of the Ibragimov Conjecture

For a stationary sequence $X = (X_k)_{k\in\mathbb{Z}}$ we define:

$$\varphi_k(X) = \varphi(\sigma(X_\ell; \ell \le 0),\ \sigma(X_j; j \ge k)).$$

We call the sequence φ-mixing if $\lim_{k\to\infty} \varphi(k) = 0$. We denote by $S_n = \sum_{k=1}^{n} X_k$.

Ibragimov [11] formulated the following conjecture:

Conjecture 2.9 Assume $(X_k)_{k\in\mathbb{Z}}$ is a stationary φ-mixing sequence such that $EX_0 = 0$ and $EX_0^2 < \infty$. Denote by $\sigma_n^2 = E(S_n^2)$ and assume $\sigma_n^2 \to \infty$. Then $(S_n/\sigma_n)_{n\ge 1}$ converges in distribution to a standard normal variable.

This conjecture was reformulated in [12] to include the functional central limit theorem. For x real denote by $[x]$ the integer part of x and introduce

$$W_n(t) = S_{[nt]}/\sigma_n, 0 \le t \le 1,$$

a random element of $D[0, 1]$, the space of functions defined on $[0, 1]$, which are continuous from the right and have left hand limits. We endow $D[0, 1]$ with uniform topology.

Conjecture 2.10 Let $(X_k)_{k\in\mathbb{Z}}$ be as above. Then W_n is weakly convergent to W, where W denotes the standard Brownian motion on $[0, 1]$.

From Peligrad [18] we know that the Ibragimov-Iosifescu conjectures hold under the additional assumption

$$\liminf_n \mathbb{E}(S_n^2)/n > 0. \tag{2.25}$$

Our study was initially motivated by this conjecture. We came short of proving it. However the results in this paper show that, from some point of view, moments of products and partial sums of a φ-mixing sequence are close to the corresponding ones of an independent sequence.

Other results related to the Ibragimov conjecture can be found, for instance, in [1, 10, 18–20, 26]. They are all based on inequalities for the maximum of partial sums and by Hoffman-Jorgensen type inequalities (see Peligrad [18]), which are valid for φ_k sufficiently small for some $k \ge 1$. These inequalities also lead to Rosenthal type inequalities.

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The Expected Norm of a Sum of Independent Random Matrices: An Elementary Approach

Joel A. Tropp

Abstract In contemporary applied and computational mathematics, a frequent challenge is to bound the expectation of the spectral norm of a sum of independent random matrices. This quantity is controlled by the norm of the expected square of the random matrix and the expectation of the maximum squared norm achieved by one of the summands; there is also a weak dependence on the dimension of the random matrix. The purpose of this paper is to give a complete, elementary proof of this important inequality.

Keywords Probability inequality • Random matrix • Sum of independent random variables

Mathematics Subject Classification (2010). 60B20; 60F10, 60G50, 60G42

1 Motivation

Over the last decade, random matrices have become ubiquitous in applied and computational mathematics. As this trend accelerates, more researchers must confront random matrices as part of their work. Classical random matrix theory can be difficult to use, and it is often silent about the questions that come up in modern applications. As a consequence, it has become imperative to develop and disseminate new tools that are easy to use and that apply to a wide range of random matrices.

J.A. Tropp (✉)
Annenberg Center, MC 305-16, Pasadena, CA, USA

California Institute of Technology, 1200 E. California Blvd., Pasadena, CA 91125-5000, USA
e-mail: jtropp@cms.caltech.edu

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1.1 Matrix Concentration Inequalities

Matrix concentration inequalities are among the most popular of these new methods. For a random matrix $\boldsymbol{Z}$ with appropriate structure, these results use simple parameters associated with the random matrix to provide bounds of the form

$$\mathbb{E}\,\|\boldsymbol{Z} - \mathbb{E}\,\boldsymbol{Z}\| \quad \leq \quad \dots \quad \text{and} \quad \mathbb{P}\,\{\|\boldsymbol{Z} - \mathbb{E}\,\boldsymbol{Z}\| \geq t\} \quad \leq \quad \dots$$

where $\|\cdot\|$ denotes the spectral norm, also known as the ℓ_2 operator norm. Matrix concentration tools have already found a place in many areas of the mathematical sciences, including

- numerical linear algebra [42]
- numerical analysis [30]
- uncertainty quantification [13]
- statistics [23]
- econometrics [6]
- approximation theory [11]
- sampling theory [2]
- machine learning [15, 26]
- learning theory [16, 31]
- signal processing [8]
- optimization [10]
- computer graphics and vision [9]
- quantum information [18]
- algorithms [12, 17]
- combinatorics [33].

These references are chosen more or less at random from a long menu of possibilities. See the monograph [44] for an overview of the main results on matrix concentration, many detailed applications, and additional background references.

1.2 The Expected Norm

The purpose of this paper is to provide a complete proof of the following important theorem. This result is adapted from [7, Theorem A.1]; see also [14, p. 6].

Theorem I (The Expected Norm of a Sum of Independent Random Matrices) *Consider an independent family* $\{\boldsymbol{S}_1, \dots, \boldsymbol{S}_n\}$ *of random* $d_1 \times d_2$ *complex-valued matrices with* $\mathbb{E}\,\boldsymbol{S}_i = \boldsymbol{0}$ *for each index i, and define*

$$\boldsymbol{Z} := \sum_{i=1}^{n} \boldsymbol{S}_i. \tag{1.1}$$

Introduce the matrix variance parameter

$$\begin{aligned} v(\boldsymbol{Z}) &:= \max\left\{\|\mathbb{E}[\boldsymbol{Z}\boldsymbol{Z}^*]\|,\ \|\mathbb{E}[\boldsymbol{Z}^*\boldsymbol{Z}]\|\right\} \\ &= \max\left\{\left\|\sum_{i=1}^{n}\mathbb{E}[\boldsymbol{S}_i\boldsymbol{S}_i^*]\right\|,\ \left\|\sum_{i=1}^{n}\mathbb{E}[\boldsymbol{S}_i^*\boldsymbol{S}_i]\right\|\right\} \end{aligned} \tag{1.2}$$

and the large deviation parameter

$$L := \left(\mathbb{E}\max_i \|\boldsymbol{S}_i\|^2\right)^{1/2}. \tag{1.3}$$

Define the dimensional constant

$$C(\boldsymbol{d}) := C(d_1, d_2) := 4\cdot\left(1 + 2\lceil \log(d_1+d_2)\rceil\right).$$

Then we have the matching estimates

$$\sqrt{c\cdot v(\boldsymbol{Z})} \;+\; c\cdot L \;\le\; \left(\mathbb{E}\,\|\boldsymbol{Z}\|^2\right)^{1/2} \;\le\; \sqrt{C(\boldsymbol{d})\cdot v(\boldsymbol{Z})} \;+\; C(\boldsymbol{d})\cdot L. \tag{1.4}$$

In the lower inequality, we can take $c := 1/4$.

The proof of this result occupies the bulk of this paper. The argument is based on the most elementary considerations possible. Indeed, we need nothing more than some simple matrix inequalities and some basic discrete probability. In contrast, all previous proofs of Theorem I rely on the noncommutative Khintchine inequality [5, 27, 36]. This paper is targeted at the high-dimensional probability community; see the arXiv version [43] for a presentation with additional details. Once the reader has digested the ideas here, it may be easier to appreciate the related—but more sophisticated—arguments based on exchangeable pairs in the papers [28, 35].

1.3 Discussion

Before we continue, some remarks about Theorem I are in order. First, although it may seem restrictive to focus on independent sums, as in (1.1), this model captures an enormous number of useful examples. See the monograph [44] for justification.

We have chosen the term *variance parameter* because the quantity (1.2) is a direct generalization of the variance of a scalar random variable. The passage from the first formula to the second formula in (1.2) is an immediate consequence of the assumption that the summands $\boldsymbol{S}_i$ are independent and have zero mean (see Sect. 4). We use the term *large-deviation parameter* because the quantity (1.3) reflects the part of the expected norm of the random matrix that is attributable to one of the summands taking an unusually large value. In practice, both parameters are easy to compute using matrix arithmetic and some basic probabilistic considerations.

In applications, it is common that we need high-probability bounds on the norm of a random matrix. Typically, the bigger challenge is to estimate the expectation of the norm, which is what Theorem I achieves. Once we have a bound for the expectation, we can use scalar concentration inequalities, such as the result [4, Theorem 6.10], to obtain high-probability bounds on the deviation between the norm and its mean value.

We have stated Theorem I as a bound on the second moment of $\|\boldsymbol{Z}\|$ because this is the most natural form of the result. Equivalent bounds hold for the first moment:

$$\sqrt{c' \cdot v(\boldsymbol{Z})} + c' \cdot L \quad \leq \quad \mathbb{E}\,\|\boldsymbol{Z}\| \quad \leq \quad \sqrt{C(\boldsymbol{d}) \cdot v(\boldsymbol{Z})} + C(\boldsymbol{d}) \cdot L.$$

We can take $c' = 1/8$. The upper bound follows easily from (1.4) and Jensen's inequality. The lower bound requires the Khintchine–Kahane inequality [24].

It is productive to interpret Theorem I as a perturbation result. Suppose that $\boldsymbol{Z} = \boldsymbol{R} - \mathbb{E}\,\boldsymbol{R}$, where $\boldsymbol{R}$ is a sum of independent random matrices. Bounds for $\mathbb{E}\,\|\boldsymbol{Z}\|$ have many useful consequences. This type of result implies that, on average, all of the singular values of $\boldsymbol{R}$ are close to the corresponding singular values of $\mathbb{E}\,\boldsymbol{R}$. On average, the singular vectors of $\boldsymbol{R}$ are close to the corresponding singular vectors of $\mathbb{E}\,\boldsymbol{R}$, provided that the associated singular values are isolated. Furthermore, we discover that, on average, each linear functional $\operatorname{tr}[\boldsymbol{CR}]$ is uniformly close to $\mathbb{E}\operatorname{tr}[\boldsymbol{CR}]$ for each fixed matrix $\boldsymbol{C} \in \mathbb{M}^{d_2 \times d_1}$ with Schatten 1-norm $\|\boldsymbol{C}\|_{S_1} \leq 1$.

Observe that the lower and upper estimates in (1.4) differ only by the factor $C(\boldsymbol{d})$. As a consequence, the lower bound has no explicit dimensional dependence, while the upper bound has only a weak dependence on the dimension. Under the assumptions of the theorem, it is not possible to make substantial improvements to either the lower bound or the upper bound. Section 6 provides examples that support this claim.

In the theory of matrix concentration, one of the major challenges is to understand what properties of the random matrix $\boldsymbol{Z}$ allow us to remove the dimensional factor $C(\boldsymbol{d})$ from the estimate (1.4). This question is largely open, but the recent papers [1, 34, 45] make some progress.

1.4 History

Variants of Theorem I have been available for some time. An early version of the upper bound appeared in Rudelson's work [39, Theorem 1]; see also [40, Theorem 3.1] and [41, Sect. 9]. The first explicit statement of the upper bound appeared in [7, Theorem A.1]; the same result was discovered independently by Dirksen [14, p. 6]. The proofs of these results all rely on the noncommutative Khintchine inequality [5, 27, 36]. In our approach, the main innovation is a particularly easy proof of a Khintchine-type inequality for matrices, patterned after [28, Corollary 7.3] and [45, Theorem 8.1].

The ideas behind the proof of the lower bound in Theorem I are older. This estimate depends on generic considerations about the behavior of a sum of independent random variables in a Banach space. These techniques are explained in detail in [25, Chap. 6]. Our presentation expands on a proof sketch that appears in the monograph [44, Sects. 5.1.2 and 6.1.2]; see also [14].

1.5 Roadmap

Section 2 contains some background material from linear algebra. To prove the upper bound in Theorem I, the key step is to establish the result for the special case of a sum of fixed matrices, each modulated by a random sign. This result appears in Sect. 3. In Sect. 4, we exploit this result to obtain the upper bound in (1.4). In Sect. 5, we present the easier proof of the lower bound in (1.4). Finally, Sect. 6 shows that it is not possible to improve (1.4) substantially.

2 Background

This section contains some background results from linear algebra and probability. Most of this material is drawn from [3, 19, 25].

2.1 Notation

We write $\mathbb{C}^d$ for the complex linear space of d-dimensional complex vectors. The symbol $\|\cdot\|$ denotes the ℓ_2 norm on $\mathbb{C}^d$. We write $\mathbb{M}^{d_1 \times d_2}$ for the complex linear space of $d_1 \times d_2$ complex matrices. The symbol $\|\cdot\|$ also denotes the spectral norm of a matrix, which is often called the ℓ_2 operator norm. The operator $\operatorname{tr}[\cdot]$ returns the trace of a square matrix; we instate the convention that powers bind before the trace. The star * refers to the conjugate transpose operation on vectors and matrices.

Next, introduce the real linear space $\mathbb{H}_d$ of $d \times d$ Hermitian matrices. The maps $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ return the algebraic minimum and maximum eigenvalues of an Hermitian matrix. We use the symbol $\preccurlyeq$ to refer to the semidefinite order on Hermitian matrices: $\boldsymbol{A} \preccurlyeq \boldsymbol{H}$ means that the matrix $\boldsymbol{H} - \boldsymbol{A}$ is positive semidefinite.

The map $\mathbb{P}\{\cdot\}$ returns the probability of an event. The operator $\mathbb{E}[\cdot]$ returns the expectation of a random variable. We only include the brackets when it is necessary for clarity, and we impose the convention that nonlinear functions bind before the expectation. The notation $\mathbb{E}_X[\cdot]$ refers to partial expectation with respect to the random variable X, with all other random variables held fixed.

2.2 Basic Spectral Theory and Some Matrix Inequalities

Each Hermitian matrix $\boldsymbol{H} \in \mathbb{H}_d$ has an *eigenvalue decomposition*

$$\boldsymbol{H} = \sum_{i=1}^{d} \lambda_i \boldsymbol{u}_i \boldsymbol{u}_i^*$$

where the eigenvalues λ_i are uniquely determined real numbers and $\{\boldsymbol{u}_i\}$ is an orthonormal basis for $\mathbb{C}^d$. For each nonnegative integer r,

$$\boldsymbol{H} = \sum_{i=1}^{d} \lambda_i \boldsymbol{u}_i \boldsymbol{u}_i^* \quad \text{implies} \quad \boldsymbol{H}^r = \sum_{i=1}^{d} \lambda_i^r \boldsymbol{u}_i \boldsymbol{u}_i^*. \tag{2.1}$$

In particular, $\boldsymbol{H}^{2p}$ is positive semidefinite for each nonnegative integer p, and $\|\boldsymbol{H}\|^{2p} = \|\boldsymbol{H}^{2p}\|$.

We need a bound for the norm of a sum of squared positive-semidefinite matrices.

Fact 2.1 (Bound for a Sum of Squares) *Consider positive-semidefinite matrices* $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_n \in \mathbb{H}_d$. *Then*

$$\left\| \sum_{i=1}^{n} \boldsymbol{A}_i^2 \right\| \le \max_i \|\boldsymbol{A}_i\| \cdot \left\| \sum_{i=1}^{n} \boldsymbol{A}_i \right\|.$$

Proof For each index i,

$$\boldsymbol{A}_i^2 \preccurlyeq M \cdot \boldsymbol{A}_i \quad \text{where } M := \max_i \lambda_{\max}(\boldsymbol{A}_i).$$

Summing these relations, we see that

$$\sum_{i=1}^{n} \boldsymbol{A}_i^2 \preccurlyeq M \cdot \sum_{i=1}^{n} \boldsymbol{A}_i.$$

Weyl's monotonicity principle [3, Corollary III.2.3] yields the inequality

$$\lambda_{\max}\left(\sum_{i=1}^{n} \boldsymbol{A}_i^2 \right) \le \lambda_{\max}\left(M \cdot \sum_{i=1}^{n} \boldsymbol{A}_i \right) = M \cdot \lambda_{\max}\left(\sum_{i=1}^{n} \boldsymbol{A}_i \right).$$

We used the fact that the maximum eigenvalue of an Hermitian matrix is positively homogeneous. Finally, the spectral norm of a positive-semidefinite matrix is equal to its maximum eigenvalue. □

We require another substantial matrix inequality, which is one of several matrix analogs of the inequality between the geometric mean and the arithmetic mean.

Fact 2.2 (GM–AM Trace Inequality) *Consider Hermitian matrices* $\boldsymbol{H}, \boldsymbol{W}, \boldsymbol{Y}$ *in* $\mathbb{H}_d$*. For integers r and q that satisfy* $0 \le q \le 2r$*,*

$$\operatorname{tr}\left[\boldsymbol{H}\boldsymbol{W}^q\boldsymbol{H}\boldsymbol{Y}^{2r-q}\right] + \operatorname{tr}\left[\boldsymbol{H}\boldsymbol{W}^{2r-q}\boldsymbol{H}\boldsymbol{Y}^q\right] \le \operatorname{tr}\left[\boldsymbol{H}^2 \cdot \left(\boldsymbol{W}^{2r} + \boldsymbol{Y}^{2r}\right)\right]. \tag{2.2}$$

In particular,

$$\sum_{q=0}^{2r} \operatorname{tr}\left[\boldsymbol{H}\boldsymbol{W}^q\boldsymbol{H}\boldsymbol{Y}^{2r-q}\right] \le \frac{2r+1}{2} \operatorname{tr}\left[\boldsymbol{H}^2 \cdot \left(\boldsymbol{W}^{2r} + \boldsymbol{Y}^{2r}\right)\right].$$

The result (2.2) is a matrix version of the following numerical inequality. For $\lambda, \mu \ge 0$,

$$\lambda^\theta \mu^{1-\theta} + \lambda^{1-\theta}\mu^\theta \le \lambda + \mu \quad \text{for each } \theta \in [0, 1]. \tag{2.3}$$

This estimate follows from the observation that the left-hand side is a convex function of θ.

Proof We will prove (2.2) as a consequence of (2.3). The case $r = 0$ is immediate, so we may assume that $r \ge 1$. Let q be an integer in the range $0 \le q \le 2r$. Introduce eigenvalue decompositions:

$$\boldsymbol{W} = \sum_{i=1}^{d} \lambda_i \boldsymbol{u}_i \boldsymbol{u}_i^* \quad \text{and} \quad \boldsymbol{Y} = \sum_{j=1}^{d} \mu_j \boldsymbol{v}_j \boldsymbol{v}_j^*.$$

Calculate that

$$\begin{aligned}
\operatorname{tr}\left[\boldsymbol{H}\boldsymbol{W}^q\boldsymbol{H}\boldsymbol{Y}^{2r-q}\right] &= \operatorname{tr}\left[\boldsymbol{H}\left(\sum_{i=1}^{d} \lambda_i^q \boldsymbol{u}_i\boldsymbol{u}_i^*\right)\boldsymbol{H}\left(\sum_{j=1}^{d} \mu_j^{2r-q} \boldsymbol{v}_j\boldsymbol{v}_j^*\right)\right] \\
&= \sum_{i,j=1}^{d} \lambda_i^q \mu_j^{2r-q} \cdot \operatorname{tr}\left[\boldsymbol{H}\boldsymbol{u}_i\boldsymbol{u}_i^*\boldsymbol{H}\boldsymbol{v}_j\boldsymbol{v}_j^*\right] \qquad (2.4) \\
&\le \sum_{i,j=1}^{d} |\lambda_i|^q \, |\mu_j|^{2r-q} \cdot \left|\boldsymbol{u}_i^*\boldsymbol{H}\boldsymbol{v}_j\right|^2.
\end{aligned}$$

The first identity relies on the formula (2.1) for the eigenvalue decomposition of a monomial. The second step depends on the linearity of the trace. In the last line, we rewrite the trace using cyclicity, and the inequality emerges when we apply absolute values.

Invoking the inequality (2.4) twice, we arrive at the bound

$$\begin{aligned}
&\operatorname{tr}\left[\boldsymbol{H}\boldsymbol{W}^{q}\boldsymbol{H}\boldsymbol{Y}^{2r-q}\right]+\operatorname{tr}\left[\boldsymbol{H}\boldsymbol{W}^{2r-q}\boldsymbol{H}\boldsymbol{Y}^{q}\right] \\
&\qquad\leq \sum_{i,j=1}^{d}\left(|\lambda_i|^{q}\,|\mu_j|^{2r-q}+|\lambda_i|^{2r-q}\,|\mu_j|^{q}\right)\cdot\left|\boldsymbol{u}_i^{*}\boldsymbol{H}\boldsymbol{v}_j\right|^{2} \\
&\qquad\leq \sum_{i,j=1}^{d}\left(\lambda_i^{2r}+\mu_j^{2r}\right)\cdot\left|\boldsymbol{u}_i^{*}\boldsymbol{H}\boldsymbol{v}_j\right|^{2}.
\end{aligned} \tag{2.5}$$

The second inequality is (2.3), with $\theta = q/(2r)$ and $\lambda = \lambda_i^{2r}$ and $\mu = \mu_j^{2r}$.

It remains to rewrite the right-hand side of (2.5) in a more recognizable form. To that end, observe that

$$\begin{aligned}
&\operatorname{tr}\left[\boldsymbol{H}\boldsymbol{W}^{q}\boldsymbol{H}\boldsymbol{Y}^{2r-q}\right]+\operatorname{tr}\left[\boldsymbol{H}\boldsymbol{W}^{2r-q}\boldsymbol{H}\boldsymbol{Y}^{q}\right] \\
&\qquad\leq \sum_{i,j=1}^{d}\left(\lambda_i^{2r}+\mu_j^{2r}\right)\cdot\operatorname{tr}\left[\boldsymbol{H}\boldsymbol{u}_i\boldsymbol{u}_i^{*}\boldsymbol{H}\boldsymbol{v}_j\boldsymbol{v}_j^{*}\right] \\
&\qquad= \operatorname{tr}\left[\boldsymbol{H}\left(\sum_{i=1}^{d}\lambda_i^{2r}\boldsymbol{u}_i\boldsymbol{u}_i^{*}\right)\boldsymbol{H}\left(\sum_{j=1}^{d}\boldsymbol{v}_j\boldsymbol{v}_j^{*}\right)\right] \\
&\qquad\qquad+\operatorname{tr}\left[\boldsymbol{H}\left(\sum_{i=1}^{d}\boldsymbol{u}_i\boldsymbol{u}_i^{*}\right)\boldsymbol{H}\left(\sum_{j=1}^{d}\mu_j^{2r}\boldsymbol{v}_j\boldsymbol{v}_j^{*}\right)\right] \\
&\qquad= \operatorname{tr}\left[\boldsymbol{H}^{2}\cdot\boldsymbol{W}^{2r}\right]+\operatorname{tr}\left[\boldsymbol{H}^{2}\cdot\boldsymbol{Y}^{2r}\right].
\end{aligned}$$

This argument just reverses the steps leading to (2.4). □

2.3 The Hermitian Dilation

Next, we introduce the *Hermitian dilation* $\mathscr{H}(\boldsymbol{B})$ of a rectangular matrix $\boldsymbol{B} \in \mathbb{M}^{d_1\times d_2}$. This is the Hermitian matrix

$$\mathscr{H}(\boldsymbol{B}) := \begin{bmatrix} \boldsymbol{0} & \boldsymbol{B} \\ \boldsymbol{B}^{*} & \boldsymbol{0} \end{bmatrix} \in \mathbb{H}_{d_1+d_2}. \tag{2.6}$$

Note that the map $\mathscr{H}$ is real-linear. By direct calculation,

$$\mathscr{H}(\boldsymbol{B})^{2} = \begin{bmatrix} \boldsymbol{B}\boldsymbol{B}^{*} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{B}^{*}\boldsymbol{B} \end{bmatrix}. \tag{2.7}$$

We also have the spectral-norm identity

$$\|\mathscr{H}(\boldsymbol{B})\| = \|\boldsymbol{B}\| . \tag{2.8}$$

This point follows by direct calculation.

2.4 Symmetrization

Symmetrization is an important technique for studying the expectation of a function of independent random variables. The idea is to inject auxiliary randomness into the function. Then we condition on the original random variables and average with respect to the extra randomness. When the auxiliary random variables are more pliable, this approach can lead to significant simplifications.

A *Rademacher* random variable ε takes the two values ± 1 with equal probability. The following result shows how we can use Rademacher random variables to study a sum of independent random matrices.

Fact 2.3 (Symmetrization) *Consider an independent family* $\{\boldsymbol{S}_1, \dots, \boldsymbol{S}_n\} \subset \mathbb{M}^{d_1 \times d_2}$ *of random matrices. Let* $\{\varepsilon_1, \dots, \varepsilon_n\}$ *be an independent family of Rademacher random variables that are also independent from the random matrices. For each* $r \geq 1$,

$$\begin{aligned}\frac{1}{2} \cdot \left(\mathbb{E} \left\| \sum_{i=1}^{n} \varepsilon_i (\boldsymbol{S}_i - \mathbb{E}\, \boldsymbol{S}_i) \right\|^r \right)^{1/r} &\leq \left(\mathbb{E} \left\| \sum_{i=1}^{n} (\boldsymbol{S}_i - \mathbb{E}\, \boldsymbol{S}_i) \right\|^r \right)^{1/r} \\ &\leq 2 \cdot \left(\mathbb{E} \left\| \sum_{i=1}^{n} \varepsilon_i \boldsymbol{S}_i \right\|^r \right)^{1/r} .\end{aligned}$$

This result holds whenever $\mathbb{E}\, \|\boldsymbol{S}_i\|^r < \infty$ *for each index i.*

See [25, Lemma 6.3] for the easy proof.

3 The Expected Norm of a Matrix Rademacher Series

To prove Theorem I, our overall strategy is to use symmetrization. This approach allows us to reduce the study of an independent sum of random matrices to the study of a sum of fixed matrices modulated by independent Rademacher random variables. This type of random matrix is called a *matrix Rademacher series*. In this section, we establish a bound on the spectral norm of a matrix Rademacher series. This is the key technical step in the proof of Theorem I.

Theorem 3.1 (Matrix Rademacher Series) *Let $\boldsymbol{H}_1, \dots, \boldsymbol{H}_n$ be fixed Hermitian matrices with dimension d. Let $\varepsilon_1, \dots, \varepsilon_n$ be independent Rademacher random variables. Then*

$$\left(\mathbb{E} \left\| \sum_{i=1}^{n} \varepsilon_i \boldsymbol{H}_i \right\|^2 \right)^{1/2} \le \sqrt{1 + 2\lceil \log d \rceil} \cdot \left\| \sum_{i=1}^{n} \boldsymbol{H}_i^2 \right\|^{1/2}. \tag{3.1}$$

The proof of Theorem 3.1 occupies the bulk of this section, beginning with Sect. 3.2. The argument is really just a fancy version of the familiar calculation of the moments of a centered standard normal random variable; see Sect. 3.8 for details.

3.1 Discussion

Before we establish Theorem 3.1, let us make a few comments. First, it is helpful to interpret the result in the same language we have used to state Theorem I. Introduce the matrix Rademacher series

$$\boldsymbol{X} := \sum_{i=1}^{n} \varepsilon_i \boldsymbol{H}_i.$$

Compute the matrix variance, defined in (1.2):

$$v(\boldsymbol{X}) := \left\| \mathbb{E}\, \boldsymbol{X}^2 \right\| = \left\| \sum_{i,j=1}^{n} \mathbb{E}[\varepsilon_i \varepsilon_j] \cdot \boldsymbol{H}_i \boldsymbol{H}_j \right\| = \left\| \sum_{i=1}^{n} \boldsymbol{H}_i^2 \right\|.$$

We may rewrite Theorem 3.1 as the statement that

$$\left(\mathbb{E}\, \|\boldsymbol{X}\|^2 \right)^{1/2} \le \sqrt{(1 + 2\lceil \log d \rceil) \cdot v(\boldsymbol{X})}.$$

In other words, Theorem 3.1 is a sharper version of the upper bound in Theorem I for the special case of a matrix Rademacher series.

Next, we have focused on bounding the second moment of $\|\boldsymbol{X}\|$ because this is the most natural form of the result. Note that we also control the first moment because of Jensen's inequality:

$$\mathbb{E} \left\| \sum_{i=1}^{n} \varepsilon_i \boldsymbol{H}_i \right\| \le \sqrt{1 + 2\lceil \log d \rceil} \cdot \left\| \sum_{i=1}^{n} \boldsymbol{H}_i^2 \right\|^{1/2}. \tag{3.2}$$

A simple variant on the proof of Theorem 3.1 provides bounds for higher moments.

Third, the dimensional factor on the right-hand side of (3.1) is asymptotically sharp. Indeed, let us write $K(d)$ for the minimum possible constant in the inequality

$$\left(\mathbb{E} \left\| \sum_{i=1}^{n} \varepsilon_i \boldsymbol{H}_i \right\|^2 \right)^{1/2} \leq K(d) \cdot \left\| \sum_{i=1}^{n} \boldsymbol{H}_i^2 \right\|^{1/2} \quad \text{for } \boldsymbol{H}_i \in \mathbb{H}_d \text{ and } n \in \mathbb{N}.$$

The example in Sect. 6.1 shows that

$$K(d) \geq \sqrt{2 \log d}.$$

In other words, (3.1) cannot be improved without making further assumptions.

Theorem 3.1 is a variant on the noncommutative Khintchine inequality, first established by Lust-Piquard [27] and later improved by Pisier [36] and by Buchholz [5]. The noncommutative Khintchine inequality gives bounds for the Schatten norm of a matrix Rademacher series, rather than for the spectral norm. Rudelson [39] pointed out that the noncommutative Khintchine inequality also implies bounds for the spectral norm of a matrix Rademacher series. In our presentation, we choose to control the spectral norm directly.

3.2 The Spectral Norm and the Trace Moments

To begin the proof of Theorem 3.1, we introduce the random Hermitian matrix

$$\boldsymbol{X} := \sum_{i=1}^{n} \varepsilon_i \boldsymbol{H}_i \tag{3.3}$$

Our goal is to bound the expected spectral norm of $\boldsymbol{X}$. We may proceed by estimating the expected trace of a power of the random matrix, which is known as a *trace moment*. Fix a positive integer p. Observe that

$$\begin{aligned} \left(\mathbb{E} \|\boldsymbol{X}\|^2 \right)^{1/2} &\leq \left(\mathbb{E} \|\boldsymbol{X}\|^{2p} \right)^{1/(2p)} \\ &= \left(\mathbb{E} \|\boldsymbol{X}^{2p}\| \right)^{1/(2p)} \leq \left(\mathbb{E} \operatorname{tr} \boldsymbol{X}^{2p} \right)^{1/(2p)}. \end{aligned} \tag{3.4}$$

The first identity is Jensen's inequality. In the last inequality, we bound the norm of the positive-semidefinite matrix $\boldsymbol{X}^{2p}$ by its trace.

Remark 3.2 (Higher Moments) It should be clear that we can also bound expected powers of the spectral norm using the same technique. For simplicity, we omit this development.

3.3 Summation by Parts

To study the trace moments of the random matrix $\boldsymbol{X}$, we rely on a discrete analog of integration by parts. This approach is clearer if we introduce some more notation. For each index i, define the random matrices

$$\boldsymbol{X}_{+i} := +\boldsymbol{H}_i + \sum_{j \neq i} \varepsilon_j \boldsymbol{H}_j \quad \text{and} \quad \boldsymbol{X}_{-i} := -\boldsymbol{H}_i + \sum_{j \neq i} \varepsilon_j \boldsymbol{H}_j$$

In other words, the distribution of $\boldsymbol{X}_{\varepsilon_i i}$ is the conditional distribution of the random matrix $\boldsymbol{X}$ given the value ε_i of the ith Rademacher variable.

Beginning with the trace moment, observe that

$$\begin{aligned}
\mathbb{E} \operatorname{tr} \boldsymbol{X}^{2p} &= \mathbb{E} \operatorname{tr} \left[\boldsymbol{X} \cdot \boldsymbol{X}^{2p-1} \right] \\
&= \sum_{i=1}^{n} \mathbb{E} \left[\mathbb{E}_{\varepsilon_i} \operatorname{tr} \left[\varepsilon_i \boldsymbol{H}_i \cdot \boldsymbol{X}^{2p-1} \right] \right] \\
&= \sum_{i=1}^{n} \mathbb{E} \left[\frac{1}{2} \operatorname{tr} \left[+ \boldsymbol{H}_i \cdot \boldsymbol{X}_{+i}^{2p-1} \right] + \frac{1}{2} \operatorname{tr} \left[- \boldsymbol{H}_i \cdot \boldsymbol{X}_{-i}^{2p-1} \right] \right] \\
&= \frac{1}{2} \sum_{i=1}^{n} \mathbb{E} \operatorname{tr} \left[\boldsymbol{H}_i \cdot \left(\boldsymbol{X}_{+i}^{2p-1} - \boldsymbol{X}_{-i}^{2p-1} \right) \right]
\end{aligned} \tag{3.5}$$

To reach the second line, we simply write out the definition (3.3) of the random matrix $\boldsymbol{X}$. Then we write the expectation as an iterated expectation. Afterward, write out the partial expectation using the notation $\boldsymbol{X}_{\pm i}$. Finally, we collect terms.

3.4 A Difference of Powers

Next, let us apply an algebraic identity to reduce the difference of powers in (3.5). For matrices $\boldsymbol{W}, \boldsymbol{Y} \in \mathbb{H}_d$, it holds that

$$\boldsymbol{W}^{2p-1} - \boldsymbol{Y}^{2p-1} = \sum_{q=0}^{2p-2} \boldsymbol{W}^{q} (\boldsymbol{W} - \boldsymbol{Y}) \boldsymbol{Y}^{2p-2-q}. \tag{3.6}$$

To check this identity, expand the matrix products and notice that the sum telescopes.

Introduce (3.6) with $\boldsymbol{W} = \boldsymbol{X}_{+i}$ and $\boldsymbol{Y} = \boldsymbol{X}_{-i}$ into (3.5) to see that

$$\begin{aligned} \mathbb{E}\,\mathrm{tr}\,\boldsymbol{X}^{2p} &= \frac{1}{2}\sum_{i=1}^{n}\mathbb{E}\,\mathrm{tr}\left[\boldsymbol{H}_i \cdot \sum_{q=0}^{2p-2}\boldsymbol{X}_{+i}^{q}\big(\boldsymbol{X}_{+i} - \boldsymbol{X}_{-i}\big)\boldsymbol{X}_{-i}^{2p-2-q}\right] \\ &= \sum_{i=1}^{n}\sum_{q=0}^{2p-2}\mathbb{E}\,\mathrm{tr}\left[\boldsymbol{H}_i\boldsymbol{X}_{+i}^{q}\boldsymbol{H}_i\boldsymbol{X}_{-i}^{2p-2-q}\right]. \end{aligned} \tag{3.7}$$

We have used the observation that $\boldsymbol{X}_{+i} - \boldsymbol{X}_{-i} = 2\boldsymbol{H}_i$.

3.5 A Bound for the Trace Moments

We are now in a position to obtain a bound for the trace moments of $\boldsymbol{X}$. Beginning with (3.7), we compute that

$$\begin{aligned} \mathbb{E}\,\mathrm{tr}\,\boldsymbol{X}^{2p} &= \sum_{i=1}^{n}\sum_{q=0}^{2p-2}\mathbb{E}\,\mathrm{tr}\left[\boldsymbol{H}_i\boldsymbol{X}_{+i}^{q}\boldsymbol{H}_i\boldsymbol{X}_{-i}^{2p-2-q}\right] \\ &\le \sum_{i=1}^{n}\frac{2p-1}{2}\,\mathbb{E}\,\mathrm{tr}\left[\boldsymbol{H}_i^2 \cdot \left(\boldsymbol{X}_{+i}^{2p-2} + \boldsymbol{X}_{-i}^{2p-2}\right)\right] \\ &= (2p-1)\cdot\sum_{i=1}^{n}\mathbb{E}\,\mathrm{tr}\left[\boldsymbol{H}_i^2 \cdot \left(\mathbb{E}_{\varepsilon_i}\boldsymbol{X}^{2p-2}\right)\right] \\ &= (2p-1)\cdot\mathbb{E}\,\mathrm{tr}\left[\left(\sum_{i=1}^{n}\boldsymbol{H}_i^2\right)\cdot\boldsymbol{X}^{2p-2}\right] \\ &\le (2p-1)\cdot\left\|\sum_{i=1}^{n}\boldsymbol{H}_i^2\right\|\cdot\mathbb{E}\,\mathrm{tr}\,\boldsymbol{X}^{2p-2}. \end{aligned} \tag{3.8}$$

The bound in the second line is Fact 2.2, with $r = p - 1$ and $\boldsymbol{W} = \boldsymbol{X}_{+i}$ and $\boldsymbol{Y} = \boldsymbol{X}_{-i}$. To reach the third line, observe that the parenthesis in the second line is twice the partial expectation of $\boldsymbol{X}^{2p-2}$ with respect to ε_i. Last, invoke the familiar spectral norm bound for the trace of a product, using the observation that $\boldsymbol{X}^{2p-2}$ is positive semidefinite.

3.6 Iteration and the Spectral Norm Bound

The expression (3.8) shows that the trace moment is controlled by a trace moment with a smaller power:

$$\mathbb{E}\operatorname{tr} \boldsymbol{X}^{2p} \leq (2p-1) \cdot \left\| \sum_{i=1}^{n} \boldsymbol{H}_i^2 \right\| \cdot \mathbb{E}\operatorname{tr} \boldsymbol{X}^{2p-2}.$$

Iterating this bound p times, we arrive at the result

$$\begin{aligned} \mathbb{E}\operatorname{tr} \boldsymbol{X}^{2p} &\leq (2p-1)!! \cdot \left\| \sum_{i=1}^{n} \boldsymbol{H}_i^2 \right\|^p \cdot \operatorname{tr} \boldsymbol{X}^0 \\ &= d \cdot (2p-1)!! \cdot \left\| \sum_{i=1}^{n} \boldsymbol{H}_i^2 \right\|^p . \end{aligned} \tag{3.9}$$

The double factorial is $(2p-1)!! := (2p-1)(2p-3)(2p-5)\cdots(5)(3)(1)$.

The expression (3.4) shows that we can control the expected spectral norm of $\boldsymbol{X}$ by means of a trace moment. Therefore, for any nonnegative integer p, it holds that

$$\mathbb{E}\|\boldsymbol{X}\| \leq \left(\mathbb{E}\operatorname{tr} \boldsymbol{X}^{2p}\right)^{1/(2p)} \leq \left(d \cdot (2p-1)!!\right)^{1/(2p)} \cdot \left\| \sum_{i=1}^{n} \boldsymbol{H}_i^2 \right\|^{1/2} . \tag{3.10}$$

The second inequality is simply our bound (3.9). All that remains is to choose the value of p to minimize the factor on the right-hand side.

3.7 Calculating the Constant

Let us develop an accurate bound for the leading factor on the right-hand side of (3.10). We claim that

$$(2p-1)!! \leq \left(\frac{2p+1}{\mathrm{e}}\right)^p . \tag{3.11}$$

Given this estimate, select $p = \lceil \log d \rceil$ to reach

$$\begin{aligned} \left(d \cdot (2p-1)!!\right)^{1/(2p)} &\leq d^{1/(2p)} \sqrt{\frac{2p+1}{\mathrm{e}}} \\ &\leq \sqrt{2p+1} = \sqrt{1 + 2\lceil \log d \rceil}. \end{aligned} \tag{3.12}$$

Introduce the inequality (3.12) into (3.10) to complete the proof of Theorem 3.1.

To check that (3.11) is valid, we use some tools from integral calculus:

$$\begin{aligned}
&\log\big((2p-1)!!\big)\\
&\quad = \sum_{i=1}^{p-1} \log(2i+1)\\
&\quad = \left[\frac{1}{2}\log(2\cdot 0+1) + \sum_{i=1}^{p-1}\log(2i+1) + \frac{1}{2}\log(2p+1)\right] - \frac{1}{2}\log(2p+1)\\
&\quad \le \int_0^p \log(2x+1)\,\mathrm{d}x - \frac{1}{2}\log(2p+1)\\
&\quad = p\log(2p+1) - p.
\end{aligned}$$

The bracket in the second line is the trapezoid rule approximation of the integral in the third line. Since the integrand is concave, the trapezoid rule underestimates the integral. Exponentiating this formula, we arrive at (3.11).

3.8 Context

The proof of Theorem 3.1 is really just a discrete, matrix version of the familiar calculation of the $(2p)$th moment of a centered normal random variable. Let us elaborate. Recall the Gaussian integration by parts formula:

$$\mathbb{E}[\gamma \cdot f(\gamma)] = \sigma^2 \cdot \mathbb{E}[f'(\gamma)] \tag{3.13}$$

where $\gamma \sim \text{NORMAL}(0, \sigma^2)$ and $f : \mathbb{R} \to \mathbb{R}$ is any function for which the integrals are finite. To compute the $(2p)$th moment of γ, we apply (3.13) repeatedly to obtain

$$\mathbb{E}\,\gamma^{2p} = \mathbb{E}\left[\gamma\cdot\gamma^{2p-1}\right] = (2p-1)\cdot\sigma^2\cdot\mathbb{E}\,\gamma^{2p-2} = \cdots = (2p-1)!!\cdot\sigma^{2p}$$

In Theorem 3.1, the matrix variance parameter $v(\boldsymbol{X})$ plays the role of the scalar variance σ^2.

In fact, the link with Gaussian integration by parts is even stronger. Consider a matrix Gaussian series

$$\boldsymbol{Y} := \sum_{i=1}^{n} \gamma_i \boldsymbol{H}_i$$

where $\{\gamma_i\}$ is an independent family of standard normal variables. If we replace the discrete integration by parts in the proof of Theorem 3.1 with Gaussian integration by parts, the argument leads to the bound

$$\left(\mathbb{E} \left\| \sum_{i=1}^{n} \gamma_i \boldsymbol{H}_i \right\|^2 \right)^{1/2} \leq \sqrt{1 + 2\lceil \log d \rceil} \cdot \left\| \sum_{i=1}^{n} \boldsymbol{H}_i^2 \right\|^{1/2}.$$

This approach requires matrix calculus, but it is slightly simpler than the argument for matrix Rademacher series in other respects. See [45, Theorem 8.1] for a proof of the noncommutative Khintchine inequality for Gaussian series along these lines. The exchangeable pairs technique for establishing the noncommutative Khintchine inequality [28, Corollary 7.1] is another realization of the same idea.

4 Upper Bounds for the Expected Norm

We are now prepared to establish the upper bound for an arbitrary sum of independent random matrices. The argument is based on the specialized result, Theorem 3.1, for matrix Rademacher series. It proceeds by steps through more and more general classes of random matrices: first positive semidefinite, then Hermitian, and finally rectangular. Here is what we will show.

Theorem 4.1 (Expected Norm: Upper Bounds) *Define the dimensional constant* $C(d) := 4(1 + 2\lceil \log d \rceil)$. *The expected spectral norm of a sum of independent random matrices satisfies the following upper bounds.*

1. ***The Positive-Semidefinite Case.*** *Consider a family* $\{\boldsymbol{T}_1, \dots, \boldsymbol{T}_n\}$ *of independent, random* $d \times d$ *positive-semidefinite matrices, and define*

$$\boldsymbol{W} := \sum_{i=1}^{n} \boldsymbol{T}_i.$$

Then

$$\mathbb{E} \|\boldsymbol{W}\| \leq \left[\|\mathbb{E}\, \boldsymbol{W}\|^{1/2} + \sqrt{C(d)} \cdot \left(\mathbb{E} \max_i \|\boldsymbol{T}_i\| \right)^{1/2} \right]^2. \tag{4.1}$$

2. ***The Centered Hermitian Case.*** *Consider a family* $\{\boldsymbol{Y}_1, \dots, \boldsymbol{Y}_n\}$ *of independent, random* $d \times d$ *Hermitian matrices with* $\mathbb{E}\, \boldsymbol{Y}_i = \boldsymbol{0}$ *for each index i, and define*

$$\boldsymbol{X} := \sum_{i=1}^{n} \boldsymbol{Y}_i.$$

Then

$$\left(\mathbb{E}\,\|\boldsymbol{X}\|^2\right)^{1/2} \le \sqrt{C(d)} \cdot \left\|\mathbb{E}\,\boldsymbol{X}^2\right\|^{1/2} + C(d) \cdot \left(\mathbb{E}\max_i \|\boldsymbol{Y}_i\|^2\right)^{1/2}. \tag{4.2}$$

3. ***The Centered Rectangular Case.*** *Consider a family* $\{\boldsymbol{S}_1, \dots, \boldsymbol{S}_n\}$ *of independent, random* $d_1 \times d_2$ *matrices with* $\mathbb{E}\,\boldsymbol{S}_i = \boldsymbol{0}$ *for each index i, and define*

$$\boldsymbol{Z} := \sum_{i=1}^{n} \boldsymbol{S}_i.$$

Then

$$\left(\mathbb{E}\,\|\boldsymbol{Z}\|^2\right)^{1/2} \le \sqrt{C(d)} \cdot \max\left\{\left\|\mathbb{E}\left[\boldsymbol{Z}\boldsymbol{Z}^*\right]\right\|^{1/2},\ \left\|\mathbb{E}\left[\boldsymbol{Z}^*\boldsymbol{Z}\right]\right\|^{1/2}\right\} + C(d) \cdot \left(\mathbb{E}\max_i \|\boldsymbol{S}_i\|^2\right)^{1/2} \tag{4.3}$$

where $d := d_1 + d_2$.

The proof of Theorem 4.1 takes up the rest of this section. The presentation includes notes about the provenance of various parts of the argument.

The upper bound in Theorem I follows instantly from Case (3) of Theorem 4.1. We just introduce the notation $v(\boldsymbol{Z})$ for the variance parameter, and we calculate that

$$\mathbb{E}\left[\boldsymbol{Z}\boldsymbol{Z}^*\right] = \sum_{i,j=1}^{n} \mathbb{E}\left[\boldsymbol{S}_i\boldsymbol{S}_j^*\right] = \sum_{i=1}^{n} \mathbb{E}\left[\boldsymbol{S}_i\boldsymbol{S}_i^*\right].$$

The first expression follows immediately from the definition of $\boldsymbol{Z}$ and the linearity of the expectation; the second identity holds because the random matrices $\boldsymbol{S}_i$ are independent and have mean zero. The formula for $\mathbb{E}\left[\boldsymbol{Z}^*\boldsymbol{Z}\right]$ is valid for precisely the same reasons.

4.1 Proof of the Positive-Semidefinite Case

Recall that $\boldsymbol{W}$ is a random $d \times d$ positive-semidefinite matrix of the form

$$\boldsymbol{W} := \sum_{i=1}^{n} \boldsymbol{T}_i \quad \text{where the } \boldsymbol{T}_i \text{ are positive semidefinite.}$$

Let us introduce notation for the quantity of interest:

$$E := \mathbb{E}\,\|\boldsymbol{W}\| = \mathbb{E}\left\|\sum_{i=1}^{n} \boldsymbol{T}_i\right\|$$

By the triangle inequality for the spectral norm,

$$E \le \left\|\sum_{i=1}^{n} \mathbb{E}\,\boldsymbol{T}_i\right\| + \mathbb{E}\left\|\sum_{i=1}^{n} (\boldsymbol{T}_i - \mathbb{E}\,\boldsymbol{T}_i)\right\| \le \left\|\sum_{i=1}^{n} \mathbb{E}\,\boldsymbol{T}_i\right\| + 2\,\mathbb{E}\left\|\sum_{i=1}^{n} \varepsilon_i \boldsymbol{T}_i\right\|.$$

The second inequality follows from symmetrization, Fact 2.3. In this expression, $\{\varepsilon_i\}$ is an independent family of Rademacher random variables, independent of $\{\boldsymbol{T}_i\}$. Conditioning on the choice of the random matrices $\boldsymbol{T}_i$, we apply Theorem 3.1 via the bound (3.2):

$$\mathbb{E}\left\|\sum_{i=1}^{n} \varepsilon_i \boldsymbol{T}_i\right\| = \mathbb{E}\left[\mathbb{E}_{\boldsymbol{\varepsilon}}\left\|\sum_{i=1}^{n} \varepsilon_i \boldsymbol{T}_i\right\|\right] \le \sqrt{1 + 2\lceil \log d \rceil} \cdot \mathbb{E}\left[\left\|\sum_{i=1}^{n} \boldsymbol{T}_i^2\right\|^{1/2}\right].$$

The operator $\mathbb{E}_{\boldsymbol{\varepsilon}}$ averages over the Rademacher random variables, with the matrices $\boldsymbol{T}_i$ fixed. Now, since the matrices $\boldsymbol{T}_i$ are positive-semidefinite,

$$\begin{aligned}
\mathbb{E}\left[\left\|\sum_{i=1}^{n} \boldsymbol{T}_i^2\right\|^{1/2}\right] &\le \mathbb{E}\left[\big(\max_i \|\boldsymbol{T}_i\|\big)^{1/2} \cdot \left\|\sum_{i=1}^{n} \boldsymbol{T}_i\right\|^{1/2}\right] \\
&\le \big(\mathbb{E}\max_i \|\boldsymbol{T}_i\|\big)^{1/2} \cdot \left(\mathbb{E}\left\|\sum_{i=1}^{n} \boldsymbol{T}_i\right\|\right)^{1/2} \\
&= \big(\mathbb{E}\max_i \|\boldsymbol{T}_i\|\big)^{1/2} \cdot E^{1/2}.
\end{aligned}$$

The first inequality is Fact 2.1, and the second is Cauchy–Schwarz. Combine the last three displays to see that

$$E \le \left\|\sum_{i=1}^{n} \mathbb{E}\,\boldsymbol{T}_i\right\| + \sqrt{4(1 + 2\lceil \log d \rceil)} \cdot \big(\mathbb{E}\max_i \|\boldsymbol{T}_i\|\big)^{1/2} \cdot E^{1/2}. \tag{4.4}$$

For any $\alpha, \beta \ge 0$, the quadratic inequality $t^2 \le \alpha + \beta t$ implies that $t \le \sqrt{\alpha} + \beta$. Applying this fact to the quadratic relation (4.4) for $E^{1/2}$, we obtain

$$E^{1/2} \le \left\|\sum_{i=1}^{n} \mathbb{E}\,\boldsymbol{T}_i\right\|^{1/2} + \sqrt{4(1 + 2\lceil \log d \rceil)} \cdot \big(\mathbb{E}\max_i \|\boldsymbol{T}_i\|\big)^{1/2}.$$

The conclusion (4.1) follows.

This argument is adapted from Rudelson's paper [39], which develops a version of this result for the case where the matrices $\boldsymbol{T}_i$ have rank one; see also [40]. The paper [41] contains the first estimates for the constants. Magen and Zouzias [29] observed that similar considerations apply when the matrices $\boldsymbol{T}_i$ have higher rank. The complete result (4.1) first appeared in [7, Appendix]. The constants in this paper are marginally better. Related bounds for Schatten norms appear in [28, Sect. 7] and in [22].

The results described in the last paragraph are all matrix versions of the classical inequalities due to Rosenthal [38, Lemma 1]. These bounds can be interpreted as polynomial moment versions of the Chernoff inequality.

4.2 Proof of the Hermitian Case

The result (4.2) for Hermitian matrices is a corollary of Theorem 3.1 and the result (4.1) for positive-semidefinite matrices. Recall that $\boldsymbol{X}$ is a $d \times d$ random Hermitian matrix of the form

$$\boldsymbol{X} := \sum_{i=1}^{n} \boldsymbol{Y}_i \quad \text{where } \mathbb{E}\, \boldsymbol{Y}_i = \boldsymbol{0}.$$

We may calculate that

$$\begin{aligned}\left(\mathbb{E}\,\|\boldsymbol{X}\|^2\right)^{1/2} &= \left(\mathbb{E}\left\|\sum_{i=1}^{n} \boldsymbol{Y}_i\right\|^2\right)^{1/2} \\ &\leq 2\left(\mathbb{E}\left[\mathbb{E}_{\boldsymbol{\varepsilon}}\left\|\sum_{i=1}^{n} \varepsilon_i \boldsymbol{Y}_i\right\|^2\right]\right)^{1/2} \\ &\leq \sqrt{4(1+2\lceil \log d\rceil)} \cdot \left(\mathbb{E}\left\|\sum_{i=1}^{n} \boldsymbol{Y}_i^2\right\|\right)^{1/2}.\end{aligned}$$

The first inequality follows from the symmetrization procedure, Fact 2.3. The second inequality applies Theorem 3.1, conditional on the choice of $\boldsymbol{Y}_i$. The remaining expectation involves a sum of independent random matrices that are positive-semidefinite. Therefore, we may invoke (4.1) with $\boldsymbol{T}_i = \boldsymbol{Y}_i^2$. We obtain

$$\mathbb{E}\left\|\sum_{i=1}^{n} \boldsymbol{Y}_i^2\right\| \leq \left[\left\|\sum_{i=1}^{n} \mathbb{E}\, \boldsymbol{Y}_i^2\right\|^{1/2} + \sqrt{4(1+2\lceil \log d\rceil)} \cdot \left(\mathbb{E} \max_i \|\boldsymbol{Y}_i^2\|\right)^{1/2}\right]^2.$$

Combine the last two displays to reach

$$\left(\mathbb{E}\,\|\boldsymbol{X}\|^2\right)^{1/2} \le \sqrt{4(1+2\lceil\log d\rceil)} \cdot \left[\left\|\sum_{i=1}^n \mathbb{E}\,\boldsymbol{Y}_i^2\right\|^{1/2} \right.$$
$$\left. + \sqrt{4(1+2\lceil\log d\rceil)} \cdot \left(\mathbb{E}\max_i \|\boldsymbol{Y}_i\|^2\right)^{1/2}\right].$$

Rewrite this expression to reach (4.2).

A version of the result (4.2) first appeared in [7, Appendix]; the constants here are marginally better. Related results for the Schatten norm appear in the papers [20–22, 28]. These bounds are matrix extensions of the scalar inequalities due to Rosenthal [38, Theorem 3] and to Rosén [37, Theorem 1]; see also Nagaev–Pinelis [32, Theorem 2]. They can be interpreted as the polynomial moment inequalities that sharpen the Bernstein inequality.

4.3 *Proof of the Rectangular Case*

Finally, we establish the rectangular result (4.3). Recall that $\boldsymbol{Z}$ is a $d_1 \times d_2$ random rectangular matrix of the form

$$\boldsymbol{Z} := \sum_{i=1}^n \boldsymbol{S}_i \quad \text{where } \mathbb{E}\,\boldsymbol{S}_i = \boldsymbol{0}.$$

Set $d := d_1 + d_2$, and form a random $d \times d$ Hermitian matrix $\boldsymbol{X}$ by dilating $\boldsymbol{Z}$:

$$\boldsymbol{X} := \mathscr{H}(\boldsymbol{Z}) = \sum_{i=1}^n \mathscr{H}(\boldsymbol{S}_i).$$

The Hermitian dilation $\mathscr{H}$ is defined in (2.6); the second relation holds because the dilation is real-linear.

Evidently, the random matrix $\boldsymbol{X}$ is a sum of independent, centered, random Hermitian matrices $\mathscr{H}(\boldsymbol{S}_i)$. Therefore, we may apply (4.2) to $\boldsymbol{X}$ to see that

$$\left(\mathbb{E}\,\|\mathscr{H}(\boldsymbol{Z})\|^2\right)^{1/2} \le \sqrt{4(1+2\lceil\log d\rceil)} \cdot \left\|\mathbb{E}\left[\mathscr{H}(\boldsymbol{Z})^2\right]\right\|^{1/2}$$
$$+ 4(1+2\lceil\log d\rceil) \cdot \left(\mathbb{E}\max_i \|\mathscr{H}(\boldsymbol{S}_i)\|^2\right)^{1/2}. \qquad (4.5)$$

Since the dilation preserves norms (2.8), the left-hand side of (4.5) is exactly what we want:

$$\left(\mathbb{E}\,\|\mathscr{H}(\boldsymbol{Z})\|^2\right)^{1/2} = \left(\mathbb{E}\,\|\boldsymbol{Z}\|^2\right)^{1/2}.$$

To simplify the first term on the right-hand side of (4.5), invoke the formula (2.7) for the square of the dilation:

$$\begin{aligned}\left\|\mathbb{E}\left[\mathscr{H}(\boldsymbol{Z})^2\right]\right\| &= \left\|\begin{bmatrix}\mathbb{E}\left[\boldsymbol{Z}\boldsymbol{Z}^*\right] & \mathbf{0} \\ \mathbf{0} & \mathbb{E}\left[\boldsymbol{Z}^*\boldsymbol{Z}\right]\end{bmatrix}\right\| \\ &= \max\left\{\left\|\mathbb{E}\left[\boldsymbol{Z}\boldsymbol{Z}^*\right]\right\|,\ \left\|\mathbb{E}\left[\boldsymbol{Z}^*\boldsymbol{Z}\right]\right\|\right\}.\end{aligned} \tag{4.6}$$

The second identity relies on the fact that the norm of a block-diagonal matrix is the maximum norm of a diagonal block. To simplify the second term on the right-hand side of (4.5), we use (2.8) again:

$$\|\mathscr{H}(\boldsymbol{S}_i)\| = \|\boldsymbol{S}_i\|.$$

Introduce the last three displays into (4.5) to arrive at the result (4.3).

The result (4.3) first appeared in the monograph [44, Eq. (6.16)] with (possibly) incorrect constants. The current paper contains the first complete presentation of the bound.

5 Lower Bounds for the Expected Norm

Finally, let us demonstrate that each of the upper bounds in Theorem 4.1 is sharp up to the dimensional constant $C(d)$. The following result gives matching lower bounds in each of the three cases.

Theorem 5.1 (Expected Norm: Lower Bounds) *The expected spectral norm of a sum of independent random matrices satisfies the following lower bounds.*

1. ***The Positive-Semidefinite Case.*** *Consider a family* $\{\boldsymbol{T}_1, \ldots, \boldsymbol{T}_n\}$ *of independent, random* $d \times d$ *positive-semidefinite matrices, and define*

$$\boldsymbol{W} := \sum_{i=1}^{n} \boldsymbol{T}_i.$$

Then

$$\mathbb{E}\,\|\boldsymbol{W}\| \geq \frac{1}{4}\left[\|\mathbb{E}\,\boldsymbol{W}\|^{1/2} + \left(\mathbb{E}\max_i \|\boldsymbol{T}_i\|\right)^{1/2}\right]^2. \tag{5.1}$$

2. ***The Centered Hermitian Case.*** *Consider a family* $\{\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_n\}$ *of independent, random* $d \times d$ *Hermitian matrices with* $\mathbb{E}\, \boldsymbol{Y}_i = \boldsymbol{0}$ *for each index i, and define*

$$\boldsymbol{X} := \sum_{i=1}^{n} \boldsymbol{Y}_i.$$

Then

$$\left(\mathbb{E}\, \|\boldsymbol{X}\|^2\right)^{1/2} \geq \frac{1}{2} \left\|\mathbb{E}\, \boldsymbol{X}^2\right\|^{1/2} + \frac{1}{4} \left(\mathbb{E} \max_i \|\boldsymbol{Y}_i\|^2\right)^{1/2}. \tag{5.2}$$

3. ***The Centered Rectangular Case.*** *Consider a family* $\{\boldsymbol{S}_1, \ldots, \boldsymbol{S}_n\}$ *of independent, random* $d_1 \times d_2$ *matrices with* $\mathbb{E}\, \boldsymbol{S}_i = \boldsymbol{0}$ *for each index i, and define*

$$\boldsymbol{Z} := \sum_{i=1}^{n} \boldsymbol{S}_i.$$

Then

$$\begin{aligned} \mathbb{E}\, \|\boldsymbol{Z}\| \geq \frac{1}{2} \max \left\{ \left\|\mathbb{E}\left[\boldsymbol{Z}\boldsymbol{Z}^*\right]\right\|^{1/2}, \ \left\|\mathbb{E}\left[\boldsymbol{Z}^*\boldsymbol{Z}\right]\right\|^{1/2} \right\} \\ + \frac{1}{4} \left(\mathbb{E} \max_i \|\boldsymbol{S}_i\|^2\right)^{1/2}. \end{aligned} \tag{5.3}$$

The rest of the section describes the proof of Theorem 5.1.

The lower bound in Theorem I is an immediate consequence of Case (3) of Theorem 5.1. We simply introduce the notation $v(\boldsymbol{Z})$ for the variance parameter.

5.1 The Positive-Semidefinite Case

The lower bound (5.1) in the positive-semidefinite case is relatively easy. Recall that

$$\boldsymbol{W} := \sum_{i=1}^{n} \boldsymbol{T}_i \quad \text{where the } \boldsymbol{T}_i \text{ are positive semidefinite.}$$

First, by Jensen's inequality

$$\mathbb{E}\, \|\boldsymbol{W}\| \geq \|\mathbb{E}\, \boldsymbol{W}\|. \tag{5.4}$$

Second, let I be the minimum value of the index i where $\max_i \|\boldsymbol{T}_i\|$ is achieved; note that I is a random variable. Since the summands $\boldsymbol{T}_i$ are positive semidefinite, it is easy to see that

$$\boldsymbol{T}_I \preccurlyeq \sum_{i=1}^{n} \boldsymbol{T}_i.$$

Using Weyl's monotonicity principle [3, Corollary III.2.3], we have

$$\max_i \|\boldsymbol{T}_i\| = \|\boldsymbol{T}_I\| = \lambda_{\max}(\boldsymbol{T}_I) \le \lambda_{\max}\left(\sum_{i=1}^{n} \boldsymbol{T}_i\right) = \left\|\sum_{i=1}^{n} \boldsymbol{T}_i\right\| = \|\boldsymbol{W}\|.$$

Take the expectation to arrive at

$$\mathbb{E}\max_i \|\boldsymbol{T}_i\| \le \mathbb{E}\|\boldsymbol{W}\|. \tag{5.5}$$

Average the two bounds (5.4) and (5.5) to obtain

$$\mathbb{E}\|\boldsymbol{W}\| \ge \frac{1}{2}\big[\,\|\mathbb{E}\,\boldsymbol{W}\| + \mathbb{E}\max_i \|\boldsymbol{T}_i\|\,\big].$$

To reach (5.1), apply the numerical fact that $2(a+b) \ge \left(\sqrt{a}+\sqrt{b}\right)^2$, valid for all $a, b \ge 0$.

5.2 Hermitian Case

The Hermitian case (5.2) is similar in spirit, but the details are a little more involved. Recall that

$$\boldsymbol{X} := \sum_{i=1}^{n} \boldsymbol{Y}_i \quad \text{where } \mathbb{E}\,\boldsymbol{Y}_i = \boldsymbol{0}.$$

First, note that

$$\left(\mathbb{E}\|\boldsymbol{X}\|^2\right)^{1/2} = \left(\mathbb{E}\|\boldsymbol{X}^2\|\right)^{1/2} \ge \left\|\mathbb{E}\,\boldsymbol{X}^2\right\|^{1/2}. \tag{5.6}$$

The second relation is Jensen's inequality. To obtain the other part of our lower bound, we use the lower bound from the symmetrization result, Fact 2.3:

$$\mathbb{E}\|\boldsymbol{X}\|^2 = \mathbb{E}\left\|\sum_{i=1}^{n} \boldsymbol{Y}_i\right\|^2 \ge \frac{1}{4}\,\mathbb{E}\left\|\sum_{i=1}^{n} \varepsilon_i \boldsymbol{Y}_i\right\|^2$$

where $\{\varepsilon_i\}$ is an independent family of Rademacher random variables, independent from $\{\boldsymbol{Y}_i\}$. Now, we condition on the choice of $\{\boldsymbol{Y}_i\}$, and we compute the partial expectation with respect to the ε_i. Let I be the minimum value of the index i where $\max_i \|\boldsymbol{Y}_i\|^2$ is achieved. By Jensen's inequality, applied conditionally,

$$\mathbb{E}_{\boldsymbol{\varepsilon}} \left\| \sum_{i=1}^{n} \varepsilon_i \boldsymbol{Y}_i \right\|^2 \geq \mathbb{E}_{\varepsilon_I} \left\| \mathbb{E} \left[\sum_{i=1}^{n} \varepsilon_i \boldsymbol{Y}_i \,\middle|\, \varepsilon_I \right] \right\|^2 = \mathbb{E}_{\varepsilon_I} \|\varepsilon_I \boldsymbol{Y}_I\|^2 = \max_i \|\boldsymbol{Y}_i\|^2 .$$

Combining the last two displays and taking a square root, we discover that

$$\left(\mathbb{E} \|\boldsymbol{X}\|^2 \right)^{1/2} \geq \frac{1}{2} \left(\mathbb{E} \max_i \|\boldsymbol{Y}_i\|^2 \right)^{1/2}. \tag{5.7}$$

Average the two bounds (5.6) and (5.7) to conclude that (5.2) is valid.

5.3 The Rectangular Case

The rectangular case (5.3) follows instantly from the Hermitian case when we apply (5.2) to the Hermitian dilation. Recall that

$$\boldsymbol{Z} := \sum_{i=1}^{n} \boldsymbol{S}_i \quad \text{where } \mathbb{E}\, \boldsymbol{S}_i = \boldsymbol{0}.$$

Define a random matrix $\boldsymbol{X}$ by applying the Hermitian dilation (2.6) to $\boldsymbol{Z}$:

$$\boldsymbol{X} := \mathscr{H}(\boldsymbol{Z}) = \sum_{i=1}^{n} \mathscr{H}(\boldsymbol{S}_i).$$

Since the random matrix $\boldsymbol{X}$ is a sum of independent, centered, random Hermitian matrices, the bound (5.2) yields

$$\left(\mathbb{E} \|\mathscr{H}(\boldsymbol{Z})\|^2 \right)^{1/2} \geq \frac{1}{2} \left\| \mathbb{E} \left[\mathscr{H}(\boldsymbol{Z})^2 \right] \right\| + \frac{1}{4} \left(\mathbb{E} \max_i \|\mathscr{H}(\boldsymbol{S}_i)\|^2 \right)^{1/2}.$$

Repeating the calculations in Sect. 4.3, we arrive at the advertised result (5.3).

6 Optimality of Theorem I

The lower bounds and upper bounds in Theorem I match, except for the dimensional factor $C(\boldsymbol{d})$. In this section, we show by example that neither the lower bounds nor the upper bounds can be sharpened substantially. More precisely, the logarithms cannot appear in the lower bound, and they must appear in the upper bound. As a consequence, unless we make further assumptions, Theorem I cannot be improved except by constant factors and, in one place, by an iterated logarithm.

6.1 Upper Bound: Variance Term

First, let us show that the variance term in the upper bound in (1.4) must contain a logarithm. This example is drawn from [44, Sect. 6.1.2].

For a large parameter n, consider the $d \times d$ random matrix

$$\boldsymbol{Z} := \sum_{i=1}^{d} \sum_{j=1}^{n} \frac{1}{\sqrt{n}} \varepsilon_{ij} \mathbf{E}_{ii}$$

As before, $\{\varepsilon_{ij}\}$ is an independent family of Rademacher random variables, and $\mathbf{E}_{ii}$ is a $d \times d$ matrix with a one in the (i, i) position and zeroes elsewhere. The variance parameter satisfies

$$v(\boldsymbol{Z}) = \left\| \sum_{i=1}^{d} \sum_{j=1}^{n} \frac{1}{n} \mathbf{E}_{ii} \right\| = \|\mathbf{I}_d\| = 1.$$

The large deviation parameter satisfies

$$L^2 = \mathbb{E} \max_{i,j} \left\| \frac{1}{\sqrt{n}} \varepsilon_{ij} \mathbf{E}_{ii} \right\|^2 = \frac{1}{n}.$$

Therefore, the variance term drives the upper bound (1.4). For this example, it is easy to estimate the norm directly. Indeed,

$$\mathbb{E} \|\boldsymbol{Z}\|^2 \approx \mathbb{E} \left\| \sum_{i=1}^{d} \gamma_i \mathbf{E}_{ii} \right\|^2 = \mathbb{E} \max_{i=1,\dots,d} |\gamma_i|^2 \approx 2 \log d.$$

Here, $\{\gamma_i\}$ is an independent family of standard normal variables, and the first approximation follows from the central limit theorem. The norm of a diagonal matrix is the maximum absolute value of one of the diagonal entries. Last, we use the well-known fact that the expected maximum among d squared standard normal variables is asymptotic to $2 \log d$. In summary,

$$\left(\mathbb{E} \|\boldsymbol{Z}\|^2 \right)^{1/2} \approx \sqrt{2 \log d \cdot v(\boldsymbol{X})}.$$

We conclude that the variance term in the upper bound must carry a logarithm. Furthermore, it follows that Theorem 3.1 is numerically sharp.

6.2 Upper Bound: Large-Deviation Term

Next, we verify that the large-deviation term in the upper bound in (1.4) must also contain a logarithm, although the bound is slightly suboptimal. This example is drawn from [44, Sect. 6.1.2].

For a large parameter n, consider the $d \times d$ random matrix

$$\boldsymbol{Z} := \sum_{i=1}^{d} \sum_{j=1}^{n} \left(\delta_{ij} - n^{-1}\right) \cdot \mathbf{E}_{ii}$$

where $\{\delta_{ij}\}$ is an independent family of $\textsc{bernoulli}\left(n^{-1}\right)$ random variables. That is, δ_{ij} takes only the values zero and one, and its expectation is n^{-1}. The variance parameter for the random matrix is

$$v(\boldsymbol{Z}) = \left\| \sum_{i=1}^{d} \sum_{j=1}^{n} \mathbb{E}\left(\delta_{ij} - n^{-1}\right)^2 \cdot \mathbf{E}_{ii} \right\| = \left\| \sum_{i=1}^{d} \sum_{j=1}^{n} n^{-1}\left(1 - n^{-1}\right) \cdot \mathbf{E}_{ii} \right\| \approx 1.$$

The large deviation parameter is

$$L^2 = \mathbb{E} \max_{i,j} \left\| \left(\delta_{ij} - n^{-1}\right) \cdot \mathbf{E}_{ii} \right\|^2 \approx 1.$$

Therefore, the large-deviation term drives the upper bound in (1.4):

$$\left(\mathbb{E} \|\boldsymbol{Z}\|^2\right)^{1/2} \leq \sqrt{4(1 + 2\lceil \log d \rceil)} + 4(1 + 2\lceil \log d \rceil).$$

On the other hand, by direct calculation

$$\begin{aligned} \left(\mathbb{E} \|\boldsymbol{Z}\|^2\right)^{1/2} &\approx \left(\mathbb{E} \left\| \sum_{i=1}^{d} (Q_i - 1) \cdot \mathbf{E}_{ii} \right\|^2\right)^{1/2} \\ &= \left(\mathbb{E} \max_{i=1,\dots,d} |Q_i - 1|^2\right)^{1/2} \approx \text{const} \cdot \frac{\log d}{\log \log d}. \end{aligned}$$

Here, $\{Q_i\}$ is an independent family of $\textsc{poisson}(1)$ random variables, and the first approximation follows from the Poisson limit of a binomial. The second approximation depends on a (messy) calculation for the expected squared maximum of a family of independent Poisson variables. We see that the large deviation term in the upper bound (1.4) cannot be improved, except by an iterated logarithm factor.

6.3 Lower Bound: Variance Term

Next, we argue that there are examples where the variance term in the lower bound from (1.4) cannot have a logarithmic factor.

Consider a $d \times d$ random matrix of the form

$$\boldsymbol{Z} := \sum_{i,j=1}^{d} \varepsilon_{ij} \mathbf{E}_{ij}.$$

Here, $\{\varepsilon_{ij}\}$ is an independent family of Rademacher random variables. The variance parameter satisfies

$$\begin{aligned} v(\boldsymbol{Z}) &= \max \left\{ \left\| \sum_{i,j=1}^{d} \left(\mathbb{E}\, \varepsilon_{ij}^2 \right) \cdot \mathbf{E}_{ij} \mathbf{E}_{ij}^* \right\|, \left\| \sum_{i,j=1}^{d} \left(\mathbb{E}\, \varepsilon_{ij}^2 \right) \cdot \mathbf{E}_{ij}^* \mathbf{E}_{ij} \right\| \right\} \\ &= \max \left\{ \| d \cdot \mathbf{I}_d \|, \| d \cdot \mathbf{I}_d \| \right\} = d. \end{aligned}$$

The large-deviation parameter is

$$L^2 = \mathbb{E} \max_{i,j} \left\| \varepsilon_{ij} \mathbf{E}_{ij} \right\|^2 = 1.$$

Therefore, the variance term controls the lower bound in (1.4):

$$\left(\mathbb{E}\, \| \boldsymbol{Z} \|^2 \right)^{1/2} \geq \sqrt{cd} + c.$$

Meanwhile, it can be shown that the norm of the random matrix $\boldsymbol{Z}$ satisfies

$$\left(\mathbb{E}\, \| \boldsymbol{Z} \|^2 \right)^{1/2} \approx \sqrt{2d}.$$

See the paper [1] for an elegant proof of this nontrivial result. We see that the variance term in the lower bound in (1.4) cannot have a logarithmic factor.

6.4 Lower Bound: Large-Deviation Term

Finally, let us produce an example where the large-deviation term in the lower bound from (1.4) cannot have a logarithmic factor.

Consider a $d \times d$ random matrix of the form

$$Z := \sum_{i=1}^{d} P_i \mathbf{E}_{ii}.$$

Here, $\{P_i\}$ is an independent family of symmetric random variables whose tails satisfy

$$\mathbb{P}\{|P_i| \geq t\} = \begin{cases} t^{-4}, & t \geq 1 \\ 1, & t \leq 1. \end{cases}$$

The key properties of these variables are that

$$\mathbb{E}\, P_i^2 = 2 \quad \text{and} \quad \mathbb{E} \max_{i=1,\dots,d} P_i^2 \approx \text{const} \cdot d^2.$$

The second expression just describes the asymptotic order of the expected maximum. We quickly compute that the variance term satisfies

$$v(Z) = \left\| \sum_{i=1}^{d} \left(\mathbb{E}\, P_i^2 \right) \mathbf{E}_{ii} \right\| = 2.$$

Meanwhile, the large-deviation factor satisfies

$$L^2 = \mathbb{E} \max_{i=1,\dots,d} \|P_i \mathbf{E}_{ii}\|^2 = \mathbb{E} \max_{i=1,\dots,d} |P_i|^2 \approx \text{const} \cdot d^2.$$

Therefore, the large-deviation term drives the lower bound (1.4):

$$\left(\mathbb{E}\, \|Z\|^2 \right)^{1/2} \gtrsim \text{const} \cdot d.$$

On the other hand, by direct calculation,

$$\left(\mathbb{E}\, \|Z\|^2 \right)^{1/2} = \left(\mathbb{E} \left\| \sum_{i=1}^{d} P_i \mathbf{E}_{ii} \right\|^2 \right)^{1/2} = \left(\mathbb{E} \max_{i=1,\dots,d} |P_i|^2 \right)^{1/2} \approx \text{const} \cdot d.$$

We conclude that the large-deviation term in the lower bound (1.4) cannot carry a logarithmic factor.

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Fechner's Distribution and Connections to Skew Brownian Motion

Jon A. Wellner

Abstract This note investigates two aspects of Fechner's two-piece normal distribution: (1) connections with the mean-median-mode inequality and (strong) log-concavity; (2) connections with skew and oscillating Brownian motion processes. The developments here have been inspired by Wallis (Stat Sci 29:106–112, 2014) and rely on Chen and Zili (Sci China Math 58:97–108, 2015).

Keywords Fechner's law • Local time • Mean • Median • Mode • Oscillating Brownian motion • Pieced half normal • Quantiles • Skewed Brownian motion

Mathematics Subject Classification (2010). Primary 62E20; Secondary 62G20, 62D99, 62N01

1 Three Two-Piece Half-Normal Distributions

The standard Gaussian density ϕ and distribution function Φ are given by

$$\phi(z) = \frac{1}{\sqrt{2\pi}} \exp(-z^2/2), \quad z \in \mathbb{R},$$

and

$$\Phi(z) = \int_{-\infty}^{z} \phi(x)dx = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)dx, \quad z \in \mathbb{R}.$$

J.A. Wellner (✉)
Department of Statistics, University of Washington, Box 354322, Seattle, WA 98195-4322, USA
e-mail: jaw@stat.washington.edu

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Now let $\sigma_+, \sigma_- > 0$ be two positive numbers with $\sigma_+ \neq \sigma_-$ in general, and consider the following three densities on $\mathbb{R}$:

$$
\begin{aligned}
f(x;\sigma_+,\sigma_-) &\equiv \begin{cases} \frac{2\sigma_-}{\sigma_+ + \sigma_-} \cdot \frac{1}{\sigma_-}\phi(x/\sigma_-), & x < 0, \\ \frac{2\sigma_+}{\sigma_+ + \sigma_-} \cdot \frac{1}{\sigma_+}\phi(x/\sigma_+), & x \geq 0; \end{cases} \\
g(x;\sigma_+,\sigma_-) &\equiv \begin{cases} \frac{1}{\sigma_-}\phi(x/\sigma_-), & x < 0, \\ \frac{1}{\sigma_+}\phi(x/\sigma_+), & x \geq 0; \end{cases} \\
h(x;\sigma_+,\sigma_-) &\equiv \begin{cases} \frac{2\sigma_+}{\sigma_+ + \sigma_-} \cdot \frac{1}{\sigma_-}\phi(x/\sigma_-), & x < 0, \\ \frac{2\sigma_-}{\sigma_+ + \sigma_-} \cdot \frac{1}{\sigma_+}\phi(x/\sigma_+), & x \geq 0. \end{cases}
\end{aligned}
\tag{1.1}
$$

It is easily seen that f, g, and h differ only in the scaling of the two half normal densities $\phi(x/\sigma_\pm)/\sigma_\pm 1_{(0,\infty)}(x\text{sign}(x))$. Thus with $\theta \equiv \sigma_-/(\sigma_- + \sigma_+)$ we have

$$
\begin{aligned}
f(x;\sigma_+,\sigma_-) &\equiv \begin{cases} 2\theta \cdot \frac{1}{\sigma_-}\phi(x/\sigma_-), & x < 0, \\ 2(1-\theta) \cdot \frac{1}{\sigma_+}\phi(x/\sigma_+), & x \geq 0; \end{cases} \\
g(x;\sigma_+,\sigma_-) &\equiv \begin{cases} \frac{1}{\sigma_-}\phi(x/\sigma_-), & x < 0, \\ \frac{1}{\sigma_+}\phi(x/\sigma_+), & x \geq 0; \end{cases} \\
h(x;\sigma_+,\sigma_-) &\equiv \begin{cases} 2(1-\theta) \cdot \frac{1}{\sigma_-}\phi(x/\sigma_-), & x < 0, \\ 2\theta \cdot \frac{1}{\sigma_+}\phi(x/\sigma_+), & x \geq 0. \end{cases}
\end{aligned}
$$

The density f is continuous on $\mathbb{R}$, while the densities g and h are discontinuous at 0. The density f is associated with [8] and "Fechner's Lagegesetz der Mittlewerte"; see [20, 25]. (Also see [9, 21, 22], and [23, Chap. 7] for further historical information about Fechner.) As noted by Wallis [25], this density (and the version thereof with an additional shift parameter) has been rediscovered repeatedly. It is interesting to note that the density f is *log-concave* (see e.g. [7]) and even *strongly log-concave* (see e.g. [28]).

The density g is the limit distribution of the median of i.i.d random variables with density p when when p is discontinuous at its median m, and then $\sigma_\pm^2 = 1/(4p(m\pm)^2)$ where $p(m\pm)$ denote the left and right limits of p at m respectively; see e.g. [27, pp. 343–354], [14, 15].

The density h is the marginal density of *oscillating Brownian motion*, see e.g. [13, p. 302]. This process, which is closely related to *skew Brownian motion* (see e.g. [3, 10, 12, 16, 19]), arises as the weak limit of random walk processes which are inhomogeneous in space: imagine letting the increment distributions change as the walk crosses through 0 with variance σ_+^2 for $x \geq 0$ and variance σ_-^2 for $x < 0$. See [13] for a first theorem of this type and [11] for further convergence results in this direction.

One point of interest here is the connection with the mean–median–mode inequality going back to Fechner and Pearson.

Fechner proved that for the density f with $\sigma_- \geq \sigma_+$ the inequality

$$\text{mean} \leq \text{median} \leq \text{mode} \tag{1.2}$$

holds true, and that strict inequalities hold when $\sigma_- > \sigma_+$. Fechner did this by examining the ratio (Med − Mode)/(Mean − Mode) and considering the limits as $\sigma_+ \nearrow \sigma_-$ and as $\sigma_+ \searrow 0$ for fixed σ_+. In our notation this ratio becomes (see Table 1)

$$\begin{aligned}\frac{\text{Med} - \text{Mode}}{\text{Mean} - \text{Mode}} &= \frac{\sigma_- \Phi^{-1}\left(\frac{\sigma_+ + \sigma_-}{4\sigma_-}\right)}{\sqrt{2/\pi}(\sigma_+ - \sigma_-)} \\ &\to \begin{cases} \pi/4, & \text{as } \sigma_+ \nearrow \sigma_-, \\ \sqrt{\pi/4}\Phi^{-1}(3/4), & \text{as } \sigma_+ \to 0, \end{cases} \\ &= \left\{\begin{matrix} 0.785398\ldots, \\ 0.845348\ldots \end{matrix}\right\} < 1.\end{aligned}$$

Apparently the phenomena of the inequalities in (1.2) was observed (but not proved) by Pearson [17] in connection with his Type III curves.

The inequalities in (1.2) are illustrated in Fig. 1.

As a result of the series of papers [2, 6, 20, 24], and counterexamples (see e.g. [1]), this phenomena is now well-understood. In particular, from [6], for distributions F with median $m = 0$ (so that, with $X \sim F$, $P(X \leq m) \geq 1/2$ and $P(X \geq m) \geq 1/2$) and $\mu = E(X)$ assumed finite, if $X^+ = \max\{X, 0\}$ and

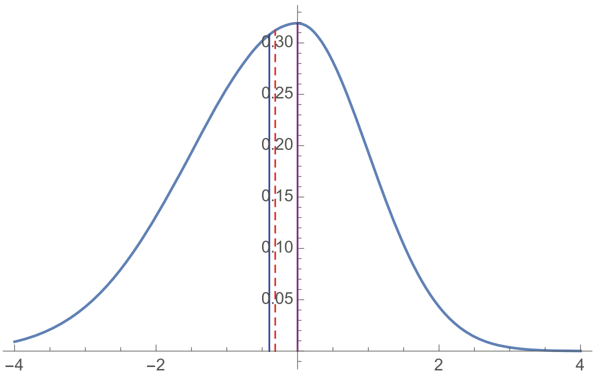

Fig. 1 Fechner's density $f(x; \sigma_-, \sigma_+)$ with $\sigma_- = 3/2$, $\sigma_+ = 1$; mean (*solid line*), median (*dashed line*)

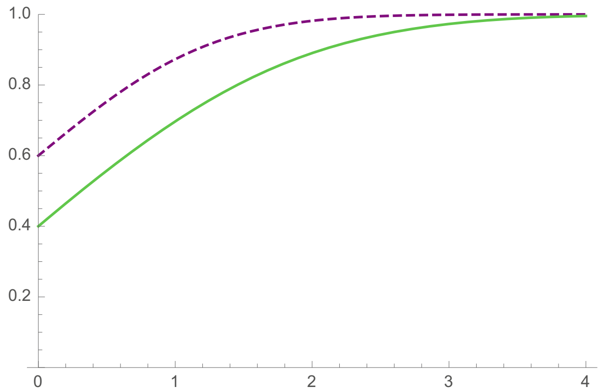

Fig. 2 Fechner stochastic order plot: F_+, *dashed curve*, F_-, *solid curve*; $\sigma_- = 3/2$, $\sigma_+ = 1$

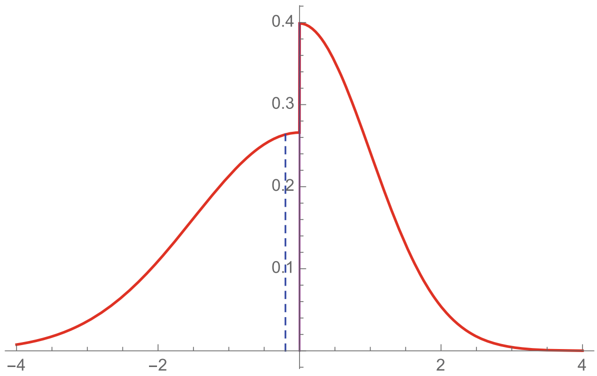

Fig. 3 Quantile limit density $g(x; \sigma_-, \sigma_+)$ with $\sigma_- = 3/2$, $\sigma_+ = 1$, mean at *dashed line*

$X^- \equiv -\min\{0, X\}$ satisfy $X^- >_s X^+$, then there is at least one mode M such that $\mu \leq 0 \leq M$. This is illustrated in Fig. 2.

Here we note that while the densities g and h also have mode at 0, the density g has median 0 and mean < 0 (when $\sigma_- > \sigma_+$), the density h has mean 0 and median > 0. Thus g gives an example of a density in which the equality median $=$ mode occurs, while h gives an example of a density for which the median fails to fall between the mean and mode, and thus, necessarily, X^- fails to be stochastically larger (or smaller) than X^+. These facts are illustrated in Figs. 3, 4, and 5, 6, respectively.

Finally, Fig. 7 gives a plot of all three of these densities together, all with $\sigma_- = 3/2$, $\sigma_+ = 1$.

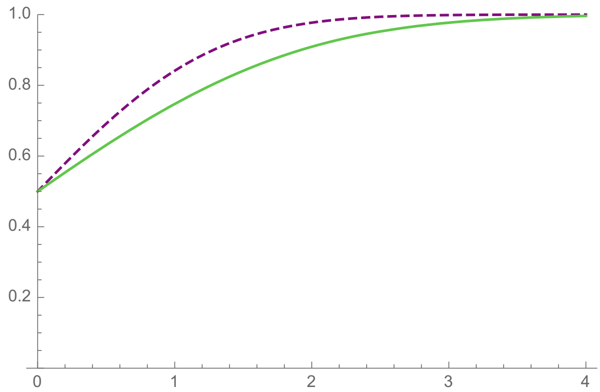

Fig. 4 Quantile stochastic order plot: G_+, *dashed curve*, G_- *solid curve*; $\sigma_- = 3/2$, $\sigma_+ = 1$

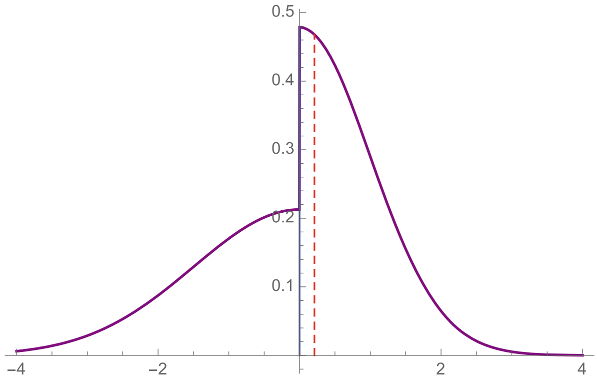

Fig. 5 Oscillating Brownian motion limit density $h(x; \sigma_-, \sigma_+)$ with $\sigma_- = 3/2$, $\sigma_+ = 1$, median at *dashed line*

2 Summary of the Properties of *f*, *g*, and *h*

Table 1 summarizes some of the properties of the densities f, g, and h. The formulas for the median are given only for the case that $\sigma_- > \sigma_+$.

In addition, the variances are given as follows:

$$Var_f(X) = \left(1 - \frac{2}{\pi}\right)(\sigma_+ - \sigma_-)^2 + \sigma_+\sigma_-,$$

$$Var_g(X) = \frac{1}{2}\left(1 - \frac{1}{\pi}\right)(\sigma_+ - \sigma_-)^2 + \sigma_+\sigma_-,$$

$$Var_h(X) = \sigma_+\sigma_-.$$

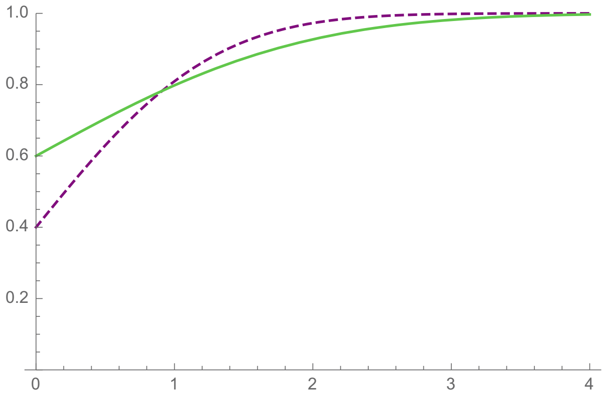

Fig. 6 Oscillating BM limit stochastic order plot: H_+, *dashed curve*, H_- *solid curve*; $\sigma_- = 3/2$, $\sigma_+ = 1$

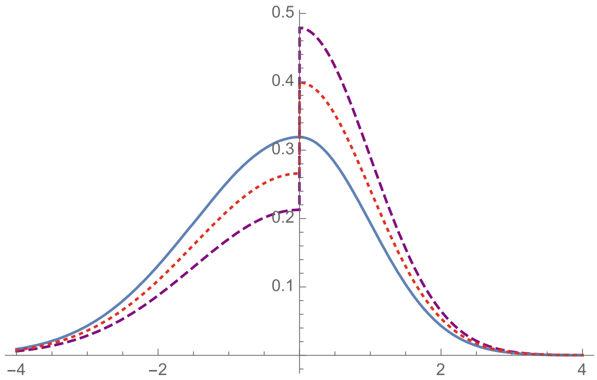

Fig. 7 The three densities f (*solid*), g (*dotted*), and h *dashed*; $\sigma_- = 3/2$, $\sigma_+ = 1$

Table 1 The mode, median, and mean of three (marginal) densities: Fechner, (nonstandard) quantile limit, and oscillating Brownian motion, as functions of σ_+ and σ_-

	Fechner	Quantile limit	Osc BM limit
Symbol	f	g	h
Mode	0	0	0
Median	$\sigma_- \Phi^{-1}\left(\frac{\sigma_+ + \sigma_-}{4\sigma_-}\right)$	0	$\sigma_+ \Phi^{-1}\left(1 - \left(\frac{4\sigma_-}{\sigma_+ + \sigma_-}\right)^{-1}\right)$
Mean	$\sqrt{\frac{2}{\pi}}(\sigma_+ - \sigma_-)$	$\frac{1}{\sqrt{2\pi}}(\sigma_+ - \sigma_-)$	0
$P(X > 0)$	$\frac{\sigma_+}{\sigma_+ + \sigma_-} = 1 - \theta$	$1/2$	$\frac{\sigma_-}{\sigma_+ + \sigma_-} = \theta$

3 Questions

We know that skew Brownian motion was studied by Walsh [26] because it provides an example of a diffusion process with discontinuous local time. We know that oscillating Brownian motion with $\sigma_+ \neq \sigma_-$ (or $q \neq p$ and $\alpha = 0$ in the notation of following sections) has both discontinuous marginal (which are scaled versions of the density h), and discontinuous local time. What are the properties of processes (if any) related to the densities f and g?

- Does Fechner's density f arise as the marginal density of a diffusion process in $\mathbb{R}$?
- Does the median zero density g arise as the marginal density of a diffusion process?
- What are the continuity properties of the marginal densities of the processes connected to the densities f and g?
- What are the continuity properties of the corresponding local time processes?

We will give answers to these questions in the next two sections.

4 A General Three-Parameter Mixture Family

Of course it is clear that f, g, and h as defined in Sect. 1 are special cases of the following mixture family: For $\theta \in [0, 1]$ and $\sigma_+, \sigma_- > 0$, let

$$q(x; \sigma_+, \sigma_-, \theta) = \theta \frac{2}{\sigma_-}\phi\left(\frac{x}{\sigma_-}\right) 1_{(-\infty,0)}(x) + (1-\theta)\frac{2}{\sigma_+}\phi\left(\frac{x}{\sigma_+}\right) 1_{[0,\infty)}(x).$$

Then

$$q(x; \sigma_+, \sigma_-, \theta) = \begin{cases} f(x; \sigma_+, \sigma_-), \text{ if } \theta = \theta_f \equiv \frac{\sigma_-}{\sigma_+ + \sigma_-}, \\ g(x; \sigma_+, \sigma_-), \text{ if } \theta = \theta_g \equiv 1/2, \\ h(x; \sigma_+, \sigma_-), \text{ if } \theta = \theta_h \equiv \frac{\sigma_+}{\sigma_+ + \sigma_-}. \end{cases}$$

For this three-parameter family, with $X \sim q$,

$$E_q X = \sqrt{\frac{2}{\pi}}\{(1-\theta)\sigma_+ - \theta\sigma_-\},$$

$$\text{median}(X) = \begin{cases} \sigma_- \Phi^{-1}\left(\frac{1}{4\theta}\right), & \text{if } \theta \geq 1/2, \\ \sigma_+ \Phi^{-1}\left(1 - \frac{1}{4(1-\theta)}\right), & \text{if } \theta < 1/2, \end{cases}$$

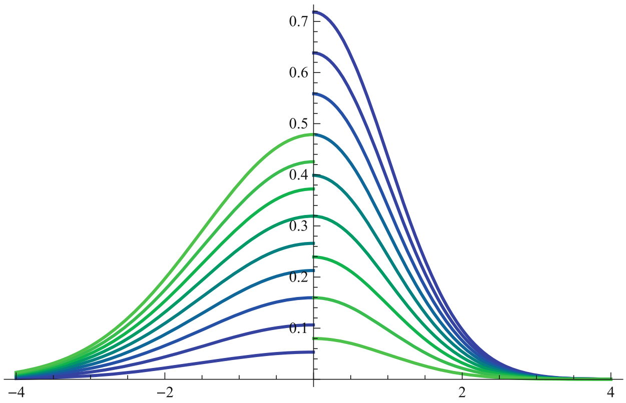

Fig. 8 The densities $q(\cdot; 3/2, 1, \theta)$ for $\theta \in \{\{.1, .2, \ldots, .9\}$

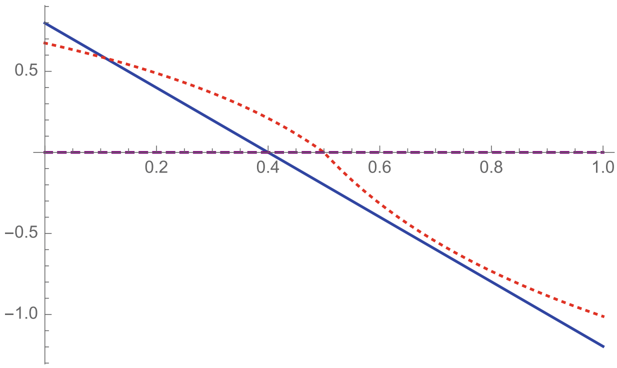

Fig. 9 Mean (*solid*), median (*dotted*), and mode (*dashed*) of the densities $q(\cdot; 3/2, 1, \theta)$ for $\theta \in (0, 1)$

$$Var_q(X) = (1-\theta)\sigma_+^2 + \theta\sigma_-^2 - ((1-\theta)\sigma_+ - \theta\sigma_-)^2 \, \frac{2}{\pi},$$
$$P_q(X > 0) = 1 - \theta.$$

Figure 8 shows the densities $q(\cdot; 3/2, 1, \theta)$ with $\theta \in \{.1, .2, \ldots, .9\}$

In Fig. 9 we see that the mean median and mode follow the inequality (1.2) for $\theta \geq 1/2$, and the reverse inequalities

$$\text{mode} \leq \text{median} \leq \text{mean} \tag{4.1}$$

for $\theta \leq .108389\ldots$, but that such inequalities fail for $\theta \in (.108389, .5)$.

5 Skew and Oscillating Brownian Motion Connections

How do these various densities connect with processes? From [19, Exercise 1.16, p. 82], we see that $q(\cdot; t, t, \theta)$ is the marginal density of *skew Brownian motion with parameter* $1-\theta$ at time t starting from 0 at $t = 0$. This process is denoted by $X_t^{1-\theta}$ in [19]. Moreover, from [19, Exercise 2.24, p. 401], $X_t^{1-\theta} = r_{1-\theta}(Y_t^{1-\theta})$ where $r_{1-\theta}(x) = (x/\theta)1_{[0,\infty)}(x)+(x/(1-\theta))1_{(-\infty,0)}(x)$. Equivalently, $Y_t^{1-\theta} = s_{1-\theta}(X_t^{1-\theta})$ where

$$s_{1-\theta}(x) = \theta x 1_{[0,\infty)}(x) + (1-\theta)x 1_{(-\infty,0)}(x).$$

Thus $Y_t^{1-\theta}$ has marginal density $h(\cdot/t, \theta, 1-\theta) = q(\cdot/t; \theta, 1-\theta, \theta)$, and it becomes clear that $Y_t^{1-\theta}$ is *oscillating Brownian motion* with $\sigma_+ = \theta, \sigma_- = 1-\theta$.

Now consider $Z_t^{1-\theta} \equiv v_\theta(X_t^{1-\theta})$ where

$$v_\theta(x) = (1-\theta)x 1_{[0,\infty)}(x) + \theta x 1_{(-\infty,0)}(x).$$

Then $Z_t^{1-\theta}$ has marginal density $f(\cdot/t, 1-\theta, \theta) = q(\cdot/t; 1-\theta, \theta, \theta)$. This is Fechner's density, and hence we call the process $Z_t^{1-\theta}$ the *Fechner process.*

6 More on the Fechner Process

Chen and Zili [5] study the following stochastic differential equation:

$$\begin{cases} dY_t^x = \left(p1_{\{Y_t^x \le 0\}} + q1_{\{0<Y_t^x \le a\}} + r1_{\{a<Y_t^x\}}\right) dB_t + \frac{\alpha}{2} dL_t^0(Y^x) + \frac{\beta}{2} dL_t^a(Y^x), \\ Y_0 = x \in \mathbb{R}, \end{cases}$$

where $\alpha, \beta \in (-\infty, 1)$, B is a one-dimensional standard Brownian motion, and for $w \in \mathbb{R}$, $L_t^w(Y^x)$ is the semimartingale local time for Y^x at level w; that is,

$$L_t^w(Y^x) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^t 1_{[w,w+\epsilon]}(Y_s^x) d\langle Y^x \rangle_s.$$

Here $\langle Y^x \rangle$ denotes the predictable quadratic variation process of Y. They note that in the special case $p = q = r = 1$ and $\beta = 0$, Y_t^x is a *skew Brownian motion* with skew parameter $1/(2-\alpha)$; and in the special case when $p = q = r = 1$ the process Y_t^x is a *double-skewed Brownian motion.* Another special case of interest is $p \ne q = r$, $\alpha = 0$, and $\beta = 0$, which corresponds to *oscillating Brownian motion* in the terminology of [13]. In the special case of $r = q$ and $\beta = 0$, Y_t^x is a *skewed oscillating Brownian motion process*, to use a combination of the terminology of [5, 13]. For further developments and applications of processes defined by the stochastic differential equation in the last display, see [18].

We are interested in a particular member of this class of processes, namely the *Fechner process* having continuous marginal densities.

