# Chapter 8
# A Review on Rough Sets and Possible World Semantics for Modal Logics

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*Dedicated to Jair Minoro Abe for his 60th birthday*

**Abstract** It is well known that rough set-based approximations of concepts and possible world semantics of modal logics are closely related. In this chapter, we review the relationships between two types of possible world semantic models, i.e., Kripke model and measure-based model, and two variation of rough sets, i.e., Pawlak's rough set and variable precision rough set.

**Keywords** Modal logic · Possible world semantics · Kripke model · Measure-based model · Rough set · Variable precision rough set

## 8.1 Introduction

Rough set theory, proposed by Pawlak [14, 15], provides a theoretical basis of set-based approximations of concepts. Lower and upper approximations by rough set theory are closely related with possible world semantics, i.e., lower approximation and necessity, and upper approximation and possibility. In this chapter, we review the relationships between two types of possible world semantic models, i.e., Kripke

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model [6] and measure-based model [11, 12], and two variations of rough sets, i.e., Pawlak's rough set [14, 15] and variable precision rough set [26].

The reminder of this chapter is structured as follows. In Sect. 8.2, language and possible world semantics for modal logics are briefly reviewed. In Sect. 8.3, lower and upper approximations in rough sets and variable precision rough sets are discussed. In Sect. 8.4, connections between possible world semantics and rough sets are discussed. Related works about rough set-based semantics for modal logics are mentioned in Sect. 8.5, and finally, we give some conclusion in Sect. 8.6.

## 8.2 Modal Logics

In this section, we review possible world semantics of modal logics. The contents of this section is mainly based on [2].

### *8.2.1 Language*

Propositional modal logic (for short, modal logic) is an extension of classical propositional logic by adding two unary operators $\Box$ and $\Diamond$, called modal operators, that express the statements $\Box p$ ($p$ is necessary) and $\Diamond p$ ($p$ is possible) for any proposition $p$.

Suppose $\mathcal{P} = \{\mathsf{p}_1, \ldots, \mathsf{p}_n(, \ldots)\}$ is a set of finite or countably infinite atomic sentences, $\top$ (truth) and $\bot$ (falsity) are constant sentences, $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (conditionality), $\leftrightarrow$ (biconditionality), and $\neg$ (negation) are logical connectives, and $\Box$ (necessity) and $\Diamond$ (possibility) are modal operators. Let $\mathcal{L}_{\mathrm{ML}}(\mathcal{P})$ be the set of sentences of modal logic constructed from the above symbols by the following construction rules:

$$\mathsf{p} \in \mathcal{P} \Rightarrow \mathsf{p} \in \mathcal{L}_{\mathrm{ML}}(\mathcal{P}),\ \top, \bot \in \mathcal{L}_{\mathrm{ML}}(\mathcal{P}),$$
$$p \in \mathcal{L}_{\mathrm{ML}}(\mathcal{P}) \Rightarrow \neg p, \Box p, \Diamond p \in \mathcal{L}_{\mathrm{ML}}(\mathcal{P}),$$
$$p, q \in \mathcal{L}_{\mathrm{ML}}(\mathcal{P}) \Rightarrow p \wedge q, p \vee q, p \rightarrow q, p \leftrightarrow q \in \mathcal{L}_{\mathrm{ML}}(\mathcal{P}).$$

We say that a sentence is a modal sentence if the sentence contains at least one modal operator, and otherwise, we say the sentence is a non-modal sentence.

### *8.2.2 Possible World Semantics for Modal Logics*

#### 8.2.2.1 Kripke Model

In this section, we consider possible world semantics to interpret sentences used in modal logic. A *Kripke model*, one of the most popular frameworks of possible world semantics, is the following triple:

$$\mathcal{M} = (U, R, v), \tag{8.1}$$

where $U$ ($\neq \emptyset$) is the set of possible worlds, $R$ is a binary relation on $U$ called an accessibility relation, and $v : \mathcal{P} \times U \to \{0, 1\}$ is a valuation function that assigns a truth value to each atomic sentence $\mathsf{p} \in \mathcal{P}$ at each world $w \in U$. We define that an atomic sentence $\mathsf{p}$ is true at a possible world $x$ by the given Kripke model $\mathcal{M}$ if and only if we have $v(\mathsf{p}, x) = 1$. We say that a Kripke model is finite if its set of possible worlds is a finite set.

