Contradictoriness, Paraconsistent Negation and Non-intended Models of Classical Logic

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Abstract Given that, by definition, two statements are contradictories if and only if it is logically impossible for both to be true and logically impossible for both to be false, some authors have argued that the negation operators of certain paraconsistent logics are not "real" negations because they allow for a statement and its negation to be true together. In this paper we argue that the same kind of argument can be levelled against the negation operator of classical propositional logic. To this end, Carnap's result that there are models of classical propositional logic with non-standard or non-normal interpretations of the connectives, and that one kind of those interpretations violate the semantical principle of non-contradiction which requires of a sentence and its negation that at least one of them be false can be used. We ponder the consequences of these arguments for the claims that paraconsistent negations are not genuine negations and that the negation of classical logic is a contradictory-forming operator and we consider the arguments that challenge the conflation between negation and contradiction.

Keywords Classical negation · Paraconsistent negations · Contradictory-forming operators · Carnap's non-standard models of classical logic

1 Introduction

It is usually accepted in the literature that negation is a contradictory-forming operator and that two statements are contradictories if and only if it is logically impossible for both to be true and logically impossible for both to be false. These two premises have

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been used by Hartley Slater [13] to argue that paraconsistent negation is not a "real" negation because a sentence and its paraconsistent negation can be true together.

In this paper we claim that a counterpart of Slater's argument can be directed against the negation operator of classical logic. Carnap's discovery that there are models of classical propositional logic with non-standard or non-normal interpretations of the connectives will be used to build such an argument. One such non-normal valuation which can be added to the set of classically admissible valuations without altering the set of theorems or the set of valid consequences assigns *true* to every well-formed formula and, therefore, assigns a designated value to every formula and its negation.

We ponder the consequences of these arguments for the claims that paraconsistent negations are not genuine negations and that the negation of classical logic is a contradictory-forming operator. To this end, we follow the arguments that Dutilh Novaes develops in [4] to challenge the conflation between negation and contradiction.

2 "Genuine" and Paraconsistent Negations

Some authors have argued that the negation operators of certain paraconsistent logics—i.e. logics which do not validate the *ex contradictione quodlibet* rule (*ECQ*): $\{A, \neg A\} \models B$, for every A and B—are not "real" negations. Given that, according to them, a "genuine" negation is a contradictory-forming operator and two statements are contradictories if and only if it is logically impossible for both to be true and logically impossible for both to be false, the negations of those logics are not "real" negations because they allow for a statement and its negation to be true together.

In a much quoted paper Hartley Slater maintains that the negation of Graham Priest's [9] paraconsistent logic *LP* (*Logic of paradox*) is not a genuine negation because in the three-valued semantics for *LP* there are two designated truth values that count as being true: t (true only), and b (both true and false), and both A and $\neg A$ can receive the designated value b in *LP*. Ironically, some years earlier Richard Routley and Graham Priest [10] had directed a similar criticism against the negation operator of da Costa's paraconsistent logic C_1 and had concluded that da Costa's negation was merely a subcontrary-forming operator—i.e. that a sentence and its da Costa's negation cannot both be false though they may both be true—.

Slater maintains that the same line of reasoning can be applied to every paraconsistent system and concludes that, properly speaking, there is no paraconsistent negation. The following argument—Slater's argument against paraconsistent negations as reconstructed by Francesco Paoli [8]—summarizes the above:

(1) Contradictories cannot be true together.
(2) A sentence and its negation are contradictories.
(3) If L is a paraconsistent logic, then, in the semantics for L, there are valuations which assign both A and $\neg A$ a designated value, for some formula A.

(4) If A and B both receive a designated value, under some valuation v, in the semantics for L, then A and B can be true together according to L.
(5) In paraconsistent logics, A and $\neg A$ may not be contradictories (from (1), (3), (4)).
(6) Thus, paraconsistent "negations" are not negations (from (2), (5)).

It can be argued that Slater obtains an easy victory because he assumes that "real" negations are, by definition, contradictory-forming operators [1]. Instead of questioning this assumption, in what follows we present an argument that uses Slater's premises to conclude that, if we accept them, not even classical negation can be considered a "genuine" negation.

3 Classical Negation and Non-standard Models of Classical Logic

In this section we will argue that the same kind of argument that Slater directs against paraconsistent negations can be levelled against the negation operator of classical propositional logic. To this end, Carnap's result that there are models of classical propositional logic with non-standard or non-normal interpretations of the connectives can be used.

