Paraconsistent Reasoning in Science and Mathematics: Introduction

Holger Andreas and Peter Verdée

In this book we present a collection of papers on the topic of applying paraconsistent logic to solve inconsistency related problems in science, mathematics and computer science. The goal is to develop, compare, and evaluate different ways of applying paraconsistent logic. After more than 60 years of mainly theoretical developments in many independent systems of paraconsistent logic, we believe the time has come to compare and apply the developed systems in order to increase our philosophical understanding of reasoning when faced with inconsistencies. This book wants to be a first step toward an application based, constructive debate to tackle the question which systems are best applied for which kind of problems and which philosophical conclusions can be drawn from such applications.

In this introduction we begin with a short but original overview and categorization of the research area of paraconsistency. We present some often heard reasons to go paraconsistent, a number of strategies to formally obtain paraconsistency and a couple of possible objections against paraconsistency. We hope that this way also readers new to the field can find their way inside a sometimes ill-structured but very interesting debate. The goal of this overview is therefore not at all encyclopaedic or historical, but we aim to enable the reader to enter and structure the field with a problem solving attitude: what are the problems paraconsistent logicians want to solve, what are the strategies they use for solving them and what are the main difficulties in the process toward the solution?

Paraconsistency is not a well defined notion. Paraconsistent reasoning could be seen as any kind of reasoning which is able to deal with inconsistencies. Paraconsistent logics propose systematic ways to reason paraconsistently. In this introduction

H. Andreas (✉)
University of British Columbia, Kelowna, Canada
e-mail: holger.andreas@ubc.ca

P. Verdée
Université catholique de Louvain, Louvain-la-Neuve, Belgium
e-mail: peter.verdee@uclouvain.be

H. Andreas and P. Verdée (eds.), *Logical Studies of Paraconsistent Reasoning in Science and Mathematics*, Trends in Logic 45,
DOI 10.1007/978-3-319-40220-8_1

we will not make a distinction between inconsistencies and contradictions. Both terms are used to indicate information from which, for some sentence A, both A and not-A can be obtained.

The only characteristic all paraconsistent logics have in common is that the *Explosion rule*, i.e. "derive B from A and $\neg A$" where $\neg$ is a negation connective, is not valid. To people who have not studied formal logic, this rule usually comes across as very awkward. It seems always unreasonable that the derivation of arbitrary conclusions is possible, no matter what the premises are. Most people will admit that contradictions and contradictory theories are false, but deny that from false information arbitrary conclusions can be obtained. There is no consensus on what should be the rational alternative to Explosion. Some people will for example rather suggest the opposite of Explosion: forbid to derive anything further once one has obtained a contradiction. But this is not the general strategy of paraconsistent logics: most of them will still allow some more innocent consequences of contradictory theories.

Although Explosion is generally not considered as pre-theoretically valid, and is (to our knowledge) never applied in actual reasoning or informal proofs, the rule is valid in the vast majority of theoretically elaborated symbolic logics (classical logic, intuitionistic logic, fuzzy logic, their extensions - most modal, deontic, temporal logics, and many more). The reason for this contrast between the counter-intuitive character and general formal validity of Explosion is its connectedness with other much more plausible principles of reasoning. Much more intuitive are the principles of Disjunctive Syllogism (From $\neg A$ and $A \vee B$, derive B) and Addition or Disjunction Introduction (From A, derive $A \vee B$). It is impossible to unrestrictedly validate both rules in a Transitive system (in which formulas derived by means of rules can be used as premises for the application of other rules), without also implicitly validating the Explosion principle. To see this, consider that Addition enables us to derive $A \vee B$, for each arbitrary formula B, from premises A and $\neg A$. If we subsequently apply Disjunctive Syllogism to this conclusion and the second premise, we immediately obtain B, which was an arbitrary formula (possibly unrelated to the premises).

However, there is no principled reason why a symbolic logic needs to validate Explosion. For various reasons one may want a logic with exactly the property not to validate Explosion. Of course one will also lose some other properties of traditional explosive symbolic logics. We believe that one should not be dogmatic about symbolic logic. The formal theory of logic is, just like any other theory, but an attempt to capture external phenomena. Such an attempt is a fallible enterprise. Even if one believes that there is one true ultimate logic, there is no absolute warrant that our present most popular logical theories have correctly captured this ultimate logic. Even if one argues that Explosion is ultimately a valid inference, one should explain how rational agents deal with inconsistencies. If this can happen in a systematic way, there is no reason why we should not explain it by means of a logic, where logic is here understood in a maximally broad sense: as any symbolic way to theorize about reasoning. Both the reasons why one wants a paraconsistent logic and the ways in which paraconsistency is obtained may be very diverse. Let us first list some of the reasons philosophers may have to develop paraconsistency.