We denote $\mathcal{M}, x \models p$ to mean that the sentence $p$ is true at the possible world $x \in U$ by the Kripke model $\mathcal{M}$. Otherwise, we denote $\mathcal{M}, x \not\models x$ to mean that $p$ is false at $x \in U$. Similar to classical propositional logic, for any non-modal sentences $p, q \in \mathcal{L}_{ML}(\mathcal{P})$ and any possible world $x \in U$, interpretation of non-modal sentences by the Kripke model $\mathcal{M}$ is defined as follows:

$$\begin{aligned}
\mathcal{M}, x \models \neg p &\iff \mathcal{M}, x \not\models p,\\
\mathcal{M}, x \models p \wedge q &\iff \mathcal{M}, x \models p \text{ and } \mathcal{M}, x \models q,\\
\mathcal{M}, x \models p \vee q &\iff \mathcal{M}, x \models p \text{ or } \mathcal{M}, x \models q,\\
\mathcal{M}, x \models p \to q &\iff \mathcal{M}, x \not\models p \text{ or } \mathcal{M}, x \models q,\\
\mathcal{M}, x \models p \leftrightarrow q &\iff \mathcal{M}, x \models p \to q \text{ and } \mathcal{M}, x \models q \to p.
\end{aligned}$$

An *accessibility relation* is used to interpret modal sentences by a Kripke model; a modal sentence $\Box p$ is true at a possible world $x \in U$ by a Kripke model $\mathcal{M}$ if and only if $p$ is true at every possible world $y$ that is accessible from $x$ in $\mathcal{M}$. On the other hand, $\Diamond p$ is true at $x$ if and only if there is at least one possible world $y$ that is accessible from $x$ and $p$ is true at $y$. Formally, interpretation of modal sentences are defined as follows:

$$\mathcal{M}, x \models \Box p \overset{\text{def}}{\iff} \forall y \in U(xRy \Rightarrow \mathcal{M}, y \models p), \tag{8.2}$$

$$\mathcal{M}, x \models \Diamond p \overset{\text{def}}{\iff} \exists y \in U(xRy \text{ and } \mathcal{M}, y \models p). \tag{8.3}$$

For any sentence $p \in \mathcal{L}_{\mathrm{ML}}(\mathcal{P})$, the truth set is the set of possible worlds at which $p$ are true by the Kripke model $\mathcal{M}$, and the truth set is defined as follows:

$$\|p\|^{\mathcal{M}} \overset{\text{def}}{=} \{x \in U \mid \mathcal{M}, x \models p\}. \tag{8.4}$$

We say that a sentence $p$ is true in a Kripke model $\mathcal{M}$ if and only if $p$ is true at every possible world in $\mathcal{M}$. We denote $\mathcal{M} \models p$ if $p$ is true in $\mathcal{M}$.

It is well known that various properties of accessible relations correspond to axiom schemas of modal systems (for details, see [2]). Table 8.1 describes the correspondence between axiom schemas in modal systems and properties of accessibility relations in Kripke models. For example, the modal system *S5* is sound and complete with respect to the class of all Kripke models that the accessibility relations are equivalence relations. The modal system *S5* consists of all inference rules and axiom

**Table 8.1** Correspondence relationship among axiom schima and accessibility relation

| Axiom schima | | Accessibility relation |
|---|---|---|
| **Df**$\Diamond$. | $\Diamond p \leftrightarrow \neg\Box\neg p$ | (No condition) |
| **M**. | $\Box(p \wedge q) \rightarrow (\Box p \wedge \Box q)$ | (No condition) |
| **C**. | $(\Box p \wedge \Box q) \rightarrow \Box(p \wedge q)$ | (No condition) |
| **N**. | $\Box\top$ | (No condition) |
| **K**. | $\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$ | (No condition) |
| **D**. | $\Box p \rightarrow \Diamond p$ | Serial |
| **P**. | $\neg\Box\bot$ | Serial |
| **T**. | $\Box p \rightarrow p$ | Reflexive |
| **B**. | $p \rightarrow \Box\Diamond p$ | Symmetric |
| **4**. | $\Box p \rightarrow \Box\Box p$ | Transitive |
| **5**. | $\Diamond p \rightarrow \Box\Diamond p$ | Euclidian |

schemas of propositional logic, the axiom schemas **Df**$\Diamond$, **K**, **T** and **5** in Table 8.1, and the following inference rule:

**RN**. from $p$ infer $\Box p$.

#### 8.2.2.2 Measure-Based Semantics

Murai et al. [11, 12] introduced *measure-based semantics* of modal logics. In the measure-based semantics, fuzzy measures assigned to each possible worlds are used to interpret modal sentences.

Let $U$ is a non-empty set. A function $\mu : 2^U \rightarrow [0, 1]$ is called a *fuzzy measure* on $U$ if the function $\mu$ satisfies the following conditions:

1. $\mu(U) = 1$,
2. $\mu(\emptyset) = 0$, and
3. $\forall X, Y \subseteq U,\ X \subseteq Y \Rightarrow \mu(X) \leq \mu(Y)$.

Formally, a fuzzy measure model $\mathcal{M}_\mu$ is the following triple:

$$\mathcal{M}_\mu = (U, \{\mu_x\}_{x \in U}, v), \tag{8.5}$$

where $U$ is a set of possible worlds, and $v$ is a valuation. $\{\mu_x\}_{x \in U}$ is a class of fuzzy measures $\mu_x$ assigned to all possible worlds $x \in U$.

In measure-based semantics of modal logic, each degree $\alpha \in (0, 1]$ of fuzzy measures corresponds to a modal operator $\Box_\alpha$ [11, 12]. In this paper, however, we fix a degree $\alpha$ and consider *$\alpha$-level fuzzy measure model*.

Similar to the case of Kripke models, $\mathcal{M}_\mu, x \models p$ indicates that the sentence $p$ is true at the possible world $x \in U$ by the $\alpha$-level fuzzy measure model $\mathcal{M}_\mu$. Interpretation of non-modal sentences is identical to that in Kripke models. On the other hand, to define the truth value of modal sentences at each world $x \in U$ in the $\alpha$-level fuzzy measure model $M_\mu$, we use the fuzzy measure $\mu_x$ assigned to the world $x$ instead of accessibility relations. Interpretation of modal sentences $\Box p$ at a world $x$ is defined as follows:

$$\mathcal{M}_\mu, x \models \Box p \overset{\text{def}}{\Longleftrightarrow} \mu_x \left(\|p\|^{\mathcal{M}_\mu}\right) \geq \alpha, \tag{8.6}$$

where $\mu_x$ is the fuzzy measure assigned to $x$. By this definition, interpretation of modal sentences $\Diamond p$ is obtained by dual fuzzy measures as follows:

$$\mathcal{M}_\mu, x \models \Diamond p \Longleftrightarrow \mu_x^* \left(\|p\|^{\mathcal{M}_\mu}\right) > 1 - \alpha, \tag{8.7}$$

where the dual fuzzy measure $\mu_x^*$ of the assigned fuzzy measure $\mu_x$ is defined as $\mu_x^*(X) \overset{\text{def}}{=} 1 - \mu_x (X^c)$ for any $X \subseteq U$.

Note that the modal system *EMNP* is sound and complete with respect to the class of all $\alpha$-level fuzzy measure models [11, 12], where the system *EMNP* consists of all inference rules and axiom schemas of propositional logic and the axiom schemas **Df**$\Diamond$, **M**, **N**, and **P** in Table 8.1 and the following inference rule:

**RE**. from $p \leftrightarrow q$ infer $\Box p \leftrightarrow \Box q$.

## 8.3 Rough Sets

### *8.3.1 Pawlak's Rough Set*

In this section, we review theoretical basis of Pawlak's rough set theory, in particular, lower and upper approximation of concepts. The contents of this section is based on [15, 17].