Chen and Zili [5] show that the resulting SDE in this latter case, namely

$$\begin{cases} dX_t^x = \left(p 1_{\{Y_t^x \le 0\}} + q 1_{\{0 < Y_t^x\}}\right) dB_t + \frac{\alpha}{2} dL_t^0(X^x)), \\ X_0^x = x \in \mathbb{R}, \end{cases} \tag{6.1}$$

has a unique strong solution, and that moreover the transition density of the diffusion X^x is given by

$$\begin{aligned} p_t^X(x,y) = &\frac{1}{\sqrt{2\pi t}} \left(\frac{1_{\{y \le 0\}}}{p} + \frac{1_{\{y>0\}}}{q} \right) \times \left\{ \exp\left(-\frac{(f(x) - f(y))^2}{2t} \right) \right. \\ &\left. + \frac{p + q(\alpha - 1)}{p - q(\alpha - 1)} \text{sign}(y) \exp\left(-\frac{(|f(x)| + |f(y)|)^2}{2t} \right) \right\} \end{aligned} \tag{6.2}$$

where $f(y) \equiv (y/p)1_{[y \le 0]} + (y/q)1_{[y>0]}$. This implies that the transition density $p_t^X(0,y)$ is given by

$$\begin{aligned} p_t^X(0,y) &= \frac{1}{\sqrt{2\pi t}} \left(\frac{1_{\{y \le 0\}}}{p} + \frac{1_{\{y>0\}}}{q} \right) \times \left\{ \exp\left(-\frac{f(y)^2}{2t} \right) \right. \\ &\quad \left. + \frac{p + q(\alpha - 1)}{p - q(\alpha - 1)} \text{sign}(y) \exp\left(-\frac{f(y)^2}{2t} \right) \right\} \\ &= \frac{1}{\sqrt{2\pi t}} \left(\frac{1_{\{y \le 0\}}}{p} + \frac{1_{\{y>0\}}}{q} \right) \times \left\{ 1 + \frac{p + q(\alpha - 1)}{p - q(\alpha - 1)} \text{sign}(y) \right\} \exp\left(-\frac{f(y)^2}{2t} \right) \\ &= \begin{cases} \frac{1}{\sqrt{2\pi t}} \cdot \frac{1}{p} \left(1 - \frac{p+q(\alpha-1)}{p-q(\alpha-1)} \right) \cdot \exp\left(-\frac{f(y)^2}{2t} \right), \ y \le 0, \\ \frac{1}{\sqrt{2\pi t}} \cdot \frac{1}{q} \left(1 + \frac{p+q(\alpha-1)}{p-q(\alpha-1)} \right) \cdot \exp\left(-\frac{f(y)^2}{2t} \right), \ y > 0. \end{cases} \end{aligned} \tag{6.3}$$

This family of marginal densities for the process $X_t^0 \equiv X_t$ is continuous at 0 if

$$\frac{1}{p} \left(1 - \frac{p + q(\alpha - 1)}{p - q(\alpha - 1)} \right) = \frac{1}{q} \left(1 + \frac{p + q(\alpha - 1)}{p - q(\alpha - 1)} \right).$$

and this is easily seen to hold if and only if

$$1 - \alpha = \frac{p^2}{q^2}, \qquad \text{or if} \qquad \alpha = 1 - \frac{p^2}{q^2} \in (-\infty, 1). \tag{6.4}$$

Then

$$p_t^X(0,y) = \begin{cases} \frac{1}{\sqrt{2\pi t}} \cdot \frac{2}{p+q} \cdot \exp\left(-\frac{y^2}{2p^2 t}\right), \ y \le 0, \\ \frac{1}{\sqrt{2\pi t}} \cdot \frac{2}{p+q} \cdot \exp\left(-\frac{y^2}{2q^2 t}\right), \ y > 0. \end{cases}$$
$$= f(y/\sqrt{t}; p, q)/\sqrt{t}$$

where $f(\cdot;\cdot,\cdot)$ is Fechner's density as given in (1.1). Again, note that f is a continuous function of its first (and all) arguments. Furthermore, the transition density $p_t^X(x,y)$ is now given by

$$p_t^X(x,y) = \begin{cases} \frac{1}{\sqrt{2\pi t}} \frac{1}{q} \left\{ \exp\left(-\frac{(x-y)^2}{2q^2 t}\right) + \frac{1-p/q}{1+p/q} \exp\left(-\frac{(x+y)^2}{2q^2 t}\right) \right\}, & x > 0, \ y > 0, \\ \frac{1}{\sqrt{2\pi t}} \frac{1}{p} \left\{ \exp\left(-\frac{(x-y)^2}{2p^2 t}\right) - \frac{1-p/q}{1+p/q} \exp\left(-\frac{(|x|+|y|)^2}{2p^2 t}\right) \right\}, & x \le 0, \ y \le 0, \\ \frac{1}{\sqrt{2\pi t}} \frac{1}{q} \left\{ \exp\left(-\frac{(\frac{x}{p}-\frac{y}{q})^2}{2t}\right) + \frac{1-p/q}{1+p/q} \exp\left(-\frac{(\frac{|x|}{p}+\frac{|y|}{q})^2}{2t}\right) \right\}, & x \le 0, \ y > 0, \\ \frac{1}{\sqrt{2\pi t}} \frac{1}{p} \left\{ \exp\left(-\frac{(\frac{x}{q}-\frac{y}{p})^2}{2t}\right) - \frac{1-p/q}{1+p/q} \exp\left(-\frac{(\frac{|x|}{q}+\frac{|y|}{p})^2}{2t}\right) \right\}, & x > 0, \ y \le 0, \end{cases}$$

which is jointly continuous as a function of (x,y). See Fig. 10. In general the transition densities of skewed oscillating Brownian motion given in (6.2) are discontinuous; see Fig. 11.

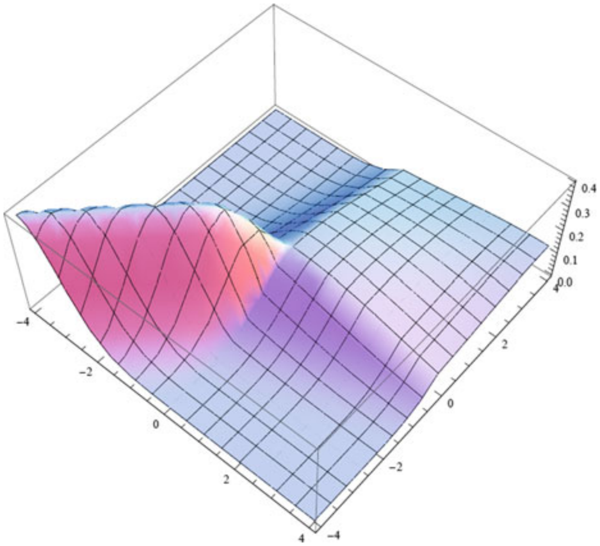

Fig. 10 Fechner process transition density $p_1^X(x,y)$ with $p = 1$ and $q = 3$

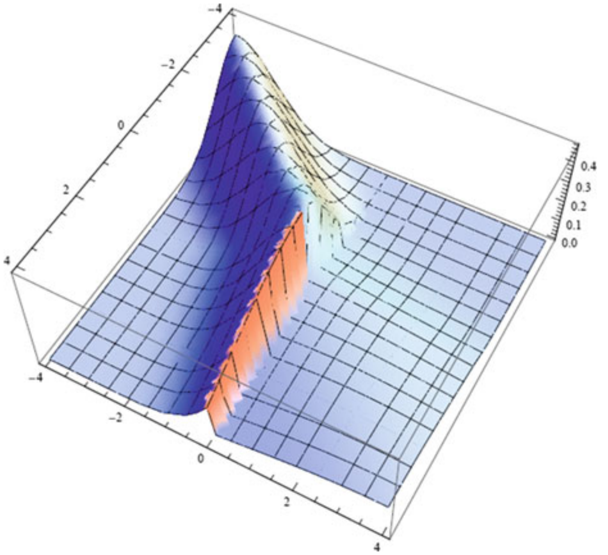

Fig. 11 Skewed oscillating Brownian motion process transition density $p_1^X(x, y)$ with $p = 1$, $q = 3$, and $\alpha = 1/2$

Question: With α related to p and q as in (6.4), does the process X_t^x have a jointly continuous local time process $L_t^w(X^x)$? (In particular is it continuous in w?)

The answer is *no* as shown by Chen [4]. Moreover, Chen [4] shows that the local time process $L_t^w(X^x)$ is jointly continuous only when $\alpha = 1 - p/q$.

Here is the proof of the two assertions from [4]. Define

$$f(y) = \begin{cases} y/p, \text{ for } y \leq 0, \\ y/q, \text{ for } y > 0. \end{cases}$$

By Chen and Zili [5, Eq. (2.9)]

$$L_t^0(X^x) = \frac{2}{2-\alpha} \widehat{L}_t^0(X^x), \tag{6.5}$$

where $\widehat{L}_t^0(X^x)$ is the symmetric local time of X^x at 0. From the proof of [5], Corollary 2.3, we see that $Z^{f(x)} \equiv f(X^x)$ is a skew driven Brownian motion driven by B starting from $f(x)$:

$$dZ_t^{f(x)} = dB_t + \frac{1}{2}\left(\frac{q(\alpha-1)}{p} + 1\right) dL_t^0(Z^{f(x)}).$$

By use of (6.5) we can rewrite the last display in term by symmetric semimartingale local time:

$$dZ_t^{f(x)} = dB_t + \frac{1}{2}\left(\frac{p+q(\alpha-1)}{p-q(\alpha-1)}\right) d\widehat{L}_t^0(Z^{f(x)}).$$

By the same computation as for (2.5) of [5], it follows that $L_t^0(X^x) = qL_t^0(Z^{f(x)})$, and hence that

$$\widehat{L}_t^0(X^x) = \frac{(2-\alpha)q}{p+q(1-\alpha)}\widehat{L}_t^0(Z^{f(x)}). \tag{6.6}$$

Since Z is a skew Brownian motion, it follows from [3, Theorem 1.2], that unless $p+q(\alpha-1)=0$ (i.e. unless $\alpha = 1-(p/q)$), the process

$$w \mapsto w + \frac{p+q(\alpha-1)}{p-q(\alpha-1)}\widehat{L}_T^0(Z^w)$$

is a discontinuous homogeneous Markov process, where $T = \inf\{t > 0 : \widehat{L}_t^0(Z^0) = 1\}$. Thus, unless $\alpha = 1-(p/q)$, by (6.6) we have $x \mapsto \widehat{L}_T^0(X^x)$ is discontinuous, and so in view of (6.5), $x \mapsto L_T^0(X^x)$ is discontinuous. For the Fechner process, $L_t^0(X^x)$ cannot be jointly continuous in (t, x), nor is it continuous in x.

When $\alpha = 1-p/q$ we see that the factors

$$\left(1 \pm \frac{p+q(\alpha-1)}{p-q(\alpha-1)}\right) = 1,$$

and hence the marginal density $p_t^X(0, y)$ in (6.3) reduces the form of g given in (1.1).

Summarizing the discussion above leads to the following proposition:

Proposition *Let $X_t^x \equiv X_t^x(p, q, \alpha)$ denote the (strong) solution of the stochastic differential equation (6.1).*

(a) For $\alpha = 1-(p/q)^2$, X_t^x has continuous transition densities and marginal densities for $x = 0$ which are scaled versions of the Fechner density f given in (1.1). On the other hand, the local time process $L_t^x(X^x)$ is discontinuous (at $x = 0$).

(b) For $\alpha = 1-p/q$, X_t^x has discontinuous transition densities and marginal densities for $x = 0$ which are scaled versions of the median zero density g given in (1.1). On the other hand, the local time process $L_t^x(X^x)$ is continuous.

Acknowledgements I owe thanks to Zheng-Qing Chen for sharing Chen and Zili [5] and Chen [4] with me. I also owe thanks to Kenneth Wallis for keeping me straight on Fechner's arguments concerning the mean-median-mode inequality (2) and the relative contributions of Fechner and Pearson.

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Part II
Limit Theorems

Erdős-Rényi-Type Functional Limit Laws for Renewal Processes

Paul Deheuvels and Joseph G. Steinebach

Abstract We prove functional limit laws for Erdős-Rényi-type increments of renewal processes.

Keywords First passage time process • Functional Erdős–Rényi law • Large deviations • Renewal process

Mathematics Subject Classification (2010). Primary 60F15, 60F17; Secondary 60F10, 60K05

1 Introduction

1.1 A Functional Limit Law

Let $\{X_n : n \geq 1\}$ be independent and identically distributed (iid) random variables (rv's). Set $S_0 := 0$, $S_n := X_1 + \ldots + X_n$ for $n \geq 1$, and $X := X_1$. Denote by $F(x) := \mathbb{P}(X \leq x)$ for $x \in \mathbb{R}$, the distribution function (df) of X, and let $\psi(t) := \mathbb{E}\left(e^{tX}\right) = \int_{\mathbb{R}} e^{tx} \mathrm{d}F(x)$ for $t \in \mathbb{R}$, denote its moment generating function (mgf). We shall assume that $(A.1\text{-}2\text{-}3)$ below are fulfilled:

$(A.1)$ $\quad \mu := \mathbb{E}(X) \in (0, \infty)$;
$(A.2)$ $\quad \mathbb{P}(X = \mu) < 1$;
$(A.3)$ $\quad -\infty \leq t_1 := \inf\{t : \psi(t) < \infty\} < 0 < t_0 := \sup\{t : \psi(t) < \infty\} \leq \infty$.

P. Deheuvels (✉)
LSTA, UPMC, t15-25, E2, 4 place Jussieu, F-75252 Paris Cedex 05, France
e-mail: paul.deheuvels@upmc.fr

J.G. Steinebach
Mathematical Institute, University of Cologne, Weyertal 86-90, 50931 Köln, Germany
e-mail: jost@math.uni-koeln.de

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At times, we shall need the additional condition $(A.4)$ or $(A.5)$.

$(A.4)$ $\mathbb{P}(X < 0) = 0$.
$(A.5)$ $\mathbb{P}(X \leq 0) = 0$.

We note that $(A.5) \Rightarrow (A.4) \Rightarrow t_1 = -\infty$. Define the *renewal process* pertaining to $\{S_n : n \geq 0\}$ by

$$N(t) := \max\{n \geq 0 : S_n \leq t\} \quad \text{for} \quad t \geq 0. \tag{1.1}$$

By $(A.1)$ and the law of large numbers, $n^{-1}S_n \to \mu > 0$ a.s. as $n \to \infty$. This implies that, a.s., $N(t) < \infty$ for all $t \geq 0$, with $t^{-1}N(t) \to 1/\mu$ as $t \to \infty$.
For each $T \in \mathbb{R}$, set $\log_+ T := \log(T \vee e)$. Fix a constant $0 < C < \infty$, and for each $x \geq 0$ consider the increment function of $t \in [0, C]$, defined by

$$\eta_{x;T}(t) := (\log_+ T)^{-1} \left(N(x + t \log_+ T) - N(x)\right) \quad \text{for} \quad t \in [0, C]. \tag{1.2}$$

We shall establish an Erdős-Rényi-type *functional limit theorem* (FLT), describing the limiting behavior as $T \to \infty$ of the random set

$$\mathcal{G}_{T;C} := \{\eta_{x;T}(\cdot) : 0 \leq x \leq T\}. \tag{1.3}$$

Our main result is stated in Theorem 1.2 at the end of Sect. 1.1. In Sects. 2 and 3, we present the proof of this theorem, together with auxiliary results on mgf's and Legendre transforms.
The following notation will be needed. Set $I := \{t : \psi(t) < \infty\}$, and define the *Legendre-Chernoff function* pertaining to $\psi(\cdot)$ by

$$\Psi(\alpha) := \sup_{t \in I}\{t\alpha - \log \psi(t)\} \quad \text{for} \quad \alpha \in \mathbb{R}. \tag{1.4}$$

Introduce the *Legendre conjugate function* $\Psi^*(\cdot)$ of $\Psi(\cdot)$, defined by

$$\Psi^*(\alpha) := \begin{cases} \alpha\Psi(1/\alpha) & \text{for} \quad \alpha > 0, \\ t_0 & \text{for} \quad \alpha = 0, \\ \infty & \text{for} \quad \alpha < 0. \end{cases} \tag{1.5}$$

We postpone until Sects. 1.2 and 2 a discussion of the properties of ψ, Ψ and Ψ^*.
We denote by $\mathcal{M}^+$ the set of nonnegative Radon measures on $\mathbb{R}^+ := [0, \infty)$, and set $\mathcal{M}^- = \{-\nu : \nu \in \mathcal{M}^+\}$. By Lebesgue-Stieltjes integration (see, e.g., Theorem 2, p. 163 in Chow and Teicher [5]), each $\nu \in \mathcal{M}^\pm$ is fully characterized by the values on a dense subset of $\mathbb{R}^+$ of its right-continuous df

$$H_\nu(t) := \int_{(-\infty,t]} \mathrm{d}\nu = \nu\left([0, t]\right) \quad \text{for} \quad t \in \mathbb{R}. \tag{1.6}$$

For $\nu \in \mathcal{M}^+$ (resp. $\nu \in \mathcal{M}^-$) H_ν belongs to the class $\mathcal{I}$ (resp. $\mathcal{D}$) of nondecreasing (resp. nonincreasing) right-continuous functions H of $t \in \mathbb{R}$ onto $\mathbb{R}^+$ (resp. $-\mathbb{R}^+$), such that $H(t) = 0$ for $t < 0$. Conversely, an arbitrary $H \in \mathcal{I}$ (resp. $H \in \mathcal{D}$) defines a unique Lebesgue-Stieltjes measure $\nu = \mathrm{d}H \in \mathcal{M}^+$ (resp. $\nu = \mathrm{d}H \in \mathcal{M}^-$) such that $H = H_\nu$. Since the map $\nu \in \mathcal{M}^+ \leftrightarrow H_\nu \in \mathcal{I}$ (resp. $\nu \in \mathcal{M}^- \leftrightarrow H_\nu \in \mathcal{D}$) is one-to-one, we shall identify $\mathcal{M}^+$ with $\mathcal{I}$ (resp. $\mathcal{M}^-$ with $\mathcal{D}$), and write the forthcoming statements in terms of either one of these sets, on the basis of convenience. For each $C > 0$, we denote by $\mathcal{M}_C^+$ the subset of $\mathcal{M}^+$ (resp. $\mathcal{M}_C^-$ the subset of $\mathcal{M}^-$), composed of all nonnegative (resp. nonpositive) Radon measures ν on $\mathbb{R}_+$, *supported* by $[0, C]$ (i.e., such that $\nu\,(\mathbb{R} - [0, C]) = 0$), and set $\mathcal{I}_C$ (resp. $\mathcal{D}_C$) for the corresponding set of df's. We endow $\mathcal{M}_C^+ \leftrightarrow \mathcal{I}_C$ (resp. $\mathcal{M}_C^- \leftrightarrow \mathcal{D}_C$), with the metric topology $\mathcal{W}$, of *weak* (or *weak**) *convergence* of nonnegative (resp. nonpositive) bounded measures. We note that any $\nu \in \mathcal{M}_C^\pm$ is bounded, since $|\nu(A)| \leq |\nu([0, C])|$ for each measurable $A \subseteq \mathbb{R}$. For $\{\nu_n : n \geq 1\} \subseteq \mathcal{M}_C^\pm$ and $\nu \in \mathcal{M}_C^\pm$, the convergence $\nu_n \to_{\mathcal{W}} \nu$ is characterized by the properties, as $n \to \infty$,

$$\int_{\mathbb{R}} \phi(t)\mathrm{d}H_{\nu_n}(t) \to \int_{\mathbb{R}} \phi(t)\mathrm{d}H_\nu(t) \text{ for each } \phi \in \mathcal{C}_B \text{ on } \mathbb{R};$$
$$\Leftrightarrow H_{\nu_n}(t) \to H_\nu(t) \text{ at each continuity point } t \in \mathbb{R} \text{ of the limit } H_\nu,$$

where $\mathcal{C}_B$ denotes the set of bounded continuous functions on $\mathbb{R}$. The weak topology $\mathcal{W}$ on $\mathcal{M}_C^+ \leftrightarrow \mathcal{I}_C$ (resp. $\mathcal{M}_C^- \leftrightarrow \mathcal{D}_C$) is metricised by the *Lévy metric* $\Delta_{\mathcal{L}}$, defined as follows. For each $H_{\nu_1}, H_{\nu_2} \in \mathcal{I}_C$, we set

$$\begin{aligned}&\Delta_{\mathcal{L}}(H_{\nu_1}, H_{\nu_2}) \\ &\quad := \inf\{\epsilon > 0 : H_{\nu_1}(t-\epsilon) - \epsilon \leq H_{\nu_2}(t) \leq H_{\nu_1}(t+\epsilon) + \epsilon,\ \forall t\}\,.\end{aligned} \tag{1.7}$$

We extend the definition of $\Delta_{\mathcal{L}}$ to $H_{\nu_1}, H_{\nu_2} \in \mathcal{D}_C$, by setting, via (1.7),

$$\Delta_{\mathcal{L}}(H_{\nu_1}, H_{\nu_2}) := \Delta_{\mathcal{L}}(-H_{\nu_1}, -H_{\nu_2}). \tag{1.8}$$

At times, we shall endow $\mathcal{M}_C^+ \leftrightarrow \mathcal{I}_C$ (resp. $\mathcal{M}_C^- \leftrightarrow \mathcal{D}_C$), with the *uniform topology* $\mathcal{U}$, induced by the distance $\Delta_{\mathcal{U}}$, defined, for each $\nu_1, \nu_2 \in \mathcal{M}_C^\pm$, by

$$\Delta_{\mathcal{U}}(H_{\nu_1}, H_{\nu_2}) = \|H_{\nu_1} - H_{\nu_2}\| := \sup_{t\in\mathbb{R}} |H_{\nu_1}(t) - H_{\nu_2}(t)|. \tag{1.9}$$

To define the Hausdorff set-metrics pertaining to $\Delta_{\mathcal{L}}$ and $\Delta_{\mathcal{U}}$, we set, for each $H \in \mathcal{I}_C$ and $\epsilon > 0$,

$$\mathcal{N}_{\epsilon;\mathcal{W}}(H) := \{G \in \mathcal{I}_C : \Delta_{\mathcal{L}}(G, H) < \epsilon\} \tag{1.10}$$
$$\mathcal{N}_{\epsilon;\mathcal{U}}(H) := \{G \in \mathcal{I}_C : \|G - H\| < \epsilon\}\,, \tag{1.11}$$

and, for each $A \subseteq \mathcal{I}_C$,

$$A^{\epsilon;\mathcal{W}} := \bigcup_{H \in A} \mathcal{N}_{\epsilon;\mathcal{W}}(H) \quad \text{and} \quad A^{\epsilon;\mathcal{U}} := \bigcup_{H \in A} \mathcal{N}_{\epsilon;\mathcal{U}}(H), \tag{1.12}$$

with the convention that $\bigcup_{\emptyset}(\cdot) := \emptyset$, when $A = \emptyset$ in (1.12). Finally, for each $A, B \subseteq \mathcal{I}_C$, we set

$$\Delta_{\mathcal{L}}(A,B) := \inf\left\{\epsilon > 0 : A \subseteq B^{\epsilon;\mathcal{W}} \text{ and } B \subseteq A^{\epsilon;\mathcal{W}}\right\}, \tag{1.13}$$

$$\Delta_{\mathcal{U}}(A,B) := \inf\left\{\epsilon > 0 : A \subseteq B^{\epsilon;\mathcal{U}} \text{ and } B \subseteq A^{\epsilon;\mathcal{U}}\right\}. \tag{1.14}$$

We extend these definitions to $A, B \in \mathcal{D}_C$, via the formal change of $\mathcal{I}_C$ into $\mathcal{D}_C$ in (1.10)–(1.14). Since $\Delta_{\mathcal{L}}(H_{\nu_1}, H_{\nu_2}) \leq \Delta_{\mathcal{U}}(H_{\nu_1}, H_{\nu_2})$, $\mathcal{U}$ is stronger than $\mathcal{W}$ on $\mathcal{I}_C$ (resp. $\mathcal{D}_C$).
Set $\lambda := \mathrm{d}x$ for the Lebesgue measure on $\mathbb{R}$. The *Lebesgue decomposition* (see, e.g., Corollary 1, p. 196 in Chow and Teicher [5]) of $\nu \in \mathcal{M}_C^{\pm}$ yields $\nu = \nu_{AC} + \nu_S$, where the *absolutely continuous* component $\nu_{AC} \in \mathcal{M}_C^{\pm}$, and the *singular* component $\nu_S \in \mathcal{M}_C^{\pm}$, of ν, fulfill $\nu_{AC} \ll \lambda$ and $\nu_S \perp \lambda$, and

$$H_\nu(t) = H_{\nu_{AC}}(t) + H_{\nu_S}(t) = \int_0^t h_\nu(t)\mathrm{d}t + H_{\nu_S}(t) \quad \text{for all} \quad t \in \mathbb{R}. \tag{1.15}$$

In (1.15), for $\nu \in \mathcal{M}_C^{\pm}$, $\pm h_\nu(t) := \pm\frac{\mathrm{d}}{\mathrm{d}t}H_{\nu_{AC}}(t) \geq 0$, is a locally integrable (with respect to λ) nonnegative function of $t \in \mathbb{R}$, with support in $[0, C]$, and uniquely defined, up to a λ-a.e. equivalence. As usual, we write $\nu_1 \ll \nu_2$ when ν_1 is absolutely continuous with respect to ν_2, and $\nu_1 \perp \nu_2$ when there exists a partition of $\mathbb{R} = A_1 \cup A_2$ into measurable sets $A_1 \cap A_2 = \emptyset$ with $\nu_1(A_2) = \nu_2(A_1) = 0$. In (1.15), $H_{\nu_S}(t) := \nu_S([0,t])$, for $t \in \mathbb{R}$, is the df of the *singular component* $\nu_S \in \mathcal{M}_C^{\pm}$ of $\nu \in \mathcal{M}_C^{\pm}$. We shall denote by $\mathcal{ACI}_C$ (resp. $\mathcal{ACD}_C$) the subset of all *absolutely continuous* $H_\nu \in \mathcal{I}_C$ (resp. $H_\nu \in \mathcal{D}_C$), fulfilling $H_{\nu_S}(t) = 0$ for all $t \in \mathbb{R}$.
We now state our main result in Theorem 1.2 below. Introduce the sets

$$K_{\Psi^*;C}^{+} := \left\{H_\nu \in \mathcal{I}_C : \int_0^C \Psi^*\left(h_\nu(t)\right)\mathrm{d}t + \Psi(0)H_{\nu_S}(C) \leq 1\right\}, \tag{1.16}$$

and

$$K_{\Psi^*;C}^{+*} := \left\{H_\nu \in \mathcal{ACI}_C : \int_0^C \Psi^*\left(h_\nu(t)\right)\mathrm{d}t \leq 1\right\}. \tag{1.17}$$

Remark 1.1

(1°) Under (A.1-2-3-4), the constant $\Psi(0)$ in (1.16) reduces to (see, e.g., Lemma 2.5 in the sequel) $\Psi(0) = -\log \mathbb{P}(X = 0)$, with the convention that $-\log 0 = \infty$, to cover the case where $\mathbb{P}(X = 0) = 0$.

(2°) In the forthcoming Sect. 1.2, we shall prove that, under (A.1-2-3-4), $K^{+}_{\Psi^*;C}$ is a compact subset of $(\mathcal{I}_C, \mathcal{W})$. If, in addition ($A$.5) holds, then $\Psi(0) = \infty$ and $K^{+}_{\Psi^*;C} = K^{+*}_{\Psi^*;C}$ is a compact subset of $(\mathcal{I}_C, \mathcal{U})$.

(3°) In Sect. 1.2, we will investigate the properties of $K^{+}_{\Psi^*;C}$, $K^{+*}_{\Psi^*;C}$. Under (A.1-2-3), by setting $\Upsilon = \Psi^*$ in (1.24)–(1.27), with Ψ as in (1.4) and Ψ^* as in (1.5), we will show that (1.16)–(1.17) are in agreement with the notation of Sect. 1.2.

Theorem 1.2 *Under* (A.1-2-3-4), *for each* $C > 0$, *we have, almost surely,*

$$\lim_{T\to\infty} \Delta_{\mathcal{L}} \left(\mathcal{G}_{T;C}, K^{+*}_{\Psi^*;C} \right) = \lim_{T\to\infty} \Delta_{\mathcal{L}} \left(\mathcal{G}_{T;C}, K^{+}_{\Psi^*;C} \right) = 0. \tag{1.18}$$

If, in addition, (A.5) *holds, then* $\Psi(0) = \infty$ *and* $K^{+*}_{\Psi^*;C} = K^{+}_{\Psi^*;C}$.

Remark 1.3 Our arguments fail to give a proof of (1.18) under the sole conditions (A.1-2-3). We conjecture that this extension holds, even though the technical details appear to be difficult to handle.

The proof of Theorem 1.2 is postponed until Sect. 3. We first present applications which motivate this result. The FLT in Theorem 1.2 is related to functional limit laws, initiated in the framework of empirical processes by Deheuvels and Mason [9], and of the following general form. Let $\mathcal{Q}$ be a class of functions on $[0, C]$, endowed with the topology induced by a metric Δ. Following (1.10)–(1.14), we extend Δ to a Hausdorff set-metric between subsets of $\mathcal{Q}$. We then consider a random set of functions $\mathcal{R}_{T;C} \subseteq \mathcal{Q}$ for $T \geq 0$, with a non-random limit set $\mathcal{R}_C \subseteq \mathcal{Q}$, in the following sense. We assume that $\mathcal{R}_C$ is a compact subset of $(\mathcal{Q}, \Delta)$, such that, a.s. as $T \to \infty$, $\Delta(\mathcal{R}_{T;C}, \mathcal{R}_C) \to 0$. Under this setup, for any functional $\Theta : \mathcal{Q} \to \mathbb{R}$, continuous in $(\mathcal{Q}, \Delta)$, we get

$$\lim_{T\to\infty} \left\{ \sup_{H\in\mathcal{R}_{T;C}} \Theta(H) \right\} = \sup_{H\in\mathcal{R}_C} \Theta(H) \quad \text{a.s.} \tag{1.19}$$

The relation (1.19) allows us to describe *globally* the strong limiting behavior of $\sup_{H\in\mathcal{R}_{T;C}} \Theta(H)$ *for all possible continuous functionals* Θ on $(\mathcal{Q}, \Delta)$. The evaluation of the (non-random) constant $\sup_{H\in\mathcal{R}_C} \Theta(H)$ reduces to an analytical problem (see, e.g., Deheuvels and Mason [8]). As an example, we apply (1.19) to $\mathcal{Q} = \mathcal{I}_C$, $\mathcal{R}_{T;C} = \mathcal{G}_{T;C}$, $\mathcal{R}_C = K^{+}_{\Psi^*;C}$ and $\Delta = \Delta_{\mathcal{L}}$. In view of Proposition 1.5, we so obtain the following corollary of Theorem 1.2.

Corollary 1.4 *Assume* (A.1-2-3-4). *Let* $\Theta : \mathcal{I}_C \to \mathbb{R}$ *be a functional, continuous with respect to the weak topology* $\mathcal{W}$ *on* $\mathcal{I}_C$. *Then, a.s.,*

$$\lim_{T\to\infty} \left\{ \sup_{0\leq x\leq T} \Theta(\eta_{x;T}) \right\} = \sup_{H\in K^{+}_{\Psi^*;C}} \Theta(H) = \sup_{H\in K^{+*}_{\Psi^*;C}} \Theta(H). \tag{1.20}$$

Example The functional $\Theta(H) = H(C)$ is continuous on $(\mathcal{I}_C, \mathcal{W})$, so that (1.20) holds. To evaluate the RHS of (1.20), we write, via (1.28),

$$\sup_{H_\nu \in \mathcal{K}^*_{\Psi^*;C}} H_\nu(C) = \sup_{H_\nu \in \mathcal{K}^*_{\Psi^*;C}} H_{\nu_{AC}}(C) = C \inf\{\alpha > 1/\mu : C\Psi^*(\alpha) \geq 1\}.$$

Consider the constants defined by

$$0 \leq \beta_C^- := \sup\{\alpha < 1/\mu : C\Psi^*(\alpha) \geq 1\} < 1/\mu \tag{1.21}$$

$$< \beta_C^+ := \inf\{\alpha > 1/\mu : C\Psi^*(\alpha) \geq 1\} < \infty. \tag{1.22}$$

By applying the above arguments to $\Theta(H) = \pm H(C)$, we infer from (1.20)–(1.22), that, under (A.1-2-3-4), a.s.,

$$\lim_{T\to\infty} \left\{ \sup_{0\leq x\leq T} \eta_{x;T}(C) \right\} = C\beta_C^+, \quad \lim_{T\to\infty} \left\{ \inf_{0\leq x\leq T} \eta_{x;T}(C) \right\} = C\beta_C^-. \tag{1.23}$$

The limit law (1.23), originally established by Steinebach [24, 25], was further refined by Bacro et al. [1], and Deheuvels and Steinebach [10].

A rough outline of our proofs is as follows. Our results rely on the description, in the forthcoming Theorem 3.7, of Erdős-Rényi-type FLT's for the increments of the partial sum process. The proof of Theorem 1.2 is then based on an inversion scheme, in the spirit of that used for quantile processes by Deheuvels and Mason [8, 9]. The details of this argument raise some huge technical difficulties. In the first place the topology involved in the FLT's plays a crucial role, together with the properties of the limit sets, which are defined as Orlicz-type classes of functions. The inversion scheme is discussed for piecewise linear functions in Sect. 2.3, and then, one needs continuity arguments to treat the general case. This is achieved through a careful analytical description of the problem, in Sects. 1.2 and 2, of interest in and of itself.

1.2 Auxiliary Results

We will consider functions $\Upsilon(\cdot) : \mathbb{R} \to [0, \infty]$, fulfilling, for some $\gamma_\Upsilon \in \mathbb{R}$:

(Υ.1) Υ is convex and nonnegative on $\mathbb{R}$;
(Υ.2) $\Upsilon(\gamma_\Upsilon) = 0$; $\Upsilon(\alpha) > 0$ for $\alpha \neq \gamma_\Upsilon$; $\Upsilon(\alpha) < \infty$ in a neighborhood of γ_Υ;
(Υ.3) $\lim_{\alpha\to-\infty} \alpha^{-1}\Upsilon(\alpha) = \rho_{1;\Upsilon} \in [-\infty, 0)$, $\lim_{\alpha\to\infty} \alpha^{-1}\Upsilon(\alpha) = \rho_{0;\Upsilon} \in (0, \infty]$.

In view of (Υ.1-2-3) and (1.15), consider the sets of functions of $\mathcal{I}_C$ and $\mathcal{D}_C$

$$K^+_{\Upsilon;C} := \left\{ H_\nu \in \mathcal{I}_C : \int_0^C \Upsilon\left(h_\nu(t)\right) \mathrm{d}t + \rho_{0;\Upsilon} H_{\nu_S}(C) \leq 1 \right\}, \tag{1.24}$$

$$K^{-}_{\Upsilon;C} := \left\{ H_\nu \in \mathcal{D}_C : \int_0^C \Upsilon\left(h_\nu(t)\right) \mathrm{d}t + \rho_{1;\Upsilon} H_{\nu_S}(C) \leq 1 \right\}. \tag{1.25}$$

Set, likewise

$$K^{+*}_{\Upsilon;C} := \left\{ H_\nu \in \mathcal{ACI}_C : \int_0^C \Upsilon\left(h_\nu(t)\right) \mathrm{d}t \leq 1 \right\}, \tag{1.26}$$

and

$$K^{-*}_{\Upsilon;C} := \left\{ H_\nu \in \mathcal{ACD}_C : \int_0^C \Upsilon\left(h_\nu(t)\right) \mathrm{d}t \leq 1 \right\}. \tag{1.27}$$

In (1.24) [resp. (1.25)], we use the convention that, when $\rho_{0;\Upsilon} = \infty$ (resp. $\rho_{1;\Upsilon} = -\infty$), the $H_\nu \in K^{+}_{\Upsilon;C}$ (resp. $H_\nu \in K^{-}_{\Upsilon;C}$) fulfill $H_{\nu_S}(C) = 0$, and are absolutely continuous on $[0, C]$. We have therefore in this case $K^{\pm}_{\Upsilon;C} = K^{\pm*}_{\Upsilon;C}$.

Proposition 1.5 *Under* (Υ.1-2-3)*, when* $0 < \rho_{0;\Upsilon}$ *(resp.* $\rho_{1;\Upsilon} < 0$*),* $K^{+}_{\Upsilon;C}$ *(resp.* $K^{-}_{\Upsilon;C}$*) is compact in* $(\mathcal{I}_C, \mathcal{W})$ *(resp.* $(\mathcal{D}_C, \mathcal{W})$*). When* $\rho_{0;\Upsilon} = \infty$ *(resp.* $\rho_{1;\Upsilon} = -\infty$*),* $K^{+}_{\Upsilon;C}$ *(resp* $K^{-}_{\Upsilon;C}$*) compact in* $(\mathcal{I}_C, \mathcal{U})$ *(resp.* $(\mathcal{D}_C, \mathcal{U})$*).*

Proof We first give proof in the "+" case, and recall some facts. A topological space **T** is *sequentially compact* iff each infinite sequence has a convergent infinite subsequence (see, e.g., Kelley [16], p. 162). In general, sequential compactness is not equivalent to compactness, however, this equivalence holds for metric spaces (see, e.g., Theorem 5, p. 138 in Kelley [16]), and our arguments will make use of this property. To be compact, a non-void subset A of a metric space needs only to be *relatively compact* (namely to be included in a compact set), and *closed*. In the present setup, this last property is equivalent to the requirement that A is *complete*. By the Helly-Bray theorem (see, e.g., Theorem 1, p. 261 in Feller [13]), a necessary and sufficient condition for $A \subseteq \mathcal{I}_C$, $A \neq \emptyset$, to be relatively compact in $(\mathcal{I}_C, \mathcal{W})$ is that:

$(C.1)$ $\quad \sup_{H \in A} H(C) < \infty$.

By the Arzelà-Ascoli theorem (see, e.g., Theorem A5, p. 369 in Rudin [20]), a necessary and sufficient condition for $A \subseteq \mathcal{I}_C$, $A \neq \emptyset$, to be relatively compact in $(\mathcal{I}_C, \mathcal{U})$, is that $(C.1\text{-}2)$ hold, where the last condition is given by:

$(C.2)$ $\quad$ The functions in A are uniformly equicontinuous.

Let now Υ fulfill (Υ.1-2-3), and let $H_\nu \in K^{+}_{\Upsilon;C}$ be as in (1.15). In view of (1.15) and (1.24), we may write the convexity inequalities

$$\Upsilon\left(\frac{1}{C}\int_0^C h_\nu(t)\mathrm{d}t\right) \leq \frac{1}{C}\int_0^C \Upsilon(h_\nu(t))\mathrm{d}t \leq \frac{1}{C}, \tag{1.28}$$

which implies that $H_{\nu_{AC}}(C) \leq M_1 := C\inf\{\alpha > \gamma_\Upsilon : \Upsilon(\alpha) \geq 1/C\} < \infty$. We note that the finiteness of $M_1 < \infty$ follows from $(\Upsilon.3)$. Since, by (1.24), $H_{\nu_S}(C) \leq M_2 := 1/\rho_{0;\Upsilon} < \infty$, we get $H_\nu(C) \leq M_1 + M_2 < \infty$. Since this inequality holds uniformly over all $H_\nu \in K^+_{\Upsilon;C}$, $A = K^+_{\Upsilon;C}$ fulfills $(C.1)$.

Let now $(\Upsilon.1\text{-}2\text{-}3)$ hold with $\rho_{0;\Upsilon} = \infty$. Fix any $\epsilon > 0$ and set $1/\infty := 0$. Our assumptions imply that $\sup_{u\geq r}(u/\Upsilon(u)) \to 0$ as $r \to \infty$. Thus, we may choose $r_\epsilon > \gamma_\Upsilon$ so large that, for all $r \geq r_\epsilon$, $\sup_{u\geq r}(u/\Upsilon(u)) < \epsilon/2$. Now, for each $\eta > 0$, $r \geq r_\epsilon$ and $H_\nu \in K^+_{\Upsilon;C} = K^{+*}_{\Upsilon;C} \subseteq \mathcal{ACI}_C$, we have the inequalities

$$\begin{aligned}
&\sup_{|t-s|\leq\eta} |H_\nu(t) - H_\nu(s)| \\
&= \sup_{|t-s|\leq\eta} \left| \int_s^t h_\nu(v)\mathrm{d}v \right| \leq \eta r + \int_0^C h_\nu(v)\mathbf{1}_{\{h_\nu(v)\geq r\}}\mathrm{d}v \\
&\leq \eta r + \sup_{u\geq r}(u/\Upsilon(u)) \int_0^C \Upsilon(h_\nu(v))\mathrm{d}v \leq \eta r + \epsilon/2.
\end{aligned}$$

Thus, whenever $|t-s| \leq \epsilon/(2r_\epsilon)$, we have $|H_\nu(t) - H_\nu(s)| \leq \epsilon$, uniformly over $H_\nu \in K^+_{\Upsilon;C}$. The choice of $\epsilon > 0$ being arbitrary, this establishes $(C.2)$ for $K^+_{\Upsilon;C}$. Making use of the just-proven fact that $K^+_{\Upsilon;C}$ fulfills $(C.1)$, we conclude that $K^+_{\Upsilon;C}$ is relatively compact in $(\mathcal{I}_C, \mathcal{U})$.

To complete our proof, we need only show that, under $(\Upsilon.1\text{-}2\text{-}3)$, $K^+_{\Upsilon;C}$ is closed in $(\mathcal{I}_C, \mathcal{W})$ (resp. $(\mathcal{I}_C, \mathcal{U})$, when $\rho_{0;\Upsilon} = \infty$). Consider the functionals

$$\mathcal{J}^+_{C;\Upsilon} : H_\nu \in \mathcal{I}_C \to \mathcal{J}^+_{C;\Upsilon}(H_\nu) = \int_0^C \Upsilon(h_\nu(t))\mathrm{d}t + \rho_{0;\Upsilon} H_{\nu_S}(C), \tag{1.29}$$

$$\mathcal{J}^-_{C;\Upsilon} : H_\nu \in \mathcal{D}_C \to \mathcal{J}^-_{C;\Upsilon}(H_\nu) = \int_0^C \Upsilon(h_\nu(t))\mathrm{d}t + \rho_{1;\Upsilon} H_{\nu_S}(C). \tag{1.30}$$

By Lemmas 3.3–3.4 of Lynch and Sethuraman [18], $\mathcal{J}^+_{C;\Upsilon}$ is *lower semi-continuous* in $(\mathcal{I}_C, \mathcal{W})$, so that, for all $\{\nu_n : n \geq 1\} \subseteq \mathcal{M}^+_C$ and $\nu \in \mathcal{M}^+_C$,

$$\lim_{n\to\infty} \Delta_{\mathcal{L}}(H_{\nu_n}, H_\nu) = 0 \quad \Rightarrow \quad \liminf_{n\to\infty} \mathcal{J}^+_{C;\Upsilon}(H_{\nu_n}) \geq \mathcal{J}_{C;\Upsilon}(H_\nu). \tag{1.31}$$

For each $n \geq 1$, $H_{\nu_n} \in K^+_{\Upsilon;C} \Leftrightarrow \mathcal{J}^+_{C;\Upsilon}(H_{\nu_n}) \leq 1$. By (1.31), this implies that $\mathcal{J}^+_{C;\Upsilon}(H_\nu) \leq 1$, whence $H_\nu \in K_{\Upsilon;C}$. This entails that $K_{\Upsilon;C}$ is complete in $(\mathcal{I}_C, \mathcal{W})$, and therefore, compact in $(\mathcal{I}_C, \mathcal{W})$. The observation that

$$\lim_{n\to\infty} \|H_{\nu_n} - H_\nu\| = 0 \quad \Rightarrow \quad \lim_{n\to\infty} \Delta_{\mathcal{L}}(H_{\nu_n}, H_\nu) = 0,$$

shows that, when $\rho_{0;\Upsilon} = \infty$, $K^+_{\Upsilon;C} = K^{+*}_{\Upsilon;C}$ is complete, and therefore, compact in $(\mathcal{I}_C, \mathcal{U})$. Refer to Varadhan [26] for a direct proof of this property.

For the proof in the "−" case, for each $\Upsilon = \Upsilon_+$ fulfilling (Υ.1-2-3), we set $\Upsilon_-(u) = \Upsilon_+(-u)$ for $u \in \mathbb{R}$, and observe that Υ_- fulfills (Υ.1-2-3), with

$$\gamma_{\Upsilon_-} = -\gamma_{\Upsilon_+}, \quad \rho_{1;\Upsilon_-} = -\rho_{0;\Upsilon_+} \quad \text{and} \quad \rho_{0;\Upsilon_-} = -\rho_{1;\Upsilon_+}.$$

For $H_\nu \in \mathcal{I}_C$ as in (1.15), we denote by $H_{(-\nu)} \in \mathcal{D}_C$ the df of $-\nu \in \mathcal{M}_C^-$. Then, for $0 \le t \le C$,

$$H_{(-\nu)}(t) := -H_\nu(t) = \int_0^t \{-h_\nu(u)\}\,\mathrm{d}u - H_{\nu_S}(t), \tag{1.32}$$

and

$$h_{(-\nu)}(t) = \frac{\mathrm{d}H_{(-\nu)}(t)}{\mathrm{d}t} = -h_\nu(t), \quad H_{(-\nu)_S}(t) = -H_{\nu_S}(t). \tag{1.33}$$

Thus, in view of (1.29)–(1.30), we see that, for any $\nu \in \mathcal{I}_C \Leftrightarrow (-\nu) \in \mathcal{D}_C$,

$$\begin{aligned} \mathcal{J}^-_{C;\Upsilon_+}(H_{(-\nu)}) &= \int_0^C \Upsilon_+(h_{(-\nu)}(t))\mathrm{d}t + \rho_{1;\Upsilon_+}H_{(-\nu)_S}(C) \\ = \mathcal{J}^+_{C;\Upsilon_-}(H_\nu) &= \int_0^C \Upsilon_-(h_\nu(t))\mathrm{d}t + \rho_{0;\Upsilon_-}H_{\nu_S}(C). \end{aligned} \tag{1.34}$$

We now make the formal change of $\Upsilon = \Upsilon_+$ into $\Upsilon = \Upsilon_-$ in (1.31). This, together with (1.7)–(1.8) and (1.34), shows that, for any $\{\nu_n : n \ge 1\} \subseteq \mathcal{M}_C^+$ and $\nu \in \mathcal{M}_C^+$ (or, for any $\{(-\nu_n) : n \ge 1\} \subseteq \mathcal{M}_C^-$ and $(-\nu) \in \mathcal{M}_C^-$), we have

$$\begin{aligned} &\lim_{n\to\infty} \Delta_{\mathcal{L}}(H_{\nu_n}, H_\nu) = 0 \Leftrightarrow \lim_{n\to\infty} \Delta_{\mathcal{L}}(H_{(-\nu_n)}, H_{(-\nu)}) = 0 \\ &\Rightarrow \liminf_{n\to\infty} \mathcal{J}^+_{C;\Upsilon_-}(H_{\nu_n}) \ge \mathcal{J}^+_{C;\Upsilon_-}(H_\nu) \\ &\Leftrightarrow \liminf_{n\to\infty} \mathcal{J}^-_{C;\Upsilon_+}(H_{(-\nu_n)}) \ge \mathcal{J}^-_{C;\Upsilon_+}(H_{(-\nu)}). \end{aligned} \tag{1.35}$$

Thus, $\mathcal{J}^-_{C;\Upsilon}$, as in (1.30), is lower semi-continuous on $(\mathcal{D}_C, \mathcal{W})$. Given this property, the remainder of the proof in the "−" case is similar to the corresponding proof in the "+" case, and details will, therefore, be omitted. □

Proposition 1.6 *Under* (Υ.1-2-3), *for each* $C > 0$, $K^+_{\Upsilon;C}$ *(resp* $K^-_{\Upsilon;C}$*) is the closure of* $K^{+*}_{\Upsilon;C}$ *(resp.* $K^*_{\Upsilon;C}$*) in* $(\mathcal{I}_C, \mathcal{W})$ *(resp.* $(\mathcal{D}_C, \mathcal{W})$*).*

Proof Following the lines of proof of Proposition 1.5, we will limit ourselves to the "+" case. For $H_\nu \in \mathcal{I}_C$ as in (1.15), set, as in (1.29),

$$\mathcal{J}^+_{C;\Upsilon}(H_\nu) := \int_0^C \Upsilon\,(h_\nu(t))\,\mathrm{d}t + \rho_{0;\Upsilon}H_{\nu_S}(C).$$

Denote by $P = \{t_0 = 1 < \ldots < t_k = C\}$ a partition of $[0, C]$, and let $\mathcal{P}_C$ denote the set of all such partitions, with $k = 1, 2, \ldots$ arbitrary. Denote by $H_\nu^P \in \mathcal{ACI}_C$ the continuous function on $[0, C]$, fulfilling:

(i) $H_\nu^P(t_i) = H_\nu(t_i)$ for $i = 0, \ldots, k$;
(ii) For $i = 1, \ldots, k$, H_ν^P is linear on (t_{i-1}, t_i), with Lebesgue derivative

$$h_\nu^P(t) = \frac{H_\nu(t_i) - H_\nu(t_{i-1})}{t_i - t_{i-1}} \quad \text{for} \quad t \in (t_{i-1}, t_i).$$

For each $P \in \mathcal{P}_C$, we denote by $\sigma(P)$ the σ-field generated by the intervals $[0, t_1], (t_1, t_2], \ldots, (t_{k-1}, C]$ of $P = \{t_0 = 1 < \ldots < t_k = C\}$. Partitions $P, P' \in \mathcal{P}_C$ are endowed with the partial order $P \preceq P' \quad \Leftrightarrow \quad \sigma(P) \subseteq \sigma(P')$. Consider now a *directed set* $(\mathcal{N}; \leq)$, namely, a non-void partially ordered set, such that any two $\eta, \eta' \in \mathcal{N}$ admit an upper bound $\eta'' \in \mathcal{N}$, in the sense that $\eta \leq \eta''$ and $\eta' \leq \eta''$. A *directed net* of partitions along $\mathcal{N}$ is a mapping $\eta \in \mathcal{N} \to P_\eta \in \mathcal{P}_C$ such that $\eta \leq \eta' \quad \Rightarrow \quad P_\eta \preceq P_{\eta'}$. Following Lynch and Sethuraman [18], we consider limits along a directed net $\mathcal{N}$, such that $\sigma(P_\eta) \to \mathcal{B}_C$. Here, $\mathcal{B}_C$ denotes the σ-algebra of Borel subsets of $[0, C]$, and convergence is meant in the sense that $\sigma\big(\bigcup_{\eta \in \mathcal{N}} P_\eta\big) = \mathcal{B}_C$. An example is given by setting $\mathcal{N} = \mathbb{N}$, and, for each $n \geq 0$, $P_n = \{Ci/2^n : i = 0, \ldots, 2^n\}$. For this $\mathcal{N}$ and $\{P_\eta : \eta \in \mathcal{N}\}$, we see that $m \leq n \quad \Rightarrow \quad P_m \subseteq P_n$, whence $\sigma(P_m) \subseteq \sigma(P_n)$. Also, it is straightforward that, in this case, $\sigma\big(\bigcup_{n \in \mathbb{N}} P_n\big) = \mathcal{B}_C$. Given this notation, in view of (1.29), and by invoking Theorem 3.2 and (3.16)–(3.25) of Lynch and Sethuraman [18], we get

$$\sup_{\eta \in \mathcal{N}} \mathcal{J}^+_{C;\Upsilon}\big(H_\nu^{P_\eta}\big) = \mathcal{J}^+_{C;\Upsilon}\big(H_\nu\big), \tag{1.36}$$

and

$$\mathcal{J}^+_{C;\Upsilon}\big(H_\nu^{P_\eta}\big) \to \mathcal{J}^+_{C;\Upsilon}\big(H_\nu\big) \quad \text{along} \quad \eta \in \mathcal{N}. \tag{1.37}$$

We consider the special case where we set in the above relations $\mathcal{N} = \mathbb{N}$ and $P_n = \{Ci/2^n : i = 0, \ldots, 2^n\}$. We so obtain that $H_\nu^{P_n} \in \mathcal{ACI}_C$ is such that $\mathcal{J}^+_{C;\Upsilon}\big(H_\nu^{P_n}\big) \to \mathcal{J}^+_{C;\Upsilon}\big(H_\nu\big)$ as $n \to \infty$. Next, we see that, for each continuity point $t \in (0, C)$, of H_ν, we have, as $n \to \infty$,

$$H_\nu\left(\frac{\lfloor 2^n Ct \rfloor}{2^n}\right) \to H_\nu(t) \quad \text{and} \quad H_\nu\left(\frac{\lfloor 2^n Ct \rfloor + 1}{2^n}\right) \to H_\nu(t).$$

Here and elsewhere, $\lfloor u \rfloor \leq u < \lfloor u \rfloor + 1$ denotes the integer part of u. In view of the definition of $H_\nu^{P_n}$, this readily implies that, as $n \to \infty$, $H_\nu^{P_n}(t) \to H_\nu(t)$ for each continuity point $t \in (0, C)$ of H_ν. Thus, as $n \to \infty$,

$$H_\nu^{P_n} \overset{\mathcal{W}}{\to} H_\nu. \tag{1.38}$$

The assumption $H_\nu \in K^{+}_{\Upsilon;C}$ is equivalent to the condition that $\mathcal{J}^{+}_{C;\Upsilon}(H_\nu) \leq 1$, which, by (1.36) implies the inequality $\mathcal{J}^{+}_{C;\Upsilon}\big(H^{P_n}_\nu\big) \leq 1$. Since $H^{P_n}_\nu \in \mathcal{ACI}_C$, we have, in turn, $H^{P_n}_\nu \in K^{*}_{\Upsilon;C}$. In view of (1.38), this suffices for our needs. □

Remark 1.7 The just-given proof improves on the statement of Proposition 1.6 by showing that, under (Υ.1-2-3), for each $C > 0$, $K^{+}_{\Upsilon;C}$ (resp $K^{-}_{\Upsilon;C}$) is the closure in $(\mathcal{I}_C, \mathcal{W})$ (resp. $(\mathcal{D}_C, \mathcal{W})$) of the subset of $K^{+*}_{\Upsilon;C}$ (resp. $H_\nu \in K^{*}_{\Upsilon;C}$), composed of piecewise linear functions.

2 Properties of Moment-Generating and Related Functions

2.1 Moment-Generating Functions and Legendre Transforms

Assume (A.1-2-3). Set $I := \{t : \psi(t) < \infty\}$. Since $\psi(0) = 1$, I is an interval with endpoints t_1 and t_0, such that $0 \in (t_1, t_0) \subseteq I \subseteq [t_1, t_0]$, and which may or may not belong to I, depending upon the law of X (see, e.g., Deheuvels [6]). The function $\psi(t) \in (0, \infty]$ is positive and log-convex on $\mathbb{R}$ (see, e.g., (2.1)–(2.2) below), analytic on (t_1, t_0) and continuous on I. (A.3) implies the existence of arbitrary moments of X, with $\frac{\mathrm{d}^m}{\mathrm{d}t^m}\,\psi(0) = \mathbb{E}(X^m)$ for all $m \in \mathbb{N}$. As in Deheuvels and Devroye [7], set, for $t_1 < t < t_0$,

$$m(t) := \frac{\mathrm{d}}{\mathrm{d}t}\log\psi(t) = \frac{\psi'(t)}{\psi(t)}, \quad \sigma^2(t) := m'(t) = \frac{\psi''(t)\psi(t) - \psi'(t)^2}{\psi(t)^2}. \tag{2.1}$$

By the Schwarz inequality, combined with (A.2-3), we get, for $t_1 < t < t_0$,

$$\psi''(t)\psi(t) - \psi'(t)^2 = \mathbb{E}(X^2 e^{tX})\mathbb{E}(e^{tX}) - \mathbb{E}(Xe^{tX})^2 > 0. \tag{2.2}$$

As follows from (2.1)–(2.2), we have $0 < m'(t) = \sigma^2(t) < \infty$ for all $t_1 < t < t_0$, together with $\sigma^2(0) = \sigma^2 := \mathrm{Var}(X)$. This entails that, under (A.1-2-3), $m(\cdot)$ is analytic and strictly increasing on (t_1, t_0), with $m(0) = \mu$. Set

$$a := \operatorname{ess\,sup}(X) = \sup\{x : F(x) < 1\}, \tag{2.3}$$

$$b := \operatorname{ess\,inf}(X) = \sup\{x : F(x) > 0\}, \tag{2.4}$$

$$-\infty \leq B := \lim_{t\downarrow t_1} m(t) < \mu = m(0) < A := \lim_{t\uparrow t_0} m(t) \leq \infty. \tag{2.5}$$

By combining (2.3)–(2.4)–(2.5) with the relation $m(t) = \mathbb{E}(Xe^{tX})/\mathbb{E}(e^{tX})$ for $t_1 < t < t_0$, we obtain the following lemma (see, e.g., Theorem 1 in Deheuvels, Devroye and Lynch [11])

Lemma 2.1 *Under* (*A*.1-2-3)*, we have*

$$-\infty \leq b \leq B < \mu < A \leq a \leq \infty. \tag{2.6}$$

Moreover, when $a < \infty$ (resp. $b > -\infty$), we have $A = a$ and $t_0 = \infty$ (resp. $B = b$ and $t_1 = -\infty$).

Proof Noting that $B < \mu < A$ follows from (2.5), we establish the inequality $A \leq a$. The proof of $b \leq B$, being similar will be omitted. There is nothing to prove if $a = \infty$, so that we limit ourselves to $a < \infty$. In this case, by the definition (2.3) of a, for each $\epsilon > 0$, we have $\mathbb{P}(X \leq a + \epsilon) = 1$. It follows that, for each $t_1 < t < t_0$,

$$m(t) = \frac{\mathbb{E}(Xe^{tX})}{\mathbb{E}(e^{tX})} \leq (a+\epsilon)\,\frac{\mathbb{E}(e^{tX})}{\mathbb{E}(e^{tX})} = a + \epsilon.$$

By letting $t \uparrow t_0$ in this inequality, we infer from (2.5) that $A \leq a + \epsilon$. Since $\epsilon > 0$ may be chosen arbitrarily small, we conclude that $A \leq a$, as sought. To complete the proof of (2.6), we observe that $a < \infty$ implies that $t_0 = \infty$, and hence, via (*A*.2), that $a > \mu > 0$. Thus, for each $0 < \epsilon < a$, we have

$$\psi(t) = \int_{-\infty}^{a} e^{tx}\mathrm{d}F(x) \leq e^{t(a-\epsilon)} + \int_{a-\epsilon}^{a} e^{tx}\mathrm{d}F(x),$$
$$\psi'(t) = \int_{-\infty}^{a} xe^{tx}\mathrm{d}F(x) \geq (a-\epsilon)\int_{a-\epsilon}^{a} e^{tx}\mathrm{d}F(x).$$

By letting $t \uparrow t_0 = \infty$, we get

$$e^{-t(a-\epsilon)}\int_{a-\epsilon}^{a} e^{tx}\mathrm{d}F(x) \geq e^{t\epsilon/2}\mathbb{P}\left(a - \tfrac{1}{2}\epsilon < X \leq a\right) \to \infty.$$

Therefore, we get, via (2.1),

$$\lim_{t\to\infty}\left\{\frac{(a-\epsilon)\int_{a-\epsilon}^{a} e^{tx}\mathrm{d}F(x)}{e^{t(a-\epsilon)} + \int_{a-\epsilon}^{a} e^{tx}\mathrm{d}F(x)}\right\} = a - \epsilon \leq \lim_{t\to\infty}\frac{\psi'(t)}{\psi(t)} = A.$$

Since our choice of $\epsilon \in (0, a)$ is arbitrary, we infer from the above relation, in combination with (2.6), that $A = a$, as sought. When $b > -\infty$, a similar argument shows that $t_1 = -\infty$ and $b = B$. □

Lemma 2.2 *Under* (*A*.1-2-3)*, when $t_0 = \infty$ (resp. $t_1 = -\infty$), we have $A = a$ (resp. $B = b$).*

Proof Assume $t_0 = \infty$. Lemma 2.1, implies that $A = a$ when $a < \infty$. We may therefore limit our proof to the case where $a = \infty$. Under this condition, we observe that, for each $c > 0$ and $t > 0$,

$$\psi(t) = \int_{-\infty}^{\infty} e^{tx} \mathrm{d}F(x) \le e^{tc} + \int_{c}^{\infty} e^{tx} \mathrm{d}F(x),$$

$$\psi'(t) = \int_{-\infty}^{\infty} x e^{tx} \mathrm{d}F(x) \ge c \int_{c}^{\infty} e^{tx} \mathrm{d}F(x).$$

Now, for each $0 < \epsilon < c$, we have, as $t \uparrow t_0 = \infty$,

$$e^{-tc} \int_{c}^{\infty} e^{tx} \mathrm{d}F(x) \ge e^{t\epsilon/2} \mathbb{P}\left(X \ge c + \tfrac{1}{2}\epsilon\right) \to \infty.$$

Therefore, we get

$$\lim_{t\to\infty} \left\{ c \int_{c}^{\infty} e^{tx} \mathrm{d}F(x) \right\} \left\{ e^{tc} + \int_{c}^{\infty} e^{tx} \mathrm{d}F(x) \right\}^{-1} = c \le \lim_{t\to\infty} \frac{\psi'(t)}{\psi(t)} = A.$$

Since $c > 0$ may be chosen arbitrarily large in this relation, we conclude that $A = \infty = a$. A similar argument shows that $B = b$ when $t_1 = -\infty$. □

Remark 2.3

(1°) Lemmas 2.1–2.2 show that $A = a$ (resp. $B = b$), unless $a = \infty$ and $t_0 < \infty$ (resp. $b = -\infty$ and $t_1 > -\infty$). Conversely, when $a = \infty$ and $t_0 < \infty$ (resp. $b = -\infty$ and $t_1 > -\infty$), we may have either $A < a$, or $A = a$ (resp. $B = b$ or $B > b$), depending upon the law of X (see, e.g., Theorem 2 in Deheuvels et al. [11]). Below are examples.

(2°) The exponential distribution, with df $F(x) = 1 - e^{-x}$ for $x \ge 0$, fulfills $\psi(t) = 1/(1-t)$ for $t < t_0 = 1 < \infty$. Since $m(t) = 1/(1-t) \to \infty$ as $t \uparrow t_0$, we so obtain a distribution for which $t_0 < \infty$, $\psi(t_0) = \infty$, and $A = a = \infty$.

(3°) Fix $0 < p < 1$, and consider the discrete rv X, with distribution given by

$$\mathbb{P}(X = k) = \frac{\mathcal{C} p^k}{k(k-1)(k-2)} \quad \text{for} \quad k = 3, 4, \dots ,$$

where the norming constant $\mathcal{C} > 0$ is given by

$$\frac{1}{\mathcal{C}} = H(p) := \sum_{k=3}^{\infty} \frac{p^k}{k(k-1)(k-2)} = \frac{1}{4}\Big(3p^2 - 2p - 2(1-p)^2 \log(1-p)\Big).$$

Here, $\psi(t) = H(pe^t)/H(p)$, $a = \infty$, $t_0 = -\log p < \infty$, so that $pe^{t_0} = 1$, and $\psi(t_0) = H(1)/H(p) = 1/(4H(p)) < \infty$. Since

$$pH'(p) = \sum_{k=3}^{\infty} \frac{p^k}{(k-1)(k-2)} = p^2 + p(1-p)\log(1-p),$$

we see that, as $t \uparrow t_0$, $pe^t \uparrow 1$, whence

$$m(t) = \frac{pe^t H'(pe^t)}{H(pe^t)} \to A = \frac{H'(1)}{H(1)} = 4.$$

We so obtain a distribution for which $t_0 = -\log p < \infty$ and $A = 1 < a = \infty$.

For a general distribution fulfilling (A.1-2-3), when $B < \alpha < A$, the equation $m(t) = \alpha$ has a unique solution $t = t^* = t^*(\alpha)$, fulfilling

$$m(t^*(\alpha)) = \alpha, \quad t_1 < t^*(\alpha) < t_0, \quad t^*(\mu) = 0. \tag{2.7}$$

It follows from (2.1), (2.5), (2.7), and the properties of $m(\cdot)$ and $\psi(\cdot)$, that $t^*(\alpha)$ is a strictly increasing analytic function of $\alpha \in (B, A)$, fulfilling

$$\lim_{\alpha\uparrow B} t^*(\alpha) = t_0 \quad \text{and} \quad \lim_{\alpha\downarrow A} t^*(\alpha) = t_1, \tag{2.8}$$

$$0 < \frac{dt^*(\alpha)}{d\alpha} = \frac{1}{m'(t^*(\alpha))} = \frac{1}{\sigma^2(t^*(\alpha))} < \infty \quad \text{for} \quad B < \alpha < A. \tag{2.9}$$

As in (1.4) let the *Legendre-Chernoff function* of ψ be given by

$$\Psi(\alpha) := \sup_{t\in I} \{t\alpha - \log \psi(t)\} \in [0, \infty] \quad \text{for} \quad \alpha \in \mathbb{R}. \tag{2.10}$$

An application of the Markov inequality entails that, for each $n \geq 1$,

$$\mathbb{P}(S_n \geq n\alpha) \leq \exp(-n\Psi(\alpha)) \quad \text{for} \quad \alpha \geq \mu, \tag{2.11}$$

$$\mathbb{P}(S_n \leq n\alpha) \leq \exp(-n\Psi(\alpha)) \quad \text{for} \quad \alpha \leq \mu. \tag{2.12}$$

By Theorem 1 of Chernoff [4], we get, under (A.1-2-3),

$$\lim_{n\to\infty} n^{-1} \log \mathbb{P}(S_n \geq n\alpha) = -\Psi(\alpha) \quad \text{for} \quad \alpha \geq \mu, \tag{2.13}$$

$$\lim_{n\to\infty} n^{-1} \log \mathbb{P}(S_n \leq n\alpha) = -\Psi(\alpha) \quad \text{for} \quad \alpha \leq \mu. \tag{2.14}$$

Some consequences of (2.13)–(2.14) are stated in the next two lemmas.

Lemma 2.4 *Under* (*A*.1-2-3)*, when* $a < \infty$ *(resp.* $b > -\infty$*), we have*

$$\Psi(\alpha) = \begin{cases} -\log \mathbb{P}(X = a) & \textit{for} \quad \alpha = a, \\ \infty & \textit{for} \quad \alpha > a, \end{cases} \tag{2.15}$$

and when $b > -\infty$

$$\Psi(b) = \begin{cases} -\log \mathbb{P}(X = b) & \textit{for} \quad \alpha = b, \\ \infty & \textit{for} \quad \alpha < b. \end{cases} \tag{2.16}$$

Proof When $a < \infty$, $\mathbb{P}(S_n \geq na) = \mathbb{P}(X = a)^n$, and, for each $\varepsilon > 0$, $\mathbb{P}(S_n \geq n(a + \varepsilon)) = 0$. Thus, by (2.13)–(2.14), taken with $\alpha = a$ (resp. $\alpha = a + \varepsilon$), we obtain that $\Psi(a) = -\log \mathbb{P}(X = a)$ (resp. $\Psi(a + \varepsilon) = \infty$), whence (2.15). By a similar argument for $\alpha \leq b$ (resp. $\alpha \leq b - \varepsilon$), we get (2.16). □

Lemma 2.5 *Under* (*A*.1-2-3-4)*, we have* $t_1 = -\infty$ *and*

$$\Psi(\alpha) = \begin{cases} -\log \mathbb{P}(X = 0) & \textit{for} \quad \alpha = 0, \\ \infty & \textit{for} \quad \alpha < 0. \end{cases} \tag{2.17}$$

Proof Under (*A*.4), we have $\mathbb{P}(S_n \leq 0) = \mathbb{P}(X = 0)^n$. Therefore, (2.17) is a direct consequence of (2.14), taken with $\alpha = 0$. □

By (2.10), $\Psi(\cdot)$ is the supremum over $t \in I$ of the set of linear (and hence, convex) functions $\{\alpha \to t\alpha - \log \psi(t) : t \in I\}$ (which by (*A*.3), includes the null function for $t = 0$). Thus, $\Psi(\cdot)$ is a (possibly infinite) convex nonnegative function on $\mathbb{R}$. It follows from (2.7) and (2.10), that, under (*A*.1-2-3),

$$\Psi(\mu) = 0, \quad \Psi(\alpha) = \alpha t^*(\alpha) - \log \psi(t^*(\alpha)) < \infty \quad \text{for} \quad B < \alpha < A, \tag{2.18}$$

As follows from (2.18) and the properties of $t^*(\alpha)$ over $\alpha \in (B, A)$, $\Psi(\alpha)$ is an analytic function of $\alpha \in (B, A)$, fulfilling, via (2.8) and (2.18),

$$\frac{\mathrm{d}\Psi(\alpha)}{\mathrm{d}\alpha} = t^*(\alpha) \quad \text{and} \quad \Psi(\alpha) = \int_\mu^\alpha t^*(u)\mathrm{d}u = \int_0^{t^*(\alpha)} tm'(t)\mathrm{d}t. \tag{2.19}$$

In particular, we infer from (2.19) that

$$\Psi(A) = \lim_{\alpha \uparrow A} \Psi(\alpha) = \int_0^{t_0} tm'(t)\mathrm{d}t, \;\; \Psi(B) = \lim_{\alpha \downarrow B} \Psi(\alpha) = \int_0^{t_1} tm'(t)\mathrm{d}t. \tag{2.20}$$

In view of (2.8) and (2.18), $\Psi(\cdot)$ is strictly convex on (B, A), decreasing on $(B, \mu]$ and increasing on $[\mu, A)$. When $A < \infty$, the definition of $\Psi(\alpha)$ for $\alpha \geq A$ is given by

$$\Psi(\alpha) = \begin{cases} \alpha t_0 - \log \psi(t_0), & \text{if} \quad A \leq \alpha < a = \infty, \quad (\Rightarrow t_0 < \infty), \\ -\log \mathbb{P}(X = a), & \text{if} \quad A = a = \alpha < \infty, \quad (\Rightarrow t_0 = \infty), \\ \infty, & \text{if} \quad A = a < \alpha < \infty, \quad (\Rightarrow t_0 = \infty). \end{cases} \tag{2.21}$$

Likewise, when $B > -\infty$, the definition of $\Psi(\alpha)$ for $\alpha \leq B$ is given by

$$\Psi(\alpha) = \begin{cases} \alpha t_1 - \log \psi(t_1), & \text{if} \quad B \geq \alpha > b = -\infty, \quad (\Rightarrow t_1 > -\infty), \\ -\log \mathbb{P}(X = b), & \text{if} \quad B = b = \alpha > -\infty, \quad (\Rightarrow t_1 = -\infty), \\ \infty, & \text{if} \quad B = b > \alpha > -\infty, \quad (\Rightarrow t_1 = -\infty). \end{cases} \tag{2.22}$$

As follows from (2.21)–(2.22), $\Psi(\cdot)$ is not strictly convex on $[A, \infty)$ when $A < a = \infty$ (resp. on $(-\infty, B]$ when $B > b = -\infty$), being linear on this interval. Set $J := \{\alpha : \Psi(\alpha) < \infty\}$. By (2.18)–(2.22), $(b, a) \subseteq J \subseteq [b, a]$ is an interval with endpoints b and a, and $\Psi(\cdot)$ is continuous on J. Moreover, we have (see, e.g., Lemma 2.1 in Deheuvels [6]),

$$\lim_{\alpha \downarrow -\infty} \alpha^{-1} \Psi(\alpha) = t_1 \quad \text{and} \quad \lim_{\alpha \uparrow \infty} \alpha^{-1} \Psi(\alpha) = t_0. \tag{2.23}$$

Remark 2.6

(1°) Let $\{a(t) : t \geq 0\}$ and $\{\tilde{a}(t) : t \geq 0\}$ be nondecreasing functions, fulfilling the conditions

(*i*) $a(0) = \tilde{a}(0) = 0$,
$T_0 := \sup\{t : a(t) < \infty\} > 0, \quad \widetilde{T}_0 := \sup\{t : \tilde{a}(t) < \infty\} > 0$;

(*ii*) $a(\cdot)$ is continuous on $I_0 := \{t \geq 0 : a(t) < \infty\}$,
$\tilde{a}(\cdot)$ is continuous on $\widetilde{I}_0 := \{t \geq 0 : \tilde{a}(t) < \infty\}$;

(*iii*) $a(\tilde{a}(t)) = t$ for $t \in \widetilde{I}_0$ and $\tilde{a}(a(t)) = t$ for $t \in I_0$.

We note that T_0 (resp. $\widetilde{T}_0$) is the upper endpoint of I_0 (resp. $\widetilde{I}_0$). Consider the *conjugate* (or *complementary*) *pair of Young functions* given by

$$A(s) = \begin{cases} \int_0^s a(u)\mathrm{d}u & \text{for} \quad 0 \leq s \leq T_0, \\ \infty & \text{for} \quad s > T_0, \end{cases} \tag{2.24}$$

$$\widetilde{A}(s) = \begin{cases} \int_0^s \tilde{a}(u)\mathrm{d}u & \text{for} \quad 0 \leq s \leq \widetilde{T}_0, \\ \infty & \text{for} \quad s > \widetilde{T}_0. \end{cases} \tag{2.25}$$

These functions fulfill Young's inequality (refer to Rao and Ren [19], p. 6)

$$st \leq A(s) + \widetilde{A}(t) \quad \text{for} \quad s \geq 0 \quad \text{and} \quad t \geq 0, \tag{2.26}$$

together with the reciprocal relations

$$A(t) = \sup_{s \in I_0}\{st - \widetilde{A}(s)\} = s\tilde{a}(s) - A(\tilde{a}(s)), \tag{2.27}$$

$$\widetilde{A}(s) = \sup_{t \in \widetilde{I_0}}\{st - A(st)\} = ta(t) - \widetilde{A}(a(t)). \tag{2.28}$$

Examples of conjugate pairs of Young functions are given, for $s > 0$, by

$$A(s) = \Psi(\mu \pm s) = \int_0^s a(u)\mathrm{d}u,\ a(u) = t^*(\mu \pm u), \tag{2.29}$$

$$\widetilde{A}(s) = \log\{\psi(\pm s)e^{\mp s\mu}\} = \int_0^s \tilde{a}(u)\mathrm{d}u,\ \tilde{a}(u) = m(\pm u) \mp \mu. \tag{2.30}$$

A consequence of (2.27)–(2.28), when applied to (2.29)–(2.30), is that the Legendre-Chernoff function $\Psi(\cdot)$ completely determines the mgf $\psi(\cdot)$ of X, whence the law of X. Since the knowledge, for all $C > 0$, of the strong limiting behavior of the maximal and minimal increments of size $C \log T$ of the partial sum process $S(\cdot)$ (see, e.g. (3.2)–(3.3) in the sequel) fully determines $\Psi(\cdot)$ (and hence, μ which is the unique solution of $\Psi(\mu) = 0$), the same holds for $\psi(\cdot)$, and hence for the law of X. We so obtain an easy proof of a characterization result due to Bártfai [2], showing that the knowledge of the limiting behavior of the Erdős-Rényi increments fully determines the underlying distribution.

(2°) For any Young function $\{A(s) : s \geq 0\}$, as in (2.24), we may define the Orlicz class (see, e.g., Rao and Ren [19], p. 45) of $A(\cdot)$ by

$$\mathcal{L}_{A;C} = \left\{ f : [0, C] \to \overline{\mathbb{R}} : \int_0^C A\left(|f(t)|\right)\mathrm{d}t < \infty \right\}.$$

In view of this definition, we may rewrite the definition (1.27) of $K^{+*}_{\Phi;C}$ into

$$K^{+*}_{\Phi;C} = \{H_\nu \in \mathcal{ACI}_C : \quad h_\nu \in \mathcal{L}_{\Phi;C}\}.$$

This last identity shows that our results are deeply rooted in the theory of Orlicz spaces (see, e.g., Krasnosel'skii and Rutickii [17], Rao and Ren [19]).

2.2 The Legendre Conjugate Function

Under (A.1-2-3), the *Legendre conjugate function* of $\Psi(\cdot)$ is given by

$$\Psi^*(\alpha) := \begin{cases} \alpha\Psi(1/\alpha) & \text{for} \quad \alpha > 0, \\ t_0 & \text{for} \quad \alpha = 0, \\ \infty & \text{for} \quad \alpha < 0. \end{cases} \tag{2.31}$$

Recall the definitions (2.3)–(2.5) and (2.7), of a, A and $t^*(\cdot)$.

Lemma 2.7 *Under* (A.1-2-3)*,* $\Psi^*(\cdot)$ *is convex and nonnegative on* $\mathbb{R}$*. It fulfills* $\Psi^*(1/\mu) = 0$*, together with*

$$\Psi^*(\alpha) = \Psi_0^*(\alpha) := \begin{cases} \sup_{t\in I}\{t - \alpha \log \psi(t)\} & \textit{for} \quad \alpha \geq 0, \\ \infty & \textit{for} \quad \alpha < 0. \end{cases} \tag{2.32}$$

Moreover, setting $1/A = 0$ *when* $A = \infty$*, for each* $0 \leq 1/A < \alpha < \infty$*,*

$$\Psi^*(\alpha) = t^*(1/\alpha) - \alpha \log \psi(t^*(1/\alpha)), \tag{2.33}$$

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\Psi^*(\alpha) = \Psi(1/\alpha) - \frac{t^*(1/\alpha)}{\alpha}, \tag{2.34}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}\alpha^2}\Psi^*(\alpha) = \frac{1}{\alpha^2\sigma^2(t^*(1/\alpha))} > 0. \tag{2.35}$$

In addition, when $A < \infty$*, we have*

$$\Psi^*(\alpha) = \begin{cases} t_0 - \alpha \log \psi(t_0), & \textit{if} \quad 0 \leq \alpha < 1/A, \ A < a = \infty, \\ -\alpha \log \mathbb{P}(X = a), & \textit{if} \quad \alpha = 1/A, \ A = a < \infty, \\ \infty, & \textit{if} \quad 0 \leq \alpha < 1/A, \ A = a < \infty. \end{cases} \tag{2.36}$$

Proof In view of (2.31)–(2.32), $\Psi^*(\alpha) = \Psi_0^*(\alpha)$ for $\alpha > 0$ follows from the definition (2.10) of $\Psi(\cdot)$. When $\alpha = 0$, (2.32) yields, via (A.3), $\Psi_0^*(0) = \sup_{t\in I} t = t_0$, which is in agreement with $\Psi^*(0) = \Psi_0^*(0) = t_0$ in (2.31). Since $\Psi^*(\alpha) = \Psi_0^*(\alpha) = \infty$ for $\alpha < 0$, we get $\Psi^*(\alpha) = \Psi_0^*(\alpha)$ for all $\alpha \in \mathbb{R}$, which establishes (2.31)–(2.32). Since, for $\alpha \geq 0$, $\Psi^*(\alpha) = \Psi_0^*(\alpha)$ is the supremum over $t \in I$ of the set of convex functions $\{t \to t - \alpha \log \psi(t)\}$ (which, via (A.3), includes the null function for $t = 0 \in I$), $\Psi^*(\cdot) = \Psi_0^*(\cdot)$ is a (possibly infinite) nonnegative convex function on $\mathbb{R}^+ := [0, \infty)$. By setting, as in (2.31)–(2.32), $\Psi^*(\alpha) = \infty$ for $\alpha < 0$, we see that the so-defined $\Psi^*(\cdot) = \Psi_0^*(\cdot)$ is convex on $\mathbb{R}$. The remainder of the proof follows from (2.8)–(2.9) and the observation that, via (2.31), that (2.33) is equivalent to (2.18) after the formal change of α into $1/\alpha$. By all this, (2.36) is a consequence of (2.21). □

As follows from (2.6) and (2.35), $\Psi^*(\cdot)$ is finite and strictly convex on $(1/A, \infty)$, decreasing on $(1/A, 1/\mu]$, increasing on $[1/\mu, \infty)$, and positive on $\mathbb{R} - \{1/\mu\}$. By (2.36), we see that, whenever $A < a = \infty$, $\Psi^*(\cdot)$ is linear (and hence, not strictly convex) on $[0, 1/A)$. Given these relations, (2.31) implies that

$$\lim_{\alpha \downarrow -\infty} \alpha^{-1} \Psi^*(\alpha) = \infty \quad \text{and} \quad \lim_{\alpha \uparrow \infty} \alpha^{-1} \Psi^*(\alpha) = \Psi(0). \tag{2.37}$$

Remark 2.8 We point out that, for all $\alpha \in \mathbb{R}$,

$$\Psi_1^*(\alpha) := \sup_{t \in I} \{t - \alpha \log \psi(t)\} \leq \Psi^*(\alpha) = \Psi_0^*(\alpha), \tag{2.38}$$

with a possibly strict inequality for $\alpha < 0$. To establish this property under (A.1-2-3-4), we recall from (2.3) that $b \geq 0$. Thus, we infer from (2.6) that $m(t) \geq b \geq 0$ for all $t \in I$. Therefore, whenever $\alpha < 0$, the function $\eta(t) := t - \alpha \log \psi(t)$ fulfills $\eta'(t) = 1 - \alpha m(t) > 0$ over $t \in I$. This yields, for $\alpha < 0$,

$$\Psi_1^*(\alpha) = \sup_{t \in I} \{t - \alpha \log \psi(t)\} = t_0 - \alpha \log \psi(t_0) \leq \infty, \tag{2.39}$$

where the inequality is strict iff $\psi(t_0) < \infty$. In the latter case, we must also have $t_0 < \infty$ (see, e.g., Remark 2.3). Therefore, whenever $\Psi_0^*(\alpha) = t_0 - \alpha \log \psi(t_0) < \infty$, we have $0 < \Psi_1^*(\alpha) < \Psi^*(\alpha) = \infty$ for $\alpha < 0$, so that the function $\Psi_1^*(\alpha)$, as defined in (2.38), differs from $\Psi^*(\alpha)$, as defined in (2.31).

2.3 An Inversion Scheme

Assume that Φ is such that $\Upsilon = \Phi$ fulfills (Υ.1-2-3), with $\gamma_\Phi > 0$. Set

$$\Phi_+(\alpha) := \begin{cases} \Phi(\alpha) & \text{for} \quad \alpha \geq 0, \\ \infty & \text{for} \quad \alpha < 0. \end{cases}$$

Observe that $\Upsilon = \Phi_+$ also fulfills (Υ.1-2-3), with $\gamma_{\Phi_+} = \gamma_\Phi > 0$. Given this property, to simplify notation, we shall set, throughout the present section, $\Phi = \Phi_+$, and work under the assumption that $\Phi(\alpha) = \infty$ for $\alpha < 0$. In agreement with (Υ.3), (2.23), and the notation (2.31), corresponding to the case where $\Phi = \Psi$, we define the *Legendre conjugate function* Φ^* of Φ by

$$\Phi^*(\alpha) = \begin{cases} \alpha \Phi(1/\alpha) & \text{for} \quad \alpha > 0, \\ \Phi^*(0) := \rho_{0;\Phi} = \lim_{u \to \infty} u^{-1} \Phi(u) & \text{for} \quad \alpha = 0, \\ \infty & \text{for} \quad \alpha < 0. \end{cases} \tag{2.40}$$

It is noteworthy that $\{\Phi^*\}^*(\alpha) = \Phi(\alpha)$ for $\alpha \geq 0$. We have, namely,

$$\Phi(\alpha) = \begin{cases} \alpha\Phi^*(1/\alpha) & \text{for} \quad \alpha > 0, \\ \Phi(0) = \rho_{0;\Phi^*} := \lim_{u\to\infty} u^{-1}\Phi^*(u) & \text{for} \quad \alpha = 0 \\ \infty & \text{for} \quad \alpha < 0. \end{cases} \tag{2.41}$$

Fix $C > 0$, and consider a non-null piecewise linear function with finitely many jumps $H_\nu \in K^+_{\Phi;C} \subseteq \mathcal{I}_C$. We denote by $\mathcal{I}^*_C$ the subset $\mathcal{I}_C$ of all such functions. We set $D := H_\nu(C) > 0$, and assume that $H_\nu \in \mathcal{I}^*_C$ is of the form

$$H_\nu(t) = \int_0^t h_\nu(s)\mathrm{d}s + \int_{[0,t]} \mathrm{d}H_{\nu_S}(t) \quad \text{for} \quad t \in \mathbb{R}, \tag{2.42}$$

where h_ν and H_{ν_S} are as follows. We introduce two nondecreasing sequences $t_0 = 0 \leq t_1 \leq \ldots \leq t_k = C$, and $T_0 = 0 \leq T_1 \leq \ldots \leq T_k = H_\nu(C)$. We set $c_i = T_i - T_{i-1}$ and $c^*_i = t_i - t_{i-1}$ for $i = 1, \ldots, k$, and set

$$\begin{aligned} \mathcal{N} &:= \{i : 1 \leq i \leq k, : t_{i-1} = t_i\} = \{i : 1 \leq i \leq k,\, c_i = 0\}, \\ \mathcal{P} &:= \{1 \leq i \leq k : t_{i-1} < t_i\} = \{i : 1 \leq i \leq k,\, c^*_i > 0\}, \\ \mathcal{N}^* &:= \{i : 1 \leq i \leq k, : T_{i-1} = T_i\} = \{i : 1 \leq i \leq k,\, c^*_i = 0\}, \\ \mathcal{P}^* &:= \{1 \leq i \leq k : T_{i-1} < T_i\} = \{i : 1 \leq i \leq k,\, c_i > 0\}. \end{aligned}$$

Denote by δ_x the Dirac measure at x. We define h_ν and H_{ν_S} in (2.42) by

$$h_\nu(t) := \sum_{i\in\mathcal{P}} \left(\frac{T_i - T_{i-1}}{t_i - t_{i-1}}\right) \mathbf{1}_{\{t_{i-1}<t\leq t_i\}} \quad \text{for} \quad t \in \mathbb{R}, \tag{2.43}$$

$$\mathrm{d}H_{\nu_S} := \sum_{i\in\mathcal{N}} (T_i - T_{i-1})\delta_{t_i}. \tag{2.44}$$

We now define the invert $H^{\mathrm{inv}}_\nu \in \mathcal{I}^*_D$ of H_ν, by

$$H^{\mathrm{inv}}_\nu(t) := \int_0^t h^{\mathrm{inv}}_\nu(s)\mathrm{d}s + \int_{[0,t]} \mathrm{d}H^{\mathrm{inv}}_{\nu_S}(t) \quad \text{for} \quad t \in \mathbb{R}, \tag{2.45}$$

where h^{inv}_ν and $H^{\mathrm{inv}}_{\nu_S}$ in (2.45) are given by

$$h^{\mathrm{inv}}_\nu(t) := \sum_{i\in\mathcal{P}^*} \left(\frac{t_i - t_{i-1}}{T_i - T_{i-1}}\right) \mathbf{1}_{\{T_{i-1}<t\leq T_i\}} \quad \text{for} \quad t \in \mathbb{R}, \tag{2.46}$$

$$\mathrm{d}H^{\mathrm{inv}}_{\nu_S} := \sum_{i\in\mathcal{N}^*} (t_i - t_{i-1})\delta_{T_i}. \tag{2.47}$$

(2.45)–(2.47) imply that $H_\nu^{\rm inv} \in \mathcal{I}_D$. By replacing $\{t_i : 1 \le i \le k\}$, H_ν, h_ν, $H_{\nu S}$ and $C > 0$ by $\{T_i : 1 \le i \le k\}$, $H_\nu^{\rm inv}$, $h_\nu^{\rm inv}$, $H_{\nu S}^{\rm inv}$ and $D > 0$, respectively, we may repeat verbatim these definitions, as to define the invert of $H_\nu^{\rm inv} \in \mathcal{I}_D$ by $\{H_\nu^{\rm inv}\}^{\rm inv} = H_\nu \in \mathcal{I}_C$. It follows that the map $\mathcal{R}_{C,D} : H_\nu \in \mathcal{I}_C^* \to H_\nu^{\rm inv} \in \mathcal{I}_D^*$ is one-to-one, with invert given by $\mathcal{R}_{D,C}$.

Lemma 2.9 *Under the assumptions above, we have*

$$H_\nu^{\rm inv}(t) = \sup\{s \le C : H_\nu(s) \le t\} \textit{ for } 0 \le t \le D = H_\nu(C), \tag{2.48}$$

$$H_\nu(t) = \sup\{s \le D : H_\nu^{\rm inv}(s) \le t\} \textit{ for } 0 \le t \le C = H_\nu^{\rm inv}(D) \tag{2.49}$$

Moreover,

$$\begin{aligned} \mathcal{J}_{C;\Phi}^+(H_\nu) &= \int_0^C \Phi(h_\nu(t))dt + \rho_{0;\Phi} H_\nu(C) \\ &= \sum_{i\in\mathcal{P}} (t_i - t_{i-1})\Phi\left(\frac{T_i - T_{i-1}}{t_i - t_{i-1}}\right) + \Phi^*(0)\sum_{i\in\mathcal{N}}(T_i - T_{i-1}) \\ &= \mathcal{J}_{D;\Phi^*}^+(H_\nu^{\rm inv}) = \int_0^C \Phi^*(h_\nu^{\rm inv}(t))dt + \rho_{0;\Phi^*} H_\nu^{\rm inv}(D) \\ &= \sum_{i\in\mathcal{P}^*} (T_i - T_{i-1})\Phi\left(\frac{t_i - t_{i-1}}{T_i - T_{i-1}}\right) + \Phi(0)\sum_{i\in\mathcal{N}^*}(t_i - t_{i-1}). \end{aligned} \tag{2.50}$$

Proof The proof of (2.48)–(2.49) is achieved by a tedious enumeration of cases, whose details are omitted. To establish (2.50), we combine (2.43)–(2.46) with the definition (1.29) of $\mathcal{J}_{C;\Phi}^+$. □

Let $C > 0$. In view of Lemma 2.9, we now define the invert of an arbitrary $H_\nu \in \mathcal{I}_C$ such that $D := H_\nu(C) > 0$ as the function $H_\nu^{\rm inv} \in \mathcal{I}_D$ fulfilling $H_\nu^{\rm inv}(D) = C$, and defined via the reciprocal equations

$$H_\nu^{\rm inv}(t) = \sup\{s \le C : H_\nu(s) \le t\} \textit{ for } 0 \le t \le D = H_\nu(C), \tag{2.51}$$

$$H_\nu(t) = \sup\{s \le D : H_\nu^{\rm inv}(s) \le t\} \text{ for } 0 \le t \le C = H_\nu^{\rm inv}(D). \tag{2.52}$$

We shall make use of the following lemma.

Lemma 2.10 *Fix $C > 0$ and $D > 0$, and let $H_{\nu_1}, H_{\nu_2} \in \mathcal{I}_C$ and $\varepsilon > 0$ be such that $\Delta_{\mathcal{L}}(H_{\nu_1}, H_{\nu_2}) < \varepsilon$. Assume that $D := H_{\nu_j}(C)$, $j = 1, 2$, and consider $H_{\nu_j}^{\rm inv} \in \mathcal{I}_D$ for $j = 1, 2$. We then have*

$$\Delta_{\mathcal{L}}(H_{\nu_1}^{\rm inv}, H_{\nu_2}^{\rm inv}) < \varepsilon. \tag{2.53}$$

Proof Fix any $0 \leq t \leq D$, and set, in view of (2.51), $0 \leq x := H_{\nu_2}^{\text{inv}}(t) \leq C$. By (2.51), we have

$$H_{\nu_2}(x-u) \leq t < H_{\nu_2}(x+u) \quad \text{for all} \quad u > 0.$$

Now, the inequality $\Delta_{\mathcal{L}}(H_{\nu_1}, H_{\nu_2}) < \varepsilon$ implies that, for all $s \in \mathbb{R}$, $H_{\nu_1}(s-\varepsilon) - \varepsilon < H_{\nu_2}(s) < H_{\nu_1}(s+\varepsilon)+\varepsilon$. By setting $s = x-u$ (resp. $s = x+u$) in this last inequality, we see that, for each $u > 0$, we have the inequalities $H_{\nu_1}(x-\varepsilon-u) < H_{\nu_2}(x-u)+\varepsilon \leq t + \varepsilon$ (resp. $t - \varepsilon < H_{\nu_2}(x+u) - \varepsilon < H_{\nu_1}(x+\varepsilon+u)$). This, in turn, implies that $H_{\nu_2}^{\text{inv}}(t) - \varepsilon = x - \varepsilon \leq H_{\nu_1}^{\text{inv}}(t+\varepsilon)$ (resp. $H_{\nu_1}^{\text{inv}}(t-\varepsilon) \leq x + \varepsilon = H_{\nu_2}^{\text{inv}}(t) + \varepsilon$). We so obtain that

$$H_{\nu_1}^{\text{inv}}(t-\varepsilon) - \varepsilon \leq H_{\nu_2}^{\text{inv}}(t) \leq H_{\nu_1}^{\text{inv}}(t+\varepsilon) + \varepsilon.$$

Our initial choice of $0 \leq t \leq D$ being arbitrary, we see that the above inequalities hold for all $0 \leq t \leq D$. This suffices for (2.53). □

The next lemma will be useful to show that, when $H_\nu \in \mathcal{I}_C$ varies in $K_{\Phi;C}^{+}$, the values of $D = H_\nu(C)$ vary within specified limits.

Lemma 2.11 *Assume that* $\Upsilon = \Phi$ *fulfills* (Υ.1-2-3). *Then, for each* $H_\nu \in K_{\Phi;C}^{+}$, *we have*

$$C\alpha_{C;\Phi}^{-} \leq H_\nu(C) \leq C\alpha_{C:\Phi}^{+}, \tag{2.54}$$

where

$$\begin{aligned} \alpha_{C;\Phi}^{-} &:= \sup\{\alpha < \gamma_\Phi : \Phi(\alpha) \geq 1/C\}, \\ \alpha_{C;\Phi}^{+} &:= \inf\{\alpha > \gamma_\Phi : \Phi(\alpha) \geq 1/C\}. \end{aligned} \tag{2.55}$$

Proof The mapping $H_\nu \in \mathcal{I}_C \to H_\nu(C)$ is continuous with respect to the weak topology $\mathcal{W}$. Therefore, by Proposition 1.6, we have

$$\sup_{H_\nu \in K_{\Phi;C}^{+}} H_\nu(C) = \sup_{H_\nu \in K_{\Phi;C}^{+*}} H_\nu(C).$$

Since $H_\nu \in K_{\Phi;C}^{+*}$ is absolutely continuous, by setting $h_\nu(t) = \frac{\mathrm{d}}{\mathrm{d}t} H_\nu(t)$, we may rewrite the convexity inequalities (1.28) into

$$\Phi\left(\frac{1}{C} H_\nu(C)\right) = \Phi\left(\frac{1}{C}\int_0^C h_\nu(t)\mathrm{d}t\right) \leq \frac{1}{C}\int_0^C \Phi(h_\nu(t))\mathrm{d}t \leq \frac{1}{C}, \tag{2.56}$$

which, in view of (2.55), implies that $H_\nu(C) \leq C\alpha_C^{+}$. The proof of the remaining inequality in (2.54) is very similar, and hence, omitted. □

The next proposition extends the validity of (2.50) to arbitrary df's.

Proposition 2.12 *Let* $\Upsilon = \Phi$*, fulfill* $(\Upsilon.1\text{-}2\text{-}3)$*, with* $\gamma_\Phi > 0$ *and* $\Phi(\alpha) = \infty$ *for* $\alpha < 0$*. Fix* $C > 0$ *and* $D > 0$*, and consider the mapping* $\mathcal{R}_{C,D} : H_{\nu_1} \in \{H_\nu \in \mathcal{I}_C,\ H_\nu(C) = D\} \to H_{\nu_1}^{\text{inv}} \in \{H_\nu \in \mathcal{I}_D,\ H_\nu(D) = C\}$*. Then the mapping* $\mathcal{R}_{C,D}$ *is continuous with respect to the weak topology* $\mathcal{W}$*, and one-to-one, with invert equal to* $\mathcal{R}_{D,C}$*. Moreover, we have the equalities*

$$\mathcal{J}^+_{C;\Phi}(H_\nu) = \mathcal{J}^+_{C;\Phi^*}(H_\nu^{\text{inv}}). \tag{2.57}$$

Proof The assertion that the mapping $\mathcal{R}_{C,D}$ is one-to-one, with invert equal to $\mathcal{R}_{D,C}$ is straightforward. The continuity of $\mathcal{R}_{C,D}$ (resp. $\mathcal{R}_{D,C}$) follows from Lemma 2.10. To complete our proof, we need only establish (2.57). For this, we fix $H_{\nu_1} \in \{H_\nu \in \mathcal{I}_C,\ H_\nu(C) = D\}$ and $H_{\nu_2} := H_{\nu_1}^{\text{inv}} \in \{H_\nu \in \mathcal{I}_D,\ H_\nu(D) = C\}$. By Proposition 1.5 and Remark 1.7, there exists a sequence $H_n \in \mathcal{I}_C^*$ such that $\Delta_{\mathcal{L}}(H_n, H_{\nu_1}) \to 0$ as $n \to \infty$. As follows from Lemma 2.10, we have also $\Delta_{\mathcal{L}}(H_n^{\text{inv}}, H_{\nu_1}^{\text{inv}}) \to 0$ as $n \to \infty$, with $H_n^{\text{inv}} \in \mathcal{I}_D^*$. Now, making use of Lemma 2.9, we see that, for each $n = 1, 2, \ldots, \mathcal{J}^+_{C;\Phi}(H_n) = \mathcal{J}^+_{C;\Phi^*}(H_n^{\text{inv}})$. This, together with an application of (1.31), shows that

$$\mathcal{J}^+_{C;\Phi}(H_\nu) \le \mathcal{L}_1 = \liminf_{n\to\infty} \mathcal{J}^+_{C;\Phi}(H_n), \tag{2.58}$$

$$\mathcal{J}^+_{C;\Phi^*}(H_\nu^{\text{inv}}) \le \mathcal{L}_1 = \liminf_{n\to\infty} \mathcal{J}^+_{C;\Phi}(H_n^{\text{inv}}). \tag{2.59}$$

To conclude, we need only choose $\{H_n : n \ge 1\}$ as in the proof of Proposition 1.6, in such a way that

$$\sup_{n\ge 1} \mathcal{J}^+_{C;\Phi}(H_n) = \mathcal{J}^+_{C;\Phi}(H_\nu).$$

By so doing, we infer from (2.58)–(2.59) that

$$\mathcal{J}^+_{C;\Phi}(H_\nu) = \mathcal{L}_1 \ge \mathcal{J}^+_{C;\Phi^*}(H_\nu^{\text{inv}}).$$

By interchanging ν_1 and ν_2, we obtain, in turn, that

$$\mathcal{J}^+_{C;\Phi}(H_\nu) \le \mathcal{J}^+_{C;\Phi^*}(H_\nu^{\text{inv}}).$$

We so obtain (2.57). □

3 Proof of Theorem 1.2

Towards proving Theorem 1.2, we recall the notation of Sects. 1, 2. For $n \geq 0$, let $S_n := \sum_{i=1}^{n} X_i$ denote the partial sums of $\{X_n : n \geq 1\}$, with the convention that $\sum_{\emptyset}(\cdot) := 0$, and, for $t \geq 0$, consider the partial sum process $S(t) := S_{\lfloor t \rfloor}$. Fix a constant $C > 0$, and for each $x \geq 0$ consider the increment function

$$\xi_{x;T}(t) := (\log_+ T)^{-1} \big(S(x + t \log_+ T) - S(x)\big), \quad t \in [0, C]. \tag{3.1}$$

The Erdős-Rényi "new law of large numbers" (Erdős and Rényi [12], Shepp [23], Deheuvels, Devroye and Lynch [6]) shows that, under (A.1-2-3), a.s.,

$$\lim_{T\to\infty} \Big\{ \sup_{0\leq x\leq T} \xi_{x;T}(C)\Big\} = C\alpha_C^+, \quad \lim_{T\to\infty} \Big\{ \inf_{0\leq x\leq T} \xi_{x;T}(C)\Big\} = C\alpha_C^-, \tag{3.2}$$

where, in view of (2.18) and (2.23),

$$\begin{aligned} -\infty < \alpha_C^- &:= \sup\{\alpha < \mu : C\Psi(\alpha) \geq 1\} < \mu \\ < \alpha_C^+ &:= \inf\{\alpha > \mu : C\Psi(\alpha) \geq 1\} < \infty. \end{aligned} \tag{3.3}$$

Consider the set of functions of $t \in [0, C]$, defined by

$$\mathcal{F}_{T;C} := \{\xi_{x;T}(\cdot) : 0 \leq x \leq T\}. \tag{3.4}$$

The original Erdős-Rényi law (3.2), describes the a.s. limiting behavior of $\sup\{\Theta(\phi) : \phi \in \mathcal{F}_{T;C}\}$, for the functional $\Theta(\phi) = \pm\phi(C)$. The variants of this result for other choices of Θ (such as $\Theta(\phi) = \pm \sup_{0\leq t\leq C} \pm\phi(t)$) discussed by Deheuvels and Devroye [7], motivate a *functional limit theorem* (FLT), describing the a.s. limiting behavior of $\mathcal{F}_{T;C}$, as $T \to \infty$. For the statement of this FLT, some complements are needed.

