# A Posteriori Error Estimates for Nonstationary Problems

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**Abstract** We apply continuous and discontinuous Galerkin time discretization together with standard finite element method for space discretization to the heat equation. For the numerical solution arising from these discretizations we present a guaranteed and fully computable a posteriori error upper bound. Moreover, we present local asymptotic efficiency estimate of this bound.

## 1 Introduction

We consider the heat equation, which represents a model problem to more general linear parabolic problems. We discretize this problem by standard finite element method in space and by either continuous or discontinuous Galerkin method in time.

Recently, time discretizations of Galerkin type start to be very popular. They represent higher order and very robust schemes for solving ordinary differential equations. When combined with classical Galerkin space discretizations, e.g. with finite element method (FEM), it is possible to analyze the complete discretization in a unified framework. For a survey about Galerkin time discretizations see [1] and [2]. A nice result presenting the connection of these discretizations to classical Runge–Kutta methods can be found in [8].

In this paper we shall focus on a posteriori error analysis of proposed problem. Our aim is to present a guaranteed, cheap and fully computable upper bound to chosen error measure that provides local efficiency at least asymptotically. To achieve these properties we use the technique of so-called equilibrated flux reconstruction, see e.g. [5]. We have been influenced by [4], where lower order time discretizations are considered, and by [2], where Galerkin time discretizations are analyzed and nodal superconvergence is derived via a posteriori error estimates.

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## 2 Continuous Problem

Let $\Omega \subset R^d$ ($d = 1, 2, 3$) be a bounded polyhedral domain with Lipschitz continuous boundary $\partial\Omega$ and $T > 0$. Let us consider the following initial–boundary value problem

$$\begin{aligned} \frac{\partial u}{\partial t} - \Delta u &= f \quad \text{in } \Omega \times (0, T), \\ u &= 0 \quad \text{in } \partial\Omega \times (0, T), \\ u &= u^0 \quad \text{in } \Omega. \end{aligned} \tag{1}$$

We assume that the right-hand side $f \in C(0, T, L^2(\Omega))$ and the initial condition $u^0 \in L^2(\Omega)$.

Let $(.,.)$ and $\|.\|$ be the $L^2(\Omega)$-scalar product and norm, respectively. Let us denote the time derivative $u' = \frac{\partial u}{\partial t}$. We define spaces $X = L^2(0, T, H_0^1(\Omega))$ and

$$\begin{aligned} Y &= \{v \in X : v' \in L^2(0, T, L^2(\Omega))\}, \\ Y_0 &= \{v \in X : v' \in L^2(0, T, L^2(\Omega)), v(0) = u^0\}. \end{aligned} \tag{2}$$

It is well known that the spaces $Y$ and $Y_0$ are subsets of $C([0, T], L^2(\Omega))$.

**Definition 1** We call $u \in Y_0$ the weak solution of problem (1), if

$$\int_0^T (f, v) - (u', v) - (\nabla u, \nabla v) dt = 0, \quad \forall v \in X. \tag{3}$$

We assume that there exists a unique weak solution of problem (3).

## 3 Discretization

We consider a space partition $\mathscr{T}_h$ consisting of a finite number of closed, $d$-dimensional simplices $K$ with mutually disjoint interiors and covering $\overline{\Omega}$, i.e. $\overline{\Omega} = \cup_{K \in \mathscr{T}_h} K$. We assume conforming properties, i.e. neighbouring elements share an entire edge or face. We set $h_K = \text{diam}(K)$ and $h = \max_K h_K$. By $\rho_K$ we denote the radius of the largest $d$-dimensional ball inscribed into $K$. We assume shape regularity of elements, i.e. $h_K/\rho_K \le C$ for all $K \in \mathscr{T}_h$, where the constant does not depend on $\mathscr{T}_h$ for $h \in (0, h_0)$.