Let $U$ be a non-empty and finite set of objects called the universe of discourse, and $E$ be an equivalence relation on $U$ called an indiscernibility relation. The ordered pair $(U, E)$ is called a *Pawlak approximation space* that is the basis of approximation in rough set theory.

For any element $x \in U$, the *equivalence class* of $x$ with respect to $E$ is defined as follows:

$$[x]_E \overset{\text{def}}{=} \{y \in U \mid xEy\}. \tag{8.8}$$

The equivalence class $[x]_E$ is the set of objects that are not discernible from $x$ with respect to $E$. The *quotient set* $U/E \overset{\text{def}}{=} \{[x]_E \mid x \in U\}$ provides a partition of $U$. According to Pawlak [15], any set $X \subseteq U$ represents a concept, and a set of concepts is called knowledge about $U$. Thus, the quotient set $U/E$ is called *$E$-basic knowledge about $U$* [15].

For any set of objects $X \subseteq U$, the *lower approximation* $\underline{E}(X)$ of $X$ and the *upper approximation* $\overline{E}(X)$ of $X$ by the equivalence relation $E$ are defined as follows, respectively:

$$\underline{E}(X) \stackrel{\text{def}}{=} \{x \in U \mid [x]_E \subseteq X\}, \tag{8.9}$$

$$\overline{E}(X) \stackrel{\text{def}}{=} \{x \in U \mid [x]_E \cap X \neq \emptyset\}. \tag{8.10}$$

The lower approximation $\underline{E}(X)$ of $X$ is the set of objects that are certainly included in $X$. On the other hand, the upper approximation $\overline{E}(X)$ of $X$ is the set of objects that may be included in $X$.

If we have $\underline{E}(X) = X = \overline{E}(X)$, we say that $X$ is $E$-definable, and otherwise, if we have $\underline{E}(X) \subset X \subset \overline{E}(X)$, we say that $X$ is $E$-rough. The concept $X$ is $E$-definable means that we can denote $X$ correctly by using background knowledge by $E$. On the other hand, $X$ is $E$-rough means that we can not denote the concept correctly based on the background knowledge.

### *8.3.2 Variable Precision Rough Set*

Variable precision rough set models (for short, VPRS) proposed by Ziarko [26] is one extension of Pawlak's rough set theory that provides a theoretical basis to treat probabilistic or inconsistent information in the framework of rough sets.

VPRS is based on the majority inclusion relation. Let $X, Y \subseteq U$ be any subsets of $U$. The majority inclusion relation is defined by the following measure $c(X, Y)$ of the relative degree of misclassification of $X$ with respect to $Y$:

$$c(X, Y) \stackrel{\text{def}}{=} \begin{cases} 1 - \dfrac{|X \cap Y|}{|X|}, & \text{if } X \neq \emptyset, \\ 0, & \text{otherwise,} \end{cases} \tag{8.11}$$

where $|X|$ represents the cardinality of the set $X$. It is easily confirmed that $X \subseteq Y$ holds if and only if $c(X, Y) = 0$ holds.

Formally, the majority inclusion relation $\stackrel{\beta}{\subseteq}$ with a fixed *precision* $\beta \in [0, 0.5)$ is defined using the relative degree of misclassification as follows:

$$X \stackrel{\beta}{\subseteq} Y \stackrel{\text{def}}{\Longleftrightarrow} c(X, Y) \leq \beta, \tag{8.12}$$

where the precision $\beta$ provides the limit of permissible misclassification [26].

Let $(U, E)$ be a Pawlak approximation space, $X \subseteq U$ be any set of objects, and the degree $\beta \in [0, 0.5)$ be a precision. The *$\beta$-lower approximation* $\underline{E}_\beta(X)$ and the *$\beta$-upper approximation* $\overline{E}_\beta(X)$ of $X$ are defined as follows:

$$\underline{E}_\beta(X) \stackrel{\text{def}}{=} \left\{ x \in U \,\middle|\, [x]_E \stackrel{\beta}{\subseteq} X \right\} \tag{8.13}$$

$$= \left\{ x \in U \,\middle|\, c\left([x]_E, X\right) \leq \beta \right\}, \tag{8.14}$$

$$\overline{E}_\beta(X) \stackrel{\text{def}}{=} \left\{ x \in U \,\middle|\, c\left([x]_E, X\right) < 1 - \beta \right\}. \tag{8.15}$$

As mentioned previously, the precision $\beta$ represents the threshold degree of misclassification of elements in the equivalence class $[x]_E$ to the set $X$. Thus, in VPRS, misclassification of elements is allowed if the ratio of misclassification is less than $\beta$. Note that the $\beta$-lower and $\beta$-upper approximations with $\beta = 0$ correspond to Pawlak's lower and upper approximations [26].