In his *Formalization of Logic* Carnap tried to solve what he called *the problem of a full formalization of (first-order) logic*, i.e. "whether—and, in what way—it is possible to construct a calculus (…) such that the principal logical signs can be interpreted only in the normal way" [2, p. 3]. After proving that the customary formalizations of first-order logic do not achieve full formalization he introduced a multiple-conclusion presentation of elementary logic that he claimed to fulfill that goal, even though in his review of Carnap's solution Alonzo Church manifested his scepticism and conjectured that "non-normal interpretations of the propositional calculus can be excluded only by semantical (as opposed to purely syntactical) rules" [3, p. 496].

Carnap proves that there exist sound bivalent valuations—with respect, for example, to the standard natural deduction rules for classical propositional logic—that do not conform to the classical truth tables for the connectives. One kind of non-normal valuations violate the semantical principle of non-contradiction, which requires of a sentence and its negation that at least one of them be false. Carnap proved that the only non-normal (sound) bivalent valuation of this type is the valuation $v_\top$ which assigns the truth-value t (*true*) to every formula, i.e. for every sentence A, $v_\top(A) = t$. Let V be the set of standard classically admissible valuations and V' an extended set of admissible bivalent valuations such that $V' = V \cup \{v_\top\}$. It is easy to show that these two different sets of admissible valuations determine the same consequence relation—in symbols, $\Gamma \models_V A$ iff $\Gamma \models_{V'} A$, for every set of formulas Γ and every formula A—and, therefore, the same set of logical truths and valid inferences.

The other kind of non-normal valuations violate the semantical rules that the negation of a false sentence must be true and that a disjunction is false if its disjuncts are both false. An example of a valuation of this second kind is the one that assigns the truth value *true* to those formulas which are theorems of classical propositional logic and *false* to those formulas which are not theorems of classical propositional logic.

The interest in Carnap's discovery of non-standard models for classical logic has recently been revived in relation with the inferentialist thesis that the meanings of the logical constants are completely determined by their introduction and elimination rules in a natural deduction system [7, 12]. But, as we will try to show in what follows, those non-standard models are also relevant for the discussion of the philosophically adequate characterization of metalogical notions—such as that of contradictoriness—and their relation with different kinds of negation.

Taking into account Carnap's results, it is possible to build the counterpart for Slater's argument against paraconsistent negations in the case of classical negation:

(1) Contradictories cannot be true together.
(2) A sentence and its negation are contradictories.
(3) There exists a (non-standard) sound and complete bivalent semantics for classical logic such that there are valuations in this semantics which assign both A and $\neg A$ the designated value, for every formula A.
(4) If A and B both receive the designated value, under some valuation v, in an adequate bivalent semantics for classical logic, then A and B can be true together.
(5) In classical logic, A and $\neg A$ may not be contradictories (from (1), (3), (4)).
(6) Thus, classical "negation" is not a negation (from (2), (5)).

4 Is Classical Negation a Contradictory-Forming Operator?

In order to ponder the consequences of Carnap's result for the case against classical negation as a contradictory-forming operator we need to fix the definitions of "negation", "contradictories"—the term "contradictories" allow us put into brackets the question about the kind of entities involved in the notion of contradiction—and "classical logic". The term "contradictories" used here allow us to postpone the question about the kind of entities that can be used to characterize the notion of contradiction. It has been pointed out that at least four different approaches to the notion of contradictories can be found in the literature [6]: semantic definitions in terms of possibility, truth and falsity; syntactic definitions in terms of form; pragmatic definitions in terms of assertion and denial; and ontological definitions in terms of states of affairs.

In his argument against paraconsistent negations Slater uses a semantic notion of contradictories and assumes that genuine negations are contradictory-forming operators. But it should be noted that the usual semantic definition of "contradictories"—

two statements (sentences, propositions, formulas) are contradictories if and only if it is logically impossible for both to be true and logically impossible for both to be false—does not involve the notions of negation or classical logic, two notions whose characterization is certainly problematic. As pointed out by Dutilh Novaes, the idea of negation as a contradictory-forming operator is a quite recent development in the history of logic and an examination of the history of this discipline shows that the syntactical notion of negation and the semantic notion of contradiction can be conceptually independent of each other. In fact, Novaes points out that the notion of contradiction in Aristotelian logic does not have a straightforward syntactical propositional counterpart because Aristotle's negation is a term-negation and, therefore, a non-propositional one. It is only in the twentieth century that the notion of negation as a contradictory-forming propositional operator has become the predominant one and its source can be found in Frege's notion of negation as a function that maps the True to the False and the False to the True. This concept of propositional negation as the syntactic counterpart of the semantic notion of contradictory propositions is clearly stated in Whitehead and Russell's *Principia Mathematica*:

> The Contradictory Function with argument p, where p is any proposition, is the proposition which is the contradictory of p, that is, the proposition asserting that p is not true. This is denoted by $\sim p$. Thus $\sim p$ is the contradictory function with p as argument and means the negation of the proposition p. It will also be referred to as the proposition not-p. Thus $\sim p$ means not-p, which means the negation of p. [15, p. 6]

Dutilh Novaes concludes that, given that most of the notions of negation that can be found throughout the history of logic are not contradictory-forming operators, Slater's argument is not sound because one of its premises is simply not true and, therefore, Priest's paraconsistent negation is, at least in principle, as genuine a negation as any other.

Dutilh Novaes defense of paraconsistent negations can be used, *mutatis mutandis*, to accommodate Carnap's non-intended interpretations of propositional logic that allow for a formula and its negation to be both true. Her point of view permits us to accept the following statement made by Slater: "…[Priest] tries to show that Boolean negation likewise involves an operator for which the truth of $\neg\alpha$ does not rule out that of α. But, even if this was true, it would merely show that Boolean negation was not a contradiction-forming operator …" [14, p. 458]. Given the premises he accepts and taking into account the existence of Carnap's non-normal valuations, this would seem a sensible conclusion for Slater to draw with respect to classical logic. Nevertheless, if contradictory-forming negations are just one kind of (real) negations, the fact that classical negation is not a contradictory-forming operator does not oblige us to accept that it is not a genuine negation. And this because it is possible to assign both A and $\neg A$, for every formula A, the designated value t within a sound and complete bivalent semantics for a natural deduction presentation of classical logic.

Of course, one can try to circumvent Carnap's results by characterizing classical propositional logic as the logic determined by standard classical models—i.e. as the set of logical truths and valid inferences determined by those models—and classical negation as the contradictory-forming connective characterized by its standard bivalent truth-table. But this strategy seems to be a question-begging one: it assumes what needs to be proved, i.e. that Carnap's non-standard semantics is not a *bona fide* (bivalent) one for classical propositional logic. But, given that Carnap's non-standard models seem to provide such a semantics—because these models determine the same set of logical truths and valid inferences as standard classical models—the burden of proof lies with those who maintain that these results do not concern classical negation or classical logic. They must make explicit the difference—and the relevance of such a difference—between the logics determined by the standard and non-standard models that justify their stance, because otherwise their strategy would seem an ad hoc application of the advice "When you meet a contradiction, make a distinction."

It might be argued that even though Carnap's valuation $v_\top$ is unobjectionable from the point of view of a formal or pure semantics, it is not possible to provide a sensible informal or intuitive account of $v_\top$. If valuations are considered as descriptions of possible worlds or states of affairs, then $v_\top$ seems to commits us to a (weak) form of trivialism according to which there is a world where every sentence holds [5, 11]. However, it is debatable whether such a world can be discarded on purely logical grounds. But, be that as it may, even if we consider only those states of affairs in which at least one proposition is false, Carnap's second kind of non-normal valuations show that the natural deduction rules for classical propositional logic do not constrain us to accept that the classical negation operator is the syntactic counterpart of the truth function which maps truth to falsehood and falsehood to truth.

5 Conclusion

Carnap's non-standard models for classical logic have been mainly discussed in relation with the inferentialist conception of the meaning of the logical constants. But, as we have tried to show in this paper, those non-standard models are also relevant for the discussion of the relation of semantic notions such as contradictoriness and its relation with different (syntactic) notions of negation. In particular, we show that Slater's argument against paraconsistent negation, which assumes that a "genuine" negation is the syntactic counterpart of the notion of contradiction, can be directed, *mutatis mutandis*, against classical negation: in view of Carnap's results, if Slater's argument were a good one then neither paraconsistent negation nor classical negation would be "real" negations. But, as the conflation between propositional negation and contradiction is not a conceptual necessity, the genuineness of classical—and paraconsistent—negation can be defended but its contradictory-forming nature—at least, according to the usual semantic characterization of the notion of contradictoriness—is doubtful.

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