The first and most obvious reason is the need to deal with inconsistent information or inconsistent theories. It is rather uncontroversial that every human attempt to obtain empirical or theoretic information is susceptible to inconsistencies. The reasons for this may be very diverse. There may be errors in the processing of information, errors in storing and retrieving information, calculation errors, errors in observation, discrepancies between theory and observations, unforeseen contradictory outcomes of theories, incompatibilities between different of our best theories about the world (possibly all empirically adequate w.r.t. past observations), inconsistent sources, inconsistent databases, defeated conclusions of non-deductive reasoning…All human epistemic methods are fallible and in case of failure there is nothing which can warrant the freedom from inconsistencies. Each part of our knowledge can in principle be wrong and then be in conflict with other parts or future observations. If this happens, it is rarely easy to solve the issue by diagnosing the problem and simply contracting the problematic sentences. These conflicts are parts of a structured web full of useful information. But even if we are able to remove mistaken information, there usually is no immediate correct alternative at hand which is harmless. As long as there is no alternative, one seems to be forced to provisionally take this inconsistent body of information as it is and continue reasoning from there, until one has found a way to solve the problem.

An Explosive logic cannot be used for this purpose, because in such a logic every inconsistent theory is interderivable (from any inconsistent theory one can derive all statements of any other theory) and thus equivalent. This means that, given an Explosive logic, all inconsistent information becomes inferentially identical and therefore entirely useless. It should not come as a surprise that this is highly undesired for the above described type of problems. Instead one may want a logic that maximally isolates (possible) inconsistencies so that the underlying problems do not infect or affect other parts of our knowledge. Or, on the other side of the spectrum, one may rather want a logic which maximally approximates an Explosive logic, but without Explosion. Similarly, one may want a logic which spreads inconsistencies to all formally related sentences to avoid potentially false assumptions of safety for indirectly affected sentences. Alternatively, one may want a mechanism to "repair" the inconsistency. Yet another project may be to devise a logic which reduces inconsistencies to more basic/primitive inconsistencies.

A second reason is dealing with inescapable, acceptable or true contradictions. This concerns several versions of dialetheism (cf. [22]). Semantic paradoxes (among which the famous Liar Paradox) show that we cannot combine traditional Explosive logic, certain parts of ordinary language (e.g. unrestricted self reference) and straightforward principles of semantics (e.g. transparent truth). Other paradoxes (set theoretic, property theoretic, related to informal mathematical proofs or definitions) show that inconsistencies are obtained by using certain intuitively very attractive principles of reasoning. In all these cases ways have been found to avoid the inconsistencies by restricting the modelled domains and the validity of the intuitive principles. But one may choose to take the intuitions behind the problematic theories seriously and so to bite the bullet and accept the inescapable inconsistency. Of course one needs a logic to reason with such an acceptable or even true inconsistency. Even if one does not want to make the inconsistency true in a strong *truth as correspondence*-sense, one may see a mathematical or semantic theory in a less realistic way and, because of external reasons, argue that an inconsistent theory is preferable over its possible consistent corrections as the most appropriate theory of a certain domain.

A third reason is dealing with the possibility of inconsistent (counterfactual) worlds. Even if inconsistent objects do not exists, one may find it useful or even metaphysically required to be able to reason with them. A Meinongian, for example, who believes that inconsistent objects do not exist, may still see them as objects that we can describe in a reasonable language. If they have inconsistent properties, an Explosive logic cannot be the underlying logic of such a language. One may also reason that logic should be maximally neutral. If the logic excludes inconsistent theories already in advance, logic seriously restricts the metaphysical possibilities. So even if one is strongly convinced that there are no true inconsistencies, one may see this as a matter of fact and not as an a priori truth determined by logic.

A fourth reason may be dealing with the entailment/implication connective as used in informal mathematics or science. Independent of one's considerations about the nature of negation and inconsistencies, logicians, such as the fathers of relevance logics (cf. e.g. [1, 12]), have attempted to give a reasonable formalization of implication connectives (closer to actual usage than material implication). They took it to be essential for implication to express a link between antecedent and consequent. There is of course no link whatsoever between p-and-not-p and q, so p-and-not-p cannot imply q, and so the object language variant of Explosion $(A \wedge \neg A) \to B$ cannot be valid in a logic based on such a view on implication. In a sense then, such logics are also paraconsistent, even if one often does not define a consequence relation but merely a set of tautologies.

A fifth reason may be the discovery of ignored domains of mathematics. In the same way as the generalization of real numbers to complex numbers turned out to be a rich broadening of mathematics, also taking inconsistent theories and inconsistent models of existing theories seriously may enlarge the mathematical domain in an interesting way. Once one has a precise paraconsistent logic to deal with inconsistencies, there is no reason why a mathematical theory could not be inconsistent, as long as it is as rigorous as the theories of classical mathematics.

For all of these reasons, paraconsistent logicians have developed a plethora of different systems in the relative short history of paraconsistency. We here list some of the most prominent approaches, divided into several categories. The categories may overlap. We certainly do not aim to give a full overview, but merely a more or less original categorisation of possible approaches.

A first category contains logics in which the consistency of sentences can be expressed formally by means of a (possibly defined) unary connective. In an Explosive logic, such a symbol would be trivial, because every sentence is supposed to be consistent in such a logic. The first examples of such logics were so called Da Costa $\mathbf{C_n}$-logics (cf. [13]) in which $\neg(A \wedge \neg A)$ is interpreted as expressing the consistency of A. Later this is generalized to the class of LFI's: Logics of Formal Inconsistency (cf. [11]). This is a general framework which contains very different paraconsistent logics with a unary consistency connective. In such logics one has Explosion for consistent formulas (and not for all other formulas). The advantage of this approach is that one can understand and model classical reasoning as well as paraconsistent reasoning, depending on which formulas are involved.