### *8.3.3 Properties of Lower and Upper Approximations*

Lower and upper approximations of Pawlak's rough set and VPRS satisfy various properties. Table 8.2 represents some properties of $\beta$-lower and upper approximations. The symbol "✓" appeared in Table 8.2 means, for each property in Table 8.2, whether the property is satisfied in the case of $\beta = 0$ and $0 < \beta < 0.5$, respectively. For example, it is easily observed that the property **C.** $\underline{E}_\beta(X) \cap \underline{E}_\beta(Y) \subseteq \underline{E}_\beta(X \cap Y)$ does not hold in VPRS with the precision $0 < \beta < 0.5$. Note that symbols assigned to properties like **C.** correspond to axiom schemas in modal logic (for detail, see [2]).

**Table 8.2** Some properties of $\beta$-lower and upper approximations [7]

| Property | | $\beta = 0$ | $0 < \beta < 0.5$ |
|---|---|---|---|
| **Df**$\Diamond$. | $\overline{E}_\beta(X) = \underline{E}_\beta(X^c)^c$ | ✓ | ✓ |
| **M**. | $\underline{E}_\beta(X \cap Y) \subseteq \underline{E}_\beta(X) \cap \underline{E}_\beta(Y)$ | ✓ | ✓ |
| **C**. | $\underline{E}_\beta(X) \cap \underline{E}_\beta(Y) \subseteq \underline{E}_\beta(X \cap Y)$ | ✓ | |
| **N**. | $\underline{E}_\beta(U) = U$ | ✓ | ✓ |
| **K**. | $\underline{E}_\beta(X^c \cup Y) \subseteq \left(\underline{E}_\beta(X)^c \cup \underline{E}_\beta(Y)\right)$ | ✓ | |
| **D**. | $\underline{E}_\beta(X) \subseteq \overline{E}_\beta(X)$ | ✓ | ✓ |
| **P**. | $\underline{E}_\beta(\emptyset) = \emptyset$ | ✓ | ✓ |
| **T**. | $\underline{E}_\beta(X) \subseteq X$ | ✓ | |
| **B**. | $X \subseteq \underline{E}_\beta(\overline{E}_\beta(X))$ | ✓ | |
| **4**. | $\underline{E}_\beta(X) \subseteq \underline{E}_\beta(\underline{E}_\beta(X))$ | ✓ | ✓ |
| **5**. | $\overline{E}_\beta(X) \subseteq \underline{E}_\beta(\overline{E}_\beta(X))$ | ✓ | ✓ |

## 8.4 Connections Between Rough Sets and Modal Logics

### 8.4.1 Pawlak Approximation Spaces as Kripke Models

As we reviewed in Sect. 8.3.1, every Pawlak approximation space $(U, E)$ consists of a finite set $U$ of objects and an equivalence relation $E$ on $U$. Hence, by regarding each object $x \in U$ as a possible world and the equivalenced relation $E$ as an accessibility relation, and by adding a valuation function $v : \mathcal{P} \times U \to \{0, 1\}$, the structure $\mathcal{M} = (U, E, v)$ induced from the Pawlak approximation space $(U, E)$ is regarded as a special case of Kripke model.