We first extend the notation of Sect. 1 to *signed measures*. Let $\mathcal{B}$ (resp. $\mathcal{B}_C$) denote the set of Borel subsets of $\mathbb{R}$ (resp. $[0, C]$). Denote by $\mathcal{M}$ (resp. $\mathcal{M}_C$) the set of *Radon measures* on $\mathbb{R}^+$ (resp., on $[0, C]$). By definition $\mathcal{M}$ (resp. $\mathcal{M}_C$) is the set of continuous linear forms on the space $\mathcal{C}(\mathbb{R}^+)$ (resp. $\mathcal{C}([0, C])$ of continuous functions with compact support in $\mathbb{R}^+$ (resp. $[0, C]$), endowed with the topology $\mathcal{U}$ of uniform convergence. Namely, $(\mathcal{C}(\mathbb{R}^+), \mathcal{U})$ (resp. $(\mathcal{C}([0, C]), \mathcal{U})$) is a Banach space, of which $\mathcal{M}$ (resp. $\mathcal{M}_C$) is the topological dual. Any $\nu \in \mathcal{M}$ (resp. $\nu \in \mathcal{M}_C$) is the difference of two nonnegative Radon measures on $\mathbb{R}^+$ (resp. $[0, C]$), in $\mathcal{M}^+$ (resp. $\mathcal{M}_C^+$). In contrast, a *Borel signed-measure* on the Borel algebra $\mathcal{B}$ of subsets of $\mathbb{R}^+$ (resp. $\mathcal{B}_C$ if subsets of $[0, C]$) is a finite countably additive set-function on $\mathcal{B}$ (resp. $\mathcal{B}_C$). Any Borel signed-measure on $\mathcal{B}$ (resp. $\mathcal{B}_C$) is the difference of two nonnegative Borel measures on $\mathcal{B}$ (resp. $\mathcal{B}_C$) (see, e.g., Theorem 1.8.1, p. 19 in Kawata [15]). A Radon measure on $\mathbb{R}^+$ is not necessarily a Borel signed-measure on $\mathcal{B}$. On the other hand, a Radon measure on $[0, C]$ (resp., on any compact subset K on $\mathbb{R}^+$), is always a

Borel signed-measure on $\mathcal{B}_C$ (resp. on the Borel algebra $\mathcal{B}_K$ of subsets of K), and conversely. Because of this, unless otherwise specified, we will work on $\mathcal{M}_C$ rather than on $\mathcal{M}$, so that the Radon measures we consider are Borel signed measures and vice-versa. Below, we shall refer, for short, to $\mathcal{M}_C$, as the set of all *Radon signed-measures* on $[0, C]$.

Denote by $\mathcal{BV}_C$ the set of all right-continuous df's H_ν, of Radon signed-measures $\nu \in \mathcal{M}_C$, of the form

$$H_\nu(x) := \nu\left((-\infty, x]\right) = \nu\left([0, x]\right) \quad \text{for} \quad x \in \mathbb{R}.$$

By Lebesgue-Stieltjes integration, the map $\nu = \mathrm{d}H_\nu \in \mathcal{M}_C \leftrightarrow H_\nu \in \mathcal{BV}_C$ is one-to-one. The set $\mathcal{BV}_C$ collects the right-continuous functions H, fulfilling $H(t) = 0$ for $t < 0$, $H(t) = H(C)$ for $t \geq C$, of *bounded (total) variation* on $[0, C]$. The *total variation* of a function ϕ on $A \subseteq \mathbb{R}$ is the supremum

$$\|\mathrm{d}\phi\|_A := \sup_P \sum_{i=1}^{n} |\phi(t_i) - \phi(t_{i-1})|,$$

over all finite $P = \{t_0 < t_1 < \ldots < t_n\} \subseteq A$. By the Hahn-Jordan decomposition theorem (see, e.g., Rudin [20], p. 173), for each $\nu \in \mathcal{M}_C$, there exists a pair of disjoint measurable subsets A_ν^+, A_ν^+ of $[0, C]$, such that $A_\nu^- \cup A_\nu^+ = [0, C]$, $A_\nu^- \cap A_\nu^+ = \emptyset$, and for which the measures defined by

$$\nu^\pm : A \in \mathcal{B}_C \quad \to \quad \nu^\pm(A) := \sup_{B \in \mathcal{B}_C} \pm\nu(A \cap B) = \pm\nu\left(A \cap A_\nu^\pm\right), \tag{3.5}$$

are such that $\nu^\pm \in \mathcal{M}_C^+$. By this construction, we have

$$\nu^- \perp \nu^+, \quad \nu = \nu^+ - \nu^- \quad \text{and} \quad H_\nu = H_{\nu^+} - H_{\nu^-}. \tag{3.6}$$

By all this, for each $A \in \mathcal{B}_C$, the total variation of H_ν on A is given by $|\nu|(A) = |\mathrm{d}H_\nu|(A) = \nu^+(A) + \nu^-(A)$. For $A = [0, C]$ and $\nu \in \mathcal{M}_C$, we get

$$|\nu|([0, C]) = \nu^+([0, C]) + \nu^-([0, C]) = H_{\nu^+}(C) + H_{\nu^-}(C) < \infty. \tag{3.7}$$

In view of (1.15), the Lebesgue decomposition of $\nu^\pm$ is as follows. For each $t \geq 0$, we have

$$H_{\nu^\pm}(t) = H_{\nu^\pm_{AC}}(t) + H_{\nu^\pm_S}(t) = \int_0^t h_{\nu^\pm}(t)dt + H_{\nu^\pm_S}(t), \tag{3.8}$$

with

$$h_{\nu^\pm}(t) =:= \frac{\mathrm{d}}{\mathrm{d}t} H^\pm_{\nu_{AC}}(t) = \begin{cases} 0 & \text{for} \quad t \in A_\nu^\mp, \\ \pm h_\nu(t) \geq 0 & \text{for} \quad t \in A_\nu^\pm, \end{cases} \tag{3.9}$$

and where the df $H_{\nu_S^{\pm}}$ of the singular component $\nu_S^{\pm}$ of $\nu^{\pm}$ fulfills $\nu_S^{\pm} \perp \lambda$. We have therefore, for all $t \in \mathbb{R}$,

$$H_\nu(t) = \int_0^t h_\nu(t)\mathrm{d}t + H_{\nu_S}(t), \tag{3.10}$$

$$h_\nu(t) = h_{\nu^+}(t) - h_{\nu^-}(t) = h_{\nu^+}(t)\mathbf{1}_{\{t\in A_\nu^+\}} - h_{\nu^-}(t)\mathbf{1}_{\{t\in A_\nu^-\}}, \tag{3.11}$$

$$H_{\nu_S}(t) = H_{\nu_S^+}(t) - H_{\nu_S^-}(t) \quad \text{and} \quad \mathrm{d}H_{\nu_S^-} \perp \mathrm{d}H_{\nu_S^+}. \tag{3.12}$$

The absolutely continuous Radon signed-measures $\nu \in \mathcal{M}_C$, with $|\nu| \ll \lambda$, compose a subset of $\mathcal{M}_C$, which we denote by $\mathcal{ACM}_C$. We denote by $\mathcal{BVAC}_C$ the set of df's H_ν of signed measures $\nu \in \mathcal{ACM}_C$.
The space $\mathcal{M}_C$ of Radon signed measures on $[0, C]$, endowed with the topology, $\mathcal{W}$, of *weak* (or *weak**) *convergence*, is denoted by $(\mathcal{M}_C, \mathcal{W})$. The topology $\mathcal{W}$ on $\mathcal{M}_C$ is defined as follows. Given a directed net $(\mathcal{N}; \leq)$ of signed measures $\nu_\eta \in \mathcal{M}_C$ for $\eta \in \mathcal{N}$, and some $\nu \in \mathcal{M}_C$, we have, along $\mathcal{N}$, $\nu_\eta \to_{\mathcal{W}} \nu$, iff, for every bounded real continuous function ϕ on $\mathbb{R}$, along $\mathcal{N}$,

$$\int_{[0,C]} \phi(t)\mathrm{d}H_{\nu_\eta}(t) \quad \to \quad \int_{[0,C]} \phi(t)\mathrm{d}H_\nu(t).$$

In contrast with the case where this convergence holds in $\mathcal{M}_C^{\pm}$, the weak convergence of ν_η to ν in $\mathcal{M}_C$ is not equivalent to the convergence of $H_{\nu_n}(t)$ to $H_\nu(t)$ at each continuity point t of H_ν. Moreover, the weak* topology $\mathcal{W}$ on the topological dual of an infinite dimensional Banach space (in the present setup, the topological dual $\mathcal{M}_C$ of $(\mathcal{C}_C, \mathcal{U})$) is, in general, not metrisable, and so, we cannot hope to endow $\mathcal{M}_C$ with a metric inducing $\mathcal{W}$. This difficulty may be overridden, thanks to the observation that the totally bounded subsets of $\mathcal{M}_C$ are weak*-metrisable. Making use of this property, we will work in the following framework. For each $M > 0$, we denote by $\mathcal{M}_{C;M}$ (resp. $\mathcal{BV}_{C;M}$) the set of all $\nu \in \mathcal{M}_C$ (resp. $H_\nu \in \mathcal{BV}_C$) such that $|\nu|([0, C]) \leq M$. We set likewise $\mathcal{M}_{C;M}^{\pm} := \mathcal{M}_C^{\pm} \cap \mathcal{M}_{C;M}$, $\mathcal{I}_{C;M} := \mathcal{BV}_{C;M} \cap \mathcal{I}_C = \{H_\nu : \nu \in \mathcal{M}_{C;M}^{+}\}$, and $\mathcal{D}_{C;M} := \mathcal{BV}_{C;M} \cap \mathcal{D}_C = \{H_\nu : \nu \in \mathcal{M}_{C;M}^{-}\}$. We invoke Corollary 1, pp. 182–183 in Högnäs [14], to show that the weak topology $\mathcal{W}$ on $\mathcal{BV}_{C;M}$ may be induced by the metric $\Delta_{\mathcal{H}}$, defined, for $H_{\nu_1}, H_{\nu_2} \in \mathcal{BV}_{C;M}$, by

$$\Delta_{\mathcal{H}}(H_{\nu_1}, H_{\nu_2}) := \int_0^C |H_{\nu_1}(t) - H_{\nu_2}(t)|\mathrm{d}t + |H_{\nu_1}(C) - H_{\nu_2}(C)|. \tag{3.13}$$

The next lemma compares the metrics $\Delta_{\mathcal{H}}$, $\Delta_{\mathcal{L}}$ and $\Delta_{\mathcal{U}}$.

Proposition 3.1 *For any $\nu_1, \nu_2 \in \mathcal{M}_C$, we have*

$$\Delta_{\mathcal{H}}(H_{\nu_1}, H_{\nu_2}) \leq (C + 1)\|H_{\nu_1} - H_{\nu_2}\|. \tag{3.14}$$

If, in addition, $\nu_1, \nu_2 \in \mathcal{M}^{\pm}_{C;M}$, *then,*

$$\Delta_{\mathcal{H}}(H_{\nu_1}, H_{\nu_2}) \leq 2(C + 1 + M)\Delta_{\mathcal{L}}(H_{\nu_1}, H_{\nu_2}), \tag{3.15}$$

$$\Delta_{\mathcal{L}}(H_{\nu_1}, H_{\nu_2}) \leq \max\left\{\Delta_{\mathcal{H}}(H_{\nu_1}, H_{\nu_2}), \Delta_{\mathcal{H}}(H_{\nu_1}, H_{\nu_2})^{1/2}\right\}. \tag{3.16}$$

Proof We infer from (1.9) and (3.13), that, for any $\nu_1, \nu_2 \in \mathcal{M}_C$,

$$\Delta_{\mathcal{H}}(H_{\nu_1}, H_{\nu_2}) \leq C\|H_{\nu_1} - H_{\nu_2}\| + \|H_{\nu_1} - H_{\nu_2}\|,$$

whence (3.14). let now $\nu_1, \nu_2 \in \mathcal{M}^{+}_{C;M}$ be such that $\Delta_{\mathcal{L}}(H_{\nu_1}, H_{\nu_2}) < \epsilon$. As follows from (1.7), we have, for all $t \in \mathbb{R}$,

$$H_{\nu_2}(t) - H_{\nu_1}(t) \leq H_{\nu_1}(t+\epsilon) - H_{\nu_1}(t) + \epsilon,$$
$$H_{\nu_1}(t) - H_{\nu_2}(t) \leq H_{\nu_2}(t+\epsilon) - H_{\nu_2}(t) + \epsilon,$$

and, for $t = C$, noting that $H_{\nu_j}(t) = H_{\nu_j}(C), j = 1, 2$, for $t \geq C$, we get

$$|H_{\nu_2}(C) - H_{\nu_1}(C)| \leq 2\epsilon + H_{\nu_1}(C+\epsilon) - H_{\nu_1}(C) + H_{\nu_2}(C+\epsilon) - H_{\nu_1}(C) = 2\epsilon.$$

Likewise, since $0 \leq H_{\nu_j}(t) \leq M, j = 1, 2$, for all $t \in \mathbb{R}$,

$$\begin{aligned}
&\int_0^C |H_{\nu_2}(t) - H_{\nu_1}(t)| \mathrm{d}t \\
&\leq \int_0^C \{H_{\nu_1}(t+\epsilon) - H_{\nu_1}(t) + \epsilon\} \,\mathrm{d}t + \int_0^C \{H_{\nu_2}(t+\epsilon) - H_{\nu_2}(t) + \epsilon\} \,\mathrm{d}t \\
&\leq 2C\epsilon + \int_C^{C+\epsilon} \{H_{\nu_1}(t) + H_{\nu_2}(t)\} \,\mathrm{d}t - \int_0^{\epsilon} \{H_{\nu_1}(t) + H_{\nu_2}(t)\} \,\mathrm{d}t \\
&\leq 2C\epsilon + 2M\epsilon.
\end{aligned}$$

By combining the above inequalities, we obtain (3.15). The proof of this statement for $\nu_1, \nu_2 \in \mathcal{M}^{-}_{C;M}$ follows along the same lines.
To establish (3.16), we let $H_{\nu_1}, H_{\nu_2} \in \mathcal{I}_{C;M}$, and set $\Delta_{\mathcal{L}}(H_{\nu_1}, H_{\nu_2}) = \epsilon'$.

(1) Consider first the case where $\epsilon' > C$. Select any $C < \epsilon < \epsilon'$. In this case, we must have, for some $t \in \mathbb{R}$, either

$$H_{\nu_1}(t-\epsilon) - \epsilon \geq H_{\nu_2}(t) \quad \text{or} \quad H_{\nu_1}(t+\epsilon) + \epsilon \leq H_{\nu_2}(t).$$

If $t \leq C$, then $H_{\nu_1}(t-\epsilon) \leq H_{\nu_1}(C-\epsilon) = 0$, so that the first inequality is impossible. We must therefore have $H_{\nu_1}(t+\epsilon)+\epsilon \leq H_{\nu_2}(t)$. This last inequality is impossible for $t < 0$, so that, we must have $0 \leq t \leq C$, in which case the inequality reduces to $H_{\nu_1}(C) + \epsilon \leq H_{\nu_2}(t)$, which, in turn, implies that

$H_{\nu_2}(C) - H_{\nu_1}(C) \geq \epsilon$. Making use of a similar argument for $t > C$, and collecting both cases, we obtain that

$$\Delta_{\mathcal{H}}(H_{\nu_2}, H_{\nu_1}) \geq |H_{\nu_2}(C) - H_{\nu_1}(C)| \geq \epsilon > \epsilon' = \Delta_{\mathcal{L}}(H_{\nu_2}, H_{\nu_1}).$$

(2) Next, we treat the case where $\epsilon' \leq C$. Then, for each $\epsilon < \epsilon' \leq C$, there exists some $t \in \mathbb{R}$ such that, either $H_{\nu_2}(t) > H_{\nu_1}(t+\epsilon)+\epsilon$, or $H_{\nu_2}(t) < H_{\nu_1}(t-\epsilon)-\epsilon$. We must have therefore one of the following cases.

(i) Assume that for some $0 \leq t \leq C-\epsilon$, $H_{\nu_2}(t) = H_{\nu_1}(t+\epsilon)+\epsilon$. In this case, we see that, for all $t \leq s \leq t+\epsilon \leq C$,

$$H_{\nu_2}(s) - H_{\nu_1}(s) \geq H_{\nu_2}(t) - H_{\nu_1}(t+\epsilon) \geq \epsilon.$$

We have therefore

$$\int_0^C |H_{\nu_2}(s) - H_{\nu_1}(s)| \mathrm{d}s \geq \int_t^{t+\epsilon} \epsilon \, \mathrm{d}s = \epsilon^2.$$

(ii) Assume now that, for some $t \geq C-\epsilon$, $H_{\nu_2}(t) = H_{\nu_1}(t+\epsilon)+\epsilon$. In this case, we see that $H_{\nu_1}(t+\epsilon) = H_{\nu_1}(C)$, so that $H_{\nu_2}(C) - H_{\nu_1}(C) \geq H_{\nu_2}(t) = H_{\nu_1}(t+\epsilon) = \epsilon$, and $|H_{\nu_2}(C) - H_{\nu_1}(C)| \geq \epsilon$.

(iii) The other cases reduce to (i-ii) via the replacement of (ν_1, ν_2) by (ν_2, ν_1).

By collecting the conclusions of Cases (i-ii-iii), we see that

$$\Delta_{\mathcal{H}}(H_{\nu_2}, H_{\nu_1}) \geq |H_{\nu_2}(C) - H_{\nu_1}(C)| \geq \min\left\{\epsilon, \epsilon^2\right\} > \min\left\{\epsilon', (\epsilon')^2\right\}$$
$$= \min\left\{\Delta_{\mathcal{L}}(H_{\nu_2}, H_{\nu_1}), \Delta_{\mathcal{L}}(H_{\nu_2}, H_{\nu_1})^2\right\},$$

from where (3.16) is straightforward. □

Remark 3.2 The inequality (3.16) provides an easy proof of Corollary 2, p. 183 of Högnäs [14], showing that, whenever $\{\nu_n : n \geq 1\} \subseteq \mathcal{M}_C^{\pm}$ and $\nu \in \mathcal{M}_C^{\pm}$ are such that $\Delta_{\mathcal{H}}(H_{\nu_n}, H_\nu) \to 0$ as $n \to \infty$, then, we also have $\Delta_{\mathcal{L}}(H_{\nu_n}, H_\nu) \to 0$ as $n \to \infty$.

Proposition 3.3 *For each $C > 0$ and $M > 0$, $\mathcal{M}_{C;M}$ is compact in $(\mathcal{M}_C, \mathcal{W})$.*

Proof Let $\{\nu_n : n \geq 1\} \subseteq \mathcal{M}_{C;M}$. For each $n \geq 1$, $\nu_n = \nu_n^+ - \nu_n^-$, and

$$|\nu_n|([0,C]) = \nu_n^+([0,C]) + \nu_n^-([0,C]) \leq M,$$

so that $\nu_n^{\pm} \in \mathcal{M}_{C;M}^+$. By the arguments in the proof of Proposition 1.5, it is readily checked that $\mathcal{M}_{C;M}^+$ is compact in $(\mathcal{M}_C^+, \mathcal{W})$. Therefore, there exist two nonnegative measures $\nu^* \in \mathcal{M}_{C;M}^+$ and $\nu^{**} \in \mathcal{M}_{C;M}^+$, together with an increasing sequence

$1 \le n_1 < n_2 < \ldots$, such that, as $j \to \infty$,

$$\Delta_{\mathcal{L}}(\nu_{n_j}^{+}, \nu^{*}) \to 0 \quad \text{and} \quad \Delta_{\mathcal{L}}(\nu_{n_j}^{-}, \nu^{**}) \to 0.$$

Setting $\nu = \nu^{*} - \nu^{**}$, we infer from (3.15), that, as $j \to \infty$,

$$\begin{aligned}\Delta_{\mathcal{H}}(\nu_{n_j}, \nu) &\le \Delta_{\mathcal{H}}(\nu_{n_j}^{+}, \nu^{*}) + \Delta_{\mathcal{H}}(\nu_{n_j}^{-}, \nu^{**})\\ &\le 2(C + M + 1)\left\{\Delta_{\mathcal{L}}(\nu_{n_j}^{+}, \nu^{*}) + \Delta_{\mathcal{L}}(\nu_{n_j}^{-}, \nu^{**})\right\} \to 0.\end{aligned}$$

We so obtain that the space $\mathcal{M}_{C;M}$, endowed with the metric topology induced by $\Delta_{\mathcal{H}}$, is sequentially compact, and hence, compact. □

Define the Hausdorff set-metric pertaining to $\Delta_{\mathcal{H}}$ as follows. For each $H_\nu \in \mathcal{M}_C$ and $\epsilon > 0$, set

$$\mathcal{N}_{\epsilon;\mathcal{H}}(H_\nu) := \{G \in \mathcal{M}_C : \Delta_{\mathcal{H}}(G, H_\nu) < \epsilon\},$$

and, for each $A \subseteq \mathcal{M}_C$ and $\epsilon > 0$, set

$$A^{\epsilon;\mathcal{H}} := \bigcup_{H_\nu \in A} \mathcal{N}_{\epsilon;\mathcal{H}}(H).$$

Finally, for each $A, B \subseteq \mathcal{M}_C$, set

$$\Delta_{\mathcal{H}}(A, B) := \inf\left\{\epsilon > 0 : A \subseteq B^{\epsilon;\mathcal{H}} \text{ and } B \subseteq A^{\epsilon;\mathcal{H}}\right\}. \tag{3.17}$$

Let $\Upsilon = \Phi$ fulfill (Υ.1-2-3), and consider the set of functions of $\mathcal{M}_C$

$$\begin{aligned}\mathcal{H}_{\Phi;C} := \Big\{H_\nu \in \mathcal{BV}_C : \int_0^C \Phi\left(h_\nu(t)\right) \mathrm{d}t\\ +\rho_{0;\Phi} H_{\nu_S^{+}}(C) + \rho_{1;\Phi} H_{\nu_S^{-}}(C) \le 1\Big\}.\end{aligned} \tag{3.18}$$

Set, likewise

$$\mathcal{H}^{*}_{\Phi;C} := \left\{H_\nu \in \mathcal{AC}_C : \int_0^C \Phi\left(h_\nu(t)\right) \mathrm{d}t \le 1\right\}. \tag{3.19}$$

In (3.18), we use the convention that, when $\rho_{0;\Phi} = \infty$ (resp. $\rho_{1;\Phi} = -\infty$), the $H_\nu \in \mathcal{H}_{\Phi;C}$ fulfill $H_{\nu_S^{+}}(C) = 0$ (resp. $H_{\nu_S^{-}}(C) = 0$). When $\rho_{0;\Phi} = -\rho_{1;\Phi} = \infty$, the $H_\nu \in H_{\Phi;C}$ are absolutely continuous on $[0, C]$, so that $\mathcal{H}_{\Phi;C} = \mathcal{H}^{*}_{\Phi;C}$.

Introduce the functional on $\mathcal{M}_C \leftrightarrow \mathcal{BV}_C$, defined by

$$\mathcal{J}_{C;\Phi}(H_\nu) := \int_0^C \Phi\left(h_\nu(t)\right) \mathrm{d}t + \rho_{0;\Phi} H_{\nu_S^+}(C) + \rho_{1;\Phi} H_{\nu_S^-}(C). \tag{3.20}$$

Proposition 3.4 *Let* $\Upsilon = \Phi$ *fulfill* (Υ.1-2-3)*, with* $\gamma_\Phi = 0$*. Then for each* $C > 0$ *and* $M > 0$*,* $\mathcal{J}_{C;\Phi}$ *is lower semi-continuous on* $(\mathcal{BV}_{C;M}, \mathcal{W})$*.*

Proof Since $(\mathcal{BV}_{C;M}, \mathcal{W})$ is metric, we need only show that, for any sequence $\{\nu_n : n \geq 1\} \subseteq \mathcal{M}_{C;M}$ and $\nu \in \mathcal{M}_{C;M}$, we have

$$\lim_{n\to\infty} \Delta_{\mathcal{H}}(H_{\nu_n}, H_\nu) = 0 \quad \Rightarrow \quad \liminf_{n\to\infty} \mathcal{J}_{C;\Phi}(H_{\nu_n}) \geq \mathcal{J}^+_{C;\Phi}(H_\nu). \tag{3.21}$$

For this, we make use of the Hahn-Jordan decomposition $\nu = \nu^+ - \nu^-$ of ν, as in (3.5)–(3.6), with $\nu_\pm \in \mathcal{M}_C^+$, and observe that the condition that $\nu \in \mathcal{BV}_{C;M}$ is equivalent, via (3.7) to

$$|\nu|([0, C]) = \nu^+([0, C]) + \nu^-([0, C]) \leq M.$$

Set, for each $A \in \mathcal{B}_C$,

$$\nu^+(A) := \nu(A \cap A_\nu^+) \quad \text{and} \quad \nu^-(A) := -\nu(A \cap A_\nu^-).$$

By this definition, $H_{\nu^\pm} \in \mathcal{I}_C^+ \cap \mathcal{BV}_{C;M+C}$, and $\nu = \nu^+ - \nu^-$. In view of (3.8)–(3.10), observe that, a.e. with respect to $u \in [0, C]$,

$$h_{\nu^\pm}(u) = \frac{\mathrm{d}\nu^\pm(u)}{\mathrm{d}u} = \begin{cases} h_\nu^\pm(u) & \text{for} \quad u \in A_\nu^\pm, \\ 0 & \text{for} \quad u \in A_\nu^\mp. \end{cases}$$

Recalling that $\Phi(0) = \Phi(\gamma_\Phi) = 0$, we infer from the above relations that

$$\mathcal{J}_{C;\Phi}(H_\nu) = \mathcal{J}^+_{C;\Psi}(H_{\nu^+}) + \mathcal{J}^+_{C;\Phi}(-H_{\nu^-}). \tag{3.22}$$

Set now $\mathcal{L} := \liminf_{n\to\infty} \mathcal{J}_{C;\Phi}(H_{\nu_n})$. There exists an increasing sequence $\{n_k : k \geq 1\}$ of integers such that

$$\mathcal{L} := \lim_{k\to\infty} \mathcal{J}_{C;\Phi}(H_{\nu_{n_k}}).$$

Consider the Hahn-Jordan decomposition $\nu_n = \nu_n^- - \nu_n^-$ of $\nu_n \in \mathcal{M}_{C;M}$. Making use of the fact (see, e.g., Proposition 3.3) that $\mathcal{M}_{C;M}$ is a compact subset of $(\mathcal{M}_C, \mathcal{W})$, and via the eventual replacement of $\{n_k : k \geq 1\}$ by a properly chosen infinite subsequence, we may assume that there exist Radon signed measures $\nu \in \mathcal{M}_{C;M}$

and $\nu_*^{\pm} \in \mathcal{M}_{C;M}^{\pm}$, such that, as $k \to \infty$

$$\nu_{n_k} \overset{\mathcal{W}}{\to} \nu, \quad \nu_{n_k}^{+} \overset{\mathcal{W}}{\to} \nu_*^{+}, \quad \nu_{n_k}^{-} = \nu_{n_k} - \nu_{n_k}^{+} \overset{\mathcal{W}}{\to} \nu_*^{-} = \nu - \nu_*^{)}. \tag{3.23}$$

This implies that $\nu = \nu_*^{+} - \nu_*^{-}$, with $\nu_*^{\pm} \in \mathcal{M}_C^{+}$. By Proposition 1.6, we infer from (3.23) that

$$\liminf_{k\to\infty} \mathcal{J}_{C;\Phi}^{+}(\pm H_{\nu_{n_k}^{\pm}}) \geq \mathcal{J}_{C;\Phi}^{+}(\pm H_{\nu_*^{\pm}}). \tag{3.24}$$

On the other hand, we infer from (3.5) that, for any $A \in \mathcal{B}_C$,

$$\begin{aligned} \nu^{+}(A) &= \nu(A \cap A_\nu^{+}) = \nu_*^{+}(A \cap A_\nu^{+}) - \nu_*^{-}(A \cap A_\nu^{+}) \leq \nu_*^{+}(A \cap A_\nu^{+}), \\ 0 &= \nu(A \cap A_\nu^{-}) \leq \nu_*^{+}(A \cap A_\nu^{-}). \end{aligned}$$

We have therefore

$$\nu^{+}(A) = \nu^{+}(A \cap A_\nu^{+}) + \nu^{+}(A \cap A_\nu^{-}) \leq \nu_*^{+}(A \cap A_\nu^{+}) + \nu_*^{+}(A \cap A_\nu^{-}) = \nu_*^{+}(A).$$

A similar argument shows that, for any $A \in \mathcal{B}_C$, $\nu^{-}(A) \leq \nu_*^{-}(A)$. This implies that the measures $\nu_*^{\pm} - \nu^{\pm}$ are nonnegative. Consider the Lebesgue decompositions of these measures, given by

$$H_{\nu_*^{\pm}}(t) - H_{\nu^{\pm}}(t) = H_{\{\nu_*^{\pm} - \nu^{\pm}\}}(t) = \int_0^t h_{\{\nu_*^{\pm} - \nu^{\pm}\}}(u)\mathrm{d}u + H_{\{\nu_*^{\pm} - \nu^{\pm}\}_S}(t) \quad \text{for} \quad t \in \mathbb{R}.$$

Since $\gamma_\Phi = 0$, Φ is nonincreasing (resp. nonincreasing) on $(-\infty, 0]$ (resp. $[0, \infty)$). By all this, we see that

$$\begin{aligned} \mathcal{J}_{C;\Phi}^{+}(H_{\nu^+}) &= \int_{[0,t]\cap A_{\nu^+}} \Phi(h_\nu(u))\mathrm{d}u + \rho_{0;\Phi} \int_{[0,t]\cap A_{\nu^+}} \mathrm{d}H_{\nu_S}(u) \\ &\leq \int_{[0,t]\cap A_{\nu^+}} \Phi(h_\nu(u) + h_{\{\nu_*^{\pm} - \nu^{\pm}\}}(u))\mathrm{d}u \\ &\quad + \rho_{0;\Phi} \int_{[0,t]\cap A_{\nu^+}} \mathrm{d}\{H_{\nu_S} + H_{\{\nu_*^{\pm} - \nu^{\pm}\}_S}\}(u) \\ &= \int_{[0,t]\cap A_{\nu^+}} \Phi(h_{\nu_*^{\pm}}(u))\mathrm{d}u + \rho_{0;\Phi} \int_{[0,t]\cap A_{\nu^+}} \mathrm{d}\{H_{\{\nu_*^{\pm}\}_S}\}(u) \\ &\leq \int_{[0,t]} \Phi(h_{\nu_*^{\pm}}(u))\mathrm{d}u + \rho_{0;\Phi} \int_{[0,t]} \mathrm{d}\{H_{\{\nu_*^{\pm}\}_S}\}(u) = \mathcal{J}_{C;\Phi}^{+}(H_{\nu_*^{+}}). \end{aligned}$$

Making use of a similar argument in the "−" case, we get, via (3.22)–(3.24),

$$\mathcal{J}_{C;\Phi}(H_\nu) = \mathcal{J}^+_{C;\Psi}(H_{\nu^+}) + \mathcal{J}^+_{C;\Phi}(-H_{\nu^-}) \le \mathcal{J}^+_{C;\Psi}(H_{\nu_*^+}) + \mathcal{J}^+_{C;\Phi}(-H_{\nu_*^-})$$

$$\le \liminf_{k\to\infty} \mathcal{J}^+_{C;\Phi}(H_{\nu^+_{n_k}}) + \liminf_{k\to\infty} \mathcal{J}^+_{C;\Phi}(-H_{\nu^-_{n_k}}) \le \liminf_{k\to\infty} \mathcal{J}_{C;\Phi}(H_{\nu_{n_k}}) = \mathcal{L}.$$

To conclude, we observe that the just-proven inequality $\mathcal{J}^+_{C;\Phi}(H_\nu) \le \mathcal{L}$ does not depend upon the choice of $\{n_k : k \ge 1\}$. □

Corollary 3.5 *Let* $\Upsilon = \Phi$ *fulfill* (Υ.1-2-3). *Then, for any* $C > 0$*, there exists an* $M > 0$ *such that* $\mathcal{H}_{\Phi;C}$ *is a compact subset of* $(\mathcal{BV}_{C;M}, \mathcal{W})$.

Proof Making use of the same arguments as in the proof of Proposition 1.6, we see that an arbitrary $H_\nu \in \mathcal{H}_{\Phi;C}$ fulfills, for all $0 \le t \le C$, the inequalities

$$-\infty < \alpha^-_{C;\Phi} := \sup\{\alpha < \gamma_\Phi : \Phi(\alpha) > 1/C\} \le H_\nu(t)$$

$$\le \alpha^+_{C;\Phi} := \inf\{\alpha > \gamma_\Phi : \Phi(\alpha) > 1/C\} < \infty.$$

This suffices to show that $H_\nu(t)$ is bounded on $[0, C]$, but does not yield the proper information on the total variation $|\mathrm{d}H_\nu|$ of H_ν on $[0, C]$. To obtain the necessary bounds, we repeat this arguments for $H_{\nu^\pm}$ and with $\Phi^\pm(\alpha) := \Phi(\pm\alpha)$ for $\alpha \le 0$, and $\Phi^\pm(\alpha) := \infty$ for $\alpha < 0$. We so obtain readily the existence of an $M > 0$ so large that $\mathcal{H}_{\Phi;C} \le \mathcal{BV}_{C;M}$.
Let $\mathbb{I}$ denote the identity. Observe that

$$\{H_\nu - \gamma_\Phi \mathbb{I} : H_\nu \in \mathcal{H}_{\Phi;C}\} = \mathcal{H}_{\Phi_0;C},$$

where $\Phi_0(\alpha) := \Phi(\alpha + \gamma_\Phi)$ for $\alpha \in \mathbb{R}$. Since then $\gamma_{\Phi_0} = 0$ we may apply Proposition 3.4 to show that $\mathcal{J}_{C;\Phi_0}$ is lower semi-continuous in $(\mathcal{BV}_{C;M}, \mathcal{W})$. This implies that $\mathcal{H}_{\Phi_0;C} = \{H_\nu \in \mathcal{BV}_{C;M} : \mathcal{J}_{C;\Phi_0}(H_\nu) \le 1\}$ is closed in $(\mathcal{BV}_{C;M}, \mathcal{W})$. Since Proposition 3.3 shows that $\mathcal{BV}_{C;M}$ is compact, the conclusion is straightforward. □

Proposition 3.6 *Assume that* $\Upsilon = \Phi$ *fulfills* (Φ.1-2-3). *Fix* $C > 0$*, and let* $M > 0$ *be so large that* $\mathcal{H}_{\Phi;C} \subseteq \mathcal{BV}_{C;M}$. *Denote by* $\mathcal{H}^{**}_{\Phi;C}$ *the subset of* $\mathcal{H}^*_{\Phi;C}$ *composed of piecewise linear functions. Then,* $\mathcal{H}^{**}_{\Phi;C}$ *is dense in* $\mathcal{H}_{\Phi;C}$*, with respect to the metric topology* $(\mathcal{BV}_{C;M}, \mathcal{W})$.

Proof Making use of the argument used in the proof of Corollary 3.5, we may limit ourselves to the case where $\gamma_\Phi = 0$. Under this assumption, we infer from Corollary 3.5 that, whenever $\{H_{\nu_n} : n \ge 1\} \subseteq \mathcal{H}^{**}_{\Phi;C} \subseteq \mathcal{H}_{\Phi;C}$, there exists an $H_\nu \in \mathcal{H}_{\Phi;C}$, together with an increasing sequence of integers $1 \le n_1 < n_2 < \ldots$, such that, as $k \to \infty$,

$$H_{\nu_k} \overset{\mathcal{W}}{\to} H_\nu.$$

Conversely, if $H_\nu \in \mathcal{H}_{\Phi;C} \subseteq \mathcal{BV}_{C;M}$ is arbitrary, we write the Hahn-Jordan decomposition $\nu = \nu^+ - \nu^-$ of ν, and make use of Proposition 1.6 and Remark 1.7 to show the existence of two sequences $\{\nu_n^\pm : n \geq 1\} \in \mathcal{M}_C^+$, such that $\nu_n^\pm \in \mathcal{H}_{\Phi;C}^{**}$, and, as $n \to \infty$

$$H_{\nu_n^\pm} \overset{\mathcal{W}}{\to} H_{\nu^\pm}.$$

Namely, with P_n as in the proof of Proposition 1.6, we set

$$H_{\nu_n^\pm} := H_{\nu^\pm}^{P_n} \quad \text{for} \quad n \geq 1.$$

Set $\nu_n = \nu_n^+ - \nu_n^-$ for $n \geq 1$. By (1.36)–(1.38), this choice ensures that

$$\begin{aligned}
&\mathcal{J}_{C;\Phi}(\nu_n) = \mathcal{J}_{C;\Phi}(\nu_n^+) + \mathcal{J}_{C;\Phi}(-\nu_n^-) \quad \text{for} \quad n \geq 1,\\
&\nu_n^\pm \overset{\mathcal{W}}{\to} \nu^\pm \quad \text{as} \quad n \to \infty,\\
&\mathcal{J}_{C;\Phi}(\pm\nu_n^\pm) \leq \mathcal{J}_{C;\Phi}(\pm\nu^\pm),\\
&\mathcal{J}_{C;\Phi}(\pm\nu_n^\pm) \to \mathcal{J}_{C;\Phi}(\pm\nu^\pm) \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Making use of the triangle inequality, we write

$$\Delta_{\mathcal{H}}(H_\nu, H_{\nu_n}) \leq \Delta_{\mathcal{H}}(H_{\nu^+}, H_{\nu_n^+}) + \Delta_{\mathcal{H}}(H_{\nu^-}, H_{\nu_n^-}) \to 0,$$

and

$$\begin{aligned}
\mathcal{J}_{C;\Phi}(\nu_n) &= \mathcal{J}_{C;\Phi}(\nu_n^+) + \mathcal{J}_{C;\Phi}(-\nu_n^-)\\
&\leq \mathcal{J}_{C;\Phi}(\nu^+) + \mathcal{J}_{C;\Phi}(-\nu^-) = \mathcal{J}_{C;\Phi}(\nu_n) \leq 1.
\end{aligned}$$

This shows that $H_{\nu_n} \in \mathcal{H}_{\Phi;C}^{**}$, and that $H_{\nu_n} \overset{\mathcal{W}}{\to} H_\nu$ as $n \to \infty$. The proof of Proposition 3.6 is, therefore, completed. □

We shall make use of Theorems 3.1 and 3.2 of Deheuvels [6], which give a general form of an Erdős-Rényi FLT for the partial sum process, stated below.

Theorem 3.7 *Under* (A.1-2-3)*, for any* $C > 0$*, there exists an* $M < \infty$ *such that, almost surely for all* T *sufficiently large,* $\mathcal{F}_{T;C} \subseteq \mathcal{BV}_{C;M}$*. Moreover, we have, almost surely,*

$$\lim_{T\to\infty} \Delta_{\mathcal{H}}\left(\mathcal{F}_{T;C}, \mathcal{H}_{\Phi;C}^*\right) = \lim_{T\to\infty} \Delta_{\mathcal{H}}\left(\mathcal{F}_{T;C}, \mathcal{H}_{\Phi;C}\right) = 0. \qquad (3.25)$$

If, in addition, $(A.4)$ *holds, then, almost surely* $\mathcal{F}_{T;C} \subseteq \mathcal{I}_C^+$*, and* $\mathcal{H}_{\Phi;C} \subseteq \mathcal{I}_C^+$*. We then have, almost surely,*

$$\lim_{T\to\infty} \Delta_{\mathcal{L}}\left(\mathcal{F}_{T;C}, \mathcal{H}^*_{\Phi;C}\right) = \lim_{T\to\infty} \Delta_{\mathcal{L}}\left(\mathcal{F}_{T;C}, \mathcal{H}_{\Phi;C}\right) = 0. \tag{3.26}$$

Under $(A.1\text{-}2\text{-}3)$*, when* $t_1 = -\infty$ *and* $t_0 = \infty$*, we have* $\mathcal{H}_{\Phi;C} = \mathcal{H}^*_{\Phi;C}$*, and*

$$\lim_{T\to\infty} \Delta_{\mathcal{U}}\left(\mathcal{F}_{T;C}, \mathcal{H}_{\Phi;C}\right) = 0. \tag{3.27}$$

Remark 3.8 The limit law (3.27), due to Borovkov [3], was shown by Deheuvels [6] to follow from (3.25). Sanchis [21, 22] rediscovered (3.27), and gave an other proof of this result.

In order to prove Theorem 1.2, we first establish the lemma:

Lemma 3.9 *Fix* $C > 0$ *and* $M > 0$*, and consider the mapping*

$$H_\nu \in \mathcal{P} : \mathcal{BC}_{C;M} \to \mathcal{P}(H_\nu) \in \mathcal{I}_C,$$

defined by

$$\mathcal{P}(H_\nu)(t) := \sup_{0\le s\le t} H_\nu(s) \quad \mathit{for} \quad 0 \le s \le t.$$

Then $\mathcal{P}$ *is a continuous mapping of* $(\mathcal{BV}_{C;M}, \mathcal{U})$ *onto* $(\mathcal{I}_C, \mathcal{W})$*.*

Remark 3.10 We conjecture that the mapping in Lemma 3.9 is continuous with respect to the weak topology. This is a crucial point in the extension of the statement of Theorem 1.2 to situations where $(A.4)$ is not fulfilled.

Proof Obviously, we have, for $H_{\nu_1}, H_{\nu_2} \in \mathcal{BV}_{C;M}$

$$\sup_{0\le t\le 1}\left|\left\{\sup_{0\le s\le t} H_{\nu_1}(s)\right\} - \left\{\sup_{0\le s\le t} H_{\nu_2}(s)\right\}\right| \le \sup_{0\le s\le 1}\left|H_{\nu_1}(s) - H_{\nu_2}(s)\right|,$$

so that $\Delta_{\mathcal{U}}\left(\mathcal{P}(H_{\nu_1}), \mathcal{P}(H_{\nu_2})\right) \le \Delta_{\mathcal{U}}\left(H_{\nu_1}, H_{\nu_2}\right)$, which completes our proof. □

Lemma 3.11 *Let* $\Upsilon = \Phi$ *fulfill* $(\Upsilon.1\text{-}2\text{-}3)$*, with* $\gamma_\Phi > 0$*, and set*

$$\Phi_+(\alpha) = \begin{cases} \Phi(\alpha) & \mathit{for} \quad \alpha \ge 0, \\ \infty & \mathit{for} \quad \alpha < 0. \end{cases} \tag{3.28}$$

Then, $\Upsilon = \Phi_+$ *fulfills* $(\Upsilon.1\text{-}2\text{-}3)$ *with* $\gamma_{\Phi_+} = \gamma_\Phi$*. Moreover, we have*

$$\mathcal{P}\left(\mathcal{H}_{\Phi;C}\right) = \mathcal{H}_{\Phi_+;C}. \tag{3.29}$$

Proof The fact that $\Upsilon = \Phi_+$ fulfills (Υ.1-2-3) with $\gamma_{\Phi_+} = \gamma_\Phi$ when $\gamma_\Phi > 0$ is straightforward. We have $\rho_{0;\Phi_+} = \rho_{0;\Phi}$ and $\rho_{0;\Phi_+} = \infty$. The remainder of the proof is straightforward and omitted. □

Proof of Theorem 1.2 Consider first the case where (A.1-2-3-4) hold. In this case, for each choice of $C > 0$ and all $T > 0$ sufficiently large, $\mathcal{F}_{T;C} \subseteq \mathcal{I}_C$, and (3.26) holds. We then combine Proposition 1.6 and Remark 1.7, with Proposition 2.12 and the inversion argument in Deheuvels and Mason [8], to obtain (1.18). For the second part of the theorem, we observe that, under the condition $\Psi(0) = \infty$, we have, by (2.37), $\rho_{0;\Psi^*} = \infty$ and $\rho_{1;\Psi^*} = -\infty$. Therefore, in this case, $K^+_{\Psi^*_C} = K^{+*}_{\Psi^*_C}$. □

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Limit Theorems for Quantile and Depth Regions for Stochastic Processes

James Kuelbs and Joel Zinn

Abstract Since contours of multi-dimensional depth functions often characterize the distribution, it has become of interest to consider structural properties and limit theorems for the sample contours [see Zuo and Serfling (Ann. Stat. 28(2):483–499, 2000) and Kong and Mizera (Stat. Sin. 22(4):1589–1610, 2012)]. In particular, Kong and Mizera have shown that for finite dimensional data, directional quantile envelopes coincide with the level sets of half-space (Tukey) depth. We continue this line of study in the context of functional data, when considering analogues of Tukey's half-space depth (Tukey, Mathematics and the picturing of data, in Proceedings of the International Congress of Mathematicians (Vancouver, BC, 1974), vol. 2 (Canadian Mathematical Congress, Montreal, QC, 1975), pp. 523–531). This includes both a functional version of the equality of (directional) quantile envelopes and quantile regions as well as limit theorems for the sample quantile regions up to $\sqrt{n}$ asymptotics.

Keywords Depth region • Quantile region

Mathematics Subject Classification (2010). Primary 60F05; Secondary 60F17, 62E20

1 Introduction

The study of depth functions and resulting depth regions provide what is called a center-outward order for multidimensional data that allows one to gain insight into the underlying probability law. Many of the results obtained are for depths in $\mathbb{R}^d$,

J. Kuelbs (✉)
Department of Mathematics, University of Wisconsin, Madison, WI 53706-1388, USA
e-mail: kuelbs@math.wisc.edu

J. Zinn
Department of Mathematics, Texas A&M University, College Station, TX 77843-3368, USA
e-mail: jzinn@math.tamu.edu

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but more recently functional data, such as that obtained from streaming data, has received considerable attention.

When trying to assess the viability of a particular statistical approach, it is important to analyze the method when applied to specific data, asking if the empirical version is consistent, and what rates of convergence one might have. In the case of functional data this often leads to considering the data functions at a discrete set of points—perhaps a large number of points in the domain of the functions. However, if the infinite dimensional process (or measure with infinite dimensional support) is unstable in some way with respect to the method one is using to order the typical functions being encountered, there is the question of what one might be modeling in these situations. In this paper, as in the development of general empirical processes, we are interested in some basic probabilistic properties of the statistics being proposed for half-space depth and the related quantile processes, when no restrictions on the domain of the functions are imposed. If one has stability for the statistic being employed to provide the infinite dimensional ordering, this alleviates some of the concerns just mentioned.

Papers that provided motivation, and some useful contrasts for what we do in this paper, include [5, 12–14]. In addition, the papers [20, 21] examine a number of unifying properties of a broad collection of such depths, and in [21] some convergence results for the related depth regions and their boundaries are established for $\mathbb{R}^d$-valued data. They also contain an extensive list of references. The paper [5] also provides results obtaining convergence of Tukey half-space depth regions (see also [18]) for $\mathbb{R}^d$-valued data under conditions that are quite different from those in [21], but in each setting something close to a "law of large numbers" is an important assumption required for the proofs. An assumption of a similar nature appears in (3.14) and (3.15) of Theorem 3.9, and can be verified in many situations by applying the empirical quantile CLTs for stochastic processes obtained in [7, 8]. These CLTs, along with the approach in [5], were a primary motivation for the various limit theorems we establish here. In particular, the results in [7, 8] allow us to obtain $\sqrt{n}$-asymptotics for the convergence of the half-space depth sets for many types of functional data. This includes data given by a broad collection of Gaussian processes, martingales, and independent increment processes. Furthermore, the limit theorems obtained have Gaussian limits uniformly over the parameter set of the data process and in the quantile levels $\alpha \in I$ for I a closed interval of $(0, 1)$, and are established directly without first introducing a corresponding half-space depth. This is in contrast with the limit theorems for empirical medians in [13, 14] based on the argmax of empirical Tukey depth processes, which have non-Gaussian limits for data in $\mathbb{R}^d$ when $d \geq 2$. A first CLT of this type was obtained in [15] for the empirical median process when the data was a sample continuous Brownian motion on $[0, 1]$, and later in [16] for the empirical α-quantile processes for each fixed $\alpha \in (0, 1)$. The proofs in these papers are quite different than those in [7] and [8], which employ empirical process theory as developed for functional data in [11], a method of Vervaat from [19], the CLT results in [1], and the necessary and sufficient conditions for sample function continuity of a Gaussian process using generic chaining as in [17].

A brief outline of the paper is as follows. In Sect. 2 we introduce basic notation. Section 3 introduces additional notation and states the main results of this paper, namely Proposition 3.7 and Theorem 3.9, which indicate how half-space depth regions for stochastic processes based on evaluation maps are uniquely determined by related upper and lower quantile functions for the process. In addition, under suitable conditions they show the empirical versions of these regions converge to the population versions with respect to a Hausdorff metric (also used for finite dimensional data in [5, 21]), and include both consistency results and $\sqrt{n}$-rates of convergence for these distances.

As mentioned above, the main assumptions required in the proof of Theorem 3.9 can be verified by applying the empirical quantile CLTs in [7, 8] for many types of functional data, but we also obtain some consistency results for empirical quantile functions in Theorem 3.16 and Corollary 3.18. They are of independent interest, and can be used in this setting to verify conditions (3.14) and (3.15) of Theorem 3.9. As is natural to expect, these consistency results are obtained under weaker conditions. However, they do not yield the $\sqrt{n}$-rates of convergence given in Theorem 3.9, which follow when one can apply the CLT results.

Theorem 3.9 applies quite generally, but the limiting regions can often be very small. This is pointed out in Sect. 4, where we examine half-space depth regions based on data obtained by independently sampling continuous Brownian motions, showing that although Theorem 3.9 applies, the quantile regions and depth regions have probability zero. That these regions may have zero probability is a problem which holds for many other processes (see Remark 4.1), and hence motivates our Proposition 4.2. This result shows that if we suitably smooth the one dimensional distributions, then one can avoid this problem for many stochastic processes. Throughout the paper when we speak of smoothing a stochastic process we mean that we are applying the smoothing of Proposition 4.2.

We postpone proofs to Sect. 5, but a number of remarks are included in earlier sections to motivate and understand how the results fit together. Section 4 also provides a brief comparison with the results used to eliminate zero depth in [3], our Proposition 4.2, and also the analogue in Proposition 4 of Kuelbs and Zinn [9].

2 Basic Notation

Throughout the paper E is a nonempty set, $D(E)$ a collection of real-valued functions on E, $\mathcal{D}_E$ is the minimal sigma-algebra making the evaluation maps $\theta_t : D(E) \to \mathbb{R}$ measurable, where

$$\theta_t(z) = z(t), t \in E, z \in D(E), \tag{2.1}$$

and μ is a probability measure on $(D(E), \mathcal{D}_E)$. Of course, the (functional) data of interest are drawn from $D(E)$ and μ is the population distribution or law on $\mathcal{D}_E$ of the data. It will also be convenient to have i.i.d. stochastic processes $X := \{X(t) :$

$t \in E\}, X_1, X_2, \cdots$ on some probability space $(\Omega, \mathcal{F}, P)$ such that the common law they induce on $(D(E), \mathcal{D}_E)$ is μ. For each $t \in E$, we denote the distribution function of $X(t)$ by $F_t(x) := F(t, x), x \in \mathbb{R}$. In addition, without loss of generality we assume that the sample paths of these processes are always in $D(E)$, and for $n \geq 1$ denote the empirical measures for μ on $(D(E), \mathcal{D}_E)$ by

$$\mu_n(\omega) = \frac{1}{n} \sum_{j=1}^{n} \delta_{X_j(\omega)}, \omega \in \Omega. \tag{2.2}$$

At this stage the set E and the set of functions $D(E)$ are purposely abstract. An important special case in [10] takes E to be the linear functions on $\mathbb{R}^d$ of Euclidean norm one, and $D(E)$ a subset (usually subspace) of the continuous functions on E containing $\{z : z(t) = t(x), t \in E, x \in \mathbb{R}^d\}$. In this setting the results of this paper apply to Tukey depth regions (and the related quantile regions), implying new empirical results for this important classical depth. The reason one does not proceed in a similar manner in the infinite dimensional case is that this choice of E is too large to always have the empirical quantile CLTs of Kuelbs and Zinn [7] applicable in this setting, and also the resulting half-space depth may be zero with probability one. For example, Proposition 3.6 of Kuelbs and Zinn [6] provides an explicit formula for half-space depth for Gaussian measures on a Banach space which is zero except for points in a set of measure zero when all continuous linear functionals or, equivalently, all continuous linear functionals of norm one, are used to define E. The papers [3, 6, 9] contain other examples where zero depths appear, but [9] also shows how to alleviate this problem using a smoothing (a random numerical shift) of the data by establishing a result similar to that in Proposition 4.2 below. In particular, Theorems 1 and 2 of Kuelbs and Zinn [9] examine other aspects of this sort of problem, and establish limit theorems for half-region depth when the set E satisfies a compactness condition. It also provides some examples which show that the smoothing method we use avoids some non-intuitive properties that the modified half-region depth proposed in other papers possesses. See Examples 1–3 in [9] for details as well as some comments in Sect. 4 below that provide additional details on the zero depth problem as handled in [3].

To describe the quantile and depth regions in our results we now recall the definition of left and right α-quantiles for real-valued random variables.

Definition 2.1 Let ξ be a real-valued random variable with Borel probability law μ_ξ, and for $x \in \mathbb{R}$ set $F_\xi(x) = P(\xi \leq x)$. Then, for $\alpha \in (0, 1)$, the left and right α-quantiles of ξ (equivalently, of the distribution function F_ξ or the probability law μ_ξ) are defined, respectively, as

$$\tau_{\alpha,l}(\xi) := \tau_{\alpha,l}(F_\xi) := \tau_{\alpha,l}(\mu_\xi) := \inf\{x : F_\xi(x) \geq \alpha\} \text{ and} \tag{2.3}$$

$$\tau_{\alpha,r}(\xi) := \tau_{\alpha,r}(F_\xi) := \tau_{\alpha,r}(\mu_\xi) := \sup\{x : F_\xi(x^-) \leq \alpha\} = \inf\{x : F_\xi(x) > \alpha\}.$$

Next we turn to the definition of the left and right α-quantile functions determined by a measure ν on $(D(E), \mathcal{D}_E)$. In Remarks 2.3 and 2.4 that follow we indicate some simplifications of this notation that we employ for our "fixed" measure μ and its empirical measures $\mu_n(\omega)$.

Definition 2.2 Let ν be a probability measure on $(D(E), \mathcal{D}_E)$, $\{\theta_t : t \in E\}$ denote the evaluation maps in (2.1), and for each $t \in E$ the distribution function of θ_t with respect to ν is F_{θ_t}. Then, for $(\alpha, t) \in (0, 1)\times E$, the left and right α-quantile functions determined by ν are

$$\tau_{\alpha,l}(t, \nu) := \tau_{\alpha,l}(\theta_t) := \tau_{\alpha,l}(F_{\theta_t}) := \inf\{x: \nu(f \in D(E): \theta_t(f) \leq x) \geq \alpha\} \text{ and} \tag{2.4}$$

$$\tau_{\alpha,r}(t, \nu) := \tau_{\alpha,r}(\theta_t) := \tau_{\alpha,r}(F_{\theta_t}) := \inf\{x: \nu(f \in D(E): \theta_t(f) \leq x) > \alpha\}. \tag{2.5}$$

Remark 2.3 If the measure ν is our "fixed" measure, μ, we simplify to

$$\tau_{\alpha,l}(t) := \tau_{\alpha,l}(t, \mu) \text{ and } \tau_{\alpha,r}(t) := \tau_{\alpha,r}(t, \mu). \tag{2.6}$$

In case we also have $\tau_{\alpha,l}(t) = \tau_{\alpha,r}(t)$ for all $t \in E$, then to denote their common value we simply write

$$\tau_\alpha(t), t \in E, \tag{2.7}$$

and note that $\tau_\alpha(t)$ is the unique function $f(t)$ on E such that for each $t \in E$, $f(t)$ is the left α-quantile of the random variable $\theta_t(\cdot)$ on $(D(E), \mathcal{D}_E, \mu)$. In addition, note that if $X := \{X(t) : t \in E\}$ is a stochastic process with sample paths in $D(E)$ that induces law μ on $(D(E), \mathcal{D}_E)$, then the left and right α-quantile functions determined by μ can be defined by the distribution functions $F_t(x) := F(t, x) = P(X(t) \leq x)$, since $F_t = F_{\mu,\theta_t}, t \in E$.

Remark 2.4 If our measure is $\mu_n(\omega)$, we usually leave out the ω and write

$$\tau^n_{\alpha,l}(t) := \tau_{\alpha,l}(t, \mu_n) \text{ and } \tau^n_{\alpha,r}(t) := \tau_{\alpha,r}(t, \mu_n). \tag{2.8}$$

However, as in the proof of Theorem 3.9, there are times when including ω is helpful, and we then write

$$\tau^n_{\alpha,l}(t, \omega) := \tau_{\alpha,l}(t, \mu_n(\omega)) \text{ and } \tau^n_{\alpha,r}(t, \omega) := \tau_{\alpha,r}(t, \mu_n(\omega)). \tag{2.9}$$

Moreover, the empirical quantiles are such that for each $\omega \in \Omega$, $t \in E$

$$\tau^n_{\alpha,l}(t, \omega) = \inf\{x: \frac{1}{n}\sum_{j=1}^{n} I_{X_j(t,\omega)\leq x)} \geq \alpha\} \text{ and} \tag{2.10}$$

$$\tau^n_{\alpha,r}(t, \omega) = \inf\{x: \frac{1}{n}\sum_{j=1}^{n} I_{X_j(t,\omega)\leq x)} > \alpha\}.$$

3 Equality of Quantile and Depth Regions, and Convergence Results

In order to state our results on half-space quantile and depth regions for functional data (or stochastic processes), the following lemma recalls some basic facts about right and left quantiles of a real-valued random variable.

Lemma 3.1 *Let ξ be a real-valued random variable with distribution function F_ξ. Then, for $\alpha \in (0, 1)$,*

$$\tau_{1-\alpha,r}(\xi) = -\tau_{\alpha,l}(-\xi), \tag{3.1}$$

and if $F_\xi(x) = \alpha$ for some $x > \tau_{\alpha,l}(\xi)$, then $F_\xi(x) = \alpha$ for all $x \in [\tau_{\alpha,l}(\xi), \tau_{\alpha,r}(\xi))$. In addition, if $x \in [\tau_{\alpha,l}(\xi), \tau_{\alpha,r}(\xi))$, then x is an α-quantile of ξ, i.e. $F_\xi(x) \geq \alpha$ and $1 - F_\xi(x^-) \geq 1 - \alpha$.

Next we turn to the definitions of α-quantile regions, half-space depths, and α-depth regions.

Definition 3.2 Let ν be a probability measure on $(D(E), \mathcal{D}_E)$ with left and right α-quantile functions as in (2.4). If $\alpha \in (0, \frac{1}{2}]$, then the α-quantile region (with respect to ν) is the subset of $D(E)$ given by

$$M_{\alpha,\nu} := \cap_{t\in E}\{z \in D(E) : \tau_{\alpha,l}(t, \nu) \leq z(t) \leq \tau_{1-\alpha,r}(t, \nu)\}. \tag{3.2}$$

Remark 3.3 If the measure ν is our "fixed" measure, μ, we simplify to

$$M_\alpha := \cap_{t\in E}\{z \in D(E) : \tau_{\alpha,l}(t) \leq z(t) \leq \tau_{1-\alpha,r}(t)\}, \tag{3.3}$$

where the left and right quantiles are (with respect to μ) as in (2.6), and again ignore its dependence on μ.

Remark 3.4 If the measure ν is $\mu_n(\omega)$, we usually leave out the ω. That is, for $\alpha \in (0, \frac{1}{2}]$ and $n \geq 1$, the empirical α-quantile region (with respect to μ_n) is denoted by

$$M_{\alpha,n} := \cap_{t\in E}\{z \in D(E) : \tau^n_{\alpha,l}(t) \leq z(t) \leq \tau^n_{1-\alpha,r}(t)\}, \tag{3.4}$$

where empirical α-quantile functions $\tau^n_{\alpha,l}(t)$ and $\tau^n_{1-\alpha,r}(t)$ are as in (2.8). Of course, these functions and $M_{\alpha,n}$ are defined for all $\omega \in \Omega$, and when that dependence needs special emphasis we will write $\tau^n_{\alpha,l}(t, \omega)$, $\tau^n_{\alpha,r}(t, \omega)$, and $M_{\alpha,n}(\omega)$, respectively. Otherwise the dependence on ω will be suppressed.

Definition 3.5 Let ν be a probability measure on $(D(E), \mathcal{D}_E)$. For any real-valued function, h, on E, we define the half-space depth of h with respect to ν and the evaluation maps $\theta_t, t \in E$, by

$$D(h, \nu) := \inf_{t \in E} \min\{\nu(z \in D(E) : z(t) \geq h(t)), \nu(z \in D(E) : z(t) \leq h(t))\}, \tag{3.5}$$

and the α-depth regions by

$$N_{\alpha,\nu} := \{h \in D(E) : D(h, \nu) \geq \alpha\}. \tag{3.6}$$

Remark 3.6 If the measure ν is $\mu_n(\omega)$, we usually leave out the ω. That is, for $n \geq 1$ the empirical half-space depths with respect to μ_n are denoted by $D(h, \mu_n)$ and the empirical α-depth regions are

$$N_{\alpha,n} := \{h \in D(E) : D(h, \mu_n) \geq \alpha\}. \tag{3.7}$$

When the dependence on ω needs special emphasis we will write $D(h, \mu_n)(\omega)$ and $N_{\alpha,n}(\omega)$, respectively. Moreover, the depth $D(h, \cdot)$ not only depends on the measure, but is defined in terms of the evaluation maps indexed by E, so in what follows we may also refer to it as the E-depth with respect to the measure involved, or simply as E-depth. In addition, when the stochastic process X induces the law μ on $(D(E), \mathcal{D}_E)$, we have

$$D(h, \mu) = \inf_{t \in E} \min\{P(X(t) \geq h(t)), P(X(t) \leq h(t))\}. \tag{3.8}$$

We write Λ^* to denote the measurable cover function of a real-valued function Λ on Ω, see [4], and for U, V subsets of $D(E)$ we denote the Hausdorff distance between U and V (with respect to the sup-norm on $D(E)$) by

$$d_H(U, V) = \inf\{\epsilon > 0 : U \subseteq V^\epsilon \text{ and } V \subseteq U^\epsilon\}, \tag{3.9}$$

where $U^\epsilon = \{z \in D(E) : \inf_{h \in U} \sup_{t \in E} |z(t) - h(t)| < \epsilon\}$. If U or V is empty, but not both, then $d_H(U, V) = \infty$.

3.1 Equality of Depth and Quantile Regions

The following proposition shows certain quantile regions are equal to related depth regions. The proposition is quite general, and also applies to the empirical quantiles and the related empirical depths.

Proposition 3.7 *Assume the notation in Sect. 2, and (3.2)–(3.7). Then, for* $\alpha \in (0, \frac{1}{2}]$ *and* ν *any probability measure on* $(D(E), \mathcal{D}_E)$ *we have*

$$M_{\alpha,\nu} = N_{\alpha,\nu}. \tag{3.10}$$

In particular, the α-quantile regions and the α-depth regions with respect to μ and also the empirical measures μ_n are such that

$$M_\alpha = N_\alpha, \tag{3.11}$$

and for $n \geq 1, \omega \in \Omega$,

$$M_{\alpha,n}(\omega) = N_{\alpha,n}(\omega). \tag{3.12}$$

Remark 3.8 Although Proposition 3.7 holds quite generally, it is important to note that there are many examples where the sets in (3.11) are small. In fact, they may have μ probability zero for all $\alpha > 0$. We present such an example in Sect. 4, but many of the examples discussed in [6] also have similar properties. Nevertheless, it is important to keep in mind that if one wants to examine quantile regions of the type in Proposition 3.7, then some variety of half-space depth emerges. Hence, we will follow this example by showing the sets involved in (3.11) are much larger for smooth versions of the data, where before smoothing they possibly had positive half space depth with probability zero. Finally, with $\tau_{0,l}(t) = -\infty$ and $\tau_{1,r}(t) = +\infty$ for all $t \in E$, which are their natural definitions, we easily see $M_0 = D(E)$. Hence (3.11) also h olds for $\alpha = 0$, since $\{h \in D(E) : D(h, \mu) \geq 0\} = D(E)$. If $D(E)$ is assumed to be a linear space, then the maps $\theta_t, t \in E$, are linear from $D(E)$ into $\mathbb{R}$, and hence M_α is convex. If $D(E)$ has a topology such that these maps are continuous, then M_α is also closed. Of course, from (3.11) the sets N_α then have similar properties.

3.2 Empirical Regions Converge

In Proposition 3.7, $D(E)$ is quite arbitrary, except that it supports the probability measure $\mu = \mathcal{L}(X)$. However, for many standard stochastic processes $X := \{X(t) : t \in E\}$ the set E is a compact interval of the real line or a compact subset of some metric space, and its sample paths may well be continuous, cadlag, or at least uniformly bounded on E. Hence, in such cases we can take $D(E)$ to be the Banach space $\ell^\infty(E)$ with sup-norm $||h||_\infty = \sup_{t\in E} |h(t)|$, or some closed linear subspace of $\ell^\infty(E)$ of smoother functions that reflect the regularity of the sample paths of X. The choice of $D(E) = \ell^\infty(E)$ is convenient in that weak convergence results for empirical processes are readily available in this setting. Moreover, if $D(E) = \ell^\infty(E)$ and the sample paths of X are in $\ell^\infty(E)$, then stochastic boundedness of X and a fairly immediate argument, see the proof of Corollary 3.11, implies for given $\alpha \in (0, \frac{1}{2}]$ and all $\omega \in \Omega$ that

$$\tau_{\alpha,l}(\cdot),\ \tau_{1-\alpha,r}(\cdot),\ \tau^n_{\alpha,l}(\cdot),\ \tau^n_{1-\alpha,r}(\cdot) \in D(E). \tag{3.13}$$

If $D(E)$ is a closed subspace of $\ell^\infty(E)$, then under suitable conditions we can still verify (3.13), but these arguments are more subtle. Hence in the next result (3.13) is an assumption, but following its statement we present some corollaries where (3.13) is verified directly.

Theorem 3.9 *Let $D(E)$ be a closed linear subspace of $\ell^\infty(E)$ with respect to the sup-norm $||\cdot||_\infty$ such that for some $\alpha \in (0, \frac{1}{2}]$ and all $\omega \in \Omega$ (3.13) holds, and the measurable cover functions*

$$\|\tau^n_{\alpha,l}(\cdot) - \tau_{\alpha,l}(\cdot)||^*_\infty \tag{3.14}$$

and

$$||\tau^n_{1-\alpha,r}(\cdot) - \tau_{1-\alpha,r}(\cdot)||^*_\infty \tag{3.15}$$

converge in probability to zero with respect to P. Then, for the given $\alpha \in (0, \frac{1}{2}]$ the sets $M_\alpha, N_\alpha, M_{\alpha,n}$, and $N_{\alpha,n}$ are non-empty, and the measurable cover functions (of the Hausdorff distances)

$$d^*_H(M_{\alpha,n}, M_\alpha) = d^*_H(N_{\alpha,n}, N_\alpha) \tag{3.16}$$

converge in probability to zero with respect to P. In addition, if $1 \leq a_n = O(\sqrt{n})$ converges to infinity, and the measurable cover functions

$$a_n||\tau^n_{\alpha,l}(\cdot) - \tau_{\alpha,l}(\cdot)||^*_\infty \text{ and } a_n||\tau^n_{1-\alpha,r}(\cdot) - \tau_{1-\alpha,r}(\cdot)||^*_\infty \tag{3.17}$$

are bounded in probability, then

$$a_n d^*_H(M_{\alpha,n}, M_\alpha) \text{ and } a_n d^*_H(N_{\alpha,n}, N_\alpha) \tag{3.18}$$

are bounded in probability with respect to P.

Remark 3.10 The assumptions (3.14) and (3.15) or (3.17) in Theorem 3.9 are non-trivial, but by applying the results in [7, 8] one can obtain a broad collection of stochastic processes for which they can be verified for all $\alpha \in (0, 1)$ with best possible a_n, namely $a_n = \sqrt{n}$. In particular, this includes Levy's area process (and other iterated self-similar processes), symmetric stable processes with stationary independent increments, and m-times iterated Brownian motion. In addition, at least in some situations, one can apply the results in [6] to identify depth regions, which when combined with Proposition 3.7 allow us to determine the related quantile regions for these processes. In Sect. 4 we examine the combined effect of Proposition 3.7 and Theorem 3.9 when they are applied to sample continuous Brownian motion. Similar results also hold for a large number of other stochastic processes, some of which are mentioned in Sect. 4.

Corollary 3.11 *Let $\mu = \mathcal{L}(X)$, $D(E) = \ell^\infty(E)$, and assume $\alpha \in (0, \frac{1}{2}]$ is such that there exists $x_\alpha \in (0, \infty)$ with*

$$\sup_{t \in E} P(|X(t)| \geq x_\alpha) < \alpha. \tag{3.19}$$

Then, (3.14) and (3.15) imply (3.16), and (3.17) implies both (3.16) and (3.18) hold.

The next two corollaries verify Theorem 3.9 for $D(E)$ a closed linear subspace of $\ell^\infty(E)$ consisting of smooth functions, provided it is assumed that the sample paths of X and both $\tau_{\alpha,l}, \tau_{1-\alpha,r}$, satisfy the same smoothness conditions. Of course, the assumption that $\tau_{\alpha,l}, \tau_{1-\alpha,r}$, are suitably smooth, can be verified by imposing suitable assumptions on the distribution functions $F(t,x) = P(X_t \leq x)$. For example, in [8] this follows immediately from the self-similarity property assumed on the stochastic process X, but it can also be verified for processes which are not self-similar by the following lemma.

Lemma 3.12 *Let (E, d) be a metric space, for each $t \in E$ assume that*

$$\lim_{s \to t} \sup_{x \in \mathbb{R}} |F(s,x) - F(t,x)| = 0, \tag{3.20}$$

and that for a given $\alpha \in (0,1)$ there exists $\theta(\alpha) > 0$ such that

$$\inf_{t \in E, |x - \tau_{\alpha,l}(t)| \leq \theta(\alpha)} f(t,x) \equiv c_\alpha > 0, \tag{3.21}$$

where $f(t,x)$ is the density of $F(t,x) := P(X_t \leq x)$. In addition assume for the given $\alpha \in (0,1)$ there exists $y_\alpha \in (0, \infty)$ such that

$$\sup_{t \in E} P(|X(t)| \geq y_\alpha) < \alpha \wedge (1 - \alpha), \tag{3.22}$$

and $C(E)$ denotes the real-valued continuous functions on (E, d). Then, $\tau_{\alpha,l}(\cdot) = \tau_{\alpha,r}(\cdot) := \tau_\alpha(\cdot) \in C(E) \cap \ell_\infty(E)$ for the given $\alpha \in (0,1)$, and $||\tau_\alpha(\cdot)||_\infty \leq y_\alpha$.

Remark 3.13 If $\alpha \in (0, \frac{1}{2}]$, then we can take $x_\alpha = y_\alpha$, where x_α is defined as in (3.19) and y_α is defined as in (3.22). The use of y_α is relevant when $\alpha \in (\frac{1}{2}, 1)$. Furthermore, if $\{X(t) : t \in E\}$ is sample path continuous, and (E, d) is a compact metric space, then $||X||_\infty$ is a measurable random function which is finite with probability one and (3.22) holds. Moreover, X sample path continuous with probability one implies it is continuous in distribution as $d(s,t) \to 0$. Hence, since $F(t,x)$ is continuous in x the convergence in distribution is uniform, which implies (3.20). Thus when (3.21) also holds we have $\tau_{\alpha,l}(\cdot) = \tau_{\alpha,r}(\cdot) \equiv \tau_\alpha(\cdot) \in C(E)$ for those $\alpha \in (0,1)$. Similar conclusions also hold if E is a compact interval of $\mathbb{R}$ and $\{X(t) : t \in E\}$ is continuous in distribution on E with cadlag sample paths.

Corollary 3.14 *Let (E, d) be a compact metric space with $D(E) = C(E)$, the space of real-valued continuous functions on (E, d), and assume for some $\alpha \in (0, \frac{1}{2}]$, $\tau_{\alpha,l}, \tau_{1-\alpha,r}$, are functions in $C(E)$. Then, for the given α, (3.14) and (3.15) imply (3.16), and (3.17) implies both (3.16) and (3.18) hold with $D(E) = C(E)$.*

Corollary 3.15 *Let E be a compact interval of the real line, and assume $D(E)$ is the space of real-valued cadlag functions on E. If for some $\alpha \in (0, \frac{1}{2}]$, $\tau_{\alpha,l}, \tau_{1-\alpha,r}$, are cadlag functions on E, then for the given α (3.14) and (3.15) imply (3.16), and (3.17) implies both (3.16) and (3.18) hold with $D(E)$ the cadlag functions on E.*

In Corollaries 3.14 and 3.15 we are assuming (3.14) and (3.15), or (3.17), and that the quantile functions $\tau_{\alpha,l}$ and $\tau_{1-\alpha,r}$ are in $D(E)$. Hence, to complete their proofs it suffices to show the empirical quantiles $\tau^n_{\alpha,l}$ and $\tau^n_{1-\alpha,r}$ are also in the corresponding $D(E)$. This follows since an argument from Lemma 3 in [8] implies the left empirical quantile functions $\tau^n_{\alpha,l}$ inherit the continuity or cadlag nature of the sample paths assumed for the process X. To obtain the same conclusion for the right empirical quantiles $\tau^n_{1-\alpha,r}$, Lemma 3.1 combined with Lemma 3 in [8] suffices.

3.3 Quantile Process Consistency

Application of the results in [7, 8] to obtain (3.14) and (3.15), or (3.17), involve CLTs for α-quantile processes which hold uniformly in $(t, \alpha) \in E \times I$, where I is a closed subinterval of $(0, 1)$. Hence, of necessity, even when E is a single point, this requires the densities $f(t, x)$ for $X(t)$ to be strictly positive and continuous on $J_t := \{x : 0 < F(t, x) < 1\}$. In the proofs of these CLTs we assumed $J_t := \mathbb{R}$, and for more general E our proofs also required that the densities $\{f(t, \cdot) : t \in E\}$ satisfying the uniform equicontinuity condition

$$\lim_{\delta \to 0} \sup_{t \in E} \sup_{|u-v| \leq \delta} |f(t, u) - f(t, v)| = 0, \tag{3.23}$$

and for every closed interval I in $(0, 1)$ there is an $\theta(I) > 0$ satisfying

$$\inf_{t \in E, \alpha \in I, |x - \tau_\alpha(t)| \leq \theta(I)} f(t, x) \equiv c_{I,\theta(I)} > 0, \tag{3.24}$$

where $\tau_\alpha(t), \alpha \in (0, 1), t \in E$, is the unique α-quantile for the distribution $F(t, \cdot)$ when its density $f(t, \cdot)$ is strictly positive.

Although these CLTs hold for a broad collection of stochastic processes, we were motivated to consider consistency results in hope of weakening these assumptions in that setting. This turns out to be the case as the condition (3.23) and that $J_t := \mathbb{R}, t \in E$, are no longer required for our consistency results. Furthermore, a local form of (3.24) holding for a fixed I rather than all I suffices [see (3.27)]. Of course, if I is a single $\alpha \in (0, 1)$ then (3.27) already appeared in (3.21) of Lemma 3.12 to verify the continuity of the function $\tau_\alpha(\cdot)$ on E.

Hence, the global strict positivity and continuity of the densities can be considerably weakened for consistency of α-empirical quantiles, making such results of independent interest. Here they provide sufficient conditions for (3.14) and (3.15), which combine to show that the measurable cover functions (of the Hausdorff distances) in (3.16) converge in probability to zero under these weakened conditions.

For $\omega \in \Omega$, $t \in E$, and $x \in \mathbb{R}$ we denote the empirical distribution functions by

$$F_n(t,x) := \frac{1}{n}\sum_{j=1}^{n} I(X_j(t,\omega) \le x) = \frac{1}{n}\sum_{j=1}^{n} I_{X_j \in C_{t,x}}, n \ge 1, C_{t,x} \in \mathcal{C}, \tag{3.25}$$

where $\mathcal{C} = \{C_{t,x} : t \in E, x \in \mathbb{R}\}$, $C_{t,x} = \{z \in D(E) : z(t) \le x\}$, and, as usual, we ignore writing that F_n depends on $\omega \in \Omega$. Then, we have

Theorem 3.16 *Let $F(t,x) = P(X(t) \le x)$ have the density $f(t,x)$ for $t \in E, x \in \mathbb{R}$. In addition, assume*

$$P^*(\limsup_{n\to\infty} \sup_{t\in E, x\in\mathbb{R}} |F_n(t,x) - F(t,x)| > 0) = 0, \tag{3.26}$$

and for I a closed interval of $(0, 1)$ there exists $\theta(I) > 0$ such that

$$\inf_{t\in E, \alpha\in I, |x-\tau_{\alpha,l}(t)|\le\theta(I)} f(t,x) \equiv c_{I,\theta(I)} > 0. \tag{3.27}$$

Then, for all $\alpha \in I$ we have $\tau_{\alpha,l}(\cdot) = \tau_{\alpha,r}(\cdot) \equiv \tau_\alpha(\cdot)$, and there is a set Ω_0 such that $P_*(\Omega_0) = 1$, and on Ω_0

$$\lim_{n\to\infty} \sup_{\alpha\in I, t\in E} |\tau^n_{\alpha,l}(t) - \tau_\alpha(t)| = 0. \tag{3.28}$$

Moreover, if $D(E)$ is a linear subspace of $\ell^\infty(E)$, $||\cdot||_\infty$ is measurable on $(D(E), \mathcal{D}_E)$, and $\tau^n_{\alpha,l}(\cdot), \tau_\alpha(\cdot) \in D(E)$, then

$$||\tau^n_{\alpha,l}(\cdot) - \tau_\alpha(\cdot)||_\infty \tag{3.29}$$

is measurable, and converges to zero with P probability one.

Remark 3.17 The condition (3.27) is used in two ways in the proof of Theorem 3.16. The first is to show for all $\alpha \in I$ we have $\tau_{\alpha,l}(\cdot) = \tau_{\alpha,r}(\cdot) \equiv \tau_\alpha(\cdot)$, and the second is to verify (5.17) and (5.18).

To prove the analogue of (3.28) and (3.29) for the processes

$$\tau^n_{1-\alpha,r}(\cdot) - \tau_{1-\alpha}(\cdot), n \ge 1, \tag{3.30}$$

we define for $\omega \in \Omega$, $t \in E$, and $x \in \mathbb{R}$ the distribution functions

$$H_n(t,x) := \frac{1}{n}\sum_{j=1}^{n} I(-X_j(t,\omega) \le x) \text{ and } H(t,x) := P(-X(t) \le x), \tag{3.31}$$

and, as usual, we ignore writing that H_n depends on $\omega \in \Omega$. Then, we have

Corollary 3.18 *Assume*

$$P^*(\limsup_{n\to\infty} \sup_{t\in E, x\in\mathbb{R}} |H_n(t,x) - H(t,x)| > 0) = 0, \tag{3.32}$$

and for $\alpha \in (0,1)$ *there exists* $\theta(\alpha) > 0$ *such that*

$$\inf_{t\in E, |x-\tau_{1-\alpha,r}(t)|\le\theta(\alpha)} f(t,x) \equiv c_{\theta(\alpha)} > 0. \tag{3.33}$$

Then, $\tau_{1-\alpha,l}(\cdot) = \tau_{1-\alpha,r}(\cdot) \equiv \tau_{1-\alpha}(\cdot)$*, and there is a set* Ω_0 *such that* $P_*(\Omega_0) = 1$*, and on* Ω_0

$$\lim_{n\to\infty} \sup_{t\in E} |\tau^n_{1-\alpha,r}(t) - \tau_{1-\alpha}(t)| = 0. \tag{3.34}$$

Moreover, if $D(E)$ *is a linear subspace of* $\ell^\infty(E)$, $||\cdot||_\infty$ *is measurable on* $(D(E), \mathcal{D}_E)$*, and* $\tau^n_{1-\alpha,r}(\cdot), \tau_{1-\alpha}(\cdot) \in D(E)$*, then*

$$||\tau^n_{1-\alpha,r}(\cdot) - \tau_{1-\alpha}(\cdot)||_\infty \tag{3.35}$$

is measurable, and converges to zero with P probability one.

Remark 3.19 In order that Theorem 3.16 and Corollary 3.18 apply to obtain conclusions from Theorem 3.9, the nontrivial assumptions (3.26) and (3.32) must be verified. However, the assumptions that the sup-norm $||\cdot||_\infty$ is measurable on $(D(E), \mathcal{D}_E)$, and the relevant quantile and empirical quantile functions are in a suitable $D(E)$, often follow more directly. For example, Lemma 3.12 and Remark 3.13 provide some sufficient conditions for this when dealing with the quantile functions, and the comments following Corollaries 3.14 and 3.15 making reference to the argument of Lemma 3 in [8] are useful for the empirical quantile functions. In particular, if $\{X(t) : t \in E\}$ is sample path continuous, and (E, d) is a compact metric space, or E is a compact interval of $\mathbb{R}$ and $\{X(t) : t \in E\}$ is continuous in distribution on E with cadlag sample paths, then the quantile functions and empirical quantile functions of Theorem 3.16 and Corollary 3.18 are in $D(E)$, and the measurability of the sup-norm holds on $D(E)$, provided $D(E)$ is the continuous functions on E (respectively, the cadlag paths on E).

4 An Example, and How to Avoid Zero Half Space Depth

As mentioned in Remark 3.8, Proposition 3.7 holds quite generally, but there are many examples where the sets in (3.11) have probability zero for all $\alpha > 0$ with respect to the probability $\mu = \mathcal{L}(X)$ on $(D(E), \mathcal{D}_E)$. Hence, we start with an example of this type. The example also satisfies the assumptions of Theorem 3.9, so even though the sets in (3.11) are small, the convergence results in Theorem 3.9 still hold. Following this we provide a smoothing result for functional data given by a stochastic process. This appears in Proposition 4.2, which shows the sets in (3.11) are much larger for smooth versions of the data, where before smoothing they possibly had positive half space depth with probability zero. The smoothing used in Proposition 4.2 is not a smoothing of the sample paths of the functional data, but of each one dimensional distribution, and in the sense made explicit in Remark 4.3, the change in the data can be made arbitrarily small. Furthermore, the half-space depth and quantile regions of the resulting smoothed process can be shown to have positive probability in many situations. Following the statement of Proposition 4.2 we provide an example for which these regions can be explicitly obtained with fairly brief arguments.

Let $\{Y(t) : t \in [0, 1]\}$ be a centered sample continuous Brownian motion with variance parameter one and $Y(0) = 0$ with probability one. Set $E = [0, 1]$ and $D(E) = C[0, 1]$, the Banach space of continuous functions on $[0, 1]$ in the sup-norm. We then have $\mathcal{D}_E$ is the Borel subsets of $C[0, 1]$, which we denote by $\mathcal{B}_{C[0,1]}$. Then, for $E = [0, 1]$ and $\mu = \mathcal{L}(Y)$ on $(C[0, 1], \mathcal{B}_{C[0,1]})$

$$\mu(h \in C[0, 1] : D(h, \mu) > 0) = 0, \tag{4.1}$$

where $D(h, \mu)$ is given by (3.5) with $\nu = \mu$ and $h \in D(E) = C[0, 1]$.

To verify (4.1) we first observe that

$$\begin{aligned} &D(h, \mu) \\ &\quad \le \inf_{0<t\le 1} \min\{\mu(z \in C[0, 1] : \frac{z(t)}{\sqrt{t}} \ge \frac{h(t)}{\sqrt{t}}), \mu(z \in C[0, 1] : \frac{z(t)}{\sqrt{t}} \le \frac{h(t)}{\sqrt{t}})\} \qquad (4.2) \\ &\quad = \inf_{0<t\le 1} \min\{1 - \Phi(\frac{h(t)}{\sqrt{t}}), \Phi(\frac{h(t)}{\sqrt{t}})\}, \end{aligned}$$

where Φ is the standard normal distribution function. Hence,

$$D(h, \mu) \le \liminf_{t \downarrow 0} \min\{1 - \Phi(\frac{h(t)}{\sqrt{t}}), \Phi(\frac{h(t)}{\sqrt{t}})\} = 0 \tag{4.3}$$

for all $h \in C[0, 1]$ such that

$$\limsup_{t \downarrow 0} \frac{h(t)}{\sqrt{t}} = \infty \text{ or } \liminf_{t \downarrow 0} \frac{h(t)}{\sqrt{t}} = -\infty. \tag{4.4}$$

Using the law of the iterated logarithm for Brownian motion at zero, we have that both terms in (4.4) hold for a set of functions $h \in C[0, 1]$ with μ-probability one.

Thus, (4.1) follows, and the sets in (3.11) are of μ-measure zero. Of course, these sets are clearly non-empty since they contain the zero function. Moreover, for standard Brownian motion on $[0, 1]$ and $\alpha \in (0, 1)$, the left and right α-quantiles for each time $t \in [0, 1]$ are equal, and $\tau_\alpha(t) = \sqrt{t}\tau_\alpha(1) = \sqrt{t}\Phi^{-1}(\alpha)$, and hence $\tau_\alpha(1) < 0$ for $0 < \alpha < \frac{1}{2}$, $\tau_{\frac{1}{2}}(1) = 0$, and $\tau_\alpha(1) > 0$ for $\frac{1}{2} < \alpha < 1$. Therefore, for Brownian motion and $\alpha \in (0, \frac{1}{2}]$ we have

$$M_\alpha = \{h \in C[0, 1] : \sqrt{t}\tau_\alpha(1) \le h(t) \le \sqrt{t}\tau_{1-\alpha}(1)\}, \tag{4.5}$$

and for $\alpha = \frac{1}{2}$ it consists only of the zero function. Since Proposition 3.7 holds, we also have for $\alpha \in (0, \frac{1}{2}]$ that

$$\{h \in C[0, 1] : D(h, \mu) \ge \alpha\} = \{h \in C[0, 1] : \sqrt{t}\tau_\alpha(1) \le h(t) \le \sqrt{t}\tau_{1-\alpha}(1)\}, \tag{4.6}$$

and by what we have shown above the sets in (4.5) and (4.6) have μ measure zero. Of course, one can also see this directly from the law of the iterated logarithm which implies both terms in (4.4) hold with μ measure one, but we thought it useful to proceed as we did because of Remark 4.1 below. In addition, applying Theorems 2 and 3 of Kuelbs and Zinn [8], we have assumptions (3.14), (3.15), and (3.17) holding for each $\alpha \in (0, 1)$. In particular, using Lemma 3 and Theorem 3 of Kuelbs and Zinn [8] one can avoid using measurable cover functions for these processes since we are assuming the input data has sample continuous or cadlag paths, and hence the conclusions of Theorem 3.9 in (3.16) and (3.18) hold, with limit sets as in (3.3) and (3.6), again without the use of measurable cover functions.

Remark 4.1 Similar ideas imply (4.1) for sample continuous fractional Brownian motions which start at zero with probability one when $t = 0$ since the LIL at zero for these processes implies (4.4) with $\sqrt{t}$ replaced by t^ρ, where ρ is the scaling parameter of the fractional Brownian motion. Also, if $\{Y(t) : t \in [0, 1]\}$ is a symmetric stable process with stationary independent increments, sample paths in the space of cadlag functions on $[0, 1]$, and $Y(0) = 0$ with probability one, then the same result holds by applying Theorem 5-iii, page 222, in [2] provided the process is not identically zero.

Now we turn to a proposition which indicates how we can smooth a broad collection of stochastic processes in order that the resulting smoothed process has strictly positive depth with probability one, and is always close to original process in a sense to be specified in Remark 4.3 below. The smoothing in Proposition 4.2 is also useful when applying the functional data results in [7, 8, 11], as many of these results depend on regularity assumptions for the distribution of $X(t)$. Hence, by choosing the density $f_Z(\cdot)$ to be suitably smooth, we can be certain the necessary regularity conditions hold. For example, it is easy to see that the processes mentioned in Remark 4.1 can be smoothed using this result.

Proposition 4.2 *Let $D(E)$ be a subset of $\ell^\infty(E)$ which contains the constant functions and is closed under addition, assume $Y := \{Y(t) : t \in E\}$ is a stochastic process with sample paths in $D(E)$ and Z is a real-valued random variable independent of Y defined on the probability space $(\Omega, \mathcal{F}, P)$, and let $X := \{X(t) : t \in E\}$, where $X(t) = Y(t) + Z$ for $t \in E$. Furthermore, assume Z has probability density $f_Z(\cdot)$ that is positive a.s. on $\mathbb{R}$ with respect to Lebesgue measure, the family of random variables $\{Y(t) : t \in E\}$ is bounded in probability, and for $h \in D(E)$ and $\mu = \mathcal{L}(X)$ on $(D(E), \mathcal{D}_E)$, the depth $D(h, \mu)$ is defined as in (3.5) with $\nu = \mu$. Then, for all $h \in D(E)$ the E-half space depth is such that*

$$D(h, \mu) > 0. \tag{4.7}$$

Remark 4.3 Since $E(||X - Y||_\infty^p) = E(|Z|^p)$ can be made small by choosing the density of Z to decrease rapidly at infinity and with probability near one in a small neighborhood of zero, the X process is close to the Y process in the L_p-norm of their sup-norm difference. Therefore, at least in this sense the smoothing does not modify the data a great deal in order that the depth be positive with probability one. In addition, if $\{Y(t) : t \in [0, 1]\}$ is standard Brownian motion and $X(t) = Y(t) + Z, t \in [0, 1]$, where Z is a mean zero Gaussian random variable independent of $\{Y(t) : t \in [0, 1]\}$ with variance $\sigma^2 > 0$, then we are able to see explicit changes in the quantile and depth regions given in (4.5) and (4.6) for $\{Y(t) : t \in [0, 1]\}$ and those for the smoothed process $\{X(t) : t \in [0, 1]\}$ obtained below. That is, since $P(X(t) \le y) = \Phi(y/\sqrt{t + \sigma^2})$ for all $y \in \mathbb{R}, t \in [0, 1]$, where Φ is the distribution function of a mean zero, variance one Gaussian random variable, we have

$$\tau_\alpha(t) = \sqrt{t + \sigma^2}\Phi^{-1}(\alpha)$$

for all $\alpha \in (0, 1)$ and $t \in [0, 1]$. Thus the α-quantile and depth regions for $\{X(t) : t \in [0, 1]\}$ and $\alpha \in (0, \frac{1}{2}]$ are given by

$$M_{\alpha,X} = \cap_{t\in[0,1]}\{z \in C[0, 1] : \sqrt{t + \sigma^2}\Phi^{-1}(\alpha) \le z(t) \le \sqrt{t + \sigma^2}\Phi^{-1}(1 - \alpha)\}.$$

Since $\sigma^2 > 0$ and $M_{\alpha,X}$ has non-empty interior with respect to the sup-norm topology on $C[0, 1]$ for all $\alpha \in (0, \frac{1}{2})$, then the Cameron-Martin formula easily implies these regions have positive probability with respect to ν, the probability law of $\{X(t) : t \in [0, 1]\}$ on $C[0, 1]$. Of course, since $M_\alpha = \{0\}$ for $\alpha = \frac{1}{2}$ then

$\nu(M_{\frac{1}{2},X}) = 0$ as one should expect. Moreover, since (3.10) implies $M_{\alpha,X} = N_{\alpha,X}$, we also have $\nu(N_{\alpha,X}) > 0$ for all $\alpha \in (0, \frac{1}{2})$ and $\nu(N_{\frac{1}{2}}) = 0$.

Remark 4.4 Although a fairly large number of possible depths have been used to provide orderings for finite dimensional data, it might be of some interest to contrast the use of Proposition 4.2 above, Proposition 4 of [9], and the results obtained in [3] to handle the zero depth problem for infinite dimensional data given by many standard stochastic processes. First it is easy to observe from the hypothesis that Proposition 4.2 applies easily, and provides positive half-space depth for a wide range of stochastic processes. Proposition 4 of [9] does the same for half-region depth. In contrast, Theorems 5 and 7 in [3] are formulated for fractional Brownian motion, but discuss other depths, and Theorem 6 depends on a series expansion of the process, so is best suited to Hilbert space. Moreover, Theorem 6 of [3] applies only to the depth $SD(x)$ defined using the Hilbert norm. So in the sentence immediately following Theorem 6 where it is proposed to apply the same depth for $C[0, 1]$-valued data using the fact that $C[0, 1] \subseteq L_2[0, 1]$, one should wonder exactly how that changes things. In particular, it is unclear how this depth defined in terms of the $L_2[0, 1]$-norm provides a natural ordering to functions whose sup-norm may be much larger then their $L_2[0, 1]$-norm.

Hence, the results in these theorems cover several depths, but they apply to far fewer processes then our Proposition 4.2, or its analogue in Proposition 4 of [9]. Moreover, the positivity of the depth for the modified processes in Theorem 5 of [3] is only shown for a modification of the original depth in two of the three depths studied. The third, an integrated depth, is already suitably modified. In contrast the results in our work show the original depth is positive once the smoothing of the one dimensional distribution functions are made. We are not claiming this for all possible depths, but for the half-space depth of this paper it follows by Proposition 4.2, and also for the half-region depth of [9] this is the case (see Proposition 4 of [9]). Furthermore, as shown in Examples 1–3 of [9], the modified half-region depth used in Theorem 5 of [3] has a number of unpleasant properties. In particular, example one shows modified half-region depth as defined in equation (4) of [3] frequently does not have a unique median (a unique function with modified half region depth 1/2 when the one dimensional distributions are continuous). Example 2 in [9] shows that there are processes symmetric about zero that have multiple medians for modified half-region depth, yet the zero function is the unique median for half-region depth of the original process and also for the process modified using our method as given in Proposition 4 of [9] or Proposition 4.2 of the current paper. Example three goes one step further and shows that under symmetry about zero, the zero function may not be among the multiple medians given by the modified half-region depth, yet it is the unique half-region depth median for the process obtained using our method for smoothing. Hence one might want to be cautious about the many new medians proposed. An addendum for complete proofs of the results mentioned for these examples is referenced in [9].

We first used the smoothing of the one dimensional distributions in our papers [7, 8], where we obtained empirical quantile CLTs for continuous time processes when the input process can come from a broad collection of Gaussian processes, martingales, and independent increment processes. These results define the quantile processes directly, and the CLTs are uniform in the quantile level $\alpha \in I$, where I is a closed interval of $(0, 1)$, and $t \in E$. Furthermore, since E is allowed to be quite general, and not just some interval of reals, these results can also be applied in a wider setting. For example, they apply to the Brownian sheet, and we also have been able to use these results to obtain new empirical quantile process limit theorems for Tukey depth on $\mathbb{R}^d$ in [10] by taking E the surface of the unit ball in $\mathbb{R}^d$. The fact E is quite general was also important for the limit theorems we obtained in [9] for half-region depth. Remark 5 there indicates that under some circumstances one might not even need to smooth the one dimensional distributions to have positive half-region depth. However, observations of a similar nature also appeared in the earlier papers [7, 8] when proving the empirical quantile CLTs.

5 Proofs of the Results

Proof of Proposition 3.7 If $h \in M_\alpha$, then $\tau_{\alpha,l}(t) \leq h(t) \leq \tau_{1-\alpha,r}(t)$ for all $t \in E$. Now $h(t) \geq \tau_{\alpha,l}(t)$ implies $\mu(z : z(t) \leq h(t)) \geq \alpha$, and similarly $h(t) \leq \tau_{1-\alpha,r}(t)$ implies $\mu(z : z(t) \geq h(t)) \geq \mu(z : z(t) \geq \tau_{1-\alpha,r}(t)) \geq \alpha$. Therefore, $D(h, \mu) \geq \min\{\alpha, \alpha\} = \alpha$.

Conversely, assume $D(h, \mu) \geq \alpha$ and $h \in D(E)$. Then, (3.5) with $\nu = \mu$ implies for every $t \in E$ that

$$\mu(z \in D(E) : z(t) \geq h(t)) \geq \alpha \tag{5.1}$$

and

$$\mu(z \in D(E) : z(t) \leq h(t)) \geq \alpha. \tag{5.2}$$

Thus, for every $t \in E$, we have from (5.1) that $h(t) \leq \tau_{1-\alpha,r}(t)$, and from (5.2) that $h(t) \geq \tau_{\alpha,l}(t)$. Therefore, for every $t \in E$ we have $\tau_{\alpha,l}(t) \leq h(t) \leq \tau_{1-\alpha,r}(t)$, which implies $h \in M_\alpha$, and the proposition is proved. □

Proof of Theorem 3.9 For $\alpha \in (0, \frac{1}{2}]$, the set M_α (respectively $M_{\alpha,n}$) is non-empty since (3.13) implies $\tau_{\alpha,l}(\cdot)$ and $\tau_{1-\alpha,r}(\cdot)$ are in M_α (respectively $\tau^n_{\alpha,l}(\cdot)$ and $\tau^n_{1-\alpha,r}(\cdot)$ are in $M_{\alpha,n}$), and Proposition 3.7 implies N_α and $N_{\alpha,n}$ are non-empty since $M_\alpha = N_\alpha$ and $M_{\alpha,n} = N_{\alpha,n}$. Hence, we observe that

$$M^\epsilon_\alpha = \{z \in D(E) : \forall t \in E,\ \tau_{\alpha,l}(t) - \epsilon < z(t) < \tau_{1-\alpha,r}(t) + \epsilon\} \tag{5.3}$$

and

$$M^\epsilon_{\alpha,n} = \{z \in D(E) : \forall t \in E,\ \tau^n_{\alpha,l}(t) - \epsilon < z(t) < \tau^n_{1-\alpha,r}(t) + \epsilon\}. \tag{5.4}$$

Now, we'll use (3.13) to show that

$$\{\omega : M_{\alpha,n}(\omega) \subseteq M_\alpha^\epsilon\} \quad (5.5)$$
$$= \{\omega : \forall t \in E,\ \tau_{\alpha,l}^n(t,\omega) > \tau_{\alpha,l}(t) - \epsilon,\ \tau_{1-\alpha,r}^n(t,\omega) < \tau_{1-\alpha,r}(t) + \epsilon\}$$

and

$$\{\omega : M_\alpha \subseteq M_{\alpha,n}(\omega)^\epsilon\} \quad (5.6)$$
$$= \{\omega : \forall t \in E,\ \tau_{\alpha,l}(t) > \tau_{\alpha,l}^n(t,\omega) - \epsilon,\ \tau_{1-\alpha,r}(t) < \tau_{1-\alpha,r}^n(t,\omega) + \epsilon\}.$$

The proofs of (5.5) and (5.6) are similar, so we'll just check (5.5). Also, it is trivial that the right-hand of (5.5) is contained in the left-hand side. To prove the other inclusion fix ω such that $M_{\alpha,n}(\omega) \subseteq M_\alpha^\epsilon$. By (3.13) we have

$$\tau_{\alpha,l}^n(\cdot,\omega), \tau_{1-\alpha,r}^n(\cdot,\omega) \in M_{\alpha,n}(\omega),$$

and since we are assuming $M_{\alpha,n}(\omega) \subseteq M_\alpha^\epsilon$, for all $t \in E$

$$\tau_{\alpha,l}(t) - \epsilon < \tau_{\alpha,l}^n(t,\omega) \le \tau_{1-\alpha,r}^n(t,\omega) < \tau_{1-\alpha,r}(t) + \epsilon,$$

which implies ω is in the right-hand side of (5.5).