A second category are the many valued logics. The idea is to allow other truth values than consistent truth and consistent falsity. An evident choice is of course the introduction of a third value indicating both true and false. Other options are: going four valued (adding a value for neither-true-nor-false) or even infinitely valued. In general Disjunctive Syllogism will not be valid, because there are non-classical truth values that make both A and $\neg A \vee B$ true without affecting the value of B. There are a lot of examples, but prominent ones are these: the three valued logic of paradox (cf. [20]), the four valued Belnap Dunn logic (cf. [14]) and, recently, paraconsistent logics based on infinitely valued fuzzy logics [15]).

A third category are systems in which negation is a modal connective. This is a diverse group, but in general one obtains paraconsistency by interpreting negation in such a way that the truth of a negated formulas is interpreted as the possible falsity of that formula. Just like it is consistent in classical modal logic that a formula is true and possibly false, there are models in which a formula and its negation are both true, given such a modal paraconsistent negation. The first and best known example is what is now known as dual intuitionistic logic (cf. [16]). In the Kripke semantics of intuitionistic logic negation is interpreted as not possibly true, were 'possibly' means provable at some future point in time. In the dual version we could read negation as possibly false or refutable at some point in time. This treatment of negation as a modal connective can be based on many other modal logics in several diverse ways of expressing the paraconsistent negation. A philosophical study of negation as a modality can be found in [8].

A fourth category are the non-adjunctive or discussive paraconsistent logics (cf. [17]). Consistencies are here possible because a sentence may be coherently held by one agent in a discussion and its negation coherently by another agent in the same discussion. From the point of view of a neutral observer of the discussion we are dealing with an inconsistency. $A \wedge \neg A$ would still explode, but there is no way to conjoin A and $\neg A$. In these logics Addition is unproblematic, but Disjunctive Syllogism is only valid in its one premise form: from $A \wedge (\neg A \vee B)$ conclude B. The Adjunction (Conjunction Introduction) rule (derive $A \wedge B$ from A and B) is blocked.

A fifth category are the non-monotonic logics. They restrict the law of Disjunctive Syllogism to formulas which could be consistent in view of the premises. For the other formulas one of the other paraconsistency strategies is used. That way only those models are selected that verify a minimal amount of inconsistencies. The advantage of this strategy is that one has the inferential richness of Explosive logics for consistent premises, but also the inconsistency tolerance when the premises are not consistent. The most well known examples are inconsistency adaptive logics [5] and minimally inconsistent **LP** (cf. [21]).

A sixth category contains logics that are not cautiously transitive. Merely blocking (cautious) transitivity makes it possible to validate both Disjunctive Syllogism and Addition (and in fact all the strength of classical logic for consistent premises) without necessarily validating the Explosion rule (cf. [6, 9]). Non-monotonic logics will usually also be non-transitive but will make sure that they are cautiously transitive (if something follows from the combination of the premises plus a conclusion, then it is also a valid conclusion of the premises alone). If one does not have cautious transitivity, one can have both unproblematic rules, as long as both are not applied in chain (one after the other). In this case one can have a monotonic structural paraconsistent logic which validates all classical consequences of consistent premises. The price to pay here is the capacity to build on earlier results. Every proof needs to start again from the basic axioms.

A seventh category are the implication revising logics. It is quite generally recognized that the material implication of classical logic (and many other logics) is far from the implication connective used in informal reasoning. A first example are the relevant logics (cf. [2] and the discussion above). Connexive logic (cf. [18]), on the other hand, are not subclassical; they really contradict classical logic. They make it false that something could imply its negation. If, by contrast, one also accepts that it is true that (some) contradictions imply their negations, one easily obtains a contradiction the logic should be able to deal with.

Still other options are logics with a non-deterministic semantics. In these systems the semantics of complex sentences are not necessarily reduced/analysed into inconsistencies concerning primitive sentences. That way the negation of a sentence A may for example be allowed to be true independent of the truth value of A. This way A and its negation may be true together. Of course one loses compositionality, but this does not need to make the logic inferentially impotent. Examples are the weaker LFIs such as **mbC** and Batens' system **CLuN** (cf. [4]).

A final category concerns the possibility to block Addition (possibly only for inconsistent formulas) in order to avoid Explosion. In a logic that merely analyses sentences into (combinations of) subsentences, of course one can never obtain an arbitrary formula. This strategy is followed in [19].

It is clear that there are a lot of paraconsistent logics with a diverse set of purposes. Although, in general, they are well-developed, both technically and philosophically, there is not much research comparing them in relation to their applicability in science and mathematics. It is surprising how little these logics are actively applied to actual scientific or mathematical theories (other than some historical reconstructions). Given how common it is that scientists have to deal with inconsistencies (between theories, between theories and observations, and inside theories) and inconsistency resolution, it is surprising that relatively little work has been done to make the involved type of reasoning logically precise.