For every Kripke model $(U, E, v)$ induced from a Pawlak approximation space $(U, E)$, it is easily confirmed that the truth conditions of modal sentence $\Box p$ by (8.2) is reformulated as follows:

$$\begin{aligned} \mathcal{M}, x \models \Box p &\iff \forall y \in U (xEy \Rightarrow \mathcal{M}, y \models p) \\ &\iff [x]_E \subseteq \|p\|^{\mathcal{M}} \quad (8.16) \\ &\iff x \in \underline{E}(\|p\|^{\mathcal{M}}). \quad (8.17) \end{aligned}$$

Similarly, the truth condition of modal sentence $\Diamond p$ by (8.3) is also reformulated as follows:

$$\begin{aligned} \mathcal{M}, x \models \Diamond p &\iff \exists y \in U (xEy \text{ and } \mathcal{M}, y \models p) \\ &\iff [x]_E \cap \|p\|^{\mathcal{M}} \neq \emptyset \quad (8.18) \\ &\iff x \in \overline{E}(\|p\|^{\mathcal{M}}). \quad (8.19) \end{aligned}$$

All axiom schemas in Table 8.1 and the inference rule **RN** are satisfied by every Kripke model with equivalence relation [2], and therefore, the knowledge represented by the Palwak approximation space $(U, E)$ are able to describe by the modal system $S5$.

As a generalization of approximation using rough sets, Yao and Li [21], Yao and Lin [22], and Yao et al. [23] have discussed generalized lower approximation and generalized upper approximation by using arbitrary binary relation $R$ on $U$ instead of the equivalence relation. A pair $(U, R)$ of a finite set $U$ of objects and a binary relation $R$ on $U$ is called an *approximation space*. For every binary relation $R$ on $U$, a set $U_R(x)$ of objects induced from an object $x \in U$ and $R$ is defined by

$$U_R(x) \stackrel{\text{def}}{=} \{y \in U \mid xRy\}. \quad (8.20)$$

Obviously, the equivalence class $[x]_E$ by an equivalence relation $E$ is a special case of the set $U_R(x)$. If we regard the set $U$ as the set of possible worlds, the set $U_R(x)$ is the set of accessible possible worlds from the possible world $x \in U$.

For any binary relation $R$ on $U$ and any set $X \subseteq U$, *generalized lower approximation* $\underline{R}(X)$ and *generalized upper approximation* $\overline{R}(X)$ are defined by

$$\underline{R}(X) \stackrel{\text{def}}{=} \{x \in U \mid U_R(x) \subseteq X\}, \tag{8.21}$$

$$\overline{R}(X) \stackrel{\text{def}}{=} \{x \in U \mid U_R(x) \cap X \neq \emptyset\}. \tag{8.22}$$

Similar reformulation of the truth condition of modal operators by (8.17) and (8.19) are also available for the set $U_R(x)$, and therefore, generalized lower and upper approximations of a truth set $\|p\|^{\mathcal{M}}$ correspond to interpretation of modal sentences $\Box p$ and $\Diamond p$ in a Kripke model $\mathcal{M} = (U, R, v)$ induced by an approximation space $(U, R)$ with arbitrary binary relation $R$:

$$\mathcal{M}, x \models \Box p \Longleftrightarrow U_R(x) \subseteq \|p\|^{\mathcal{M}} \tag{8.23}$$

$$\Longleftrightarrow x \in \underline{R}(\|p\|^{\mathcal{M}}), \tag{8.24}$$

$$\mathcal{M}, x \models \Diamond p \Longleftrightarrow U_R(x) \cap \|p\|^{\mathcal{M}} \neq \emptyset \tag{8.25}$$

$$\Longleftrightarrow x \in \overline{R}(\|p\|^{\mathcal{M}}). \tag{8.26}$$

This fact illustrates close connection between various modal systems and generalized lower and upper approximations, and properties of the binary relation $R$ used for generalized lower and upper approximations correspond to axiom schemas of modal systems as shown in Tables 8.1 and 8.2. Note that Yao [20] also studied theoretical aspects of generalized rough sets induced by arbitrary binary relations.

### 8.4.2 Possible World Semantics with Variable Precision Rough Sets

Kudo et al. [8] discussed a possible world semantics of modal logics using VPRS by introducing $\alpha$-level fuzzy measure models based on background knowledge. The original purpose of this model is to provide a unified framework of deduction, induction, and abduction using granularity of possible worlds based on VPRS and measure-based semantics for modal logic.