Therefore, both $\{\omega : M_{\alpha,n}(\omega) \subseteq M_\alpha^\epsilon\}$ and $\{\omega : M_\alpha \subseteq M_{\alpha,n}(\omega)^\epsilon\}$ contain the set A_n, where

$$A_n = \{\omega : \sup_{t\in E} |\tau_{\alpha,l}^n(t,\omega) - \tau_{\alpha,l}(t)| < \epsilon\} \cap \{\sup_{t\in E} |\tau_{1-\alpha,r}^n(t,\omega) - \tau_{1-\alpha,r}(t)| < \epsilon\}. \quad (5.7)$$

Now

$$\{\omega : d_H(M_{\alpha,n}(\omega), M_\alpha) < \epsilon\} = \{\omega : M_{\alpha,n}(\omega) \subseteq M_\alpha^\epsilon\} \cap \{\omega : M_\alpha \subseteq M_{\alpha,n}(\omega)^\epsilon\}, \quad (5.8)$$

and hence

$$\{\omega : d_H(M_{\alpha,n}(\omega), M_\alpha) \ge \epsilon\} \quad (5.9)$$
$$= \{\omega : M_{\alpha,n}(\omega) \subseteq M_\alpha^\epsilon\}^c \cup \{\omega : M_\alpha \subseteq M_{\alpha,n}(\omega)^\epsilon\}^c \subseteq A_n^c.$$

Therefore,

$$P^*(d_H(M_{\alpha,n}, M_\alpha) \ge \epsilon) \le P^*(\sup_{t\in E} |\tau_{\alpha,l}^n(t) - \tau_{\alpha,l}(t)| \ge \epsilon) \quad (5.10)$$
$$+ P^*(\sup_{t\in E} |\tau_{1-\alpha,r}^n(t) - \tau_{1-\alpha,r}(t)| \ge \epsilon)$$

where P^* denotes the outer probability given by P. Since $P^*(\Lambda > c) = P(\Lambda^* > c)$ for all constants c, we thus have from (3.14) and (3.15) that the measurable cover function of the left term in (3.16) converges in probability to zero. Furthermore, applying (3.11) and (3.12) of Proposition 3.7, we also have the equality in (3.16), and hence the measurable cover function of the right term in (3.16) also converges in probability to zero. Therefore, the first part of the theorem holds.

Since (5.10) holds for every $\epsilon > 0$ we have with $\epsilon = L/a_n > 0$ that

$$P^*(a_n d_H(M_{\alpha,n}, M_\alpha) \geq L) \leq P^*(a_n \sup_{t\in E} |\tau^n_{\alpha,l}(t) - \tau_{\alpha,l}(t)| \geq L) \tag{5.11}$$

$$+P^*(a_n \sup_{t\in E} |\tau^n_{1-\alpha,r}(t) - \tau_{1-\alpha,r}(t)| \geq L),$$

and hence (5.10) and the relationship between P-outer probability and measurable cover functions mentioned above implies $a_n d^*_H(M_{\alpha,n}, M_\alpha)$ is bounded in probability. Again, by applying (3.11) and (3.12) of Proposition 3.7, we also have $a_n d^*_H(N_{\alpha,n}, N_\alpha)$ is bounded in probability. Thus (3.18) holds, and the theorem follows. □

Proof of Corollary 3.11 Since we are assuming (3.14),(3.15), and (3.17) to hold, it suffices to show the stochastic boundedness assumption in (3.19) implies (3.13) with $D(E) = \ell^\infty(E)$. Since the sample paths of X are assumed to be in $D(E) = \ell^\infty(E)$, we have for $\alpha \in (0, \frac{1}{2}], n \geq 1, t \in E$ that

$$\tau^n_{\alpha,l}(t) = \inf\{x : \frac{1}{n}\sum_{j=1}^n I(X_j(t) \leq x) \geq \alpha\} \geq -\beta$$

and

$$\tau^n_{1-\alpha,r}(t) = \inf\{x : \frac{1}{n}\sum_{j=1}^n I(X_j(t) \leq x) > 1-\alpha\} \leq \beta,$$

where $\beta := \sup_{1\leq j\leq n} \sup_{t\in E} |X_j(t)| = \sup_{1\leq j\leq n} ||X_j||_\infty < \infty$. If $\alpha \in (0, \frac{1}{2})$, then for every $t \in E$

$$-\beta \leq \tau^n_{\alpha,l}(t) \leq \tau^n_{\alpha,r}(t) \leq \tau^n_{1-\alpha,l}(t) \leq \tau^n_{1-\alpha,r}(t) \leq \beta,$$

and hence $\tau^n_{\alpha,l}(\cdot), \tau^n_{1-\alpha,r}(\cdot) \in \ell^\infty(E)$. Furthermore, if $\alpha = \frac{1}{2}$, then by deleting the middle two terms in the previous inequality we have $\tau^n_{\frac{1}{2},l}(\cdot), \tau^n_{\frac{1}{2},r}(\cdot) \in \ell^\infty(E)$.

Similarly, if $\alpha \in (0, \frac{1}{2}]$ and (3.19) holds, then for all $t \in E$

$$\tau_{\alpha,l}(t) = \inf\{x : P(X(t) \leq x) \geq \alpha\} \geq -x_\alpha$$

and since $0 < \alpha \leq \frac{1}{2} \leq 1 - \alpha < 1$,

$$\tau_{1-\alpha,r}(t) = \inf\{x : P(X(t) \leq x) > 1 - \alpha\} \leq x_\alpha.$$

Since $t \in E$ is arbitrary, arguing as above we also have for the given $\alpha \in (0, \frac{1}{2}]$ that $\tau_{\alpha,l}(\cdot), \tau_{1-\alpha,r}(\cdot) \in \ell^\infty(E)$. Therefore, (3.13) holds when $D(E) = \ell^\infty(E)$. □

Proof of Lemma 3.12 Since we are assuming (3.21), for each $t \in E$ the distribution function $F(t,x)$ is strictly increasing and continuous in x on the open interval $(\tau_{\alpha,l}(t) - \theta(\alpha), \tau_{\alpha,l}(t) + \theta(\alpha))$ and hence for the given α we have $\tau_{\alpha,l}(\cdot) = \tau_{\alpha,r}(\cdot) \equiv \tau_\alpha(\cdot)$. Moreover, if $\alpha \in (0, \frac{1}{2}])$ and (3.22) holds, then as in the proof of Corollary 3.11 for the given $\alpha \in (0, \frac{1}{2}])$ we have $\sup_{t \in E} |\tau_\alpha(t)| \leq y_\alpha < \infty$, and $\tau_\alpha(\cdot) \in \ell^\infty(E)$. Furthermore, if $\alpha \in (\frac{1}{2}, 1)$, then $1-\alpha \in (0, \frac{1}{2})$ and arguing as before we have for $t \in E$ that

$$-y_\alpha \leq \tau_{1-\alpha,l}(t) \leq \tau_{1-\alpha,r}(t) \leq \tau_{\alpha,l}(t) \leq \tau_{\alpha,r}(t) \leq y_\alpha.$$

Therefore, we again have $\sup_{t \in E} |\tau_\alpha(t)| \leq y_\alpha < \infty$, and $\tau_\alpha(\cdot) \in \ell^\infty(E)$.

Hence it suffices to show $\tau_\alpha(\cdot) \in C(E)$, so fix $t \in E, \alpha \in (0,1), \epsilon \in (0, \theta(\alpha))$, and take $\delta > 0$ such that $d(s,t) \leq \delta$ and (3.20) implies

$$\sup_{(s,x):d(s,t) \leq \delta, x \in \mathbb{R}} |F(s,x) - F(t,x)| \leq \Delta, \tag{5.12}$$

where $\Delta < c_\alpha \epsilon$ and c_α is as in (3.21). Then, $d(s,t) \leq \delta$ and (5.12) implies

$$F(t, \tau_\alpha(s)) \leq F(t, \tau_\alpha(t)) + \Delta = \alpha + \Delta < F(t, \tau_\alpha(t) + \epsilon) \tag{5.13}$$

and

$$F(t, \tau_\alpha(s)) \geq F(s, \tau_\alpha(s)) - \Delta = \alpha - \Delta > F(t, \tau_\alpha(t) - \epsilon) \tag{5.14}$$

since (3.21) implies

$$F(t, \tau_\alpha(t) + \epsilon) = \alpha + \int_{\tau_\alpha(t)}^{\tau_\alpha(t)+\epsilon} f(t,x)dx \geq \alpha + c_\alpha \epsilon > \alpha + \Delta$$

and

$$F(t, \tau_\alpha(t) - \epsilon) = \alpha - \int_{\tau_\alpha(t)-\epsilon}^{\tau_\alpha(t)} f(t,x)dx \geq \alpha - c_\alpha \epsilon < \alpha - \Delta.$$

Now (5.13) and (5.14) combine to imply

$$\tau_\alpha(t) - \epsilon \leq \tau_\alpha(s) \leq \tau_\alpha(t) + \epsilon$$

for $d(s,t) \le \delta$. Since $\epsilon > 0$ can be taken arbitrarily small, $\tau_\alpha(\cdot) \in C(E)$ and the lemma is verified. □

Proof of Theorem 3.16 Since we are assuming (3.27), for each $t \in E$ the distribution function $F(t,x)$ is strictly increasing and continuous in x on the open interval $(\tau_{\alpha,l}(t) - \theta(I), \tau_{\alpha,l}(t) + \theta(I))$, and hence for $\alpha \in I$ we have $\tau_{\alpha,l}(\cdot) = \tau_{\alpha,r}(\cdot) \equiv \tau_\alpha(\cdot)$.

To establish (3.28) we use (3.26) to take Ω_0 such that $P_*(\Omega_0) = 1$, and on Ω_0

$$\limsup_{n\to\infty} \sup_{C_{t,x}\in\mathcal{C}} |F_n(t,x) - F(t,x)| = 0. \tag{5.15}$$

Since we are also assuming (3.27), fix $I \subseteq (0,1)$, $0 < \epsilon \le \theta(I)$, and $\omega \in \Omega_0$. Then, by (5.15) and our definition of Ω_0 there exists $\delta \equiv \delta(\omega,\epsilon) > 0$ such that for $n \ge n(\delta(\omega,\epsilon))$ we have

$$\sup_{C_{t,x}\in\mathcal{C}} |F_n(t,x) - F(t,x)| < \delta, \tag{5.16}$$

Furthermore, if $\delta = \delta(\omega,\epsilon) < \epsilon c_{I,\theta(I)}$, our choice of $\epsilon > 0$ and (3.27) implies

$$\sup_{\alpha\in I, t\in E} F(t, \tau_\alpha(t) - \epsilon) < \alpha - \delta, \tag{5.17}$$

and

$$\inf_{\alpha\in I, t\in E} F(t, \tau_\alpha(t) + \epsilon) > \alpha + \delta. \tag{5.18}$$

That is, (5.17) holds since

$$\sup_{\alpha\in I, t\in E} F(t, \tau_\alpha(t) - \epsilon) \le \alpha - \inf_{\alpha\in I, t\in E} \int_{\tau_\alpha(t)-\epsilon}^{\tau_\alpha(t)} f(t,x)dx,$$

and (3.27) and our choice of $\delta = \delta(\omega,\epsilon) < \epsilon c_{I,\theta(I)}$ then implies

$$\inf_{\alpha\in I, t\in E} \int_{\tau_\alpha(t)-\epsilon}^{\tau_\alpha(t)} f(t,x)dx \ge \epsilon c_{I,\theta(I)} > \delta.$$

Similarly, (5.18) holds since

$$\inf_{\alpha\in I, t\in E} F(t, \tau_\alpha(t) + \epsilon) \ge \alpha + \inf_{\alpha\in I, t\in E} \int_{\tau_\alpha(t)}^{\tau_\alpha(t)+\epsilon} f(t,x)dx,$$

and (3.27) and our choice of δ implies

$$\inf_{\alpha\in I, t\in E}\int_{\tau_\alpha(t)}^{\tau_\alpha(t)+\epsilon} f(t,x)dx \geq \epsilon c_{I,\theta(I)} > \delta.$$

Thus for $n \geq n(\delta(\omega,\epsilon))$, (5.16), and by definition of $\tau^n_{\alpha,l}(t)$ that

$$\inf_{\alpha\in I, t\in E} F_n(t,\tau^n_{\alpha,l}(t)) \geq \alpha,$$

we have on Ω_0 that

$$\inf_{\alpha\in I, t\in E} F(t,\tau^n_{\alpha,l}(t)) \geq \inf_{\alpha\in I, t\in E} F_n(t,\tau^n_{\alpha,l}(t)) - \delta \geq \alpha - \delta.$$

Therefore, by (5.17) for all $t \in E$, all $\alpha \in I$, and $\omega \in \Omega_0$

$$\tau^n_{\alpha,l}(t) \geq \tau_\alpha(t) - \epsilon, \tag{5.19}$$

provided $n \geq n(\delta(\omega,\epsilon))$.

Similarly, for all $x < \tau^n_{\alpha,l}(t)$ we have for all $t \in E$, all $\alpha \in I$, and $\omega \in \Omega_0$

$$F(t,x) < F_n(t,x) + \delta(\omega,\epsilon) \leq \alpha + \delta < F(t,\tau_\alpha(t)+\epsilon),$$

provided $n \geq n(\delta(\omega,\epsilon))$, where the first inequality follows from (5.16), the second by definition of $\tau^n_\alpha(t)$ and that $x < \tau^n_\alpha(t)$, and the third by (5.18). Thus $\tau_\alpha(t) + \epsilon > x$ for all $x < \tau^n_\alpha(t)$, and combining this with (5.19) we have for all $t \in E$, all $\alpha \in I$, and $\omega \in \Omega_0$ that

$$\tau_\alpha(t) - \epsilon \leq \tau^n_\alpha(t) \leq \tau_\alpha(t) + \epsilon, \tag{5.20}$$

provided $n \geq n(\delta(\omega,\epsilon))$. Since $\epsilon > 0$ was arbitrary, letting $n \to \infty$ implies (3.28). $\square$

Proof of Corollary 3.18 Let $\nu = \mathcal{L}(-X)$, $\nu_n = \frac{1}{n}\sum_{j=1}^n \delta_{-X_j}$ on $(D(E),\mathcal{D}_E)$, and for $\alpha \in (0,1)$ define the left α-quantiles $\tau_{\alpha,l}(t,\nu)$ and $\tau_{\alpha,l}(t,\nu_n)$ as in Definition 2.2. Then, since (3.32) holds, Theorem 3.16 implies

$$\lim_{n\to\infty}\sup_{t\in E}|\tau_{\alpha,l}(t,\nu_n) - \tau_{\alpha,l}(t,\nu)| = 0 \tag{5.21}$$

on a subset E_0 of Ω such that $P_*(E_0) = 1$ provided for the given α there exists $\theta(\alpha) > 0$ such that

$$\inf_{t\in E, |x-\tau_{\alpha,l}(t,\nu)|\leq\theta(\alpha)} h(t,x) \equiv c_{\theta(\alpha)} > 0 \tag{5.22}$$

where $h(t,x)$ is the density of $-X(t)$.

Now for each $t \in E$ we can take $h(t, x) = f(t, -x)$ for all $x \in \mathbb{R}$, and hence Lemma 3.1 implies $\tau_{\alpha,l}(t, \nu) = -\tau_{1-\alpha,r}(t), t \in E$, where as usual $\tau_{1-\alpha,r}(t)$ is defined using $\mathcal{L}(X)$. Thus, (5.22) is equivalent to

$$\inf_{t\in E, |x+\tau_{1-\alpha,r}(t)|\le\theta(\alpha)} f(t, -x) \equiv c_{\theta(\alpha)} > 0, \tag{5.23}$$

and setting $u = -x$ we have (5.23) equivalent to

$$\inf_{t\in E, |-u+\tau_{1-\alpha,r}(t)|\le\theta(\alpha)} f(t, u) \equiv c_{\theta(\alpha)} > 0, \tag{5.24}$$

which follows from (3.33). Therefore, (5.21), holds, and since Lemma 3.1 also implies $\tau_{\alpha,l}(t, \nu_n) = -\tau^n_{1-\alpha,r}(t), t \in E$, we thus have (3.34). Of course, (3.35) then follows immediately, and Corollary 3.18 is proved. □

Proof of Proposition 4.2 Since $\{Y(t) : t \in E\}$ is bounded in probability there exists $c \in (0, \infty)$ such that $\sup_{t\in E} P(|Y(t)| > c) < \frac{1}{2}$. Then, for $t \in E$

$$P(X(t) \ge h(t)) = \int_{\mathbb{R}} P(Y(t) \ge h(t) - u) f_Z(u)du, \tag{5.25}$$

and for $u \ge c + ||h||_\infty$ we have

$$P(Y(t) \ge h(t) - u) \ge P(Y(t) \ge ||h||_\infty - (c + ||h||_\infty)) \ge P(Y(t) \ge -c). \tag{5.26}$$

Since our choice of c implies $P(Y(t) \ge -c) \ge P(|Y(t)| \le c) > \frac{1}{2}$, (5.26) implies

$$P(Y(t) \ge h(t) - u) > \frac{1}{2}, \tag{5.27}$$

and combining (5.25) and (5.27)

$$\inf_{t\in E} P(X(t) \ge h(t)) \ge \frac{1}{2} P(Z \ge c + ||h||_\infty) \equiv \delta_1(c, ||h||_\infty) > 0. \tag{5.28}$$

We also have for $t \in E$ that

$$P(X(t) \le h(t)) = \int_{\mathbb{R}} P(Y(t) \le h(t) - u) f_Z(u)du, \tag{5.29}$$

and for $t \in E$ and $u \le -(c + ||h||_\infty)$ that

$$P(Y(t) \le h(t) - u) \ge P(Y(t) \le -||h||_\infty + (c + ||h||_\infty)) \ge P(Y(t) \le c). \tag{5.30}$$

Our choice of c also implies $P(Y(t) \leq c) \geq P(|Y(t)| \leq c) > \frac{1}{2}$, and hence (5.30) implies

$$P(Y(t) \leq h(t) - u) > \frac{1}{2}. \tag{5.31}$$

Combining (5.29) and (5.31) we thus have

$$\inf_{t \in E} P(X(t) \leq h(t)) \geq \frac{1}{2} P(Z \leq -(c + ||h||_\infty)) \equiv \delta_2(c, ||h||_\infty) > 0. \tag{5.32}$$

Hence, (5.28) and (5.32) imply (4.7), and the proposition is proved. □

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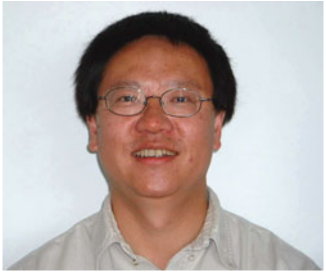
In Memory of Wenbo V. Li's Contributions

Qi-Man Shao

Abstract Wenbo V. Li was Professor of Mathematical Sciences at the University of Delaware. He died of a heart attack in January 2013. Wenbo made significant contributions to many of the areas in which he worked, especially to small value probability estimates. This note is a brief survey of Wenbo V. Li's contributions, as well as discussion of the open problems he posed or found of interest.

Keywords Comparison inequality • Correlation inequality • Metric entropy • Open questions • Small value probabilities

Mathematics Subject Classification (2010). Primary 60G15; Secondary 60F10



Q.-M. Shao (✉)
Department of Statistics, The Chinese University of Hong Kong, Shatin, N.T., Hong Kong SAR, China
e-mail: qmshao@cuhk.edu.hk

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1 Introduction

Wenbo V. Li, Professor of Mathematical Sciences at the University of Delaware, died suddenly of a heart attack on January 26, 2013, near his home in Newark, Delaware. He was 49 years old. Wenbo is survived by his wife Sunny and his son James.

Wenbo was born on October 27, 1963 in Harbin, China. After obtaining his Bachelor's degree in applied mathematics from Jilin University in Changchun, China, he came to the United States in 1986 and studied probability theory at the University of Wisconsin–Madison, under the supervision of James Kuelbs. Wenbo obtained his PhD in Mathematics in 1992 and joined the University of Delaware as an Assistant Professor, being promoted to Associate Professor in 1996 and Full Professor in 2002.

Wenbo was a probabilist with a broad range of research interests, including Gaussian processes, random polynomials and matrices, stochastic inequalities, probability on Banach spaces, geometric functional analysis, stochastic partial differential equations and random dynamics. He was a leading expert on small value probability estimates. In 2006, he was elected Fellow of the IMS, "for his distinguished research in the theory of Gaussian processes and in using this theory to solve many important problems in diverse areas of probability". He served as an associate editor for several probability journals, including *The Annals of Probability*, *Journal of Theoretical Probability*, *Journal of Mathematical Research and Exposition*, and *International Journal of Stochastic Processes*. He organized or co-organized many international conferences/workshops, including High Dimensional Probability (2005), IMS-China International Conference on Statistics and Probability (2009), International Conference on Small Deviation Probabilities (2003, 2005).

In this short note, we briefly review some of Wenbo's main contributions and some open questions Wenbo posed or found of interest.

2 Selected Wenbo's Main Contributions

Wenbo had many research interests and made significant contributions to various topics, especially to the estimation of small value probabilities. In this section, we give a brief review of some of his main contributions.

2.1 Small Ball Probability and the Metric Entropy

Let μ be a centered Gaussian measure on a real separable Banach space B with norm $\|\cdot\|$ and dual B^*. The reproducing Hilbert space of μ, denoted by $(H_\mu, \|\cdot\|_\mu)$, is the completion of the range of the mapping $S : B^* \to B$ defined via the Bochner integral

$$Sf = \int_B xf(x)d\mu(x), \quad f \in B^*,$$

and the completion is with respect to the norm defined by the inner product

$$\langle Sf, Sg\rangle = \int_B f(x)g(x)d\mu(x), \; f, g \in B^*.$$

If K is a compact set in a metric space (E, d), then the metric entropy of K is defined as $H(K, \epsilon) = \log N(K, \epsilon)$, where $N(K, \epsilon)$ is minimum number of balls in E with radius ϵ needed to cover K.

One of the most significant contributions that Wenbo made to small ball theory was achieved with Jim Kuelbs. This was to disclose the precise link between the small problem for a Gaussian measure μ and the metric entropy of the unit ball of the reproducing Hilbert space generated by μ. The notation in this result is as in [7] and [12].

Theorem 2.1 (Kuelbs and Li [7]) *Let μ be a centered Gaussian measure on a real separable Banach space B and let*

$$\log \mu(B_\epsilon) = -\phi(\epsilon),$$

where $B_\epsilon = \{x \in B : \|x\| < \epsilon\}$. Let K_μ be the unit ball of the reproducing Hilbert space H_μ and $H(K_\mu, \epsilon)$ be the metric entropy of K_μ.

(i) Then

$$H(K_\mu, \epsilon/(2\phi(\epsilon))^{1/2}) \succeq \phi(2\epsilon).$$

In particular, if $\phi(\epsilon) \preceq \phi(2\epsilon)$ and $\phi(\epsilon) \succeq \epsilon^{-\alpha}J(\epsilon^{-1})$, $\alpha > 0$ and J is a slowly varying function at infinity such that $J(x) \approx J(x^\rho)$ for each $\rho > 0$, then

$$H(K_\mu, \epsilon) \succeq \epsilon^{-2\alpha/(2+\alpha)}J(1/\epsilon)^{2/(2+\alpha)} \quad as \;\; \epsilon \to 0.$$

(ii) If $\phi(\epsilon) \preceq f(\epsilon)$, where $f(1/x)$ is a regularly varying function at infinity, then

$$H(K_\mu, \epsilon/f(\epsilon)^{1/2}) \preceq f(\epsilon).$$

In particular, if $f(\epsilon) = \epsilon^{-\alpha}J(\epsilon^{-1})$, $\alpha > 0$ and J is as in (i), then

$$H(K_\mu, \epsilon) \preceq \epsilon^{-2\alpha/(2+\alpha)}J(1/\epsilon)^{2/(2+\alpha)} \quad as \;\; \epsilon \to 0.$$

(iii) If $H(K_\mu, \epsilon) \succeq g(\epsilon)$, *where* $g(1/x)$ *is a regularly varying function at infinity, then*

$$\phi(\epsilon) \succeq g(\epsilon/\phi(\epsilon)^{1/2}).$$

In particular, if $g(\epsilon) = \epsilon^{-\beta} J(\epsilon^{-1})$, $0 < \beta < 2$ *and* J *is as in (i), then*

$$\phi(\epsilon) \succeq \epsilon^{-2\beta/(2-\beta)} J(1/\epsilon)^{2/(2-\beta)}.$$

(iv) If $H(K_\mu, \epsilon) \preceq g(\epsilon)$, *where* $g(1/x)$ *is a regularly varying function at infinity, then*

$$\phi(2\epsilon) \preceq g(\epsilon/\phi(\epsilon)^{1/2}).$$

In particular, if $g(\epsilon) = \epsilon^{-\beta} J(\epsilon^{-1})$, $0 < \beta < 2$ *and* J *is as in (i), then*

$$\phi(\epsilon) \preceq \epsilon^{-2\beta/(2-\beta)} J(1/\epsilon)^{2/(2-\beta)}.$$

The last part in (iv) was proved by Li and Linde [12]. The link established in Theorem 2.1 permits the application of tools and results from functional analysis to attack important problems of estimating the small ball probabilities, which are of special interest in probability theory, and vice versa. We refer to Li [11] for latest development and applications.

2.2 Li's Weak Correlation Inequality

Let μ be a centered Gaussian measure on a real separable Banach space. Let A and B be any two symmetric convex sets. The Gaussian correlation conjecture says that

$$\mu(A \cap B) \geq \mu(A)\mu(B).$$

Wenbo worked for many years on the Gaussian correlation conjecture and proved the following weak correlation inequality:

Theorem 2.2 (Li [9]) *For* A *and* B *symmetric convex sets and* $0 < \lambda < 1$

$$\mu(A \cap B) \geq \mu(\lambda A)\mu\big((1-\lambda^2)^{1/2} B\big).$$

The Gaussian correlation conjecture is equivalent to: for any centered Gaussian random vector $(X_1, \cdots, X_n)$ and for $1 \leq k < n$

$$P(|X_i| \leq x_i, 1 \leq i \leq n) \geq P(|X_i| \leq x_i, 1 \leq i \leq k) P(|X_j| \leq x_j, k < j \leq n) \tag{2.1}$$

for $x_1, \cdots, x_n > 0$.

It is known that (2.1) holds for $k = 1$. There have been several claims of a proof of the Gaussian correlation conjecture.

2.3 Li's Comparison Theorem

Let $\xi_n,\ n \geq 1$ be i.i.d. standard normal random variables, $\{a_n\}$ and $\{b_n\}$ be sequences of strictly positive real numbers with $\sum_{n=1}^{\infty} a_n < \infty$, $\sum_{n=1}^{\infty} b_n < \infty$.

Theorem 2.3 (Li [8]) *If*

$$\sum_{n=1}^{\infty} |1 - a_n/b_n| < \infty, \tag{2.2}$$

then as $\epsilon \to 0$

$$P\Big(\sum_{n=1}^{\infty} a_n \xi_n^2 \leq \epsilon^2\Big) \sim \Big(\prod_{n=1}^{\infty}(b_n/a_n)\Big)^{1/2} P\Big(\sum_{n=1}^{\infty} b_n \xi_n^2 \leq \epsilon^2\Big). \tag{2.3}$$

Theorem 2 of Gao et al. [5] removes assumption (2.2) and shows (2.3) provided $\prod_{n=1}^{\infty}(b_n/a_n) < \infty$.

2.4 A Reversed Slepian Type Inequality

Let $\{X_i, 1 \leq i \leq n\}$ and $\{Y_i, 1 \leq i \leq n\}$ be centered Gaussian random vectors. The classical Slepian [15] inequality states that

If $EX_i^2 = EY_i^2$ *and* $E(X_iX_j) \leq E(Y_iY_j)$ *for all* $1 \leq i,j \leq n$, *then for any* x

$$P\Big(\max_{1\leq i\leq n} X_i \leq x\Big) \leq P\Big(\max_{1\leq i\leq n} Y_i \leq x\Big)$$

The Slepian inequality has played an important role in various probability estimates for Gaussian measure. Li and Shao [13] established the following reversed inequality, which is a special case of Theorem 2.2 in [13].

Theorem 2.4 (Li and Shao [13]) *Assume* $EX_i^2 = EY_i^2 = 1$ *and* $0 \leq E(X_iX_j) \leq E(Y_iY_j)$ *for all* $1 \leq i,j \leq n$. *Then for* $x \geq 0$

$$\begin{aligned} P\Big(\max_{1\leq i\leq n} X_i \leq x\Big) &\leq P\Big(\max_{1\leq i\leq n} Y_i \leq x\Big) \\ &\leq P\Big(\max_{1\leq i\leq n} X_i \leq x\Big) \times \\ &\quad \prod_{1\leq i<j\leq n} \Big(\frac{\pi - 2\arcsin(EX_iX_j)}{\pi - 2\arcsin(EY_iY_j)}\Big)^{\exp\left(-x^2/(1+EY_iY_j)\right)}. \end{aligned}$$

2.5 The First Exit Time of a Brownian Motion from an Unbounded Convex Domain

Let $B(t) = (B_1(t), \cdots, B_d(t)) \in R^d, t \geq 0$ be a standard d-dimensional Brownian motion, where $B_i(t), 1 \leq i \leq d$ are independent standard Brownian motions. Let

$$D = \{(x, y) \in R^{d+1} : \; y > f(x), \; x \in R^d\},$$

where $f(x)$ is a convex function on R^d. The first exit time τ_D of a $(d+1)$-dimensional Brownian motion from D starting at the point $(x_0, f(x_0) + 1)$ is defined by

$$\tau_D = \inf\{t \geq 0 : \quad (x_0 + B(t), f(x_0) + 1 + B_0(t)) \notin D\},$$

where $B_0(t)$ is a standard Brownian motion independent of $B(t)$. Bañuelos et al. [1] proved that If $d = 1, f(x) = |x|^2$, then

$$\log P(\tau_D \geq t) \approx -t^{1/3} \;\text{ as }\; t \to \infty.$$

Li [10] gave a very general estimate for $\log P(\tau_D > t)$. In particular, for $f(x) = \exp(\|x\|^p), \; p > 0$,

$$\lim_{t \to \infty} t^{-1} (\log t)^{2/p} \log P(\tau_D \geq t) = -j_v^2/2,$$

where $v = (d-2)/2$ and j_v is the smallest positive zero of the Bessel function J_v.

2.6 Lower Tail Probabilities

Let $\{X_t, \; t \in T\}$ be a real valued Gaussian process indexed by T with $\mathbf{E}X_t = 0$. Lower tail probability refers to

$$P\Big(\sup_{t \in T}(X_t - X_{t_0}) \leq x\Big) \;\text{ as }\; x \to 0, \; t_0 \in T$$

or

$$P\Big(\sup_{t \in T} X_t \leq x\Big) \;\text{ as }\; |T| \to \infty.$$

Li and Shao [14] obtained a general result for the lower tail probability of non-stationary Gaussian process

$$\mathbb{P}\Big(\sup_{t \in T}(X_t - X_{t_0}) \leq x\Big) \;\text{ as }\; x \to 0,$$

Special cases include:

(a) Let $\{X(t), t \in [0,1]^d\}$ be a centered Gaussian process with $X(0) = 0$ and stationary increments, that is

$$\forall\ t, s \in [0,1]^d,\ \ E(X_t - X_s)^2 = \sigma^2(\|t - s\|).$$

If there are $0 < \alpha \le \beta < 1$ such that $\sigma(h)/h^\alpha$ is non-decreasing and $\sigma(h)/h^\beta$ non-increasing, then as $\epsilon \to 0$

$$\ln P(\sup_{t\in[0,1]^d} X(t) \le \sigma(\epsilon)) \approx -\log\frac{1}{\epsilon}.$$

(b) Let $\{X(t), t \in [0,1]^d\}$ be a centered Gaussian process with $X(0) = 0$ and

$$E(X_t X_s) = \prod_{i=1}^{d} \frac{1}{2}(\sigma^2(t_i) + \sigma^2(s_i) - \sigma^2(|t_i - s_i|)).$$

If there are $0 < \alpha \le \beta < 1$ such that $\sigma(h)/h^\alpha$ is non-decreasing and $\sigma(h)/h^\beta$ non-increasing, then as $\epsilon \to 0$, then as $\epsilon \to 0$

$$\ln P(\sup_{t\in[0,1]^d} X(t) \le \sigma^d(\epsilon)) \approx -\ln^d\frac{1}{\epsilon}.$$

2.7 Large Deviations for Self-Interaction Local Times

Let $B(t), t \ge 0$ be a one-dimensional Brownian motion and $k \ge 2$. The self-interaction local time given by

$$\beta_t = \int_{[0,t]^k} 1_{\{B(s_1)=B(s_2)=\cdots=B(s_k)\}} ds_1 \cdots ds_k$$

measures the intensity of the k-multiple self-intersection of the Brownian path. It is known that

$$\beta_t = \int_{-\infty}^{\infty} L^k(t,x)dx,$$

where

$$L(t,x) = \int_0^t \delta_x(B(s))ds$$

is the local time of the Brownian motion.

Chen and Li [3] proved that the following holds:

$$\lim_{x\to\infty} x^{-2/(k-1)} \log P(\beta_1 \geq x) = -\frac{1}{4(k-1)}\Big(\frac{k+1}{2}\Big)^{(3-k)/(k-1)} B\Big(\frac{1}{k-1}, \frac{1}{2}\Big)^2,$$

where $B(\cdot,\cdot)$ is the beta function.

This is a special case of Theorem 1.1 in [3].

2.8 Ten Lectures on Small Value Probabilities and Applications

Wenbo delivered ten comprehensive lectures on small value probabilities and applications at NSF/CBMS Regional Research Conference in the Mathematical Sciences, University of Alabama in Huntsville, June 04–08, 2012. We highly recommend them to anyone who is interested in this topic.

3 Wenbo's Open Problems

From time to time Wenbo raised many interesting open questions. In this section we summarize a selection of them, some of which might not be originally due to him. We refer to Wenbo's ten lectures for details.

1. **Gaussian products conjecture:**

   For any centered Gaussian vector $(X_1, \cdots, X_n)$, it holds

   $$E(X_1^{2m} \cdots X_n^{2m}) \geq E(X_1^{2m}) \cdots E(X_n^{2m})$$

   for each integer $m \geq 1$.

   It is known it is true when $m = 1$ (Frenkel [4]) .
2. **Gaussian minimum conjecture**:

   Let $(X_i, 1 \leq i \leq n)$ be a centered Gaussian random vector. Then

   $$E \min_{1\leq i\leq n} |X_i| \geq E \min_{1\leq i\leq n} |X_i^*|,$$

   where X_i^* are independent Gaussian random variables with $E(X_i^{*2}) = E(X_i^2)$.

   A weak result was proved by Gordon et al. [6]:

   $$E \min_{1\leq i\leq n} |X_i| \geq (1/2) E \min_{1\leq i\leq n} |X_i^*|.$$

3. **Conjecture**: Let ϵ_i, $1 \le i \le n$ be i.i.d. r.v.'s $P(\epsilon_i = \pm 1) = 1/2$. Then for any $\{a_i\}$ satisfying $\sum_{i=1}^n a_i^2 = 1$

$$P\Big(|\sum_{i=1}^n a_i \epsilon_i| \le 1\Big) \ge 1/2.$$

The best known lower bound is $3/8$.

In the following open questions 4–6, let ϵ_{ij} be i.i.d. Bernoulli random variables with $P(\epsilon_{ij} = \pm 1) = 1/2$.

4. **Determinant of Bernoulli matrices:**

Let $M_n = (\epsilon_{ij})_{n\times n}$. It is easy to show that

$$E(\det(M_n)^2) = n!.$$

It was proved by Tao and Vu [16] that

$$P\Big(|\det(M_n)| \le \sqrt{n!}\exp(-29(n\log n)^{1/2})\Big) = o(1).$$

Conjecture: For $0 < \delta < 1$,

$$P\Big(|\det(M_n)| \le (1-\delta)\sqrt{n!}\Big) = o(1)$$

and with probability tending to 1

$$|\det(M_n)| = n^{O(1)}\sqrt{n!}.$$

5. **Singularity probability of random Bernoulli matrices:**

Let $M_n = (\epsilon_{ij})_{n\times n}$. Clearly, one has

$$P(\det(M_n) = 0) \ge (1 + o(1))n^2 2^{1-n}.$$

It is conjectured that

$$P(\det(M_n) = 0) = \Big(\frac{1}{2} + o(1)\Big)^n.$$

The best known result is due to Bourgain et al. [2]:

$$P(\det(M_n) = 0) \le \Big(\frac{1}{\sqrt{2}} + o(1)\Big)^n.$$

6. **Gaussian Hadamard conjecture**:
   The Hadamard conjecture can be restated as

$$P\Big(\max_{1\le j\ne k\le n} |\sum_{i=1}^{n} \epsilon_{ij}\epsilon_{ik}| < 1\Big) \ge 2^{-n^2},$$

where $n = 4m$

The **Gaussian Hadamard conjecture** is: Let ξ_{ij} be i.i.d. standard normal random variables. Then

$$\ln P\Big(\max_{1\le j\ne k\le n} |\sum_{i=1}^{n} \xi_{ij}\xi_{ik}| < 1\Big) \approx -n^2.$$

7. **The traveling salesman problem:**
   Let

$$L_n = \min_{\sigma} \sum_{i=1}^{n-1} |X_{\sigma(i+1)} - X_{\sigma(i)}|$$

be the shortest tour of n i.i.d. uniform points $\{X_i, 1 \le i \le n\} \subset [0,1]^d$, where σ denotes a permutation of $\{1, \cdots, n\}$. It is known that

$$E(L_n)/n^{(d-1)/d} \to \beta(d).$$

Open question: What is the value of $\beta(d)$? Does the central limit theorem hold?

8. **Two-sample matching:**
   Let $\{X_i\}$ and $\{Y_i\}$ be i.i.d. uniformly distributed on $[0,1]^2$. Consider

$$M_n = \min_{\sigma} \sum_{i=1}^{n} |X_i - Y_{\sigma(i)}|, \quad M_n^* = \min_{\sigma} \max_{1\le i\le n} |X_i - Y_{\sigma(i)}|.$$

It is known that there exist $0 < c_0 < c_1 < \infty$ such that

$$c_0 \le \frac{EM_n}{\sqrt{n\log n}} \le c_1, \quad c_0 \le \frac{EM_n^*}{n^{-1/2}(\log n)^{3/4}} \le c_1.$$

Open question: What are the exact limits? What are the limiting distributions of $M_n/\sqrt{n\log n}$ and $M_n^*/(n^{-1/2}(\log n)^{3/4})$?

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Part III
Stochastic Processes

Orlicz Integrability of Additive Functionals of Harris Ergodic Markov Chains

Radosław Adamczak and Witold Bednorz

Abstract For a Harris ergodic Markov chain $(X_n)_{n\geq 0}$, on a general state space, started from the small measure or from the stationary distribution, we provide optimal estimates for Orlicz norms of sums $\sum_{i=0}^{\tau} f(X_i)$, where τ is the first regeneration time of the chain. The estimates are expressed in terms of other Orlicz norms of the function f (with respect to the stationary distribution) and the regeneration time τ (with respect to the small measure). We provide applications to tail estimates for additive functionals of the chain (X_n) generated by unbounded functions as well as to classical limit theorems (CLT, LIL, Berry-Esseen).

Keywords Limit theorems • Markov chains • Orlicz spaces • Tail inequalities • Young functions

Mathematics Subject Classification (2010). Primary 60J05, 60E15; Secondary 60K05, 60F05

1 Introduction and Notation

Consider a Polish space $\mathcal{X}$ with the Borel σ-field $\mathcal{B}$ and let $(X_n)_{n\geq 0}$ be a time homogeneous Markov chain on $\mathcal{X}$ with a transition function $P\colon \mathcal{X} \times \mathcal{B} \to [0, 1]$. Throughout the article we will assume that the chain is Harris ergodic, i.e. that there exists a unique probability measure π on $(\mathcal{X}, \mathcal{B})$ such that

$$\|P^n(x, \cdot) - \pi\|_{TV} \to 0$$

for all $x \in \mathcal{X}$, where $\|\cdot\|_{TV}$ denotes the total variation norm, i.e. $\|\mu\|_{TV} = \sup_{A\in\mathcal{B}} |\mu(A)|$ for any signed measure μ.

R. Adamczak (✉) • W. Bednorz
Institute of Mathematics, University of Warsaw, Banacha 2, 02–097 Warszawa, Poland
e-mail: radamcz@mimuw.edu.pl; wbednorz@mimuw.edu.pl

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One of the best known and most efficient tools of studying such chains is the regeneration technique [4, 30], which we briefly recall bellow. We refer the reader to the monographs [28, 31] and [12] for extensive description of this method and restrict ourselves to the basics which we will need to formulate and prove our results.

One can show that under the above assumptions there exists a set (usually called *small set*) $C \in \mathcal{E}^+ = \{A \in \mathcal{B}: \pi(A) > 0\}$, a positive integer m, $\delta > 0$ and a Borell probability measure ν on $\mathcal{X}$ (*small measure*) such that

$$P^m(x, \cdot) \geq \delta\nu(\cdot) \tag{1.1}$$

for all $x \in C$. Moreover one can always choose m and ν in such a way that $\nu(C) > 0$.

Existence of the above objects allows to redefine the chain (possibly on an enlarged probability space) together with an auxiliary regeneration structure. More precisely, one defines the sequence $(\tilde{X}_n)_{n\geq 0}$ and a sequence $(Y_n)_{n\geq 0}$ of $\{0, 1\}$-random variables by requiring that $\tilde{X}_0$ have the same distribution as X_0 and specifying the conditional probabilities (see [28, Chap. 17.3.1]) as follows. Denote $\mathcal{F}^{\tilde{X}}_{km} = \sigma((\tilde{X}_i)_{i\leq km})$ and $\mathcal{F}^Y_{k-1} = \sigma((Y_i)_{i\leq k-1})$. For $x \in C$ let

$$r(x, y) = \frac{\delta\nu(dy)}{P^m(x, dy)}.$$

Note that the density in the definition of r is well-defined by (1.1) and the Radon-Nikodym theorem. Moreover it does not exceed δ^{-1} and so $r(x, y) \leq 1$. Now for $A_1, \ldots, A_m \in \mathcal{B}$ set

$$\begin{aligned}
&\mathbb{P}\Big(\{Y_k = 1\} \cap \bigcap_{i=1}^m \{\tilde{X}_{km+i} \in A_i\} | \mathcal{F}^{\tilde{X}}_{km}, \mathcal{F}^Y_{k-1}, \tilde{X}_{km} = x\Big) \\
&= \mathbb{P}\Big(\{Y_0 = 1\} \cap \bigcap_{i=1}^m \{\tilde{X}_i \in A_i\} | \tilde{X}_0 = x\Big) \\
&= \mathbf{1}_{\{x\in C\}} \int_{A_1} \cdots \int_{A_m} r(x, x_m) P(x_{m-1}, dx_m) P(x_{m-2}, dx_{m-1}) \cdots P(x, dx_1)
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{P}(\{Y_k = 0\} \cap \bigcap_{i=1}^m \{\tilde{X}_{km+i} \in A_i\} | \mathcal{F}^{\tilde{X}}_{km}, \mathcal{F}^Y_{k-1}, \tilde{X}_{km} = x) \\
&= \mathbb{P}(\{Y_0 = 0\} \cap \bigcap_{i=1}^m \{\tilde{X}_i \in A_i\} | \tilde{X}_0 = x) \\
&= \int_{A_1} \cdots \int_{A_m} \Big(1 - \mathbf{1}_{\{x\in C\}} r(x, x_m)\Big) P(x_{m-1}, dx_m) \cdots P(x, dx_1).
\end{aligned}$$

Note that if $\tilde{X}_{km} = x \notin C$, then (conditionally) almost surely $Y_k = 0$ and the conditional distribution of $(\tilde{X}_{km+1}, \ldots, \tilde{X}_{(k+1)m})$ given $\mathcal{F}^{\tilde{X}}_{km}, \mathcal{F}^{Y}_{k-1}, \tilde{X}_{km}$ is the same as the conditional distribution of $(X_{km+1}, \ldots, X_{(k+1)m})$ given $X_{km} = x$.

The process $(\tilde{X}_n, Y_n)_{n\geq 0}$ is usually referred to as the split chain (although some authors reserve this name for the process $(\tilde{X}_{nm}, Y_n)_{n\geq 0}$). In the special case of $m = 1$ the above construction admits a nice 'algorithmic' interpretation: if $\tilde{X}_n = x \in C$, then one tosses a coin with probability of heads equal to δ; if one gets heads, then the point $\tilde{X}_{n+1}$ is generated from the measure ν (which is independent of x) and one sets $Y_n = 1$, otherwise the new point $\tilde{X}_{n+1}$ is generated from the transition function $(P(x,\cdot) - \delta\nu(\cdot))/(1-\delta)$ and one sets $Y_n = 0$; if $\tilde{X}_n = x \notin C$, then $Y_n = 0$ and $\tilde{X}_{n+1}$ is generated from $P(x,\cdot)$. In the general case this interpretation works for the process $(\tilde{X}_{nm}, Y_n)$ and the above formulas allow to fill in the missing values of $\tilde{X}_{nm+i}$ in a consistent way.

One can easily check that $(\tilde{X}_n)_{n\geq 0}$ has the same distribution as $(X_n)_{n\geq 0}$ and so we may and will identify the two sequences (we will suppress the tilde). The auxiliary variables Y_n can be used to introduce some independence which allows to recover many results for Markov chains from corresponding statements for the independent (or one-dependent) case. Indeed, observe that if we define the stopping times

$$\tau(0) = \inf\{k \geq 0, Y_k = 1\},\ \tau(i) = \inf\{k > \tau(i-1)\colon Y_k = 1\},\ i = 1, 2, \ldots,$$

then the blocks

$$R_0 = (X_0, \ldots, X_{\tau(0)m+m-1}), R_i = (X_{m(\tau(i-1)+1)}, \ldots, X_{m\tau(i)+m-1})$$

are one-dependent, i.e. for all k, $\sigma(R_i, i < k)$ is independent of $\sigma(R_i, i > k)$. In the special case, when $m = 1$ (the *strongly aperiodic case*) the blocks R_i are independent. Moreover, for $i \geq 1$ the blocks R_i form a stationary sequence.

In particular for any function $f\colon \mathcal{X} \to \mathbb{R}$, the corresponding additive functional $\sum_{i=0}^{n} f(X_i)$ can be split (modulo the initial and final segment) into a sum (of random length) of one-dependent (independent for $m = 1$) identically distributed summands

$$s_i(f) = \sum_{j=m(\tau(i)+1)}^{m\tau(i+1)+m-1} f(X_j).$$

A crucial and very useful fact is the following equality, which follows from Pitman's occupation measure formula ([35, 36], see also Theorem 10.0.1 in [28]), i.e. for any measurable $F\colon \mathcal{X} \times \{0,1\} \to \mathbb{R}$,

$$\mathbb{E}_\nu \sum_{i=0}^{\tau(0)} F(X_{mi}, Y_i) = \delta^{-1}\pi(C)^{-1}\mathbb{E}_\pi F(X_0, Y_0), \tag{1.2}$$

where by $\mathbb{E}_\mu$ we denote the expectation for the process with X_0 distributed according to the measure μ. More precisely, if one side of (1.2) is well defined, then so is the other side and the equality holds.

It is also worth noting that independently of μ, the $\mathbb{P}_\mu$-distribution of $s_i(f)$ is equal to the $\mathbb{P}_\nu$-distribution of

$$S = S(f) = \sum_{i=0}^{\tau(0)m+m-1} f(X_i).$$

In particular, by (1.2) this easily implies that for any initial distribution μ,

$$\mathbb{E}_\mu s_i(f) = \mathbb{E}_\nu S = \delta^{-1}\pi(C)^{-1}m\int_{\mathcal{X}} fd\pi. \quad (1.3)$$

The above technique of decomposing additive functionals of Markov chains into independent or almost independent summands has proven to be very useful in studying limit theorems for Markov chains (see e.g. [8, 9, 12, 20, 28, 31, 40]), as well as in obtaining non-asymptotic concentration inequalities (see e.g. [1, 2, 13, 14]). The basic difficulty of this approach is providing proper integrability for the variable S. This is usually achieved either via pointwise drift conditions (e.g. [2, 5, 14, 28]), especially important in Markov Chain Monte Carlo algorithms or other statistical applications, when not much information regarding the behaviour of f with respect to the stationary measure is available. Such drift conditions are also useful for quantifying the ergodicity of the chain, measured in terms of integrability of the regeneration time $T = \tau(1) - \tau(0)$ (which via coupling constructions can be translated in the language of total variation norms or mixing coefficients). Classical assumptions about integrability of T are of the form $\mathbb{E}T^\alpha < \infty$ or $\mathbb{E}\exp(\theta T) < \infty$, which corresponds to polynomial or geometric ergodicity of the chain. However, recently new, modified drift conditions have been introduced [14, 15], which give other orders of integrability of T, corresponding to various subgeometric rates of ergodicity. Chains satisfying such drift conditions appear naturally in Markov Chain Monte Carlo algorithms or analysis of nonlinear autoregressive models [15].

Another line of research concerns the behaviour of the stationary chain. It is then natural to impose conditions concerning integrability of f with respect to the measure π and to assume some order of ergodicity of the chain. Such an approach has both advantages and limitations. On the one hand, investigating chains started from a point provides us with more precise information, on the other hand it requires more a priori knowledge on the relation between the function f and the transition function P (in particular one can show that in certain situations the subtle drift criteria involving the function f cannot be avoided, see e.g. [2, 21]). When dealing with the stationary case one in turn relies just on the integrability of f with respect to π and with this restricted information it is possible to obtain properties of the additive functional valid for all stationary chains admitting π as a stationary measure and possessing a prescribed order of ergodicity.

From this point of view it is natural to ask questions concerning more general notions of integrability of the variable S. In this note we will focus on Orlicz integrability. Recall that $\varphi\colon [0,\infty) \to \mathbb{R}_+$ is called a Young functions if it is strictly increasing, convex and $\varphi(0) = 0$. For a real random variable X we define the Orlicz norm corresponding to φ as

$$\|X\|_\varphi = \inf\{C > 0\colon \mathbb{E}\varphi(|X|/C) \le 1\}.$$

The Orlicz space associated to φ is the set L_φ of random variables X such that $\|X\|_\varphi < \infty$.

In what follows, we will deal with various underlying measures on the state space $\mathcal{X}$ or on the space of trajectories of the chain. To stress the dependence of the Orlicz norm on the initial distribution μ of the chain (X_n) we will denote it by $\|\cdot\|_{\mu,\varphi}$, e.g. $\|S\|_{\pi,\varphi}$ will denote the φ-Orlicz norm of the functional S for the stationary chain, whereas $\|S\|_{\nu,\varphi}$—the φ-Orlicz norm of the same functional for the chain started from initial distribution ν. We will also denote by $\|f\|_{\mu,\rho}$ the ρ-Orlicz norm of the function $f\colon \mathcal{X} \to \mathbb{R}$ when the underlying probability measure is μ. Although the notation is the same for Orlicz norms of functionals of the Markov chains and functions on $\mathcal{X}$, the meaning will always be clear from the context and thus should not lead to misunderstanding.

Remarks

1. Note that the distribution of $T = \tau(1) - \tau(0)$ is independent of the initial distribution of the chain and is equal to the distribution of $\tau(0) + 1$ for the chain starting from the measure ν. Thus $\|T\|_\psi = \|\tau(0) + 1\|_{\nu,\psi}$.
2. In [33], the authors consider ergodicity of order ψ of a Markov chain, for a special class of nondecreasing functions $\psi\colon \mathbb{N} \to \mathbb{R}_+$. They call a Markov chain ergodic of order ψ iff $\mathbb{E}_\nu \psi^\circ(T) < \infty$, where $\psi^\circ(n) = \sum_{i=1}^n \psi(i)$. Since ψ° can be extended to a convex increasing function, one can easily see that this notion is closely related to the finiteness of a proper Orlicz norm of T (related to a certain shift of the function ψ°).

We will be interested in the following two, closely related, questions.

Question 1 Given two Young functions φ and ψ and a Markov chain (X_n) such that $\|T\|_\psi < \infty$, what do we have to assume about $f\colon \mathcal{X} \to \mathbb{R}$ to guarantee that $\|S\|_{\nu,\varphi} < \infty$ (resp. $\|S\|_{\pi,\varphi} < \infty$)?

Question 2 Given two Young functions ρ and ψ, a Markov chain (X_n) such that $\|T\|_\psi < \infty$ and $f\colon \mathcal{X} \to \mathbb{R}$, such that $\|f\|_{\pi,\rho} < \infty$, what can we say about the integrability of S for the chain started from ν or from π?

As it turns out, the answers to both questions are surprisingly explicit and elementary. We present them in Sect. 2 (Theorems 2, 10, Corollaries 7, 15). The upper estimates have very short proofs, which rely only on elementary properties of Orlicz functions and the formula (1.3). They are also optimal as can be seen from Propositions 4, 11 and Theorem 5, which are proven in Sect. 3. The proofs of the optimality results are obtained by a construction of a general class of examples of Markov chains with appropriate integrability properties.

We would like to stress that despite being elementary, both the estimates and the counterexamples have non-trivial applications (some of which we present in the last section) and therefore are of considerable interest. For example when specialized to $\varphi(x) = x^2$, the estimates give optimal conditions for the CLT or LIL for Markov chains under assumptions concerning the rate of ergodicity and integrability of the test functions in the stationary case. To our best knowledge these corollaries have not been presented before in this generality, even though it is likely that they can be also obtained from general results for strongly mixing sequences derived in [16, 38, 39] (we provide a more detailed discussion in the remark after Theorem 17). Also, we are able to obtain exponential tail estimates for additive functionals of chains started from the small measure or from the stationary distribution. They may find applications both in the statistical analysis of stationary chains and in some MCMC algorithms as in some situations one is able to sample from the small measure.

In the following sections of the article we present our main estimates, demonstrate their optimality and provide applications to limit theorems and tail estimates. For the reader's convenience we gather all the basic facts about Orlicz spaces, which are used in the course of the proof, in the appendix (we refer the reader to the monographs [22, 25, 37] for a more detailed account on this class of Banach spaces).

2 Main Estimates

To simplify the notation, in what follows we will write τ instead of $\tau(0)$.

2.1 The Chain Started from ν

Assumption (A) We will assume that

$$\lim_{x\to 0} \psi(x)/x = 0 \text{ and } \psi(1) \geq 1.$$

Since any Young function on a probability space is equivalent to a function satisfying this condition (see the definition of domination and equivalence of functions below), it will not decrease the generality of our estimates, while allowing to describe them in a more concise manner (in the general case one simply needs to adjust appropriately the constants in the inequalities). In particular, the assumption (A) guarantees the correctness of the following definition (where by a generalized Young function we mean a nondecreasing convex function $\rho\colon [0,\infty) \to [0,\infty]$ with $\rho(0) = 0$, $\lim_{x\to\infty} \rho(x) = \infty$).

Definition 1 Let φ and ψ be Young functions. Assume that ψ satisfies the assumption (A). Define the generalized Young function $\rho = \rho_{\varphi,\psi}$ by the formula

$$\rho(x) = \sup_{y\geq 0} \frac{\varphi(xy) - \psi(y)}{y}.$$

Theorem 2 *Let φ and ψ be Young functions. Assume that ψ satisfies the assumption (A). Let $\rho = \rho_{\varphi,\psi}$. Then for any Harris ergodic Markov chain (X_n), a small set C, a measure ν satisfying (1.1), and all functions $f : \mathcal{X} \to \mathbb{R}$, we have*

$$\Big\| \sum_{j=0}^{m\tau+m-1} f(X_j) \Big\|_{\nu,\varphi} \le 2m\|\tau + 1\|_{\nu,\psi}\|f\|_{\pi,\rho}. \tag{2.1}$$

Proof Let $a = \|\tau + 1\|_{\nu,\psi}$, $b = \|f\|_{\pi,\rho}$. We have

$$\begin{aligned}
\mathbb{E}_\nu\varphi\Big(\frac{S}{abm}\Big) &= \mathbb{E}_\nu\varphi\Big(\frac{\sum_{j=0}^{\tau m+m-1} f(X_j)}{abm}\Big)\\
&\le \mathbb{E}_\nu \sum_{j=0}^{\tau m+m-1} \frac{\varphi(f(X_j)b^{-1}(\tau+1)a^{-1})}{(\tau+1)m}\\
&\le \mathbb{E}_\nu \sum_{j=0}^{\tau m+m-1} \frac{\rho(f(X_j)b^{-1})}{am} + \mathbb{E}_\nu \sum_{j=0}^{\tau m+m-1} \frac{\psi((\tau+1)a^{-1})}{(\tau+1)m}\\
&= \delta^{-1}\pi(C)^{-1}a^{-1}\mathbb{E}_\pi\rho(f(X_0)b^{-1}) + \mathbb{E}_\nu\psi((\tau+1)a^{-1}),
\end{aligned}$$

where the first inequality follows from convexity of φ, the second one from the definition of the function ρ and the last equality from (1.3). Let us now notice that another application of (1.3) gives

$$\mathbb{E}_\nu(\tau + 1) = \delta^{-1}\pi(C)^{-1}.$$

Thanks to the assumption $\psi(1) \ge 1$, we have $\mathbb{E}_\nu\psi((\tau + 1)\delta\pi(C)) \ge \psi(\mathbb{E}_\nu(\tau + 1)\delta\pi(C)) = \psi(1) \ge 1$, which implies that $a \ge \delta^{-1}\pi(C)^{-1}$. Combined with the definition of a and b this gives

$$\mathbb{E}_\nu\varphi\Big(\frac{S}{abm}\Big) \le 2$$

and hence $\mathbb{E}_\nu\varphi(S/(2abm)) \le \mathbb{E}_\nu 2^{-1}\varphi(S/abm) \le 1$, which ends the proof. □

Let us point out that the only ingredient of the above proof, which is related to Markov chains is the use of formula (1.3). In particular, a simple rephrasing of the proof leads to the following result, which may be of independent interest.

Theorem 3 *Let $(U_n)_{n\ge 0}$ be a sequence of real random variables and let T be a positive integer-valued random variable. Assume that for all functions $f : \mathbb{R} \to [0, \infty)$,*

$$\mathbb{E}\sum_{i=0}^{T-1} f(U_i) \le \mathbb{E}T\mathbb{E}f(U_0).$$

Let φ and ψ be two Young functions and assume that ψ satisfies the assumption (A). Let $\rho = \rho_{\varphi,\psi}$. Then

$$\|\sum_{i=0}^{T-1} U_i\|_\varphi \leq 2\|T\|_\psi \|U_0\|_\rho.$$

Note that the hypotheses of the above theorem are satisfied e.g. when U_i are i.i.d. and $T-1$ is a stopping time.

As one can see, the proof of Theorem 2 is very simple. At the same time, it turns out that the estimate given in Theorem 2 is optimal (up to constants) and thus answers completely Question 1 for the chain starting from ν. Below we present two results on optimality of Theorem 2 whose proofs are postponed to the next section.

Domination and Equivalence of Functions Consider two non-decreasing functions $\rho_1, \rho_2 \colon [0,\infty) \to [0,\infty]$, such that $\rho_1(0) = \rho_2(0) = 0$. As is classical in the theory of Orlicz spaces with respect to probabilistic measures, we say that ρ_2 dominates ρ_1 (denoted by $\rho_1 \preceq \rho_2$) if there exist positive constants C_1, C_2 and x_0, such that

$$\rho_1(x) \leq C_1\rho_2(C_2 x) \tag{2.2}$$

for $x \geq x_0$. Assume now for a while that the underlying probability space is rich enough, so that the Orlicz spaces considered are infinite-dimensional. One can easily check that if ρ_i are Young functions, then $\rho_1 \preceq \rho_2$ iff there is an inclusion and comparison of norms between the corresponding Orlicz spaces. We will say that ρ_1 and ρ_2 are equivalent ($\rho_1 \simeq \rho_2$) iff $\rho_1 \preceq \rho_2$ and $\rho_2 \preceq \rho_1$. One can also easily check that two Young functions are equivalent iff they define equivalent Orlicz norms (and the same remains true for functions equivalent to Young functions). Note also that if (2.2) holds and ρ_2 is a Young function, then $\rho_1(x) \leq \rho_2(\max(C_1, 1)C_2 x)$ for all $x \geq x_0$.

Our first optimality result is

Proposition 4 (Weak Optimality of Theorem 2) *Let φ and ψ be as in Theorem 2. Assume that a Young function ρ has the property that for every $\mathcal{X}$, every Harris ergodic Markov chain (X_n) on $\mathcal{X}$, a small set C, a small measure ν with $\|\tau\|_{\nu,\psi} < \infty$ and every function $f \colon \mathcal{X} \to \mathbb{R}$ such that $\|f\|_{\pi,\rho} < \infty$, we have $\|S(f)\|_{\nu,\varphi} < \infty$. Then $\rho_{\varphi,\psi} \preceq \rho$.*

It turns out that if we assume something more about the functions φ and ψ, the above proposition can be considerably strengthened.

Theorem 5 (Strong Optimality of Theorem 2) *Let φ, ψ and ρ be as in Theorem 2. Assume additionally that $\varphi^{-1} \circ \psi$ is equivalent to a Young function. Let Y be a random variable such that $\|Y\|_\rho = \infty$. Then there exists a Harris ergodic Markov chain (X_n) on some Polish space $\mathcal{X}$, with stationary distribution π, a small set C, a* *small measure* ν *and a function* $f: \mathcal{X} \to \mathbb{R}$*, such that the distribution of* f *under* π *is equal to the law of* Y*,* $\|\tau\|_{\nu,\psi} < \infty$ *and* $\|S(f)\|_{\nu,\varphi} = \infty$.

Remarks

1. In the last section we will see that the above theorem for $\varphi(x) = x^2$ can be used to construct examples of chains violating the central limit theorem.
2. We do not know if the additional assumption on convexity of $\varphi^{-1} \circ \psi$ is needed in the above theorem.
3. In fact in the construction we provide, the set C is an atom for the chain (i.e. in the minorization condition, $m = 1$ and $\delta = 1$).

The above results give a fairly complete answer to Question 1 for a chain started from a small measure. We will now show that Theorem 2 can be also used to derive the answer to Question 2.

Recall that the Legendre transform of a function $\rho: [0, \infty) \to \mathbb{R}_+ \cup \{\infty\}$ is defined as $\rho^* = \sup\{xy - \rho(y): y \geq 0\}$. Our answer to Question 2 is based on the following observation (which will also be used in the proof of Theorem 5).

Proposition 6 *Let* φ, ψ *be Young functions. Assume that* ψ *satisfies the assumption (A) and* $\lim_{x\to\infty} \psi(x)/x = \infty$*. Then the function* $\rho = \rho_{\varphi,\psi}$ *is equivalent to* η^**, where* $\eta = (\psi^*)^{-1} \circ \varphi^*$*. More precisely, for any* $x \geq 0$*,*

$$2\eta^*(2^{-1}x) \leq \rho(x) \leq 2^{-1}\eta^*(2x). \tag{2.3}$$

Before we prove the above proposition, let us formulate an immediate corollary, whose optimality will also be shown in the next section.

Corollary 7 *Let* ρ *and* ψ *be two Young functions. Assume that* ψ *satisfies the assumption (A) and* $\lim_{x\to\infty} \psi(x)/x = \infty$*. Then for any Harris ergodic Markov chain, small set* C*, small measure* ν *and any* $f: \mathcal{X} \to \mathbb{R}$ *we have*

$$\|S\|_{\nu,\tilde{\varphi}} \leq 4m\|(\tau + 1)\|_{\nu,\psi}\|f\|_{\pi,\rho},$$

where $\tilde{\varphi} = (\psi^* \circ \rho^*)^*$.

Proof of Proposition 6 Using the fact that $\varphi^{**} = \varphi$ we get

$$\begin{aligned}\rho(x) &= \sup_{y\geq 0} \frac{\varphi(xy) - \psi(y)}{y} = \sup_{y\geq 0}\sup_{z\geq 0} \frac{xyz - \varphi^*(z) - \psi(y)}{y} \\ &= \sup_{z\geq 0}\left(xz - \inf_{y\geq 0} \frac{\varphi^*(z) + \psi(y)}{y}\right) = \tilde{\eta}^*(x),\end{aligned}$$

where $\tilde{\eta}(z) = \inf_{y\geq 0}(\varphi^*(z) + \psi(y))y^{-1}$. Note that if $\varphi^*(z) < \infty$, then as a function of y, $\varphi^*(z)y^{-1}$ decreases, whereas $\psi(y)y^{-1}$ increases, so for all $z \geq 0$ we have

$$\frac{\varphi^*(z)}{y_0} \leq \tilde{\eta}(z) \leq 2\frac{\varphi^*(z)}{y_0},$$

where y_0 is defined by the equation $\varphi^*(z) = \psi(y_0)$, i.e. $y_0 = \psi^{-1}(\varphi^*(z))$. In combination with Lemma 22 from the Appendix, this yields

$$\frac{1}{2}\eta(z) \leq \tilde{\eta}(z) \leq 2\eta(z). \tag{2.4}$$

Note that this is also true if $\varphi^*(z) = \infty$. Now, (2.4) easily implies that $2\eta^*(x/2) \leq \rho(x) \leq 2^{-1}\eta^*(2x)$ and thus ends the proof of the proposition. □

We also have the following proposition whose proof is deferred to Sect. 3.

Proposition 8 *Let ψ and ρ be as in Corollary 7 and let φ be a Young function such that for every $\mathcal{X}$, every Markov chain (X_n) on $\mathcal{X}$, a small set C, a small measure ν and $f\colon \mathcal{X} \to \mathbb{R}$ with $\|\tau\|_{\nu,\psi} < \infty$ and $\|f\|_{\pi,\rho} < \infty$, we have $\|S\|_{\nu,\varphi} < \infty$. Then $\varphi \preceq (\psi^* \circ \rho^*)^*$.*

Examples Let us now take a closer look at consequences of our theorems for classical Young functions. The following examples are straightforward and rely only on theorems presented in the last two sections and elementary formulas for Legendre transforms of classical Young functions. The formulas we present here will be used in Sect. 4. We also note that below we consider functions of the form $x \mapsto \exp(x^\alpha) - 1$ for $\alpha \in (0, 1)$. Formally such functions are not Young functions but it is easy to see that they can be modified for small values of x in such a way that they become Young functions. It is customary to define $\|X\|_{\psi_\alpha} = \inf\{C > 0\colon \mathbb{E}\exp((|X|/C)^\alpha) \leq 2\}$. Under such definition $\|\cdot\|_{\psi_\alpha}$ is a quasi-norm, which can be shown to be equivalent to the Orlicz norm corresponding to the modified function.

1. If $\varphi(x) = x^p$ and $\psi(x) = x^r$, where $r > p \geq 1$, then $\rho_{\varphi,\psi}(x) \simeq x^{\frac{p(r-1)}{r-p}}$.
2. If $\varphi(x) = \exp(x^\alpha) - 1$ and $\psi(x) \simeq \exp(x^\beta) - 1$, where $\beta \geq \alpha$, then $\rho_{\varphi,\psi}(x) \simeq \exp(x^{\frac{\alpha\beta}{\beta-\alpha}}) - 1$.
3. If $\varphi(x) = x^p$ and $\psi(x) \simeq \exp(x^\beta) - 1$, where $\beta > 0$, then $\rho_{\varphi,\psi}(x) \simeq x^p \log^{(p-1)/\beta} x$.
4. If $\psi(x) = x^r$ $(r > 1)$ and $\rho(x) = x^p$ $(p \geq 1)$, then $\varphi(x) \simeq x^{\frac{rp}{r+p-1}}$.
5. If $\psi(x) \simeq \exp(x^\beta) - 1$ and $\rho(x) = \exp(x^\alpha) - 1$ $(\alpha, \beta > 0)$, then $\varphi(x) \simeq \exp(x^{\frac{\alpha\beta}{\alpha+\beta}}) - 1$.
6. If $\psi(x) \simeq \exp(x^\beta) - 1$ $(\beta > 0)$ and $\rho(x) = x^p$ $(p \geq 1)$, then $\varphi(x) \simeq \frac{x^p}{\log^{(p-1)/\beta} x}$.

2.2 The Stationary Case

We will now present answers to Questions 1 and 2 in the stationary case. Let us start with the following

Definition 9 Let φ and ψ be Young functions. Assume that $\lim_{x\to 0}\psi(x)/x = 0$ and define the generalized Young function $\zeta = \zeta_{\varphi,\psi}$ by the formula

$$\zeta(x) = \sup_{y\geq 0}(\varphi(xy) - y^{-1}\psi(y)).$$

In the stationary case the function ζ will play a role analogous to the one of function ρ for the chain started from the small measure.

Theorem 10 *Let φ and ψ be Young functions,* $\lim_{x\to 0}\psi(x)/x = 0$. *Let $\zeta = \zeta_{\varphi,\zeta}$. Then for any Harris ergodic Markov chain (X_n), small set C, small measure ν and all functions $f\colon \mathcal{X} \to \mathbb{R}$, we have*

$$\Big\| \sum_{j=0}^{m\tau+m-1} f(X_j)\Big\|_{\pi,\varphi} \leq m\|\tau+1\|_{\nu,\psi}\Big(1+\delta\pi(C)\|\tau+1\|_{\nu,\psi}\Big)\|f\|_{\pi,\zeta}. \tag{2.5}$$

Proof The proof is very similar to the proof of Theorem 2, however it involves one more application of Pitman's formula to pass from the stationary case to the case of the chain started from ν.

Consider any functional $F\colon \mathcal{X}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}} \to \mathbb{R}$ (measurable with respect to the product σ-field) on the space of the trajectories of the process $(X_n, Y_n)_{n\geq 0}$ (recall from the introduction that we identify X_n and $\tilde{X}_n$). By the definition of the split chain, independently of the distribution of X_0, we have for any $i \in \mathbb{N}$,

$$\mathbb{E}(F((X_j)_{j\geq im}, (Y_j)_{j\geq i})|\mathcal{F}^X_{im}, \mathcal{F}^Y_i) = G(X_{im}, Y_i),$$

where $G(x,y) = \mathbb{E}_{(x,y)}F((X_i)_{i\geq 0}, (Y_i)_{i\geq 0}) = \mathbb{E}F((X_i)_{i\geq 0}, (Y_i)_{i\geq 0}|X_0 = x, Y_0 = y)$. In particular for the functional

$$F((X_i)_{i\geq 0}, (Y_i)_{i\geq 0}) = \varphi\Big((abm)^{-1}\sum_{i=0}^{m\tau+m-1} f(X_i)\Big),$$

where $a = \|\tau+1\|_{\nu,\psi}$ and $b = \|f\|_{\pi,\zeta}$, we have

$$\mathbb{E}_\pi\varphi((abm)^{-1}\sum_{i=0}^{m\tau+m-1} f(X_i)) = \mathbb{E}_\pi G(X_0, Y_0) = \delta\pi(C)\mathbb{E}_\nu\sum_{i=0}^{\tau} G(X_{im}, Y_i)$$

$$= \delta\pi(C)\sum_{i=0}^{\infty}\mathbb{E}_\nu G(X_{im}, Y_i)\mathbf{1}_{\{i\leq\tau\}}$$

$$
\begin{aligned}
&= \delta\pi(C) \sum_{i=0}^{\infty} \mathbb{E}_\nu \mathbb{E}\Big(F((X_j)_{j\geq im}, (Y_j)_{j\geq i}) | \mathcal{F}^X_{im}, \mathcal{F}^Y_i \Big) \mathbf{1}_{\{i\leq\tau\}} \\
&= \delta\pi(C) \sum_{i=0}^{\infty} \mathbb{E}_\nu \varphi\Big((abm)^{-1} \sum_{j=im}^{m\tau+m-1} f(X_j) \Big) \mathbf{1}_{\{i\leq\tau\}} \\
&= \delta\pi(C)\mathbb{E}_\nu \sum_{i=0}^{\tau} \varphi\Big((abm)^{-1} \sum_{j=im}^{m\tau+m-1} f(X_j) \Big) \\
&\leq \delta\pi(C)\mathbb{E}_\nu \sum_{i=0}^{\tau} \sum_{j=im}^{m\tau+m-1} \frac{1}{m(\tau-i+1)} \varphi((ab)^{-1}(\tau-i+1)f(X_j)) \\
&= \delta\pi(C)\mathbb{E}_\nu \sum_{j=0}^{m\tau+m-1} \sum_{i=0}^{\lfloor j/m \rfloor} \frac{1}{m(\tau-i+1)} \varphi((ab)^{-1}(\tau-i+1)f(X_j)) \\
&\leq \delta\pi(C)\mathbb{E}_\nu \sum_{j=0}^{m\tau+m-1} \frac{\lfloor jm^{-1} \rfloor + 1}{m(\tau+1)} \varphi((ab)^{-1}(\tau+1)f(X_j)),
\end{aligned}
$$

where the second equality follows from (1.2) and the two last inequalities from the convexity of φ.

We thus obtain

$$
\begin{aligned}
\mathbb{E}_\pi \varphi((abm)^{-1}S(f)) &\leq \delta\pi(C)m^{-1}\mathbb{E}_\nu \sum_{i=0}^{m\tau+m-1} \varphi((ab)^{-1}(\tau+1)f(X_i)) \\
&\leq \delta\pi(C)m^{-1}\mathbb{E}_\nu \sum_{i=0}^{m\tau+m-1} \zeta(b^{-1}f(X_i)) + \delta\pi(C)a\mathbb{E}_\nu\psi(a^{-1}(\tau+1)) \\
&\leq \mathbb{E}_\pi\zeta(b^{-1}f(X_0)) + \delta\pi(C)a\mathbb{E}_\nu\psi(a^{-1}(\tau+1)) \leq 1 + \delta\pi(C)a,
\end{aligned}
$$

where we used (1.3). This ends the proof of the theorem. □

Remark The dependence of the estimates presented in the above theorem on $\|\tau+1\|_{\nu,\psi}$ cannot be improved in the case of general Orlicz functions, since for $\varphi(x) = x$, $\psi(x) = x^2$, and $f \equiv 1$ we have $\|S(f)\|_{\pi,\varphi} = \mathbb{E}_\pi(\tau+1) \simeq \mathbb{E}_\nu(\tau+1)^2 = \|\tau+1\|^2_{\nu,\psi}$. However, under additional assumptions on the growth of φ, one can obtain a better estimate and replace the factor $1 + \delta\pi(C)\|\tau+1\|_{\nu,\psi}$ by $g(1 + \delta\pi(C)\|\tau+1\|_{\nu,\psi})$, where $g(r) = \sup_{x>0} x/\varphi^{-1}(\varphi(x)/r)$. For rapidly growing φ and large $\|\tau+1\|_{\nu,\psi}$ this may be an important improvement. It is also elementary to check that for $\varphi(x) = \exp(x^\gamma) - 1$, we can use $g(r) \simeq \log^{1/\gamma}(r)$.

Just as in the case of Theorem 2, the estimates given in Theorem 10 are optimal. Below we state the corresponding optimality results, deferring their proofs to Sect. 3.

Proposition 11 (Weak Optimality of Theorem 10) *Let φ and ψ be as in Theorem 10. Assume that a Young function ζ has the property that for every $\mathcal{X}$, every Harris ergodic Markov chain (X_n) on $\mathcal{X}$, small set C and small measure ν with $\|\tau\|_{\nu,\psi} < \infty$ and every function $f: \mathcal{X} \to \mathbb{R}$ such that $\|f\|_{\pi,\zeta} < \infty$, we have $\|S(f)\|_{\pi,\varphi} < \infty$. Then $\zeta_{\varphi,\psi} \preceq \zeta$.*

Theorem 12 (Strong Optimality of Theorem 10) *Let φ, ψ and ζ be as in Theorem 10. Let $\tilde{\psi}(x) = \psi(x)/x$ and assume that $\tilde{\psi}$ is strictly increasing, $\tilde{\psi}(0) = 0$, $\tilde{\psi}(\infty) = \infty$. Assume additionally that the function $\eta = \varphi^{-1} \circ \tilde{\psi}$ is equivalent to a Young function. Let Y be a random variable such that $\|Y\|_\zeta = \infty$. Then there exists a Harris ergodic Markov chain (X_n) on some Polish space $\mathcal{X}$ with stationary distribution π, small set C, small measure ν and a function $f: \mathcal{X} \to \mathbb{R}$, such that the distribution of f under π is equal to the law of Y, $\|\tau\|_{\nu,\varphi} < \infty$ and $\|S(f)\|_{\pi,\varphi} = \infty$.*

Our next goal is to provide the answer to Question 2 in the case of stationary chains. For this we will need an equivalent expression for the function ζ, given in the following

Proposition 13 *For any Young functions φ, ψ such that $\lim_{x\to 0} \psi(x)/x = 0$, the function $\zeta = \zeta_{\varphi,\psi}$ is equivalent to $\varphi \circ \eta^*$, where $\eta(x) = \varphi^{-1}(\psi(x)/x)$. More precisely, for all $x \geq 0$,*

$$\varphi(\eta^*(x)) \leq \zeta(x) \leq \frac{1}{2}\varphi(\eta^*(2x)).$$

Proof Thanks to the assumption on ψ, we have $\lim_{y\to 0}(\varphi(xy) - \psi(y)/y) = 0$, so we can restrict our attention to $y > 0$, such that $\varphi(xy) > \psi(y)/y$ (note that if there are no such y, then $\eta^*(x) = \zeta(x) = 0$ and the inequalities of the proposition are trivially true). For such y, by convexity of φ, we obtain

$$\varphi(xy) - \psi(y)/y \geq \varphi(xy - \varphi^{-1}(\psi(y)/y))$$

and

$$\varphi(xy) - \psi(y)/y \leq \varphi(xy) - \psi(y)/(2y) \leq \frac{1}{2}\varphi(2xy - \varphi^{-1}(\psi(y)/y)),$$

which, by taking the supremum over y, proves the proposition. □

We will also need the following

Lemma 14 *Assume that ζ and ψ are Young functions, $\tilde{\psi}(x) = \psi(x)/x$ is strictly increasing and $\tilde{\psi}(0) = 0$, $\tilde{\psi}(\infty) = \infty$. Let the function κ be defined by*

$$\kappa^{-1}(x) = \zeta^{-1}(x)\tilde{\psi}^{-1}(x) \tag{2.6}$$

for all $x \geq 0$. *Then there exist constants* $K, x_0 \in (0, \infty)$ *such that for all* $x \geq x_0$,

$$K^{-1}x \leq (\vartheta^*)^{-1}(x)\tilde{\psi}^{-1}(\kappa(x)) \leq 2x \tag{2.7}$$

where $\vartheta = \kappa^{-1} \circ \tilde{\psi}$.

Moreover the function $\tilde{\zeta} = \kappa \circ \vartheta^*$ *is equivalent to* ζ.

Proof Note first that $\vartheta(x) = \zeta^{-1}(\tilde{\psi}(x))x$, and so ϑ is equivalent to a Young function (e.g. by Lemma 21 in the Appendix). The inequalities (2.7) follow now by Lemma 22 from the Appendix.

Moreover

$$\tilde{\zeta}^{-1}(x) = (\vartheta^*)^{-1}(\kappa^{-1}(x))$$

and thus by (2.7) for x sufficiently large,

$$K^{-1}\zeta^{-1}(x) = K^{-1}\frac{\kappa^{-1}(x)}{\tilde{\psi}^{-1}(x)} \leq \tilde{\zeta}^{-1}(x) \leq 2\frac{\kappa^{-1}(x)}{\tilde{\psi}^{-1}(x)} = 2\zeta^{-1}(x),$$

which clearly implies that $\tilde{\zeta}(K^{-1}x) \leq \zeta(x) \leq \tilde{\zeta}(2x)$ for x large enough. □

If we now start from two Young functions ψ and ζ and assume that φ is a Young function such that $\varphi \preceq \kappa$ [with κ given by (2.6)], then $\varphi((\varphi^{-1} \circ \tilde{\psi})^*) \preceq \kappa((\kappa^{-1} \circ \tilde{\psi})^*) \simeq \zeta$. Thus the above Lemma, together with Theorem 10 and Proposition 13 immediately give the following Corollary, which gives the answer to Question 2 for the stationary chain (to distinguish it from the case of the chain started from ν, the role of ρ is now played by ζ).

Corollary 15 *Assume that* ζ *and* ψ *are Young functions,* $\tilde{\psi}(x) = \psi(x)/x$ *is strictly increasing,* $\tilde{\psi}(0) = 0$, $\tilde{\psi}(\infty) = \infty$. *Let the function* κ *be defined by (2.6). If* φ *is a Young function such that* $\varphi \preceq \kappa$, *then there exists* $K < \infty$, *such that for any Harris ergodic Markov chain* (X_n) *on* $\mathcal{X}$, *small set* C, *small measure* ν *and* $f: \mathcal{X} \to \mathbb{R}$,

$$\|S(f)\|_{\pi,\varphi} \leq K\|\tau + 1\|_{\nu,\psi}\Big(1 + \delta\pi(C)\|\tau + 1\|_{\nu,\psi}\Big)\|f\|_{\pi,\zeta}. \tag{2.8}$$

Remark For slowly growing functions ψ and ζ there may be no Orlicz function φ, such that $\varphi \preceq \kappa$. This is not surprising, since as we will see from the construction presented in Sect. 3.1, the π-integrability of $S(f)$ is closely related to integrability of functions from a point-wise product of Orlicz spaces. As a consequence, $S(f)$ may not even be integrable.

We have the following optimality result corresponding to Corollary 15. Its proof will be presented in the next section.

Proposition 16 *Assume that* ζ *and* ψ *are Young functions,* $\tilde{\psi}(x) = \psi(x)/x$ *is strictly increasing,* $\tilde{\psi}(0) = 0$, $\tilde{\psi}(\infty) = \infty$. *Let the function* κ *be defined by (2.6)* *and let φ be a Young function such that for every $\mathcal{X}$, every Harris ergodic Markov chain (X_n) on $\mathcal{X}$, small set C, small measure ν and $f: \mathcal{X} \to \mathbb{R}$ with $\|\tau\|_{\nu,\psi} < \infty$ and $\|f\|_{\pi,\zeta} < \infty$, we have $\|S(f)\|_{\pi,\varphi} < \infty$. Then $\varphi \preceq \kappa$.*

Remark By convexity of φ, the condition $\varphi \preceq \kappa$ holds iff there exists a constant $K < \infty$ and $x_0 > 0$, such that

$$K\varphi^{-1}(x) \geq \tilde{\psi}^{-1}(x)\zeta^{-1}(x)$$

for $x > x_0$. Thus, under the assumptions that $\tilde{\psi}$ is strictly increasing $\tilde{\psi}(0) = 0$, $\tilde{\psi}(\infty) = \infty$, the above condition characterizes the triples of Young functions such that $\|f\|_{\pi,\zeta} < \infty$ implies $\|S(f)\|_{\pi,\varphi} < \infty$ for all Markov chains with $\|\tau\|_{\nu,\psi} < \infty$.

Examples Just as in the previous section, we will now present some concrete formulas for classical Young functions, some of which will be used in Sect. 4 to derive tail inequalities for additive functionals of stationary Markov chains.

1. If $\varphi(x) = x^p$ and $\psi(x) = x^r$, where $r > p + 1 \geq 2$, then $\zeta_{\varphi,\psi}(x) \simeq x^{\frac{p(r-1)}{r-p-1}}$.
2. If $\varphi(x) = \exp(x^\alpha) - 1$ and $\psi(x) = \exp(x^\beta) - 1$, where $\beta \geq \alpha$, then $\zeta_{\varphi,\psi}(x) \simeq \exp(x^{\frac{\alpha\beta}{\beta-\alpha}}) - 1$.
3. If $\varphi(x) = x^p$ and $\psi(x) = \exp(x^\beta) - 1$, where $\beta > 0$, then $\zeta_{\varphi,\psi}(x) \simeq x^p \log^{p/\beta} x$.
4. If $\psi(x) = x^r$ and $\zeta(x) = x^p$ $(r \geq 2, p \geq (r-1)/(r-2))$, then $\varphi(x) \simeq x^{\frac{(r-1)p}{r+p-1}}$.
5. If $\psi(x) = \exp(x^\beta) - 1$ and $\zeta(x) = \exp(x^\alpha) - 1$ $(\alpha, \beta > 0)$, then $\varphi(x) \simeq \exp(x^{\frac{\alpha\beta}{\alpha+\beta}}) - 1$.
6. If $\psi(x) = \exp(x^\beta) - 1$ $(\beta > 0)$ and $\zeta(x) = x^p$ $(p > 1)$, then $\varphi(x) \simeq \frac{x^p}{\log^{p/\beta} x}$.

3 Proofs of Optimality

3.1 Main Counterexample

We will now introduce a general construction of a Markov chain, which will serve as an example in proofs of all our optimality theorems.

Let $\mathcal{S}$ be a Polish space and let α be a Borel probability measure on $\mathcal{S}$. Consider two Borel measurable functions $\tilde{f}: \mathcal{S} \to \mathbb{R}$ and $h: \mathcal{S} \to \mathbb{N} \setminus \{0\}$. We will construct a Markov chain on some Polish space $\mathcal{X} \supseteq \mathcal{S}$, a small set $C \subseteq \mathcal{X}$, a probability measure ν and a function $f: \mathcal{X} \to \mathbb{R}$, possessing the following properties.

Properties of the chain

(i) The condition (1.1) is satisfied with $m = 1$ and $\delta = 1$ (in other words C is an atom for the chain),
(ii) $\nu(\mathcal{S}) = 1$,
(iii) for any $x \in \mathcal{S}$, $\mathbb{P}_x(\tau + 1 = h(x)) = 1$,
(iv) for any $x \in \mathcal{S}$, $\mathbb{P}_x(S(f) = \tilde{f}(x)h(x)) = 1$,

(v) for any function $G: \mathbb{R} \to \mathbb{R}$ we have

$$\mathbb{E}_\nu G(S(f)) = R \int_{\mathcal{S}} G(\tilde{f}(x)h(x))h(x)^{-1}\alpha(dx)$$

and

$$\mathbb{E}_\nu G(\tau + 1) = R \int_{\mathcal{S}} G(h(x))h(x)^{-1}\alpha(dx),$$

where $R = (\int_C h(y)^{-1}\alpha(dy))^{-1}$,

(vi) (X_n) admits a unique stationary distribution π and the law of f under π is the same as the law of $\tilde{f}$ under α,

(vii) for any nondecreasing function $F: \mathcal{X} \to \mathbb{R}$,

$$\mathbb{E}_\pi F(|S(f)|) \geq \frac{1}{2}\int_{\mathcal{S}} F(h(x)|\tilde{f}(x)|/2)\alpha(dx).$$

(viii) if $\alpha(\{x: h(x) = 1\}) > 0$, then the chain is Harris ergodic.

Construction of the Chain Let $\mathcal{X} = \bigcup_{n=1}^{\infty}\{x \in \mathcal{S}: h(x) \geq n\} \times \{n\}$. As a disjoint union, it clearly possesses a natural structure of a measurable space inherited from $\mathcal{S}$. By Theorem 3.2.4. in [41] this structure is compatible with some Polish topology on $\mathcal{X}$. Formally, $\mathcal{S} \not\subseteq \mathcal{X}$ but it does not pose a problem as we can clearly identify $\mathcal{S}$ with $\mathcal{S} \times \{1\} = \{x \in \mathcal{S}: h(x) \geq 1\} \times \{1\}$.

The dynamics of the chain will be very simple.

- If $X_n = (x, i)$ and $h(x) > i$, then with probability one $X_{n+1} = (x, i + 1)$.
- If $X_n = (x, i)$ and $h(x) = i$, then $X_{n+1} = (y, 1)$, where y is distributed according to the probability measure

$$\nu(dy) = Rh(y)^{-1}\alpha(dy). \tag{3.1}$$

More formally, the transition function of the chain is given by

$$P((x,i), A) = \begin{cases} \delta_{(x,i+1)}(A) & \text{if } i < h(x) \\ \nu(\{y \in \mathcal{S}: (y,1) \in A\}) & \text{if } i = h(x). \end{cases}$$

In other words, the chain describes a particle, which after departing from a point $(x, 1) \in \mathcal{S}$ changes its 'level' by jumping deterministically to points $(x, 2), \ldots, (x, h(x))$ and then goes back to 'level' one by selecting the first coordinate according to the measure ν.

Clearly, $\nu(\mathcal{S}) = 1$ and so condition (ii) is satisfied. Note that α and ν are formally measures on $\mathcal{S}$, but we may and will sometimes treat them as measures on $\mathcal{X}$.

Let now $C = \{(x,i) \in \mathcal{X}: h(x) = i\}$. Then $P((x,i), A) = \nu(A)$ for any $(x,i) \in C$ and a Borel subset A of $\mathcal{X}$, which shows that (1.1) holds with $m = 1$ and $\delta = 1$.

Let us now prove condition (iii). Since C is an atom for the chain, $Y_n = 1$ iff $X_n \in C$. Moreover, if $X_0 = (x,1) \simeq x \in \mathcal{S}$, then $X_i = (x, i+1)$ for $i + 1 \leq h(x)$ and $\tau = \inf\{i \geq 0: X_i \in C\} = \inf\{i \geq 0: i+1 = h(x)\} = h(x) - 1$, which proves property (iii).

To assure that property (iv) holds, it is enough to define

$$f((x,i)) = \tilde{f}(x),$$

since then $X_0 = (x,1)$ implies that $f(X_n) = \tilde{f}(x)$ for $n \leq \tau$.

Condition (v) follows now from properties (ii), (iii) and (iv) together with formula (3.1).

We will now pass to conditions (vi) and (vii).

By the construction of the chain it is easy to prove that the chain admits a unique stationary measure π given by

$$\pi(A \times \{k\}) = \alpha(A) n^{-1}$$

for $A \subseteq \{x \in \mathcal{S}: h(x) = n\}$ and any $k \leq n$. Thus for any Borel set $B \subseteq \mathbb{R}$ we have

$$\begin{aligned}
\pi(\{(x,i) \in \mathcal{X}: f((x,i)) \in B\}) &= \pi(\{(x,i) \in \mathcal{X}: \tilde{f}(x) \in B\}) \\
&= \sum_{n \geq 1} \pi(\{(x,i) \in \mathcal{X}: h(x) = n, \tilde{f}(x) \in B\} \\
&= \sum_{n \geq 1} n \cdot n^{-1} \alpha(\{x \in \mathcal{S}: h(x) = n, \tilde{f}(x) \in B\}) \\
&= \alpha(\{x \in \mathcal{S}: \tilde{f}(x) \in B\}).
\end{aligned}$$

As for (vii), $X_0 = (x,i)$ implies that

$$S(f) = (h(x) - i + 1)\tilde{f}(x).$$

Thus, letting $A_{n,k} = \{(x,k) \in \mathcal{X}: h(x) = n\}$, $B_n = \{x \in \mathcal{S}: h(x) = n\}$, we get

$$\begin{aligned}
\mathbb{E}_\pi F(|S(f)|) &= \int_{\mathcal{X}} F((h(x) - i + 1)|\tilde{f}(x)|)\pi(d(x,i)) \\
&= \sum_{n,k} \int_{A_{n,k}} F((n - k + 1)|\tilde{f}(x)|)\pi(d(x,i)) \\
&= \sum_{n} \sum_{k \leq n} \int_{B_n} n^{-1} F((n - k + 1)|\tilde{f}(x)|)\alpha(dx)
\end{aligned}$$

$$\geq \sum_n \int_{B_n} \frac{1}{2} F(n|\tilde{f}(x)|/2)\alpha(dx)$$
$$= \frac{1}{2} \int_{\mathcal{S}} F(h(x)|\tilde{f}(x)|/2)\alpha(dx),$$

proving (vii).

Now we will prove (viii). Note that $A := \{x \in \mathcal{S}: h(x) = 1\} \subseteq C$. Thus, if $\alpha(A) > 0$, then also $\nu(C) > 0$, which proves that the chain is strongly aperiodic (see e.g. Chap. 5 of [28] or Chap. 2 of [31]). Moreover one can easily see that π is an irreducibility measure for the chain and the chain is Harris recurrent. Thus, by Proposition 6.3. of [31] the chain is Harris ergodic (in fact in [31] ergodicity is defined as aperiodicity together with positivity and Harris recurrence; Proposition 6.3. states that this is equivalent to convergence of n-step probabilities for any initial point).

What remains to be proven is condition (i). Since $\pi(C) > 0$ we have $C \in \mathcal{E}^+$, whereas inequality (1.1) for $m = \delta = 1$ is satisfied by the construction.

3.2 The Chain Started from ν

We will start with the proof of Proposition 4. The chain constructed above will allow us to reduce it to elementary techniques from the theory of Orlicz spaces.

Proof of Proposition 4 Assume that the function ρ does not satisfy the condition $\rho_{\varphi,\psi} \preceq \rho$. Thus, there exists a sequence of numbers $x_n \to \infty$ such that

$$\rho(x_n) < \rho_{\varphi,\psi}(x_n 2^{-n}).$$

By the definition of $\rho_{\varphi,\psi}$ this means that there exists a sequence $t_n > 0$ such that

$$\frac{\varphi(x_n t_n 2^{-n})}{t_n} \geq \rho(x_n) + \frac{\psi(t_n)}{t_n}.$$

One can assume that $t_n \geq 2$. Indeed, for all n large enough if $t_n \leq 2$, then

$$\frac{\varphi((x_n 2^{-1})2^{-(n-1)} \cdot 2)}{2} \geq \frac{\varphi(x_n t_n 2^{-n})}{t_n} \geq \rho(x_n) \geq 2\rho(x_n 2^{-1})$$
$$\geq \rho(x_n 2^{-1}) + \frac{\psi(2)}{2}.$$

Set $\tau_n = \lfloor t_n \rfloor$ for $n \geq 1$ and $\tau_0 = 1$. We have for $n \geq 1$,

$$\frac{\varphi(x_n \tau_n 2^{1-n})}{\tau_n} \geq \frac{\varphi(x_n t_n 2^{-n})}{t_n} \geq \rho(x_n) + \frac{\psi(t_n)}{t_n} \geq \rho(x_n) + \frac{\psi(\tau_n)}{\tau_n} \geq 1, \tag{3.2}$$

where in the last inequality we used the assumption (A). Define now $p_n = C2^{-n}(\psi(\tau_n)/\tau_n + \rho(x_n))^{-1}$, where C is a constant, so that $\sum_{n\geq 0} p_n = 1$. Consider a Polish space $\mathcal{S}$ with a probability measure α, a partition $\mathcal{S} = \bigcup_{n\geq 0} A_n$, $\alpha(A_n) = p_n$ and two functions h and $\tilde{f}$, such that $\tilde{f}(x) = x_n$ and $h(x) = \tau_n$ for $x \in A_n$.

Let $(X_n)_{n\geq 0}$ be the Markov chain obtained by applying to $\mathcal{S}$, $\tilde{f}$ and h the main construction introduced in Sect. 3.1. By property (viii) and the condition $\tau_0 = 1$, the chain is Harris ergodic. By property (v), we have

$$\mathbb{E}_\nu \psi(\tau + 1) = R\int_{\mathcal{S}} \psi(h(x))h(x)^{-1}\alpha(dx) = R\sum_{n\geq 0} \frac{\psi(\tau_n)}{\tau_n} p_n \leq 2RC,$$

where the inequality follows from the definition of p_n. Thus, the chain (X_n) satisfies $\|\tau\|_{\nu,\psi} < \infty$.

By property (vi) we get

$$\mathbb{E}_\pi \rho(f) = \int_{\mathcal{S}} \rho(\tilde{f}(x))\alpha(dx) = \sum_{n\geq 0} \rho(x_n)p_n \leq 2C.$$

On the other hand for any $\theta > 0$, we have by property (v), the construction of functions $\tilde{f}$, g and (3.2),

$$\begin{aligned}
\mathbb{E}_\nu \varphi(\theta|S(f)|) &= R\int_{\mathcal{S}} \varphi(\theta|\tilde{f}(x)|h(x))h(x)^{-1}\alpha(dx) \\
&\geq R\sum_{n\geq 1} \frac{\varphi(2^{n-1}\theta x_n \tau_n 2^{1-n})}{\tau_n} p_n \\
&\geq R\sum_{n:2^{n-1}\theta\geq 1} 2^{n-1}\theta \frac{\varphi(x_n\tau_n 2^{1-n})}{\tau_n} p_n \\
&\geq R\sum_{n:2^{n-1}\theta\geq 1} 2^{n-1}\theta \Big(\rho(x_n) + \frac{\psi(\tau_n)}{\tau_n}\Big) p_n = \infty,
\end{aligned}$$

which shows that $\|S(f)\|_{\nu,\varphi} = \infty$ and proves the proposition. □

Proof of Theorem 5 Let $\mathcal{S}$ be a Polish space, α a probability measure on $\mathcal{S}$ and $\tilde{f}: \mathcal{S} \to \mathbb{R}$ a function whose law under α is the same as the law of Y.

We will consider in detail only the case when $\lim_{x\to\infty} \varphi(x)/x = \infty$. It is easy to see, using formula (1.3) and the construction below, that in the case $\varphi \simeq id$ the theorem also holds (note that in this case also $\rho \simeq id$).

By the convexity assumption and Lemma 23 from the Appendix, we obtain that $\eta = (\psi^*)^{-1} \circ \varphi^*$ is equivalent to a Young function. Thus by Proposition 6 and Lemma 22 from the Appendix we get

$$\rho^*(\cdot) \simeq (\psi^*)^{-1} \circ \varphi^*(\cdot) \simeq \frac{\varphi^*(\cdot)}{\psi^{-1} \circ \varphi^*(\cdot)}. \tag{3.3}$$

By Lemma 20 in the Appendix (or in the case when $\rho^* \simeq id$ by the well known facts about the spaces L_1 and L_∞), there exists a function $g\colon \mathcal{S} \to \mathbb{R}_+$ such that

$$\int_{\mathcal{S}} \frac{\varphi^*(g(x))}{\psi^{-1}(\varphi^*(g(x)))}\alpha(dx) < \infty \text{ and } \int_{\mathcal{S}} |\tilde{f}(x)|g(x)\alpha(dx) = \infty. \tag{3.4}$$

Define the function $h\colon \mathcal{S} \to \mathbb{N} \setminus \{0\}$ by $h(x) = \lfloor \psi^{-1}(\varphi^*(g(x)))\rfloor + 1$.

Let now $\mathcal{X}$, (X_n) and f be the Polish space, Markov chain and function obtained from $\mathcal{S}, \alpha, \tilde{f}, h$ according to the main construction of Sect. 3.1. Note that we can assume that $\alpha(\{x\colon h(x) = 1\}) > 0$ and thus by property (viii) this chain is Harris ergodic.

Note that by the definition of h, if $h(x) \ge 2$, then $h(x) \le 2\psi^{-1}(\varphi^*(g(x)))$. Thus, by property (v) and (3.4), we get

$$\mathbb{E}_\nu \psi((\tau+1)/2) = R\int_{\mathcal{S}} \psi(h(x)/2)h(x)^{-1}\alpha(dx)$$
$$\le R\psi(1/2) + R\int_{\mathcal{S}} \frac{\varphi^*(g(x))}{\psi^{-1}(\varphi^*(g(x)))}\alpha(dx) < \infty,$$

which implies that $\|\tau\|_{\nu,\psi} < \infty$. Recall now the definition of ν, given in (3.1). By the property (v), for all $a > 0$, we have

$$\mathbb{E}_\nu \varphi(|S(f)|/a) = \int_{\mathcal{S}} \varphi(|\tilde{f}(x)|h(x)/a)\nu(dx),$$

which implies that $\|S(f)\|_{\nu,\varphi} < \infty$ iff $\|\tilde{f}h\|_{\nu,\varphi} < \infty$ (note that on the left hand side ν is treated as a measure on $\mathcal{X}$ and on the right hand side as a measure on $\mathcal{S}$).

Note however, that by (3.4) we have

$$\int_{\mathcal{S}} \varphi^*(g(x))\nu(dx) = R\int_{\mathcal{S}} \frac{\varphi^*(g(x))}{h(x)}\alpha(dx) \le R\int_{\mathcal{S}} \frac{\varphi^*(g(x))}{\psi^{-1}(\varphi^*(g(x)))}\alpha(dx) < \infty,$$

which gives $\|g\|_{\nu,\varphi^*} < \infty$, but

$$\int_{\mathcal{S}} |\tilde{f}(x)|h(x)g(x)\nu(dx) = R\int_{\mathcal{S}} |\tilde{f}(x)|g(x)\alpha(dx) = \infty.$$

This shows that $\|\tilde{f}h\|_{\nu,\varphi} = \infty$ and ends the proof. □

Proof of Proposition 8 Note that for any function f (not necessarily equivalent to a Young function) we have $f^{**} \le f$. Thus, by Proposition 6, we have

$$\rho_{\varphi,\psi} \preceq \rho \iff ((\psi^*)^{-1} \circ \varphi^*)^* \preceq \rho$$
$$\implies \rho^* \preceq (\psi^*)^{-1} \circ \varphi^* \iff \psi^* \circ \rho^* \preceq \varphi^* \iff \varphi \preceq (\psi^* \circ \rho^*)^*,$$

which ends the proof by Proposition 4. □

3.3 The Stationary Case

For the proofs of results concerning optimality of our estimates for the chain started from π we will also use the general construction from Sect. 3.1. As already mentioned, in this case the problem turns out to be closely related to the classical theory of point-wise multiplication of Orlicz spaces (we refer to [34] for an overview).