There may be several reasons for this. Let us summarize some of the possible objections one may have against adopting a paraconsistent logic for concrete applications in science and mathematics. Many people are reluctant to use paraconsistent logics because when adopting them, one loses the strongest possible argument to reject problematic hypotheses and theories, i.e. the fact that they are inconsistent. In a paraconsistent context, logic alone does not suffice to reject inconsistent theories. Consequently, in such a context new information will never, by pure force of logic, necessitate the revision of old information. Logic no longer excludes the possibility to keep piling up all kinds of inconsistent information without ever contracting old information. Belief revision is therefore no longer a logical requirement. There may of course be many other reasonable criteria for rejecting problematic theories (incoherence, vagueness, lack of elegance, lack of explanatory power, empirical inadequacy etc.) and consistency may in a paraconsistent context still be a locally valid extra-logical requirement, or a property one may want to satisfy as much as possible. Nevertheless it is dialectically very attractive to possess a logical criterion to dismiss every theory from which one can derive an inconsistency.

Moreover, the discussion above shows that (most) paraconsistent logics need to lose some a priori attractive principles of reason. We have become so used to classical logic that all of the above considered principles (Disjunctive Syllogism, Addition, Adjunction, Monotonicity and Transitivity) seem very natural principles of logical consequence often successfully applied in informal mathematical or scientific reasoning. If such principles are no longer logically valid, one needs to explain the discrepancy between logic and practice. Are the apparently successful applications of the invalid principles mistakes, locally correct applications of a generally speaking invalid principle, or the result of a mistaken formalization? Moreover, if not merely metalogical principles (Monotonicity, Transitivity) but actual logic rules (Disjunctive Syllogism, Addition, Adjunction) are blocked, one loses inferential power to the effect that many theories become much weaker. This may be desirable for those parts of the considered theory where one is confronted with actual inconsistencies, but problematic where everything seems to behave consistently.

Logics that have more inferential power but require specific treatment for consistent parts of theories are often computationally highly complex. In order to find out whether a specific application of a rule is valid, one needs to know already whether an inconsistency is derivable from certain involved formulas in relation to the rest of the theory. Calculating this may be very difficult (cf. [25]). Suppose that, inside a given theory $\Gamma \cup \{A, \neg A \vee B\}$, one wants to apply Disjunctive Syllogism to obtain B. If the paraconsistent logic only allows this rule if A is consistent, one needs to find out whether $\neg A$ is derivable from Γ, before one is able to correctly apply this single rule. If Γ and A have some non-logical vocabulary in common and Γ is a large set or forms a complex theory, finding this out may be immensely time consuming or even undecidable. Computational complexity is not a conclusive reason to reject a logic, because the logic may be seen as merely the ideal but difficult or unreachable standard of reasoning. But in that case one should explain how human agents can at least approximately deal with the unreachable ideal reasoning standards in practice.

Yet another possible objection is the question whether formal logic is applicable in an inconsistent context. Those who do accept the importance of inconsistency tolerant reasoning may object that this type of reasoning (largely) happens extralogically (cf. [24]). Important extra-logical factors involved in dealing with inconsistencies are: the sources of information, the priority ordering of information and its sources, the goals of reasoning, social and dialectical dynamics of reasoning and arguing, and fallible diagnostic reasoning. Nobody will deny that such factors play a role in dealing with (at least some) inconsistencies, but that does not mean that one cannot also say something with logical generality based solely on the form of involved expressions.

Even those who accept that paraconsistent logics correctly formalize some phenomena, may still claim that, to the extent that they are useful, paraconsistent logics can be translated into more traditional explosive logics. Either one claims that what paraconsistent logicians would formalize as an inconsistency should actually be formalized differently (possibly with the same syntactic consequences). If an agent receives information A and information $\neg A$ from two equally reliable sources, it makes sense to formalize this as 'agent 1 believes A' and 'agent 2 believes $\neg A$', using doxastic or epistemic modalities. Even in explosive logics this pair does not explode. Similar modal solutions work for every kind of inconsistency coming from different origins (incompatible axioms, theories, observations). In case the inconsistency originates from one indissoluble (inconsistent) body of information this strategy does not work, but one could then argue that the body of information is simply unreliable and should not be used for doing further reasoning.

Another alternative is to consider the paraconsistent negation as a coherent unary connective that can be added to classical logic, but not as *the* negation. One can often use the usually classical semantics of the paraconsistent negation connective to add it to classical logic as a conservative extension. Or one can define inconsistency tolerant databases or inconsistent properties/collections inside a purely classical context as well-defined mathematical objects. Compare it to fuzzy set theory. People speak of fuzzy *sets* (cf. [26]), but they are merely useful classical mathematics tools, which are defined by means of ordinary sets and real number theory. They behave in such a way that they are more subtle generalizations of ordinary sets (ones to which the elements belong to a certain degree, instead of just in or out). They are no alternatives to classical set theory, but mere extensions of it. People who accept the usefulness of some paraconsistent logics can claim the same thing about a paraconsistent negation; a useful tool that can be defined in a rich enough classical logic (plus perhaps some parts of classical set theory). For many applications of paraconsistent logic it seems indeed unnecessary to really revise classical logic; it is often sufficient to realize that classical logic is not the appropriate tool to approach inconsistent collections of information. But classical logic was never meant for this purpose anyway. The idea would be that one could keep using classical logic with its inconsistency intolerant negation for all the more foundational/justifying purposes it was meant for. This may be a reasonable position if it concerns rather practical applications of paraconsistent logic, but it does not work for more fundamental applications about the very basics of mathematics, philosophy and logic.