As we reviewed in previous sections, each equivalence class $[x]_E$ represents a concept and the set of concepts, i.e., the quotient set $U/E$, describe knowledge by the given Pawak approximation space $(U, E)$. Suppose a Pawlak approximation space $(U, E)$ is given and a Kripke model $\mathcal{M} = (U, E, v)$ induced from $(U, E)$ and a valuation $v$ is considered. In the Kripke model $\mathcal{M}$, any non-modal sentence $p$ that represents a fact is characterized by its truth set $\|p\|^{\mathcal{M}}$. By using the background knowledge, when we consider the fact represented by the non-modal sentence $p$, we may not need to consider *all* possible worlds in the truth set $\|p\|^{\mathcal{M}}$ and we often consider only *typical situations* about the fact $p$.

To describe such typical situations of the fact $p$, the $\beta$-lower approximation of the truth set $\|p\|^{\mathcal{M}}$ by the equivalence relation $E$ is examined and regard each possible world in the $\beta$-lower approximation of the truth set $\|p\|^{\mathcal{M}}$ as a typical situation about $p$ based on background knowledge $U/E$. It enables us to regard situations that are not in the $\beta$-lower approximation as exceptions of the fact $p$. Thus, using background knowledge by the quotient set $U/E$, the following two sets of possible worlds about a fact $p$ are considerable [8]:

- $\|p\|^{\mathcal{M}}$: correct representation of the fact $p$
- $\underline{E}_{\beta}(\|p\|^{\mathcal{M}})$: the set of typical situations about $p$ (situations that are not typical may also be included)

Using the given Kripke model as background knowledge, an $\alpha$-level fuzzy measure model to treat typical situations about facts as $\beta$-lower approximations in the framework of modal logic are introduced [8]. Let $\mathcal{M} = (U, E, v)$ be a Kripke model induced from a Pawlak approximation space $(U, E)$ and a valuation function $v : \mathcal{P} \times U \to \{0, 1\}$, and $\alpha \in (0.5, 1]$ be a fixed degree. An *$\alpha$-level fuzzy measure model $\mathcal{M}_{\alpha}^{E}$ based on background knowledge* is the following triple:

$$\mathcal{M}_{\alpha}^{E} \stackrel{\text{def}}{=} (U, \{\mu_x^E\}_{x \in U}, v), \tag{8.27}$$

where $U$ and $v$ are the same as in $\mathcal{M}$. The fuzzy measure $\mu_x^E : 2^U \to [0, 1]$ assigned to each $x \in U$ is a *rough membership fucntion* [16], i.e., a probability measure based on the equivalence class $[x]_E$ with respect to $E$, defined by

$$\mu_x^E(X) \stackrel{\text{def}}{=} \frac{|[x]_E \cap X|}{|[x]_E|}, \ \forall X \subseteq U. \tag{8.28}$$

Similar to the case of Kripke-style models, we denote that a sentence $p$ is true at a world $x \in U$ by an $\alpha$-level fuzzy measure model $\mathcal{M}_{\alpha}^{E}$ by $\mathcal{M}_{\alpha}^{E}, x \models p$. According to the truth valuation of modal sentences in the measure-based semantics by (8.6) and (8.7), truth valuation of modal sentences, $\Box p$ and $\Diamond p$, by the $\alpha$-level fuzzy measure model $\mathcal{M}_{\alpha}^{E}$ is defined by

$$\mathcal{M}_{\alpha}^{E}, x \models \Box p \stackrel{\text{def}}{\Longleftrightarrow} \mu_x^E\left(\|p\|^{\mathcal{M}_{\alpha}^{E}}\right) \geq \alpha, \tag{8.29}$$

$$\mathcal{M}_{\alpha}^{E}, x \models \Diamond p \stackrel{\text{def}}{\Longleftrightarrow} \mu_x^E\left(\|p\|^{\mathcal{M}_{\alpha}^{E}}\right) > 1 - \alpha. \tag{8.30}$$

The truth set of a sentence $p$ in the $\alpha$-level fuzzy measure model $\mathcal{M}_{\alpha}^{E}$ is defined by

$$\|p\|^{\mathcal{M}_{\alpha}^{E}} \stackrel{\text{def}}{=} \{x \in U \mid \mathcal{M}_{\alpha}^{E}, x \models p\}. \tag{8.31}$$

The constructed $\alpha$-level fuzzy measure model $\mathcal{M}_{\alpha}^{E}$ from the given Kripke model $\mathcal{M}$ has the following properties.