Proof of Proposition 11 Assume that the function ζ does not satisfy the condition $\zeta_{\varphi,\psi} \preceq \zeta$. Thus, there exists a sequence of numbers $x_n \to \infty$, such that $\zeta(x_n) < \zeta_{\varphi,\psi}(x_n 2^{-n})$, i.e. for some sequence $t_n > 0$, $n = 1, 2, \ldots$, we have

$$\varphi(x_n 2^{-n} t_n) \geq \zeta(x_n) + \psi(t_n)/t_n.$$

Similarly as in the proof of Proposition 4, we show that without loss of generality we can assume that t_n are positive integers and thus the right hand side above is bounded from below. Let us additionally define $t_0 = 1$.

Let $p_n = C2^{-n}(\zeta(x_n) + \psi(t_n)/t_n)^{-1}$, where C is such that $\sum_{n\geq 0} p_n = 1$, and consider a probability space $(\mathcal{S}, \alpha)$, where $S = \bigcup_{n=0}^{\infty} A_n$ with A_n disjoint and $\alpha(A_n) = p_n$, together with two functions $\tilde{f}: \mathcal{S} \to \mathbb{R}$ and $h: \mathcal{S} \to \mathbb{R}$ such that for $x \in A_n$, we have $\tilde{f}(x) = x_n$, $h(x) = t_n$.

By applying to $\mathcal{S}, \tilde{f}$ and h the general construction of Sect. 3.1, we get a Harris ergodic Markov chain and a function f, which by properties (v) and (vi) satisfy

$$\mathbb{E}_\nu \psi(\tau + 1) = R \sum_{n\geq 0} \frac{\psi(t_n)}{t_n} p_n \leq 2RC,$$

$$\mathbb{E}_\pi \zeta(f) = \sum_{n\geq 0} \zeta(x_n) p_n \leq C.$$

However, by property (vii) we get for any $\theta > 0$,

$$\begin{aligned}
\mathbb{E}_\pi \varphi(\theta |S(f)|) &\geq \frac{1}{2} \int_{\mathcal{S}} \varphi(\theta h(x) |\tilde{f}(x)|/2) \alpha(dx) = \frac{1}{2} \sum_{n\geq 0} \varphi(\theta x_n t_n/2) p_n \\
&\geq \frac{1}{2} \sum_{n\geq 1} \varphi(2^{n-1} \theta x_n t_n 2^{-n}) p_n \\
&\geq \frac{1}{2} \sum_{n: 2^{n-1}\theta \geq 1} 2^{n-1} \theta \varphi(x_n t_n 2^{-n}) p_n \\
&\geq \frac{1}{2} \sum_{n: 2^{n-1}\theta \geq 1} 2^{n-1} \theta (\zeta(x_n) + \psi(t_n)/t_n) p_n = \infty,
\end{aligned}$$

which ends the proof. □

Proof of Theorem 12 Consider first the case $\lim_{x\to\infty} \eta(x)/x = \infty$.

Note that under the assumptions of the lemma, $\tilde{\psi}$ is also equivalent to a Young function. We will show that for some constant C and x large enough we have

$$\varphi^{-1}(x) \le C\zeta^{-1}(x)\tilde{\psi}^{-1}(x). \tag{3.5}$$

We have $\eta^{-1}(x) = \tilde{\psi}^{-1}(\varphi(x))$ and thus, by the assumption on η and Lemma 22 from the Appendix, we get $\varphi^{-1}(x) \le C(\eta^*)^{-1}(\varphi^{-1}(x))\tilde{\psi}^{-1}(x)$ for some constant $C < \infty$ and x large enough. But by Proposition 13, $(\eta^*)^{-1}(\varphi^{-1}(x)) \le 2\zeta^{-1}(x)$ and thus (3.5) follows.

If $\lim_{x\to\infty} \eta(x)/x < \infty$, then (3.5) also holds if we interpret ζ^{-1} as the generalized inverse (note that in this case $L_\zeta = L_\infty$)

Theorem 1 from [26] states that if $\varphi, \zeta, \tilde{\psi}$ are Young functions such that (3.5) holds for all $x \in [0,\infty)$ and Y is a random variable such that $\|Y\|_\zeta = \infty$, then there exists a random variable X, such that $\|X\|_{\tilde{\psi}} < \infty$ and $\|XY\|_\varphi = \infty$. One can easily see that the functions $\varphi, \zeta, \tilde{\psi}$ can be modified to equivalent Young functions such that (3.5) holds for all $x \ge 0$ (possibly with a different C). Thus, there exists X satisfying the above condition. Clearly, one can assume that with probability one X is a positive integer and $\mathbb{P}(X = 1) > 0$. Consider now a Polish space $(\mathcal{S}, \alpha)$ and $\tilde{f}, h: \mathcal{S} \to \mathbb{R}$ such that $(\tilde{f}, h)$ is distributed with respect to α as (Y, X). Let (X_n) be the Markov chain given by the construction of Sect. 3.1. By property (v) we have

$$\mathbb{E}_\nu \psi\Big(\frac{\tau+1}{a}\Big) = R\int_{\mathcal{S}} \psi\Big(\frac{h(x)}{a}\Big)h(x)^{-1}\alpha(dx) = \frac{R}{a}\mathbb{E}\tilde{\psi}\Big(\frac{X}{a}\Big) < \infty$$

for a large enough, since $\|X\|_{\tilde{\psi}} < \infty$. By property (vi), the law of f under π is equal to the law of Y. Finally, by property (vii), for every $a > 0$,

$$\mathbb{E}_\pi \varphi\Big(\frac{|S(f)|}{a}\Big) \ge 2^{-1}\mathbb{E}\varphi\Big(\frac{XY}{2a}\Big) = \infty,$$

which proves that $\|S(f)\|_{\pi,\varphi} = \infty$. □

Proof of Proposition 16 Let $\eta = \varphi^{-1} \circ \tilde{\psi}$. By Propositions 11, 13 and Lemma 14 we have

$$\varphi \circ \eta^* \preceq \zeta \simeq \kappa \circ \vartheta^*, \tag{3.6}$$

where $\vartheta = \kappa^{-1} \circ \tilde{\psi}$. In particular η^* is finite and so $\lim_{x\to\infty} \eta(x)/x \ge \lim_{x\to\infty} \eta^{**}(x)/x = \infty$. By (3.6), $(\vartheta^*)^{-1} \circ \kappa^{-1} \preceq (\eta^*)^{-1} \circ \varphi^{-1}$. Another application of Lemma 14 together with Lemma 22 in the Appendix yields for some constant $C \in (1,\infty)$ and x large enough,

$$\begin{aligned}\kappa^{-1}(x) &\le C(\vartheta^*)^{-1}(\kappa^{-1}(x))\tilde{\psi}^{-1}(x) \le C^2(\eta^*)^{-1}(\varphi^{-1}(Cx))\tilde{\psi}^{-1}(Cx)\\ &= C^2(\eta^*)^{-1}(\varphi^{-1}(Cx))\eta^{-1}(\varphi^{-1}(Cx)) \le 2C^2\varphi^{-1}(Cx),\end{aligned}$$

which implies that $\varphi \preceq \kappa$. □

4 Applications

4.1 Limit Theorems for Additive Functionals

It is well known that for a Harris ergodic Markov chain and a function f, the CLT

$$\frac{f(X_0) + \ldots + f(X_{n-1})}{\sqrt{n}} \xrightarrow{d} \mathcal{N}(0, \sigma_f^2) \tag{4.1}$$

holds in the stationary case iff it holds for any initial distribution.

Moreover (see [12] and [6]) under the assumption that $\mathbb{E}_\pi f^2 < \infty$, the above CLT holds iff $\mathbb{E}_\nu S(f) = 0$, $\mathbb{E}_\nu (S(f))^2 < \infty$ and the asymptotic variance is given by $\sigma_f^2 = \delta\pi(C)m^{-1}(\mathbb{E}s_1(f)^2 + 2\mathbb{E}s_1(f)s_2(f))$. If the chain has an atom, this equivalence holds without the assumption $\mathbb{E}_\pi f^2 < \infty$.

It is also known (see [12]) that the condition $\mathbb{E}_\pi f = 0$, $\mathbb{E}_\nu S(|f|)^2 < \infty$ implies the law of the iterated logarithm

$$-\sigma_f = \liminf_{n\to\infty} \frac{\sum_{i=0}^{n-1} f(X_i)}{\sqrt{n \log\log n}} \leq \limsup_{n\to\infty} \frac{\sum_{i=0}^{n-1} f(X_i)}{\sqrt{n \log\log n}} = \sigma_f \ a.s. \tag{4.2}$$

Moreover for chains with an atom $\limsup_{n\to\infty} \frac{|\sum_{i=0}^{n-1} f(X_i)|}{\sqrt{n \log\log n}} < \infty$ *a.s.* implies the CLT (see [12], Theorem 2.2. and Remark 2.3).

Our results from Sect. 2.1 can be thus applied to give optimal conditions for CLT and LIL in terms of ergodicity of the chain (expressed by Orlicz integrability of the regeneration time) and integrability of f with respect to the stationary measure.

The following theorem is an immediate consequence of Theorems 2, 5 and Proposition 4.

Theorem 17 *Consider a Harris ergodic Markov chain (X_n) on a Polish space $\mathcal{X}$ and a function $f: \mathcal{X} \to \mathbb{R}$, $\mathbb{E}_\pi f = 0$. Let ψ be a Young function such that $\lim_{x\to 0} \psi(x)/x = 0$ and assume that $\|\tau\|_{\nu,\psi} < \infty$. Let finally $\rho(x) = \tilde{\psi}^*(x^2)$, where $\tilde{\psi}(x) = \psi(x)/x$. If $\|f\|_{\pi,\rho} < \infty$, then the CLT (4.1) and LIL (4.2) hold.*

Moreover, every Young function $\tilde{\rho}$ such that $\|f\|_{\pi,\tilde{\rho}}$ implies the CLT (or LIL) for all Harris ergodic Markov chains with $\|\tau\|_{\nu,\psi} < \infty$ satisfies $\rho \preceq \tilde{\rho}$.

If the function $x \mapsto \sqrt{\psi(x)}$ is equivalent to a Young function, then for every random variable Y with $\|Y\|_\rho = \infty$ one can construct a stationary Harris ergodic Markov chain (X_n) and a function f such that $f(X_n)$ has the same law as Y, $\|\tau\|_{\nu,\psi} < \infty$ and both (4.1) and (4.2) fail.

Remark As noted in [20], in the case of geometric ergodicity, i.e. when $\psi(x) = \exp(x) - 1$, the CLT part of the above theorem can be obtained from results in [16], giving very general and optimal conditions for CLT under α-mixing. The integrability condition for f is in this case $\mathbb{E}_\pi f^2 \log_+(|f|) < \infty$. The sufficiency for the LIL part can be similarly deduced from [38].

The equivalence of the exponential decay of mixing coefficients with geometric ergodicity of Markov chains (measured in terms of ψ) follows from [32, 33]. Optimality of the condition follows from examples given in [10]. Examples of geometrically ergodic Markov chains and a function f such that $\mathbb{E}f^2 < \infty$ and the CLT fails have been also constructed in [11, 18]. Let us point out that if the Markov chain is reversible and geometrically ergodic, then $\|f\|_{\pi,2} < \infty$ implies the CLT and thus also $\mathbb{E}_\nu S(f)^2 < \infty$. Thus under this additional assumptions our formulas for $\psi(x) = \exp(x) - 1$ and $\rho(x) = x^2$ are no longer optimal (our example from Sect. 3.1 is obviously non-reversible). It would be of interest to derive counterparts of theorems from Sect. 2 under the assumption of reversibility.

It is possible that in a more general case Theorem 17 can also be recovered from the results in [10, 11, 16, 38, 39], however we have not attempted to do this in full generality (we have only verified that such an approach works in the case of $\psi(x) = x^p$). To be more precise, let us briefly explain what route one may take to give a parallel proof of the Central Limit Theorem. The results in [16, 39] provide optimal condition for CLT of stationary sequences $(Y_i)_{i=0}^{\infty}$ in terms of strong mixing coefficients and appropriate integrability of Y_0. The integrability condition requires that Y_0 belong to a certain rearrangement invariant function space, characterized in terms of mixing coefficients. On the other hand, it is well known that Harris ergodic Markov chains are strongly mixing and the mixing sequence can be bounded from above in terms of tails of the coupling time, which is closely related to the regeneration time, considered in this paper (see e.g. the formula (9.22) in [39]). Since for Harris chains it is enough to study the CLT for the stationary case, to derive a CLT in the spirit of Theorem 17 from results of [16, 39], one would need to (1) obtain bounds on the coupling time in terms of the regeneration time, (2) deduce the speed of mixing for the chain from the integrability of the coupling time, (3) relate the rearrangement invariant function space given by the speed of mixing to an appropriate Orlicz space. Clearly, there are well known tools, which can be applied at each step, however carrying out all the calculations related to steps (1) and (3) in the case of general Young functions may be a nontrivial, technical task.

Let us also remark that to our best knowledge, so far there has been no 'regeneration' proof of Theorem 17 even in the case of geometric ergodicity. In our opinion such proofs are of interest since they are usually more elementary than results for general mixing sequences, moreover they provide more 'geometric' insight into the structure of the Markov chain dynamics. The question of a corresponding 'regeneration' proof of the CLT for uniformly ergodic chains (answered in [6]) had been proposed as an open problem by Roberts and Rosenthal [40].

Berry-Esseen Type Theorems Similarly, we can use a result by Bolthausen [8, 9] to derive Berry-Esseen type bounds for additive functionals of stationary chains. More specifically, Lemma 2 in [9], together with Theorem 2 give

Theorem 18 *Let (X_n) be a stationary strongly aperiodic Harris ergodic Markov chain on $\mathcal{X}$, such that $\|\tau\|_{\nu,\psi} < \infty$, where ψ is a Young function satisfying $(x \mapsto x^3) \preceq \psi$ and $\lim_{x\to 0} \psi(x)/x = 0$. Let $\rho = \Psi^*(x^3)$, where $\Psi(x) = \psi(\sqrt{x})/\sqrt{x}$.*

Then for every $f\colon \mathcal{X} \to \mathbb{R}$ *such that* $\mathbb{E}_\pi f = 0$, $\|f\|_{\pi,\rho} < \infty$ *and* $\sigma_f^2 := \mathbb{E}(S(f))^2 > 0$, *we have*

$$\Big|\mathbb{P}\Big(\frac{\sum_{i=0}^{n-1} f(X_i) - \mathbb{E}_\pi f}{\sigma_f\sqrt{n}} \leq x\Big) - \Phi(x)\Big| = \mathcal{O}(n^{-1/2}),$$

where $\Phi(x) = (2\pi)^{-1/2}\int_{-\infty}^{x} \exp(-y^2/2)dy$.

4.2 Tail Estimates

The last application we develop concerns tail inequalities for additive functionals. The approach we take is by now fairly standard (see e.g. [1, 2, 7, 13, 15, 23, 24]) and relies on splitting the additive functional into a sum of independent (or one-dependent blocks) and using inequalities for sums of independent random variables. Our results on Orlicz integrability imply inequalities for the chain started from the small measure (an atom) or from the stationary distribution. The former case may have potential applications in MCMC algorithms in situations when small measure is known explicitly and one is able to sample from it. In what follows we denote $\psi_\alpha = \exp(x^\alpha) - 1$.

Theorem 19 *Let* $(X_n)_{n\geq 0}$ *be a Harris ergodic Markov chain on* $\mathcal{X}$. *Assume that* $\|\tau\|_{\nu,\psi_\alpha} < \infty$ *for some* $\alpha \in (0,1]$. *Let* $f\colon \mathcal{X} \to \mathbb{R}$ *be a measurable function, such that* $\mathbb{E}_\pi f = 0$. *If* $\|f\|_{\pi,\psi_\beta} < \infty$ *for some* $\beta > 0$, *then for all* $t \geq 0$,

$$\begin{aligned}
&\mathbb{P}_\nu(|f(X_0) + \ldots + f(X_{n-1})| \geq t) \qquad (4.3)\\
&\leq K\exp\Big(-\frac{t^2}{Kn\delta\pi(C)\mathbb{E}_\nu S(f)^2}\Big) + K\exp\Big(-\frac{t}{K\|f\|_{\pi,\psi_\beta}\|\tau+1\|^3_{\nu,\psi_\alpha}}\Big)\\
&+ K\exp\Big(-\frac{t^\gamma}{K(\|f\|_{\pi,\psi_\beta}\|\tau+1\|_{\nu,\psi_\alpha})^\gamma \log n}\Big)
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{P}_\pi(|f(X_0) + \ldots + f(X_{n-1})| \geq t)\\
&\leq K\exp\Big(-\frac{t^2}{Kn\delta\pi(C)\mathbb{E}_\nu S(f)^2}\Big) + K\exp\Big(-\frac{t}{K\|f\|_{\pi,\psi_\beta}\|\tau+1\|^3_{\nu,\psi_\alpha}}\Big)\\
&+ K\exp\Big(-\frac{t^\gamma}{K(\|f\|_{\pi,\psi_\beta}\|\tau+1\|_{\nu,\psi_\alpha})^\gamma \log(\|\tau+1\|_{\nu,\psi_\alpha})}\Big)\\
&+ K\exp\Big(-\frac{t^\gamma}{K(\|f\|_{\pi,\psi_\beta}\|\tau+1\|_{\nu,\psi_\alpha})^\gamma \log n}\Big),
\end{aligned}$$

where $\gamma = \frac{\alpha\beta}{\alpha+\beta}$ *and* K *depends only on* α, β *and* m *in the formula (1.1).*

Remarks

1. The proof of the above theorem is similar to those presented in [1, 2], therefore we will present only a sketch.
2. When $m = 1$, $\delta\pi(C)\mathbb{E}_\nu S(f)^2$ is the variance of the limiting Gaussian distribution for the additive functional.
3. If one does not insist on having the limiting variance in the case $m = 1$ as the subgaussian coefficient and instead replaces it by $\mathbb{E}_\nu S(f)^2$, one can get rid of the second summand on the right hand sides of the estimates (i.e. the summand containing $\|\tau + 1\|^3$).
4. One can also obtain similar results for suprema of empirical processes of a Markov chain (or equivalently for additive functionals with values in a Banach space). The difference is that one obtains then bounds on deviation above expectation and not from zero. A proof is almost the same, it simply requires a suitable generalization of an inequality for real valued summands, relying on the celebrated Talagrand's inequality, and an additional argument to take care of the expectation. Since our goal is rather to illustrate the consequences of results from Sect. 2, than to provide the most general inequalities, we do not state the details and refer the reader to [1, 2] for the special case of geometrically ergodic Markov chains. For the same reason we will not try to evaluate constants in the inequalities.
5. In a similar way one can obtain tail estimates in the polynomial case (i.e. when the regeneration time or the function f are only polynomially integrable). One just needs to use the explicit formulas for ϕ from other examples that have been discussed in Sect. 2. The estimate of the bounded part (after truncation) comes again from Bernstein's inequality, whereas the unbounded part can be handled with the Hoffman-Joergensen inequality [or its easy modifications for functions of the form $x \mapsto x^p/(\log^\beta x)$], just as e.g. in [17].
6. Similar inequalities for α-mixing sequences (however with a different subgaussian coefficient) were proved in [27]. We refer to [2] for a detailed discussion of the difference between the results of [27] and results in the spirit of Theorem 19.

Proof of Theorem 19 Below we will several times use known bounds for sums of independent random variables in the one-dependent case. Clearly, it may be done at the cost of worsening the constants by splitting the sum into sums of odd and even terms, therefore we will just write the final result without further comments. In the proof we will use the letter K to denote constants depending on α, β, m. Their values may change from one occurrence to another.

Setting $N = \inf\{i \colon m\tau(i) + m - 1 \geq n - 1\}$ we may write

$$
\begin{aligned}
&|f(X_0) + \ldots + f(X_{n-1})| \\
&\leq \sum_{i=0}^{(m\tau(0)+m-1)} |f(X_i)| + |\sum_{i=1}^{N} s_{i-1}(f)| + \sum_{i=n}^{m\tau(N)+m-1} |f(X_i)| \\
&=: I + II + III,
\end{aligned}
$$

where each of the sums on the right hand side may be interpreted as empty.

The first and last terms can be taken care of by Chebyshev's inequalities corresponding to proper Orlicz norms, using estimates of Corollaries 7 and 15 [note that $\mathbb{P}(III \geq t) \leq \mathbb{P}(I \geq t) + n\mathbb{P}(|s_0(|f|)| \geq t)$].

We will consider only the case of the chain started from ν. The stationary case is similar, simply to bound I we use the estimates of Orlicz norms for the chain started from π, given in Theorem 10 (together with the remark following it to get a better dependence on $\|\tau + 1\|_{\nu,\psi_\alpha}$).

By Corollary 7 and examples provided in Sect. 2, for $\varphi = \psi_\gamma$ we obtain

$$\|s_i(f)\|_\varphi \leq \|s_i(|f|)\|_\varphi = \|S(|f|)\|_{\nu,\psi_\gamma} \leq K\|\tau + 1\|_{\nu,\psi_\alpha}\|f\|_{\pi,\psi_\beta}.$$

Thus,

$$\mathbb{P}(I \geq t) + \mathbb{P}(III \geq t) \leq 2n\exp\Big(-\Big(\frac{t}{K\|\tau + 1\|_{\nu,\psi_\alpha}\|f\|_{\pi,\psi_\beta}}\Big)^\gamma\Big). \tag{4.4}$$

The second term can be split into $II_1 + II_2$, where

$$II_1 = |\sum_{i=1}^{N}(s_i(f)\mathbf{1}_{\{|s_i(f)|\leq a\}} - \mathbb{E}s_i(f)\mathbf{1}_{\{|s_i(f)|\leq a\}})|,$$

$$II_2 = |\sum_{i=1}^{N}(s_i(f)\mathbf{1}_{\{|s_i(f)|> a\}} - \mathbb{E}s_i(f)\mathbf{1}_{\{|s_i(f)|> a\}})|.$$

Setting $a = K_{\alpha,\beta}\|s_i(f)\|_{\psi_\gamma}\log^{1/\gamma} n \leq K_{\alpha,\beta}m\|f\|_{\pi,\psi_\beta}\|\tau+1\|_{\nu,\psi_\alpha}\log^{1/\gamma} n$, we can proceed as in Lemma 3 of [2] to get

$$\mathbb{P}(II_2 \geq t) \leq 2\exp\Big(-\Big(\frac{t}{K_{\alpha,\beta}a}\Big)^\gamma\Big). \tag{4.5}$$

It remains to bound the term II_1. Introduce the variables $T_i = \tau(i) - \tau(i-1)$, $i \geq 1$ and note that $\mathbb{E}T_i = \delta^{-1}\pi(C)^{-1}$. Recall also that T_i is distributed as $\tau + 1$ for the chain started from ν. For $4nm^{-1}\pi(C)\delta \geq 2$, we have

$$\begin{aligned}
&\mathbb{P}(N > 4nm^{-1}\pi(C)\delta) \leq \mathbb{P}\Big(\sum_{i=1}^{\lfloor 4nm^{-1}\pi(C)\delta\rfloor} T_i \leq n/m\Big)\\
&= \mathbb{P}\Big(\sum_{i=1}^{\lfloor 4nm^{-1}\pi(C)\delta\rfloor}(T_i - \mathbb{E}T_i) \leq n/m - 2nm^{-1}\pi(C)\delta\mathbb{E}T_i\Big)\\
&= \mathbb{P}\Big(\sum_{i=1}^{\lfloor 4nm^{-1}\pi(C)\delta\rfloor}(T_i - \mathbb{E}T_i) \leq -n/m\Big)\\
&\leq k(n/m),
\end{aligned} \tag{4.6}$$

where

$$k(t) = \min\Big(K_\alpha \exp\Big(-\frac{1}{K_\alpha}\min\Big(\frac{t^2 m}{n\|\tau+1\|_{\nu,\psi_\alpha}^2}, \frac{t^\alpha}{\|\tau+1\|_{\nu,\psi_\alpha}^\alpha}\Big)\Big), 1\Big).$$

The last bound for $\alpha = 1$ follows from Bernstein's ψ_1 inequality and for $\alpha < 1$ from results in [19] (as shown in Sect. 3.2.2 of [3]).

Note that if $4nm^{-1}\pi(C)\delta < 2$, then

$$\begin{aligned} k(n/m) &\geq \exp(-K_\alpha nm^{-1}\|\tau+1\|_{\nu,\psi_\alpha}^{-2}) \geq \exp(-K_\alpha nm^{-1}(\mathbb{E}\tau+1)^{-2}) \\ &= \exp(-K_\alpha nm^{-1}\delta^2\pi(C)^2) \geq \exp(-K_\alpha/2). \end{aligned}$$

Thus the above tail estimate for N remains true also in this case (after adjusting the constant K_α).

Therefore, by Bernstein's bounds on suprema of partial sums of a sequence of independent random variables bounded by $2a$, we get

$$\begin{aligned} &\mathbb{P}(II_1 \geq t) \leq \mathbb{P}(II_1 \geq t \ \& \ N \leq 4nm^{-1}\pi(C)\delta) + k(n/m) \\ &\leq K\exp\Big(-\frac{1}{K}\min\Big(\frac{t^2}{nm^{-1}\pi(C)\delta\mathbb{E}_\nu S(f)^2}, \frac{t}{\|f\|_{\pi,\psi_\beta}\|\tau+1\|_{\nu,\psi_\alpha}\log^{1/\gamma} n}\Big)\Big) \\ &+ k(n/m). \end{aligned} \tag{4.7}$$

Note that the right-hand sides of (4.4) and (4.5) as well as the first term on the right-hand side of (4.7) are dominated by the right hand-side of (4.3) (provided that the latter does not exceed one and after adjusting the constants). Now, if $k(n/m) \leq K\exp(-(t/K\|f\|_{\pi,\psi_\beta}\|\tau+1\|_{\nu,\psi_\alpha})^\alpha)$, then also $k(n/m)$ is dominated by the right-hand side of (4.3), which gives the desired estimate.

We will now provide an alternate bound on II, which will work if

$$k(n/m) > K\exp\Big(-\Big(\frac{t}{K\|f\|_{\pi,\psi_\beta}\|\tau+1\|_{\nu,\psi_\alpha}}\Big)^\alpha\Big) \tag{4.8}$$

for sufficiently large K, which will allow us to complete the proof.

Since $N \leq n/m$, by the same inequalities we used to derive (4.6) (i.e. ψ_γ Bernstein's bounds) and Lévy type inequalities (like in [1]), we obtain

$$\begin{aligned} &\mathbb{P}(II \geq t) \\ &\leq K_\gamma \exp\Big(-\frac{1}{K_\gamma}\min\Big(\frac{t^2 m}{n(\|f\|_{\pi,\psi_\beta}\|\tau+1\|_{\nu,\psi_\alpha})^2}, \Big(\frac{t}{\|f\|_{\pi,\psi_\beta}\|\tau+1\|_{\nu,\psi_\alpha}}\Big)^\gamma\Big)\Big), \end{aligned} \tag{4.9}$$

By (4.8) and the definition of k, we have

$$K\exp\Big(-\Big(\frac{t}{K\|f\|_{\pi,\psi_\beta}\|\tau+1\|_{\nu,\psi_\alpha}}\Big)^{\alpha}\Big)\le k(n/m)$$
$$\le K\exp\Big(-\Big(\frac{n}{Km\|\tau+1\|^2_{\nu,\psi_\alpha}}\Big)^{\alpha}\Big)$$

(we use here the fact that for our purposes we can assume that the constant K in (4.8) is large enough). Thus,

$$\frac{t^2m}{n(\|f\|_{\pi,\psi_\beta}\|\tau+1\|_{\nu,\psi_\alpha})^2}\ge\frac{t}{\|f\|_{\pi,\psi_\beta}\|\tau+1\|^3_{\nu,\psi_\alpha}},$$

which ends the proof by (4.9), (4.4) and (4.5). □

Appendix. Some Generalities on Orlicz Young Functions and Orlicz Spaces

All the lemmas presented below are standard facts from the theory of Orlicz spaces, we present them here for the reader's convenience. For the proof of the first lemma below see e.g. [22, 37].

Lemma 20 *If φ is a Young function, then $X\in L_\varphi$ if and only if $\mathbb{E}|XY|<\infty$ for all Y such that $\mathbb{E}\varphi^*(Y)\le 1$. Moreover, the norm*

$$\|X\|=\sup\{\mathbb{E}XY\colon \mathbb{E}\varphi^*(Y)\le 1\}$$

is equivalent to $\|X\|_\varphi$.

The next lemma is a modification of Lemma 5.4. in [29]. In the original formulation it concerns the notion of equivalence of functions (and not asymptotic equivalence relevant in our probabilistic setting). One can however easily see that the proof from [29] yields the version stated below.

Lemma 21 *Consider two increasing continuous functions $F,G\colon[0,\infty)\to[0,\infty)$ with $F(0)=G(0)=0$, $F(\infty)=G(\infty)=\infty$. The following conditions are equivalent:*

(i) $F\circ G^{-1}$ is equivalent to a Young function.
(ii) There exist positive constants C,x_0 such that

$$F\circ G^{-1}(sx)\ge C^{-1}sF\circ G^{-1}(x)$$

for all $s\ge 1$ and $x\ge x_0$.

(iii) There exist positive constants C, x_0 *such that*

$$\frac{F(sx)}{F(x)} \geq C^{-1}\frac{G(sx)}{G(x)}$$

for all $s \geq 1$, $x \geq x_0$.

Lemma 22 *For any Young function* ψ *such that* $\lim_{x\to\infty}\psi(x)/x = \infty$ *and any* $x \geq 0$,

$$x \leq (\psi^*)^{-1}(x)\psi^{-1}(x) \leq 2x.$$

Moreover, the right hand side inequality holds for any strictly increasing continuous function $\psi\colon [0,\infty) \to [0,\infty)$, *such that* $\psi(0) = 0$, $\psi(\infty) = \infty$, $\lim_{x\to\infty}\psi(x)/x = \infty$.

Lemma 23 *Let* φ *and* ψ *be two Young functions. Assume that*

$$\lim_{x\to\infty}\varphi(x)/x = \infty.$$

If $\varphi^{-1}\circ\psi$ *is equivalent to a Young function, then so is* $(\psi^*)^{-1}\circ\varphi^*$.

Proof It is easy to see that under the assumptions of the lemma we also have $\lim_{x\to\infty}\psi(x)/x = \infty$ and thus $\varphi^*(x)$, $\psi^*(x)$ are finite for all $x \geq 0$. Applying Lemma 21 with $F = \varphi^{-1}$, $G = \psi^{-1}$, we get that

$$\frac{\varphi^{-1}(sx)}{\psi^{-1}(sx)} \geq C^{-1}\frac{\varphi^{-1}(x)}{\psi^{-1}(x)}$$

for some $C > 0$, all $s \geq 1$ and x large enough. By Lemma 22 we obtain for x large enough,

$$\frac{(\psi^*)^{-1}(sx)}{(\varphi^*)^{-1}(sx)} \geq (4C)^{-1}\frac{(\psi^*)^{-1}(x)}{(\varphi^*)^{-1}(x)},$$

which ends the proof by another application of Lemma 21. □

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Bounds for Stochastic Processes on Product Index Spaces

Witold Bednorz

Abstract In this paper we discuss the question of how to bound the supremum of a stochastic process with an index set of a product type. It is tempting to approach the question by analyzing the process on each of the marginal index sets separately. However it turns out that it is necessary to also study suitable partitions of the entire index set. We show what can be done in this direction and how to use the method to reprove some known results. In particular we point out that all known applications of the Bernoulli Theorem can be obtained in this way. Moreover we use the shattering dimension to slightly extend the application to VC classes. We also show some application to the regularity of paths of processes which take values in vector spaces. Finally we give a short proof of the Mendelson–Paouris result on sums of squares for empirical processes.

Keywords Shattering dimension • Stochastic inequalities • VC classes

Mathematics Subject Classification (2010). Primary 60G15; Secondary 60G17

1 Introduction

In this paper I denotes a countable set and $(F, \|\cdot\|)$ a separable Banach space. Consider the class $\mathcal{A}$ of subsets of I. We say that the class $\mathcal{A}$ *satisfies the maximal inequality* if for any symmetric independent random variables X_i, $i \in I$ taking values

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W. Bednorz (✉)
Institute of Mathematics, University of Warsaw, Banacha 2, 02-097 Warsaw, Poland
e-mail: wbednorz@mimuw.edu.pl

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in F the following inequality holds

$$\mathbf{E}\sup_{A\in\mathcal{A}}\left\|\sum_{i\in A}X_i\right\|\leqslant K\mathbf{E}\left\|\sum_{i\in I}X_i\right\|, \tag{1.1}$$

where K depends on $\mathcal{A}$ only. We point out that in this paper K will be used to denote constants that appear in the formulation of our results and may depend on them. We use c, C, L, M to denote absolute constants, which may change their values from line to line by numerical factors. Also we write $\sim$ to express that two quantities are comparable up to a universal constant. This will help us to reduce the notation in this paper. It is an easy observation to see that (1.1) is equivalent to

$$\mathbf{E}\sup_{A\in\mathcal{A}}\left\|\sum_{i\in A}v_i\varepsilon_i\right\|\leqslant K\mathbf{E}\left\|\sum_{i\in I}v_i\varepsilon_i\right\|, \tag{1.2}$$

where $(v_i)_{i\in I}$, consists of vectors in F and $(\varepsilon_i)_{i\in I}$ is a Bernoulli sequence, i.e. a sequence of independent r.v.'s such that $\mathbf{P}(\varepsilon_i=\pm 1)=\frac{1}{2}$.

To understand what is the proper characterization of such classes $\mathcal{A}$ we recall here the notion of VC dimension. We say that $\mathcal{A}$ has VC dimension d if there exists a set $B\subset I$, $|B|=d$ such that $|\{B\cap A:\ A\in\mathcal{A}\}|=2^d$ but for all $B\subset I$, $|B|>d$, $|\{B\cap A:\ A\in\mathcal{A}\}|<2^{d+1}$. It means that $\mathcal{A}$ shatters some set B of cardinality d, but does not shatter any set of cardinality $d+1$. The result which has been proved in [1] as a corollary of the Bernoulli Theorem states that finite VC dimension is the necessary and sufficient condition for the class $\mathcal{A}$ to have the property (1.1). Since our paper refers often to the Bernoulli Theorem we recall its formulation. We begin by mentioning Talagrand's result for Gaussian's processes. In order to find two-sided bounds for supremum of the process $G(t)=\sum_{i\in I}t_ig_i$, where $t\in T\subset\ell^2(I)$ and $(g_i)_{i\in I}$ is a Gaussian sequence, i.e. a sequence of independent standard Gaussian r.v.'s we need Talagrand's $\gamma_2(T)$ numbers, cf. Definition 2.2.19 in [15] or (3.1) below. By the well known Theorem 2.4.1 in [15] we have

$$\mathbf{E}\sup_{t\in T}G(t)\sim\gamma_2(T). \tag{1.3}$$

The Bernoulli Theorem, i.e. Theorem 1.1 in [1], concerns a similar question for processes of random signs.

Theorem 1.1 *Suppose that* $T\subset\ell^2(I)$*. Then*

$$\mathbf{E}\sup_{t\in T}\sum_{i\in I}t_i\varepsilon_i\sim\inf_{T\subset T_1+T_2}\left(\sup_{t\in T_1}\|t\|_1+\gamma_2(T_2)\right).$$

where the infimum is taken over all decompositions $T_1+T_2=\{t_1+t_2:\ t_1\in T_1, t_2\in T_2\}$ *that contain the set* T *and* $\|t\|_1=\sum_{i\in I}|t_i|$.

Note that if $0 \in T$ then we can also require that $0 \in T_2$ in the above result. The consequence of Theorem 1.1 to our problem with maximal inequalities is as follows.

Theorem 1.2 *The class $\mathcal{A}$ satisfies (1.1) with a finite constant K if and only if $\mathcal{A}$ is a VC class of a finite dimension. Moreover the square root of the dimension is up to a universal constant comparable with the optimal value of K.*

Observe that part of the result is obvious. Namely one can easily show that if $\mathcal{A}$ satisfies the maximal inequality then it is necessarily a VC class of a finite dimension. Indeed let $(\varepsilon_i)_{i\in I}$ be a Bernoulli sequence. Suppose that set $B \subset I$ is shattered. Let $x_i = 1$ for $i \in B$ and $x_i = 0$, $i \notin B$. Obviously

$$\mathbf{E}\left|\sum_{i\in I} x_i\varepsilon_i\right| = \mathbf{E}\left|\sum_{i\in B}\varepsilon_i\right| \leqslant \sqrt{|B|}$$

and on the other hand

$$\mathbf{E}\sup_{A\in\mathcal{A}}\left|\sum_{i\in A} x_i\varepsilon_i\right| = \mathbf{E}\sup_{A\in\mathcal{A}}\left|\sum_{i\in A\cap B} x_i\varepsilon_i\right| \geqslant \mathbf{E}\sum_{i\in B}\varepsilon_i 1_{\varepsilon_i=1} = |B|/2.$$

Consequently if (1.1) holds then $K \geqslant \sqrt{|B|}/2$. Therefore (1.1) implies that the cardinality of B must be smaller or equal $4K^2$.

Much more difficult is to prove the converse statement, i.e. that for each VC class $\mathcal{A}$ of dimension d inequality (1.2) holds with K comparable with $\sqrt{d}$. In order to prove this result one has to first replace the basic formulation of the maximal inequality—(1.2) by its equivalent version

$$\mathbf{E}\sup_{A\in\mathcal{A}}\sup_{t\in T}\left|\sum_{i\in A} t_i\varepsilon_i\right| \leqslant K\mathbf{E}\sup_{t\in T}\left|\sum_{i\in I} t_i\varepsilon_i\right|, \tag{1.4}$$

where $(\varepsilon_i)_{i\in I}$ is a Bernoulli sequence and $0 \in T \subset \ell^2(I)$. Note that we use absolute values since part of our work concerns complex spaces. However it is important to mention that in the real case $\mathbf{E}\sup_{t\in T}\sum_{i\in I} t_i\varepsilon_i$ is comparable with $\mathbf{E}\sup_{t\in T}|\sum_{i\in I} t_i\varepsilon_i|$ if $0 \in T$ and therefore we often require in this paper that $0 \in T \subset \ell^2(I)$. Let us denote

$$b(T) = \mathbf{E}\sup_{t\in T}\left|\sum_{i\in I} t_i\varepsilon_i\right|, \quad g(T) = \mathbf{E}\sup_{t\in T}\left|\sum_{i\in I} t_i g_i\right|,$$

where $(\varepsilon_i)_{i\in I}$, $(g_i)_{i\in I}$ are respectively Bernoulli and Gaussian sequence. We recall that, what was known for a long time [5, 7], (1.4) holds when Bernoulli random variables are replaced by Gaussians, i.e.

$$\mathbf{E}\sup_{A\in\mathcal{A}}\sup_{t\in T}\left|\sum_{i\in A} g_i t_i\right| \leqslant C\sqrt{d}\mathbf{E}\sup_{t\in T}\left|\sum_{i\in I} t_i g_i\right| = C\sqrt{d}g(T), \tag{1.5}$$

for any $0 \in T \subset \ell^2(I)$. Due to Theorem 1.1 one can cover the set T by $T_1 + T_2$, where $0 \in T_2$ and

$$\max\left\{\sup_{t\in T_1}\|t\|_1, g(T_2)\right\} \leqslant Lb(T). \tag{1.6}$$

Therefore using (1.5) and (1.6)

$$\begin{aligned}
&\mathbf{E}\sup_{A\in\mathcal{A}}\sup_{t\in T}\left|\sum_{i\in A}\varepsilon_i t_i\right| \\
&\leqslant \sup_{t\in T_1}\|t\|_1 + \mathbf{E}\sup_{A\in\mathcal{A}}\sup_{t\in T_2}|\sum_{i\in A}\varepsilon_i t_i| \\
&\leqslant \sup_{t\in T_1}\|t\|_1 + \sqrt{\frac{\pi}{2}}\mathbf{E}\sup_{A\in\mathcal{A}}\sup_{t\in T_2}|\sum_{i\in A} t_i\varepsilon_i \mathbf{E}|g_i|| \\
&\leqslant \sup_{t\in T_1}\|t\|_1 + \sqrt{\frac{\pi}{2}}\mathbf{E}\sup_{A\in\mathcal{A}}\sup_{t\in T_2}|\sum_{i\in A} t_i g_i| \\
&\leqslant \sup_{t\in T_1}\|t\|_1 + \sqrt{\frac{\pi}{2}}g(T_2) \leqslant CL\sqrt{d}b(T).
\end{aligned}$$

This proves Theorem 1.2.

Here is another example in which a similar approach works. Let G be a compact Abelian group and $(v_i)_{i\in I}$ a sequence of vectors taking values in F. Let χ_i, $i \in I$ be characters on G. A deep result of Fernique [4] is

$$\mathbf{E}\sup_{h\in G}\left\|\sum_{i\in I} v_i\chi_i(h)g_i\right\| \leqslant C\left(\mathbf{E}\left\|\sum_{i\in I} v_i g_i\right\| + \sup_{\|x^*\|\leqslant 1}\mathbf{E}\sup_{h\in G}\left|\sum_{i\in I} x^*(v_i)\chi_i(h)g_i\right|\right).$$

This can be rewritten similarly as (1.5), i.e. for any $0 \in T \subset \ell^2(I)$ (which is a complex space in this case)

$$\mathbf{E} \sup_{h\in G} \sup_{t\in T} \left| \sum_{i\in I} t_i \chi_i(h) g_i \right| \leqslant C \left(g(T) + \sup_{t\in T} \mathbf{E} \sup_{h\in G} \left| \sum_{i\in I} t_i \chi_i(h) g_i \right| \right). \tag{1.7}$$

Once again the Bernoulli Theorem permits us to prove a similar result for Bernoulli sequences. Namely by Theorem 1.1 we get the decomposition $T \subset T_1 + T_2$, $0 \in T_2$ such that

$$\max \left\{ \sup_{t\in T_1} \|t\|_1, g(T_2) \right\} \leqslant Lb(T). \tag{1.8}$$

Consequently using (1.7), (1.8) and $|\chi_i(h)| \leqslant 1$ we get

$$\begin{aligned}
&\mathbf{E} \sup_{h\in G} \sup_{t\in T} \left| \sum_{i\in I} t_i \chi_i(h) \varepsilon_i \right| && (1.9)\\
&\leqslant \sup_{t\in T_1} \|t\|_1 + \mathbf{E} \sup_{h\in G} \sup_{t\in T_2} \left| \sum_{i\in A} \varepsilon_i t_i \chi_i(h) \right| \\
&\leqslant \sup_{t\in T_1} \|t\|_1 + \sqrt{\frac{\pi}{2}} \mathbf{E} \sup_{h\in G} \sup_{t\in T_2} \left| \sum_{i\in I} t_i \chi_i(h) g_i \right| \\
&\leqslant \sup_{t\in T_1} \|t\|_1 + C \left(g(T_2) + \sup_{t\in T_2} \mathbf{E} \sup_{h\in G} \left| \sum_{i\in I} t_i \chi_i(h) g_i \right| \right) \\
&\leqslant CL \left(b(T) + \sup_{t\in T_2} \mathbf{E} \sup_{h\in G} |\sum_{i\in I} t_i \chi_i(h) g_i| \right). && (1.10)
\end{aligned}$$

The final step is the Marcus–Pisier estimate [12] (see Theorem 3.2.12 in [15])

$$\sup_{t\in T_2} \mathbf{E} \sup_{h\in G} \left| \sum_{i\in I} t_i \chi_i(h) g_i \right| \leqslant M \sup_{t\in T_2} \mathbf{E} \sup_{h\in G} \left| \sum_{i\in I} t_i \chi_i(h) \varepsilon_i \right|. \tag{1.11}$$

Note that (1.11) is deeply based on the translational invariance of the distance

$$d_t(g,h) = \left(\sum_{i\in I} |t_i|^2 |\chi_i(g) - \chi_i(h)|^2 \right)^{\frac{1}{2}} \quad g, h \in G. \tag{1.12}$$

Since we may assume that $T_2 \subset T - T_1$ we get

$$\begin{aligned}
&\sup_{t\in T_2}\mathbf{E}\sup_{h\in G}\left|\sum_{i\in I} t_i\chi_i(h)\varepsilon_i\right| \\
&\leqslant \sup_{t\in T}\mathbf{E}\sup_{h\in G}\left|\sum_{i\in I} t_i\chi_i(h)\varepsilon_i\right| + \sup_{t\in T_1}\mathbf{E}\sup_{h\in G}\left|\sum_{i\in I} t_i\chi_i(h)\varepsilon_i\right| \\
&\leqslant \sup_{t\in T}\mathbf{E}\sup_{h\in G}\left|\sum_{i\in I} t_i\chi_i(h)\varepsilon_i\right| + Lb(T). \qquad (1.13)
\end{aligned}$$

Combining (1.9) with (1.11) and (1.13) we get the following result.

Theorem 1.3 *Suppose that* $0 \in T \subset \ell^2(I)$. *For any compact group G and a collection of vectors* $v_i \in F$ *in a complex Banach space* $(F, \|\cdot\|)$ *and characters* χ_i *on G the following holds*

$$\mathbf{E}\sup_{h\in G}\sup_{t\in T}\left|\sum_{i\in I} t_i\chi_i(h)\varepsilon_i\right| \leqslant K\left(b(T) + \sup_{t\in T}\mathbf{E}\sup_{h\in G}\left|\sum_{i\in I} t_i\chi_i(h)\varepsilon_i\right|\right).$$

The aim of this note is to explore the questions described above in a unified language. We consider random processes $X(u,t)$, $(u,t) \in U \times T$ with values in $\mathbb{R}$ or $\mathbb{C}$, which means that we study stochastic processes defined on product index sets. In particular we cover all canonical processes in this way. Indeed, suppose that $U \subset \mathbb{R}^I$ or $\mathbb{C}^I$ and $T \subset \mathbb{R}^I$ or $\mathbb{C}^I$ are such that for any $u \in U$ and $t \in T$ we have that $\sum_{i\in I}|u_i t_i|^2 < \infty$. Then for any family of independent random variables X_i such that $\mathbf{E}X_i = 0$, $\mathbf{E}|X_i|^2 = 1$,

$$X(u,t) = \sum_{i\in I} u_i t_i X_i, \quad u \in U, t \in T$$

is a well defined process. As we have already mentioned, our main class of examples includes Gaussian canonical processes, where $X_i = g_i$, $i \in I$ are standard normal variables and Bernoulli canonical processes, where $X_i = \varepsilon_i$, $i \in I$ are random signs. In particular, our goal is to find bounds for $\mathbf{E}\sup_{u\in U}\|\sum_{i\in I} u_i v_i \varepsilon_i\|$, where $v_i \in F$, $i \in I$, formulated in terms of $\mathbf{E}\|\sum_{i\in I} v_i\varepsilon_i\|$. One of our results is an application of the shattering dimension introduced by Mendelson and Vershynin [14], which enables us to generalize Theorem 1.2. In this way we deduce that under mild conditions on $U \subset \mathbb{R}^I$ we have

$$\mathbf{E}\sup_{u\in U}\left\|\sum_{i\in I} u_i v_i X_i\right\| \leqslant K\mathbf{E}\left\|\sum_{i\in I} v_i X_i\right\|$$

for any independent symmetric r.v.'s X_i, $i \in I$. We show how to apply the result to the analysis of convex bodies and their volume in high dimensional spaces. On the other hand we can use our approach to study processes $X(t) = (X_i(t))_{i\in I}$, $t \in [0, 1]$, which take values in $\mathbb{R}^I$ or $\mathbb{C}^I$. For example to check whether paths $t \to X(t)$ belong to ℓ^2 we should consider

$$X(u,t) = \sum_{i\in I} u_i X_i(t), \quad u = (u_i)_{i\in I} \in U, t \in [0,1],$$

where U is the unit ball in $\ell^2(I)$, i.e. $U = \{u \in \mathbb{R}^I : \sum_{i\in I} |u_i|^2 \leqslant 1\}$. The finiteness of $\|X(t)\|_2 < \infty$ is equivalent to the finiteness of $\sup_{u\in U} |X(u,t)|$. Similarly we can treat a well known question in the theory of empirical processes. Suppose that $(\mathcal{E}, \mathcal{B})$ is a measurable space and $\mathcal{F}$ a countable family of measurable real functions on $\mathcal{E}$. Let $X_1, X_2, \ldots, X_N$ be independent random variables, which take values in $(\mathcal{E}, \mathcal{B})$, we may define

$$X(u,f) = \sum_{i=1}^{N} u_i f(X_i), \quad u = (u_i)_{i=1}^{N} \in U, f \in \mathcal{F},$$

where $U = B_N(0,1) = \{u \in \mathbb{R}^N : \sum_{i=1}^N |u_i|^2 \leqslant 1\}$. Then it is clear that

$$\sup_{u\in U} |X(u,f)|^2 = \sum_{i=1}^{N} |f(X_i)|^2, \quad \text{for all } f \in \mathcal{F}.$$

In the last section we give a short proof of Mendelson–Paouris result [13] that provides an upper bound for $\mathbf{E} \sup_{u\in U} \sup_{f\in\mathcal{F}} |X(u,t)|$.

2 Upper Bounds

For the sake of exposition we shall give an idea how to bound stochastic processes. The approach we present slightly extends results of Latala [9, 10] and Mendelson–Paouris [13]. Suppose that $\mathbf{E}|X(t) - X(s)| < \infty$ for all $s,t \in T$. For each $s,t \in T$ and $n \geqslant 0$ we define $\bar{q}_n(s,t)$ as the smallest $q > 0$ such that

$$\begin{aligned} F_{q,n}(s,t) &= \mathbf{E} q^{-1}(|X(t) - X(s)| - q)_+ \\ &= \int_1^\infty \mathbf{P}(|X(t) - X(s)| > qt)dt \leqslant N_n^{-1}. \end{aligned} \tag{2.1}$$

We shall prove the following observation.

'Note that the theory we describe below can be extended to the case of arbitrary increasing numbers N_n.' but for our purposes it is better to work in the type exponential case where $N_n = 2^{2^n}$ for $n > 0$ and $N_0 = 1$.

Lemma 2.1 *Function $\bar{q}_n(s,t)$, $s,t \in T$ is a distance on T, namely is symmetric, satisfies the triangle inequality and $\bar{q}_n(s,t) = 0$ if and only if $X(s) = X(t)$ a.s.*

Proof Obviously $\bar{q}_n(s,t)$ is finite and symmetric $\bar{q}_n(s,t) = \bar{q}_n(t,s)$. To see that it equals 0 if and only if $\mathbf{P}(|X(t) - X(s)| > 0) > 0$ note that if $X(s) \neq X(t)$ then $\mathbf{E}|X(t) - X(s)| > 0$. The function $q \to F_{q,n}(s,t)$ is decreasing continuous and a.s. $F_{q,n}(s,t) \to \infty$ if $q \to 0$ and $\mathbf{E}|X(t) - X(s)| > 0$. Moreover $F_{q,n}(s,t)$ is strictly decreasing on the interval $\{q > 0 : \ F_{q,n}(s,t) > 0\}$ and consequently $\bar{q}(s,t)$ is the unique solution of the equation $F_{q,n}(s,t) = N_n^{-1}$, namely

$$\mathbf{E}(\bar{q}_n(s,t))^{-1}(|X(t) - X(s)| - \bar{q}_n(s,t))_+ = N_n^{-1}.$$

Finally we show that $\bar{q}$ satisfies the triangle inequality. Indeed for any $u, v, w \in T$ either $\bar{q}(u,v) = 0$ or $\bar{q}(v,w) = 0$ or $\bar{q}(u,w) = 0$ and the inequality is trivial or all the quantities are positive and then

$$F_{\bar{q}_n(u,v),n}(u,v) = F_{\bar{q}_n(v,w),n}(v,w) = F_{\bar{q}_n(u,w),n}(u,w) = N_n^{-1}.$$

It suffices to observe

$$\begin{aligned}
&\frac{1}{\bar{q}_n(u,v) + \bar{q}_n(v,w)}\mathbf{E}\left(|X(u) - X(w)| - \bar{q}_n(u,v) - \bar{q}_n(w,v)\right)_+ \\
&\leqslant \mathbf{E}\left(\frac{|X(u) - X(w)| + |X(w) - X(v)|}{\bar{q}_n(u,v) + \bar{q}_n(w,v)} - 1\right)_+ .
\end{aligned}$$

The function $x \to (x-1)_+$ is convex which implies that

$$(px + qy - 1)_+ \leqslant p(x-1)_+ + q(y-1)_+ \tag{2.2}$$

for $p, q \geqslant 0$, $p + q = 1$ and $x, y > 0$. We use (2.2) for

$$x = \frac{|X(u) - X(v)|}{\bar{q}_n(u,v)}, \quad y = \frac{|X(v) - X(w)|}{\bar{q}_n(v,w)}$$

and

$$p = \frac{\bar{q}_n(u,v)}{\bar{q}_n(u,v) + \bar{q}_n(w,v)}, \quad q = \frac{\bar{q}_n(v,w)}{\bar{q}_n(u,v) + \bar{q}_n(w,v)}.$$

Therefore

$$\begin{aligned}
&\frac{1}{\bar{q}_n(u,v) + \bar{q}_n(v,w)}\mathbf{E}\left(|X(u) - X(w)| - \bar{q}_n(u,v) - \bar{q}_n(w,v)\right)_+ \\
&\leqslant p\mathbf{E}\left(\frac{|X(u) - X(v)|}{\bar{q}_n(u,v)} - 1\right)_+ + q\mathbf{E}\left(\frac{|X(v) - X(w)|}{\bar{q}_n(v,w)} - 1\right)_+ \\
&\leqslant pN_n^{-1} + qN_n^{-1} = N_n^{-1}
\end{aligned}$$

which by definition gives that

$$\bar{q}_n(u,v) + \bar{q}_n(v,w) \geqslant \bar{q}_n(v,w).$$

□

Obviously usually we do not need to use the optimal distances $\bar{q}_n$ and can replace the construction by one that is sufficient for our purposes. We say that a family of distances q_n, $n \geqslant 0$ on T is *admissible* if $q_n(s,t) \geqslant \bar{q}_n(s,t)$ and $q_{n+1}(s,t) \geqslant q_n(s,t)$. For example Latala [9, 10] and Mendelson–Paouris [13] used moments, namely if $\|X(t) - X(s)\|_p = (\mathbf{E}|X(t) - X(s)|^p)^{\frac{1}{p}} < \infty$, $p \geqslant 1$ then we may take $q_n(s,t) = 2\|X(t) - X(s)\|_{2^n}$. Indeed observe that

$$\mathbf{E}\left(\frac{|X(t) - X(s)|}{2\|X(t) - X(s)\|_{2^n}} - 1\right)_+ \leqslant \frac{\mathbf{E}|X(t) - X(s)|^{2^n}}{(2\|X(t) - X(s)\|_{2^n})^{2^n}} \leqslant \frac{1}{N_n}.$$

Following Talagrand we say that a sequence of partitions $\mathcal{A}_n$, $n \geqslant 0$ of a set T is *admissible* if it is increasing, $\mathcal{A}_0 = \{T\}$ and $|\mathcal{A}_n| \leqslant N_n$. Let us also define admissible sequences of partitions $\mathcal{A} = (\mathcal{A}_n)_{n\geqslant 0}$ of the set T, which means nested sequences of partitions such that $\mathcal{A}_0 = \{T\}$ and $|\mathcal{A}_n| \leqslant N_n$. For each $A \in \mathcal{A}_n$ we define

$$\Delta_n(A) = \sup_{s,t\in A} q_n(s,t).$$

Let $A_n(t)$ denote the element of $\mathcal{A}_n$ that contains point t. For each $A \in \mathcal{A}_n$, $n \geqslant 0$ we define t_A to be an arbitrary point in A. We may and will assume that if $t_A \in B \in \mathcal{A}_{n+1}$, $B \subset A \in \mathcal{A}_n$ then $t_B = t_A$. Let $\pi_n(t) = t_{A_n(t)}$, then $\pi_0(t) = t_T$ is a fixed point in T. Let $T_n = \{\pi_n(t) : \ t \in T\}$ for $n \geqslant 0$. Clearly the T_n, $n \geqslant 0$ are nested, namely $T_n \subset T_{n+1}$ for $n \geqslant 0$. For each stochastic process $X(t)$, $t \in T$ and $\tau \geqslant 0$ we may define

$$\gamma_X^\tau(T) = \inf_{\mathcal{A}} \sup_{t\in T} \sum_{n=0}^{\infty} \Delta_{n+\tau}(A_n(t)).$$

We prove that for $\tau \geqslant 2$, $\gamma_X^\tau(T)$ is a good upper bound for $\mathbf{E}\sup_{s,t\in T} |X(t) - X(s)|$.

Theorem 2.2 *For $\tau \geqslant 2$ the following inequality holds*

$$\mathbf{E}\sup_{s,t\in T} |X(t) - X(s)| \leqslant 4\gamma_X^\tau(T).$$

Proof Note that

$$\begin{aligned} &|X(t) - X(\pi_0(t))| \\ &\leqslant \sum_{n=0}^{\infty} q_{n+\tau}(\pi_{n+1}(t), \pi_n(t)) \end{aligned}$$

$$
+\sum_{n=0}^{\infty}(|X(\pi_{n+1}(t))-X(\pi_n(t))|-q_{n+\tau}(\pi_{n+1}(t),\pi_n(t)))_+
$$

$$
\leqslant \sum_{n=0}^{\infty}\Delta_{n+\tau}(A_n(t))+\sum_{n=0}^{\infty}\sum_{u\in T_n}\sum_{v\in A_n(u)\cap T_{n+1}}(|X(u)-X(v)|-q_{n+\tau}(u,v))_+ .
$$

For any $\varepsilon > 0$ one can find nearly optimal admissible partition $(\mathcal{A}_n)_{n\geqslant 0}$ such that

$$
\sup_{t\in T}\sum_{n=0}^{\infty}\Delta_{n+\tau}(A_n(t)) \leqslant (1+\varepsilon)\gamma_X^{\tau}(T)
$$

and therefore

$$
\begin{aligned}
&\mathbf{E}\sup_{t\in T}|X(t)-X(\pi_0(t))| \\
&\leqslant \gamma_X^{\tau}(T)+\varepsilon+\sum_{n=0}^{\infty}\sum_{u\in T_n}\sum_{v\in A_n(u)\cap T_{n+1}}\frac{q_{n+\tau}(u,v)}{N_{n+\tau}} \\
&\leqslant (1+\varepsilon)\gamma_X^{\tau}(T)+\sum_{n=0}^{\infty}\sum_{u\in T_n}\Delta_{n+\tau}(A_n(u))\frac{N_{n+1}}{N_{n+\tau}} \\
&\leqslant \gamma_X^{\tau}(T)\left(1+\varepsilon+\sum_{n=0}^{\infty}\frac{N_nN_{n+1}}{N_{n+\tau}}\right) \\
&\leqslant (1+\varepsilon)\gamma_X^{\tau}(T)+\gamma_X^{\tau}(T)\left(\frac{N_0N_1}{N_2}+\sum_{n=1}^{\infty}\frac{1}{N_n}\right) \\
&\leqslant \gamma_X^{\tau}(T)\left(1+\varepsilon+\sum_{n=0}^{\infty}2^{-n-1}\right),
\end{aligned}
$$

where in the last line we used $N_n^2 = N_{n+1}$, $N_n \geqslant 2^{n+1}$ for $n \geqslant 1$ and $N_0N_1/N_2 = 1/4 < 1/2$. Since ε is arbitrary small we infer that $\mathbf{E}\sup_{t\in T}|X(t)-X(\pi_0(t))| \leqslant 2\gamma_X^{\tau}(T)$ and hence $\mathbf{E}\sup_{s,t}|X(t)-X(s)| \leqslant 4\gamma_X^{\tau}(T)$. □

The basic question is how to construct admissible sequences of partitions. The simplest way to do this goes through entropy numbers. Recall that for a given distance ρ on T the quantity $N(T,\rho,\varepsilon)$ denotes the smallest number of balls of radius ε with respect to ρ necessary to cover the set T. Consequently for a given $\tau \geqslant 0$ we define entropy numbers as

$$
e_n^{\tau} = \inf\{\varepsilon > 0:\ N(T,q_{n+\tau},\varepsilon) \leqslant N_n\},\quad n \geqslant 0.
$$

Having the construction ready we may easily produce an admissible sequence of partitions. At each level n there exists at most N_n sets of $q_{n+\tau}$ diameter 2ε that covers T. To obtain a nested sequence of partitions we must intersect all the sets constructed at levels $0, 1, \ldots, n-1$. The partition $\mathcal{A}_n$ has no more than $N_0 N_1 \ldots N_{n-1} \leqslant N_n$ elements. Moreover for each set $A \in \mathcal{A}_n$ we have

$$\Delta_{n+\tau-1}(A) \leqslant 2e_{n-1}^{\tau} \quad \text{for } n \geqslant 1.$$

Obviously $\Delta_{\tau-1}(T) \leqslant \Delta_{\tau}(T) \leqslant 2e_0^{\tau}$. Let $\mathcal{E}_X^{\tau}(T) = \sum_{n=0}^{\infty} e_n^{\tau}$, then for any $\tau \geqslant 1$

$$\gamma_X^{\tau-1}(T) \leqslant 2e_0^{\tau} + 2\sum_{n=1}^{\infty} e_{n-1}^{\tau} \leqslant 4\mathcal{E}_X^{\tau}(T).$$

and hence by Theorem 2.2 with $\tau \geqslant 3$

$$\mathbf{E} \sup_{s,t\in T} |X(t) - X(s)| \leqslant 16\mathcal{E}_X^{\tau}(T). \tag{2.3}$$

We turn to our main question of processes on product spaces. For $X(u,t)$, $u \in U$, $t \in T$ we define two different families of distances $q_{n,t}$ and $q_{n,u}$, which are admissible to control the marginal processes, respectively, $u \to X(u,t)$ on U and $t \to X(u,t)$ on T.

First for a given $t \in T$, let $q_{n,t}$, $n \geqslant 0$ be a family of distances on U admissible for the process $u \to X(u,t)$ and let $e_{n,t}^{\tau}$, $n \geqslant 0$ be entropy numbers on U constructed for $q_{n,t}$, $n \geqslant 0$. By (2.3) we infer that for $\tau \geqslant 3$

$$\mathbf{E} \sup_{u,v\in U} |X(u,t) - X(v,t)| \leqslant 16\sum_{n=0}^{\infty} e_{n,t}^{\tau}. \tag{2.4}$$

If we define $\mathcal{E}_{X,t}^{\tau}(U) = \sum_{n=0}^{\infty} e_{n,t}^{\tau}$ and $\mathcal{E}_{X,T}^{\tau}(U) = \sup_{t\in T} \mathcal{E}_{X,t}^{\tau}(U)$, then we may rewrite (2.4) as

$$\sup_{t\in T} \mathbf{E} \sup_{u,v\in U} |X(u,t) - X(v,t)| \leqslant 16\mathcal{E}_{X,T}^{\tau}(U),$$

and in this way the entropy numbers may be used to bound the family of processes $u \to X(u,t)$, where $t \in T$.

On the other hand for a given $u \in U$ let $q_{n,u}$, $n \geqslant 0$ be a family of distances on T admissible for $t \to X(u,t)$. Obviously $q_{n,U} = \sup_{u\in U} q_{n,u}$ is a good upper bound for all distances $q_{n,u}$. Let

$$\gamma_{X,U}^{\tau}(T) = \inf_{\mathcal{A}} \sup_{t\in T} \sum_{n=0}^{\infty} \Delta_{n+\tau,U}(A_n(t)),$$

where the infimum is taken over all admissible partitions $\mathcal{A} = (\mathcal{A}_n)_{n\geqslant 0}$ of T and

$$\Delta_{n,U}(A) = \sup_{s,t\in A} q_{n,U}(s,t) = \sup_{s,t\in A}\sup_{u\in U} q_{n,u}(s,t).$$

Theorem 2.2 applied to the distances $q_{n,U}$, $n \geqslant 0$ yields that

$$\sup_{u\in U}\mathbf{E}\sup_{s,t\in T}|X(u,t)-X(u,s)| \leqslant 4\gamma^{\tau}_{X,U}(T).$$

We prove that the two quantities $\mathcal{E}^{\tau}_{X,T}(U)$ and $\gamma^{\tau}_{X,U}(T)$ suffice to control the process $X(u,t)$, $u\in U$, $t\in T$.

We state our main result, which extends the idea described in Theorem 3.3.1 in [15].

Theorem 2.3 *For any* $\tau \geqslant 4$ *the following inequality holds*

$$\mathbf{E}\sup_{u,v\in U}\sup_{s,t\in T}|X(u,t)-X(v,s)| \leqslant 24(\gamma^{\tau}_{X,U}(T)+\mathcal{E}^{\tau}_{X,T}(U)).$$

Proof We first observe that if $\gamma^{\tau}_{X,U}(T) < \infty$ then, by the definition, for any $\varepsilon > 0$ there exists an admissible partition sequence $\mathcal{C} = (\mathcal{C}_n)_{n\geqslant 0}$ of T, which satisfies

$$(1+\varepsilon)\gamma^{\tau}_{X,U}(T) \geqslant \sup_{t\in T}\sum_{n=0}^{\infty}\Delta_{n+\tau,U}(C_n(t)).$$

Let us fix $C\in\mathcal{C}_n$ and let $\pi_n(C)$ be a point in T such that

$$e^{\tau}_{n,\pi_n(C)} \leqslant (1+\varepsilon)\inf\{e^{\tau}_{n,t}:\ t\in C\}.$$

Consequently, for any $n\geqslant 2$ there exists a partition $\mathcal{B}_{C,n-2}$ of the set U into at most N_{n-2} sets B that satisfy

$$\Delta_{n+\tau-2,\pi_{n-2}(C)}(B) \leqslant 2e^{\tau}_{n-2,\pi_{n-2}(C)},\quad \text{where } \Delta_{n,t}(B) = \sup_{u,v\in B} q_{n,t}(u,v).$$

Using sets $B\times C$ for $B\in\mathcal{B}_{C,n-2}$ and $C\in\mathcal{C}_{n-2}$ we get a partition $\mathcal{A}'_{n-2}$ of $U\times T$ into at most $N^2_{n-2}\leqslant N_{n-1}$ sets. Finally intersecting all the constructed sets in $\mathcal{A}'_0,\mathcal{A}'_1,\ldots,\mathcal{A}'_{n-2}$ we obtain a nested sequence of partitions $(\mathcal{A}_n)_{n\geqslant 2}$ such that $|\mathcal{A}_n|\leqslant N_n$. We complete the sequence by $\mathcal{A}_0 = \mathcal{A}_1 = \{U\times T\}$. In this way $\mathcal{A} = (\mathcal{A}_n)_{n\geqslant 0}$ is an admissible sequence of partitions for $U\times T$. Let $A_n(u,t)$ be the element of $\mathcal{A}_n$ that contains point (u,t). Clearly

$$A_n(u,t)\subset B\times C,$$

where $C = C_{n-2}(t)$ and $u \in B \in \mathcal{B}_{C,n-2}$. Therefore for $n \geqslant 2$,

$$\sup_{s,s' \in C} q_{n+\tau-2,U}(s,s') \leqslant \Delta_{n+\tau-2,U}(C)$$

and

$$\sup_{v,v' \in B} q_{n+\tau-2,\pi_{n-2}(C)}(v,v') \leqslant 2e^{\tau}_{n-2,\pi_{n-2}(C)} \leqslant 2(1+\varepsilon)e^{\tau}_{n-2,t}.$$

We turn to the analysis of optimal quantiles $\bar{q}_n$ for the process $X(u,t)$, $u \in U$, $t \in T$. We show that for any $x, y, z \in T$ and $v, w \in U$

$$\bar{q}_n((v,x),(w,y)) \leqslant q_{n,U}(x,z) + q_{n,U}(y,z) + q_{n,z}(v,w).$$

This holds due to the triangle inequality

$$\begin{aligned}
&\bar{q}_n((v,x),(w,y)) \\
&\leqslant \bar{q}_n((v,x),(v,z)) + \bar{q}_n((w,y),(w,z)) + \bar{q}_n((v,z),(w,z)) \\
&\leqslant q_{n,v}(x,z) + q_{n,w}(y,z) + q_{n,z}(v,w) \\
&\leqslant q_{n,U}(x,z) + q_{n,U}(y,z) + q_{n,z}(v,w).
\end{aligned}$$

In particular it implies that for any $(v,s) \in B \times C$

$$\begin{aligned}
&\bar{q}_{n+\tau-2}((u,t),(v,s)) \\
&\leqslant q_{n+\tau-2,U}(t,\pi_{n-2}(C)) + q_{n+\tau-2,U}(s,\pi_{n-2}(C)) + q_{n+\tau-2,\pi_{n-2}(C)}(u,v)
\end{aligned}$$

and hence

$$\Delta_{n+\tau-2}(B \times C) \leqslant 2\Delta_{n+\tau-2,U}(C_{n-2}(t)) + 2(1+\varepsilon)e^{\tau}_{n-2,t}.$$

If $\tau \geqslant 2$ then also for $n = 0, 1$,

$$\Delta_{n+\tau-2}(U \times T) \leqslant \Delta_{\tau}(U \times T) \leqslant 2\Delta_{\tau}(T) + 2(1+\varepsilon)\sup_{t \in T} e^{\tau}_{0,t}.$$

It implies that for any $\tau \geqslant 2$

$$\gamma^{\tau-2}_X(U \times T) \leqslant 4\Delta_{\tau}(T) + 4(1+\varepsilon)\sup_{t \in T} e^{\tau}_{0,t} + 2(1+\varepsilon)\gamma^{\tau}_{X,U}(T) + 2(1+\varepsilon)\mathcal{E}^{\tau}_{X,T}(U).$$

Therefore for any $\tau \geqslant 4$ we may apply Theorem 1.3 for distances $\bar{q}_n$ and in this way prove our result with the constant 24. □

In the next sections we analyze various applications of our result.

3 Gaussian Case

As we have said, the main application of our theory is to Gaussian canonical processes. Let $T \subset \ell^2(I)$ and recall that

$$G(t) = \sum_{i \in I} t_i g_i, \quad t \in T,$$

where $(g_i)_{i \in I}$ is a sequence of i.i.d. standard normal variables. For the process $G(t)$, $t \in T$ the natural distance is

$$d(s,t) = (\mathbf{E}|G(t) - G(s)|^2)^{\frac{1}{2}} = \|t - s\|_2, \quad s,t \in T.$$

It is easy to see that the optimal quantities for G satisfy

$$\bar{q}_n(s,t) \sim 2^{\frac{n}{2}} d(s,t), \quad \text{for all } s,t \in T.$$

Consequently denoting $q_n(s,t) = C 2^{\frac{n}{2}} d(s,t)$ for large enough C we get an admissible sequence of distances. Moreover,

$$\gamma_G(T) \sim \gamma_2(T,d) = \inf_{\mathcal{A}} \sup_{t \in T} \sum_{n=0}^{\infty} 2^{\frac{n}{2}} \Delta(A_n(t)), \tag{3.1}$$

where the infimum is taken over all admissible $\mathcal{A} = (\mathcal{A}_n)_{n \geqslant 0}$ sequences of partitions and $\Delta(A) = \sup_{s,t \in A} d(s,t)$. Since d is the canonical distance for $\ell^2(I)$ we usually suppress d in $\gamma_2(T,d)$ and simply write $\gamma_2(T)$. As we have pointed out in the introduction that using (1.3) we get

$$K^{-1}\gamma_2(T) \leqslant \mathbf{E} \sup_{s,t \in T} |G(t) - G(s)| \leqslant K\gamma_2(T).$$

Let us also define

$$e_n = \inf\{\varepsilon : \ N(T,d,\varepsilon) \leqslant N_n\}, \quad n \geqslant 0.$$

Obviously $e_n^{\tau} \leqslant C 2^{\frac{n+\tau}{2}} e_n$ and hence

$$\mathcal{E}_G^{\tau}(T) \leqslant C 2^{\frac{\tau}{2}} \sum_{n \geqslant 0} 2^{\frac{n}{2}} e_n.$$

Let $\mathcal{E}(T,d) = \sum_{n=0}^{\infty} 2^{\frac{n}{2}} e_n$. It is a well known fact (Theorem 3.1.1 in [15]) that

$$K^{-1}\mathcal{E}(T,d) \leqslant \int_0^{\infty} \sqrt{\log(N(T,d,\varepsilon))} d\varepsilon \leqslant K\mathcal{E}(T,d). \tag{3.2}$$

Again since d is the canonical distance on $\ell^2(I)$ we will suppress d in $\mathcal{E}(T,d)$ and write $\mathcal{E}(T)$. From (2.3) we infer Dudley's bound

$$\mathbf{E}\sup_{s,t\in T}|G(t)-G(s)| \leqslant 16C2^{\frac{\tau}{2}}\mathcal{E}(T).$$

Turning our attention to product spaces let us recall that for $U \subset \mathbb{R}^I$ or $\mathbb{C}^I$, $T \subset \mathbb{R}^I$ or C^I such that $ut = (u_i t_i)_{i\in I} \in \ell^2(I)$ for all $u \in U$ and $t \in T$ we may define

$$G(u,t) = \sum_{i\in I} u_i t_i g_i, \quad u \in U, t \in T,$$

where $(g_i)_{i\in I}$ is a Gaussian sequence. Note that for all $s,t \in T$, $u,v \in U$

$$\bar{q}_n((u,t),(v,s)) \leqslant C2^{\frac{n}{2}}\|ut - vs\|_2.$$

For each $u \in U$ and $s,t \in T$ let $d_u(s,t) = \|u(t-s)\|_2$. We may define

$$q_{n,u}(s,t) = C2^{\frac{n}{2}}d_u(s,t), \quad q_{n,U}(s,t) = C2^{\frac{n}{2}}\sup_{u\in U} d_u(s,t).$$

In particular

$$q_{n,U}(s,t) \leqslant C\sup_{u\in U}\|u\|_\infty d(s,t)$$

and therefore

$$\gamma^\tau_{G,U}(T) \leqslant C2^{\frac{\tau}{2}}\sup_{u\in U}\|u\|_\infty \gamma_2(T,d). \tag{3.3}$$

On the other hand we define for all $t \in T$ and $u,v \in U$

$$q_{n,t}(u,v) = C2^{\frac{n}{2}}\|t(u-v)\|_2. \tag{3.4}$$

For each $t \in T$ let us denote by d_t the distance on U given by

$$d_t(u,v) = \left(\sum_{i\in I}|t_i|^2|u_i - v_i|^2\right)^{\frac{1}{2}}, \quad u,v \in U.$$

Using these distances we may rewrite (3.4) as $q_{n,t} = C2^{\frac{n}{2}}d_t$. Let

$$e_{n,t} = \inf\{\varepsilon :\ N(U,d_t,\varepsilon) \leqslant N_n\}, \quad n \geqslant 0 \ \text{ and } \ \mathcal{E}(U,d_t) = \sum_{n=0}^{\infty} 2^{\frac{n}{2}}e_{n,t}.$$

We obtain from (3.4)

$$e^{\tau}_{n,t} \leqslant C2^{\frac{n+\tau}{2}} e_{n,t} \ \text{ and } \ \mathcal{E}^{\tau}_{X,T}(U) \leqslant C2^{\frac{\tau}{2}} \sup_{t\in T} \mathcal{E}(U, d_t).$$

Using (3.2) we have

$$K^{-1}\mathcal{E}(U, d_t) \leqslant \int_0^{\infty} \sqrt{\log(N(U, d_t, \varepsilon))}d\varepsilon \leqslant K\mathcal{E}(U, d_t).$$

We recall also that if $0 \in T$ then by (1.3) $\gamma_2(T) \sim g(T) = \mathbf{E} \sup_{t\in T} |G(t)|$. We may state the following corollary of Theorem 2.3, which extends slightly Theorem 3.3.1 in [15].

Corollary 3.1 *For any* $\tau \geqslant 4$

$$\mathbf{E} \sup_{u,v\in U} \sup_{s,t\in T} |G(u, t) - G(v, s)|$$

$$\leqslant 32(\gamma^{\tau}_{G,U}(T) + \mathcal{E}^{\tau}_{G,T}(U)) \leqslant 32C2^{\frac{\tau}{2}} \left(\sup_{u\in U} \|u\|_{\infty}\gamma_2(T) + \sup_{t\in T} \mathcal{E}(U, d_t)\right).$$

Moreover

$$\mathcal{E}(U, d_t) \sim \int_0^{\infty} \sqrt{\log N(U, d_t, \varepsilon)}d\varepsilon \ \ \textit{and} \ \ \gamma_2(T) \sim g(T) \ \ \textit{if}\ 0 \in T.$$

It is tempting to replace $\sup_{u\in U} \|u\|_{\infty}\gamma_2(T)$ and $\sup_{t\in T} \mathcal{E}(U, d_t)$, respectively, by $\sup_{u\in U} \gamma_2(T, d_u)$ and $\sup_{t\in T} \gamma_2(U, d_t)$. We shall show that this approach cannot work. To this aim let us consider the following toy example, where T and U are usual ellipsoids, i.e.

$$U = \left\{u \in \mathbb{R}^I : \sum_{i\in I} \frac{|u_i|^2}{|x_i|^2} \leqslant 1\right\}. \tag{3.5}$$

and

$$T = \left\{t \in \mathbb{R}^I : \sum_{i\in I} \frac{|t_i|^2}{|y_i|^2} \leqslant 1\right\}.$$

Obviously

$$\mathbf{E} \sup_{u\in U, t\in T} |G(u, t)| = \mathbf{E} \max_{i\in I} |x_i y_i g_i|.$$

On the other hand

$$\sup_{u\in U}\mathbf{E}\sup_{t\in T}|G(u,t)| \sim \sup_{u\in U}\|uy\|_2 = \max_{i\in I}|x_iy_i| = \|xy\|_\infty.$$

Similarly, $\sup_{t\in T}\mathbf{E}\sup_{u\in U}|G(u,t)| \sim \|xy\|_\infty$. However $\|xy\|_\infty \leqslant 1$ does not guarantee that $\mathbf{E}\max_{i\in I}|x_iy_ig_i|$ is finite, for example if $x_i = y_i = 1$. On the other hand Corollary 3.1 implies the following result for the ellipsoid U.

Remark 3.2 Suppose that U is given by (3.5). Then for any set $0 \in T \subset \ell^2(I)$,

$$\mathbf{E}\sup_{u\in U}\sup_{t\in T}|G(u,t)| \leqslant K(\|x\|_\infty g(T) + \Delta(T)\|x\|_2).$$

Proof For the sake of simplicity we assume that $I = \mathbb{N}$. We use Corollary 3.1 together with the following observation. For each $t \in T$ points $(u_it_i)_{i\in\mathbb{N}}$, $u \in U$ forms an ellipsoid $U_t = \{a \in \mathbb{R}^{\mathbb{N}} : \sum_{i\in\mathbb{N}} \frac{|u_i|^2}{|t_i|^2|x_i|^2} \leqslant 1\}$ and therefore by Proposition 2.5.2 in [15]

$$\int_0^\infty \sqrt{\log N(U,d_t,\varepsilon)}d\varepsilon \leqslant L\sum_{n=0}^\infty 2^{\frac{n}{2}}|x_{2^n}t_{2^n}|,$$

where L is a universal constant, assuming that we have rearranged $(x_it_i)_{i\in\mathbb{N}}$ in such a way that the sequence $|x_it_i|$, $i \in \mathbb{N}$ is non-increasing. It suffices to note that by the Schwartz inequality

$$\sum_{n=0}^\infty 2^{\frac{n}{2}}(|x_{2^n}t_{2^n}|^2) \leqslant 2|x_1t_1| + 2\sum_{n=1}^\infty\left(\sum_{i=2^{n-1}+1}^{2^n}|x_it_i|^2\right)^{\frac{1}{2}}$$

$$\leqslant 2|x_1t_1| + 2\sum_{n=1}^\infty \max_{2^{n-1}<i\leqslant 2^n}|x_i|\left(\sum_{i=2^{n-1}+1}^{2^n}|t_i|^2\right)^{\frac{1}{2}} \leqslant 2\|x\|_2\|t\|_2.$$

Consequently,

$$\mathbf{E}\sup_{u\in U}\sup_{t\in T}|G(u,t)| \leqslant K(\|x\|_\infty\gamma_2(T) + \Delta(T)\|x\|_2).$$

It remains to apply (1.3), i.e. $g(T) \sim \gamma_2(T)$. □

Note that $\|x\|_\infty = \sup_{u\in U}\|u\|_\infty$ and $\|x\|_2 \sim \gamma_2(U)$. It is clear that this result can be slightly improved if $\mathcal{E}(U) < \infty$. Indeed similarly to (3.3) one can show that

$$\mathcal{E}^\tau_{G,U}(T) \leqslant C2^{\frac{\tau}{2}}\sup_{t\in T}\|t\|_\infty\mathcal{E}(U)$$

and hence by Corollary 3.1 for any set U such that $\mathcal{E}(U) < \infty$

$$\mathbf{E}\sup_{u\in U}\sup_{t\in T}|G(u,t)| \leqslant K\left(\|x\|_{\infty}g(T) + \sup_{t\in T}\|t\|_{\infty}\mathcal{E}(U)\right).$$

It is sort of general rule that we expect certain regularity of U, whereas the set T is usually assumed to be unknown.

We next turn to showing that Corollary 3.1 is sufficiently strong to answer both of the questions about VC classes and Fourier series that we posed in the introduction, in the Gaussian type formulation. Namely we prove that these problems are related to certain properties of entropy functionals $\int_0^{\infty}\sqrt{\log(N(U,d_t,\varepsilon))}d\varepsilon$, $t \in T$.

We begin with a result on VC classes of finite dimension. In this case $\mathcal{A}$ consists of $a \in \mathbb{R}^I$, of the form $a = 1_A$ for some $A \subset I$ and T is any subset of $\ell^2(I)$ that contains 0. By Theorem 14.12 in [11] we have that for any $t \in \ell^2(I)$ such that $\|t\|_2 = 1$ and given VC class $\mathcal{A}$ of dimension d

$$\log N(\mathcal{A},d_t,\varepsilon) \leqslant Ld\left(1+\log\frac{1}{\varepsilon}\right), \quad 0<\varepsilon<1.$$

Consequently

$$\sup_{t\in T}\int_0^{\infty}\sqrt{\log N(\mathcal{A},d_t,\varepsilon)}d\varepsilon \leqslant \sqrt{Ld}\Delta(T)\int_0^1(1+\log\frac{1}{\varepsilon})^{\frac{1}{2}}d\varepsilon \leqslant M\sqrt{d}\Delta(T)$$

and hence $\mathcal{E}(\mathcal{A},d_t) \leqslant M\sqrt{d}\Delta(T)$ for any $t \in T$. Since clearly $\Delta(T) \leqslant \gamma_2(T)$ we may use Corollary 3.1 with $\tau = 4$ and deduce

$$\mathbf{E}\sup_{a\in\mathcal{A}}\sup_{t\in T}|G(a,t)| \leqslant 64C(\gamma_2(T) + M\sqrt{d}\Delta(T)) \leqslant K\sqrt{d}\gamma_2(T),$$

which by (1.3) implies (1.5).

Our next step is to show that the Gaussian version of the problem on Fourier series is also related to Corollary 3.1. In this case U consists of $u \in \mathbb{C}^I$ of the form $u_i = \chi_i(h)$ for $h \in G$, where χ_i, $i \in I$ are characters on the compact Abelian group G and T is any subset of $\ell^2(I)$ that contains 0. Recall that the crucial observation for our study is that distances d_t, $t \in T$ defined in (1.12) are translationally invariant on the group G, i.e.

$$d_t(f\cdot h, g\cdot h) = d_t(f,g) = (\sum_{i\in I}|t_i|^2|\chi_i(f)-\chi_i(g)|^2)^{\frac{1}{2}}, \quad \text{for any } f,g,h\in G.$$

Therefore by a deep result of Fernique (Theorem 3.1.1 in [15]) we have

$$K^{-1}\mathcal{E}(G,d_t) \leqslant \mathbf{E}\sup_{g\in G}|G(g,t)| \leqslant K\mathcal{E}(G,d_t).$$

Consequently by Corollary 3.1 and (1.3) we can deduce (1.7), which is the Gaussian version of Theorem 1.3.

The first new consequence of Corollary 3.1 concerns shattering dimension of U. Suppose that U is the class of real functions bounded by 1, i.e. $U \subset [-1, 1]^I$. We say that a subset B of I is ε-shattered if there exists a level function v on B such that given any subset A of B one can find a function $u \in U$ with $u_i \leqslant v_i$ if $i \in A$ and $u_i \geqslant v_i + \varepsilon$ if $i \in B \setminus A$. The shattering dimension of U, denoted by $\mathrm{vc}(U, \varepsilon)$, is the maximal cardinality of a set ε-shattered by U. A deep result of Mendelson and Vershynin [14] is that for any $t \in \ell_2(I)$ such that $\|t\|_2 = 1$ we have

$$\log N(U, d_t, \varepsilon) \leqslant L\mathrm{vc}(U, c\varepsilon) \log\left(\frac{2}{\varepsilon}\right), \quad 0 < \varepsilon < 1, \tag{3.6}$$

where L and c are positive absolute constants. This leads to the following corollary.

Corollary 3.3 *Suppose that* $0 \in T \subset \ell^2(I)$ *and* $\sup_{u \in U} \|u\|_\infty$. *The following inequality holds*

$$\mathbf{E} \sup_{u \in U} \sup_{t \in T} |G(u, t)| \leqslant K\left(g(T) + \Delta(T) \int_0^1 \sqrt{\mathrm{vc}(U, \varepsilon) \log(2/\varepsilon)}\, d\varepsilon\right).$$

Proof It suffices to apply (3.6) then Corollary 3.1 and finally (1.3). □

It is worth mentioning what this result says about the shattering dimension of convex bodies. Suppose that $U \subset [-1, 1]^d$ is a convex and symmetric body then $\mathrm{vc}(U, \varepsilon)$ is the maximal cardinality of a subset J of $\{1, 2, \ldots, d\}$ such that $P_J(U) \supset [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]^J$, where P_J is the orthogonal projection from $\mathbb{R}^d$ on $\mathbb{R}^J$. For example, suppose that U is a unit ball in $\mathbb{R}^d$ then $\mathrm{vc}(U, \varepsilon) = k - 1$ for any $\varepsilon \in [\frac{2}{\sqrt{k}}, \frac{2}{\sqrt{k-1}})$ and $k = 1, 2, \ldots d$ and, moreover, $\mathrm{vc}(U, \varepsilon) = d$ for $\varepsilon < \frac{2}{\sqrt{d}}$. Consequently

$$\int_0^1 \sqrt{\mathrm{vc}(U, \varepsilon) \log(2/\varepsilon)}\, d\varepsilon \leqslant K\sqrt{d \log d}.$$

Note that for $t \in \mathbb{R}^d$, $t_i = 1/\sqrt{d}$ we have that $\int_0^\infty \sqrt{\log N(U, d_t, \varepsilon)}\, d\varepsilon$ is up to a constant smaller than $\sqrt{d}$. Hence the above estimate is not far from this.

4 Bernoulli Case

Our next aim is to obtain a version of Corollary 3.1 in the setting of Bernoulli processes. We recall that by Bernoulli processes we mean

$$X(t) = \sum_{i \in I} t_i \varepsilon_i, \quad \text{for } t \in T \subset \ell^2(I),$$

where $(\varepsilon_i)_{i\in I}$ is a Bernoulli sequence. Note that $b(T) = \mathbf{E}\sup_{t\in T}|X(t)|$. If $0 \in T$ then by Theorem 1.1 we have a geometrical characterization of $b(T)$.

We turn to the analysis of Bernoulli processes on product spaces, namely we consider

$$X(u,t) = \sum_{i\in I} u_i t_i \varepsilon_i, \quad t \in T, u \in U,$$

where $T \subset \mathbb{R}^I$ or $\mathbb{C}^I$, $U \subset \mathbb{R}^I$ or $\mathbb{C}^I$ and $(\varepsilon_i)_{i\in I}$ is a Bernoulli sequence. Our approach is to use Theorem 1.1 in order to extend Corollary 3.1 to the case of random signs.

Theorem 4.1 *Suppose that* $0 \in T$. *Then there exists* $\pi : T \to \ell^2$ *such that* $\|\pi(t)\|_1 \leqslant Lb(T)$ *for all* $t \in T$, $\pi(0) = 0$ *and*

$$\mathbf{E}\sup_{u\in U}\sup_{t\in T}|X(u,t)| \leqslant K\left(\sup_{u\in U}\|u\|_\infty b(T) + \sup_{t\in T}\mathcal{E}(U, d_{t-\pi(t)})\right),$$

where

$$\mathcal{E}(U, d_{t-\pi(t)}) \sim \int_0^\infty \sqrt{\log N(U, d_{t-\pi(t)}, \varepsilon)}\, d\varepsilon.$$

Proof Obviously we may assume that $b(T) = \mathbf{E}\sup_{t\in T}|\sum_{i\in I} t_i\varepsilon_i| < \infty$. Therefore by Theorem 1.1 there exists a decomposition $T \subset T_1 + T_2$, $T_1, T_2 \subset \ell^2(I)$ which satisfies $0 \in T_2$ and

$$\max\left\{\sup_{t\in T_1}\|t\|_1, g(T_2)\right\} \leqslant Lb(T), \tag{4.1}$$

where L is a universal constant. Hence combining Corollary 3.1 with (4.1)

$$\begin{aligned}
&\mathbf{E}\sup_{u\in U}\sup_{t\in T}|X(u,t)| \\
&\leqslant C\left(\mathbf{E}\sup_{u\in U}\sup_{t\in T_1}|X(u,t)| + \mathbf{E}\sup_{u\in U}\sup_{t\in T_2}|X(u,t)|\right) \\
&\leqslant C\left(\sup_{u\in U}\|u\|_\infty \sup_{t\in T_1}\|t\|_1 + \sqrt{\frac{\pi}{2}}\mathbf{E}\sup_{u\in U}\sup_{t\in T_2}|G(u,t)|\right) \\
&\leqslant CL\left(\sup_{u\in U}\|u\|_\infty b(T) + \sup_{t\in T_2}\int_0^\infty \sqrt{\log N(U, d_t, \varepsilon)}\, d\varepsilon\right).
\end{aligned}$$

The decomposition of T into $T_1 + T_2$ can be defined in a way that $T_1 = \{\pi(t) : t \in T\}$ and $T_2 = \{t - \pi(t) : t \in T\}$ where $\pi : T \to \ell^2$ is such that $\pi(0) = 0$, $\|\pi(t)\|_1 \leqslant Lb(T)$ and $\gamma_2(T_2) \leqslant Lb(T)$. It completes the proof. □

Our remarks on the entropy function from the previous section show that Theorem 4.1 solves both our questions from the introduction. In fact we can easily extend our result for the functional shattering dimension.

Corollary 4.2 *Suppose that* $\sup_{u\in U}\|u\|_\infty \leqslant 1$ *and* U *is of* ε*-shattering dimension* $\mathrm{vc}(U,\varepsilon)$ *for* $0<\varepsilon<1$*, then*

$$\mathbf{E}\sup_{u\in U}\sup_{t\in T}|X(u,t)| \leqslant Kb(T)\int_0^1 \sqrt{\mathrm{vc}(U,\varepsilon)\log(2/\varepsilon)}d\varepsilon.$$

Proof Obviously by Theorem 4.1,

$$\begin{aligned}&\mathbf{E}\sup_{u\in U}\sup_{t\in T}|X(u,t)|\\ &\leqslant K\left(\sup_{u\in U}\|u\|_\infty b(T)+\sup_{t\in T}\|t-\pi(t)\|_2\int_0^1\sqrt{\mathrm{vc}(U,\varepsilon)\log(2/\varepsilon)}d\varepsilon\right),\end{aligned}$$

where $\|\pi(t)\|_1 \leqslant Lb(T)$. Since

$$\sup_{t\in T}\|t-\pi(t)\|_2 \leqslant Lb(T) \text{ and } \sup_{u\in U}\|u\|_\infty \leqslant 1$$

it implies that

$$\mathbf{E}\sup_{u\in U}\sup_{t\in T}|X(u,t)| \leqslant Kb(T)\int_0^1 \sqrt{\mathrm{vc}(U,\varepsilon)\log(2/\varepsilon)}d\varepsilon.$$

□

Corollary 4.3 *Suppose that* $\sup_{u\in U}\|u\|_\infty \leqslant 1$ *and* $v_i\in F$*,* $i\in I$*, then*

$$\mathbf{E}\sup_{u\in U}\left\|\sum_{i\in I}u_i v_i\varepsilon_i\right\| \leqslant K\mathbf{E}\left\|\sum_{i\in I}v_i\varepsilon_i\right\|\int_0^1\sqrt{\mathrm{vc}(U,\varepsilon)\log(2/\varepsilon)}d\varepsilon.$$

Note that in the same way as for the maximal inequality we may ask what is the right characterization of $U\subset\mathbb{R}^I$, $\sup_{u\in U}\|u\|_\infty\leqslant 1$ for which the inequality

$$\mathbf{E}\sup_{u\in U}\left\|\sum_{i\in I}u_iX_i\right\| \leqslant K\left\|\sum_{i\in I}X_i\right\|, \tag{4.2}$$

holds for any sequence of independent symmetric r.v.'s X_i, $i\in I$, which take values in a separable Banach space F. With the same proof as the first part of Theorem 1.2 one can show that U should satisfy the condition

$$\sup_{0<\varepsilon<1}\varepsilon\sqrt{\mathrm{vc}(U,\varepsilon)}<\infty.$$

On the other hand Corollary 4.3 implies that

$$\int_0^1 \sqrt{\mathrm{vc}(U,\varepsilon)\log(2/\varepsilon)}d\varepsilon < \infty$$

is a sufficient condition for the inequality (4.2). The precise answer to this question seems at the moment to be beyond our reach.

We revisit our toy example, where U is the ellipsoid

$$U = \left\{ u \in \mathbb{R}^I : \sum_{i\in I} \frac{|u_i|^2}{|x_i|^2} \leqslant 1 \right\},$$

where $|x_i| > 0$ are given numbers. One can show the following result.

Remark 4.4 Suppose that U is the usual ellipsoid. Then for any set $0 \in T \subset \ell^2(I)$

$$\mathbf{E} \sup_{u\in U} \sup_{t\in T} |X(u,t)| \leqslant K\|x\|_2 b(T).$$

Proof We may argue in a similar way as in Remark 3.2 using this time Theorem 4.1 and obtain

$$\mathbf{E} \sup_{u\in U} \sup_{t\in T} |X(u,t)| \leqslant K \left(\|x\|_\infty b(T) + \sup_{t\in T} \|t-\pi(t)\|_2 \|x\|_2 \right).$$

Since $\Delta(T) \leqslant Lb(T)$ and $\|\pi(t)\|_1 \leqslant Lb(T)$ we get $\sup_{t\in T} \|t-\pi(t)\|_2 \leqslant Lb(T)$ and therefore

$$\mathbf{E} \sup_{u\in U} \sup_{t\in T} |X(u,t)| \leqslant K\|x\|_2 b(T).$$

□

On the other hand by Schwarz's inequality

$$\mathbf{E} \sup_{u\in U} \sup_{t\in T} |X(u,t)| = \sup_{t\in T} \mathbf{E} \sup_{u\in U} |X(u,t)| = \sup_{t\in T} \|tx\|_2$$

Consequently in this case the expectation of the supremum of the Bernoulli process over the product of index sets can be explained by the analysis of one of its marginal processes.

We remark that once we prove the comparability of moments like (4.2) one can deduce also the comparability of tails.

Remark 4.5 Suppose that $\sup_{u\in U}\|u\|_\infty \leqslant 1$. If for any X_i, $i \in I$ symmetric independent random variables which take values in a separable Banach space $(F, \|\cdot\|)$

$$\mathbf{E}\sup_{u\in U}\left\|\sum_{i\in I} u_i X_i\right\| \leqslant L\mathbf{E}\left\|\sum_{i\in I} X_i\right\|, \tag{4.3}$$

then there exists an absolute constant K such that

$$\mathbf{P}\left(\sup_{u\in U}\left\|\sum_{i\in I} u_i X_i\right\| \geqslant Kt\right) \leqslant K\mathbf{P}\left(\left\|\sum_{i\in I} X_i\right\| \geqslant t\right).$$

Proof It suffices to prove the inequality for $X_i = v_i\varepsilon_i$, where $v_i \in F$ and ε_i are independent Bernoulli variables. The general result follows then from the Fubini theorem. By the result of Dilworth and Montgomery-Smith [2] for all $p \geqslant 1$

$$\left\|\sup_{u\in U}\left\|\sum_{i\in I} u_i v_i \varepsilon_i\right\|\right\|_p \leqslant C\left(\left\|\sup_{u\in U}\left\|\sum_{i\in I} u_i v_i \varepsilon_i\right\|\right\|_1 + \sup_{u\in U}\sup_{\|x^*\|\leqslant 1}\left\|\sum_{i\in I} u_i x_i^*(v_i)\varepsilon_i\right\|_p\right).$$

Therefore by the Markov inequality and the assumption $\sup_{u\in U}\|u\|_\infty \leqslant 1$ we obtain for all $p \geqslant 1$

$$\mathbf{P}\left(\sup_{u\in U}\left\|\sum_{i\in I} u_i v_i \varepsilon_i\right\| \geqslant C\left(\mathbf{E}\left\|\sum_{i\in I} v_i\varepsilon_i\right\| + \sup_{\|x^*\|\leqslant 1}\left\|\sum_{i\in I} x^*(v_i)\varepsilon_i\right\|_p\right)\right) \leqslant e^{-p}. \tag{4.4}$$

On the other hand it is known (e.g. [8]) that for any functional $x^* \in F^*$

$$\mathbf{P}\left(\left|\sum_{i\in I} x^*(v_i)\varepsilon_i\right| \geqslant M^{-1}\left\|\sum_{i\in I} x^* v_i \varepsilon_i\right\|_p\right) \geqslant \min\{c, e^{-p}\}, \tag{4.5}$$

where $M \geqslant 1$. Hence

$$\mathbf{P}\left(\sup_{u\in U}\left\|\sum_{i\in I} u_i v_i \varepsilon_i\right\| \geqslant C\left(\mathbf{E}\left\|\sum_{i\in I} v_i\varepsilon_i\right\| + Mt\right)\right)$$
$$\leqslant \sup_{\|x^*\|\leqslant 1} c^{-1}\mathbf{P}\left(\left|\sum_{i\in I} x^*(v_i)\varepsilon_i\right| > t\right) \leqslant c^{-1}\mathbf{P}\left(\left\|\sum_{i\in I} v_i\varepsilon_i\right\| > t\right).$$

We end the proof considering two cases. If $t \geqslant M^{-1}\mathbf{E}\left\|\sum_{i\in I} v_i\varepsilon_i\right\|$ then

$$\mathbf{P}\left(\sup_{u\in U}\left\|\sum_{i\in I} u_i v_i \varepsilon_i\right\| \geqslant 2CMt\right) \leqslant c^{-1}\mathbf{P}\left(\|\sum_{i\in I} v_i\varepsilon_i\| > t\right).$$

If $t \leqslant M^{-1}\mathbf{E}\|\sum_{i\in I} v_i\varepsilon_i\|$ then by Paley-Zygmund inequality and the Kahane inequality with the optimal constant [6]

$$\mathbf{P}\left(\left\|\sum_{i\in I} v_i\varepsilon_i\right\| > t\right) \geqslant M^{-2}\frac{\left(\mathbf{E}\left\|\sum_{i\in I}\varepsilon_i v_i\right\|\right)^2}{\mathbf{E}\left\|\sum_{i\in I}\varepsilon_i v_i\right\|^2} \geqslant \frac{1}{2}M^{-2}.$$

It shows that

$$\mathbf{P}\left(\sup_{u\in U}\left\|\sum_{i\in I} u_i v_i \varepsilon_i\right\| \geqslant Kt\right) \leqslant K\mathbf{P}\left(\left\|\sum_{i\in I} v_i\varepsilon_i\right\| > t\right),$$

which completes the proof. □

5 Applications

As our first application to processes on $\mathcal{A} = [0, 1]$ with values in $\mathbb{R}^I$. In order to discuss whether or not $X(a)$, $a \in \mathcal{A}$, has its paths in $\ell^2(I)$ space we must verify the condition $\|X(a)\|_2 < \infty$ a.s. for all $a \in \mathcal{A}$. This is the same question as

$$\sup_{\|x^*\|\leqslant 1}\sup_{a\in\mathcal{A}}\langle x^*, X(a)\rangle < \infty.$$

Hence it suffices to check the condition $\mathbf{E}\sup_{a\in\mathcal{A}}\sup_{t\in T}|X(a,t)| < \infty$, where $T = \{t \in \ell^2(I) : \ \|t\|_2 \leqslant 1\}$ and

$$X(a,t) = \langle t, X(a)\rangle, \quad t \in T, a \in \mathcal{A}.$$

Note that in this way we match each random vector $X(a)$ with the process $X(a,t)$, $t \in T$. By Theorem 2.3 (note that $X(0,s) = 0$)

$$\mathbf{E} \sup_{a\in\mathcal{A}} \sup_{t\in T} |X(a,t)| \leqslant 32(\gamma^{\tau}_{X,\mathcal{A}}(T) + \mathcal{E}^{\tau}_{X,T}(\mathcal{A})).$$

In this setting we usually expect that

$$q_{n,a}(s,t) \leqslant q_{n,1}(s,t), \quad a \in \mathcal{A}, \tag{5.1}$$

which means that on average the increments increase with time. Condition (5.1) yields $\gamma^{\tau}_{X,\mathcal{A}}(T) \leqslant \gamma^{\tau}_{X(1)}(T)$. Under certain assumptions we may also expect that $\gamma^{\tau}_{X(1)}(T)$ is equivalent to $\mathbf{E}\|X(1)\|_2$. For example this is the case when $X(1)$ is a centred Gaussian vector and also if $X(1)$ consists of entries $X(1)_i$, $i \in I$ that are independent random variables that satisfy some technical assumptions as stated in [10]. Moreover if there exists an increasing family of functions $\eta_n : \mathbb{R}_+ \to \mathbb{R}_+$, which are continuous, increasing, $\eta_n(0) = 0$ such that for all $t \in T$ and $a, b \in \mathcal{A}$

$$q_{n,t}(a,b) \leqslant \eta_n(|a-b|) \ \text{ and } \ \sum_{n=0}^{\infty} \eta^{-1}_{n+\tau}(N_n) < \infty \tag{5.2}$$

then

$$\mathcal{E}^{\tau}_{X,T}(\mathcal{A}) \leqslant \sum_{n=0}^{\infty} \eta^{-1}_{n+\tau}(N_n) < \infty.$$

In this way we obtain the following remark, which is a generalization results in [3, 4].