We have listed a number of often heard objections to the usefulness of paraconsistent logic as an alternative to classical logic. None of these objections are sufficient arguments to reject the usefulness of paraconsistent logics, but those defending paraconsistent logics need to specify how to overcome these issues. This itself is an interesting debate and the possible answers depend a lot on which logic and which application one has in mind.

The reader understands by now that paraconsistency is a diverse phenomenon with different *raisons d'être*, different technical solution and different ways to respond to criticism, all of which have to do with the specific application one has in mind. Nevertheless there is also quite a lot of common ground. Similar techniques have been used, similar arguments have been given against Explosion and against the critics of paraconsistency, and similar inconsistent theories have been studied, all of this often independently from one another and inside different schools of paraconsistency. Nevertheless there is relatively little study about the similarities and differences of the different currents of paraconsistency in relation to the intended real life applications of paraconsistency.

In order to open the debate on how the different formalisms relate to their real life applications in the philosophy of science and mathematics, we decided to organize a conference on this topic in Munich, Germany: the conference *Paraconsistent Reasoning in Science and Mathematics* (June 11–13, 2014) in the beautiful setting of the Carl-Friedrich-von-Siemens-Stiftung. Our aim was to bring the different schools of paraconsistency together to open the debate on how the different formalisms relate to their real life applications in the philosophy of science and mathematics.

The level of the talks and the quality of the debate was so high that the participants of the conference were all in favour of publishing a volume on the topic of the conference, aiming toward a written and more detailed follow up of this debate. The present book is the result. We hope the reader will find that it lives up to the expectations. In what follows, we give a brief summary of every paper of our collection.

1 Holger Andreas and Peter Verdée: Adaptive Proofs for Networks of Partial Structures

According to Carnap [10], we interpret and understand the theoretical terms of a theory T in such a manner that the axioms of T come out true. If, however, T is classically inconsistent, this semantic doctrine fails to work properly. The theoretical terms remain uninterpreted in this case. This is not satisfactory insofar as numerous scientific theories turned out to be inconsistent in some way or other – science is full of inconsistencies. Hence, it is desirable to have a semantics of theoretical terms that also applies to inconsistent theories.

Holger Andreas and Peter Verdée, consequently, develop a paraconsistent generalization of the semantic doctrine in question: we interpret and understand the theoretical terms of a theory T in such a manner that the axioms of T are satisfied to a maximal extent. Formally, we describe such interpretations in terms of a network of partial structures, and thereby define a preferred models semantics of paraconsistent scientific reasoning. This semantics respectively defines an inference relation for flat and prioritized axiomatic theories.

A preferred models semantics by itself does not give us a proof-theoretic account of scientific reasoning with theoretical terms. For this to be achieved, the framework of adaptive logic with its dynamic proof-theory suggests itself. Hence, we present a flat and a lexicographic adaptive logic which are proven to capture the inference relation for flat and prioritized axiomatic theories. Because the adaptive logics belong to the category of standard (lexicographic) adaptive logics, the adaptive characterization immediately gives rise to an adequate dynamic proof theory for the inference relations. The paper concludes with a demonstration of how we can derive sensible conclusions from Bohr's model of the atom using adaptive proofs.

2 Franzesco Berto: Ceteris Paribus Imagination

Franzesco Berto explores impossible worlds for an analysis of ceteris paribus imagination. An impossible world is one where the laws of classical logic may be violated by the truth-value assignment to atomic and complex formulas. Hence, an impossible world may verify a set of sentences that is classically inconsistent. Impossible worlds, therefore, may serve as a model of inconsistent beliefs.

Why should we want to model inconsistent beliefs? The underlying motivation derives from the limitations of our logical capacities. We are unable to grasp all logical consequences of a set of explicit beliefs, and we may even fail to recognize inconsistencies in our explicit beliefs. As is well known, this happened to Frege when he developed his *Basic Laws of Arithmetic*. In brief, we are not logically omniscient.

Ceteris paribus imagination is modelled by a conditional: if an agent explicitly conceives A to be the case, then B is part of the imagined scenario. In formal terms: $[A]B$, where $[*]$ is a modal operator, defined by an accessibility relation on the set of possible and impossible worlds. [A] B holds true if B is verified by all worlds (possible and impossible ones) that are reachable from the actual world and in which A holds true.

Having defined a worlds semantics of $[A]B$, Berto investigates which axioms envisioned for variably strict conditionals remain valid for ceteris paribus imagination. Notably, $[A \wedge \neg A]B$ fails to hold for arbitrary B. Imagining an inconsistent scenario does not mean that we trivialize what we conceive. In this respect, the ceteris paribus conditional behaves like a paraconsistent consequence relation.