**Theorem 8.1** [8] *Let $\mathcal{M}$ be a finite Kripke model such that its accessibility relation $E$ is an equivalence relation and $\mathcal{M}_\alpha^E$ be the $\alpha$-level fuzzy measure model based on the background knowledge $\mathcal{M}$ defined by (8.27). For any non-modal sentence $p \in \mathcal{L}_{ML}(\mathcal{P})$ and any sentence $q \in \mathcal{L}_{ML}(\mathcal{P})$, the following equations hold:*

$$\|p\|^{\mathcal{M}_\alpha^E} = \|p\|^{\mathcal{M}}, \tag{8.32}$$

$$\|\Box q\|^{\mathcal{M}_\alpha^E} = \underline{E}_{1-\alpha}\left(\|q\|^{\mathcal{M}_\alpha^E}\right), \tag{8.33}$$

$$\|\Diamond q\|^{\mathcal{M}_\alpha^E} = \overline{E}_{1-\alpha}\left(\|q\|^{\mathcal{M}_\alpha^E}\right). \tag{8.34}$$

**Theorem 8.2** (Soundness [8]) *For any $\alpha$-level fuzzy measure model $\mathcal{M}_\alpha^E$ defined by (8.27) based on any finite Kripke model $\mathcal{M}$ such that its accessibility relation $E$ is an equivalence relation, the following soundness properties are satisfied in the case of $\alpha = 1$ and $\alpha \in (0.5, 1)$, respectively:*

- *If $\alpha = 1$, then all theorems of the system S5 are true in $\mathcal{M}_\alpha^E$.*
- *If $\alpha \in (0.5, 1)$, then all theorems of the system EMND45 are true in $\mathcal{M}_\alpha^E$,*

*where the system EMND45 consists of the inference rules and axiom schemas of the system EMNP and the axiom schemas* **D**, **4**, *and* **5**.

This result enables us to represent facts and rules in reasoning processes as non-modal sentences and typical situations of facts and rules as lower approximations of truth sets of non-modal sentences [8]. From (8.32) and (8.33) in Theorem 8.1, the $\alpha$-level fuzzy measure model $\mathcal{M}_\alpha^E$ based on background knowledge $\mathcal{M}$ exhibits the characteristics of correct representations of facts by the truth sets of non-modal sentences and typical situations of the facts by the $(1-\alpha)$-lower approximations of truth sets of sentences. Thus, a modal sentence $\Box p$ is interpreted as *typically* $p$, and typical situations are used to characterize semantical aspects of deduction, induction, and abduction in a granularity-based framework [8].

## 8.5 Related Works

Connections between generalized rough sets and modal logics have been widely discussed with various approaches; Thiele [19] discussed an approach to generalize rough set theory based on arbitrary binary relations and modal logics. Kondo [5] and Zhu [25] discussed some fundamental properties of generalized rough set induced by binary relations. Järvinen et al. [4] discussed connections among modal logic, rough set, and Galois connection. Liau [9, 10] discussed modal logics semantics with probabilistic approximation spaces.

Various kinds of rough-set-based modal logics have also been introduced (e.g. [13]). As one example, Balbiani et al. [1] introduced a modal logic for Pawlak's approximation space with rough cardinality $n$.

## 8.6 Conclusion

In this chapter, we reviewed close relationships between rough set-based lower and upper approximations of concepts and possible world semantics of modal logics. We concentrated the relationships between two types of possible world semantic models, i.e., Kripke model and measure-based model, and two types of rough sets, i.e., Pawlak's rough set and VPRS. Relationships between possible world semantics and other various types of rough sets, i.e., covering-based rough set [24], dominance-based rough set [3], and Bayesian rough set [18], will be explored in future issues.

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