Remark 5.1 Suppose that $X(a,t)$, $a \in \mathcal{A}$, $t \in T$ satisfies (5.1), (5.2) and $\mathbf{E}\|X(1)\|_2$ is comparable with $\gamma_{X(1)}(T)$ then

$$\mathbf{E} \sup_{a\in\mathcal{A}} \sup_{t\in T} |X(a,t)| \leqslant K(\mathbf{E}\|X(1)\|_2 + 1).$$

As our second application we consider empirical processes. Let $(\mathcal{X}, \mathcal{B})$ be a measurable space, $\mathcal{F}$ be a countable family of measurable functions $f : \mathcal{X} \to \mathbb{R}$ such that $0 \in \mathcal{F}$and $X_1, X_2, \ldots, X_N$ be family of independent random variables that satisfy type Bernstein inequality

$$\mathbf{P}\left(|(f-g)(X_i)| > d_1(f,g)t + d_2(f,g)t^{\frac{1}{2}}\right) \leqslant 2\exp(-t), \quad \text{for all } t > 1. \tag{5.3}$$

We shall analyze the case when $d_2(f,g) \geqslant d_1(f,g)$. By Exercise 9.3.5 in [15] for any centred independent random variables $Y_1, Y_2, \ldots, Y_n$, which satisfy

$$\mathbf{P}\left(|Y_i| \geqslant At + Bt^{\frac{1}{2}}\right) \leqslant 2\exp(-t) \quad \text{for all } i = 1,2,\ldots,N \text{ and } t > 1,$$

where $A \leqslant B$ and for any numbers $u_1, u_2, \ldots, u_N$ we have

$$\mathbf{P}\left(\left|\sum_{i=1}^{N} u_i Y_i\right| \geqslant L\left(A\|u\|_\infty t + B\|u\|_2 t^{\frac{1}{2}}\right)\right) \leqslant 2\exp(-t) \quad \text{for all } t > 1. \tag{5.4}$$

Observe that if we define

$$X(u,f) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \varepsilon_i u_i f(X_i), \quad u \in U, f \in \mathcal{F},$$

where U is a unit ball in $\mathbb{R}^N$ and $(\varepsilon_i)_{i=1}^N$ is a Bernoulli sequence independent of X_i, $i = 1,2,\ldots,N$, then

$$\mathbf{E}\sup_{u\in U}\sup_{f\in\mathcal{F}} |X(u,f)| = \mathbf{E}\sup_{f\in\mathcal{F}}\left(\frac{1}{N}\sum_{i=1}^{N}|f(X_i)|^2\right)^{\frac{1}{2}}.$$

Clearly by (5.4) applied to $Y_i = \varepsilon_i(f-g)(X_i)$ we get for all $t > 1$

$$\mathbf{P}\left(|X(u,f) - X(u,g)| \geqslant \frac{L}{\sqrt{N}}\left(d_1(f,g)\|u\|_\infty t + d_2(f,g)\|u\|_2 t^{\frac{1}{2}}\right)\right)$$
$$\leqslant 2\exp(-t). \tag{5.5}$$

Thus, in particular, we can use

$$q_{n,u}(f,g) = L(d_1(f,g)\|u\|_\infty 2^n + d_2(f,g)\|u\|_2 2^{\frac{n}{2}}).$$

Then

$$q_{n,U}(f,g) = \frac{L}{\sqrt{N}}(d_1(f,g)2^n + d_2(f,g)2^{\frac{n}{2}}). \tag{5.6}$$

Let

$$\gamma_i(\mathcal{F}, d_i) = \inf_{\mathcal{A}}\sup_{f\in\mathcal{F}}\sum_{n=0}^{\infty} 2^{\frac{n}{i}}\Delta_i(A_n(f)),$$

where the infimum is taken over all admissible partitions $\mathcal{A} = (\mathcal{A}_n)_{n\geqslant 0}$ and $\Delta_i(A) = \sup_{f,g\in A} d_i(f,g)$. It is easy to construct admissible partitions, which work for both $\gamma_1(\mathcal{F}, d_1)$ and $\gamma_2(\mathcal{F}, d_2)$. Namely we consider partitions $\mathcal{A}^1 = (\mathcal{A}^1_n)_{n\geqslant 0}$ and $\mathcal{A}^2 = (\mathcal{A}^2_n)_{n\geqslant 0}$ such that

$$(1+\varepsilon)\gamma_2(\mathcal{F}, d_i) \geqslant \sup_{f\in\mathcal{F}} \sum_{n=0}^{\infty} 2^{\frac{n}{i}} \Delta_i(A^i_n(f)), \quad i = 1, 2,$$

for some arbitrary small $\varepsilon > 0$ and then define $\mathcal{A} = (\mathcal{A}_n)_{n\geqslant 0}$ by $\mathcal{A}_n = \mathcal{A}^1_{n-1} \cap \mathcal{A}^2_{n-1}$ for $n \geqslant 1$ and $\mathcal{A}_0 = \{\mathcal{F}\}$. Obviously $\mathcal{A}$ is admissible. Moreover,

$$\sup_{f\in\mathcal{F}} \sum_{n=0}^{\infty} 2^{\frac{n}{i}} \Delta_i(A_n(f)) \leqslant (1+\varepsilon) 2^{\frac{1}{i}} \gamma_i(\mathcal{F}, d_i), \quad i = 1, 2. \tag{5.7}$$

Using the partition $\mathcal{A}$ we derive from (5.6)

$$\gamma^{\tau}_{X,U}(\mathcal{F}) \leqslant \frac{2^{\tau+1}(1+\varepsilon)}{\sqrt{N}} (\gamma_1(\mathcal{F}, d_1) + \gamma_2(\mathcal{F}, d_2)).$$

On the other hand using that $X(u,f) - X(v,f) = X(u-v,f) - X(0,f)$, similarly to (5.5), we get for all $t > 1$

$$\mathbf{P}\left(|X(u,f) - X(v,f)| \geqslant \frac{L}{\sqrt{N}} \left(d_1(f,0)\|u-v\|_\infty t + d_2(f,0)\|u-v\|_2 t^{\frac{1}{2}}\right)\right) \leqslant 2\exp(-t).$$

Hence we may define

$$q_{n,f}(u,v) = \frac{L}{\sqrt{N}} \left(d_1(f,0)\|u-v\|_\infty 2^n + d_2(f,0)\|u-v\|_2 2^{\frac{n}{2}}\right). \tag{5.8}$$

Our aim is to compute the entropy numbers $e^{\tau}_{n,f}$, $n \geqslant 0$. Let $n_0 \geqslant 1$ be such that $2^{n_0} \geqslant N \geqslant 2^{n_0-1}$. It is straightforward that we only have to consider the case when N is suitably large. We claim that for any $n \geqslant n_0$ and suitably chosen $\sigma \geqslant 1$ it is possible to cover $B_N(0,1)$, a unit ball in $\mathbb{R}^N$, by at most $N_{n+\sigma}$ cubes of ℓ^∞ diameter at most $N^{-\frac{1}{2}}(N_n)^{-\frac{1}{N}}$. Indeed we can apply the volume type argument. It is possible to cover $B_N(0,1)$ with M disjoint cubes of ℓ^∞ diameter t with disjoint interiors in $B_N(0, 1+t\sqrt{N})$. Since $|B_N(0,1)| \sim (C/\sqrt{N})^N$ we get

$$M \leqslant \frac{|B_N(0,1+t\sqrt{N})|}{t^N} \leqslant \left(\frac{C}{\sqrt{N}}\left(\frac{1+t\sqrt{N}}{t}\right)\right)^N \leqslant C^N \left(1 + \frac{1}{t\sqrt{N}}\right)^N.$$

We may choose $t = N^{-\frac{1}{2}} N_n^{-\frac{1}{N}}$, then

$$C^N \left(1 + \frac{1}{t\sqrt{N}}\right)^N = C^N \left(1 + N_n^{\frac{1}{N}}\right)^N \leqslant (N_{n+\sigma})^{\frac{1}{N}},$$

where the last inequality uses the assumption that $n \geqslant n_0$. In this way we cover $B_N(0,1)$ by at most $N_{n+\sigma}$ sets of ℓ^∞ diameter at most $N^{-\frac{1}{2}}(N_n)^{-\frac{1}{N}}$ and ℓ^2 diameter at most $N_n^{-\frac{1}{N}}$. By (5.8) we infer the following entropy bound

$$e^\tau_{n+\sigma,f} \leqslant L2^\tau \left(d_1(f,0)N^{-1}(N_n)^{-\frac{1}{N}} + d_2(f,0)N^{-\frac{1}{2}}(N_n)^{-\frac{1}{N}}\right).$$

We recall that the constant L is absolute but may change its value from line to line up to a numerical factor. This implies the bound

$$\begin{aligned}&\sum_{n=n_0+\sigma}^{\infty} e^\tau_{n,f}\\ &\leqslant L2^\tau \left(\sum_{n=n_0+\sigma}^{\infty}\right)\\ &\leqslant L2^\tau \left(d_1(f,0) + d_2(f,0)\right).\end{aligned}$$

The second step is to consider $n \leqslant n_0/2 + \sigma$. In this case we can simply use the trivial covering of $B_N(0,1)$ by a single set, which obviously has ℓ^∞ and ℓ^2 diameter equal 2, and hence

$$e^\tau_{n,f} = \frac{L2^\tau}{\sqrt{N}}(d_1(f,0)2^n + d_2(f,0)2^{\frac{n}{2}})$$

and

$$\sum_{n=0}^{n_0/2+\sigma} e^\tau_{n,f} \leqslant L2^\tau \left(d_1(f,0) + d_2(f,0)\right).$$

The most difficult case is when $n_0/2+\sigma \leqslant n \leqslant n_0+\sigma$. In this setting we will cover $B_N(0,1)$ with cubes of ℓ^∞ diameter $2t$, where on $t = \frac{1}{\sqrt{m_n}}$ and $m_n \leqslant N$. We will not control ℓ^2 diameter, we simply use that it is always bounded by 2. Note that if $x \in B_N(0,1)$ there are only m_n coordinates such that $|x_i| > t$. Therefore we can cover $B_N(0,1)$ with cubes in $\mathbb{R}^N$ of ℓ^∞ diameter $2t$ if for each subset $J \subset \{1,2,\ldots,N\}$ such that $|J| = m_n$ we cover $B_J(0,1) \subset \mathbb{R}^J$ with cubes in $\mathbb{R}^J$ of ℓ^∞ diameter $2t$. In this way we cover the situation, where all coordinates but those in J stay in the cube

$[-t,t]^{J^c}$. By our volume argument one needs at most

$$M_J \leqslant C^J \left(1 + \frac{2t}{\sqrt{|J|}}\right)^{|J|} = C^{m_n}\left(1 + \frac{2t}{\sqrt{m_n}}\right)^{m_n}$$

$2t$-cubes in $\mathbb{R}^J$ to cover $B_J(0,1)$. Our choice of t guarantees that $M_J \leqslant (3C/2)^{m_n}$. Therefore one needs $\binom{N}{m_n}(3C/2)^{m_n}$ cubes of ℓ^∞ diameter $2t$ to cover $B_N(0,1)$. We require that

$$\binom{N}{m_n}\left(\frac{3C}{2}\right)^{m_n} \leqslant N_n.$$

It remains to find m_n that satisfies the above inequality. First observe that

$$\binom{N}{m_n} \leqslant \left(\frac{eN}{m_n}\right)^{m_n} = \exp(m_n \log(eN/m_n)).$$

Following Talagrand we define m_n as the smallest integer such that

$$2^{n-\sigma} \leqslant m_n \log(eN/m_n).$$

Clearly if $n \geqslant n_0/2 + \sigma$ then $m_n > 1$ and thus by Lemma 9.3.12 in [15] we deduce that $m_n \log(eN/m_n) \leqslant 2^{n-\sigma+1}$ and hence

$$\binom{N}{m_n}\left(\frac{3C}{2}\right)^{m_n} \leqslant \exp(2^{n-\sigma+1})\left(\frac{3C}{2}\right)^{m_n} \leqslant N_n,$$

for sufficiently large σ. Again by Lemma 9.3.12 in [15] we have $\frac{1}{8}m_{n+1} \leqslant m_n$ for all $n \in \{[n_0/2]+\sigma,\ldots,n_0+\sigma-1\}$ and $m_{n_0+\sigma} = 2^{n_0} \geqslant N$. It implies that

$$m_n \geqslant N\left(\frac{1}{8}\right)^{n_0+\sigma-n}. \tag{5.9}$$

Recall that each of the covering cubes has ℓ^2 diameter 2 and therefore by the definition of m_n and then by (5.9)

$$\begin{aligned}
e^\tau_{n,f} &\leqslant \frac{L2^\tau}{\sqrt{N}}\left(d_1(f,0)\frac{2^n}{\sqrt{m_n}} + d_2(f,0)2^{\frac{n}{2}}\right)\\
&\leqslant L2^{\tau+\frac{n-n_0}{2}}\left(d_1(f,0)\sqrt{\log(eN/m_n)} + d_2(f,0)\right)\\
&\leqslant L2^{\tau+\frac{n-n_0}{2}}(d_1(f,0)\sqrt{1+(n_0+\sigma-n)\log(8)} + d_2(f,0)).
\end{aligned}$$

Again we derive the bound

$$\sum_{n=n_0/2+\sigma}^{n_0+\sigma} e_{n,f}^{\tau} \leqslant L2^{\tau}(d_1(f,0)+d_2(f,0)).$$

We have established that

$$\mathcal{E}_{X,f}^{\tau}(U)=\sum_{n=0}^{\infty} e_{n,f}^{\tau} \leqslant L2^{\tau}(d_1(f,0)+d_2(f,0))$$

and consequently

$$\mathcal{E}_{X,\mathcal{F}}^{\tau}(U)=\sup_{f\in\mathcal{F}} \mathcal{E}_{X,f}^{\tau}(U) \leqslant L2^{\tau}\sup_{f\in\mathcal{F}}(d_1(f,0)+d_2(f,0)).$$

By Theorem 2.3 we get

$$\begin{aligned}&\mathbf{E}\sup_{f\in\mathcal{F}}\Big(\frac{1}{\sqrt{N}}\sum_{i=1}^{N}|f(X_i)|^2\Big)^{\frac{1}{2}}\\&\leqslant K\left(\frac{1}{\sqrt{N}}(\gamma_1(\mathcal{F},d_1)+\gamma_2(\mathcal{F},d_2))+\Delta_1(\mathcal{F})+\Delta_2(\mathcal{F})\right).\end{aligned}$$

Note that our assumption that d_2 dominates d_1 is not necessary for the result. Clearly using $\bar{d}_2=\max(d_1,d_2)$ instead of d_2 we only have to observe that our admissible partition $\mathcal{A}$ works for $\gamma_1(\mathcal{F},d_1)$ and $\gamma_2(\mathcal{F},d_2)$ in the sense of (5.7) one can use the following inequality

$$\sum_{n=0}^{\infty} 2^{\frac{n}{2}}\bar{\Delta}_2(A_n(f)) \leqslant \sum_{n=0}^{\infty} 2^{n}\Delta_1(A_n(f))+\sum_{n=0}^{\infty} 2^{\frac{n}{2}}\Delta_2(A_n(f)),$$

where $\bar{\Delta}_2(A)$ is the diameter of A with respect to $\bar{d}_2$ distance. In the same way $\bar{\Delta}_2(\mathcal{F})\leqslant\Delta_1(\mathcal{F})+\Delta_2(\mathcal{F})$. We have proved the following result.

Theorem 5.2 *Suppose that* $0\in\mathcal{F}$ *and* $\mathcal{F}$ *satisfies (5.3). Then*

$$\begin{aligned}&\mathbf{E}\sup_{f\in\mathcal{F}}\frac{1}{\sqrt{N}}\left(\sum_{i=1}^{N}|f(X_i)|^2\right)^{\frac{1}{2}}\\&\leqslant K\left(\frac{1}{\sqrt{N}}(\gamma_1(\mathcal{F},d_1)+\gamma_2(\mathcal{F},d_2))+\Delta_1(\mathcal{F})+\Delta_2(\mathcal{F})\right).\end{aligned}$$

This result is due to Mendelson and Paouris [13] (see Theorem 9.3.1 in [15]) and concerns a slightly more general situation. The proof we have shown is different and much less technically involved.

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Permanental Vectors and Selfdecomposability

Nathalie Eisenbaum

Abstract Exponential variables, gamma variables or squared centered Gaussian variables, are always selfdecomposable. Does this property extend to multivariate gamma distributions? We show here that for any d-dimensional centered Gaussian vector $(\eta_1, \ldots, \eta_d)$ with a nonsingular covariance, the vector $(\eta_1^2, \ldots, \eta_d^2)$ is not selfdecomposable unless its components are independent. More generally, permanental vectors with nonsingular kernels are not selfdecomposable unless their components are independent.

Keywords Exponential variable • Gamma variable • Gaussian process • Infinite divisibility • Permanental process • Selfdecomposability

Mathematics Subject Classification (2010). 60J25, 60J55, 60G15, 60E07

1 Introduction

A d-dimensional vector X is selfdecomposable if for every b in [0, 1] there exists a d-dimensional vector X_b, independent of X such that

$$X \overset{(\text{law})}{=} bX + X_b.$$

Exponential variables are well-known examples of one dimensional selfdecomposable variables. Equivalently, gamma variables and squared centered Gaussian variables are selfdecomposable. It is hence natural to ask whether this property extends to multivariate gamma distributions. We answer here the following question: given $(\eta_1, \ldots, \eta_d)$ a centered Gaussian vector with a nonsingular covariance, is the vector $(\eta_1^2, \ldots, \eta_d^2)$ selfdecomposable?

N. Eisenbaum (✉)
Laboratoire de probabilités et modèles aléatoires, UMR 7599 - CNRS, Université Paris 6, 4, Place Jussieu - Case 188, 75252 Paris Cedex 05, France
e-mail: nathalie.eisenbaum@upmc.fr

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This question is actually in keeping with the older question first raised by Lévy in 1948 [5] of the infinite divisibility of (η_1^2, η_2^2). Indeed selfdecomposition implies infinite divisibility and several answers have since been made to Lévy's question. To Lévy's doubts on the possible infinite divisibility of (η_1^2, η_2^2), when η_1 and η_2 are not independent, Vere-Jones [7] answers that this couple is always infinitely divisible. In dimension $d > 2$, Griffiths [4] has established criteria (later rewritten by Bapat [1] and Eisenbaum and Kaspi [2]) for the realization of that property. For example a fractional Brownian motion satisfies this criteria iff its index is lower than the Brownian motion index.

Hence the property of infinite divisibility does not really separate the one-dimensional case from the higher dimensional cases. Actually the stronger property of selfdecomposability does. Indeed we show here that unless its components are independent, $(\eta_1^2, \dots, \eta_d^2)$ is never selfdecomposable.

Our answer easily extends to permanental vectors. Permanental vectors represent a natural extension of squared Gaussian vectors. They have marginals with gamma distribution without the Gaussian structure. More precisely, a vector $(\psi_1, \psi_2, \dots, \psi_n)$ with nonnegative components is a permanental vector with kernel $G = (G(i,j), 1 \le i,j \le n)$ and index $\beta > 0$, if its Laplace transform has the following form, for every $\alpha_1, \alpha_2, \dots, \alpha_n$ in $\mathbb{R}_+$

$$I\!E(\exp\{-\frac{1}{2}\sum_{i=1}^{n}\alpha_i\psi_i\}) = \det(I + \alpha G)^{-1/\beta} \tag{1.1}$$

where I is the $n \times n$-identity matrix and α is the diagonal matrix with diagonal entries.

Vere-Jones [8] has formulated clear necessary and sufficient conditions for the existence of such vectors. In particular, symmetry of the matrix G is not necessary. The interest of permanental vectors with non-symmetric kernel is mostly due to their connection with local times of Markov processes (see [3]).

Note that when G is symmetric positive definite and $\beta = 2$, (1.1) gives the Laplace transform of a vector $(\eta_1^2, \eta_2^2, \dots, \eta_n^2)$ where $(\eta_1, \eta_2, \dots, \eta_n)$ is a centered Gaussian vector with covariance G.

Section 2 presents the proof of the following theorem.

Theorem 1.1 *Unless its components are independent, a permanental vector of dimension* $d > 1$ *with a nonsingular kernel is never selfdecomposable.*

2 Proof of Theorem 1.1

Let (ψ_x, ψ_y) be a permanental couple with a kernel G and index β. Vere-Jones necessary and sufficient condition of existence of a permanental couple translates into:

$$G(x, x) \geq 0, \; G(y, y) \geq 0, \; G(x, y)G(y, x) \geq 0,$$

$$G(x, x)G(y, y) \geq G(x, y)G(y, x).$$

This condition is independent of β, hence (ψ_x, ψ_y) is always infinitely divisible. We choose to take $\beta = 2$ and note then that (ψ_x, ψ_y) has the same law as (η_x^2, η_y^2) where (η_x, η_y) is a centered Gaussian couple with covariance

$$\tilde{G} = \begin{bmatrix} G(x, x) & \sqrt{G(x, y)G(y, x)} \\ \sqrt{G(x, y)G(y, x)} & G(y, y) \end{bmatrix}$$

The two cases "$G(x, y)G(y, x) = 0$" and "$G(x, x)G(y, y) = G(x, y)G(y, x)$" correspond respectively to η_x and η_y independent, and to the existence of a constant $c > 0$ such that $\eta_x^2 = c\eta_y^2$. In this two cases, the couple (η_x^2, η_y^2) is obviously selfdecomposable.

Excluding these two obvious cases, we show now that (η_x^2, η_y^2) is not selfdecomposable by using the following necessary and sufficient condition for selfdecomposition (see Theorem 15.10 in Sato's book [6]). The unit sphere $\{z \in \mathbb{R}^2 : |z| = 1\}$ is denoted by S.

A given infinitely divisible distribution on $\mathbb{R}^2$ *with Lévy measure* ν *is selfdecomposable iff*

$$\nu(B) = \int_S \lambda(dz) \int_0^\infty 1_B(rz)k_z(r)\frac{dr}{r} \tag{2.1}$$

with a finite measure λ *on* S *and a nonnegative function* $k_z(r)$ *measurable in* $z \in S$ *and decreasing in* $r > 0$.

Without loss of generality we can assume that $\tilde{G} = \begin{bmatrix} 1 & \sigma \\ \sigma & 1 \end{bmatrix}$, with $\sigma \in (0, 1)$.

The Lévy measure ν of (η_x^2, η_y^2) is computed in [7]:

$$\nu(dxdy) = \frac{1}{x}e^{-\frac{x}{(1-\sigma^2)}}1_{(x>0)}dx\,\delta_0(dy) + \frac{1}{y}e^{-\frac{y}{(1-\sigma^2)}}1_{(y>0)}\,\delta_0(dx)dy$$
$$+ \frac{\sigma^2}{1-\sigma^2}e^{-\frac{1}{1-\sigma^2}(x+y)}(\sigma^2 xy)^{-1/2}I_1\left(\frac{2\sigma\sqrt{xy}}{1-\sigma^2}\right)1_{(x>0,y>0)}dxdy$$

where I_1 is the modified Bessel function of order 1.

Hence for B a Borel subset of $\mathbb{R}_+^2$

$$\nu(B) = \int_0^{\frac{\pi}{2}} \delta_0(d\theta) \int_0^\infty 1_B(r,0)\exp\{-\frac{r}{1+\sigma^2}\}dr + \int_0^{\frac{\pi}{2}} \delta_{\frac{\pi}{2}}(d\theta) \int_0^\infty 1_B(0,r))\exp\{-\frac{r}{1+\sigma^2}\}dr + \int_0^{\frac{\pi}{2}} d\theta \int_0^\infty 1_B(r\cos\theta, r\sin\theta)k_\theta(r)\frac{dr}{r}$$

with $k_\theta(r) = r\frac{1}{\sigma\sqrt{\cos\theta\sin\theta}}\exp\{-\frac{r}{1-\sigma^2}(\cos\theta+\sin\theta)\}I_1(\frac{2\sigma r}{1-\sigma^2}\sqrt{\cos\theta\sin\theta})$.

We can write $\nu(B)$ in the form (2.1) identifying $[0, \pi/2]$ with the corresponding arc of S and putting $k_z(r) = k_\theta(r)$ when z is not on the coordinate axes. Since I_1 is an increasing function and re^{-r} is increasing for small r, $k_\theta(r)$ is increasing in r in a neighborhood of 0. Therefore, $k_z(r)$ is not decreasing in $r > 0$ when z is not in the coordinate axes.

Suppose that ν satisfies another decomposition with the following form:

$\nu(B) = \int_S \tilde{\lambda}(dz) \int_0^\infty 1_B(rz)\tilde{k}_z(r)\frac{dr}{r}$ with $\tilde{\lambda}$ finite measure on S and $\tilde{k}_z(r)$ nonnegative measurable function on $S\times\mathbb{R}^+$. Taking inspiration from Remark 15.12 of Sato's book [6], one obtains then the existence of a measurable strictly positive function h on S such that: $\tilde{\lambda}(dz) = h(z)\lambda(dz)$ and $\tilde{k}_z(r) = \frac{1}{h(z)}k_z(r)$ $\lambda(dz)$-almost every z. Hence $\tilde{k}_z(r)$ would not be a decreasing function of r either.

Consequently there is no decomposition of ν satisfying (2.1). □

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Permanental Random Variables, M-Matrices and α-Permanents

Michael B. Marcus and Jay Rosen

Abstract We explore some properties of a recent representation of permanental vectors which expresses them as sums of independent vectors with components that are independent gamma random variables.

Keywords Infinitely divisible processes • M-matrices • Permanental random variables

Mathematics Subject Classification (2010). Primary 15B99, 60E07, 60J55; Secondary 60G17

1 Introduction

An R^n valued α-permanental random variable $X = (X_1, \ldots, X_n)$ is a random variable with Laplace transform

$$E\left(e^{-\sum_{i=1}^{n} s_i X_i}\right) = \frac{1}{|I + RS|^{\alpha}} \tag{1.1}$$

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M.B. Marcus (✉)
CUNY Graduate Center, 253 West 73rd St., Apt. 2E, New York, NY 10023, USA
e-mail: mbmarcus@optonline.net

J. Rosen
Department of Mathematics, College of Staten Island, CUNY, 2800 Victory Boulevard, Staten Island, NY 10314, USA
e-mail: jrosen30@optimum.net

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for some $n \times n$ matrix R and diagonal matrix S with entries s_i, $1 \le i \le n$, and $\alpha > 0$. Permanental random variables were introduced by Vere-Jones [3], who called them multivariate gamma distributions. (Actually he considered the moment generating function.)

In [2, Lemma 2.1] we obtain a representation for permanental random variables with the property that $A = R^{-1}$ is an M-matrix. A matrix $A = \{a_{i,j}\}_{1\le i,j\le n}$ is said to be a nonsingular M-matrix if

(1) $a_{i,j} \le 0$ for all $i \ne j$.
(2) A is nonsingular and $A^{-1} \ge 0$.

The representation depends on the α-perminant of the off diagonal elements of A which we now define.

The α-perminant of $n \times n$ matrix M is

$$|M|_\alpha = \begin{vmatrix} M_{1,1} & \cdots & M_{1,n} \\ \cdots & & \cdots \\ M_{n,1} & \cdots & M_{n,n} \end{vmatrix}_\alpha = \sum_\pi \alpha^{c(\pi)} M_{1,\pi(1)} M_{2,\pi(1)} \cdots M_{n,\pi(n)}. \tag{1.2}$$

Here the sum runs over all permutations π on $[1,n]$ and $c(\pi)$ is the number of cycles in π.

We use boldface, such as $\mathbf{x}$, to denote vectors. Let $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{N}^n$ and $|\mathbf{k}| = \sum_{l=1}^n k_l$. For $1 \le m \le |\mathbf{k}|$, set $i_m = j$, where

$$\sum_{l=1}^{j-1} k_l < m \le \sum_{l=1}^{j} k_l. \tag{1.3}$$

For any $n \times n$ matrix $C = \{c_{i,j}\}_{1\le i,j\le n}$ we define

$$C(\mathbf{k}) = \begin{bmatrix} c_{i_1,i_1} & c_{i_1,i_2} & \cdots & c_{i_1,i_{|\mathbf{k}|}} \\ c_{i_2,i_1} & c_{i_2,i_2} & \cdots & c_{i_2,i_{|\mathbf{k}|}} \\ \cdots & & \cdots & \\ c_{i_{|\mathbf{k}|},i_1} & c_{i_{|\mathbf{k}|},i_2} & \cdots & c_{i_{|\mathbf{k}|},i_{|\mathbf{k}|}} \end{bmatrix}, \tag{1.4}$$

and $C(\mathbf{0}) = 1$. For example, if $\mathbf{k} = (0,2,3)$ then $|\mathbf{k}| = 5$ and $i_1 = i_2 = 2$ and $i_3 = i_4 = i_5 = 3$,

$$C(0,2,3) = \begin{bmatrix} c_{2,2} & c_{2,2} & c_{2,3} & c_{2,3} & c_{2,3} \\ c_{2,2} & c_{2,2} & c_{2,3} & c_{2,3} & c_{2,3} \\ c_{3,2} & c_{3,2} & c_{3,3} & c_{3,3} & c_{3,3} \\ c_{3,2} & c_{3,2} & c_{3,3} & c_{3,3} & c_{3,3} \\ c_{3,2} & c_{3,2} & c_{3,3} & c_{3,3} & c_{3,3} \end{bmatrix}. \tag{1.5}$$

Here is an alternate description of $C(\mathbf{k})$. For any $n \times n$ matrix $C = \{c_{i,j}\}_{1\le i,j\le n}$ the matrix $C(\mathbf{k})$ is an $|\mathbf{k}| \times |\mathbf{k}|$ matrix with its first k_1 diagonal elements equal to $c_{1,1}$, its next k_2 diagonal elements equal to $c_{2,2}$, and so on. The general element $C(\mathbf{k})_{p,q} = c_{\bar{p},\bar{q}}$, where $\bar{p}$ is equal to either index of diagonal element in row p (the diagonal element has two indices but they are the same), and $\bar{q}$ equal to either index of the diagonal element in column q. Thus in the above example we see that $C(0,2,3)_{4,1} = c_{3,2}$.

Suppose that A is an $n \times n$ M-matrix. Set $a_i = a_{i,i}$ and write

$$A = D_A - B, \tag{1.6}$$

where D is a diagonal matrix with entries $a_1, \ldots, a_n$ and all the elements of B are non-negative. (Note that all the diagonal elements of B are equal to zero.) In addition set

$$\overline{A} = D_A^{-1} A = I - D_A^{-1} B := I - \overline{B}. \tag{1.7}$$

The next lemma is [2, Lemma 2.1].

Lemma 1.1 *Let $A = R^{-1}$ be an $n \times n$ nonsingular M-matrix with diagonal entries $a_1, \ldots, a_n$ and S be an $n \times n$ diagonal matrix with entries $(s_1, \ldots, s_n)$. Then (1.1) is equal to*

$$\begin{aligned} &\frac{|A|^\alpha}{\prod_{i=1}^n a_i^\alpha} \sum_{\mathbf{k}=(k_1,\ldots,k_n)} \frac{|B(\mathbf{k})|_\alpha}{\prod_{i=1}^n a_i^{k_i} k_i!} \frac{1}{(1+(s_1/a_1))^{\alpha+k_1} \cdots (1+(s_n/a_n))^{\alpha+k_n}} \\ &= |\overline{A}|^\alpha \sum_{\mathbf{k}=(k_1,\ldots,k_n)} \frac{|\overline{B}(\mathbf{k})|_\alpha}{\prod_{i=1}^n k_i!} \frac{1}{(1+(s_1/a_1))^{\alpha+k_1} \cdots (1+(s_n/a_n))^{\alpha+k_n}}. \end{aligned} \tag{1.8}$$

where the sum is over all $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{N}^n$. (The series converges for all $s_1, \ldots, s_n \in R_+^n$ for all $\alpha > 0$.)

Setting $S = 0$ we see that

$$\frac{|A|^\alpha}{\prod_{i=1}^n a_i^\alpha} \sum_{\mathbf{k}=(k_1,\ldots,k_n)} \frac{|B(\mathbf{k})|_\alpha}{\prod_{i=1}^n a_i^\alpha k_i!} = |\overline{A}|^\alpha \sum_{\mathbf{k}=(k_1,\ldots,k_n)} \frac{|\overline{B}(\mathbf{k})|_\alpha}{\prod_{i=1}^n k_i!} = 1. \tag{1.9}$$

Let $Z_{\alpha,\overline{B}}$ be an n-dimensional integer valued random variable with

$$P\left(Z_{\alpha,\overline{B}} = (k_1, \ldots, k_n)\right) = |\overline{A}|^\alpha \frac{|\overline{B}(\mathbf{k})|_\alpha}{\prod_{i=1}^n k_i!}. \tag{1.10}$$

(We omit writing the subscript $\alpha, \overline{B}$ when they are fixed from term to term.)

The sum in (1.8) is the Laplace transform of the α-permanental random variable X. Therefore,

$$X \overset{law}{=} \sum_{\mathbf{k}=(k_1,\ldots,k_n)} I_{k_1,\ldots,k_n}(Z)\left(\xi_{\alpha+k_1,a_1},\ldots,\xi_{\alpha+k_n,a_n}\right) \tag{1.11}$$
$$\overset{law}{=} \left(\xi_{\alpha+Z_1,a_1},\ldots,\xi_{\alpha+Z_n,a_n}\right),$$

where Z and all the gamma distributed random variables, $\xi_{u,v}$ are independent. Recall that the probability density function of $\xi_{u,v}$ is

$$f(u,v;x) = \frac{v^u x^{u-1} e^{-vx}}{\Gamma(u)} \quad \text{for } x > 0 \text{ and } u,v > 0, \tag{1.12}$$

and equal to 0 for $x \le 0$. We see that when X has probability density $\xi_{u,v}$, vX has probability density $\xi_{u,1}$. It is easy to see that

$$E(\xi_{u,1}^p) = \frac{\Gamma(p+u)}{\Gamma(u)}. \tag{1.13}$$

It follows from (1.11) that for measurable functions f on R_+^n,

$$E(f(X)) = \sum_{\mathbf{k}=(k_1,\ldots,k_n)} P\left(Z=(k_1,\ldots,k_n)\right) E\left(f\left(\xi_{\alpha+k_1,a_1},\ldots,\xi_{\alpha+k_n,a_n}\right)\right)$$
$$= E\left(f\left(\xi_{\alpha+Z_1,a_1},\ldots,\xi_{\alpha+Z_n,a_n}\right)\right). \tag{1.14}$$

Since

$$\xi_{\alpha+\beta,a} \overset{law}{=} \xi_{\alpha,a} + \xi_{\beta,a}, \tag{1.15}$$

it follows from (1.14) that for all increasing functions f

$$E(f(X)) \ge E\left(f\left(\xi_{\alpha,a_1},\ldots,\xi_{\alpha,a_n}\right)\right). \tag{1.16}$$

We explain in [2] that in some respects (1.16) is a generalization of the Sudakov Inequality for Gaussian processes and use it to obtain sufficient conditions for permanental processes to be unbounded.

A permanental process is a process with finite joint distributions that are permanental random variables. For example, let $G = \{G(t), t \in R\}$ be a Gaussian process with covariance $\tilde{R}(s,t)$. Then for all n and all $t_1,\ldots,t_n$ in R^n, $(G^2(t_1)/2,\ldots,G^2(t_n)/2)$ is an n-dimensional 1/2-permanental random variable, with R in (1.1) equal to the kernel $\{\tilde{R}(t_i,t_j)\}_{i,j=1}^n$. The stochastic process $G^2/2 = \{G^2(t)/2, t \in R\}$ is a 1/2-permanental process. In [2] we consider permanental processes defined for all $\alpha > 0$ and for kernels $R(s,t)$ that need not be symmetric.

In the first part of this paper we give some properties of the random variable Z. It turns out that it is easy to obtain the Laplace transform of Z.

Lemma 1.2

$$E\left(e^{-\sum_{i=1}^n s_i Z_i}\right) = \frac{|\overline{A}|^\alpha}{|I-(\overline{B}E(\mathbf{s}))|^\alpha} \tag{1.17}$$

where $E(\mathbf{s})$ is an $n \times n$ diagonal matrix with entries e^{-s_i}, $i = 1, \ldots, n$.

Proof

$$\begin{aligned} E\left(e^{-\sum_{i=1}^n s_i Z_i}\right) &= \sum_{\mathbf{k}=(k_1,\ldots,k_n)} e^{-\sum_{i=1}^n s_i k_i} P\left(Z = (k_1, \ldots, k_n)\right) \\ &= |\overline{A}|^\alpha \sum_{\mathbf{k}=(k_1,\ldots,k_n)} \prod_{i=1}^n e^{-s_i k_i} \frac{|\overline{B}(\mathbf{k})|_\alpha}{\prod_{i=1}^n k_i!}. \end{aligned} \tag{1.18}$$

Note that

$$\prod_{i=1}^n e^{-s_i k_i} \frac{|\overline{B}(\mathbf{k})|_\alpha}{\prod_{i=1}^n k_i!} = \frac{|(\overline{B}E(\mathbf{s}))(\mathbf{k})|_\alpha}{\prod_{i=1}^n k_i!}. \tag{1.19}$$

By (1.9) with $\overline{B}(0)$ replaced by $\overline{B}E(\mathbf{s})$ for each fixed $\mathbf{s}$

$$|I-\overline{B}E(\mathbf{s})|^\alpha \sum_{\mathbf{k}=(k_1,\ldots,k_n)} \frac{|(\overline{B}E(\mathbf{s}))(\mathbf{k})|_\alpha}{\prod_{i=1}^n k_i!} = 1. \tag{1.20}$$

We get (1.17) from this and (1.18). □

A significant property of permanental random variables that are determined by kernels that are inverse M-matrices is that they are infinitely divisible. Similarly it follows from (1.17) that for all $\alpha, \beta > 0$

$$Z_{\alpha+\beta,\overline{B}} \overset{law}{=} Z_{\alpha,\overline{B}} + Z_{\beta,\overline{B}}. \tag{1.21}$$

We can differentiate (1.17) to give a simple formula for the moments of the components of $Z_{\alpha,\overline{B}}$, which we simply denote by Z in the following lemma.

Lemma 1.3 *For any integer $m \ge 1$ and $1 \le p \le n$,*

$$E(Z_p^m) = \sum_{\substack{j_0+\cdots j_l=m, j_i\ge 1 \\ l=0,1,\ldots,m-1}} (-1)^{l+m+1} \alpha^{j_0} (\alpha+1)^{j_1} \cdots (\alpha+l)^{j_l} (R_{p,p}A_{p,p})^l (R_{p,p}A_{p,p}-1). \tag{1.22}$$

($A_{p,p}$ is also referred to as a_p elsewhere in this paper.)

Proof To simplify the notation we take $p = 1$. Note that by (1.17) and the fact that $\overline{A} = I - (\overline{B}E(\mathbf{0}))$

$$E(Z_1^m) = (-1)^m \frac{\partial^m}{\partial s_1^m} \left(\frac{|\overline{A}|^\alpha}{|I - (\overline{B}E(\mathbf{s}))|^\alpha} \right) \Bigg|_{\mathbf{s}=0} \tag{1.23}$$

$$= (-1)^m |I - (\overline{B}E(\mathbf{0}))|^\alpha \frac{\partial^m}{\partial s_1^m} \left(\frac{1}{|I - (\overline{B}E(\mathbf{s}))|^\alpha} \right) \Bigg|_{\mathbf{s}=0}.$$

Hence to prove (1.22) it suffices to show that for any m

$$\frac{\partial^m}{\partial s_1^m} \left(\frac{1}{|I - (\overline{B}E(\mathbf{s}))|^\alpha} \right) \Bigg|_{\mathbf{s}=0} = \sum_{\substack{j_0 + \cdots j_l = m, j_i \geq 1 \\ l = 0,1,\ldots,m-1}} (-1)^{l+1} \tag{1.24}$$

$$\alpha^{j_0} (\alpha + 1)^{j_1} \cdots (\alpha + l)^{j_l} \frac{(R_{1,1} A_{1,1})^{\,l}}{|I - (\overline{B}E(\mathbf{0}))|^\alpha} (R_{1,1} A_{1,1} - 1).$$

Note that for any $\gamma > 0$ we have

$$\frac{\partial}{\partial s_1} \left(\frac{1}{|I - (\overline{B}E(\mathbf{s}))|^\gamma} \right) = -\frac{\gamma}{|I - (\overline{B}E(\mathbf{s}))|^{\gamma+1}} \frac{\partial}{\partial s_1} |I - (\overline{B}E(\mathbf{s}))|. \tag{1.25}$$

We expand the determinant by the first column. Since $\overline{b}_{1,1} = 0$ we have

$$|I - (\overline{B}E(\mathbf{s}))| = M_{1,1} - \overline{b}_{2,1} e^{-s_1} M_{2,1} + \overline{b}_{3,1} e^{-s_1} M_{3,1} \cdots \pm \overline{b}_{n,1} e^{-s_1} M_{n,1} \tag{1.26}$$

where $M_{i,1}$ are minors of $I - (\overline{B}E(\mathbf{s}))$ and the last sign is plus or minus according to whether n is odd or even. Note that the terms $M_{i,1}$ are not functions of s_1. Using (1.26) we see that

$$\frac{\partial}{\partial s_1} |I - (\overline{B}E(\mathbf{s}))| = \overline{b}_{2,1} e^{-s_1} M_{2,1} - \overline{b}_{3,1} e^{-s_1} M_{3,1} \cdots \mp \overline{b}_{n,1} e^{-s_1} M_{n,1}$$

$$= -|I - (\overline{B}E(\mathbf{s}))| + M_{1,1}. \tag{1.27}$$

Using this we get

$$\frac{\partial}{\partial s_1} \left(\frac{1}{|I - (\overline{B}E(\mathbf{s}))|^\gamma} \right) = \frac{\gamma}{|I - (\overline{B}E(\mathbf{s}))|^\gamma} - \frac{\gamma M_{1,1}}{|I - (\overline{B}E(\mathbf{s}))|^{\gamma+1}} \tag{1.28}$$

$$= -\frac{\gamma}{|I - (\overline{B}E(\mathbf{s}))|^\gamma} \left(\frac{M_{1,1}}{|I - (\overline{B}E(\mathbf{s}))|} - 1 \right).$$

We now show by induction on m that

$$\frac{\partial^m}{\partial s_1^m}\left(\frac{1}{|I-(\overline{B}E(\mathbf{s}))|^\alpha}\right)=\sum_{\substack{j_0+\cdots j_l=m, j_i\geq 1\\ l=0,1,\ldots,m-1}}(-1)^{l+1} \tag{1.29}$$

$$\alpha^{j_0}(\alpha+1)^{j_1}\cdots(\alpha+l)^{j_l}\frac{M_{1,1}^l}{|I-(\overline{B}E(\mathbf{s}))|^{\alpha+l}}\left(\frac{M_{1,1}}{|I-(\overline{B}E(\mathbf{s}))|}-1\right).$$

It is easy to see that for $m=1$ this agrees with (1.28) for $\gamma=\alpha$. Assume that (1.29) holds for m. We show it holds for $m+1$. We take another derivative with respect to s_1. It follows from (1.28) that

$$\frac{\partial}{\partial s_1}\left(\frac{M_{1,1}^l}{|I-(\overline{B}E(\mathbf{s}))|^{\alpha+l}}\left(\frac{M_{1,1}}{|I-(\overline{B}E(\mathbf{s}))|}-1\right)\right) \tag{1.30}$$

$$=\frac{\partial}{\partial s_1}\left(\frac{M_{1,1}^{l+1}}{|I-(\overline{B}E(\mathbf{s}))|^{\alpha+l+1}}-\frac{M_{1,1}^l}{|I-(\overline{B}E(\mathbf{s}))|^{\alpha+l}}\right)$$

$$=-\frac{(\alpha+l+1)M_{1,1}^{l+1}}{|I-(\overline{B}E(\mathbf{s}))|^{\alpha+l+1}}\left(\frac{M_{1,1}}{|I-(\overline{B}E(\mathbf{s}))|}-1\right)$$

$$+\frac{(\alpha+l)M_{1,1}^l}{|I-(\overline{B}E(\mathbf{s}))|^{\alpha+l}}\left(\frac{M_{1,1}}{|I-(\overline{B}E(\mathbf{s}))|}-1\right).$$

Let us consider the term corresponding $l=k$ when we take another derivative with respect to s_1. Two sets of terms in (1.29) contribute to this. One set are the terms in which $j_0+\cdots+j_k=m-1, j_k\geq 1$ which become terms in which $j_0+\cdots+(j_k+1)=m, j_k\geq 1$, when $\frac{M_{1,1}^l}{|I-(\overline{B}E(\mathbf{s}))|^{\alpha+l}}\left(\frac{M_{1,1}}{|I-(\overline{B}E(\mathbf{s}))|}-1\right)$ is replaced by the last line of (1.30). This almost gives us all we need. We are only lacking $j_0+\cdots+j_k=m, j_k=1$. This comes from the next to last line of (1.30) multiplying the terms in (1.29) in which $l=k-1$. One can check that the sign of the terms for $l=k$ for $m+1$ is different from the sign of the terms for $l=k$ for m which is what we need. This completes the proof by induction.

Recall that $\overline{A}=I-(\overline{B}E(\mathbf{0}))$ and that $M_{1,1}$ is actually a function of $\mathbf{s}$. Therefore,

$$\frac{g(\mathbf{0})}{|I-(\overline{B}E(\mathbf{0}))|}=\frac{M_{1,1}(\mathbf{0})}{|\overline{A}|}=((\overline{A})^{-1})_{1,1}=R_{1,1}A_{1,1}, \tag{1.31}$$

by (1.7). Combining this with (1.29) we get (1.24). □

Recall that

$$\overline{R}_{p,p}=R_{p,p}A_{p,p}. \tag{1.32}$$

The next lemma gives relationship between the moments of the components of a permanental random variables X and the components of the corresponding random variables Z.

Lemma 1.4 *For* $m_j \geq 1$, *We have*

$$E\left(\prod_{j=1}^{n}(a_jX_j)^{m_j}\right) = E\left(\prod_{j=1}^{n}\prod_{l=0}^{m_j-1}\left(\alpha + Z_j + l\right)\right). \tag{1.33}$$

or, equivalently

$$|\overline{R}(\mathbf{m})|_\alpha = E\left(\prod_{j=1}^{n}\prod_{l=0}^{m_j-1}\left(\alpha + Z_j + l\right)\right). \tag{1.34}$$

Proof Let $a_1, \ldots, a_n$ denote the diagonal elements of A and set $Y = (a_1X_1, \ldots, a_nX_n)$. Then

$$Y \stackrel{law}{=} (\xi_{\alpha+Z_1,1}, \ldots, \xi_{\alpha+Z_n,1}) \tag{1.35}$$

The left-hand side of (1.33) is $E\left(\prod_{j=1}^{n}(Y_j)^{m_j}\right)$. Therefore, by (1.14) it is equal to

$$E\left(\prod_{j=1}^{n}\xi_{\alpha+Z_j,1}^{m_j}\right) = E\left(\prod_{j=1}^{n}\frac{\Gamma(\alpha + Z_j + m_j)}{\Gamma(\alpha + Z_j)}\right), \tag{1.36}$$

from which we get (1.33).

It follows from [3, Proposition 4.2] that for any $\mathbf{m} = (m_1, \ldots, m_n)$

$$E\left(\prod_{j=1}^{n}X_j^{m_j}\right) = |R(\mathbf{m})|_\alpha. \tag{1.37}$$

Since $|R(\mathbf{m})|_\alpha \prod_{j=1}^{n} a_j^{m_j} = |\overline{R}(\mathbf{m})|_\alpha$ we get (1.34). □

One can use the approach of Lemma 1.3 or try to invert (1.33) to find mixed moments of Z_i. Either approach seems difficult. However, it is easy to make a little progress in this direction.

Lemma 1.5 *For all* i *and* j, *including* $i = j$,

$$Cov\, Z_iZ_j = Cov\, a_iX_ia_jX_j = \alpha\, a_1a_2R_{i,j}R_{j,i}. \tag{1.38}$$

Proof By Lemma 1.4

$$E(Z_i) + \alpha = \alpha a_i R_{i,i} \tag{1.39}$$

and

$$E(a_i a_j X_i X_j) = E((\alpha + Z_i)(\alpha + Z_j)). \tag{1.40}$$

We write

$$\begin{aligned} E((\alpha + Z_i)(\alpha + Z_j)) &= \alpha^2 + \alpha E(Z_i) + \alpha E(Z_j) + E(Z_i)E(Z_j) \\ &= \alpha^2 + \alpha E(Z_i) + \alpha E(Z_j) + Cov(Z_i Z_j) + E(Z_i)E(Z_j) \\ &= (E(Z_i) + \alpha)(E(Z_j) + \alpha) + Cov(Z_i Z_j) \\ &= (\alpha a_i R_{i,i})(\alpha a_j R_{j,j}) + Cov(Z_i Z_j), \end{aligned} \tag{1.41}$$

where we use (1.39) for the last line. Using (1.41) and calculating the left-hand side of (1.40) we get the equality of the first and third terms in (1.38). To find the equality of the second and third terms in (1.38) we differentiate the Laplace transform of (X_1, X_2) in (1.1). □

If $\mathbf{m} = (0, \dots, m_j, 0, \dots 0) := \tilde{\mathbf{m}}$ it follows from (1.34) that

$$|\overline{R}(\tilde{\mathbf{m}})|_\alpha = E\left(\prod_{l=0}^{m_j-1} \left(\alpha + Z_j + l\right)\right). \tag{1.42}$$

Note that

$$|\overline{R}(\tilde{\mathbf{m}})|_\alpha = \overline{R}_{j,j}^{m_j} |E_{m_j}|_\alpha \tag{1.43}$$

where E_{m_j} is an $m_j \times m_j$ matrix with all entries equal to 1. Therefore, by [3, Proposition 3.6]

$$|\overline{R}(\tilde{\mathbf{m}})|_\alpha = \overline{R}_{j,j}^{m_j} \prod_{l=0}^{m_j-1} (\alpha + l). \tag{1.44}$$

Combining (1.42) and (1.44) we get the following inversion of (1.22):

Lemma 1.6

$$\overline{R}_{j,j}^{m_j} = E\left(\prod_{l=0}^{m_j-1} \frac{\alpha + Z_j + l}{\alpha + l}\right). \tag{1.45}$$

As a simple example of (1.45) or (1.33) we have

$$\alpha \overline{R}_{i,i} = \alpha + E\,(Z_i)\,. \tag{1.46}$$

Adding this up for $i = 1, \ldots, n$ we get

$$E\,(\|Z\|_{\ell^1}) = \alpha \left(\sum_{i=1}^{n} \overline{R}_{i,i} - n \right). \tag{1.47}$$

In the next section we give some formulas relating the ℓ^1 norms of permanental random variables to the ℓ^1 norms of the corresponding random variables Z.

We give an alternate form of Lemma 1.3 in which the proof uses Lemma 1.6.

Lemma 1.7 *For any m and $1 \leq p \leq n$,*

$$E(Z_p^m) = \sum_{l=0}^{m} \sum_{(j_0, j_1, \ldots, j_l) \in J_m(l)} (-1)^{l+m} \alpha^{j_0} (\alpha+1)^{j_1} \cdots (\alpha+l)^{j_l} (R_{p,p} A_{p,p})^{\,l} \tag{1.48}$$

where

$$J_m(l) = \{(j_0, j_1, \ldots, j_l) \,|\, j_0 + \cdots j_l = m;\, j_i \geq 1, i = 0, \ldots, l-1; j_l \geq 0\}. \tag{1.49}$$

Proof To simplify the notation we take $p = 1$. It follows from (1.45) that for each m,

$$E\left(\prod_{i=0}^{m-1} (\alpha + i + Z_1) \right) = (R_{1,1} A_{1,1})^m \prod_{i=0}^{m-1} (\alpha + i)\,. \tag{1.50}$$

When $m = 1$ this gives

$$E\,(Z_1) = \alpha R_{1,1} A_{1,1} - \alpha, \tag{1.51}$$

which proves (1.48) when $m = 1$.

Expanding the left hand side of (1.50) gives

$$\sum_{k=0}^{m} \sum_{\substack{U \subseteq [0, m-1] \\ |U| = m-k}} \prod_{i \in U} (\alpha + i)\, E\left(Z_1^k\right) = (R_{1,1} A_{1,1})^m \prod_{i=0}^{m-1} (\alpha + i)\,. \tag{1.52}$$

We prove (1.48) inductively. We have just seen that when $m = 1$ (1.52) holds when $E(Z_1)$ takes the value given in (1.48). Therefore, if we show that (1.52) holds when $m = 2$ and $E(Z_1)$ and $E(Z_1^2)$ take the value given in (1.48), it follows that (1.48) gives the correct value of $E(Z_1^2)$ when $m = 2$. We now assume that we have shown this up to $m - 1$ and write out the left-hand side of (1.52), replacing each $E(Z_1^k)$,

$k = 1, \ldots, m$ by the right-hand side of (1.48). Doing this we obtain terms, depending on k and U, which, up to their sign, are of the form

$$I(j_1, \ldots, j_{m-1}; l) = \prod_{i=0}^{m-1} (\alpha + i)^{j_i} (R_{1,1}A_{1,1})^l, \tag{1.53}$$

where $\sum_{i=0}^{m-1} j_i = m$; $0 \le l \le m$; $j_i \ge 1, i = 0, \ldots, l-1$; $j_l \ge 0$ and $0 \le j_i \le 1, i = l+1, \ldots, m-1$.

The terms in (1.53) may come from the term $\prod_{i \in U} (\alpha + i)$ in (1.52) or they may come from the expression for $E\left(Z_1^k\right)$ in (1.48). Suppose that for some $i = 0, \ldots, l-1$ we have $j_i > 1$ and $i \in U$ and $k = \bar{k} \ge l$. Consider what this term is in (1.53). Note that we obtain the same term with a change of sign when U is replaced by $U - \{i\}$ and $k = \bar{k} + 1$. The same observation holds in reverse. Furthermore, both these arguments also apply when $j_l > 0$.

Because of all this cancelation, when we add up all the terms which, up to their sign, are of the form (1.53), and take their signs into consideration we only get non-zero contributions when all $j_i = 1, i = 0, \ldots, l-1$ and $j_l = 0$. That is, we only get non-zero contributions when

$$\sum_{i=0}^{l} j_i = l \tag{1.54}$$

But recall that we have $\sum_{i=0}^{m-1} j_i = m$ in (1.53) so for this to hold we must have $\sum_{i=l+1}^{m-1} j_i = m - l$ with $0 \le j_i \le 1, i = l+1, \ldots, m-1$. This is not possible because there are only $m - l - 1$ terms in this sum. Therefore we must have $l = m$ in (1.54), which can also be written as $\sum_{i=0}^{m-1} j_i = m$ because $j_l = j_m = 0$. This shows that summing all the terms on the left-hand side of (1.52) gives the right hand side of (1.52). This completes the induction step and establishes (1.48). □

Example 1.1 It is interesting to have some examples. We have already pointed out that for any $1 \le p \le n$

$$E(Z_p) = \alpha(A_{p,p}R_{p,p} - 1). \tag{1.55}$$

Using (1.48) we get

$$\begin{aligned} E(Z_p^4) &= \left[\alpha(\alpha+1)(\alpha+2)(\alpha+3)\right](A_{p,p}R_{p,p})^4 \\ &\quad - \left[\alpha^2(\alpha+1)(\alpha+2) + \alpha(\alpha+1)^2(\alpha+2)\right. \\ &\qquad \left. + \alpha(\alpha+1)(\alpha+2)^2 + \alpha(\alpha+1)(\alpha+2)(\alpha+3)\right](A_{p,p}R_{p,p})^3 \\ &\quad + \left[\alpha^2(\alpha+1)(\alpha+2) + \alpha(\alpha+1)^2(\alpha+2) + \alpha(\alpha+1)(\alpha+2)^2\right. \\ &\qquad \left. + \alpha^3(\alpha+1) + \alpha^2(\alpha+1)^2 + \alpha(\alpha+1)^3\right](A_{p,p}R_{p,p})^2 \\ &\quad - \left[\alpha^3(\alpha+1) + \alpha^2(\alpha+1)^2 + \alpha(\alpha+1)^3 + \alpha^4\right](A_{p,p}R_{p,p}) + \alpha^4. \end{aligned} \tag{1.56}$$

Using (1.22) we get

$$
\begin{aligned}
E(Z_p^4) = {} & \big[\alpha(\alpha+1)(\alpha+2)(\alpha+3)\big](A_{p,p}R_{p,p})^3(A_{p,p}R_{p,p}-1) \\
& -\big[\alpha^2(\alpha+1)(\alpha+2)+\alpha(\alpha+1)^2(\alpha+2) \\
& \qquad +\alpha(\alpha+1)(\alpha+2)^2\big](A_{p,p}R_{p,p})^2(A_{p,p}R_{p,p}-1) \\
& +\big[\alpha^3(\alpha+1)+\alpha^2(\alpha+1)^2+\alpha(\alpha+1)^3\big](A_{p,p}R_{p,p})(A_{p,p}R_{p,p}-1) \\
& -\alpha^4(A_{p,p}R_{p,p}-1).
\end{aligned}
\tag{1.57}
$$

2 Some Formulas for the ℓ^1 Norm of Permanental Random Variables

We can use (1.14) and properties of independent gamma random variables to obtain some interesting formulas for functions of permanental random variables. Taking advantage of the infinitely divisibility of the components of Y, defined in (1.35), we see that

$$
\|Y\|_{\ell^1} \overset{law}{=} \xi_{\alpha n+\|Z\|_{\ell^1},1}. \tag{2.1}
$$

Note that

$$
P(\|Z\|_{\ell^1}=j) = \sum_{\substack{\mathbf{k}=(k_1,\dots,k_n)\\ |\mathbf{k}|=j}} P\left(Z=(k_1,\dots,k_n)\right), \tag{2.2}
$$

where $|\mathbf{k}| = \sum_{i=1}^n k_i$. The following lemma is an immediate consequence of (2.1):

Lemma 2.1 *Let Φ be a positive real valued function. Then*

$$
E(\Phi(\|Y\|_{\ell^1})) = E(\Phi(\xi_{n\alpha+\|Z\|_{\ell^1},1})). \tag{2.3}
$$

Example 2.1 It follows from (2.3) and (1.13) that for any $p>0$,

$$
E(\|Y\|_1^p) = E\left(\frac{\Gamma\left(\|Z\|_{\ell^1}+n\alpha+p\right)}{\Gamma\left(\|Z\|_{\ell^1}+n\alpha\right)}\right). \tag{2.4}
$$

Clearly, $E(\|Y\|_{\ell^1}) = \sum_{i=1}^n a_i E(X_i)$ and $E(X_i)=\alpha R_{i,i}$. Therefore, (1.47) follows from (2.4) with $p=1$.

Using these results we get a formula for the expectation of the ℓ^2 norm of certain n-dimensional Gaussian random variables.

Corollary 2.1 *Let $\eta = (\eta_1, \ldots, \eta_n)$ be a mean zero Gaussian random variable with covariance matrix R. Assume that $A = R^{-1}$ exists and is an M-matrix. Let $\{a_i\}_{i=1}^n$ denote the diagonal elements of A. Set $\mathbf{a^{1/2}}\boldsymbol{\eta} = (a_1^{1/2}\,\eta_1, \ldots, a_n^{1/2}\,\eta_n)$. Then*

$$\left\| \frac{\mathbf{a^{1/2}}\boldsymbol{\eta}}{\sqrt{2}} \right\|_{\ell^2}^2 \overset{law}{=} \xi_{n/2+\|Z\|_{\ell^1},1} \,. \tag{2.5}$$

and

$$E\left(\left\| \frac{\mathbf{a^{1/2}}\boldsymbol{\eta}}{\sqrt{2}} \right\|_2 \right) = E\left(\frac{\Gamma\left(\|Z\|_{\ell^1} + (n+1)/2\right)}{\Gamma\left(\|Z\|_{\ell^1} + n/2\right)} \right). \tag{2.6}$$

Proof The statement in (2.5) is simply (2.1) with $Y = (a_1^{1/2}\,\eta_1, \ldots, a_n^{1/2}\,\eta_n)$ and $\alpha = 1/2$. The statement in (2.6) is simply (2.4) with $p = 1/2$. □

3 Symmetrizing *M*-Matrices

It follows from [1, p. 135, G_{20}; see also p. 150, E_{11}] that a symmetric M-matrix is positive definite. Therefore when the M-matrix A is symmetric and $\alpha = 1/2$, (1.1), with $R = A^{-1}$, is the Laplace transform of a vector with components that are the squares of the components of a Gaussian vector. For this reason we think of symmetric M-matrices as being special. Therefore, given an M-matrix, we ask ourselves how does the permanental vector it defines compare with the permanental vector defined by a symmetrized version of the M-matrix.

When the M-matrix is symmetric and $\alpha = 1/2$, (1.1), with $R = M^{-1}$, is the Laplace transform of a vector with components that are the squares of the components of a Gaussian vector. For this reason we think of symmetric M-matrices as being special. Therefore, given an M-matrix, we ask ourselves how does the permanental vector it defines compare with the permanental vector defined by a symmetrized version of the M-matrix.

For a positive $n \times n$ matrix C with entries $\{c_{i,j}\}$ we define $S(C)$ to be the $n \times n$ matrix with entries $\{(c_{i,j}c_{j,i})^{1/2}\}$. When A is an $n \times n$ non-singular M-matrix of the form $A = D - B$, as in (1.6), we set $A_{sym} = D - S(B)$. We consider the relationship of permanental vectors determined by A and A_{sym}, i.e. by $R = A^{-1}$ and $R_{sym} := A_{sym}^{-1}$ as in (1.1). In Remark 3.2 we explain how this can be used in the study of sample path properties of permanental processes.

Lemma 3.1 *Let A be a non-singular M-matrix, with diagonal elements $\{a_i\}$, $i = 1, \ldots, n$. Then*

$$|\overline{A}| \leq 1. \tag{3.1}$$

Proof This follows from (1.9) since $B(\mathbf{0}) = 1$. □

The series expansion in (1.8) gives the following relationships between two non-singular M-matrices A and A' subject to certain regularity conditions.

Lemma 3.2 *Let A and A' be $n \times n$ non singular M-matrices and define $\overline{A}$ and $\overline{A}'$ as in (1.7). Assume that $\overline{B}' \geq \overline{B}$. Then*

$$\overline{A}' \leq \overline{A} \qquad \text{and} \qquad (\overline{A})^{-1} \leq (\overline{A}')^{-1}. \tag{3.2}$$

Proof The first inequality in (3.2) follows immediately from (1.7).

To obtain the second statement in (3.2) we write $A = D_A(I - D_A^{-1}B)$, so that by [1, Chap. 7, Theorem 5.2]

$$A^{-1}D_A = (I - \overline{B})^{-1} = \sum_{j=0}^{\infty} (\overline{B})^j \quad \text{and} \quad A'^{-1}D_A = (I - \overline{B}')^{-1} = \sum_{j=0}^{\infty} (\overline{B}')^j \tag{3.3}$$

both converge. Therefore $A^{-1}D_A \leq A'^{-1}D_{A'}$ which is the same as the second inequality in (3.2). □

Lemma 3.3 *When A is an $n \times n$ non-singular M-matrix, A_{sym}, and $\overline{A}_{sym}$ are $n \times n$ non-singular M-matrices and*

$$|A_{sym}| \geq |A| \quad \text{and} \quad |\overline{A}_{sym}| \geq |\overline{A}|. \tag{3.4}$$

Proof We prove this for A and A_{sym}. Given this it is obvious that the lemma also holds for $\overline{A}$ and $\overline{A}_{sym}$.

It follows from [1, p. 136, H_{25}] that we can find a positive diagonal matrix $E = \text{diag}\,(e_1, \ldots, e_n)$ such that

$$EAE^{-1} + E^{-1}A^tE \tag{3.5}$$

is strictly positive definite. We use this to show that A_{sym} is strictly positive definite.

We write $A = D - B$ as in (1.6). For any $x = (x_1, \ldots, x_n)$, by definition,

$$\begin{aligned} \sum_{i,j=1}^{n} (A_{sym})_{i,j} x_i x_j &= \sum_{i=1}^{n} a_i x_i^2 - \sum_{i,j=1}^{n} (b_{i,j} b_{j,i})^{1/2} x_i x_j \\ &\geq \sum_{i=1}^{n} a_i x_i^2 - \sum_{i,j=1}^{n} (b_{i,j} b_{j,i})^{1/2} |x_i|\,|x_j| \\ &= \sum_{i=1}^{n} a_i x_i^2 - \sum_{i,j=1}^{n} (e_i b_{i,j} e_j^{-1} e_j b_{j,i} e_i^{-1})^{1/2} |x_i|\,|x_j|, \end{aligned} \tag{3.6}$$

where, the first equality uses the facts that $B \geq 0$ and has $b_{i,i} = 0$, $1 \leq i \leq n$.

Using the inequality between the geometric mean and arithmetic mean of numbers we see that the last line of (3.6)

$$\geq \sum_{i=1}^{n} a_i x_i^2 - \frac{1}{2}\sum_{i,j=1}^{n}(e_i b_{i,j} e_j^{-1} + e_j b_{j,i} e_i^{-1})|x_i|\ |x_j| \tag{3.7}$$

$$= \sum_{i=1}^{n} a_i x_i^2 - \frac{1}{2}\sum_{i,j=1}^{n}(e_i b_{i,j} e_j^{-1} + e_i^{-1} b_{i,j}^t e_j)|x_i|\ |x_j|$$

$$= \frac{1}{2}\sum_{i,j=1}^{n}((EAE^{-1})_{i,j} + (E^{-1}A^t E)_{i,j})|x_i|\ |x_j| > 0,$$

by (3.5). Therefore A_{sym} is strictly positive definite and by definition, A_{sym} has non-positive off diagonal elements. Since the eigenvalues of A_{sym} are real and strictly positive we see by [1, p. 135, G_{20}] that A_{sym} is a non-singular M-matrix.

To get (3.4) we note that by (1.9)

$$|A|^{\alpha} \sum_{\mathbf{k}=(k_1,\dots,k_n)} \frac{|B(\mathbf{k})|_{\alpha}}{\prod_{i=1}^{n} a_i^k k_i!} = |A_{sym}|^{\alpha} \sum_{\mathbf{k}=(k_1,\dots,k_n)} \frac{|S(B)(\mathbf{k})|_{\alpha}}{\prod_{i=1}^{n} a_i^k k_i!}. \tag{3.8}$$

Using (3.9) in the next lemma, we get (3.4). □

Lemma 3.4 *Let C be a positive $n \times n$ matrix. Then*

$$|S(C)|_{\alpha} \leq |C|_{\alpha} \quad \textit{and} \quad |S(C)(\mathbf{k})|_{\alpha} \leq |C(\mathbf{k})|_{\alpha}. \tag{3.9}$$

Proof Consider two terms on the right-hand side of (1.2) for $|C|_{\alpha}$,

$$\alpha^{c(\pi)} c_{1,\pi(1)} c_{2,\pi(2)} \cdots c_{n,\pi(n)} \tag{3.10}$$

and

$$\alpha^{c(\pi^{-1})} c_{1,\pi^{-1}(1)} c_{2,\pi^{-1}(2)} \cdots c_{n,\pi^{-1}(n)} = \alpha^{c(\pi)} c_{\pi(1),1} c_{\pi(2),2} \cdots c_{\pi(n),n} \tag{3.11}$$

The sum of these terms is

$$\alpha^{c(\pi)} \left(c_{1,\pi(1)} c_{2,\pi(2)} \cdots c_{n,\pi(n)} + c_{\pi(1),1} c_{\pi(2),2} \cdots c_{\pi(n),n}\right). \tag{3.12}$$

The corresponding sum of these terms for $|S(C)|_{\alpha}$ is

$$\alpha^{c(\pi)} 2 (c_{1,\pi(1)} c_{2,\pi(1)} \cdots c_{n,\pi(n)} c_{\pi(1),1} c_{\pi(2),2} \cdots c_{\pi(n),n})^{1/2}. \tag{3.13}$$

Considering the inequality between the geometric mean and arithmetic mean of numbers we see that the term in (3.12) is greater than or equal to the term in (3.13).

The same inequality holds for all the other terms on the right-hand side of (1.2). Therefore we have the first inequality in (3.9). A similar analysis gives the second inequality. □

Theorem 3.1 *Let X and $\tilde{X}$ be permanental vectors determined by A and A_{sym} and f be a positive function on R^n. Then*

$$E(f(X)) \geq \frac{|\overline{A}|^{\alpha}}{|\overline{A}_{sym}|^{\alpha}} E(f(\tilde{X})). \tag{3.14}$$

Proof Using Lemma 3.4 and (1.14) we have

$$\begin{aligned} &E(f(X)) \qquad (3.15)\\ &= |\overline{A}| \sum_{\mathbf{k}=(k_1,\ldots,k_n)} \frac{|\overline{B}(\mathbf{k})|_{\alpha}}{\prod_{i=1}^{n} k_i!} E\left(f\left(\xi_{\alpha+k_1,a_1},\ldots,\xi_{\alpha+k_n,a_n}\right)\right)\\ &\geq |\overline{A}| \sum_{\mathbf{k}=(k_1,\ldots,k_n)} \frac{|\overline{B}_{sym}(\mathbf{k})|_{\alpha}}{\prod_{i=1}^{n} k_i!} E\left(f\left(\xi_{\alpha+k_1,a_1},\ldots,\xi_{\alpha+k_n,a_n}\right)\right)\\ &= \frac{|\overline{A}|^{\alpha}}{|\overline{A}_{sym}|^{\alpha}} E(f(\tilde{X})). \end{aligned}$$

□

This leads to an interesting two sided inequality.

Corollary 3.1 *Let X and $\tilde{X}$ be permanental vectors determined by A and A_{sym}. Then for all functions g of X and $\tilde{X}$ and sets $\mathcal{B}$ in the range of g*

$$\begin{aligned} \frac{|A|^{\alpha}}{|A_{sym}|^{\alpha}} P\left(g(\tilde{X}) \in \mathcal{B}\right) &\leq P\left(g(X) \in \mathcal{B}\right) \qquad (3.16)\\ &\leq \left(1 - \frac{|A|^{\alpha}}{|A_{sym}|^{\alpha}}\right) + \frac{|A|^{\alpha}}{|A_{sym}|^{\alpha}} P\left(g(\tilde{X}) \in \mathcal{B}\right). \end{aligned}$$

Proof The first inequality follows by taking $f(X) = I_{g(X)\in\mathcal{B}}(\cdot)$ in (3.14) and, similarly, the second inequality follows by taking $f(X) = I_{g(X)\in\mathcal{B}^c}(\cdot)$ in (3.14). □

Corollary 3.2 *Under the hypotheses of Corollary 3.1*

$$P\left(g(X) \in \mathcal{B}\right) = 1 \implies P\left(g(\tilde{X}) \in \mathcal{B}\right) = 1. \tag{3.17}$$

Proof It follows from the first inequality in (3.16) that

$$P\left(g(X) \in \mathcal{B}^c\right) = 0 \implies P\left(g(\tilde{X}) \in \mathcal{B}^c\right) = 0. \tag{3.18}$$

We get (3.17) by taking complements. □

Remark 3.1 A useful application of Corollaries 3.1 and 3.2 is to take $g(X) = \|X\|_\infty$.

Remark 3.2 When $\{R(s,t), s,t \in S\}$ is the potential density of a transient Markov process with state space S, for all $(s_1,\dots,s_n)$ in S, the matrix $\{R(s_i,s_j)\}_{i,j=1}^n$ is invertible and its inverse $A(s_1,\dots,s_n)$ is a non-singular M-matrix. For all $(s_1,\dots,s_n)$ in S consider $A_{sym}(s_1,\dots,s_n)$. If

$$\inf_{\forall t_1,\dots,t_n,\forall n} \frac{|A(t_1,\dots,t_n)|}{|A_{sym}(t_1,\dots,t_n)|} > 0 \tag{3.19}$$

it follows from Corollary 3.1 that $\sup_{t\in T} X_t < \infty$ almost surely if and only if $\sup_{t\in T} \tilde{X}_t < \infty$ almost surely. Here we also use the fact that these are tail events.

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Convergence in Law Implies Convergence in Total Variation for Polynomials in Independent Gaussian, Gamma or Beta Random Variables

Ivan Nourdin and Guillaume Poly

Abstract Consider a sequence of polynomials of bounded degree evaluated in independent Gaussian, Gamma or Beta random variables. We show that, if this sequence converges in law to a nonconstant distribution, then (1) the limit distribution is necessarily absolutely continuous with respect to the Lebesgue measure and (2) the convergence automatically takes place in the total variation topology. Our proof, which relies on the Carbery–Wright inequality and makes use of a diffusive Markov operator approach, extends the results of Nourdin and Poly (Stoch Proc Appl 123:651–674, 2013) to the Gamma and Beta cases.

Keywords Absolute continuity • Carbery–Wright inequality • Convergence in law • Convergence in total variation • Log-concave distribution • Orthogonal polynomials

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I. Nourdin (✉)
Faculté des Sciences, de la Technologie et de la Communication, Université du Luxembourg, UR en Mathématiques, 6, rue Richard Coudenhove-Kalergi, Esch-sur-Alzette L-1359, Luxembourg
e-mail: ivan.nourdin@uni.lu

G. Poly
Université de Rennes 1, Institut de Recherche Mathématiques de Rennes (IRMAR), 263 Avenue du General Leclerc, CS 74205, F-35042 Rennes Cedex, France
e-mail: guillaume.poly@univ-rennes1.fr

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1 Introduction and Main Results

The Fortet-Mourier distance between (laws of) random variables, defined as

$$d_{FM}(F,G) = \sup_{\substack{\|h\|_\infty \le 1 \\ \|h'\|_\infty \le 1}} \big|E[h(F)] - E[h(G)]\big| \tag{1.1}$$

is well-known to metrize the convergence in law, see, e.g., [5, Theorem 11.3.3]. In other words, one has that $F_n \overset{\text{law}}{\to} F_\infty$ if and only if $d_{FM}(F_n, F_\infty) \to 0$ as $n \to \infty$. But there is plenty of other distances that allows one to measure the proximity between laws of random variables. For instance, one may use the Kolmogorov distance:

$$d_{Kol}(F,G) = \sup_{x\in\mathbb{R}} \big|P(F \le x) - P(G \le x)\big|.$$

Of course, if $d_{Kol}(F_n, F_\infty) \to 0$ then $F_n \overset{\text{law}}{\to} F_\infty$. But the converse implication is wrong in general, meaning that the Kolmogorov distance does *not* metrize the convergence in law. Nevertheless, it becomes true when the target law is *continuous* (that is, when the law of F_∞ has a density with respect to the Lebesgue measure), a fact which can be easily checked by using (for instance) Dini's second theorem. Yet another popular distance for measuring the distance between laws of random variables, which is even stronger than the Kolmogorov distance, is the total variation distance:

$$d_{TV}(F,G) = \sup_{A\in\mathcal{B}(\mathbb{R})} \big|P(F \in A) - P(G \in A)\big|. \tag{1.2}$$

One may prove that

$$d_{TV}(F,G) = \frac{1}{2} \sup_{\|h\|_\infty \le 1} \big|E[h(F)] - E[h(G)]\big|, \tag{1.3}$$

or, whenever F and G both have a density (noted f and g respectively)

$$d_{TV}(F,G) = \frac{1}{2}\int_{\mathbb{R}} |f(x) - g(x)|dx.$$

Unlike the Fortet-Mourier or Kolmogorov distances, it can happen that $F_n \overset{\text{law}}{\to} F_\infty$ for continuous F_n and F_∞ without having that $d_{TV}(F_n, F_\infty) \to 0$. For an explicit counterexample, one may consider $F_n \sim \frac{2}{\pi}\cos^2(nx)\mathbf{1}_{[0,\pi]}(x)dx$; indeed, it is immediate to check that $F_n \overset{\text{law}}{\to} F_\infty \sim \mathcal{U}_{[0,\pi]}$ but $d_{TV}(F_n, F_\infty) \not\to 0$ (it is indeed a strictly positive quantity that does not depend on n).

As we just saw, the convergence in total variation is very strong and therefore it cannot be expected from the mere convergence in law without further assumptions. For instance, in our case, it is crucial that the random variables under consideration are in the domain of suitable differential operators. Let us give three representative results in this direction. Firstly, there is a celebrated theorem of Ibragimov (see, e.g., Reiss [10]) according to which, if F_n, F_∞ are continuous random variables with densities f_n,f_∞ that are *unimodal*, then $F_n \overset{\text{law}}{\to} F_\infty$ if and only if $d_{TV}(F_n, F_\infty) \to 0$. Secondly, let us quote the paper [11], in which necessary and sufficient conditions are given (in term of the absolute continuity of the laws) so that the classical Central Limit Theorem holds in total variation. Finally, let us mention [1] or [6] for conditions ensuring the convergence in total variation for random variables in Sobolev or Dirichlet spaces. Although all the above examples are related to very different frameworks, they have in common the use of a particular structure of the involved variables; loosely speaking, this structure allows to derive a kind of "non-degeneracy" in an appropriate sense which, in turn, enables to reinforce the convergence, from the Fortet-Mourier distance to the total variation one.

Our goal in this short note is to exhibit another instance where convergence in law and in total variation are equivalent. More precisely, we shall prove the following result, which may be seen as an extension to the Gamma and Beta cases of our previous results in [9].

Theorem 1.0.1 *Assume that one of the following three conditions is satisfied:*

(1) $X \sim N(0, 1)$*;*
(2) $X \sim \Gamma(r, 1)$ *with* $r \geq 1$*;*
(3) $X \sim \beta(a, b)$ *with* $a, b \geq 1$.

Let $X_1, X_2, \ldots$ *be independent copies of* X. *Fix an integer* $d \geq 1$ *and, for each* n, *let* m_n *be a positive integer and let* $Q_n \in \mathbb{R}[x_1, \ldots, x_{m_n}]$ *be a multilinear polynomial of degree at most* d*; assume further that* $m_n \to \infty$ *as* $n \to \infty$. *Finally, suppose that* F_n *has the form*

$$F_n = Q_n(X_1, \ldots, X_{m_n}), \quad n \geq 1,$$

and that it converges in law as $n \to \infty$ *to a non-constant random variable* F_∞. *Then the law of* F_∞ *is absolutely continuous with respect to the Lebesgue measure and* F_n *actually converges to* F_∞ *in total variation.*

In the statement of Theorem 1.0.1, by 'multilinear polynomial of degree at most d' we mean a polynomial $Q \in \mathbb{R}[x_1, \ldots, x_m]$ of the form

$$Q(x_1, \ldots, x_m) = \sum_{S \subset \{1,\ldots,m\},\, |S| \leq d} a_S \prod_{i \in S} x_i,$$

for some real coefficients a_S and with the usual convention that $\prod_{i \in \emptyset} x_i = 1$.

Before providing the proof of Theorem 1.0.1, let us comment a little bit about why we are 'only' considering the three cases (1), (2) and (3). This is actually due to our method of proof. Indeed, the two main ingredients we are using for showing Theorem 1.0.1 are the following.

(a) We will make use of a Markov semigroup approach. More specifically, our strategy relies on the use of orthogonal polynomials, which are also eigenvectors of diffusion operators. In dimension 1, up to affine transformations only the Hermite (case (1)), Laguerre (case (2)) and Jacobi (case (3)) polynomials are of this form, see [7].
(b) We will make use of the Carbery–Wright inequality (Theorem 2.1). The main assumption for this inequality to hold is the log-concavity property. This impose some further (weak) restrictions on the parameters in the cases (2) and (3).

The rest of the paper is organized as follows. In Sect. 2, we gather some useful preliminary results. Theorem 1.0.1 is shown in Sect. 3.

2 Preliminaries

From now on, we shall write m instead of m_n for the sake of simplicity.

2.1 Markov Semigroup

In this section, we introduce the framework we will need to prove Theorem 1.0.1. We refer the reader to [2] for the details and missing proofs. Fix an integer m and let μ denote the distribution of the random vector $(X_1, \ldots, X_m)$, with $X_1, \ldots, X_m$ being independent copies of X, for X satisfying either (1), (2) or (3). In these three cases, there exists a reversible Markov process on $\mathbb{R}^m$, with semigroup P_t, equilibrium measure μ and generator $\mathcal{L}$. The operator $\mathcal{L}$ is selfadjoint and negative semidefinite. We define the Dirichlet form $\mathcal{E}$ associated to $\mathcal{L}$ and acting on some domain $\mathcal{D}(\mathcal{L})$ such that, for any $f, g \in \mathcal{D}(\mathcal{L})$,

$$\mathcal{E}(f, g) = -\int f\mathcal{L}g d\mu = -\int g\mathcal{L}f d\mu.$$

When $f = g$, we simply write $\mathcal{E}(f)$ instead of $\mathcal{E}(f, f)$. The carré du champ operator Γ will be also of interest; it is the operator defined as

$$\Gamma(f, g) = \frac{1}{2}\big(\mathcal{L}(fg) - f\mathcal{L}g - g\mathcal{L}f\big).$$

Similarly to $\mathcal{E}$, when $f = g$ we simply write $\Gamma(f)$ instead of $\Gamma(f,f)$. Since $\int \mathcal{L}f\, d\mu = 0$, we observe the following link between the Dirichlet form $\mathcal{E}$ and the carré du champ operator Γ:

$$\int \Gamma(f,g)d\mu = \mathcal{E}(f,g).$$

An important property which is satisfied in the three cases (1), (2) and (3) is that Γ is diffusive in the following sense:

$$\Gamma(\phi(f),g) = \phi'(f)\Gamma(f,g). \tag{2.1}$$

Besides, and it is another important property shared by (1), (2), (3), the eigenvalues of $-\mathcal{L}$ may be ordered as a countable sequence like $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$, with a corresponding sequence of orthonormal eigenfunctions u_0, u_1, u_2, $\cdots$ where $u_0 = 1$; in addition, this sequence of eigenfunctions forms a complete orthogonal basis of $L^2(\mu)$. For completeness, let us give more details in each of our three cases (1), (2), (3).

(1) The case where $X \sim N(0,1)$. We have

$$\mathcal{L}f(x) = \Delta f(x) - x \cdot \nabla f(x), \quad x \in \mathbb{R}^m, \tag{2.2}$$

where Δ is the Laplacian operator and ∇ is the gradient. As a result,

$$\Gamma(f,g) = \nabla f \cdot \nabla g. \tag{2.3}$$

We can compute that $\mathrm{Sp}(-\mathcal{L}) = \mathbb{N}$ and that $\mathrm{Ker}(\mathcal{L} + kI)$ (with I the identity operator) is composed of those polynomials $R(x_1,\ldots,x_m)$ having the form

$$R(x_1,\ldots,x_m) = \sum_{i_1+i_2+\cdots+i_m=k} \alpha(i_1,\cdots,i_m) \prod_{j=1}^{m} H_{i_j}(x_j).$$

Here, H_i stands for the Hermite polynomial of degree i.

(2) The case where $X \sim \Gamma(r,1)$. The density of X is $f_X(t) = t^{r-1}\frac{e^{-t}}{\Gamma(r)}$, $t \geq 0$, with Γ the Euler Gamma function; it is log-concave for $r \geq 1$. Besides, we have

$$\mathcal{L}f(x) = \sum_{i=1}^{m} \Big(x_i \partial_{ii} f + (r+1-x_i)\partial_i f\Big), \quad x \in \mathbb{R}^m. \tag{2.4}$$

As a result,

$$\Gamma(f,g)(x) = \sum_{i=1}^{m} x_i \partial_i f(x) \partial_i g(x), \quad x \in \mathbb{R}^m. \tag{2.5}$$

We can compute that $\mathrm{Sp}(-\mathcal{L}) = \mathbb{N}$ and that $\mathrm{Ker}(\mathcal{L} + kI)$ is composed of those polynomial functions $R(x_1, \ldots, x_m)$ having the form

$$R(x_1, \ldots, x_m) = \sum_{i_1+i_2+\cdots+i_m=k} \alpha(i_1, \cdots, i_m) \prod_{j=1}^{m} L_{i_j}(x_j).$$

Here $L_i(X)$ stands for the ith Laguerre polynomial of parameter r defined as

$$L_i(x) = \frac{x^{-r} e^x}{i!} \frac{d^i}{dx^i} \left\{ e^{-x} x^{i+r} \right\}, \quad x \in \mathbb{R}.$$

(3) The case where $X \sim \beta(a,b)$. In this case, X is continuous with density

$$f_X(t) = \begin{cases} \dfrac{t^{a-1}(1-t)^{b-1}}{\int_0^1 u^{a-1}(1-u)^{b-1}\,du} & \text{if } t \in [0,1] \\ 0 & \text{otherwise} \end{cases}.$$

The density f_X is log-concave when $a, b \geq 1$. Moreover, we have

$$\mathcal{L}f(x) = \sum_{i=1}^{m} \Big((1 - x_i^2) \partial_{ii} f + (b - a - (b+a)x_i) \partial_i f \Big), \quad x \in \mathbb{R}^m. \tag{2.6}$$

As a result,

$$\Gamma(f,g)(x) = \sum_{i=1}^{m} (1 - x_i^2) \partial_i f(x) \partial_i g(x), \quad x \in \mathbb{R}^m. \tag{2.7}$$

Here, the structure of the spectrum turns out to be a little bit more complicated than in the two previous cases (1) and (2). Indeed, we have that

$$\begin{aligned} &\mathrm{Sp}(-\mathcal{L}) \\ &= \{ i_1(i_1 + a + b - 1) + \cdots + i_m(i_m + a + b - 1) \mid i_1, \ldots, i_m \in \mathbb{N} \}. \end{aligned}$$

Note in particular that the first nonzero element of $\mathrm{Sp}(-\mathcal{L})$ is $\lambda_1 = a+b-1 > 0$. Also, one can compute that, when $\lambda \in \mathrm{Sp}(-\mathcal{L})$, then $\mathrm{Ker}(\mathcal{L} + \lambda I)$ is composed of those polynomial functions $R(x_1, \ldots, x_m)$ having the form

$$R(x_1, \ldots, x_m) = \sum_{i_1(i_1+a+b-1)+\cdots+i_m(i_m+a+b-1)=\lambda} \alpha(i_1, \cdots, i_{n_m}) J_{i_1}(x_1) \cdots J_{i_m}(x_m).$$

Here $J_i(X)$ is the ith Jacobi polynomial defined, for $x \in \mathbb{R}$, as

$$J_i(x) = \frac{(-1)^i}{2^i i!}(1-x)^{1-a}(1+x)^{1-b}\frac{d^i}{dx^i}\left\{(1-x)^{a-1}(1+x)^{b-1}(1-x^2)^i\right\}.$$

To end up with this quick summary, we stress that a Poincaré inequality holds true in the three cases (1), (2) and (3). This is well-known and easy to prove, by using the previous facts together with the decomposition

$$L^2(\mu) = \bigoplus_{\lambda \in \mathrm{Sp}(-\mathcal{L})} \mathrm{Ker}(\mathcal{L} + \lambda I).$$

Namely, with $\lambda_1 > 0$ the first nonzero eigenvalue of $-\mathcal{L}$, we have

$$\mathrm{Var}_\mu(f) \le \frac{1}{\lambda_1}\mathcal{E}(f). \tag{2.8}$$

2.2 Carbery–Wright Inequality

The proof of Theorem 1.0.1 will rely, among others, on the following crucial inequality due to Carbery and Wright [4, Theorem 8]. We state it here for convenience.

Theorem 2.1 (Carbery–Wright) *There exists an absolute constant $c > 0$ such that, if $Q : \mathbb{R}^m \to \mathbb{R}$ is a polynomial of degree at most k and μ is a log-concave probability measure on $\mathbb{R}^m$ then, for all $\alpha > 0$,*

$$\left(\int Q^2 d\mu\right)^{\frac{1}{2k}} \times \mu\{x \in \mathbb{R}^m : |Q(x)| \le \alpha\} \le c\,k\,\alpha^{\frac{1}{k}}. \tag{2.9}$$

2.3 Absolute Continuity

There is a celebrated result of Borell [3] according to which, if X_1, X_2, ... are independent, identically distributed and X_1 has an absolute continuous law, then any nonconstant polynomial in the X_i's has an absolute continuous law, too. In the particular case where the common law satisfies either (1), (2) or (3) in Theorem 1.0.1, one can recover Borell's theorem as a consequence of the Carbery–Wright inequality. We provide the proof of this fact here, since it may be seen as a first step towards the proof of Theorem 1.0.1.

Proposition 2.1.1 *Assume that one of the three conditions (1), (2) or (3) of Theorem 1.0.1 is satisfied. Let $X_1, X_2, \ldots$ be independent copies of X. Consider two integers $m, d \geq 1$ and let $Q \in \mathbb{R}[x_1, \ldots, x_m]$ be a polynomial of degree d. Then the law of $Q(X_1, \ldots, X_m)$ is absolutely continuous with respect to the Lebesgue measure if and only if its variance is not zero.*

Proof Write μ for the distribution of $(X_1, \ldots, X_m)$ and assume that the variance of $Q(X_1, \ldots, X_m))$ is strictly positive. We shall prove that, if A is a Borel set of $\mathbb{R}$ with Lebesgue measure zero, then $P(Q(X_1, \ldots, X_m) \in A) = 0$. This will be done in three steps.

Step 1. Let $\varepsilon > 0$ and let B be a *bounded* Borel set. We shall prove that

$$\int \mathbf{1}_{\{Q\in B\}} \frac{\Gamma(Q)}{\varepsilon + \Gamma(Q)} d\mu \tag{2.10}$$
$$= \int \left(\int_{-\infty}^{Q} \mathbf{1}_B(u)du \times \left\{ \frac{-\mathcal{L}Q}{\Gamma(Q)+\varepsilon} + \frac{\Gamma(Q, \Gamma(Q))}{(\Gamma(Q)+\varepsilon)^2} \right\} \right) d\mu.$$

Indeed, let $h : \mathbb{R} \to [0, 1]$ be C^∞ with compact support. We can write, using among other (2.1),

$$\int \left(\int_{-\infty}^{Q} h(u)du \times \frac{-\mathcal{L}Q}{\Gamma(Q)+\varepsilon} \right) d\mu = \mathcal{E}\left(\int_{-\infty}^{Q} h(u)du \times \frac{1}{\Gamma(Q)+\varepsilon}, Q \right)$$
$$= \int \left(h(Q)\frac{\Gamma(Q)}{\Gamma(Q)+\varepsilon} - \int_{-\infty}^{Q} h(u)du\, \frac{\Gamma(Q, \Gamma(Q))}{(\Gamma(Q)+\varepsilon)^2} \right) d\mu.$$

Applying Lusin's theorem allows one, by dominated convergence, to pass from h to $\mathbf{1}_B$ in the previous identity; this leads to the desired conclusion (2.10).
Step 2. Let us apply (2.10) to $B = A \cap [-n, n]$. Since $\int_{-\infty}^{\cdot} \mathbf{1}_B(u)du$ is zero almost everywhere, one deduces that, for all $\varepsilon > 0$ and all $n \in \mathbb{N}^*$,

$$\int \mathbf{1}_{\{Q\in A\cap[-n,n]\}} \frac{\Gamma(Q)}{\varepsilon + \Gamma(Q)} d\mu = 0.$$

By monotone convergence ($n \to \infty$) it comes that, for all $\varepsilon > 0$,

$$\int \mathbf{1}_{\{Q \in A\}} \frac{\Gamma(Q)}{\varepsilon + \Gamma(Q)} d\mu = 0. \tag{2.11}$$

Step 3. Observe that $\Gamma(Q)$ is a polynomial of degree at most $2d$, see indeed (2.3), (2.5) or (2.7). We deduce from the Carbery–Wright inequality (2.9), together with the Poincaré inequality (2.8), that $\Gamma(Q)$ is strictly positive almost everywhere. Thus, by dominated convergence ($\varepsilon \to 0$) in (2.11) we finally get that $\mu\{Q \in A\} = P(Q(X_1, \dots, X_m) \in A) = 0$.

□

3 Proof of Theorem 1.0.1

We are now in a position to show Theorem 1.0.1. We will split its proof in several steps.

Step 1. For any $p \in [1, \infty)$ we shall prove that

$$\sup_n \int |Q_n|^p d\mu_m < \infty. \tag{3.1}$$

(Let us recall our convention about m from the beginning of Sect. 2.) Indeed, using (for instance) Propositions 3.11, 3.12 and 3.16 of [8] (namely, a hypercontractivity property), one first observes that, for any $p \in [2, \infty)$, there exists a constant $c_p > 0$ such that, for all n,

$$\int |Q_n|^p d\mu_m \le c_p \left(\int Q_n^2 d\mu_m \right)^{p/2}. \tag{3.2}$$

(This is for obtaining (3.2) that we need Q_n to be *multilinear*.) On the other hand, one can write

$$\begin{aligned}
&\int Q_n^2 d\mu_m \\
&= \int Q_n^2 \mathbf{1}_{\{Q_n^2 \ge \frac{1}{2} \int Q_n^2 d\mu_m\}} d\mu_m + \int Q_n^2 \mathbf{1}_{\{Q_n^2 < \frac{1}{2} \int Q_n^2 d\mu_m\}} d\mu_m \\
&\le \sqrt{\int Q_n^4 d\mu_m} \sqrt{\mu_m \left\{ x : Q_n(x)^2 \ge \frac{1}{2} \int Q_n^2 d\mu_m \right\}} + \frac{1}{2} \int Q_n^2 d\mu_m,
\end{aligned}$$

so that, using (3.2) with $p = 4$,

$$\mu_m \left\{ x : \, Q_n(x)^2 \geq \frac{1}{2} \int Q_n^2 d\mu_m \right\} \geq \frac{\left(\int Q_n^2 d\mu_m\right)^2}{4 \int Q_n^4 d\mu_m} \geq \frac{1}{4c_4}.$$

But $\{Q_n\}_{n\geq 1}$ is tight as $\{F_n\}_{n\geq 1}$ converges in law. As a result, there exists $M > 0$ such that, for all n,

$$\mu_m \left\{ x : \, Q_n(x)^2 \geq M \right\} < \frac{1}{4c_4}.$$

We deduce that $\int Q_n^2 d\mu_m \leq 2M$ which, together with (3.2), leads to the claim (3.1).

Step 2. We shall prove the existence of a constant $c > 0$ such that, for any $u > 0$ and any $n \in \mathbb{N}^*$,

$$\mu_m \{ x : \, \Gamma(Q_n) \leq u \} \leq c \, \frac{u^{\frac{1}{2d}}}{\mathrm{Var}_{\mu_m}(Q_n)^{\frac{1}{2d}}}. \tag{3.3}$$

Observe first that $\Gamma(Q_n)$ is a polynomial of degree at most $2d$, see indeed (2.3), (2.5) or (2.7). On the other hand, since X has a log-concave density, the probability μ_m is absolutely continuous with a log-concave density as well. As a consequence, Carbery–Wright inequality (2.9) applies and yields the existence of a constant $c > 0$ such that

$$\mu_m \{ x : \, \Gamma(Q_n) \leq u \} \leq c \, u^{\frac{1}{2d}} \left(\int \Gamma(Q_n) d\mu_m \right)^{-\frac{1}{2d}}.$$

To get the claim (3.3), it remains one to apply the Poincaré inequality (2.8).

Step 3. We shall prove the existence of $n_0 \in \mathbb{N}^*$ and $\kappa > 0$ such that, for any $\varepsilon > 0$,

$$\sup_{n \geq n_0} \int \frac{\varepsilon}{\Gamma(Q_n) + \varepsilon} d\mu_m \leq \kappa \, \varepsilon^{\frac{1}{2d+1}}. \tag{3.4}$$

Indeed, thanks to the result shown in Step 2 one can write

$$\begin{aligned} \int \frac{\varepsilon}{\Gamma(Q_n) + \varepsilon} d\mu_m &\leq \frac{\varepsilon}{u} + \mu_m \{ x : \, \Gamma(Q_n) \leq u \} \\ &\leq \frac{\varepsilon}{u} + c \, \frac{u^{\frac{1}{2d}}}{\mathrm{Var}_{\mu_m}(Q_n)^{\frac{1}{2d}}}. \end{aligned}$$

But, by Step 1 and since $\mu_m \circ Q_n^{-1}$ converges to some probability measure η, one has that $\mathrm{Var}_{\mu_m}(Q_n)$ converges to the variance of η as $n \to \infty$. Moreover this variance is strictly positive by assumption. We deduce the existence of $n_0 \in \mathbb{N}^*$ and $\delta > 0$ such that

$$\sup_{n \geq n_0} \int \frac{\varepsilon}{\Gamma(Q_n) + \varepsilon} d\mu_m \leq \frac{\varepsilon}{u} + \delta\, u^{\frac{1}{2d}}.$$

Choosing $u = \varepsilon^{\frac{2d}{2d+1}}$ leads to the desired conclusion (3.4).