3 Bryson Brown: On the Preservation of Reliability

Science is full of inconsistencies: first, we have scientific theories that are internally inconsistent and thus imply a contradiction. Second, we have scientific theories that make assumptions inconsistent with other accepted scientific theories. Third, we have numerous approximations and idealizations that are known to be inconsistent with what we strictly believe about the respective theoretical entities. Fourth, scientific theories are often times inconsistent with certain predecessor theories, while preserving many of their empirical predictions. These inconsistencies strongly suggest the need for a paraconsistent treatment of scientific reasoning.

Bryson Brown attempts to provide methodological foundations for a paraconsistent approach to scientific reasoning. His proposal is to view reliability-in place of truth-as the property to be preserved by proper scientific reasoning, as well as in the replacement of earlier scientific theories by new ones. The main focus of Brown's paper is on reliable inference patterns in the history of science, including Planck's treatment of black body radiation and Bohr's theory of the hydrogen atom; work by Nancy Cartwright and Bas C. van Fraassen is also discussed, leading up to an account of a modestly paraconsistent approach to scientific reasoning.

4 Luis Estrada-González: Prospects for Triviality

This paper studies triviality in mathematical theories, an important enemy of most paraconsistent logicians. Paraconsistent logics (want to) serve as the underlying logic of inconsistent theories, exactly because they can avoid triviality. Trivial theories are usually seen as meaningless and even disastrous. This position was among others defended by Chris Mortensen.

The author of this paper discusses the central question whether triviality is always so bad and wants to answer it in the negative, against Mortensen's position. He argues that there is a case of an extremely simple mathematical category theoretic universe, a degenerate topos, in which everything is true. This universe is therefore trivial, but it is not inherently problematic.

Mortensen's case is built on a trivialisation result for real number theory. González shows that either one of the premises of the trivialization result cannot obtain (from a point of view external to the universe) and thus the argument is unsound, or that it obtains in calculations internal to such a trivial universe. In the latter case the calculations in the trivial universe are possible and meaningful albeit extremely simple. Our actual universe is probably not as simple as and so does not correspond to this degenerate topos, but that does not mean that what is done inside the universe is meaningless.

5 Andreas Kapsner: On Gluts in Mathematics and Science

This paper discusses the question whether truth value gluts (both true and false) should be designated in an analysis of mathematical and scientific reasoning. Practically speaking the question is whether one should assert sentences that are true and false and whether they should be used as basis for decisions and actions and as premises of arguments. The traditional paraconsistent view is that there are truth value gluts and that they should be designated. In some sense the converse goes against the very basic starting point of paraconsistency: a non-designated glut will not block the Explosion rules.

Kapsner defends the view that it is often, but not always, unreasonable to assert glutted statements. He presents a clear case: if two costumer reviews contradict each other on the quality of a product, one should not assert the contradicting information obtained by reading the reviews. Subsequently he presents some cases from the history of science (the infinitesimal calculus and the Darwin–Kelvin debate on the age of the earth) to indicate that also in these case it may be more reasonable not to designate gluts.

6 Carlos A. Oller: Contradictoriness, Paraconsistent Negation and Non-intended Models of Classical Logic

This paper concerns the often heard argument that paraconsistent negation is not a real negation because a sentence and its negation should never be true together. The author attacks the argument by showing that it could also be used to show that classical logic's negation is by the same standards not a real negation either.

Classical logic has certain unavoidable non-intended models. Carnap was the first to point out that adding a trivial model (in which all formulas are true) to the semantics of classical logic does not affect the set of valid consequences. In such a model of course formulas and their negations are both true. It seems thus that it is impossible even in classical logic to exclude the possibility that a formula and its negation are both true.

7 Hitoshi Omori: From Paraconsistent Logic to Dialetheic Logic

This paper proposes a new approach to paraconsistent logic to be applied as the underlying logic of a dialetheic version of naive set theory and naive truth. The author proposes an attractive logic which is not only paraconsistent in that it can tolerate inconsistent formulas, but also dialetheic, in the sense that it also makes certain inconsistent formulas into tautologies.

Omori returns to the modern origins of paraconsistent logic and proposes a paraconsistent account of negation in line with some ideas by Jaśkowski: that a good negation should be a connective such that each formula and its negation form contradictory pairs. This is realized by requiring that a formula is true iff its negation is false, and false iff its negation is true. A necessary condition for paraconsistent logics respecting this account of negation is that Double Negation rules are valid.

In order for the logic to work with prototypical inconsistent mathematical theories such as naive set theory, one needs a weak enough biconditional to non-trivially express axiom schemas like the axiom of Abstraction. To this purpose a strategy suggested by Laura Goodship is used: opt for the material biconditional of **LP**. Omori adds to this concept some ingredients of LFIs (a consistency operator) and connexive logics (the conditional is false when the antecedent is not true or the consequent is false). The result is a dialetheic logic with a functionally complete three valued semantics.

8 Martin Pleitz: Paradoxes of Expression

The history of the paradoxes and attempted solutions thereof shows many cases where a certain solution falls prey to another, more refined variant of the original paradox. The revenge liar is the most famous instance of such cases. Martin Pleitz adds another chapter to this history of attempted solutions and recurrent paradoxes.