Step 4. Let m' be shorthand for $m_{n'}$ and recall the Fortet-Mourier distance (1.1) as well as the total variation distance (1.3) from the Introduction. We shall prove that, for any $n, n' \geq n_0$ (with n_0 and κ given by Step 3), any $0 < \alpha \leq 1$ and any $\varepsilon > 0$,

$$\begin{aligned} d_{TV}(F_n, F_{n'}) \leq \frac{1}{\alpha} d_{FM}(F_n, F_{n'}) + 4\kappa\, \varepsilon^{\frac{1}{2d+1}} \\ + 2\sqrt{\frac{2}{\pi}}\, \frac{\alpha}{\varepsilon^2} \sup_{n \geq n_0} \left(\int \Gamma(Q_n, \Gamma(Q_n)) d\mu_m + \int \left|\mathcal{L} Q_n\right| d\mu_m \right). \end{aligned} \tag{3.5}$$

Indeed, set $p_\alpha(x) = \frac{1}{\alpha\sqrt{2\pi}} e^{-\frac{x^2}{2\alpha^2}}$, $x \in \mathbb{R}$, $0 < \alpha \leq 1$, and let $g \in C_c^\infty$ be bounded by 1. It is immediately checked that

$$\|g * p_\alpha\|_\infty \leq 1 \leq \frac{1}{\alpha} \quad \text{and} \quad \|(g * p_\alpha)'\|_\infty \leq \frac{1}{\alpha}. \tag{3.6}$$

Let $n, n' \geq n_0$ be given integers. Using Step 3 and (3.6) we can write

$$\begin{aligned} &\left| \int g\, d(\mu_m \circ Q_n^{-1}) - \int g\, d(\mu_{m'} \circ Q_{n'}^{-1}) \right| \\ &= \left| \int g \circ Q_n\, d\mu_m - \int g \circ Q_{n'}\, d\mu_{m'} \right| \\ &\leq \left| \int (g * p_\alpha) \circ Q_n d\mu_m - \int (g * p_\alpha) \circ Q_{n'} d\mu_{m'} \right| \\ &\quad + \left| \int (g - g * p_\alpha) \circ Q_n \times \left(\frac{\Gamma(Q_n)}{\Gamma(Q_n) + \varepsilon} + \frac{\varepsilon}{\Gamma(Q_n) + \varepsilon} \right) d\mu_m \right| \\ &\quad + \left| \int (g - g * p_\alpha) \circ Q_{n'} \times \left(\frac{\Gamma(Q_{n'})}{\Gamma(Q_{n'}) + \varepsilon} + \frac{\varepsilon}{\Gamma(Q_{n'}) + \varepsilon} \right) d\mu_{m'} \right| \\ &\leq \frac{1}{\alpha} d_{FM}(F_n, F_{n'}) + 2 \int \frac{\varepsilon}{\Gamma(Q_n) + \varepsilon} d\mu_m + 2 \int \frac{\varepsilon}{\Gamma(Q_{n'}) + \varepsilon} d\mu_{m'} \end{aligned}$$

$$
\begin{aligned}
&+\left|\int (g-g*p_\alpha)\circ Q_n \times \frac{\Gamma(Q_n)}{\Gamma(Q_n)+\varepsilon} d\mu_m\right| \\
&+\left|\int (g-g*p_\alpha)\circ Q_{n'} \times \frac{\Gamma(Q_{n'})}{\Gamma(Q_{n'})+\varepsilon} d\mu_{m'}\right| \\
\leq\ & \frac{1}{\alpha} d_{FM}(F_n, F_{n'}) + 4\kappa\, \varepsilon^{\frac{1}{2d+1}} \\
&+2\sup_{n\geq n_0}\left|\int (g-g*p_\alpha)\circ Q_n \times \frac{\Gamma(Q_n)}{\Gamma(Q_n)+\varepsilon} d\mu_m\right|.
\end{aligned}
$$

Now, set $\Psi(x)=\int_{-\infty}^x g(s)ds$ and let us apply (2.1). We obtain

$$
\begin{aligned}
&\left|\int (g-g*p_\alpha)\circ Q_n \times \frac{\Gamma(Q_n)}{\Gamma(Q_n)+\varepsilon} d\mu_m\right| \\
=\ &\left|\int \frac{1}{\Gamma(Q_n)+\varepsilon}\, \Gamma\big((\Psi-\Psi*p_\alpha)\circ Q_n, Q_n\big) d\mu_m\right| \\
=\ &\left|\int (\Psi-\Psi*p_\alpha)\circ Q_n \times \left(\Gamma\big(Q_n, \frac{1}{\Gamma(Q_n)+\varepsilon}\big) + \frac{\mathcal{L}Q_n}{\Gamma(Q_n)+\varepsilon}\right) d\mu_m\right| \\
=\ &\left|\int (\Psi-\Psi*p_\alpha)\circ Q_n \times \left(-\frac{\Gamma(Q_n,\Gamma(Q_n))}{(\Gamma(Q_n)+\varepsilon)^2} + \frac{\mathcal{L}Q_n}{\Gamma(Q_n)+\varepsilon}\right) d\mu_m\right| \\
\leq\ & \frac{1}{\varepsilon^2}\int |(\Psi-\Psi*p_\alpha)\circ Q_n| \times \big(\Gamma(Q_n,\Gamma(Q_n)) + \big|\mathcal{L}Q_n\big|\big) d\mu_m.
\end{aligned} \tag{3.7}
$$

On the other hand,

$$
\begin{aligned}
|\Psi(x)-\Psi*p_\alpha(x)| &= \left|\int_{\mathbb{R}} p_\alpha(y)\left(\int_{-\infty}^x (g(u)-g(u-y))\,du\right)dy\right| \\
&\leq \int_{\mathbb{R}} p_\alpha(y)\left|\int_{-\infty}^x g(u)du - \int_{-\infty}^x g(u-y)du\right| dy \\
&\leq \int_{\mathbb{R}} p_\alpha(y)\left|\int_{x-y}^x g(u)du\right| dy \leq \int_{\mathbb{R}} p_\alpha(y)\,|y|\,dy \leq \sqrt{\frac{2}{\pi}}\alpha,
\end{aligned} \tag{3.8}
$$

so the desired conclusion (3.5) now follows easily.
Step 5. We shall prove that

$$
\sup_{n\geq n_0}\left(\int \Gamma(Q_n,\Gamma(Q_n))d\mu_m + \int \big|\mathcal{L}Q_n\big| d\mu_m\right) < \infty. \tag{3.9}
$$

First, relying on the results of Sect. 2.1 we have that

$$Q_n \in \bigoplus_{\alpha \le \lambda_{2d}} \mathrm{Ker}(\mathcal{L} + \alpha I).$$

Since $\mathcal{L}$ is a bounded operator on the space $\bigoplus_{\alpha \le \lambda_{2d}} \mathrm{Ker}(\mathcal{L} + \alpha I)$ and Q_n is bounded in $L^2(\mu_m)$, we deduce immediately that $\sup_n \int \mathcal{L}(Q_n)^2 d\mu_m < \infty$, implying in turn that

$$\sup_n \int |\mathcal{L}(Q_n)| d\mu_m < \infty.$$

Besides, one has $\Gamma = \frac{1}{2}(\mathcal{L} + 2\lambda I)$ on $\mathrm{Ker}(\mathcal{L} + \lambda I)$ and one deduces for the same reason as above that

$$\sup_n \int \Gamma(Q_n, \Gamma(Q_n)) d\mu_m < \infty.$$

The proof of (3.9) is complete.
Step 6: Conclusion. The Fortet-Mourier distance d_{FM} metrizing the convergence in distribution, our assumption ensures that $d_{FM}(F_n, F_{n'}) \to 0$ as $n, n' \to \infty$. Therefore, combining (3.9) with (3.5), letting $n, n' \to \infty$, then $\alpha \to 0$ and then $\varepsilon \to 0$, we conclude that $\lim_{n,n' \to \infty} d_{TV}(F_n, F_{n'}) = 0$, meaning that F_n is a Cauchy sequence in the total variation topology. But the space of bounded measures is complete for the total variation distance, so the distribution of F_n must converge to some distribution, say η, in the total variation distance. Of course, η must coincide with the law of F_∞. Moreover, let A be a Borel set of Lebesgue measure zero. By Proposition 2.1.1, we have $P(F_n \in A) = 0$ when n is large enough. Since $d_{TV}(F_n, F_\infty) \to 0$ as $n \to \infty$, we deduce that $P(F_\infty \in A) = 0$ as well, thus proving that the law of F_∞ is absolutely continuous with respect to the Lebesgue measure by the Radon-Nikodym theorem. The proof of Theorem 1.0.1 is now complete. □

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Part IV
High Dimensional Statistics

Perturbation of Linear Forms of Singular Vectors Under Gaussian Noise

Vladimir Koltchinskii and Dong Xia

Abstract Let $A \in \mathbb{R}^{m\times n}$ be a matrix of rank r with singular value decomposition (SVD) $A = \sum_{k=1}^{r} \sigma_k (u_k \otimes v_k)$, where $\{\sigma_k, k = 1, \ldots, r\}$ are singular values of A (arranged in a non-increasing order) and $u_k \in \mathbb{R}^m, v_k \in \mathbb{R}^n, k = 1, \ldots, r$ are the corresponding left and right orthonormal singular vectors. Let $\tilde{A} = A + X$ be a noisy observation of A, where $X \in \mathbb{R}^{m\times n}$ is a random matrix with i.i.d. Gaussian entries, $X_{ij} \sim \mathcal{N}(0, \tau^2)$, and consider its SVD $\tilde{A} = \sum_{k=1}^{m\wedge n} \tilde{\sigma}_k (\tilde{u}_k \otimes \tilde{v}_k)$ with singular values $\tilde{\sigma}_1 \geq \ldots \geq \tilde{\sigma}_{m\wedge n}$ and singular vectors $\tilde{u}_k, \tilde{v}_k, k = 1, \ldots, m \wedge n$.

The goal of this paper is to develop sharp concentration bounds for linear forms $\langle \tilde{u}_k, x\rangle, x \in \mathbb{R}^m$ and $\langle \tilde{v}_k, y\rangle, y \in \mathbb{R}^n$ of the perturbed (empirical) singular vectors in the case when the singular values of A are distinct and, more generally, concentration bounds for bilinear forms of projection operators associated with SVD. In particular, the results imply upper bounds of the order $O\left(\sqrt{\frac{\log(m+n)}{m\vee n}}\right)$ (holding with a high probability) on

$$\max_{1\leq i\leq m} \left|\left\langle \tilde{u}_k - \sqrt{1+b_k}\,u_k, e_i^m \right\rangle\right| \quad \text{and} \quad \max_{1\leq j\leq n} \left|\left\langle \tilde{v}_k - \sqrt{1+b_k}\,v_k, e_j^n \right\rangle\right|,$$

where b_k are properly chosen constants characterizing the bias of empirical singular vectors $\tilde{u}_k, \tilde{v}_k$ and $\{e_i^m, i = 1, \ldots, m\}, \{e_j^n, j = 1, \ldots, n\}$ are the canonical bases of $\mathbb{R}^m, \mathbb{R}^n$, respectively.

Keywords Gaussian noise • Perturbation • Random matrix • Singular vector

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V. Koltchinskii (✉) • D. Xia
School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332, USA
e-mail: vlad@math.gatech.edu; dxia7@math.gatech.edu

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1 Introduction and Main Results

Analysis of perturbations of singular vectors of matrices under a random noise is of importance in a variety of areas including, for instance, digital signal processing, numerical linear algebra and spectral based methods of community detection in large networks (see [2–5, 7, 8, 10, 12] and references therein). Recently, random perturbations of singular vectors have been studied in Vu [14], Wang [16], O'Rourke et al. [9], Benaych-Georges and Nadakuditi [1]. However, up to our best knowledge, this paper proposes first sharp results concerning concentration of *the components* of singular vectors of randomly perturbed matrices. At the same time, there has been interest in the recent literature in so called "delocalization" properties of eigenvectors of random matrices, see Vu and Wang [15], Rudelson and Vershynin [11] and references therein. In this case, the "information matrix" A is equal to zero, $\tilde{A} = X$ and, under certain regularity conditions, it is proved that the magnitudes of the components for the eigenvectors of X (in the case of symmetric square matrix) are of the order $O\big(\frac{\log(n)}{\sqrt{n}}\big)$ with a high probability. This is somewhat similar to the results on "componentwise concentration" of singular vectors of $\tilde{A} = A + X$ proved in this paper, but the analysis in the case when $A \neq 0$ is quite different (it relies on perturbation theory and on the condition that the gaps between the singular values are sufficiently large).

Later in this section, we provide a formal description of the problem studied in the current paper. Before this, we introduce the notations that will be used throughout the paper. For nonnegative K_1, K_2, the notation $K_1 \lesssim K_2$ (equivalently, $K_2 \gtrsim K_1$) means that there exists an absolute constant $C > 0$ such that $K_1 \leq CK_2$; $K_1 \asymp K_2$ is equivalent to $K_1 \lesssim K_2$ and $K_2 \lesssim K_1$ simultaneously. In the case when the constant C might depend on γ, we provide these symbols with subscript γ : say, $K_1 \lesssim_\gamma K_2$. There will be many constants involved in the arguments that may evolve from line to line.

In what follows, $\langle \cdot, \cdot \rangle$ denotes the inner product of finite-dimensional Euclidean spaces. For $N \geq 1$, $e_j^N, j = 1, \ldots, N$ denotes the canonical basis of the space $\mathbb{R}^N$. If P is the orthogonal projector onto a subspace $L \subset \mathbb{R}^N$, then $P^\perp$ denotes the projector onto the orthogonal complement $L^\perp$. With a minor abuse of notation, $\|\cdot\|$ denotes both the l_2-norm of vectors in finite-dimensional spaces and the operator norm of matrices (i.e., their largest singular value). The Hilbert-Schmidt norm of matrices is denoted by $\|\cdot\|_2$. Finally, $\|\cdot\|_\infty$ is adopted for the l_∞-norm of vectors.

In what follows, $A' \in \mathbb{R}^{n\times m}$ denotes the transpose of a matrix $A \in \mathbb{R}^{m\times n}$. The following mapping $\Lambda : \mathbb{R}^{m\times n} \mapsto \mathbb{R}^{(m+n)\times(m+n)}$ will be frequently used:

$$\Lambda(A) := \begin{pmatrix} 0 & A \\ A' & 0 \end{pmatrix}, A \in \mathbb{R}^{m\times n}.$$

Note that the image $\Lambda(A)$ is a symmetric $(m+n)\times(m+n)$ matrix.

Vectors $u \in \mathbb{R}^m$, $v \in \mathbb{R}^n$, etc. will be viewed as column vectors (or $m \times 1, n \times 1$, etc. matrices). For $u \in \mathbb{R}^m$, $v \in \mathbb{R}^n$, denote by $u \otimes v$ the matrix $uv' \in \mathbb{R}^{m\times n}$. In other words, $u \otimes v$ can be viewed as a linear transformation from $\mathbb{R}^n$ into $\mathbb{R}^m$ defined as follows: $(u \otimes v)x = u\langle v, x\rangle, x \in \mathbb{R}^n$.

Let $A \in \mathbb{R}^{m\times n}$ be an $m \times n$ matrix and let

$$A = \sum_{i=1}^{m\wedge n} \sigma_i (u_i \otimes v_i)$$

be its singular value decomposition (SVD) with singular values $\sigma_1 \geq \ldots \geq \sigma_{m\wedge n} \geq 0$, orthonormal left singular vectors $u_1, \ldots, u_{m\wedge n} \in \mathbb{R}^m$ and orthonormal right singular vectors $v_1, \ldots, v_{m\wedge n} \in \mathbb{R}^n$. If A is of rank $\mathrm{rank}(A) = r \leq m \wedge n$, then $\sigma_i = 0, i > r$ and the SVD can be written as $A = \sum_{i=1}^r \sigma_i(u_i \otimes v_i)$. Note that in the case when there are repeated singular values σ_i, the singular vectors are not unique. In this case, let $\mu_1 > \ldots \mu_d > 0$ with $d \leq r$ be distinct singular values of A arranged in decreasing order and denote $\Delta_k := \{i : \sigma_i = \mu_k\}, k = 1, \ldots, d$. Let $\nu_k := \mathrm{card}(\Delta_k)$ be the multiplicity of $\mu_k, k = 1, \ldots, d$. Denote

$$P_k^{uv} := \sum_{i\in\Delta_k} (u_i \otimes v_i),\ P_k^{vu} := \sum_{i\in\Delta_k} (v_i \otimes u_i),$$
$$P_k^{uu} := \sum_{i\in\Delta_k} (u_i \otimes u_i),\ P_k^{vv} := \sum_{i\in\Delta_k} (v_i \otimes v_i).$$

It is straightforward to check that the following relationships hold:

$$(P_k^{uu})' = P_k^{uu},\ (P_k^{uu})^2 = P_k^{uu},\ P_k^{vu} = (P_k^{uv})',\ P_k^{uv}P_k^{vu} = P_k^{uu}. \tag{1.1}$$

This implies, in particular, that the operators P_k^{uu}, P_k^{vv} are orthogonal projectors (in the spaces $\mathbb{R}^m, \mathbb{R}^n$, respectively). It is also easy to check that

$$P_k^{uu}P_{k'}^{uu} = 0,\ P_k^{vv}P_{k'}^{vv} = 0,\ P_k^{vu}P_{k'}^{uv} = 0,\ P_k^{uv}P_{k'}^{vu} = 0,\ k \neq k'. \tag{1.2}$$

The SVD of matrix A can be rewritten as $A = \sum_{k=1}^d \mu_k P_k^{uv}$ and it can be shown that the operators $P_k^{uv}, k = 1, \ldots, d$ are uniquely defined. Let

$$B = \Lambda(A) = \begin{pmatrix} 0 & A \\ A' & 0 \end{pmatrix} = \sum_{k=1}^d \mu_k \begin{pmatrix} 0 & P_k^{uv} \\ P_k^{vu} & 0 \end{pmatrix}.$$

For $k = 1, \ldots, d$, denote

$$P_k := \frac{1}{2}\begin{pmatrix} P_k^{uu} & P_k^{uv} \\ P_k^{vu} & P_k^{vv} \end{pmatrix},\ P_{-k} := \frac{1}{2}\begin{pmatrix} P_k^{uu} & -P_k^{uv} \\ -P_k^{vu} & P_k^{vv} \end{pmatrix},$$

and also

$$\mu_{-k} := -\mu_k.$$

Using relationships (1.1), (1.2), it is easy to show that $P_k P_{k'} = P_{k'} P_k = \mathbb{1}(k = k')P_k$ for all $k, k', 1 \le |k| \le d, 1 \le |k'| \le d$. Since the operators $P_k : \mathbb{R}^{m+n} \mapsto \mathbb{R}^{m+n}, 1 \le |k| \le d$ are also symmetric, they are orthogonal projectors onto mutually orthogonal subspaces of $\mathbb{R}^{m+n}$. Note that, by a simple algebra, $B = \sum_{1\le|k|\le d} \mu_k P_k$, implying that μ_k are distinct eigenvalues of B and P_k are the corresponding eigenprojectors. Note also that if $2\sum_{k=1}^d \nu_k < m+n$, then zero is also an eigenvalue of B (that will be denoted by μ_0) of multiplicity $\nu_0 := n + m - 2\sum_{k=1}^d \nu_k$. Representation $A \mapsto B = \Lambda(A) = \begin{pmatrix} 0 & A \\ A' & 0 \end{pmatrix}$ will play a crucial role in what follows since it allows to reduce the analysis of SVD for matrix A to the spectral representation $B = \sum_{1\le|k|\le d} \mu_k P_k$. In particular, the operators P_k^{uv} involved in the SVD $A = \sum_{k=1}^d \mu_k P_k^{uv}$ can be recovered from the eigenprojectors P_k of matrix B (hence, they are uniquely defined). Define also $\theta_i := \frac{1}{\sqrt{2}}\begin{pmatrix} u_i \\ v_i \end{pmatrix}$ and $\theta_{-i} := \frac{1}{\sqrt{2}}\begin{pmatrix} u_i \\ -v_i \end{pmatrix}$ for $i = 1, \dots, r$ and let $\Delta_{-k} := \{-i : i \in \Delta_k\}, k = 1, \dots, d$. Then, $\theta_i, 1 \le |i| \le r$ are orthonormal eigenvectors of B (not necessarily uniquely defined) corresponding to its non-zero eigenvalues $\sigma_1 \ge \dots \ge \sigma_r > 0 > \sigma_{-r} \ge \dots \ge \sigma_{-1}$ with $\sigma_{-i} = -\sigma_i$ and

$$P_k = \sum_{i\in\Delta_k} (\theta_i \otimes \theta_i), 1 \le |k| \le d.$$

It will be assumed in what follows that A is perturbed by a random matrix $X \in \mathbb{R}^{m\times n}$ with i.i.d. entries $X_{ij} \sim \mathcal{N}(0, \tau^2)$ for some $\tau > 0$. Given the SVD of the perturbed matrix

$$\tilde{A} = A + X = \sum_{j=1}^{m\wedge n} \tilde{\sigma}_i (\tilde{u}_i \otimes \tilde{v}_i),$$

our main interest lies in estimating singular vectors u_i and v_i of the matrix A in the case when its singular values σ_i are distinct, or, more generally, in estimating the operators $P_k^{uu}, P_k^{uv}, P_k^{vu}, P_k^{vv}$. To this end, we will use the estimators

$$\tilde{P}_k^{uu} := \sum_{i\in\Delta_k} (\tilde{u}_i \otimes \tilde{u}_i),\ \tilde{P}_k^{uv} := \sum_{i\in\Delta_k} (\tilde{u}_i \otimes \tilde{v}_i),$$

$$\tilde{P}_k^{vu} := \sum_{i\in\Delta_k} (\tilde{v}_i \otimes \tilde{u}_i),\ \tilde{P}_k^{vv} := \sum_{i\in\Delta_k} (\tilde{v}_i \otimes \tilde{v}_i),$$

and our main goal will be to study the fluctuations of the bilinear forms of these random operators around the bilinear forms of operators $P_k^{uu}, P_k^{uv}, P_k^{vu}, P_k^{vv}$. In the case when the singular values of A are distinct, this would allow us to study the fluctuations of linear forms of singular vectors $\tilde{u}_i, \tilde{v}_i$ around the corresponding linear forms of u_i, v_i which would provide a way to control the fluctuations of components of "empirical" singular vectors in a given basis around their true counterparts. Clearly, the problem can be and will be reduced to the analysis of spectral representation of a symmetric random matrix

$$\tilde{B} = \Lambda(\tilde{A}) = \begin{pmatrix} 0 & \tilde{A} \\ \tilde{A}' & 0 \end{pmatrix} = B + \Gamma, \quad \text{where } \Gamma = \Lambda(X) = \begin{pmatrix} 0 & X \\ X' & 0 \end{pmatrix}, \tag{1.3}$$

that can be viewed as a random perturbation of the symmetric matrix B. The spectral representation of this matrix can be written in the form

$$\tilde{B} = \sum_{1 \le |i| \le (m \wedge n)} \tilde{\sigma}_i (\tilde{\theta}_i \otimes \tilde{\theta}_i),$$

where

$$\tilde{\sigma}_{-i} = -\tilde{\sigma}_i,\ \tilde{\theta}_i := \frac{1}{\sqrt{2}} \begin{pmatrix} \tilde{u}_i \\ \tilde{v}_i \end{pmatrix},\ \tilde{\theta}_{-i} := \frac{1}{\sqrt{2}} \begin{pmatrix} \tilde{u}_i \\ -\tilde{v}_i \end{pmatrix},\ i = 1, \ldots, (m \wedge n).$$

If the operator norm $\|\Gamma\|$ of the "noise" matrix Γ is small enough comparing with the "spectral gap" of the k-th eigenvalue μ_k of B (for some $k = 1, \ldots, d$), then it is easy to see that $\tilde{P}_k := \sum_{i \in \Delta_k} (\tilde{\theta}_i \otimes \tilde{\theta}_i)$ is the orthogonal projector on the direct sum of eigenspaces of $\tilde{B}$ corresponding to the "cluster" $\{\tilde{\sigma}_i : i \in \Delta_k\}$ of its eigenvalues localized in a neighborhood of μ_k. Moreover, $\tilde{P}_k = \frac{1}{2} \begin{pmatrix} \tilde{P}_k^{uu} & \tilde{P}_k^{uv} \\ \tilde{P}_k^{vu} & \tilde{P}_k^{vv} \end{pmatrix}$. Thus, it is enough to study the fluctuations of bilinear forms of random orthogonal projectors $\tilde{P}_k$ around the corresponding bilinear form of the spectral projectors P_k to derive similar properties of operators $\tilde{P}_k^{uu}, \tilde{P}_k^{uv}, \tilde{P}_k^{vu}, \tilde{P}_k^{vv}$.

We will be interested in bounding the bilinear forms of operators $\tilde{P}_k - P_k$ for $k = 1, \ldots, d$. To this end, we will provide separate bounds on the random error $\tilde{P}_k - \mathbb{E}\tilde{P}_k$ and on the bias $\mathbb{E}\tilde{P}_k - P_k$. For $k = 1, \ldots, d$, $\bar{g}_k$ denotes the distance from the eigenvalue μ_k to the rest of the spectrum of A (the eigengap of μ_k). More specifically, for $2 \le k \le d-1$, $\bar{g}_k = \min(\mu_k - \mu_{k+1}, \mu_{k-1} - \mu_k)$, $\bar{g}_1 = \mu_1 - \mu_2$ and $\bar{g}_d = \min(\mu_{d-1} - \mu_d, \mu_d)$.

The main assumption in the results that follow is that $\mathbb{E}\|X\| < \frac{\bar{g}_k}{2}$ (more precisely, $\mathbb{E}\|X\| \le (1-\gamma)\frac{\bar{g}_k}{2}$ for a positive γ). In view of the concentration inequality of Lemma 2.1 in the next section, this essentially means that the operator norm of the random perturbation matrix $\|\Gamma\| = \|X\|$ is strictly smaller than one half of the spectral gap $\bar{g}_k$ of singular value μ_k. Since, again by Lemma 2.1, $\mathbb{E}\|X\| \asymp \tau\sqrt{m \vee n}$, this assumption also means that $\bar{g}_k \gtrsim \tau\sqrt{m \vee n}$ (so, the spectral gap $\bar{g}_k$ is sufficiently large). Our goal is to prove that, under this assumption, the values of bilinear form $\langle \tilde{P}_k x, y\rangle$ of random spectral projector $\tilde{P}_k$ have tight concentration around their means (with the magnitude of deviations of the order $\sqrt{\frac{1}{m\vee n}}$). We will also show that the bias $\mathbb{E}\tilde{P}_k - P_k$ of the spectral projector $\tilde{P}_k$ is "aligned" with the spectral projector P_k (up to an error of the order $\sqrt{\frac{1}{m\vee n}}$ in the operator norm). More precisely, the following results hold.

Theorem 1.1 *Suppose that for some $\gamma \in (0,1)$, $\mathbb{E}\|X\| \le (1-\gamma)\frac{\bar{g}_k}{2}$. There exists a constant $D_\gamma > 0$ such that, for all $x, y \in \mathbb{R}^{m+n}$ and for all $t \ge 1$, the following inequality holds with probability at least $1 - e^{-t}$:*

$$\left|\left\langle(\tilde{P}_k - \mathbb{E}\tilde{P}_k)x, y\right\rangle\right| \le D_\gamma \frac{\tau\sqrt{t}}{\bar{g}_k}\Bigl(\frac{\tau\sqrt{m\vee n} + \tau\sqrt{t}}{\bar{g}_k} + 1\Bigr)\|x\|\|y\|. \tag{1.4}$$

Assuming that $t \lesssim m \vee n$ and taking into account that $\tau\sqrt{m\vee n} \asymp \mathbb{E}\|X\| \le \bar{g}_k$, we easily get from the bound of Theorem 1.1 that

$$\left|\left\langle(\tilde{P}_k - \mathbb{E}\tilde{P}_k)x, y\right\rangle\right| \lesssim_\gamma \frac{\tau\sqrt{t}}{\bar{g}_k}\|x\|\|y\| \lesssim_\gamma \sqrt{\frac{t}{m\vee n}}\|x\|\|y\|,$$

so, the fluctuations of $\langle \tilde{P}_k x, y\rangle$ around its expectation are indeed of the order $\sqrt{\frac{1}{m\vee n}}$.

The next result shows that the bias $\mathbb{E}\tilde{P}_k - P_k$ of $\tilde{P}_k$ can be represented as a sum of a "low rank part" $P_k(\mathbb{E}\tilde{P}_k - P_k)P_k$ and a small remainder.

Theorem 1.2 *The following bound holds with some constant $D > 0$:*

$$\left\|\mathbb{E}\tilde{P}_k - P_k\right\| \le D\frac{\tau^2(m\vee n)}{\bar{g}_k^2}. \tag{1.5}$$

Moreover, suppose that for some $\gamma \in (0,1)$, $\mathbb{E}\|X\| \le (1-\gamma)\frac{\bar{g}_k}{2}$. Then, there exists a constant $C_\gamma > 0$ such that

$$\left\|\mathbb{E}\tilde{P}_k - P_k - P_k(\mathbb{E}\tilde{P}_k - P_k)P_k\right\| \le C_\gamma \frac{\nu_k\tau^2\sqrt{m\vee n}}{\bar{g}_k^2}. \tag{1.6}$$

Since, under the assumption $\mathbb{E}\|X\| \le (1-\gamma)\frac{\bar{g}_k}{2}$, we have $\bar{g}_k \gtrsim \tau\sqrt{m\vee n}$, bound (1.6) implies that the following representation holds

$$\mathbb{E}\tilde{P}_k - P_k = P_k(\mathbb{E}\tilde{P}_k - P_k)P_k + T_k$$

with the remainder T_k satisfying the bound

$$\|T_k\| \lesssim_\gamma \frac{\tau^2\sqrt{m \vee n}}{\bar{g}_k^2} \lesssim_\gamma \frac{\nu_k}{\sqrt{m \vee n}}.$$

We will now consider a special case when μ_k has multiplicity 1 ($\nu_k = 1$). In this case, $\Delta_k = \{i_k\}$ for some $i_k \in \{1, \ldots, (m \wedge n)\}$ and $P_k = \theta_{i_k} \otimes \theta_{i_k}$. Let $\tilde{P}_k := \tilde{\theta}_{i_k} \otimes \tilde{\theta}_{i_k}$. Note that on the event $\|\Gamma\| = \|X\| < \frac{\bar{g}_k}{2}$ that is assumed to hold with a high probability, the multiplicity of $\tilde{\sigma}_{i_k}$ is also 1 (see the discussion in the next section after Lemma 2.2). Note also that the unit eigenvectors $\theta_{i_k}, \tilde{\theta}_{i_k}$ are defined only up to their signs. Due to this, we will assume without loss of generality that $\langle \tilde{\theta}_{i_k}, \theta_{i_k} \rangle \geq 0$.

Since $P_k = \theta_{i_k} \otimes \theta_{i_k}$ is an operator of rank 1, we have

$$P_k(\mathbb{E}\tilde{P}_k - P_k)P_k = b_k P_k,$$

where

$$b_k := \Big\langle (\mathbb{E}\tilde{P}_k - P_k)\theta_{i_k}, \theta_{i_k} \Big\rangle = \mathbb{E}\langle \tilde{\theta}_{i_k}, \theta_{i_k} \rangle^2 - 1.$$

Therefore,

$$\mathbb{E}\tilde{P}_k = (1 + b_k)P_k + T_k$$

and b_k turns out to be the main parameter characterizing the bias of $\tilde{P}_k$. Clearly, $b_k \in [-1, 0]$ (note that $b_k = 0$ is equivalent to $\tilde{\theta}_{i_k} = \theta_{i_k}$ a.s. and $b_k = -1$ is equivalent to $\tilde{\theta}_{i_k} \perp \theta_{i_k}$ a.s.). On the other hand, by bound (1.5) of Theorem 1.2,

$$|b_k| \leq \Big\| \mathbb{E}\tilde{P}_k - P_k \Big\| \lesssim \frac{\tau^2(m \vee n)}{\bar{g}_k^2}. \tag{1.7}$$

In the next theorem, it will be assumed that the bias is not too large in the sense that b_k is bounded away by a constant $\gamma > 0$ from -1.

Theorem 1.3 *Suppose that, for some $\gamma \in (0, 1)$, $\mathbb{E}\|X\| \leq (1-\gamma)\frac{\bar{g}_k}{2}$ and $1 + b_k \geq \gamma$. Then, for all $x \in \mathbb{R}^{m+n}$ and for all $t \geq 1$ with probability at least $1 - e^{-t}$,*

$$\Big|\langle \tilde{\theta}_{i_k} - \sqrt{1 + b_k}\theta_{i_k}, x\rangle\Big| \lesssim_\gamma \frac{\tau\sqrt{t}}{\bar{g}_k}\Big(\frac{\tau\sqrt{m \vee n} + \tau\sqrt{t}}{\bar{g}_k} + 1\Big)\|x\|.$$

Assuming that $t \lesssim m \vee n$, the bound of Theorem 1.3 implies that

$$\Big|\langle \tilde{\theta}_{i_k} - \sqrt{1 + b_k}\theta_{i_k}, x\rangle\Big| \lesssim_\gamma \frac{\tau\sqrt{t}}{\bar{g}_k}\|x\| \lesssim_\gamma \sqrt{\frac{t}{m \vee n}}\|x\|.$$

Therefore, the fluctuations of $\langle\tilde{\theta}_{i_k}, x\rangle$ around $\sqrt{1+b_k}\langle\theta_{i_k}, x\rangle$ are of the order $\sqrt{\frac{1}{m\vee n}}$.

Recall that $\theta_{i_k} := \frac{1}{\sqrt{2}}\begin{pmatrix} u_{i_k} \\ v_{i_k}\end{pmatrix}$, where u_{i_k}, v_{i_k} are left and right singular vectors of A corresponding to its singular value μ_k. Theorem 1.3 easily implies the following corollary.

Corollary 1.4 *Under the conditions of Theorem 1.3, with probability at least* $1 - \frac{1}{m+n}$,

$$\max\Big\{\big\|\tilde{u}_{i_k} - \sqrt{1+b_k}u_{i_k}\big\|_\infty, \big\|\tilde{v}_{i_k} - \sqrt{1+b_k}v_{i_k}\big\|_\infty\Big\} \lesssim \sqrt{\frac{\log(m+n)}{m\vee n}}.$$

For the proof, it is enough to take $t = 2\log(m+n),\ x = e_i^{m+n}, i = 1,\ldots,(m+n)$ and to use the bound of Theorem 1.3 along with the union bound. Then recalling that $\theta_{i_k} = \frac{1}{\sqrt{2}}(u'_{i_k}, v'_{i_k})'$, Theorem 1.3 easily implies the claim.

Theorem 1.3 shows that the "naive estimator" $\langle\tilde{\theta}_{i_k}, x\rangle$ of linear form $\langle\theta_{i_k}, x\rangle$ could be improved by reducing its bias that, in principle, could be done by its simple rescaling $\langle\tilde{\theta}_{i_k}, x\rangle \mapsto \langle(1+b_k)^{-1/2}\tilde{\theta}_{i_k}, x\rangle$. Of course, the difficulty with this approach is related to the fact that the bias parameter b_k is unknown. We will outline below a simple approach based on repeated observations of matrix A. More specifically, let $\tilde{A}^1 = A + X^1$ and $\tilde{A}^2 = A + X^2$ be two independent copies of $\tilde{A}$ and denote $\tilde{B}^1 = \Lambda(\tilde{A}^1), \tilde{B}^2 = \Lambda(\tilde{A}^2)$. Let $\tilde{\theta}^1_{i_k}$ and $\tilde{\theta}^2_{i_k}$ be the eigenvectors of $\tilde{B}^1$ and $\tilde{B}^2$ corresponding to their eigenvalues $\tilde{\sigma}^1_{i_k}, \tilde{\sigma}^2_{i_k}$. The signs of $\tilde{\theta}^1_{i_k}$ and $\tilde{\theta}^2_{i_k}$ are chosen so that $\big\langle\tilde{\theta}^1_{i_k}, \tilde{\theta}^2_{i_k}\big\rangle \geq 0$. Let

$$\tilde{b}_k := \big\langle\tilde{\theta}^1_{i_k}, \tilde{\theta}^2_{i_k}\big\rangle - 1. \tag{1.8}$$

Given $\gamma > 0$, define

$$\hat{\theta}^{(\gamma)}_{i_k} := \frac{\tilde{\theta}^1_{i_k}}{\sqrt{1+\tilde{b}_k \vee \frac{\sqrt{\gamma}}{2}}}.$$

Corollary 1.5 *Under the assumptions of Theorem 1.3, there exists a constant* $D_\gamma > 0$ *such that for all* $x \in \mathbb{R}^{m+n}$ *and all* $t \geq 1$ *with probability at least* $1 - e^{-t}$,

$$|\hat{b}_k - b_k| \leq D_\gamma \frac{\tau\sqrt{t}}{\bar{g}_k}\Big[\frac{\tau\sqrt{m\vee n} + \tau\sqrt{t}}{\bar{g}_k} + 1\Big] \tag{1.9}$$

and

$$|\langle\hat{\theta}^{(\gamma)}_{i_k} - \theta_{i_k}, x\rangle| \leq D_\gamma \frac{\tau\sqrt{t}}{\bar{g}_k}\Big[\frac{\tau\sqrt{m\vee n} + \tau\sqrt{t}}{\bar{g}_k} + 1\Big]\|x\|. \tag{1.10}$$

Note that $\hat{\theta}_{i_k}^{(\gamma)}$ is not necessarily a unit vector. However, its linear form provides a better approximation of the linear forms of θ_{i_k} than in the case of vector $\tilde{\theta}_{i_k}^{1}$ that is properly normalized. Clearly, the result implies similar bounds for the singular vectors $\hat{u}_{i_k}^{(\gamma)}$ and $\hat{v}_{i_k}^{(\gamma)}$.

2 Proofs of the Main Results

The proofs follow the approach of Koltchinskii and Lounici [6] who did a similar analysis in the problem of estimation of spectral projectors of sample covariance. We start with discussing several preliminary facts used in what follows. Lemmas 2.1 and 2.2 below provide moment bounds and a concentration inequality for $\|\Gamma\| = \|X\|$. The bound on $\mathbb{E}\|X\|$ of Lemma 2.1 is available in many references (see, e.g., Vershynin [13]). The concentration bound for $\|X\|$ is a straightforward consequence of the Gaussian concentration inequality. The moment bounds of Lemma 2.2 can be easily proved by integrating out the tails of the exponential bound that follows from the concentration inequality of Lemma 2.1.

Lemma 2.1 *There exist absolute constants* $c_0, c_1, c_2 > 0$ *such that*

$$c_0 \tau \sqrt{m \vee n} \le \mathbb{E}\|X\| \le c_1 \tau \sqrt{m \vee n}$$

and for all $t > 0$*,*

$$\mathbb{P}\left\{\left|\|X\| - \mathbb{E}\|X\|\right| \ge c_2 \tau \sqrt{t}\right\} \le e^{-t}.$$

Lemma 2.2 *For all* $p \ge 1$*, it holds that*

$$\mathbb{E}^{1/p}\|X\|^p \asymp \tau \sqrt{m \vee n}$$

According to a well-known result that goes back to Weyl, for symmetric (or Hermitian) $N \times N$ matrices C, D

$$\max_{1 \le j \le N} \left|\lambda_j^{\downarrow}(C) - \lambda_j^{\downarrow}(D)\right| \le \|C - D\|,$$

where $\lambda^{\downarrow}(C), \lambda^{\downarrow}(D)$ denote the vectors consisting of the eigenvalues of matrices C, D, respectively, arranged in a non-increasing order. This immediately implies that, for all $k = 1, \ldots, d$,

$$\max_{j \in \Delta_k} |\tilde{\sigma}_j - \mu_k| \le \|\Gamma\|$$

and

$$\min_{j \in \cup_{k' \neq k} \Delta_{k'}} |\tilde{\sigma}_j - \mu_k| \geq \bar{g}_k - \|\Gamma\|.$$

Assuming that $\|\Gamma\| < \frac{\bar{g}_k}{2}$, we get that $\{\tilde{\sigma}_j : j \in \Delta_k\} \subset (\mu_k - \bar{g}_k/2, \mu_k + \bar{g}_k/2)$ and the rest of the eigenvalues of $\tilde{B}$ are outside of this interval. Moreover, if $\|\Gamma\| < \frac{\bar{g}_k}{4}$, then the cluster of eigenvalues $\{\tilde{\sigma}_j : j \in \Delta_k\}$ is localized inside a shorter interval $(\mu_k - \bar{g}_k/4, \mu_k + \bar{g}_k/4)$ of radius $\bar{g}_k/4$ and its distance from the rest of the spectrum of $\tilde{B}$ is $> \frac{3}{4}\bar{g}_k$. These simple considerations allow us to view the projection operator $\tilde{P}_k = \sum_{j \in \Delta_k} (\tilde{\theta}_j \otimes \tilde{\theta}_j)$ as a projector on the direct sum of eigenspaces of $\tilde{B}$ corresponding to its eigenvalues located in a "small" neighborhood of the eigenvalue μ_k of B, which makes $\tilde{P}_k$ a natural estimator of P_k.

Define operators C_k as follows:

$$C_k = \sum_{s \neq k} \frac{1}{\mu_s - \mu_k} P_s.$$

In the case when $2 \sum_{k=1}^{d} \nu_k < m+n$ and, hence, $\mu_0 = 0$ is also an eigenvalue of B, it will be assumed that the above sum includes $s = 0$ with P_0 being the corresponding spectral projector.

The next simple lemma can be found, for instance, in Koltchinskii and Lounici [6]. Its proof is based on a standard perturbation analysis utilizing Riesz formula for spectral projectors.

Lemma 2.3 *The following bound holds:*

$$\|\tilde{P}_k - P_k\| \leq 4 \frac{\|\Gamma\|}{\bar{g}_k}.$$

Moreover,

$$\tilde{P}_k - P_k = L_k(\Gamma) + S_k(\Gamma),$$

where $L_k(\Gamma) := C_k \Gamma P_k + P_k \Gamma C_k$ *and*

$$\|S_k(\Gamma)\| \leq 14 \left(\frac{\|\Gamma\|}{\bar{g}_k} \right)^2.$$

Proof of Theorem 1.1 Since $\mathbb{E}L_k(\Gamma) = 0$, it is easy to check that

$$\tilde{P}_k - \mathbb{E}\tilde{P}_k = L_k(\Gamma) + S_k(\Gamma) - \mathbb{E}S_k(\Gamma) =: L_k(\Gamma) + R_k(\Gamma). \tag{2.1}$$

We will first provide a bound on the bilinear form of the remainder $\langle R_k(\Gamma)x, y\rangle$. Note that

$$\langle R_k(\Gamma)x, y\rangle = \langle S_k(\Gamma)x, y\rangle - \langle \mathbb{E}S_k(\Gamma)x, y\rangle$$

is a function of the random matrix $X \in \mathbb{R}^{m\times n}$ since $\Gamma = \Lambda(X)$ [see (1.3)]. When we need to emphasize this dependence, we will write Γ_X instead of Γ. With some abuse of notation, we will view X as a point in $\mathbb{R}^{m\times n}$ rather than a random variable.

Let $0 < \gamma < 1$ and define a function $h_{x,y,\delta}(\cdot) : \mathbb{R}^{m\times n} \to \mathbb{R}$ as follows:

$$h_{x,y,\delta}(X) := \langle S_k(\Gamma_X)x, y\rangle\, \phi\left(\frac{\|\Gamma_X\|}{\delta}\right),$$

where ϕ is a Lipschitz function with constant $\frac{1}{\gamma}$ on $\mathbb{R}_+$ and $0 \le \phi(s) \le 1$. More precisely, assume that $\phi(s) = 1, s \le 1,\ \phi(s) = 0, s \ge (1+\gamma)$ and ϕ is linear in between. We will prove that the function $X \mapsto h_{x,y,\delta}(X)$ satisfy the Lipschitz condition. Note that

$$|\langle (S_k(\Gamma_{X_1}) - S_k(\Gamma_{X_2}))\, x, y\rangle| \le \|S_k(\Gamma_{X_1}) - S_k(\Gamma_{X_2})\|\, \|x\|\, \|y\|.$$

To control the norm $\|S_k(\Gamma_{X_1}) - S_k(\Gamma_{X_2})\|$, we need to apply Lemma 4 from [6]. It is stated below without the proof.

Lemma 2.4 *Let $\gamma \in (0,1)$ and suppose that $\delta \le \frac{1-\gamma}{1+\gamma}\frac{\bar{g}_k}{2}$. There exists a constant $C_\gamma > 0$ such that, for all symmetric $\Gamma_1, \Gamma_2 \in \mathbb{R}^{(m+n)\times(m+n)}$ satisfying the conditions $\|\Gamma_1\| \le (1+\gamma)\delta$ and $\|\Gamma_2\| \le (1+\gamma)\delta$,*

$$\|S_k(\Gamma_1) - S_k(\Gamma_2)\| \le C_\gamma \frac{\delta}{\bar{g}_k^2}\|\Gamma_1 - \Gamma_2\|.$$

We now derive the Lipschitz condition for the function $X \mapsto h_{x,y,\delta}(X)$.

Lemma 2.5 *Under the assumption that $\delta \le \frac{1-\gamma}{1+\gamma}\frac{\bar{g}_k}{2}$, there exists a constant $C_\gamma > 0$,*

$$\left|h_{x,y,\delta}(X_1) - h_{x,y,\delta}(X_2)\right| \le C_\gamma \frac{\delta\|X_1 - X_2\|_2}{\bar{g}_k^2}\|x\|\,\|y\|. \tag{2.2}$$

Proof Suppose first that $\max(\|\Gamma_{X_1}\|, \|\Gamma_{X_2}\|) \le (1+\gamma)\delta$. Using Lemma 2.4 and Lipschitz properties of function ϕ, we get

$$\begin{aligned}
|h_{x,y,\delta}(X_1) - h_{x,y,\delta}(X_2)| &= \left|\langle S_k(\Gamma_{X_1})x, y\rangle \phi\left(\frac{\|\Gamma_{X_1}\|}{\delta}\right) - \langle S_k(\Gamma_{X_2})x, y\rangle \phi\left(\frac{\|\Gamma_{X_2}\|}{\delta}\right)\right| \\
&\le \|S_k(\Gamma_{X_1}) - S_k(\Gamma_{X_2})\| \|x\| \|y\| \phi\left(\frac{\|\Gamma_{X_1}\|}{\delta}\right) \\
&+ \|S_k(\Gamma_{X_2})\| \left|\phi\left(\frac{\|\Gamma_{X_1}\|}{\delta}\right) - \phi\left(\frac{\|\Gamma_{X_2}\|}{\delta}\right)\right| \|x\| \|y\| \\
&\le C_\gamma \frac{\delta \|\Gamma_{X_1} - \Gamma_{X_2}\|}{\bar{g}_k^2} \|x\| \|y\| + \frac{14(1+\gamma)^2\delta^2}{\bar{g}_k^2} \frac{\|\Gamma_{X_1} - \Gamma_{X_2}\|}{\gamma\delta} \|x\| \|y\| \\
&\lesssim_\gamma \frac{\delta \|\Gamma_{X_1} - \Gamma_{X_2}\|}{\bar{g}_k^2} \|x\| \|y\| \lesssim_\gamma \frac{\delta \|X_1 - X_2\|_2}{\bar{g}_k^2} \|x\| \|y\|.
\end{aligned}$$

In the case when $\min(\|\Gamma_{X_1}\|, \|\Gamma_{X_2}\|) \ge (1+\gamma)\delta$, we have $h_{x,y,\delta}(X_1) = h_{x,y,\delta}(X_2) = 0$, and (2.2) trivially holds. Finally, in the case when $\|\Gamma_{X_1}\| \le (1+\gamma)\delta \le \|\Gamma_{X_2}\|$, we have

$$\begin{aligned}
|h_{x,y,\delta}(X_1) - h_{x,y,\delta}(X_2)| &= \left|\langle S_k(\Gamma_{X_1})x, y\rangle \phi\left(\frac{\|\Gamma_{X_1}\|}{\delta}\right)\right| \\
&= \left|\langle S_k(\Gamma_{X_1})x, y\rangle \phi\left(\frac{\|\Gamma_{X_1}\|}{\delta}\right) - \langle S_k(\Gamma_{X_1})x, y\rangle \phi\left(\frac{\|\Gamma_{X_2}\|}{\delta}\right)\right| \\
&\le \|S_k(\Gamma_{X_1})\| \left|\phi\left(\frac{\|\Gamma_{X_1}\|}{\delta}\right) - \phi\left(\frac{\|\Gamma_{X_2}\|}{\delta}\right)\right| \|x\| \|y\| \\
&\le 14\left(\frac{(1+\gamma)\delta}{\bar{g}_k}\right)^2 \frac{\|\Gamma_{X_1} - \Gamma_{X_2}\|}{\gamma\delta} \|x\| \|y\| \\
&\lesssim_\gamma \frac{\delta \|X_1 - X_2\|_2}{\bar{g}_k^2} \|x\| \|y\|.
\end{aligned}$$

The case $\|\Gamma_{X_2}\| \le (1+\gamma)\delta \le \|\Gamma_{X_1}\|$ is similar. □

Our next step is to apply the following concentration bound that easily follows from the Gaussian isoperimetric inequality.

Lemma 2.6 *Let $f : \mathbb{R}^{m\times n} \mapsto \mathbb{R}$ be a function satisfying the following Lipschitz condition with some constant $L > 0$:*

$$|f(A_1) - f(A_2)| \le L\|A_1 - A_2\|_2, A_1, A_2 \in \mathbb{R}^{m\times n}$$

Suppose X is a random $m \times n$ matrix with i.i.d. entries $X_{ij} \sim \mathcal{N}(0, \tau^2)$. Let M be a real number such that

$$\mathbb{P}\big\{f(X) \geq M\big\} \geq \frac{1}{4} \text{ and } \mathbb{P}\big\{f(X) \leq M\big\} \geq \frac{1}{4}.$$

Then there exists some constant $D_1 > 0$ such that for all $t \geq 1$,

$$\mathbb{P}\Big\{\big|f(X) - M\big| \geq D_1 L\tau\sqrt{t}\Big\} \leq e^{-t}.$$

The next lemma is the main ingredient in the proof of Theorem 1.1. It provides a Bernstein type bound on the bilinear form $\langle R_k(\Gamma)x, y\rangle$ of the remainder R_k in the representation (2.1).

Lemma 2.7 *Suppose that, for some $\gamma \in (0, 1)$, $\mathbb{E}\|\Gamma\| \leq (1-\gamma)\frac{\bar{g}_k}{2}$. Then, there exists a constant $D_\gamma > 0$ such that for all $x, y \in \mathbb{R}^{m+n}$ and all $t \geq \log(4)$, the following inequality holds with probability at least $1 - e^{-t}$*

$$|\langle R_k(\Gamma)x, y\rangle| \leq D_\gamma \frac{\tau\sqrt{t}}{\bar{g}_k}\left(\frac{\tau\sqrt{m \vee n} + \tau\sqrt{t}}{\bar{g}_k}\right)\|x\|\|y\|.$$

Proof Define $\delta_{n,m}(t) := \mathbb{E}\|\Gamma\| + c_2\tau\sqrt{t}$. By the second bound of Lemma 2.1, with a proper choice of constant $c_2 > 0$, $\mathbb{P}\{\|\Gamma\| \geq \delta_{n,m}(t)\} \leq e^{-t}$. We first consider the case when $c_2\tau\sqrt{t} \leq \frac{\gamma}{2}\frac{\bar{g}_k}{2}$, which implies that

$$\delta_{n,m}(t) \leq (1 - \gamma/2)\frac{\bar{g}_k}{2} = \frac{1-\gamma'}{1+\gamma'}\frac{\bar{g}_k}{2}$$

for some $\gamma' \in (0, 1)$ depending only on γ. Therefore, it enables us to use Lemma 2.5 with $\delta := \delta_{n,m}(t)$. Recall that $h_{x,y,\delta}(X) = \big\langle S_k(\Gamma)x, y\big\rangle\phi\left(\frac{\|\Gamma\|}{\delta}\right)$ and let $M := \text{Med}\big(\big\langle S_k(\Gamma)x, y\big\rangle\big)$. Observe that, for $t \geq \log(4)$,

$$\begin{aligned}
&\mathbb{P}\{h_{x,y,\delta}(X) \geq M\} \geq \mathbb{P}\{h_{x,y,\delta}(X) \geq M, \|\Gamma\| \leq \delta_{n,m}(t)\} \\
&\geq \mathbb{P}\{\big\langle S_k(\Gamma)x, y\big\rangle \geq M\big\} - \mathbb{P}\{\|\Gamma\| > \delta_{n,m}(t)\} \geq \frac{1}{2} - e^{-t} \geq \frac{1}{4}
\end{aligned}$$

and, similarly. $\mathbb{P}(h_{x,y,\delta}(X) \leq M) \geq \frac{1}{4}$. Therefore, by applying Lemmas 2.5 and 2.6, we conclude that with probability at least $1 - e^{-t}$,

$$\big|h_{x,y,\delta}(X) - M\big| \lesssim_\gamma \frac{\delta_{n,m}(t)\tau\sqrt{t}}{\bar{g}_k^2}\|x\|\|y\|$$

Since, by the first bound of Lemma 2.1, $\delta_{n,m}(t) \lesssim \tau(\sqrt{m \vee n} + \sqrt{t})$, we get that with the same probability

$$\left|h_{x,y,\delta}(X) - M\right| \lesssim_\gamma \frac{\tau\sqrt{t}}{\bar{g}_k} \frac{\tau\sqrt{m \vee n} + \tau\sqrt{t}}{\bar{g}_k} \|x\| \|y\|.$$

Moreover, on the event $\{\|\Gamma\| \leq \delta_{n,m}(t)\}$ that holds with probability at least $1 - e^{-t}$, $h_{x,y,\delta}(X) = \langle S_k(\Gamma)x, y\rangle$. Therefore, the following inequality holds with probability at least $1 - 2e^{-t}$:

$$\left|\langle S_k(\Gamma)x, y\rangle - M\right| \lesssim_\gamma \frac{\tau\sqrt{t}}{\bar{g}_k} \frac{\tau\sqrt{m \vee n} + \tau\sqrt{t}}{\bar{g}_k} \|x\| \|y\|. \tag{2.3}$$

We still need to prove a similar inequality in the case $c_2\tau\sqrt{t} \geq \frac{\gamma}{2}\frac{\bar{g}_k}{2}$. In this case,

$$\mathbb{E}\|\Gamma\| \leq (1-\gamma)\frac{\bar{g}_k}{2} \leq \frac{2c_2(1-\gamma)}{\gamma}\tau\sqrt{t},$$

implying that $\delta_{n,m}(t) \lesssim_\gamma \tau\sqrt{t}$. It follows from Lemma 2.3 that

$$\left|\langle S_k(\Gamma)x, y\rangle\right| \leq \|S_k(\Gamma)\| \|x\| \|y\| \lesssim \frac{\|\Gamma\|^2}{\bar{g}_k^2} \|x\| \|y\|$$

This implies that with probability at least $1 - e^{-t}$,

$$\left|\langle S_k(\Gamma)x, y\rangle\right| \lesssim \frac{\delta_{n,m}^2(t)}{\bar{g}_k^2} \|x\| \|y\| \lesssim_\gamma \frac{\tau^2 t}{\bar{g}_k^2} \|x\| \|y\|.$$

Since $t \geq \log(4)$ and $e^{-t} \leq 1/4$, we can bound the median M of $\langle S_k(\Gamma)x, y\rangle$ as follows:

$$M \lesssim_\gamma \frac{\tau^2 t}{\bar{g}_k^2} \|x\| \|y\|,$$

which immediately implies that bound (2.3) holds under assumption $c_2\tau\sqrt{t} \geq \frac{\gamma}{2}\frac{\bar{g}_k}{2}$ as well. By integrating out the tails of exponential bound (2.3), we obtain that

$$\left|\mathbb{E}\langle S_k(\Gamma)x, y\rangle - M\right| \leq \mathbb{E}\left|\langle S_k(\Gamma)x, y\rangle - M\right| \lesssim_\gamma \frac{\tau^2\sqrt{m \vee n}}{\bar{g}_k^2} \|x\| \|y\|,$$

which allows us to replace the median by the mean in concentration inequality (2.3). To complete the proof, it remains to rewrite the probability bound $1 - 2e^{-t}$ as $1 - e^{-t}$ by adjusting the value of the constant D_γ. □

Recalling that $\tilde{P}_k - \mathbb{E}\tilde{P}_k = L_k(\Gamma) + R_k(\Gamma)$, it remains to study the concentration of $\langle L_k(\Gamma)x, y\rangle$.

Lemma 2.8 *For all* $x, y \in \mathbb{R}^{m+n}$ *and* $t > 0$,

$$\mathbb{P}\left(\left|\langle L_k(\Gamma)x, y\rangle\right| \geq 4\frac{\tau\|x\|\|y\|\sqrt{t}}{\bar{g}_k}\right) \leq e^{-t}.$$

Proof Recall that $L_k(\Gamma) = P_k\Gamma C_k + C_k\Gamma P_k$ implying that

$$\langle L_k(\Gamma)x, y\rangle = \langle \Gamma P_k x, C_k y\rangle + \langle \Gamma C_k x, P_k y\rangle.$$

If $x = \binom{x_1}{x_2}, y = \binom{y_1}{y_2}$, where $x_1, y_1 \in \mathbb{R}^m, x_2, y_2 \in \mathbb{R}^n$, then it is easy to check that

$$\langle \Gamma x, y\rangle = \langle X x_2, y_1\rangle + \langle X y_2, x_1\rangle.$$

Clearly, the random variable $\langle \Gamma x, y\rangle$ is normal with mean zero and variance

$$\mathbb{E}\langle \Gamma x, y\rangle^2 \leq 2\Big[\mathbb{E}\langle X x_2, y_1\rangle^2 + \mathbb{E}\langle X y_2, x_1\rangle^2\Big].$$

Since X is an $m \times n$ matrix with i.i.d. $\mathcal{N}(0, \tau^2)$ entries, we easily get that

$$\mathbb{E}\langle X x_2, y_1\rangle^2 = \mathbb{E}\langle X, y_1 \otimes x_2\rangle^2 = \tau^2\|y_1 \otimes x_2\|_2^2 = \tau^2\|x_2\|^2\|y_1\|^2$$

and, similarly,

$$\mathbb{E}\langle X y_2, x_1\rangle^2 = \tau^2\|x_1\|^2\|y_2\|^2.$$

Therefore,

$$\begin{aligned}\mathbb{E}\langle \Gamma x, y\rangle^2 \leq & 2\tau^2\Big[\|x_2\|^2\|y_1\|^2 + \|x_1\|^2\|y_2\|^2\Big] \\ \leq & 2\tau^2\Big[(\|x_1\|^2 + \|x_2\|^2)(\|y_1\|^2 + \|y_2\|^2)\Big] = 2\tau^2\|x\|^2\|y\|^2.\end{aligned}$$

As a consequence, the random variable $\langle L_k(\Gamma)x, y\rangle$ is also normal with mean zero and its variance is bounded from above as follows:

$$\begin{aligned}\mathbb{E}\langle L_k(\Gamma)x, y\rangle^2 \leq & 2\Big[\mathbb{E}\langle \Gamma P_k x, C_k y\rangle^2 + \mathbb{E}\langle \Gamma C_k x, P_k y\rangle^2\Big] \\ \leq & 4\tau^2\Big[\|P_k x\|^2\|C_k y\|^2 + \|C_k x\|^2\|P_k y\|^2\Big].\end{aligned}$$

Since $\|P_k\| \le 1$ and $\|C_k\| \le \frac{1}{\bar{g}_k}$, we get that

$$\mathbb{E}\langle L_k(\Gamma)x, y\rangle^2 \le \frac{8\tau^2}{\bar{g}_k^2}\|x\|^2\|y\|^2.$$

The bound of the lemma easily follows from standard tail bounds for normal random variables. □

The upper bound on $|\langle(\tilde{P}_k - \mathbb{E}\tilde{P}_k)x, y\rangle|$ claimed in Theorem 1.1 follows by combining Lemmas 2.7 and 2.8. □

Proof of Theorem 1.2 Note that, since $\tilde{P}_k - P_k = L_k(\Gamma) + S_k(\Gamma)$ and $\mathbb{E}L_k(\Gamma) = 0$, we have

$$\mathbb{E}\tilde{P}_k - P_k = \mathbb{E}S_k(\Gamma).$$

It follows from the bound on $\|S_k(\Gamma)\|$ of Lemma 2.3 that

$$\left\|\mathbb{E}\tilde{P}_k - P_k\right\| \le \mathbb{E}\|S_k(\Gamma)\| \le 14\frac{\mathbb{E}\|\Gamma\|^2}{\bar{g}_k^2} \tag{2.4}$$

and the bound of Lemma 2.2 implies that

$$\left\|\mathbb{E}\tilde{P}_k - P_k\right\| \lesssim \frac{\tau^2(m \vee n)}{\bar{g}_k^2},$$

which proves (1.5).

Let

$$\delta_{n,m} := \mathbb{E}\|\Gamma\| + c_2\tau\sqrt{\log(m+n)}.$$

It follows from Lemma 2.1 that, with a proper choice of constant $c_2 > 0$,

$$\mathbb{P}\left(\|\Gamma\| \ge \delta_{n,m}\right) \le \frac{1}{m+n}.$$

In the case when $c_2\tau\sqrt{\log(m+n)} > \frac{\gamma}{2}\frac{\bar{g}_k}{2}$, the proof of bound (1.6) is trivial. Indeed, in this case

$$\left\|\mathbb{E}\tilde{P}_k - P_k\right\| \le \mathbb{E}\|\tilde{P}_k\| + \|P_k\| \le 2 \lesssim_\gamma \frac{\tau^2\log(m+n)}{\bar{g}_k^2} \lesssim \frac{\nu_k\tau^2\sqrt{m\vee n}}{\bar{g}_k^2}.$$

Since $\left\|P_k(\mathbb{E}\tilde{P}_k - P_k)P_k\right\| \le \left\|\mathbb{E}\tilde{P}_k - P_k\right\|$, bound (1.6) of the theorem follows when $c_2\tau\sqrt{\log(m+n)} > \frac{\gamma}{2}\frac{\bar{g}_k}{2}$.

In the rest of the proof, it will be assumed that $c_2\tau\sqrt{\log(m+n)} \le \frac{\gamma}{2}\frac{\bar{g}_k}{2}$ which, together with the condition $\mathbb{E}\|\Gamma\| = \mathbb{E}\|X\| \le (1-\gamma)\frac{\bar{g}_k}{2}$, implies that $\delta_{n,m} \le (1-\gamma/2)\frac{\bar{g}_k}{2}$. On the other hand, $\delta_{n,m} \lesssim \tau\sqrt{m \vee n}$. The following decomposition of the bias $\mathbb{E}\tilde{P}_k - P_k$ is obvious:

$$\begin{aligned}\mathbb{E}\tilde{P}_k - P_k &= \mathbb{E}S_k(\Gamma) = \mathbb{E}P_kS_k(\Gamma)P_k \\ &+\mathbb{E}\left(P_k^{\perp}S_k(\Gamma)P_k + P_kS_k(\Gamma)P_k^{\perp} + P_k^{\perp}S_k(\Gamma)P_k^{\perp}\right)\mathbb{1}(\|\Gamma\| \le \delta_{n,m}) \qquad (2.5)\\ &+\mathbb{E}\left(P_k^{\perp}S_k(\Gamma)P_k + P_kS_k(\Gamma)P_k^{\perp} + P_k^{\perp}S_k(\Gamma)P_k^{\perp}\right)\mathbb{1}(\|\Gamma\| > \delta_{n,m})\end{aligned}$$

We start with bounding the part of the expectation in the right hand side of (2.5) that corresponds to the event $\{\|\Gamma\| \le \delta_{n,m}\}$ on which we also have $\|\Gamma\| < \frac{\bar{g}_k}{2}$. Under this assumption, the eigenvalues μ_k of B and $\sigma_j(\tilde{B}), j \in \Delta_k$ of $\tilde{B}$ are inside the circle γ_k in $\mathbb{C}$ with center μ_k and radius $\frac{\bar{g}_k}{2}$. The rest of the eigenvalues of $B, \tilde{B}$ are outside of γ_k. According to the Riesz formula for spectral projectors,

$$\tilde{P}_k = -\frac{1}{2\pi i}\oint_{\gamma_k} R_{\tilde{B}}(\eta)d\eta,$$

where $R_T(\eta) = (T-\eta I)^{-1}, \eta \in \mathbb{C}\setminus\sigma(T)$ denotes the resolvent of operator T ($\sigma(T)$ being its spectrum). It is also assumed that the contour γ_k has a counterclockwise orientation. Note that the resolvents will be viewed as operators from $\mathbb{C}^{m+n}$ into itself. The following power series expansion is standard:

$$\begin{aligned}R_{\tilde{B}}(\eta) =& R_{B+\Gamma}(\eta) = (B+\Gamma-\eta I)^{-1}\\ =&[(B-\eta I)(I+(B-\eta I)^{-1}\Gamma)]^{-1}\\ =&(I+R_B(\eta)\Gamma)^{-1}R_B(\eta) = \sum_{r\ge 0}(-1)^r[R_B(\eta)\Gamma]^rR_B(\eta),\end{aligned}$$

where the series in the last line converges because $\|R_B(\eta)\Gamma\| \le \|R_B(\eta)\|\|\Gamma\| < \frac{2}{\bar{g}_k}\frac{\bar{g}_k}{2} = 1$. The inequality $\|R_B(\eta)\| \le \frac{2}{\bar{g}_k}$ holds for all $\eta \in \gamma_k$. One can easily verify that

$$\begin{aligned}P_k =& -\frac{1}{2\pi i}\oint_{\gamma_k} R_B(\eta)d\eta,\\ L_k(\Gamma) =& \frac{1}{2\pi i}\oint_{\gamma_k} R_B(\eta)\Gamma R_B(\eta)d\eta,\\ S_k(\Gamma) =& -\frac{1}{2\pi i}\oint_{\gamma_k}\sum_{r\ge 2}(-1)^r[R_B(\eta)\Gamma]^rR_B(\eta)d\eta.\end{aligned}$$

The following spectral representation of the resolvent will be used

$$R_B(\eta) = \sum_s \frac{1}{\mu_s - \eta} P_s,$$

where the sum in the right hand side includes $s = 0$ in the case when $\mu_0 = 0$ is an eigenvalue of B (equivalently, in the case when $2\sum_{k=1}^d \nu_k < m + n$). Define

$$\tilde{R}_B(\eta) := R_B(\eta) - \frac{1}{\mu_k - \eta} P_k = \sum_{s \neq k} \frac{1}{\mu_s - \eta} P_s.$$

Then, for $r \geq 2$,

$$\begin{aligned} P_k^{\perp}[R_B(\eta)\Gamma]^r R_B(\eta) P_k = & \frac{1}{\mu_k - \eta} P_k^{\perp}[R_B(\eta)\Gamma]^r P_k \\ = & \frac{1}{(\mu_k - \eta)^2} \sum_{s=2}^{r} (\tilde{R}_B(\eta)\Gamma)^{s-1} P_k \Gamma (R_B(\eta)\Gamma)^{r-s} P_k + \frac{1}{\mu_k - \eta} (\tilde{R}_B(\eta)\Gamma)^r P_k. \end{aligned}$$

The above representation easily follows from the following simple observation: let $a := \frac{P_k}{\mu_k - \eta}\Gamma$ and $b := \tilde{R}_B(\eta)\Gamma$. Then

$$\begin{aligned} (a+b)^r = & a(a+b)^{r-1} + b(a+b)^{r-1} \\ = & a(a+b)^{r-1} + ba(a+b)^{r-2} + b^2(a+b)^{r-2} \\ = & a(a+b)^{r-1} + ba(a+b)^{r-2} + b^2 a(a+b)^{r-3} + b^3(a+b)^{r-3} \\ = & \ldots = \sum_{s=1}^{r} b^{s-1} a(a+b)^{r-s} + b^r. \end{aligned}$$

As a result,

$$\begin{aligned} P_k^{\perp} S_k(\Gamma) P_k = & -\sum_{r \geq 2} (-1)^r \frac{1}{2\pi i} \oint_{\gamma_k} \Bigg[\frac{1}{(\mu_k - \eta)^2} \sum_{s=2}^{r} (\tilde{R}_B(\eta)\Gamma)^{s-1} P_k \Gamma (R_B(\eta)\Gamma)^{r-s} P_k \\ & + \frac{1}{\mu_k - \eta} (\tilde{R}_B(\eta)\Gamma)^r P_k \Bigg] d\eta \end{aligned} \tag{2.6}$$

Let $P_k = \sum_{l\in\Delta_k} \theta_l \otimes \theta_l$, where $\{\theta_l, l \in \Delta_k\}$ are orthonormal eigenvectors corresponding to the eigenvalue μ_k. Therefore, for any $y \in \mathbb{R}^{m+n}$,

$$(\tilde{R}_B(\eta)\Gamma)^{s-1} P_k \Gamma (R_B(\eta)\Gamma)^{r-s} P_k y = \sum_{l\in\Delta_k} (\tilde{R}_B(\eta)\Gamma)^{s-1} \theta_l \otimes \theta_l \Gamma (R_B(\eta)\Gamma)^{r-s} P_k y$$

$$= \sum_{l\in\Delta_k} \langle \Gamma (R_B(\eta)\Gamma)^{r-s} P_k y, \theta_l\rangle \, (\tilde{R}_B(\eta)\Gamma)^{s-2} \tilde{R}_B(\eta)\Gamma\theta_l \tag{2.7}$$

Since $|\langle \Gamma (R_B(\eta)\Gamma)^{r-s} P_k y, \theta_l\rangle| \le \|\Gamma\|^{r-s+1} \|R_B(\eta)\|^{r-s} \|y\|$, we get

$$\mathbb{E} |\langle \Gamma (R_B(\eta)\Gamma)^{r-s} P_k y, \theta_l\rangle|^2 \mathbb{1}(\|\Gamma\| \le \delta_{n,m}) \le \delta_{n,m}^{2(r-s+1)} \left(\frac{2}{\bar{g}_k}\right)^{2(r-s)} \|y\|^2.$$

Also, for any $x \in \mathbb{R}^{m+n}$, we have to bound

$$\mathbb{E} \left|\left\langle (\tilde{R}_B(\eta)\Gamma)^{s-2} \tilde{R}_B(\eta)\Gamma\theta_l, x\right\rangle\right|^2 \mathbb{1}(\|\Gamma\| \le \delta_{n,m}). \tag{2.8}$$

In what follows, we need some additional notations. Let $X_1^c, \dots, X_n^c \sim \mathcal{N}(0, \tau^2 I_m)$ be the i.i.d. columns of X and $(X_1^r)', \dots, (X_n^r)' \sim \mathcal{N}(0, \tau^2 I_n)$ be its i.i.d. rows (here I_m and I_n are $m \times m$ and $n \times n$ identity matrices). For $j = 1, \dots, n$, define the vector $\check{X}_j^c = ((X_j^c)', 0)' \in \mathbb{R}^{m+n}$, representing the $(m + j)$-th column of matrix Γ. Similarly, for $i = 1, \dots, m$, $\check{X}_i^r = (0, (X_i^r)')' \in \mathbb{R}^{m+n}$ represents the i-th row of Γ. With these notations, the following representations of Γ holds

$$\Gamma = \sum_{j=1}^n e_{m+j}^{m+n} \otimes \check{X}_j^c + \sum_{j=1}^n \check{X}_j^c \otimes e_{m+j}^{m+n},$$

$$\Gamma = \sum_{i=1}^m \check{X}_i^r \otimes e_i^{m+n} + \sum_{i=1}^m e_i^{m+n} \otimes \check{X}_i^r,$$

and, moreover,

$$\sum_{j=1}^n e_{m+j}^{m+n} \otimes \check{X}_j^c = \sum_{i=1}^m \check{X}_i^r \otimes e_i^{m+n}, \quad \sum_{j=1}^n \check{X}_j^c \otimes e_{m+j}^{m+n} = \sum_{i=1}^m e_i^{m+n} \otimes \check{X}_i^r.$$

Therefore,

$$\Big\langle (\tilde R_B(\eta)\Gamma)^{s-2}\tilde R_B(\eta)\Gamma\theta_l, x\Big\rangle = \sum_{j=1}^n \Big\langle \check X_j^c, \theta_l\Big\rangle \Big\langle (\tilde R_B(\eta)\Gamma)^{s-2}\tilde R_B(\eta) e_{m+j}^{m+n}, x\Big\rangle$$
$$+\sum_{j=1}^n \Big\langle e_{m+j}^{m+n}, \theta_l\Big\rangle \Big\langle (\tilde R_B(\eta)\Gamma)^{s-2}\tilde R_B(\eta)\check X_j^c, x\Big\rangle =: I_1(x) + I_2(x),$$

and we get

$$\begin{aligned}\mathbb{E}\left|\Big\langle (\tilde R_B(\eta)\Gamma)^{s-2}\tilde R_B(\eta)\Gamma\theta_l, x\Big\rangle\right|^2 \mathbb{1}(\|\Gamma\|\le \delta_{n,m}) \\ \le 2\mathbb{E}(|I_1(x)|^2 + |I_2(x)|^2)\mathbb{1}(\|\Gamma\|\le\delta_{n,m}).\end{aligned} \tag{2.9}$$

Observe that the random variable $(\tilde R_B(\eta)\Gamma)^{s-2}\tilde R_B(\eta)$ is a function of $\{P_t\check X_j^c, t\neq k, j = 1,\dots,n\}$. Indeed, since $\tilde R_B(\eta)$ is a linear combination of operators $P_t, t\neq k$, it is easy to see that $(\tilde R_B(\eta)\Gamma)^{s-2}\tilde R_B(\eta)$ can be represented as a linear combination of operators

$$(P_{t_1}\Gamma P_{t_2})(P_{t_2}\Gamma P_{t_3})\dots(P_{t_{s-2}}\Gamma P_{t_{s-1}})$$

with $t_j\neq k$ and with non-random complex coefficients. On the other hand,

$$P_{t_k}\Gamma P_{t_{k+1}} = \sum_{j=1}^n P_{t_k} e_{m+j}^{m+n}\otimes P_{t_{k+1}}\check X_j^c + \sum_{j=1}^n P_{t_k}\check X_j^c\otimes P_{t_{k+1}} e_{m+j}^{m+n}.$$

These two facts imply that $(\tilde R_B(\eta)\Gamma)^{s-2}\tilde R_B(\eta)$ is a function of $\{P_t\check X_j^c, t\neq k, j = 1,\dots,n\}$. Similarly, it is also a function of $\{P_t\check X_i^r, t\neq k, i = 1,\dots,m\}$.

It is easy to see that random variables $\{P_k\check X_j^c, j = 1,\dots,n\}$ and $\{P_t\check X_j^c, j = 1,\dots,n, t\neq k\}$ are independent. Since they are mean zero normal random variables and $\check X_j^c, j = 1,\dots,n$ are independent, it is enough to check that, for all $j = 1,\dots,n,\ t\neq k,\ P_k\check X_j^c$ and $P_t\check X_j^c$ are uncorrelated. To this end, observe that

$$\begin{aligned}\mathbb{E}(P_k\check X_j^c\otimes P_t\check X_j^c) &= P_k\mathbb{E}(\check X_j^c\otimes\check X_j^c)P_t \\ &= \frac14\begin{pmatrix}P_k^{uu} & P_k^{uv}\\ P_k^{vu} & P_k^{vv}\end{pmatrix}\begin{pmatrix}I_m & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}P_t^{uu} & P_t^{uv}\\ P_t^{vu} & P_t^{vv}\end{pmatrix} \\ &= \frac14\begin{pmatrix}P_k^{uu}P_t^{uu} & P_k^{uu}P_t^{uv}\\ P_k^{vu}P_t^{uu} & P_k^{vu}P_t^{uv}\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix},\end{aligned}$$

where we used orthogonality relationships (1.2). Quite similarly, one can prove independence of $\{P_k\check X_i^r, i = 1,\dots,m\}$ and $\{P_t\check X_i^r, i = 1,\dots,m, t\neq k\}$.

We will now provide an upper bound on $\mathbb{E}|I_1(x)|^2\mathbb{1}(\|\Gamma\| \le \delta_{n,m})$. To this end, define

$$\omega_j(x) = \left\langle (\tilde{R}_B(\eta)\Gamma)^{s-2}\tilde{R}_B(\eta)e_{m+j}^{m+n}, x \right\rangle, \quad j = 1, \dots, n$$
$$= \omega_j^{(1)}(x) + i\omega_j^{(2)}(x) \in \mathbb{C}.$$

Let $I_1(x) = \kappa^{(1)}(x) + i\kappa^{(2)}(x) \in \mathbb{C}$. Then, conditionally on $\{P_t\check{X}_j^c : t \neq k, j = 1, \dots, n\}$, the random vector $(\kappa^{(1)}(x), \kappa^{(2)}(x))$ has the same distribution as mean zero Gaussian random vector in $\mathbb{R}^2$ with covariance,

$$\left(\sum_{j=1}^{n} \frac{\tau^2}{2}\omega_j^{k_1}(x)\omega_j^{k_2}(x)\right), k_1, k_2 = 1, 2$$

(to check the last claim, it is enough to compute conditional covariance of $(\kappa^{(1)}(x), \kappa^{(2)}(x))$ given $\{P_t\check{X}_j^c : t \neq k, j = 1, \dots, n\}$ using the fact that $(\tilde{R}_B(\eta)\Gamma)^{s-2}\tilde{R}_B(\eta)$ is a function of $\{P_t\check{X}_j^c, t \neq k, j = 1, \dots, n\}$). Therefore,

$$\mathbb{E}\left(|I_1(x)|^2 \middle| P_t\check{X}_j^c : t \neq k, j = 1, \dots, n\right)$$
$$= \mathbb{E}\left((\kappa^{(1)}(x))^2 + (\kappa^{(2)}(x))^2 \middle| P_t\check{X}_j^c : t \neq k, j = 1, \dots, n\right)$$
$$= \frac{\tau^2}{2}\sum_{j=1}^{n}\left((\omega_j^{(1)}(x))^2 + (\omega_j^{(2)}(x))^2\right) = \frac{\tau^2}{2}\sum_{j=1}^{n}|\omega_j(x)|^2.$$

Furthermore,

$$\sum_{j=1}^{n}\tau^2|\omega_j(x)|^2 = \tau^2\sum_{j=1}^{n}|\omega_j(x)|^2$$
$$= \tau^2\sum_{j=1}^{n}\left|\left\langle \tilde{R}_B(\eta)(\Gamma\tilde{R}_B(\eta))^{s-2}x, e_{m+j}^{m+n}\right\rangle\right|^2$$
$$= \tau^2\left\langle \tilde{R}_B(\eta)(\Gamma\tilde{R}_B(\eta))^{s-2}x, \tilde{R}_B(\eta)(\Gamma\tilde{R}_B(\eta)\Gamma)^{s-2}x\right\rangle$$
$$\le \tau^2\|\tilde{R}_B(\eta)\|^{2(s-1)}\|\Gamma\|^{2(s-2)}\|x\|^2.$$

Under the assumption $\delta_{n,m} < \frac{\bar{g}_k}{2}$, the following inclusion holds:

$$\{\|\Gamma\| \le \delta_{n,m}\} \subset \left\{\sum_{j=1}^{n}\tau^2|\omega_j(x)|^2 \le \tau^2\left(\frac{2}{\bar{g}_k}\right)^{2(s-1)}\delta_{n,m}^{2(s-2)}\|x\|^2\right\} =: G$$

Therefore,

$$\mathbb{E}|I_1(x)|^2\mathbb{1}(\|\Gamma\| \le \delta_{n,m}) \le \mathbb{E}|I_1(x)|^2\mathbb{1}_G = \mathbb{E}\mathbb{E}\biggl(|I_1(x)|^2\biggl|P_t\check{X}_j^c, t \ne k, j = 1, \dots, n\biggr)\mathbb{1}_G$$
$$=\mathbb{E}\mathbb{E}\biggl(\sum_{j=1}^n \tau^2|\omega_j(x)|^2\biggl|P_t\check{X}_j^c, t \ne k, j = 1, \dots, n\biggr)\mathbb{1}_G \le \tau^2\left(\frac{2}{\bar{g}_k}\right)^{2(s-1)}\delta_{n,m}^{2(s-2)}\|x\|^2. \tag{2.10}$$

A similar bound holds also for $\mathbb{E}|I_2(x)|^2\mathbb{1}(\|\Gamma\| \le \delta_{n,m})$:

$$\mathbb{E}|I_2(x)|^2\mathbb{1}(\|\Gamma\| \le \delta_{n,m}) \le \tau^2\left(\frac{2}{\bar{g}_k}\right)^{2(s-1)}\delta_{n,m}^{2(s-2)}\|x\|^2. \tag{2.11}$$

For the proof, it is enough to observe that

$$I_2(x) = \sum_{j=1}^n \left\langle e_{m+j}^{m+n}, \theta_l\right\rangle \left\langle (\tilde{R}_B(\eta)\Gamma)^{s-2}\tilde{R}_B(\eta)\check{X}_j^c, x\right\rangle$$
$$= \left\langle (\tilde{R}_B(\eta)\Gamma)^{s-2}\tilde{R}_B(\eta)\left(\textstyle\sum_{j=1}^n \check{X}_j^c \otimes e_{m+j}^{m+n}\right)\theta_l, x\right\rangle$$
$$= \left\langle (\tilde{R}_B(\eta)\Gamma)^{s-2}\tilde{R}_B(\eta)\left(\textstyle\sum_{i=1}^m e_i^{m+n} \otimes \check{X}_i^r\right)\theta_l, x\right\rangle$$
$$= \sum_{i=1}^m \left\langle \check{X}_i^r, \theta_l\right\rangle \left\langle (\tilde{R}_B(\eta)\Gamma)^{s-2}\tilde{R}_B(\eta)e_i^{m+n}, x\right\rangle$$

and to repeat the previous conditioning argument (this time, given $\{P_t\check{X}_i^r : t \ne k, i = 1, \dots, m\}$).

Combining bounds (2.10), (2.11) and (2.9), we get

$$\mathbb{E}\left|\left\langle (\tilde{R}_B(\eta)\Gamma)^{s-2}\tilde{R}_B(\eta)\Gamma\theta_l, x\right\rangle\right|^2\mathbb{1}(\|\Gamma\| \le \delta_{n,m}) \le 2\tau^2\left(\frac{2}{\bar{g}_k}\right)^{2(s-1)}\delta_{n,m}^{2(s-2)}\|x\|^2.$$

Then, it follows that

$$\left|\mathbb{E}\left\langle \Gamma(R_B(\eta)\Gamma)^{r-s}P_k y, \theta_l\right\rangle \left\langle (\tilde{R}_B(\eta)\Gamma)^{s-2}\tilde{R}_B(\eta)\Gamma\theta_l, x\right\rangle \mathbb{1}(\|\Gamma\| \le \delta_{n,m})\right|$$
$$\le \left(\mathbb{E}\left|\left\langle \Gamma(R_B(\eta)\Gamma)^{r-s}P_k y, \theta_l\right\rangle\right|^2\mathbb{1}(\|\Gamma\| \le \delta_{n,m})\right)^{1/2}$$
$$\times \left(\mathbb{E}\left|\left\langle (\tilde{R}_B(\eta)\Gamma)^{s-2}\tilde{R}_B(\eta)\Gamma\theta_l, x\right\rangle\right|^2\mathbb{1}(\|\Gamma\| \le \delta_{n,m})\right)^{1/2}$$
$$\le \sqrt{2}\tau\left(\frac{2\delta_{n,m}}{\bar{g}_k}\right)^{r-1}\|x\|\|y\|,$$

which, taking into account (2.7), implies that

$$\left|\mathbb{E}\left\langle(\tilde{R}_B(\eta)\Gamma)^{s-1}P_k\Gamma(R_B(\eta)\Gamma)^{r-s}P_ky, x\right\rangle \mathbb{1}(\|\Gamma\| \le \delta_{n,m})\right| \le \sqrt{2}\nu_k\tau\left(\frac{2\delta_{n,m}}{\bar{g}_k}\right)^{r-1}\|x\|\|y\|$$

Since $(\tilde{R}_B(\eta)\Gamma)^rP_k = (\tilde{R}_B(\eta)\Gamma)^{r-1}\tilde{R}_B(\eta)\Gamma P_k$, it can be proved by a similar argument that

$$\left|\mathbb{E}\left\langle(\tilde{R}_B(\eta)\Gamma)^rP_ky, x\right\rangle \mathbb{1}(\|\Gamma\| \le \delta_{n,m})\right| \le \sqrt{2}\nu_k\tau\frac{2}{\bar{g}_k}\left(\frac{2\delta_{n,m}}{\bar{g}_k}\right)^{r-1}\|x\|\|y\|.$$

Therefore, substituting the above bounds in (2.6) and taking into account that $|\mu_k - \eta| = \frac{\bar{g}_k}{2}$, $\eta \in \gamma_k$ and that the length of the contour of integration γ_k is equal to $2\pi\frac{\bar{g}_k}{2}$, we get

$$\left|\mathbb{E}\left\langle P_k^{\perp}S_k(\Gamma)P_ky, x\right\rangle \mathbb{1}(\|\Gamma\| \le \delta_{n,m})\right| \le \sum_{r\ge 2}\frac{r\bar{g}_k}{2}\left(\frac{2}{\bar{g}_k}\right)^2\sqrt{2}\nu_k\tau\left(\frac{2\delta_{n,m}}{\bar{g}_k}\right)^{r-1}\|x\|\|y\|$$

$$=\frac{2}{\bar{g}_k}\sqrt{2}\nu_k\tau\sum_{r\ge 2}r\left(\frac{2\delta_{n,m}}{\bar{g}_k}\right)^{r-1}\|x\|\|y\| \lesssim_\gamma \nu_k\tau\frac{\delta_{n,m}}{\bar{g}_k^2}\|x\|\|y\|,$$

where we also used the condition $\delta_{n,m} \le (1-\gamma/2)\frac{\bar{g}_k}{2}$ implying that $\frac{2\delta_{n,m}}{\bar{g}_k} \le 1-\gamma/2$. Clearly, this implies that

$$\left\|\mathbb{E}P_k^{\perp}S_k(\Gamma)P_k\right\|\mathbb{1}(\|\Gamma\| \le \delta_{n,m}) \lesssim_\gamma \nu_k\tau\frac{\delta_{n,m}}{\bar{g}_k^2} \lesssim_\gamma \frac{\nu_k\tau\sqrt{m\vee n}}{\bar{g}_k^2}.$$

Furthermore, the same bound, obviously, holds for

$$\left\|\mathbb{E}\langle P_kS_k(\Gamma)P_k^{\perp}y, x\rangle\mathbb{1}(\|\Gamma\| \le \delta_{n,m})\right\| = \left\|\mathbb{E}\langle P_k^{\perp}S_k(\Gamma)P_kx, y\rangle\mathbb{1}(\|\Gamma\| \le \delta_{n,m})\right\|$$

and, by similar arguments, it can be demonstrated that it also holds for

$$\left\|\mathbb{E}P_k^{\perp}S_k(\Gamma)P_k^{\perp}\right\|\mathbb{1}(\|\Gamma\| \le \delta_{n,m})$$

(the only different term in this case is $(\tilde{R}_B(\eta)\Gamma)^r\tilde{R}_B(\eta)$, but, since $\{\mu_t, t \ne k\}$ are outside of the circle γ_k, it simply leads to $\oint_{\gamma_k}(\tilde{R}_B(\eta)\Gamma)^r\tilde{R}_B(\eta)d\eta = 0$).

It remains to observe that

$$
\begin{aligned}
&\Big\|\mathbb{E}\left(P_k^{\perp}S_k(\Gamma)P_k + P_kS_k(\Gamma)P_k^{\perp} + P_k^{\perp}S_k(\Gamma)P_k^{\perp}\right)\mathbb{1}(\|\Gamma\| > \delta_{n,m})\Big\| \\
\leq&\mathbb{E}\Big\|P_k^{\perp}S_k(\Gamma)P_k + P_kS_k(\Gamma)P_k^{\perp} + P_k^{\perp}S_k(\Gamma)P_k^{\perp}\Big\|\mathbb{1}(\|\Gamma\| > \delta_{n,m}) \\
\leq&\mathbb{E}\|S_k(\Gamma)\|\mathbb{1}(\|\Gamma\| > \delta_{n,m}) \\
\leq&(\mathbb{E}\|S_k(\Gamma)\|^2)^{1/2}\mathbb{P}^{1/2}(\|\Gamma\| > \delta_{n,m}) \\
\lesssim&\mathbb{E}^{1/2}\left(\frac{\|\Gamma\|}{\bar{g}_k}\right)^4\mathbb{P}^{1/2}(\|\Gamma\| > \delta_{n,m}) \lesssim \frac{1}{\sqrt{m\vee n}}\frac{\tau^2(m\vee n)}{\bar{g}_k^2} \lesssim \frac{\tau^2\sqrt{m\vee n}}{\bar{g}_k^2}
\end{aligned}
$$

and to substitute the above bounds to identity (2.5) to get that

$$
\Big\|\mathbb{E}\tilde{P}_k - P_k - P_k\mathbb{E}S_k(\Gamma)P_k\Big\| \lesssim_\gamma \frac{\nu_k\tau^2\sqrt{m\vee n}}{\bar{g}_k^2},
$$

which implies the claim of the theorem. □

Proof of Theorem 1.3 By a simple computation (see Lemma 8 and the derivation of (6.6) in [6]), the following identity holds

$$
\begin{aligned}
\left\langle\tilde{\theta}_{i_k} - \sqrt{1+b_k}\theta_{i_k}, x\right\rangle =& \frac{\rho_k(x)}{\sqrt{1+b_k+\rho_k(x)}} \\
&-\frac{\sqrt{1+b_k}}{\sqrt{1+b_k+\rho_k(x)}\left(\sqrt{1+b_k+\rho_k(x)}+\sqrt{1+b_k}\right)}\rho_k(\theta_{i_k})\,\langle\theta_{i_k}, x\rangle\,,
\end{aligned}
\tag{2.12}
$$

where $\rho_k(x) := \left\langle(\tilde{P}_k - (1+b_k)P_k)\theta_{i_k}, x\right\rangle, x \in \mathbb{R}^{m+n}$. In what follows, assume that $\|x\| = 1$. By the bounds of Theorems 1.1 and 1.2, with probability at least $1 - e^{-t}$:

$$
|\rho_k(x)| \leq D_\gamma\frac{\tau\sqrt{t}}{\bar{g}_k}\Big(\frac{\tau\sqrt{m\vee n}+\tau\sqrt{t}}{\bar{g}_k}+1\Big).
$$

The assumption $\mathbb{E}\|X\| \leq (1-\gamma)\frac{\bar{g}_k}{2}$ implies that $\tau\sqrt{m\vee n} \lesssim \bar{g}_k$. Therefore, if t satisfies the assumption $\frac{\tau\sqrt{t}}{\bar{g}_k} \leq c_\gamma$ for a sufficiently small constant $c_\gamma > 0$, then we have $|\rho_k(x)| \leq \gamma/2$. By the assumption that $1 + b_k \geq \gamma$, this implies that $1 + b_k + \rho_k(x) \geq \gamma/2$. Thus, it easily follows from identity (2.12) that with probability at least $1 - 2e^{-t}$

$$
\left|\left\langle\tilde{\theta}_{i_k} - \sqrt{1+b_k}\theta_{i_k}, x\right\rangle\right| \lesssim_\gamma \frac{\tau\sqrt{t}}{\bar{g}_k}\Big(\frac{\tau\sqrt{m\vee n}+\tau\sqrt{t}}{\bar{g}_k}+1\Big).
$$

It remains to show that the same bound holds when $\frac{\tau\sqrt{t}}{\bar{g}_k} > c_\gamma$. In this case, we simply have that

$$\left|\langle \tilde{\theta}_{i_k} - \sqrt{1+b_k}\theta_{i_k}, x\rangle\right| \le \|\tilde{\theta}_{i_k}\| + (1+b_k)\|\theta_{i_k}\| \le 2 \lesssim_\gamma \frac{\tau^2 t}{\bar{g}_k^2},$$

which implies the bound of the theorem. □

Proof of Corollary 1.5 By a simple algebra,

$$\begin{aligned}|\tilde{b}_k - b_k| =& \left|\langle \tilde{\theta}^1_{i_k}, \tilde{\theta}^2_{i_k}\rangle - (1+b_k)\right| \le \left|\sqrt{1+b_k}\langle \tilde{\theta}^1_{i_k} - \sqrt{1+b_k}\theta_{i_k}, \theta_{i_k}\rangle\right| \\ &+\left|\sqrt{1+b_k}\langle \tilde{\theta}^2_{i_k} - \sqrt{1+b_k}\theta_{i_k}, \theta_{i_k}\rangle\right| + \left|\langle \tilde{\theta}^1_{i_k} - \sqrt{1+b_k}\theta_{i_k}, \tilde{\theta}^2_{i_k} - \sqrt{1+b_k}\theta_{i_k}\rangle\right|.\end{aligned}$$

Corollary 1.5 implies that with probability at least $1 - e^{-t}$

$$\left|\sqrt{1+b_k}\langle \tilde{\theta}^1_{i_k} - \sqrt{1+b_k}\theta_{i_k}, \theta_{i_k}\rangle\right| \lesssim_\gamma \frac{\tau\sqrt{t}}{\bar{g}_k}\Big[\frac{\tau\sqrt{m\vee n} + \tau\sqrt{t}}{\bar{g}_k} + 1\Big],$$

where we also used the fact that $1 + b_k \in [0, 1]$. A similar bound holds with the same probability for

$$\left|\sqrt{1+b_k}\langle \tilde{\theta}^2_{i_k} - \sqrt{1+b_k}\theta_{i_k}, \theta_{i_k}\rangle\right|.$$

To control the remaining term

$$\left|\langle \tilde{\theta}^1_{i_k} - \sqrt{1+b_k}\theta_{i_k}, \tilde{\theta}^2_{i_k} - \sqrt{1+b_k}\theta_{i_k}\rangle\right|,$$

note that $\tilde{\theta}^1_{i_k}$ and $\tilde{\theta}^2_{i_k}$ are independent. Thus, applying the bound of Theorem 1.3 conditionally on $\tilde{\theta}^2_{i_k}$, we get that with probability at least $1 - e^{-t}$

$$\left|\langle \tilde{\theta}^1_{i_k} - \sqrt{1+b_k}\theta_{i_k}, \tilde{\theta}^2_{i_k} - \sqrt{1+b_k}\theta_{i_k}\rangle\right| \lesssim_\gamma \frac{\tau\sqrt{t}}{\bar{g}_k}\Big[\frac{\tau\sqrt{m\vee n} + \tau\sqrt{t}}{\bar{g}_k} + 1\Big]\|\tilde{\theta}^2_{i_k} - \sqrt{1+b_k}\theta_{i_k}\|.$$

It remains to observe that

$$\|\tilde{\theta}^2_{i_k} - \sqrt{1+b_k}\theta_{i_k}\| \le 2$$

to complete the proof of bound (1.9).

Assume that $\|x\| \le 1$. Recall that under the assumptions of the corollary, $\tau\sqrt{m\vee n} \lesssim_\gamma \bar{g}_k$ and, if $\frac{\tau\sqrt{t}}{\bar{g}_k} \le c_\gamma$ for a sufficiently small constant $c_{\gamma,}$ then bound (1.9) implies that $|\tilde{b}_k - b_k| \le \gamma/4$ (on the event of probability at least $1 - e^{-t}$).

Since $1 + b_k \geq \gamma/2$, on the same event we also have $1 + \tilde{b}_k \geq \gamma/4$ implying that $\hat{\theta}_{i_k}^{(\gamma)} = \frac{\tilde{\theta}_{i_k}^1}{\sqrt{1+\tilde{b}_k}}$. Therefore,

$$\left|\left\langle \hat{\theta}_{i_k}^{(\gamma)} - \theta_{i_k}, x\right\rangle\right| = \frac{1}{\sqrt{1+\tilde{b}_k}}\left|\left\langle \tilde{\theta}_{i_k}^1 - \sqrt{1+\tilde{b}_k}\theta_{i_k}, x\right\rangle\right| \tag{2.13}$$

$$\lesssim_\gamma \left|\left\langle \tilde{\theta}_{i_k}^1 - \sqrt{1+b_k}\theta_{i_k}, x\right\rangle\right| + \left|\sqrt{1+b_k} - \sqrt{1+\tilde{b}_k}\right|.$$

The first term in the right hand side can be bounded using Theorem 1.3 and, for the second term,

$$\left|\sqrt{1+b_k} - \sqrt{1+\tilde{b}_k}\right| = \frac{|\tilde{b}_k - b_k|}{\sqrt{1+b_k} + \sqrt{1+\tilde{b}_k}} \lesssim_\gamma |\tilde{b}_k - b_k|,$$

so bound (1.9) can be used. Substituting these bounds in (2.13), we derive (1.10) in the case when $\frac{\tau\sqrt{t}}{\bar{g}_k} \leq c_\gamma$.

In the opposite case, when $\frac{\tau\sqrt{t}}{\bar{g}_k} > c_\gamma$, we have

$$\left|\left\langle \hat{\theta}_{i_k}^{(\gamma)} - \theta_{i_k}, x\right\rangle\right| \leq \|\hat{\theta}_{i_k}^{(\gamma)}\| + \|\theta_{i_k}\| \leq \frac{1}{\sqrt{1+\tilde{b}_k} \vee \frac{\sqrt{\gamma}}{2}} + 1 \leq \frac{2}{\sqrt{\gamma}} + 1.$$

Therefore,

$$\left|\left\langle \hat{\theta}_{i_k}^{(\gamma)} - \theta_{i_k}, x\right\rangle\right| \lesssim_\gamma \frac{\tau\sqrt{t}}{\bar{g}_k},$$

which implies (1.10) in this case. □

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Optimal Kernel Selection for Density Estimation

Matthieu Lerasle, Nelo Molter Magalhães, and Patricia Reynaud-Bouret

Abstract We provide new general kernel selection rules thanks to penalized least-squares criteria. We derive optimal oracle inequalities using adequate concentration tools. We also investigate the problem of minimal penalty as described in Birgé and Massart (2007, Probab. Theory Relat. Fields, 138(1–2):33–73).

Keywords Density estimation • Kernel estimators • Minimal penalty • Optimal penalty • Oracle inequalities

1 Introduction

Concentration inequalities are central in the analysis of adaptive nonparametric statistics. They lead to sharp penalized criteria for model selection [20], to select bandwidths and even approximation kernels for Parzen's estimators in high dimension [17], to aggregate estimators [24] and to properly calibrate thresholds [9].

In the present work, we are interested in the selection of a general kernel estimator based on a least-squares density estimation approach. The problem has been considered in L^1-loss by Devroye and Lugosi [8]. Other methods combining log-likelihood and roughness/smoothness penalties have also been proposed in [10–12]. However these estimators are usually quite difficult to compute in practice. We propose here to minimize penalized least-squares criteria and obtain from them more easily computable estimators. Sharp concentration inequalities for U-statistics [1, 16, 18] control the variance term of the kernel estimators, whose asymptotic behavior has been precisely described, for instance in [7, 21, 22]. We derive from these bounds (see Proposition 4.1) a penalization method to select a kernel which

M. Lerasle • P. Reynaud-Bouret (✉)
Université Côte d'Azur, CNRS, LJAD, UMR 7351, 06100 Nice, France
e-mail: mlerasle@unice.fr; Patricia.REYNAUD-BOURET@unice.fr

N.M. Magalhães
INRIA, Select Project, Departement de Mathematiques d'Orsay, Université Paris-Sud 11, 91405 Orsay Cedex, France
e-mail: nelo.moltermagalhaes@gmail.com

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satisfies an asymptotically optimal oracle inequality, i.e. with leading constant asymptotically equal to 1.

In the spirit of [14], we use an extended definition of kernels that allows to deal simultaneously with classical collections of estimators as projection estimators, weighted projection estimators, or Parzen's estimators. This method can be used for example to select an optimal model in model selection (in accordance with [20]) or to select an optimal bandwidth together with an optimal approximation kernel among a finite collection of Parzen's estimators. In this sense, our method deals, in particular, with the same problem as that of Goldenshluger and Lepski [17] and we establish in this framework that a leading constant 1 in the oracle inequality is indeed possible.

Another main consequence of concentration inequalities is to prove the existence of a minimal level of penalty, under which no oracle inequalities can hold. Birgé and Massart shed light on this phenomenon in a Gaussian setting for model selection [5]. Moreover in this setting, they prove that the optimal penalty is twice the minimal one. In addition, there is a sharp phase transition in the dimension of the selected models leading to an estimate of the optimal penalty in their case (which is known up to a multiplicative constant). Indeed, starting from the idea that in many models the optimal penalty is twice the minimal one (this is the slope heuristic), Arlot and Massart [3] propose to detect the minimal penalty by the phase transition and to apply the "×2" rule (this is the slope algorithm). They prove that this algorithm works at least in some regression settings.

In the present work, we also show that minimal penalties exist in the density estimation setting. In particular, we exhibit a sharp "phase transition" of the behavior of the selected estimator around this minimal penalty. The analysis of this last result is not standard however. First, the "slope heuristic" of [5] only holds in particular cases as the selection of projection estimators, see also [19]. As in the selection of a linear estimator in a regression setting [2], the heuristic can sometimes be corrected: for example for the selection of a bandwidth when the approximation kernel is fixed. In general since there is no simple relation between the minimal penalty and the optimal one, the slope algorithm of [3] shall only be used with care for kernel selection. Surprisingly our work reveals that the minimal penalty can be negative. In this case, minimizing an unpenalized criterion leads to oracle estimators. To our knowledge, such phenomenon has only been noticed previously in a very particular classification setting [13]. We illustrate all of these different behaviors by means of a simulation study.

In Sect. 2, after fixing the main notation, providing some examples and defining the framework, we explain our goal, describe what we mean by an *oracle inequality* and state the exponential inequalities that we shall need. Then we derive optimal penalties in Sect. 3 and study the problem of minimal penalties in Sect. 4. All of these results are illustrated for our three main examples : projection kernels, approximation kernels and weighted projection kernels. In Sect. 5, some simulations are performed in the approximation kernel case. The main proofs are detailed in Sect. 6 and technical results are discussed in the appendix.

2 Kernel Selection for Least-Squares Density Estimation

2.1 *Setting*

Let $X, Y, X_1, \dots, X_n$ denote i.i.d. random variables taking values in the measurable space $(\mathbb{X}, \mathcal{X}, \mu)$, with common distribution P. Assume P has density s with respect to μ and s is uniformly bounded. Hence, s belongs to L^2, where, for any $p \geq 1$,

$$L^p := \left\{ t : \mathbb{X} \to \mathbb{R}, \text{ s.t. } \|t\|_p^p := \int |t|^p \, d\mu < \infty \right\} \quad .$$

Moreover, $\|\cdot\| = \|\cdot\|_2$ and $\langle \cdot, \cdot \rangle$ denote respectively the L^2-norm and the associated inner product and $\|\cdot\|_\infty$ is the supremum norm. We systematically use $x \vee y$ and $x \wedge y$ for $\max(x, y)$ and $\min(x, y)$ respectively, and denote $|A|$ the cardinality of the set A. Recall that $x_+ = x \vee 0$ and, for any $y \in \mathbb{R}^+$, $\lfloor y \rfloor = \sup\{n \in \mathbb{N} \text{ s.t. } n \leq y\}$.

Let $\{k\}_{k \in \mathcal{K}}$ denote a collection of symmetric functions $k : \mathbb{X}^2 \to \mathbb{R}$ indexed by some given finite set $\mathcal{K}$ such that

$$\sup_{x \in \mathbb{X}} \int_{\mathbb{X}} k(x, y)^2 d\mu(y) \ \vee \sup_{(x,y) \in \mathbb{X}^2} |k(x, y)| < \infty \quad .$$

A function k satisfying these assumptions is called a *kernel*, in the sequel. A kernel k is associated with an estimator $\hat{s}_k$ of s defined for any $x \in \mathbb{X}$ by

$$\hat{s}_k(x) := \frac{1}{n} \sum_{i=1}^{n} k(X_i, x) \quad .$$

Our aim is to select a "good" $\hat{s}_{\hat{k}}$ in the family $\{\hat{s}_k, k \in \mathcal{K}\}$. Our results are expressed in terms of a constant $\Gamma \geq 1$ such that for all $k \in \mathcal{K}$,

$$\sup_{x \in \mathbb{X}} \int_{\mathbb{X}} k(x, y)^2 d\mu(y) \ \vee \sup_{(x,y) \in \mathbb{X}^2} |k(x, y)| \leq \Gamma n \quad . \tag{2.1}$$

This condition plays the same role as $\int |k(x, y)| s(y) d\mu(y) < \infty$, the milder condition used in [8] when working with L^1-losses. Before describing the method, let us give three examples of such estimators that are used for density estimation, and see how they can naturally be associated to some kernels. Section A of the appendix gives the computations leading to the corresponding Γ's.

Example 1 (Projection Estimators) Projection estimators are among the most classical density estimators. Given a linear subspace $S \subset L^2$, the projection estimator on S is defined by

$$\hat{s}_S = \arg\min_{t \in S} \left\{ \|t\|^2 - \frac{2}{n} \sum_{i=1}^{n} t(X_i) \right\} \quad .$$

Let $\mathcal{S}$ be a family of linear subspaces S of L^2. For any $S \in \mathcal{S}$, let $(\varphi_\ell)_{\ell \in \mathcal{I}_S}$ denote an orthonormal basis of S. The projection estimator $\hat{s}_S$ can be computed and is equal to

$$\hat{s}_S = \sum_{\ell \in \mathcal{I}_S} \left(\frac{1}{n} \sum_{i=1}^{n} \varphi_\ell(X_i) \right) \varphi_\ell \ .$$

It is therefore easy to see that it is the estimator associated to the *projection kernel* k_S defined for any x and y in $\mathbb{X}$ by

$$k_S(x, y) := \sum_{\ell \in \mathcal{I}_S} \varphi_\ell(x)\varphi_\ell(y) \ .$$

Notice that k_S actually depends on the basis $(\varphi_\ell)_{\ell \in \mathcal{I}_S}$ even if $\hat{s}_S$ does not. In the sequel, we always assume that some orthonormal basis $(\varphi_\ell)_{\ell \in \mathcal{I}_S}$ is given with S. Given a finite collection $\mathcal{S}$ of linear subspaces of L^2, one can choose the following constant Γ in (2.1) for the collection $(k_S)_{S \in \mathcal{S}}$

$$\Gamma = 1 \vee \frac{1}{n} \sup_{S \in \mathcal{S}} \sup_{f \in S, \|f\|=1} \|f\|_\infty^2 \ . \tag{2.2}$$

Example 2 (Parzen's Estimators) Given a bounded symmetric integrable function $K : \mathbb{R} \to \mathbb{R}$ such that $\int_{\mathbb{R}} K(u)du = 1$ and a bandwidth $h > 0$, the Parzen estimator is defined by

$$\forall x \in \mathbb{R}, \quad \hat{s}_{K,h}(x) = \frac{1}{nh} \sum_{i=1}^{n} K\left(\frac{x - X_i}{h} \right) \ .$$

It can also naturally be seen as a kernel estimator, associated to the function $k_{K,h}$ defined for any x and y in $\mathbb{R}$ by

$$k_{K,h}(x, y) := \frac{1}{h} K\left(\frac{x - y}{h} \right) \ .$$

We shall call the function $k_{K,h}$ an approximation or Parzen kernel.

Given a finite collection of pairs $(K, h) \in \mathcal{H}$, one can choose $\Gamma = 1$ in (2.1) if,

$$h \geq \frac{\|K\|_\infty \|K\|_1}{n} \quad \text{for any } (K, h) \in \mathcal{H} \ . \tag{2.3}$$

Example 3 (Weighted Projection Estimators) Let $(\varphi_i)_{i=1,\dots,p}$ denote an orthonormal system in L^2 and let $w = (w_i)_{i=1,\dots,p}$ denote real numbers in $[0, 1]$. The associated weighted kernel projection estimator of s is defined by

$$\hat{s}_w = \sum_{i=1}^{p} w_i \left(\frac{1}{n} \sum_{j=1}^{n} \varphi_i(X_j) \right) \varphi_i \ .$$

These estimators are used to derive very sharp adaptive results. In particular, Pinsker's estimators are weighted kernel projection estimators (see for example [23]). When $w \in \{0, 1\}^p$, we recover a classical projection estimator. A weighted projection estimator is associated to the *weighted projection kernel* defined for any x and y in $\mathbb{X}$ by

$$k_w(x, y) := \sum_{i=1}^{p} w_i \varphi_i(x) \varphi_i(y) \ .$$

Given any finite collection $\mathcal{W}$ of weights, one can choose in (2.1)

$$\Gamma = 1 \vee \left(\frac{1}{n} \sup_{x \in \mathbb{X}} \sum_{i=1}^{p} \varphi_i(x)^2 \right) \ . \tag{2.4}$$

2.2 Oracle Inequalities and Penalized Criterion

The goal is to estimate s in the best possible way using a finite collection of kernel estimators $(\hat{s}_k)_{k \in \mathcal{K}}$. In other words, the purpose is to select among $(\hat{s}_k)_{k \in \mathcal{K}}$ an estimator $\hat{s}_{\hat{k}}$ from the data such that $\left\| \hat{s}_{\hat{k}} - s \right\|^2$ is as close as possible to $\inf_{k \in \mathcal{K}} \| \hat{s}_k - s \|^2$. More precisely our aim is to select $\hat{k}$ such that, with high probability,

$$\left\| \hat{s}_{\hat{k}} - s \right\|^2 \le C_n \inf_{k \in \mathcal{K}} \| \hat{s}_k - s \|^2 + R_n \ , \tag{2.5}$$

where $C_n \ge 1$ is the leading constant and $R_n > 0$ is usually a remaining term. In this case, $\hat{s}_{\hat{k}}$ is said to satisfy an *oracle inequality*, as long as R_n is small compared to $\inf_{k \in \mathcal{K}} \| \hat{s}_k - s \|^2$ and C_n is a bounded sequence. This means that the selected estimator does as well as the best estimator in the family up to some multiplicative constant. The best case one can expect is to get C_n close to 1. This is why, when $C_n \to_{n \to \infty} 1$, the corresponding oracle inequality is called *asymptotically optimal*. To do so, we study minimizers of *penalized least-squares criteria*. Note that in our three examples choosing $\hat{k} \in \mathcal{K}$ amounts to choosing the smoothing parameter, that is respectively to choosing $\widehat{S} \in \mathcal{S}$, $(\widehat{K}, \hat{h}) \in \mathcal{H}$ or $\hat{w} \in \mathcal{W}$.

Let P_n denote the empirical measure, that is, for any real valued function t,

$$P_n(t) := \frac{1}{n} \sum_{i=1}^{n} t(X_i) \ .$$

For any $t \in L^2$, let also $P(t) := \int_{\mathbb{X}} t(x) s(x) d\mu(x)$.

The *least-squares contrast* is defined, for any $t \in L^2$, by

$$\gamma(t) := \|t\|^2 - 2t \ .$$

Then for any given function pen : $\mathcal{K} \to \mathbb{R}$, the *least-squares penalized criterion* is defined by

$$\mathcal{C}_{\text{pen}}(k) := P_n\gamma(\hat{s}_k) + \text{pen}(k) \ . \tag{2.6}$$

Finally the selected $\hat{k} \in \mathcal{K}$ is given by any minimizer of $\mathcal{C}_{\text{pen}}(k)$, that is,

$$\hat{k} \in \arg\min_{k\in\mathcal{K}} \left\{ \mathcal{C}_{\text{pen}}(k) \right\} \ . \tag{2.7}$$

As $P\gamma(t) = \|t - s\|^2 - \|s\|^2$, it is equivalent to minimize $\|\hat{s}_k - s\|^2$ or $P\gamma(\hat{s}_k)$. As our goal is to select $\hat{s}_{\hat{k}}$ satisfying an oracle inequality, an ideal penalty pen_{id} should satisfy $\mathcal{C}_{\text{pen}_{\text{id}}}(k) = P\gamma(\hat{s}_k)$, i.e. criterion (2.6) with

$$\text{pen}_{\text{id}}(k) := (P - P_n)\gamma(\hat{s}_k) = 2(P_n - P)(\hat{s}_k) \ .$$

To identify the main quantities of interest, let us introduce some notation and develop $\text{pen}_{\text{id}}(k)$. For all $k \in \mathcal{K}$, let

$$s_k(x) := \int_{\mathbb{X}} k(y,x)s(y)d\mu(y) = \mathbb{E}\left[k(X,x)\right], \qquad \forall x \in \mathbb{X} \ ,$$

and

$$U_k := \sum_{i\neq j=1}^{n} \left(k(X_i,X_j) - s_k(X_i) - s_k(X_j) + \mathbb{E}\left[k(X,Y)\right] \right) \ .$$

Because those quantities are fundamental in the sequel, let us also define $\Theta_k(x) = A_k(x,x)$ where for $(x,y) \in \mathbb{X}^2$

$$A_k(x,y) := \int_{\mathbb{X}} k(x,z)k(z,y)d\mu(z) \ . \tag{2.8}$$

Denoting

$$\text{for all } x \in \mathbb{X}, \qquad \chi_k(x) = k(x,x) \ ,$$

the ideal penalty is then equal to

$$\begin{aligned}\text{pen}_{\text{id}}(k) &= 2(P_n - P)(\hat{s}_k - s_k) + 2(P_n - P)s_k \\ &= 2\left(\frac{P\chi_k - Ps_k}{n} + \frac{(P_n - P)\chi_k}{n} + \frac{U_k}{n^2} + \left(1 - \frac{2}{n}\right)(P_n - P)s_k \right) \ .\end{aligned} \tag{2.9}$$

The main point is that by using concentration inequalities, we obtain:

$$\mathrm{pen}_{\mathrm{id}}(k) \simeq 2\left(\frac{P\chi_k - Ps_k}{n}\right) \ .$$

The term Ps_k/n depends on s which is unknown. Fortunately, it can be easily controlled as detailed in the sequel. Therefore one can hope that the choice

$$\mathrm{pen}(k) = 2\frac{P\chi_k}{n}$$

is convenient. In general, this choice still depends on the unknown density s but it can be easily estimated in a data-driven way by

$$\mathrm{pen}(k) = 2\frac{P_n\chi_k}{n} \ .$$

The goal of Sect. 3 is to prove this heuristic and to show that $2P\chi_k/n$ and $2P_n\chi_k/n$ are optimal choices for the penalty, that is, they lead to an asymptotically optimal oracle inequality.

2.3 Concentration Tools

To derive sharp oracle inequalities, we only need two fundamental concentration tools, namely a weak Bernstein's inequality and the concentration bounds for degenerate U-statistics of order two. We cite them here under their most suitable form for our purpose.

A Weak Bernstein's Inequality

Proposition 2.1 *For any bounded real valued function f and any $X_1,\ldots,X_n$ i.i.d. with distribution P, for any $u > 0$,*

$$\Pr\left((P_n - P)f \geq \sqrt{\frac{2P\left(f^2\right)u}{n}} + \frac{\|f\|_\infty u}{3n}\right) \leq \exp(-u) \ .$$

The proof is straightforward and can be derived from either Bennett's or Bernstein's inequality [6].

Concentration of Degenerate U-Statistics of Order 2

Proposition 2.2 *Let $X, X_1, \ldots X_n$ be i.i.d. random variables defined on a Polish space $\mathbb{X}$ equipped with its Borel σ-algebra and let $\left(f_{i,j}\right)_{1\leq i\neq j\leq n}$ denote bounded* *real valued symmetric measurable functions defined on* $\mathbb{X}^2$*, such that for any* $i \neq j$*,* $f_{i,j} = f_{j,i}$ *and*

$$\forall\ i,j \text{ s.t. } 1 \leq i \neq j \leq n, \qquad \mathbb{E}\left[f_{i,j}(x,X)\right] = 0 \qquad \text{for a.e. } x \text{ in } \mathbb{X}\ . \tag{2.10}$$

Let U *be the following totally degenerate* U*-statistic of order* 2*,*

$$U = \sum_{1\leq i\neq j\leq n} f_{i,j}(X_i, X_j)\ .$$

Let A *be an upper bound of* $\left|f_{i,j}(x,y)\right|$ *for any* i,j,x,y *and*

$$B^2 = \max\left(\sup_{i,x\in\mathbb{X}} \sum_{j=1}^{i-1} \mathbb{E}\left[f_{i,j}(x,X_j)^2\right], \sup_{j,t\in\mathbb{X}} \sum_{i=j+1}^{n} \mathbb{E}\left[f_{i,j}(X_i,t)^2\right]\right)$$

$$C^2 = \sum_{1\leq i\neq j\leq n} \mathbb{E}\left[f_{i,j}(X_i,X_j)^2\right]$$

$$D = \sup_{(a,b)\in\mathcal{A}} \mathbb{E}\left[\sum_{1\leq i<j\leq n} f_{i,j}(X_i,X_j)a_i(X_i)b_j(X_j)\right]\ ,$$

where $\mathcal{A} = \left\{(a,b),\ \text{s.t. } \mathbb{E}\left[\sum_{i=1}^{n-1} a_i(X_i)^2\right] \leq 1,\ \mathbb{E}\left[\sum_{j=2}^{n} b_j(X_j)^2\right] \leq 1\right\}$.

Then there exists some absolute constant $\kappa > 0$ *such that for any* $u > 0$*, with probability larger than* $1 - 2.7e^{-u}$*,*

$$U \leq \kappa\left(C\sqrt{u} + Du + Bu^{3/2} + Au^2\right)\ .$$

The present result is a simplification of Theorem 3.4.8 in [15], which provides explicit constants for any variables defined on a Polish space. It is mainly inspired by Houdré and Reynaud-Bouret [18], where the result therein has been stated only for real variables. This inequality actually dates back to Giné et al. [16]. This result has been further generalized by Adamczak to U-statistics of any order [1], though the constants are not explicit.

3 Optimal Penalties for Kernel Selection

The main aim of this section is to show that $2P\chi_k/n$ is a theoretical optimal penalty for kernel selection, which means that if $\mathrm{pen}(k)$ is close to $2P\chi_k/n$, the selected kernel $\hat{k}$ satisfies an asymptotically optimal oracle inequality.

3.1 Main Assumptions

To express our results in a simple form, a positive constant Υ is assumed to control, for any k and k' in $\mathcal{K}$, all the following quantities.

$$(\Gamma(1 + \|s\|_\infty)) \vee \sup_{k \in \mathcal{K}} \|s_k\|^2 \leq \Upsilon \ , \tag{3.11}$$

$$P\left(\chi_k^2\right) \leq \Upsilon n P\Theta_k \ , \tag{3.12}$$

$$\|s_k - s_{k'}\|_\infty \leq \Upsilon \vee \sqrt{\Upsilon n}\,\|s_k - s_{k'}\| \ , \tag{3.13}$$

$$\mathbb{E}\left[A_k(X,Y)^2\right] \leq \Upsilon P\Theta_k \ , \tag{3.14}$$

$$\sup_{x \in \mathbb{X}} \mathbb{E}\left[A_k(X,x)^2\right] \leq \Upsilon n \ , \tag{3.15}$$

$$v_k^2 := \sup_{t \in \mathbb{B}_k} Pt^2 \leq \Upsilon \vee \sqrt{\Upsilon P\Theta_k} \ , \tag{3.16}$$

where $\mathbb{B}_k$ is the set of functions t that can be written $t(x) = \int a(z)k(z,x)d\mu(z)$ for some $a \in L^2$ with $\|a\| \leq 1$.

These assumptions may seem very intricate. They are actually fulfilled by our three main examples under very mild conditions (see Sect. 3.3).

3.2 The Optimal Penalty Theorem

In the sequel, $\square$ denotes a positive absolute constant whose value may change from line to line and if there are indices such as $\square_\theta$, it means that this is a positive function of θ and only θ whose value may change from line to line.

Theorem 3.1 *If Assumptions (3.11)–(3.16) hold, then, for any $x \geq 1$, with probability larger than $1 - \square|\mathcal{K}|^2 e^{-x}$, for any $\theta \in (0,1)$, any minimizer $\hat{k}$ of the penalized criterion (2.6) satisfies the following inequality*

$$\forall k \in \mathcal{K}, \qquad (1-4\theta)\left\|s - \hat{s}_{\hat{k}}\right\|^2 \leq (1+4\theta)\|s - \hat{s}_k\|^2 + \left(\mathrm{pen}(k) - 2\frac{P\chi_k}{n}\right) - \left(\mathrm{pen}\left(\hat{k}\right) - 2\frac{P\chi_{\hat{k}}}{n}\right) + \square\frac{\Upsilon x^2}{\theta n} \ . \tag{3.17}$$

Assume moreover that there exists $C > 0$, $\delta' \geq \delta > 0$ and $r \geq 0$ such that for any $x \geq 1$, with probability larger than $1 - Ce^{-x}$, for any $k \in \mathcal{K}$,

$$(\delta - 1)\frac{P\Theta_k}{n} - \square r\frac{\Upsilon x^2}{n} \leq \mathrm{pen}(k) - \frac{2P\chi_k}{n} \leq (\delta' - 1)\frac{P\Theta_k}{n} + \square r\frac{\Upsilon x^2}{n} \ . \tag{3.18}$$

Then for all $\theta \in (0,1)$ *and all* $x \geq 1$*, the following holds with probability at least* $1 - \square(C + |\mathcal{K}|^2)e^{-x}$*,*

$$\frac{(\delta \wedge 1) - 5\theta}{(\delta' \vee 1) + (4 + \delta')\theta} \left\| s - \hat{s}_{\hat{k}} \right\|^2 \leq \inf_{k \in \mathcal{K}} \|s - \hat{s}_k\|^2 + \square \left(r + \frac{1}{\theta^3} \right) \frac{\Upsilon x^2}{n} \quad .$$

Let us make some remarks.

- First, this is an oracle inequality (see (2.5)) with leading constant C_n and remaining term R_n given by

$$C_n = \frac{(\delta' \vee 1) + (4 + \delta')\theta}{(\delta \wedge 1) - 5\theta} \quad \text{and} \quad R_n = \square C_n (r + \theta^{-3}) \frac{\Upsilon x^2}{n} \quad ,$$

  as long as

  - θ is small enough for C_n to be positive,
  - x is large enough for the probability to be large and
  - n is large enough for R_n to be negligible.

  Typically, $r, \delta, \delta', \theta$ and Υ are bounded w.r.t. n and x has to be of the order of $\log(|\mathcal{K}| \vee n)$ for the remainder to be negligible. In particular, $\mathcal{K}$ may grow with n as long as (i) $\log(|\mathcal{K}| \vee n)^2$ remains negligible with respect to n and (ii) Υ does not depend on n.
- If $\mathrm{pen}(k) = 2P\chi_k/n$, that is if $\delta = \delta' = 1$ and $r = C = 0$ in (3.18), the estimator $\hat{s}_{\hat{k}}$ satisfies an asymptotically optimal oracle inequality i.e. $C_n \to_{n\to\infty} 1$ since θ can be chosen as close to 0 as desired. Take for instance, $\theta = (\log n)^{-1}$.
- In general $P\chi_k$ depends on the unknown s and this last penalty cannot be used in practice. Fortunately, its empirical counterpart $\mathrm{pen}(k) = 2P_n\chi_k/n$ satisfies (3.18) with $\delta = 1 - \theta$, $\delta' = 1 + \theta$, $r = 1/\theta$ and $C = 2|\mathcal{K}|$ for any $\theta \in (0,1)$ and in particular $\theta = (\log n)^{-1}$ (see (6.34) in Proposition B.1). Hence, the estimator $\hat{s}_{\hat{k}}$ selected with this choice of penalty also satisfies an asymptotically optimal oracle inequality, by the same argument.
- Finally, we only get an oracle inequality when $\delta > 0$, that is when $\mathrm{pen}(k)$ is larger than $(2P\chi_k - P\Theta_k)/n$ up to some residual term. We discuss the necessity of this condition in Sect. 4.

3.3 Main Examples

This section shows that Theorem 3.1 can be applied in the examples. In addition, it provides the computation of $2P\chi_k/n$ in some specific cases of special interest.

Example 1 (Continued)

Proposition 3.2 *Let $\{k_S, S \in \mathcal{S}\}$ be a collection of projection kernels. Assumptions* (3.11)*,* (3.12)*,* (3.14)*,* (3.15) *and* (3.16) *hold for any $\Upsilon \geq \Gamma(1 + \|s\|_\infty)$, where Γ is given by* (2.2)*. In addition, Assumption* (3.13) *is satisfied under either of the following classical assumptions (see [20, Chap. 7]):*

$$\forall S, S' \in \mathcal{S}, \qquad \text{either } S \subset S' \text{ or } S' \subset S \ , \tag{3.19}$$

or

$$\forall S \in \mathcal{S}, \qquad \|s_{k_S}\|_\infty \leq \frac{\Upsilon}{2} \ . \tag{3.20}$$

These particular projection kernels satisfy for all $(x, y) \in \mathbb{X}^2$

$$A_{k_S}(x, y) = \int_{\mathbb{X}} k_S(x, z) k_S(y, z) d\mu(z)$$
$$= \sum_{(i,j) \in \mathcal{I}_S^2} \varphi_i(x)\varphi_j(y) \int_{\mathbb{X}} \varphi_i(z)\varphi_j(z) d\mu(z) = k_S(x, y) \ .$$

In particular, $\Theta_{k_S} = \chi_{k_S} = \sum_{i \in \mathcal{I}_S} \varphi_i^2$ and $2P\chi_{k_S} - P\Theta_{k_S} = P\chi_{k_S}$.

Moreover, it appears that the function Θ_{k_S} is constant in some linear spaces S of interest (see [19] for more details). Let us mention one particular case studied further on in the sequel. Suppose $\mathcal{S}$ is a collection of regular histogram spaces S on $\mathbb{X}$, that is, any $S \in \mathcal{S}$ is a space of piecewise constant functions on a partition $\mathcal{I}_S$ of $\mathbb{X}$ such that $\mu(i) = 1/D_S$ for any i in $\mathcal{I}_S$. Assumption (3.20) is satisfied for this collection as soon as $\Upsilon \geq 2\|s\|_\infty$. The family $(\varphi_i)_{i \in \mathcal{I}_S}$, where $\varphi_i = \sqrt{D_S}\mathbf{1}_i$ is an orthonormal basis of S and

$$\chi_{k_S} = \sum_{i \in \mathcal{I}_S} \varphi_i^2 = D_S \ .$$

Hence, $P\chi_{k_S} = D_S$ and $2D_S/n$ can actually be used as a penalty to ensure that the selected estimator satisfies an asymptotically optimal oracle inequality. Moreover, in this example it is actually necessary to choose a penalty larger than D_S/n to get an oracle inequality (see [19] or Sect. 4 for more details).

Example 2 (Continued)

Proposition 3.3 *Let $\{k_{K,h}, (K, h) \in \mathcal{H}\}$ be a collection of approximation kernels. Assumptions* (3.11)–(3.16) *hold with $\Gamma = 1$, for any*

$$\Upsilon \geq \max_K \left\{ \frac{|K(0)|}{\|K\|^2} \vee \left(1 + 2\|s\|_\infty \|K\|_1^2\right) \right\} \ ,$$

as soon as (2.3) *is satisfied.*

These approximation kernels satisfy, for all $x \in \mathbb{R}$,

$$\chi_{k_{K,h}}(x) = k_{K,h}(x,x) = \frac{K(0)}{h} \ ,$$
$$\Theta_{k_{K,h}}(x) = A_{k_{K,h}}(x,x) = \frac{1}{h^2}\int_{\mathbb{R}} K\left(\frac{x-y}{h}\right)^2 dy = \frac{\|K\|^2}{h} \ .$$

Therefore, the optimal penalty $2P\chi_{k_{K,h}}/n = 2K(0)/(nh)$ can be computed in practice and yields an asymptotically optimal selection criterion. Surprisingly, the lower bound $2P\chi_{k_{K,h}}/n - P\Theta_{k_{K,h}}/n = (2K(0) - \|K\|^2)/(nh)$ can be negative if $\|K\|^2 > 2K(0)$ and even if $K(0) > 0$, which is usually the case for Parzen kernels. In this case, a minimizer of (2.6) satisfies an oracle inequality, even if this criterion is not penalized. This remarkable fact is illustrated in the simulation study in Sect. 5.

Example 3 (Continued)

Proposition 3.4 *Let $\{k_w, w \in \mathcal{W}\}$ be a collection of weighted projection kernels. Assumption* (3.11) *is valid for $\Upsilon \geq \Gamma(1 + \|s\|_\infty)$, where Γ is given by* (2.4)*. Moreover* (3.11) *and* (2.1) *imply* (3.12)–(3.16)*.*

For these weighted projection kernels, for all $x \in \mathbb{X}$

$$\chi_{k_w}(x) = \sum_{i=1}^{p} w_i \varphi_i(x)^2, \qquad \text{hence} \qquad P\chi_{k_w} = \sum_{i=1}^{p} w_i P\varphi_i^2 \qquad \text{and}$$
$$\Theta_{k_w}(x) = \sum_{i,j=1}^{p} w_i w_j \varphi_i \varphi_j \int_{\mathbb{X}} \varphi_i(x)\varphi_j(x) d\mu(x) = \sum_{i=1}^{p} w_i^2 \varphi_i(x)^2 \leq \chi_{k_w}(x) \ .$$

In this case, the optimal penalty $2P\chi_{k_w}/n$ has to be estimated in general. However, in the following example it can still be directly computed.

Let $\mathbb{X} = [0, 1]$, let μ be the Lebesgue measure. Let $\varphi_0 \equiv 1$ and, for any $j \geq 1$,

$$\varphi_{2j-1}(x) = \sqrt{2}\cos(2\pi j x), \qquad \varphi_{2j}(x) = \sqrt{2}\sin(2\pi j x) \ .$$

Consider some odd p and a family of weights $\mathcal{W} = \{w_i, i = 0, \ldots, p\}$ such that, for any $w \in \mathcal{W}$ and any $i = 1, \ldots, p/2$, $w_{2i-1} = w_{2i} = \tau_i$. In this case, the values of the functions of interest do not depend on x

$$\chi_{k_w} = w_0 + \sum_{j=1}^{p/2} \tau_j, \qquad \Theta_{k_w} = w_0^2 + \sum_{j=1}^{p/2} \tau_j^2 \ .$$

In particular, this family includes Pinsker's and Tikhonov's weights.

4 Minimal Penalties for Kernel Selection

The purpose of this section is to see whether the lower bound $\mathrm{pen}_{\min}(k) := (2P\chi_k - P\Theta_k)/n$ is sharp in Theorem 3.1. To do so we first need the following result which links $\|s - \hat{s}_k\|$ to deterministic quantities, thanks to concentration tools.

4.1 Bias-Variance Decomposition with High Probability

Proposition 4.1 *Assume $\{k\}_{k\in\mathcal{K}}$ is a finite collection of kernels satisfying Assumptions (3.11)–(3.16). For all $x > 1$, for all η in $(0, 1]$, with probability larger than $1 - \square|\mathcal{K}|e^{-x}$*

$$\|s_k - \hat{s}_k\|^2 \le (1+\eta)\frac{P\Theta_k}{n} + \square\frac{\Upsilon x^2}{\eta n} \quad ,$$

$$\frac{P\Theta_k}{n} \le (1+\eta)\,\|s_k - \hat{s}_k\|^2 + \square\frac{\Upsilon x^2}{\eta n} \quad .$$

Moreover, for all $x > 1$ and for all η in $(0, 1)$, with probability larger than $1 - \square|\mathcal{K}|e^{-x}$, for all $k \in \mathcal{K}$, each of the following inequalities hold

$$\|s - \hat{s}_k\|^2 \le (1+\eta)\left(\|s - s_k\|^2 + \frac{P\Theta_k}{n}\right) + \square\frac{\Upsilon x^2}{\eta^3 n} \quad ,$$

$$\|s - s_k\|^2 + \frac{P\Theta_k}{n} \le (1+\eta)\,\|s - \hat{s}_k\|^2 + \square\frac{\Upsilon x^2}{\eta^3 n} \quad .$$

This means that not only in expectation but also with high probability can the term $\|s - \hat{s}_k\|^2$ be decomposed in a bias term $\|s - s_k\|^2$ and a "variance" term $P\Theta_k/n$. The bias term measures the capacity of the kernel k to approximate s whereas $P\Theta_k/n$ is the price to pay for replacing s_k by its empirical version $\hat{s}_k$. In this sense, $P\Theta_k/n$ measures the complexity of the kernel k in a way which is completely adapted to our problem of density estimation. Even if it does not seem like a natural measure of complexity at first glance, note that in the previous examples, it is indeed always linked to a natural complexity. When dealing with regular histograms defined on $[0, 1]$, $P\Theta_{k_S}$ is the dimension of the considered space S, whereas for approximation kernels $P\Theta_{k_{K,h}}$ is proportional to the inverse of the considered bandwidth h.

4.2 Some General Results About the Minimal Penalty

In this section, we assume that we are in the asymptotic regime where the number of observations $n \to \infty$. In particular, the asymptotic notations refers to this regime.

From now on, the family $\mathcal{K} = \mathcal{K}_n$ may depend on n as long as both Γ and Υ remain absolute constants that do not depend on it. Indeed, on the previous examples, this seems a reasonable regime. Since $\mathcal{K}_n$ now depends on n, our selected $\hat{k} = \hat{k}_n$ also depends on n.

To prove that the lower bound $\mathrm{pen}_{\min}(k)$ is sharp, we need to show that the estimator chosen by minimizing (2.6) with a penalty smaller than $\mathrm{pen}_{\min}$ does not satisfy an oracle inequality. This is only possible if the $\|s - \hat{s}_k\|^2$'s are not of the same order and if they are larger than the remaining term $\square(r + \theta^{-3})\Upsilon x^2/n$. From an asymptotic point of view, we rewrite this thanks to Proposition 4.1 as for all $n \geq 1$, there exist $k_{0,n}$ and $k_{1,n}$ in $\mathcal{K}_n$ such that

$$\left\|s - s_{k_{1,n}}\right\|^2 + \frac{P\Theta_{k_{1,n}}}{n} \gg \left\|s - s_{k_{0,n}}\right\|^2 + \frac{P\Theta_{k_{0,n}}}{n} \gg \square \left(r + \frac{1}{\theta^3}\right) \frac{\Upsilon x^2}{n} \ , \tag{4.21}$$

where $a_n \gg b_n$ means that $b_n/a_n \to_{n\to\infty} 0$. More explicitly, denoting by $\mathrm{o}(1)$ a sequence only depending on n and tending to 0 as n tends to infinity and whose value may change from line to line, one assumes that there exists c_s and c_R positive constants such that for all $n \geq 1$, there exist $k_{0,n}$ and $k_{1,n}$ in $\mathcal{K}_n$ such that

$$\left\|s - s_{k_{0,n}}\right\|^2 + \frac{P\Theta_{k_{0,n}}}{n} \leq c_s \, \mathrm{o}(1) \left(\left\|s - s_{k_{1,n}}\right\|^2 + \frac{P\Theta_{k_{1,n}}}{n} \right) \tag{4.22}$$

$$\frac{(\log(|\mathcal{K}_n| \vee n))^3}{n} \leq c_R \, \mathrm{o}(1) \left(\left\|s - s_{k_{0,n}}\right\|^2 + \frac{P\Theta_{k_{0,n}}}{n} \right) \ . \tag{4.23}$$

We put a log-cube factor in the remaining term to allow some choices of $\theta = \theta_n \to_{n\to\infty} 0$ and $r = r_n \to_{n\to\infty} +\infty$.

But (4.22) and (4.23) (or (4.21)) are not sufficient. Indeed, the following result explains what happens when the bias terms are always the leading terms.

Corollary 4.2 *Let $(\mathcal{K}_n)_{n\geq 1}$ be a sequence of finite collections of kernels k satisfying Assumptions (3.11)–(3.16) for a positive constant Υ independent of n and such that*

$$\frac{1}{n} = c_b \, \mathrm{o}(1) \inf_{k\in\mathcal{K}_n} \frac{\|s - s_k\|^2}{P\Theta_k} \ , \tag{4.24}$$

for some positive constant c_b.

Assume that there exist real numbers of any sign $\delta' \geq \delta$ and a sequence $(r_n)_{n\geq 1}$ of nonnegative real numbers such that, for all $n \geq 1$, with probability larger than $1 - \Box/n^2$, for all $k \in \mathcal{K}_n$,

$$\delta \frac{P\Theta_k}{n} - \Box_{\delta,\delta',\Upsilon} \frac{r_n \log(n \vee |\mathcal{K}_n|)^2}{n} \leq \mathrm{pen}(k) - \frac{2P\chi_k - P\Theta_k}{n} \leq \delta' \frac{P\Theta_k}{n} + \Box_{\delta,\delta',\Upsilon} \frac{r_n \log(n \vee |\mathcal{K}_n|)^2}{n} .$$

Then, with probability larger than $1 - \Box/n^2$,

$$\left\| s - \hat{s}_{\hat{k}_n} \right\|^2 \leq (1 + \Box_{\delta,\delta',\Upsilon,c_b} \, \mathrm{o}(1)) \inf_{k \in \mathcal{K}_n} \|s - \hat{s}_k\|^2 + \Box_{\delta,\delta',\Upsilon} \left(r_n + \log n \right) \frac{\log(n \vee |\mathcal{K}_n|)^2}{n} .$$

The proof easily follows by taking $\theta = (\log n)^{-1}$ in (3.17), $\eta = 2$ for instance in Proposition 4.1 and by using Assumption (4.24) and the bounds on $\mathrm{pen}(k)$. This result shows that the estimator $\hat{s}_{\hat{k}_n}$ satisfies an asymptotically optimal oracle inequality when condition (4.24) holds, whatever the values of δ and δ' even when they are negative. This proves that the lower bound $\mathrm{pen}_{\min}$ is not sharp in this case.

Therefore, we have to assume that at least one bias $\|s - s_k\|^2$ is negligible with respect to $P\Theta_k/n$. Actually, to conclude, we assume that this happens for $k_{1,n}$ in (4.21).

Theorem 4.3 *Let $(\mathcal{K}_n)_{n\geq 1}$ be a sequence of finite collections of kernels satisfying Assumptions (3.11)–(3.16), with Υ not depending on n. Each $\mathcal{K}_n$ is also assumed to satisfy (4.22) and (4.23) with a kernel $k_{1,n} \in \mathcal{K}_n$ in (4.22) such that*

$$\left\| s - s_{k_{1,n}} \right\|^2 \leq c \, \mathrm{o}(1) \frac{P\Theta_{k_{1,n}}}{n} , \tag{4.25}$$

for some fixed positive constant c. Suppose that there exist $\delta \geq \delta' > 0$ and a sequence $(r_n)_{n\geq 1}$ of nonnegative real numbers such that $r_n \leq \Box \log(|\mathcal{K}_n| \vee n)$ and such that for all $n \geq 1$, with probability larger than $1 - \Box n^{-2}$, for all $k \in \mathcal{K}_n$,

$$\frac{2P\chi_k - P\Theta_k}{n} - \delta \frac{P\Theta_k}{n} - \Box_{\delta,\delta',\Upsilon} \frac{r_n \log(|\mathcal{K}_n| \vee n)^2}{n} \leq \mathrm{pen}(k) \leq \frac{2P\chi_k - P\Theta_k}{n} - \delta' \frac{P\Theta_k}{n} + \Box_{\delta,\delta',\Upsilon} \frac{r_n \log(|\mathcal{K}_n| \vee n)^2}{n} . \tag{4.26}$$

Then, with probability larger than $1 - \square/n^2$*, the following holds*

$$P\Theta_{\hat{k}_n} \geq \left(\frac{\delta'}{\delta} + \square_{\delta,\delta',\Upsilon,c,c_s,c_R}\, \mathrm{o}(1) \right) P\Theta_{k_{1,n}} \quad \textit{and} \tag{4.27}$$

$$\left\| s - \hat{s}_{\hat{k}_n} \right\|^2 \geq \left(\frac{\delta'}{\delta} + \square_{\delta,\delta',\Upsilon,c,c_s,c_R}\, \mathrm{o}(1) \right) \left\| s - \hat{s}_{k_{1,n}} \right\|^2 \gg \left\| s - \hat{s}_{k_{0,n}} \right\|^2 \geq \inf_{k \in \mathcal{K}_n} \left\| s - \hat{s}_k \right\|^2 \ . \tag{4.28}$$

By (4.28), under the conditions of Theorem 4.3, the estimator $\hat{s}_{\hat{k}_n}$ cannot satisfy an oracle inequality, hence, the lower bound $(2P\chi_k - P\Theta_k)/n$ in Theorem 3.1 is sharp. This shows that $(2P\chi_k - P\Theta_k)/n$ is a minimal penalty in the sense of [5] for kernel selection. When

$$\mathrm{pen}(k) = \frac{2P\chi_k - P\Theta_k}{n} + \kappa \frac{P\Theta_k}{n} \ ,$$

the complexity $P\Theta_{\hat{k}_n}$ presents a sharp phase transition when κ becomes positive. Indeed, when $\kappa < 0$ it follows from (4.27) that the complexity $P\Theta_{\hat{k}_n}$ is asymptotically larger than $P\Theta_{k_{1,n}}$. But on the other hand, as a consequence of Theorem 3.1, when $\kappa > 0$, this complexity becomes smaller than

$$\square_\kappa n \inf_{k \in \mathcal{K}_n} \left(\left\| s - s_k \right\|^2 + \frac{P\Theta_k}{n} \right) \leq \square_\kappa \left(n \left\| s - s_{k_{0,n}} \right\|^2 + P\Theta_{k_{0,n}} \right) \ll \square_\kappa \left(n \left\| s - s_{k_{1,n}} \right\|^2 + P\Theta_{k_{1,n}} \right) \leq \square_\kappa P\Theta_{k_{1,n}} \ .$$

4.3 Examples

Example 1 (Continued) Let $\mathcal{S} = \mathcal{S}_n$ be the collection of spaces of regular histograms on $[0, 1]$ with dimensions $\{1, \ldots, n\}$ and let $\hat{S} = \hat{S}_n$ be the selected space thanks to the penalized criterion. Recall that, for any $S \in \mathcal{S}_n$, the orthonormal basis is defined by $\varphi_i = \sqrt{D_S}\mathbf{1}_i$ and $P\Theta_{k_S} = D_S$. Assume that s is α-Hölderian, with $\alpha \in (0, 1]$ with α-Hölderian norm L. It is well known (see for instance Section 1.3.3. of [4]) that the bias is bounded above by

$$\left\| s - s_{k_S} \right\|^2 \leq \square_L D_S^{-2\alpha} \ .$$

In particular, if $D_{S_1} = n$,

$$\left\| s - s_{k_{S_1}} \right\|^2 \leq \square_L n^{-2\alpha} \ll 1 = \frac{D_{S_1}}{n} = \frac{P\Theta_{k_{S_1}}}{n} \ .$$

Thus, (4.25) holds for kernel k_{S_1}. Moreover, if $D_{S_0} = \lfloor\sqrt{n}\rfloor$,

$$\frac{(\log(n \vee |\mathcal{S}_n|))^3}{n} \ll \left\|s - s_{k_{S_0}}\right\|^2 + \frac{D_{S_0}}{n} \leq \square_L\left(\frac{1}{n^\alpha} + \frac{1}{\sqrt{n}}\right) \ll \left\|s - s_{k_{S_1}}\right\|^2 + \frac{D_{S_1}}{n} \ .$$

Hence, (4.21) holds with $k_{0,n} = k_{S_0}$ and $k_{1,n} = k_{S_1}$. Therefore, Theorem 4.3 and Theorem 3.1 apply in this example. If $\mathrm{pen}(k_S) = (1-\delta)D_S/n$, the dimension $D_{k_{\widehat{S_n}}} \geq \square_\delta n$ and $\hat{s}_{k_{\widehat{S_n}}}$ is not consistent and does not satisfy an oracle inequality. On the other hand, if $\mathrm{pen}(k_S) = (1+\delta)D_S/n$,

$$D_{\widehat{S_n}} \leq \square_{L,\delta}\left(n^{1-\alpha} + \sqrt{n}\right) \ll D_{S_1} = n$$

and $\hat{s}_{k_{\widehat{S_n}}}$ satisfies an oracle inequality which implies that, with probability larger than $1 - \square/n^2$,

$$\left\|s - \hat{s}_{k_{\widehat{S_n}}}\right\|^2 \leq \square_{\alpha,L,\delta}\, n^{-2\alpha/(2\alpha+1)} \ ,$$

by taking $D_S \simeq n^{1/(2\alpha+1)}$. It achieves the minimax rate of convergence over the class of α-Hölderian functions.

From Theorem 3.1, the penalty $\mathrm{pen}(k_S) = 2D_S/n$ provides an estimator $\hat{s}_{k_{\widehat{S_n}}}$ that achieves an asymptotically optimal oracle inequality. Therefore the optimal penalty is equal to 2 times the minimal one. In particular, the slope heuristics of [5] holds in this example, as already noticed in [19].

Finally to illustrate Corollary 4.2, let us take $s(x) = 2x$ and the collection of regular histograms with dimension in $\{1, \ldots, \lfloor n^\beta \rfloor\}$, with $\beta < 1/3$. Simple calculations show that

$$\frac{\|s - s_{k_S}\|^2}{D_S} \geq \square D_S^{-3} \geq \square n^{-3\beta} \gg n^{-1}.$$

Hence (4.24) applies and the penalized estimator with penalty $\mathrm{pen}(k_S) \simeq \delta\frac{D_S}{n}$ always satisfies an oracle inequality even if $\delta = 0$ or $\delta < 0$. This was actually expected since it is likely to choose the largest dimension which is also the oracle choice in this case.

Example 2 (Continued) Let K be a fixed function, let $\mathcal{H} = \mathcal{H}_n$ denote the following grid of bandwidths

$$\mathcal{H} = \left\{\frac{\|K\|_\infty \|K\|_1}{i} \quad / \quad i = 1, \ldots, n\right\}$$

and let $\hat{h} = \hat{h}_n$ be the selected bandwidth. Assume as before that s is a density on $[0, 1]$ that belongs to the Nikol'ski class $\mathcal{N}(\alpha, L)$ with $\alpha \in (0, 1]$ and $L > 0$. By Proposition 1.5 in [25], if K satisfies $\int |u|^\alpha |K(u)| \, du < \infty$

$$\left\| s - s_{k_{K,h}} \right\|^2 \leq \square_{\alpha,K,L} h^{2\alpha} \quad .$$

In particular, when $h_1 = \|K\|_\infty \|K\|_1 / n$,

$$\left\| s - s_{k_{K,h_1}} \right\|^2 \leq \square_{\alpha,K,L} n^{-2\alpha} \ll \frac{P\Theta_{k_{K,h_1}}}{n} = \frac{\|K\|^2}{\|K\|_\infty \|K\|_1} \quad .$$

On the other hand, for $h_0 = \|K\|_\infty \|K\|_1 / \lfloor \sqrt{n} \rfloor$,

$$\begin{aligned} \frac{(\log n \vee |\mathcal{H}_n|)^2}{n} &\ll \left\| s - s_{k_{K,h_0}} \right\|^2 + \frac{P\Theta_{k_{K,h_0}}}{n} \\ &\leq \square_{K,\alpha,L} \left(\frac{1}{n^\alpha} + \frac{1}{\sqrt{n}} \right) \ll \left\| s - s_{k_{K,h_1}} \right\|^2 + \frac{P\Theta_{k_{K,h_1}}}{n} \quad . \end{aligned}$$

Hence, (4.21) and (4.25) hold with kernels $k_{0,n} = k_{K,h_0}$ and $k_{1,n} = k_{K,h_1}$. Therefore, Theorems 4.3 and 3.1 apply in this example. If for some $\delta > 0$ we set $\text{pen}(k_{K,h}) = (2K(0) - \|K\|^2 - \delta \|K\|^2)/(nh)$, then $\hat{h}_n \leq \square_{\delta,K} n^{-1}$ and $\hat{s}_{k_{K,\hat{h}_n}}$ is not consistent and does not satisfy an oracle inequality. On the other hand, if $\text{pen}(k_{K,h}) = (2K(0) - \|K\|^2 + \delta \|K\|^2)/(nh)$, then

$$\hat{h}_n \geq \square_{\delta,K,L} \left(n^{1-\alpha} + \sqrt{n} \right)^{-1} \gg \square_{\delta,K,L} n^{-1} \quad ,$$

and $\hat{s}_{K,k_{\hat{h}_n}}$ satisfies an oracle inequality which implies that, with probability larger than $1 - \square/n^2$,

$$\left\| s - \hat{s}_{k_{K,\hat{h}_n}} \right\|^2 \leq \square_{\alpha,K,L,\delta} n^{-2\alpha/(2\alpha+1)} \quad ,$$

for $h = \|K\|_\infty \|K\|_1 / \lfloor n^{1/(2\alpha+1)} \rfloor \in \mathcal{H}$. In particular it achieves the minimax rate of convergence over the class $\mathcal{N}(\alpha, L)$. Finally, if $\text{pen}(k_{K,h}) = 2K(0)/(nh)$, $\hat{s}_{k_{K,\hat{h}_n}}$ achieves an asymptotically optimal oracle inequality, thanks to Theorem 3.1.

The minimal penalty is therefore

$$\text{pen}_{\min}(k_{K,h}) = \frac{2K(0) - \|K\|^2}{nh} \quad .$$

In this case, the optimal penalty $\mathrm{pen}_{\mathrm{opt}}(k_{K,h}) = 2K(0)/(nh)$ derived from Theorem 3.1 is not twice the minimal one, but one still has, if $2K(0) \neq \|K\|^2$,

$$\mathrm{pen}_{\mathrm{opt}}(k_{K,h}) = \frac{2K(0)}{2K(0) - \|K\|^2}\, \mathrm{pen}_{\min}(k_{K,h}) \ ,$$

even if they can be of opposite sign depending on K. This type of nontrivial relationship between optimal and minimal penalty has already been underlined in [2] in regression framework for selecting linear estimators.

Note that if one allows two kernel functions K_1 and K_2 in the family of kernels such that $2K_1(0) \neq \|K_1\|^2$, $2K_2(0) \neq \|K_2\|^2$ and

$$\frac{2K_1(0)}{2K_1(0) - \|K_1\|^2} \neq \frac{2K_2(0)}{2K_2(0) - \|K_2\|^2} \ ,$$

then there is no absolute constant multiplicative factor linking the minimal penalty and the optimal one.

5 Small Simulation Study

In this section we illustrate on simulated data Theorems 3.1 and 4.3. We focus on approximation kernels only, since projection kernels have been already discussed in [19].

We observe an $n = 100$ i.i.d. sample of standard gaussian distribution. For a fixed parameter $a \geq 0$ we consider the family of kernels

$$k_{K_a,h}(x,y) = \frac{1}{h} K_a\left(\frac{x-y}{h}\right) \qquad \text{with} \quad h \in \mathcal{H} = \left\{ \frac{1}{2i},\ i = 1,\ldots,50 \right\} \ ,$$

where for $x \in \mathbb{R}$, $\quad K_a(x) = \dfrac{1}{2\sqrt{2\pi}}\left(e^{-\frac{(x-a)^2}{2}} + e^{-\frac{(x+a)^2}{2}} \right)$.

In particular the kernel estimator with $a = 0$ is the classical Gaussian kernel estimator. Moreover

$$K_a(0) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{a^2}{2}\right) \quad \text{and} \quad \|K_a\|^2 = \frac{1 + e^{-a^2}}{4\sqrt{\pi}} \ .$$

Thus, depending on the value of a, the minimal penalty $(2K_a(0) - \|K_a\|^2)/(nh)$ may be negative. We study the behavior of the penalized criterion

$$\mathcal{C}_{\mathrm{pen}}\left(k_{K_a,h}\right) = P_n\gamma(\hat{s}_{k_{K_a,h}}) + \mathrm{pen}(k_{K_a,h})$$

with penalties of the form

$$\mathrm{pen}\,(k_{K_a,h}) = \frac{2K_a(0) - \|K_a\|^2}{nh} + \kappa \frac{\|K_a\|^2}{nh} \ , \tag{5.29}$$

for different values of κ ($\kappa = -1, 0, 1$) and a ($a = 0, 1.5, 2, 3$). On Fig. 1 are represented the selected estimates by the optimal penalty $2K_a(0)/(nh)$ for the different values of a and on Fig. 2 one sees the evolution of the different penalized criteria as a function of $1/h$. The contrast curves for $a = 0$ are classical on Fig. 2. Without penalization, the criterion decreases and leads to the selection of the smallest bandwidth. At the minimal penalty, the curve is flat and at the optimal penalty one selects a meaningful bandwidth as shown on Fig. 1.

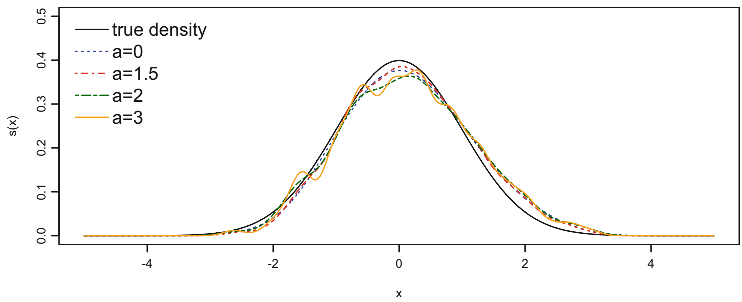

Fig. 1 Selected approximation kernel estimators when the penalty is the optimal one, i.e. $2K_a(0)/(nh)$

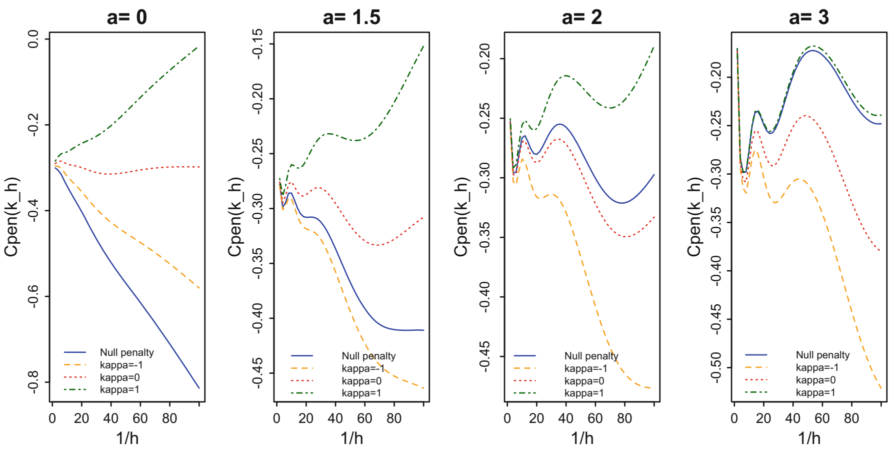

Fig. 2 Behavior of $P_n\gamma(\hat{s}_{k_{K_a,h}})$ (*solid blue line*) and $\mathcal{C}_{\mathrm{pen}}\left(k_{K_a,h}\right)$ as a function of $1/h$, which is proportional to the complexity $P\Theta_{k_{K_a,h}}$

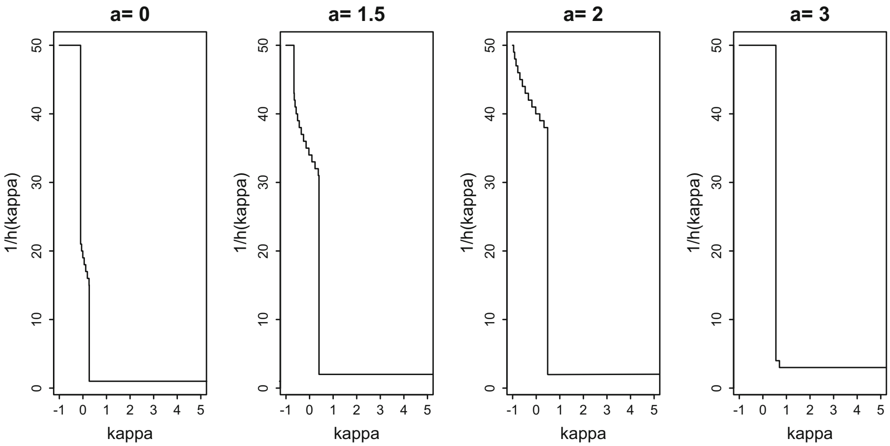

Fig. 3 Behavior of $1/\hat{h}$, which is proportional to the complexity $P\Theta_{k_{K_a,h}}$, for the estimator selected by the criterion whose penalty is given by (5.29), as a function of κ

When $a > 0$, despite the choice of those unusual kernels, the reconstructions on Fig. 1 for the optimal penalty are also meaningful. However when $a = 2$ or $a = 3$, the criterion with minimal penalty is smaller than the unpenalized criterion, meaning that minimizing the latter criterion leads by Theorem 3.1 to an oracle inequality. In our simulation, when $a = 3$, the curves for the optimal criterion and the unpenalized one are so close that the same estimator is selected by both methods.

Finally Fig. 3 shows that there is indeed in all cases a sharp phase transition around $\kappa = 0$ i.e. at the minimal penalty for the complexity of the selected estimate.

6 Proofs

6.1 *Proof of Theorem 3.1*

The starting point to prove the oracle inequality is to notice that any minimizer $\hat{k}$ of $\mathcal{C}_{\text{pen}}$ satisfies

$$\left\|s - \hat{s}_{\hat{k}}\right\|^2 \le \|s - \hat{s}_k\|^2 + (\text{pen}(k) - \text{pen}_{\text{id}}(k)) - \left(\text{pen}\left(\hat{k}\right) - \text{pen}_{\text{id}}\left(\hat{k}\right)\right) \ .$$

Using the expression of the ideal penalty (2.9) we find

$$\left\|s-\hat{s}_{\hat{k}}\right\|^2 \le \|s-\hat{s}_k\|^2 + \left(\mathrm{pen}(k) - 2\frac{P\chi_k}{n}\right) - \left(\mathrm{pen}\left(\hat{k}\right) - 2\frac{P\chi_{\hat{k}}}{n}\right)$$
$$+ 2\frac{P(s_k - s_{\hat{k}})}{n} + 2\left(1-\frac{2}{n}\right)(P_n - P)(s_{\hat{k}} - s_k)$$
$$+ 2\frac{(P_n-P)(\chi_{\hat{k}} - \chi_k)}{n} + 2\frac{U_{\hat{k}} - U_k}{n^2} \quad . \tag{6.30}$$

By Proposition B.1 (see the appendix), for all $x > 1$, for all θ in $(0,1)$, with probability larger than $1-(7.4|\mathcal{K}| + 2|\mathcal{K}|^2)e^{-x}$,

$$\left\|s-\hat{s}_{\hat{k}}\right\|^2 \le \|s-\hat{s}_k\|^2 + \left(\mathrm{pen}(k) - 2\frac{P\chi_k}{n}\right) - \left(\mathrm{pen}\left(\hat{k}\right) - 2\frac{P\chi_{\hat{k}}}{n}\right)$$
$$+ \theta\left\|s - s_{\hat{k}}\right\|^2 + \theta\|s-s_k\|^2 + \square\frac{\Upsilon}{\theta n}$$
$$+ \left(1-\frac{2}{n}\right)\theta\left\|s-s_{\hat{k}}\right\|^2 + \left(1-\frac{2}{n}\right)\theta\|s-s_k\|^2 + \square\frac{\Upsilon x^2}{\theta n}$$
$$+ \theta\frac{P\Theta_k}{n} + \theta\frac{P\Theta_{\hat{k}}}{n} + \square\frac{\Upsilon x}{\theta n} + \theta\frac{P\Theta_k}{n} + \theta\frac{P\Theta_{\hat{k}}}{n} + \square\frac{\Upsilon x^2}{\theta n}$$

Hence

$$\left\|s-\hat{s}_{\hat{k}}\right\|^2 \le \|s-\hat{s}_k\|^2 + \left(\mathrm{pen}(k) - 2\frac{P\chi_k}{n}\right) - \left(\mathrm{pen}\left(\hat{k}\right) - 2\frac{P\chi_{\hat{k}}}{n}\right)$$
$$+ 2\theta\left[\left\|s-s_{\hat{k}}\right\|^2 + \frac{P\Theta_{\hat{k}}}{n}\right] + 2\theta\left[\|s-s_k\|^2 + \frac{P\Theta_k}{n}\right] + \square\frac{\Upsilon x^2}{\theta n} \quad .$$

This bound holds using (3.11)–(3.13) only. Now by Proposition 4.1 applied with $\eta = 1$, we have for all $x > 1$, for all $\theta \in (0,1)$, with probability larger than $1-(16.8|\mathcal{K}| + 2|\mathcal{K}|^2)e^{-x}$,

$$\left\|s-\hat{s}_{\hat{k}}\right\|^2 \le \|s-\hat{s}_k\|^2 + \left(\mathrm{pen}(k) - 2\frac{P\chi_k}{n}\right) - \left(\mathrm{pen}\left(\hat{k}\right) - 2\frac{P\chi_{\hat{k}}}{n}\right)$$
$$+ 4\theta\left\|s-\hat{s}_{\hat{k}}\right\|^2 + 4\theta\|s-\hat{s}_k\|^2 + \square\frac{\Upsilon x^2}{\theta n} \quad .$$

This gives the first part of the theorem.

For the second part, by the condition (3.18) on the penalty, we find for all $x > 1$, for all θ in $(0, 1)$, with probability larger than $1 - (C + 16.8|\mathcal{K}| + 2|\mathcal{K}|^2)e^{-x}$,

$$(1-4\theta)\left\|s-\hat{s}_{\hat{k}}\right\|^2 \leq$$
$$(1+4\theta)\left\|s-\hat{s}_k\right\|^2+(\delta'-1)_+\frac{P\Theta_k}{n}+(1-\delta)_+\frac{P\Theta_{\hat{k}}}{n}+\square\left(r+\frac{1}{\theta}\right)\frac{\Upsilon x^2}{n} \ .$$

By Proposition 4.1 applied with $\eta = \theta$, we have with probability larger than $1 - (C + 26.2|\mathcal{K}| + 2|\mathcal{K}|^2)e^{-x}$,

$$(1-4\theta)\left\|s-\hat{s}_{\hat{k}}\right\|^2 \leq (1+4\theta)\left\|s-\hat{s}_k\right\|^2+(\delta'-1)_+(1+\theta)\|s-\hat{s}_k\|^2$$
$$+(1-\delta)_+(1+\theta)\left\|s-\hat{s}_{\hat{k}}\right\|^2+\square\left(r+\frac{1}{\theta^3}\right)\frac{\Upsilon x^2}{n} \ ,$$

that is

$$\left((\delta\wedge 1)-\theta(4+(1-\delta)_+)\right)\left\|s-\hat{s}_{\hat{k}}\right\|^2$$
$$\leq\left((\delta'\vee 1)+\theta(4+(\delta'-1)_+)\right)\|s-\hat{s}_k\|^2+\square\left(r+\frac{1}{\theta^3}\right)\frac{\Upsilon x^2}{n} \ .$$

Hence, because $1 \leq [(\delta' \vee 1) + (4 + (\delta' - 1)_+)\theta] \leq (\delta' \vee 1) + (4 + \delta')\theta$, we obtain the desired result.

6.2 *Proof of Proposition 4.1*

First, let us denote for all $x \in \mathbb{X}$

$$F_{A,k}(x) := \mathbb{E}\left[A_k(X,x)\right], \qquad \zeta_k(x) := \int \left(k(y,x)-s_k(y)\right)^2 d\mu(y) \ ,$$

and

$$U_{A,k} := \sum_{i\neq j=1}^{n}\left(A_k(X_i,X_j)-F_{A,k}(X_i)-F_{A,k}(X_j)+\mathbb{E}\left[A_k(X,Y)\right]\right) \ .$$

Some easy computations then provide the following useful equality

$$\|s_k-\hat{s}_k\|^2=\frac{1}{n}P_n\zeta_k+\frac{1}{n^2}U_{A,k} \ .$$

We need only treat the terms on the right-hand side, thanks to the probability tools of Sect. 2.3. Applying Proposition 2.1, we get, for any $x \geq 1$, with probability larger than $1 - 2|\mathcal{K}|\, e^{-x}$,

$$|(P_n - P)\zeta_k| \leq \sqrt{\frac{2x}{n} P\zeta_k^2} + \frac{\|\zeta_k\|_\infty x}{3n} \ .$$

One can then check the following link between ζ_k and Θ_k

$$P\zeta_k = \int (k(y,x) - s_k(x))^2 s(y) d\mu(x) d\mu(y) = P\Theta_k - \|s_k\|^2 \ .$$

Next, by (2.1) and (3.11)

$$\begin{aligned}\|\zeta_k\|_\infty &= \sup_{y \in \mathbb{X}} \int (k(y,x) - \mathbb{E}[k(X,x)])^2 d\mu(x) \\ &\leq 4 \sup_{y\in\mathbb{X}} \int k(y,x)^2 d\mu(x) \leq 4\Upsilon n \ .\end{aligned}$$

In particular, since $\zeta_k \geq 0$,

$$P\zeta_k^2 \leq \|\zeta_k\|_\infty P\zeta_k \leq 4\Upsilon n P\Theta_k \ .$$

It follows from these computations and from (3.11) that there exists an absolute constant $\square$ such that, for any $x \geq 1$, with probability larger than $1 - 2|\mathcal{K}|\, e^{-x}$, for any $\theta \in (0,1)$,

$$|P_n\zeta_k - P\Theta_k| \leq \theta P\Theta_k + \square \frac{\Upsilon x}{\theta} \ .$$

We now need to control the term $U_{A,k}$. From Proposition 2.2, for any $x \geq 1$, with probability larger than $1 - 5.4|\mathcal{K}|\, e^{-x}$,

$$\frac{|U_{A,k}|}{n^2} \leq \frac{\square}{n^2}\left(C\sqrt{x} + Dx + Bx^{3/2} + Ax^2\right) \ .$$

By (2.1), (3.11) and Cauchy-Schwarz inequality,

$$A = 4 \sup_{(x,y)\in\mathbb{X}^2} \int k(x,z)k(y,z)d\mu(z) \leq 4 \sup_{x\in\mathbb{X}} \int k(x,z)^2 d\mu(z) \leq 4\Upsilon n \ .$$

In addition, by (3.15), $B^2 \leq 16 \sup_{x\in\mathbb{X}} \mathbb{E}\left[A_k(X,x)^2\right] \leq 16\Upsilon n$.

Moreover, applying the Assumption (3.14),

$$C^2 \le \sum_{i \neq j=1}^{n} \mathbb{E}\left[A_k(X_i, X_j)^2\right] = n^2 \mathbb{E}\left[A_k(X, Y)^2\right] \le n^2 \Upsilon P\Theta_k \ .$$

Finally, applying the Cauchy-Schwarz inequality and proceeding as for C^2, the quantity used to define D can be bounded above as follows:

$$\mathbb{E}\left[\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} a_i(X_i)b_j(X_j)A_k(X_i, X_j)\right] \le n\sqrt{\mathbb{E}\left[A_k(X,Y)^2\right]} \le n\sqrt{\Upsilon P\Theta_k} \ .$$

Hence for any $x \ge 1$, with probability larger than $1 - 5.4\,|\mathcal{K}|\,e^{-x}$,

$$\text{for any } \theta \in (0,1), \quad \frac{|U_{A,k}|}{n^2} \le \theta \frac{P\Theta_k}{n} + \square \frac{\Upsilon x^2}{\theta n} \ .$$

Therefore, for all $\theta \in (0, 1)$,

$$\left| \|\hat{s}_k - s_k\|^2 - \frac{P\Theta_k}{n} \right| \le 2\theta \frac{P\Theta_k}{n} + \square \frac{\Upsilon x^2}{\theta n} \ ,$$

and the first part of the result follows by choosing $\theta = \eta/2$. Concerning the two remaining inequalities appearing in the proposition, we begin by developing the loss. For all $k \in \mathcal{K}$

$$\|\hat{s}_k - s\|^2 = \|\hat{s}_k - s_k\|^2 + \|s_k - s\|^2 + 2\langle \hat{s}_k - s_k, s_k - s\rangle \ .$$

Then, for all $x \in \mathbb{X}$

$$\begin{aligned} F_{A,k}(x) - s_k(x) &= \int s(y) \int k(x,z)k(z,y)d\mu(z)d\mu(y) - \int s(z)k(z,x)d\mu(z) \\ &= \int \left(\int s(y)k(z,y)d\mu(y) - s(z) \right) k(x,z)d\mu(z) \\ &= \int \left(s_k(z) - s(z) \right) k(z,x)d\mu(z) \ . \end{aligned}$$

Moreover, since $PF_{A,k} = \|s_k\|^2$, we find

$$\begin{aligned} \langle \hat{s}_k - s_k, s_k - s\rangle &= \int \left(\hat{s}_k(x) \left(s_k(x) - s(x) \right) \right) d\mu(x) + \mathbb{E}\left[s_k(X)\right] - \|s_k\|^2 \\ &= \frac{1}{n}\sum_{i=1}^{n} \int \left(k(x, X_i) \left(s_k(x) - s(x) \right) \right) d\mu(x) + P(s_k - F_{A,k}) \end{aligned}$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left(F_{A,k}(X_i) - s_k(X_i)\right) + P(s_k - F_{A,k})$$
$$= (P_n - P)(F_{A,k} - s_k) \ .$$

This expression motivates us to apply again Proposition 2.1 to this term. We find by (2.1), (3.11) and Cauchy-Schwarz inequality

$$\sup_{x\in\mathbb{X}} |F_{A,k}(x) - s_k(x)| \le \|s - s_k\| \sup_{x\in\mathbb{X}} \int \frac{|s(z) - s_k(z)|}{\|s - s_k\|} k(x,z)d\mu(z)$$
$$\le \|s - s_k\| \sqrt{\sup_{x\in\mathbb{X}} \int k(x,z)^2 d\mu(z)} \le \|s - s_k\| \sqrt{\Upsilon n} \ .$$

Moreover,

$$P\left(F_{A,k} - s_k\right)^2 \le \|s - s_k\|^2 P\left(\int \frac{|s(z) - s_k(z)|}{\|s - s_k\|} k(.,z)d\mu(z)\right)^2$$
$$\le \|s - s_k\|^2 v_k^2 \ .$$

Thus by (3.16), for any $\theta, u > 0$,

$$\sqrt{\frac{2P\left(F_{A,k} - s_k\right)^2 x}{n}} \le \theta \|s - s_k\|^2 + \frac{\left(\Upsilon \vee \sqrt{\Upsilon P\Theta_k}\right)x}{2\theta n}$$
$$\le \theta \|s - s_k\|^2 + \frac{\Upsilon x}{\theta n} \vee \left(\frac{u}{\theta}\frac{P\Theta_k}{n} + \frac{\Upsilon x^2}{16\theta u n}\right) \ .$$

Hence, for any $\theta \in (0,1)$ and $x \ge 1$, taking $u = \theta^2$

$$\sqrt{\frac{2P\left(F_{A,k} - s_k\right)^2 x}{n}} \le \theta\left(\|s - s_k\|^2 + \frac{P\Theta_k}{n}\right) + \square\frac{\Upsilon x^2}{\theta^3 n} \ .$$

By Proposition 2.1, for all θ in $(0,1)$, for all $x > 0$ with probability larger than $1 - 2|\mathcal{K}|e^{-x}$,

$$2\left|\langle \hat{s}_k - s_k, s_k - s\rangle\right| \le 2\sqrt{\frac{2P\left(F_{A,k} - s_k\right)^2 x}{n}} + 2\|s - s_k\| \sqrt{\Upsilon n}\frac{x}{3n}$$
$$\le 3\theta\left(\|s - s_k\|^2 + \frac{P\Theta_k}{n}\right) + \square\frac{\Upsilon x^2}{\theta^3 n} \ .$$

Putting together all of the above, one concludes that for all θ in (0, 1), for all $x > 1$, with probability larger than $1 - 9.4|\mathcal{K}|e^{-x}$

$$\|\hat{s}_k - s\|^2 - \|s_k - s\|^2 \leq 3\theta \|s - s_k\|^2 + (1 + 4\theta)\frac{P\Theta_k}{n} + \square \frac{\Upsilon x^2}{\theta^3 n}$$

and

$$\|\hat{s}_k - s\|^2 - \|s_k - s\|^2 \geq -3\theta \left(\|s - s_k\|^2 + \frac{P\Theta_k}{n} \right) + (1 - \theta)\frac{P\Theta_k}{n} - \square \frac{\Upsilon x^2}{\theta^3 n} \ .$$

Choosing, $\theta = \eta/4$ leads to the second part of the result.

6.3 Proof of Theorem 4.3

It follows from (3.17) (applied with $\theta = \square(\log n)^{-1}$ and $x = \square \log(n \vee |\mathcal{K}_n|)$) and Assumption (4.26) that with probability larger than $1 - \square n^{-2}$ we have for any $k \in \mathcal{K}$ and any $n \geq 2$

$$\left\| \hat{s}_{\hat{k}_n} - s \right\|^2 \leq \left(1 + \frac{\square}{\log n} \right) \|\hat{s}_k - s\|^2 - (1 + \delta') \left(1 + \frac{\square}{\log n} \right) \frac{P\Theta_k}{n} + (1 + \delta) \left(1 + \frac{\square}{\log n} \right) \frac{P\Theta_{\hat{k}_n}}{n} + \square_{\delta,\delta',\Upsilon} \frac{\log(|\mathcal{K}_n| \vee n)^3}{n} \ . \tag{6.31}$$

Applying this inequality with $k = k_{1,n}$ and using Proposition 4.1 with $\eta = \square(\log n)^{-1/3}$ and $x = \square \log(|\mathcal{K}_n| \vee n)$ as a lower bound for $\left\| \hat{s}_{\hat{k}_n} - s \right\|^2$ and as an upper bound for $\left\| \hat{s}_{k_{1,n}} - s \right\|^2$, we obtain asymptotically that with probability larger than $1 - \square n^{-2}$,

$$- \delta(1 + \square_\delta \, \mathrm{o}(1))\frac{P\Theta_{\hat{k}_n}}{n} \leq (1 + \mathrm{o}(1)) \left\| s_{k_{1,n}} - s \right\|^2 - \delta'(1 + \square_{\delta'} \, \mathrm{o}(1))\frac{P\Theta_{k_{1,n}}}{n} + \square_{\delta,\delta',\Upsilon} \frac{\log(|\mathcal{K}_n| \vee n)^3}{n} \ .$$

By Assumption (4.25), $\left\| s_{k_{1,n}} - s \right\|^2 \leq c \, \mathrm{o}(1)\frac{P\Theta_{k_{1,n}}}{n}$ and by (4.22),

$$\frac{(\log(|\mathcal{K}_n| \vee n))^3}{n} \leq c_R c_s \, \mathrm{o}(1)\frac{P\Theta_{k_{1,n}}}{n} \ .$$

This gives (4.27). In addition, starting with the event where (6.31) holds and using Proposition 4.1, we also have with probability larger than $1 - \square n^{-2}$,

$$\left\|\hat{s}_{\hat{k}_n} - s\right\|^2 \leq \left(1 + \frac{\square}{\log n}\right)\left\|\hat{s}_{k_{1,n}} - s\right\|^2 - (1+\delta')\frac{P\Theta_{k_{1,n}}}{n} + (1+\delta)\,(1+\mathrm{o}(1))\left\|\hat{s}_{\hat{k}_n} - s\right\|^2 + \square_{\delta,\delta',\Upsilon}\frac{\log(|\mathcal{K}_n| \vee n)^3}{n} \enspace .$$

Since $\left\|\hat{s}_{k_{1,n}} - s\right\|^2 \simeq \frac{P\Theta_{k_{1,n}}}{n}$, this leads to

$$(-\delta + \square_\delta\ \mathrm{o}(1))\left\|\hat{s}_{\hat{k}} - s\right\|^2 \leq - (\delta' + \square_{\delta',c}\ \mathrm{o}(1))\left\|\hat{s}_{k_{1,n}} - s\right\|^2 + \square_{\delta,\delta',\Upsilon}\frac{\log(|\mathcal{K}_n| \vee n)^3}{n} \enspace .$$

This leads to (4.28) by (4.21).

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Appendix 1: Proofs for the Examples

Computation of the Constant Γ *for the Three Examples*

We have to show for each family $\{k\}_{k\in\mathcal{K}}$ (see (2.8) and (2.1)) that there exists a constant $\Gamma \geq 1$ such that for all $k \in \mathcal{K}$

$$\sup_{x\in\mathbb{X}} |\Theta_k(x)| \leq \Gamma n, \quad \text{and} \quad \sup_{(x,y)\in\mathbb{X}^2} |k(x,y)| \leq \Gamma n \enspace .$$

Example 1 (Projection Kernels) First, notice that from Cauchy-Schwarz inequality we have for all $(x,y) \in \mathbb{X}^2$ $|k_S(x,y)| \leq \sqrt{\chi_{k_S}(x)\chi_{k_S}(y)}$ and by orthonormality, for any $(x,x') \in \mathbb{X}^2$,

$$A_{k_S}(x,x') = \sum_{(i,j)\in\mathcal{I}_S^2} \varphi_i(x)\varphi_j(x')\int_{\mathbb{X}} \varphi_i(y)\varphi_j(y)d\mu(y) = k_S(x,x') \enspace .$$

In particular, for any $x \in \mathbb{X}$, $\Theta_{k_S}(x) = \chi_{k_S}(x)$. Hence, projection kernels satisfy (2.1) for $\Gamma = 1 \vee n^{-1}\sup_{S\in\mathcal{S}}\|\chi_{k_S}\|_\infty$. We conclude by writing

$$\|\chi_{k_S}\|_\infty = \sup_{x\in\mathbb{X}}\sum_{i\in\mathcal{I}_S}\varphi_i(x)^2 = \sup_{\substack{(a_i)_{i\in\mathcal{I}}\ \text{s.t.}\\ \sum_{i\in\mathcal{I}_S} a_i^2 = 1}}\ \sup_{x\in\mathbb{X}}\left(\sum_{i\in\mathcal{I}_S} a_i\varphi_i(x)\right)^2 \enspace .$$

For $f \in S$ we have $\|f\|^2 = \sum_{i\in\mathcal{I}} \langle f, \varphi_i\rangle^2$. Hence with $a_i = \langle f, \varphi_i\rangle$,

$$\|\chi_{k_S}\|_\infty = \sup_{f\in S, \|f\|=1} \|f\|_\infty^2 \quad .$$

Example 2 (Approximation Kernels) First, $\sup_{(x,y)\in\mathbb{X}^2} |k_{K,h}(x,y)| \le \|K\|_\infty / h$. Second, since $K \in L^1$

$$\Theta_{k_{K,h}}(x) = \frac{1}{h^2}\int_{\mathbb{X}} K\left(\frac{x-y}{h}\right)^2 dy = \frac{\|K\|^2}{h} \le \frac{\|K\|_\infty \|K\|_1}{h} \quad .$$

Now $K \in L^1$ and $\int K(u)du = 1$ implies $\|K\|_1 \ge 1$, hence (2.1) holds with $\Gamma = 1$ if one assumes that $h \ge \|K\|_\infty \|K\|_1 / n$.

Example 3 (Weighted Projection Kernels) For all $x \in \mathbb{X}$

$$\Theta_{k_w}(x) = \sum_{i,j=1}^{p} w_i\varphi_i(x) w_j\varphi_j(x) \int_{\mathbb{X}} \varphi_i(y)\varphi_j(y) d\mu(y) = \sum_{i=1}^{p} w_i^2 \varphi_i(x)^2 \quad .$$

From Cauchy-Schwarz inequality, for any $(x,y) \in \mathbb{X}^2$,

$$|k_w(x,y)| \le \sqrt{\Theta_{k_w}(x)\Theta_{k_w}(y)} \quad .$$

We thus find that k_w verifies (2.1) with $\Gamma \ge 1 \vee n^{-1} \sup_{w\in\mathcal{W}} \|\Theta_{k_w}\|_\infty$. Since $w_i \le 1$ we find the announced result which is independent of $\mathcal{W}$.

Proof of Proposition 3.2

Since $\|s_{k_S}\|^2 \le \|s\|^2 \le \|s\|_\infty$, we find that (3.11) only requires $\Upsilon \ge \Gamma(1 + \|s\|_\infty)$. Assumption (3.12) holds: this follows from $\Upsilon \ge \Gamma$ and

$$\mathbb{E}\left[\chi_{k_S}(X)^2\right] \le \|\chi_{k_S}\|_\infty P\chi_{k_S} \le \Gamma n P\Theta_{k_S} \quad .$$

Now for proving Assumption (3.14), we write

$$\begin{aligned}
\mathbb{E}\left[A_{k_S}(X,Y)^2\right] = \mathbb{E}\left[k_S(X,Y)^2\right] &= \int_{\mathbb{X}} \mathbb{E}\left[k_S(X,x)^2\right] s(x) d\mu(x) \\
&\le \|s\|_\infty \sum_{(i,j)\in\mathcal{I}_S^2} \mathbb{E}\left[\varphi_i(X)\varphi_j(X)\right] \int_{\mathbb{X}} \varphi_i(x)\varphi_j(x) d\mu(x) \\
&= \|s\|_\infty P\Theta_{k_S} \le \Upsilon P\Theta_{k_S} \quad .
\end{aligned}$$

In the same way, Assumption (3.15) follows from $\|s\|_\infty \Gamma \leq \Upsilon$. Suppose (3.19) holds with $S = S + S'$ so that the basis $(\varphi_i)_{i\in\mathcal{I}}$ of S' is included in the one $(\varphi_i)_{i\in\mathcal{J}}$ of S. Since $\|\chi_{k_S}\|_\infty \leq \Gamma n$ we have

$$s_{k_S}(x) - s_{k_{S'}}(x) = \sum_{j\in\mathcal{J}\setminus\mathcal{I}} \left(P\varphi_j\right)\varphi_j(x) \leq \sqrt{\sum_{j\in\mathcal{J}\setminus\mathcal{I}} \left(P\varphi_j\right)^2 \sum_{j\in\mathcal{J}\setminus\mathcal{I}} \varphi_j(x)^2}$$
$$\leq \left\|s_{k_S} - s_{k_{S'}}\right\| \|\chi_{k_S}\|_\infty^{1/2} \leq \left\|s_{k_S} - s_{k_{S'}}\right\| \sqrt{\Gamma n} \ .$$

Hence, (3.13) holds in this case. Assuming (3.20) implies that (3.13) holds since

$$\left\|s_{k_S} - s_{k_{S'}}\right\|_\infty \leq \|s_{k_S}\|_\infty + \left\|s_{k_{S'}}\right\|_\infty \leq \Upsilon \ .$$

Finally for (3.16), for any $a \in L^2$,

$$\int_{\mathbb{X}} a(x)k_S(x,y)d\mu(x) = \sum_{i\in\mathcal{I}} \langle a, \varphi_i\rangle\varphi_i(y) = \Pi_S(a) \ .$$

is the orthogonal projection of a onto S. Therefore, $\mathbb{B}_{k_S}$ is the unit ball in S for the L^2-norm and, for any $t \in \mathbb{B}_{k_S}$, $\mathbb{E}\left[t(X)^2\right] \leq \|s\|_\infty \|t\|^2 \leq \|s\|_\infty$.

Proof of Proposition 3.3

First, since $\|K\|_1 \geq 1$

$$\left\|s_{k_{K,h}}\right\|^2 = \int_{\mathbb{X}} \left(\int_{\mathbb{X}} s(y)\frac{1}{h}K\left(\frac{x-y}{h}\right)dy\right)^2 dx$$
$$= \int_{\mathbb{X}} \left(\int_{\mathbb{X}} s(x+hz)K(z)\,dz\right)^2 dx$$
$$\leq \|K\|_1^2 \int_{\mathbb{X}} \left(\int_{\mathbb{X}} s(x+hz)\frac{|K(z)|}{\|K\|_1}dz\right)^2 dx$$
$$\leq \|K\|_1^2 \int_{\mathbb{X}^2} s(x+hz)^2\frac{|K(z)|}{\|K\|_1}dxdz \leq \|s\|_\infty\|K\|_1^2 \ .$$

Hence, Assumption (3.11) holds if $\Upsilon \geq 1 + \|s\|_\infty \|K\|_1^2$. Now, we have

$$P\left(\chi^2_{k_{K,h}}\right) = \frac{K(0)^2}{h^2} = P\Theta_{k_{K,h}}\frac{K(0)^2}{\|K\|^2 h} \leq nP\Theta_{k_{K,h}}\frac{K(0)^2}{\|K\|^2\|K\|_\infty\|K\|_1} \ ,$$

so it is sufficient to have $\Upsilon \geq |K(0)| / \|K\|^2$ (since $|K(0)| \leq \|K\|_\infty$) to ensure (3.12). Moreover, for any $h \in \mathcal{H}$ and any $x \in \mathbb{X}$,

$$s_{k_{K,h}}(x) = \int_{\mathbb{X}} s(y) \frac{1}{h} K\left(\frac{x-y}{h}\right) dy = \int_{\mathbb{X}} s(x+zh) K(z) dz \leq \|s\|_\infty \|K\|_1 \quad .$$

Therefore, Assumption (3.13) holds for $\Upsilon \geq 2 \|s\|_\infty \|K\|_1$. Then on one hand

$$\begin{aligned}
\left|A_{k_{K,h}}(x,y)\right| &\leq \frac{1}{h^2} \int_{\mathbb{X}} \left| K\left(\frac{x-z}{h}\right) K\left(\frac{y-z}{h}\right) \right| dz \\
&\leq \frac{1}{h} \int_{\mathbb{X}} \left| K\left(\frac{x-y}{h} - u\right) K(u) \right| du \\
&\leq \frac{\|K\|^2}{h} \wedge \frac{\|K\|_\infty \|K\|_1}{h} \leq P\Theta_{k_{K,h}} \wedge n \quad .
\end{aligned}$$

And on the other hand

$$\begin{aligned}
\mathbb{E}\left[\left|A_{k_{K,h}}(X,x)\right|\right] &\leq \frac{1}{h} \int_{\mathbb{X}^2} \left| K\left(\frac{x-y}{h} - u\right) K(u) \right| du \, s(y) dy \\
&= \int_{\mathbb{X}^2} |K(v) K(u)| \, s(x + h(v-u)) du dv \leq \|s\|_\infty \|K\|_1^2 \quad .
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sup_{x \in \mathbb{X}} \mathbb{E}\left[A_{k_{K,h}}(X,x)^2\right] &\leq \sup_{(x,y) \in \mathbb{X}^2} \left|A_{k_{K,h}}(x,y)\right| \sup_{x \in \mathbb{X}} \mathbb{E}\left[\left|A_{k_{K,h}}(X,x)\right|\right] \\
&\leq \left(P\Theta_{k_{K,h}} \wedge n\right) \|s\|_\infty \|K\|_1^2 \quad ,
\end{aligned}$$

and $\mathbb{E}\left[A_{k_{K,h}}(X,Y)^2\right] \leq \sup_{x \in \mathbb{X}} \mathbb{E}\left[A_{k_{K,h}}(X,x)^2\right] \leq \|s\|_\infty \|K\|_1^2 P\Theta_{k_{K,h}}$. Hence Assumption (3.14) and (3.15) hold when $\Upsilon \geq \|s\|_\infty \|K\|_1^2$. Finally let us prove that Assumption (3.16) is satisfied. Let $t \in \mathbb{B}_{k_{K,h}}$ and $a \in L^2$ be such that $\|a\| = 1$ and $t(y) = \int_{\mathbb{X}} a(x) \frac{1}{h} K\left(\frac{x-y}{h}\right) dx$ for all $y \in \mathbb{X}$. Then the following follows from Cauchy-Schwarz inequality

$$t(y) \leq \frac{1}{h} \sqrt{\int_{\mathbb{X}} a(x)^2 dx} \sqrt{\int_{\mathbb{X}} K\left(\frac{x-y}{h}\right)^2 dx} \leq \frac{\|K\|}{\sqrt{h}} \quad .$$

Thus for any $t \in \mathbb{B}_{k_{K,h}}$

$$Pt^2 \leq \|t\|_\infty \langle |t|, s \rangle \leq \frac{\|K\|}{\sqrt{h}} \|s\| = \|s\| \sqrt{P\Theta_{k_{K,h}}} \leq \sqrt{\Upsilon P\Theta_{k_{K,h}}} \quad .$$

We conclude that all the assumptions hold if

$$\Upsilon \geq \left(|K(0)| / \|K\|^2 \right) \vee \left(1 + 2 \|s\|_\infty \|K\|_1^2 \right) \ .$$

Proof of Proposition 3.4

Let us define for convenience $\Phi(x) := \sum_{i=1}^p \varphi_i(x)^2$, so $\Gamma \geq 1 \vee n^{-1} \|\Phi\|_\infty$. Then we have for these kernels: $\Phi(x) \geq \chi_{k_w}(x) \geq \Theta_{k_w}(x)$ for all $x \in \mathbb{X}$. Moreover, denoting by Πs the orthogonal projection of s onto the linear span of $(\varphi_i)_{i=1,\ldots,p}$,

$$\|s_{k_w}\|^2 = \sum_{i=1}^p w_i^2 (P\varphi_i)^2 \leq \|\Pi s\|^2 \leq \|s\|^2 \leq \|s\|_\infty \ .$$

Assumption (3.11) holds for this family if $\Upsilon \geq \Gamma(1 + \|s\|_\infty)$. We prove in what follows that all the remaining assumptions are valid using only (2.1) and (3.11).

First, it follows from Cauchy-Schwarz inequality that, for any $x \in \mathbb{X}$, $\chi_{k_w}(x)^2 \leq \Phi(x)\Theta_{k_w}(x)$. Assumption (3.12) is then automatically satisfied from the definition of Γ

$$\mathbb{E}\left[\chi_{k_w}(X)^2 \right] \leq \|\Phi\|_\infty P\Theta_{k_w} \leq \Gamma n P\Theta_{k_w} \ .$$

Now let w and w' be any two vectors in $[0, 1]^p$, we have

$$s_{k_w} = \sum_{i=1}^p w_i (P\varphi_i)\varphi_i, \qquad s_{k_w} - s_{k_{w'}} = \sum_{i=1}^p (w_i - w_i') (P\varphi_i) \varphi_i \ .$$

Hence $\left\| s_{k_w} - s_{k_{w'}} \right\|^2 = \sum_{i=1}^p (w_i - w_i')^2 (P\varphi_i)^2$ and, by Cauchy-Schwarz inequality, for any $x \in \mathbb{X}$,

$$\left| s_{k_w}(x) - s_{k_{w'}}(x) \right| \leq \left\| s_{k_w} - s_{k_{w'}} \right\| \sqrt{\Phi(x)} \leq \left\| s_{k_w} - s_{k_{w'}} \right\| \sqrt{\Gamma n} \ .$$

Assumption (3.13) follows using (3.11). Concerning Assumptions (3.14) and (3.15), let us first notice that by orthonormality, for any $(x, x') \in \mathbb{X}^2$,

$$A_{k_w}(x, x') = \sum_{i=1}^p w_i^2 \varphi_i(x)\varphi_i(x') \ .$$

Therefore, Assumption (3.15) holds since

$$\begin{aligned}\mathbb{E}\left[A_{k_w}(X,x)^2\right] &= \int_{\mathbb{X}}\left(\sum_{i=1}^{p} w_i^2\varphi_i(y)\varphi_i(x)\right)^2 s(y)d\mu(y)\\ &\le \|s\|_\infty \sum_{1\le i,j\le p} w_i^2 w_j^2 \varphi_i(x)\varphi_j(x)\int_{\mathbb{X}}\varphi_i(y)\varphi_j(y)d\mu(y)\\ &= \|s\|_\infty \sum_{i=1}^{p} w_i^4\varphi_i(x)^2 \le \|s\|_\infty \Phi(x) \le \|s\|_\infty \Gamma n \ .\end{aligned}$$

Assumption (3.14) also holds from similar computations:

$$\begin{aligned}\mathbb{E}\left[A_{k_w}(X,Y)^2\right] &= \int_{\mathbb{X}}\mathbb{E}\left[\left(\sum_{i=1}^{p} w_i^2\varphi_i(X)\varphi_i(x)\right)^2\right] s(x)d\mu(x)\\ &\le \|s\|_\infty \sum_{1\le i,j\le p} w_i^2 w_j^2 \mathbb{E}\left[\varphi_i(X)\varphi_j(X)\right]\int_{\mathbb{X}}\varphi_i(x)\varphi_j(x)d\mu(x)\\ &\le \|s\|_\infty P\Theta_{k_w} \ .\end{aligned}$$

We finish with the proof of (3.16). Let us prove that $\mathbb{B}_{k_w} = \mathcal{E}_{k_w}$, where

$$\mathcal{E}_{k_w} = \left\{ t = \sum_{i=1}^{p} w_i t_i \varphi_i, \ \text{ s.t. } \sum_{i=1}^{p} t_i^2 \le 1 \right\} \ .$$

First, notice that any $t \in \mathbb{B}_{k_w}$ can be written

$$\int_{\mathbb{X}} a(x) k_w(x,y) d\mu(x) = \sum_{i=1}^{p} w_i \langle a, \varphi_i\rangle \varphi_i(y) \ .$$

Then, consider some $t \in \mathcal{E}_{k_w}$. By definition, there exists a collection $(t_i)_{i=1,\dots,p}$ such that $t = \sum_{i=1}^{p} w_i t_i \varphi_i$, and $\sum_{i=1}^{p} t_i^2 \le 1$. If $a = \sum_{i=1}^{p} t_i\varphi_i$, $\|a\|^2 = \sum_{i=1}^{p} t_i^2 \le 1$ and $\langle a, \varphi_i\rangle = t_i$, hence $t \in \mathbb{B}_{k_w}$. Conversely, for $t \in \mathbb{B}_{k_w}$, there exists some function $a \in L^2$ such that $\|a\|^2 \le 1$, and $t = \sum_{i=1}^{p} w_i\langle a, \varphi_i\rangle\varphi_i$. Since $(\varphi_i)_{i=1,\dots,p}$ is an orthonormal system, one can take $a = \sum_{i=1}^{p}\langle a, \varphi_i\rangle\varphi_i$. With $t_i = \langle a, \varphi_i\rangle$, we find $\|a\|^2 = \sum_{i=1}^{p} t_i^2$ and $t \in \mathcal{E}_{k_w}$. For any $t \in \mathbb{B}_{k_w} = \mathcal{E}_{k_w}$, $\|t\|^2 = \sum_{i=1}^{p} w_i^2 t_i^2 \le \sum_{i=1}^{p} t_i^2 \le 1$. Hence $Pt^2 \le \|s\|_\infty \|t\|^2 \le \|s\|_\infty$.

Appendix 2: Concentration of the Residual Terms

The following proposition gathers the concentration bounds of the remaining terms appearing in (6.30).

Proposition B.1 *Let* $\{k\}_{k\in\mathcal{K}}$ *denote a finite collection of kernels satisfying* (2.1) *and suppose that Assumptions* (3.11)–(3.13) *hold. Then*

$$\forall \theta \in (0,1), \qquad 2\frac{P(s_{\hat{k}} - s_k)}{n} \le \theta \left\| s - s_{\hat{k}} \right\|^2 + \theta \left\| s - s_k \right\|^2 + \frac{2\Upsilon}{\theta n} \ . \tag{6.32}$$

For any $x \ge 1$*, with probability larger than* $1 - 2\left|\mathcal{K}\right|^2 e^{-x}$*, for any* $(k, k') \in \mathcal{K}^2$*, for any* $\theta \in (0,1)$*,*

$$\left|2(P_n - P)(s_k - s_{k'})\right| \le \theta \left(\left\| s - s_{k'} \right\|^2 + \left\| s - s_k \right\|^2 \right) + \square \frac{\Upsilon x^2}{\theta n} \ . \tag{6.33}$$

For any $x \ge 1$*, with probability larger than* $1 - 2\left|\mathcal{K}\right| e^{-x}$*, for any* $k \in \mathcal{K}$*,*

$$\forall \theta \in (0,1), \qquad \left|2(P_n - P)\chi_k\right| \le \theta P\Theta_k + \square \frac{\Upsilon x}{\theta} \ . \tag{6.34}$$

For any $x \ge 1$*, with probability larger than* $1 - 5.4\left|\mathcal{K}\right| e^{-x}$*, for any* $k \in \mathcal{K}$*,*

$$\forall \theta \in (0,1), \qquad \frac{2\left|U_k\right|}{n^2} \le \theta \frac{P\Theta_k}{n} + \square \frac{\Upsilon x^2}{\theta n} \ . \tag{6.35}$$

Proof First for (6.32), notice that, by (3.13), for any $\theta \in (0,1)$

$$\begin{aligned} 2\frac{P(s_{\hat{k}} - s_k)}{n} &\le 2\frac{\left\| s_{\hat{k}} - s_k \right\|_\infty}{n} \le \frac{2}{n}\left(\Upsilon \vee \left(\frac{\theta}{4} n \left\| s_k - s_{\hat{k}} \right\|^2 + \frac{\Upsilon}{\theta} \right) \right) \\ &\le \frac{\theta}{2}\left\| s_k - s_{\hat{k}} \right\|^2 + \frac{2\Upsilon}{\theta n} \le \theta \left\| s - s_{\hat{k}} \right\|^2 + \theta \left\| s - s_k \right\|^2 + \frac{2\Upsilon}{\theta n} \ . \end{aligned}$$

Then, by Proposition 2.1, with probability larger than $1 - \left|\mathcal{K}\right|^2 e^{-x}$,

$$\text{for any } (k, k') \in \mathcal{K}^2, \quad (P_n - P)(s_k - s_{k'}) \le \sqrt{\frac{2P\left(s_k - s_{k'}\right)^2 x}{n}} + \frac{\left\| s_k - s_{k'} \right\|_\infty x}{3n} \ .$$

Since by (3.11) $P\left(s_k - s_{k'}\right)^2 \le \left\| s \right\|_\infty \left\| s_k - s_{k'} \right\|^2 \le \Upsilon \left\| s_k - s_{k'} \right\|^2$,

$$\sqrt{\frac{2P\left(s_k - s_{k'}\right)^2 x}{n}} \le \frac{\theta}{4}\left\| s_k - s_{k'} \right\|^2 + \frac{2\Upsilon x}{\theta n} \ .$$

Moreover, by (3.13) $\frac{\|s_k - s_{k'}\|_\infty x}{3n} \leq \frac{\theta}{4}\|s_k - s_{k'}\|^2 + \square\frac{\Upsilon x^2}{\theta n}$. Hence, for $x \geq 1$, with probability larger than $1 - |\mathcal{K}|^2 e^{-x}$

$$(P_n - P)(s_k - s_{k'}) \leq \frac{\theta}{2}\|s_k - s_{k'}\|^2 + \square\frac{\Upsilon x^2}{\theta n}$$
$$\leq \theta\left(\|s - s_{k'}\|^2 + \|s - s_k\|^2\right) + \square\frac{\Upsilon x^2}{\theta n} \quad ,$$

which gives (6.33). Now, using again Proposition 2.1, with probability larger than $1 - |\mathcal{K}| e^{-x}$, for any $k \in \mathcal{K}$,

$$(P_n - P)\chi_k \leq \sqrt{\frac{2P(\chi_k)^2 x}{n}} + \frac{\|\chi_k\|_\infty x}{3n} \quad .$$

By (2.1) and (3.11), for any $k \in \mathcal{K}$, $\|\chi_k\|_\infty \leq \sup_{(x,y)\in\mathbb{X}^2}|k(x,y)| \leq \Gamma n \leq \Upsilon n$.

Concerning (6.34), we get by (3.12), $P\chi_k^2 \leq \Upsilon n P\Theta_k$, hence, for any $x \geq 1$ we have with probability larger than $1 - |\mathcal{K}| e^{-x}$

$$(P_n - P)\chi_k \leq \theta P\Theta_k + \left(\frac{1}{3} + \frac{1}{2\theta}\right)\Upsilon x \quad .$$

For (6.35), we apply Proposition 2.2 to obtain with probability larger than $1 - 2.7|\mathcal{K}| e^{-x}$, for any $k \in \mathcal{K}$,

$$\frac{U_k}{n^2} \leq \frac{\square}{n^2}\left(C\sqrt{x} + Dx + Bx^{3/2} + Ax^2\right) \quad ,$$

where A, B, C, D are defined accordingly to Proposition 2.2. Let us evaluate all these terms. First, $A \leq 4\sup_{(x,y)\in\mathbb{X}^2}|k(x,y)| \leq 4\Upsilon n$ by (2.1) and (3.11). Next, $C^2 \leq \square n^2\mathbb{E}\left[k(X,Y)^2\right] \leq \square n^2\|s\|_\infty P\Theta_k \leq \square n^2\Upsilon P\Theta_k$.

Using (2.1), we find $B^2 \leq 4n\sup_{x\in\mathbb{X}}\int k(x,y)^2 s(y)d\mu(y) \leq 4n\|s\|_\infty\Gamma$.

By (3.11), we consequently have $B^2 \leq 4\Upsilon n$. Finally, using Cauchy-Schwarz inequality and proceeding as for C^2,

$$\mathbb{E}\left[\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} a_i(X_i)b_j(X_j)k(X_i,X_j)\right] \leq n\sqrt{\mathbb{E}\left[k(X,Y)^2\right]} \leq n\sqrt{\Upsilon P\Theta_k} \quad .$$

Hence, $D \leq n\sqrt{\Upsilon P\Theta_k}$ which gives (6.35).

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