The focus is on a recent proposal by Graham Priest to solve the semantic and set-theoretic paradoxes using a biconditional that does not detach, i.e., that fails to satisfy *modus ponens*. A detachable truth schema, however, is needed for what has been described as *blind endorsement*. For example, if one holds that everything that the Bible says is true, one blindly endorses all the claims made in the Bible. To solve this problem, Priest entertains the idea of introducing a detachable conditional, together with an expression predicate that allows us to say that certain propositions are expressed by certain sentences.

Martin Pleitz formulates a very reasonable principle that an expression operator, licensing blind endorsements, should satisfy: any meaningful sentence should be synonymous with itself. Based on an axiomatic formulation of this principle, he shows that variants of the Liar and the Curry paradox can be formulated. Hence, we have a contradiction and a way to trivialize the system envisioned by Priest. As triviality is unbearable even in a paraconsistent setting, this casts serious doubt on Priest's proposed solution.

9 Corry Shores: Dialetheism in the Structure of Phenomenal Time

The very idea of motion seems to be contradictory: if we say that an object is moving, we imply that it is at different places at different times. So far, things are consistent. If, however, we want to say that an object is moving *right now*, we seem to ascribe the property of changing positions to a specific time point. At a specific time point, however, an object can only be at one place. Drawing on Zeno's paradox of the arrow and assuming that only the present time point has reality, we can thus argue that no object is really moving. Motion is not part of reality. Likewise, the flow of time is an unreal phenomenon.

Alternatively, we can accept that motion and time are contradictory but real, thereby embracing some form dialetheism. This alternative is investigated and sympathetically entertained by Corry Shores in his contribution. Besides the work of Zeno, he draws on Husserl's phenomenology and subsequent phenomenological research to motivate a dialetheist account of change and time. Dialetheist ideas about the phenomenology of time are thus brought together with recent work in theoretical psychology.

10 Fenner S. Tanswell: Saving Proof from Paradox: Gödel's Paradox and the Inconsistency of Informal Mathematics

In this paper two of the most popular arguments (by Beall resp. Priest) in favour of the inconsistency of (informal) mathematics (and so the need to formalize it with a paraconsistent logic) are discussed. A first argument is based on what is sometimes called Gödel's paradox, i.e. a sentence expressing that it is not provable. Accepting the existence of such a sentence leads to a contradiction in mathematics. The second argument is based on the incompatibility of completeness and consistency established by Gödel's incompleteness theorems. Arguing in favour of the completeness of informal mathematics, thus also forms an argument against the consistency of mathematics.

Tanswell argues against these arguments that the necessary distinctions between formality and informality are often ignored. The author also points at problems with the assumption of the unity of informal mathematics.

11 Heinrich Wansing and Sergei Odintsov: On the Methodology of Paraconsistent Logic

When we decided to organize our conference on *Paraconsistent Reasoning in Science and Mathematics*, we wanted to stimulate discussion, exchange of ideas, and further research on the desiderata that a paraconsistent logic should satisfy. We argued there to be three core desiderata: (1) A paraconsistent logic ought to capture the inferential use of inconsistent but non-trivial theories. (2) A paraconsistent approach should explain how one can weaken the underlying logic of classical logic to get rid of the explosion principle and still have enough inferential power to be successful. (3) It is desirable to have a philosophical motivation for the deviation from classical logic in terms of epistemological and, possibly, also metaphysical considerations.

We have then been very pleased to see that Heinrich Wansing and Sergei Odintsov directly address the question of which desiderata a paraconsistent logic should satisfy. While investigating the historical roots of the above desiderata, they cast some doubt on desiderata (2) and (3). More precisely, they question that the reference logic of a paraconsistent logic should be classical logic, arguing that the choice of classical logic as reference logic is at least difficult to justify. As for the philosophical motivation for developing a specific paraconsistent logic, the notion of information should play a central role rather than epistemological and metaphysical considerations. As information about whatever domain is rarely complete and often times inconsistent, an informational methodology of paraconsistent logic may lead us to choosing a non-bivalent logic as reference logic.

The paper further discusses in great detail the maximality condition that a paraconsistent logic should satisfy with reference to classical logic, thus drawing on work by Arieli et al. [3]. Moreover, it examines methodological considerations on the desiderata of a paraconsistent logic that have been suggested by Priest and Routley [23]. Finally, Wansing and Odintsov sketch a universal approach to constructing a paraconsistent logic for a given reference logic that may well not be classical logic.

12 Zach Weber: Paraconsistent Computation and Dialetheic Machines

This paper concerns the application of paraconsistent logic and dialetheism to theoretic computer science. The question is asked whether there are algorithms which are essentially paraconsistent, in the sense that only paraconsistent logic can recognize them. While the question may seem counterintuitive, it is clear that certain objects can exists in paraconsistent mathematics which cannot exist otherwise (for example the Russell set or the set of all ordinals). So it is not unlikely that also the concept of an algorithm should be reconsidered in a paraconsistent setting in order for classically unknowable but useful objects to be recognized and studied.

The author argues in favour of the existence of such properly paraconsistent computations. Arguments by Sylvan and Copeland, Routley, and Priest support this view. One of the arguments goes as follows: in view of a straightforward diagonalization, the algorithm that enumerates all algorithms (intuitively) is but at the same time cannot be an algorithm. If it is an algorithm (and it sure seems to be one) it has to be an inconsistent algorithm.

Subsequently Weber investigates the ways in which one could formulate paraconsistent algorithms in a dialetheic mathematical metalanguage. He discusses the properties of so called dialetheic machines and their relation with finiteness and the halting problem.

References

1. Ackermann, W. (1956). Begründung einer strengen Implikation. *Journal of Symbolic Logic*, *21*, 113–128.
2. Anderson, A. R., & Jr Belnap, N. D. (1975). *Entailment. The logic of relevance and necessity* (Vol. 1). Princeton: Princeton University Press.
3. Arieli, O., Avron, A., & Zamansky, A. (2011). Ideal paraconsistent logics. *Studia Logica*, *99*, 31–60.
4. Batens, D. (1999). Inconsistency-adaptive logics. In E. Orłowska (Ed.), *Logic at work. Essays dedicated to the memory of Helena Rasiowa* (pp. 445–472). Heidelberg, New York: Physica Verlag (Springer).
5. Batens, D. (2000). A survey of inconsistency-adaptive logics. In [7] (pp. 49–73).
6. Batens, D. (2012). It might have been classical logic. *Logique et Analyse*, *55*(218), 241.
7. Batens, D., Mortensen, C., Priest, G., & Van Bendegem, J. P. (Eds.). (2000). *Frontiers of paraconsistent logic*. Baldock, UK: Research Studies Press.
8. Berto, F. (2015). A modality called ‘negation’. *Mind*, *124*(495), 761–793. doi:10.1093/mind/fzv026.

9. Besnard, P., & Hunter, A. (1995). Quasi-classical logic: Non-trivializable classical reasoning from inconsistent information. *Symbolic and quantitative approaches to reasoning and uncertainty* (pp. 44–51). Heidelberg: Springer.
10. Carnap, R. (1958). Beobachtungssprache und theoretische Sprache. *Dialectica*, *12*, 236–248.
11. Carnielli, W. A., Coniglio, M. E., & Marcos, J. (2007). Logics of formal inconsistency. In D. Gabbay & F. Guenthner (Eds.), *Handbook of philosophical logic* (Vol. 14, pp. 1–93). Heidelberg: Springer.
12. Church, A. (1951). The weak theory of implication. In: A. Menne, A. Wilhemy & H. Angsil (Eds.), *Kontrolliertes Denken. Untersuchungen zum Logikkalkül und zur Logik der Einzelwissenschaften* (pp. 22–37). Karl Alber, München. (abstract: *JSL*, 16, 1951, p. 239).
13. da Costa, N. C. (1963). Calculs propositionnels pour les systèmes formels inconsistants. *Comptes rendus de l'Académie des sciences de Paris*, *259*, 3790–3792.
14. Dunn, J. M. (1976). Intuitive semantics for first-degree entailments and 'coupled trees'. *Philosophical Studies*, *29*(3), 149–168.
15. Ertola, R., Esteva, F., Flaminio, T., Godo, L., & Noguera, C. (2015). Paraconsistency properties in degree-preserving fuzzy logics. *Soft Computing*, *19*(3), 531–546.
16. Goodman, N. D. (1981). The logic of contradiction. *Mathematical Logic Quarterly*, *27*(8–10), 119–126. doi:10.1002/malq.19810270803.
17. Jaśkowski, S. (1969). Propositional calculus for contradictory deductive systems. *Studia Logica*, *24*, 243–257.
18. Mccall, S. (1966). Connexive implication. *The Journal of Symbolic Logic*, *31*, 415–433. doi:10.2307/2270458. http://journals.cambridge.org/article_S0022481200066822.
19. Meheus, J. (2000). An extremely rich paraconsistent logic and the adaptive logic based on it. In [7] (pp. 189–201).
20. Priest, G. (1979). The logic of paradox. *Journal of Philosophical Logic*, *8*, 219–241.
21. Priest, G. (1991). Minimally inconsistent LP. *Studia Logica*, *50*, 321–331.
22. Priest, G. (2006). *In contradiction. A study of the transconsistent* (2nd ed.). Oxford: Oxford University Press. (first edition 1987).
23. Priest, G., & Routley, R. (1989). Systems of paraconsistent logic. In F. Priest, R. Routley, & J. Norman (Eds.), *Paraconsistent logic: Essays on the inconsistent* (pp. 142–155). Munich: Philosophia Verlag.
24. Urbaniak, R., & Siniło, P. (2014). The inapplicability of (selected) paraconsistent logics. *Journal of Applied Non-Classical Logics*, *24*(4), 368–383.
25. Verdée, P. (2009). Adaptive logics using the minimal abnormality strategy are Π^1_1-complex. *Synthese*, *167*, 93–104.
26. Zadeh, L. (1965). Fuzzy sets. Information. *Control*, *8*, 338–353.