We set the space for the semidiscrete solution

$$X_h = \{v \in H_0^1(\Omega) : v|_K \in P^p(K)\}, \tag{4}$$

where $P^p(K)$ denotes the space of polynomials up to the degree $p \geq 1$ on $K$. We define $\Pi_\Omega^p : H_0^1(\Omega) \to X_h$ to be the $L^2$-orthogonal projection.

In order to discretize problem (3) in time, we consider a time partition $0 = t_0 < t_1 < \ldots < t_r = T$ with time intervals $I_m = (t_{m-1}, t_m)$, time steps $\tau_m = |I_m| = t_m - t_{m-1}$ and $\tau = \max_{m=1,\ldots,r} \tau_m$. Let $(.,.)_{K,m}$ and $(.,.)_K$ be the local $L^2$-scalar products over $K \times I_m$ and $K$, respectively, and $\|.\|_{K,m}$ be the local $L^2(K \times I_m)$-norm. In the forthcoming discretization process we will assume two variants of the time discretization, the conforming and the nonconforming one. In the conforming case, the approximate solution will be sought in the spaces of piecewise polynomial functions

$$Y_{0h}^\tau = \{v \in Y : v|_{I_m} = \sum_{j=0}^{q+1} v_{j,m} t^j,\ v_{j,m} \in X_h, v(0) = \Pi_\Omega^p u^0\} \tag{5}$$

and in the nonconforming case in the space

$$X_h^\tau = \{v \in X : v|_{I_m} = \sum_{j=0}^{q} v_{j,m} t^j,\ v_{j,m} \in X_h\}. \tag{6}$$

The spaces $Y_{0h}^\tau$ and $X_h^\tau$ represent natural discrete spaces to $Y_0$ and $X$, respectively. The space $Y_{0h}^\tau$ consists of functions that are one degree higher in time than the functions from the space $X_h^\tau$. On the other hand the functions from $Y_{0h}^\tau$ are continuous with respect to time with fixed starting value at 0. Altogether, both these spaces have the same dimension $r(q+1)\dim X_h$.

For a function $v \in X_h^\tau$ we define the one–sided limits

$$v_\pm^m = v(t_m\pm) = \lim_{t \to t_m \pm} v(t) \tag{7}$$

and the jumps

$$\{v\}_m = v_+^m - v_-^m, \quad m \geq 1 \quad \text{and} \quad \{v\}_0 = v_+^0 - u^0. \tag{8}$$

We omit the subscript $\pm$ for continuous functions $v \in Y$, since $v(t_m\pm) = v(t_m)$.

Now, we are able to formulate two variants of discrete schemes – the conforming version:

**Definition 2** We say that the function $u_h^\tau \in Y_{0h}^\tau$ is the discrete solution of problem (3) obtained by time continuous Galerkin – finite element method (cG–FEM), if the following conditions are satisfied

$$\int_{I_m} ((u_h^\tau)', v) + (\nabla u_h^\tau, \nabla v) dt = \int_{I_m} (f, v) dt \tag{9}$$
$$\forall m = 1, \ldots, r,\ \forall v \in X_h^\tau,$$

and the nonconforming version:

**Definition 3** We say that the function $u_h^\tau \in X_h^\tau$ is the discrete solution of problem (3) obtained by time discontinuous Galerkin – finite element method (dG–FEM), if the following conditions are satisfied

$$\int_{I_m} ((u_h^\tau)', v) + (\nabla u_h^\tau, \nabla v) dt + (\{u_h^\tau\}_{m-1}, v_+^{m-1}) = \int_{I_m} (f, v) dt \qquad (10)$$
$$\forall m = 1, \dots, r, \ \forall v \in X_h^\tau.$$

It is evident that the exact solution $u \in Y_0$ defined by (3) satisfies both relations (9) and (10).

The methods (9) and (10) can be viewed as a generalization of classical one–step methods for parabolic problems. It is possible to show that setting $q = 0$, i.e. piecewise linear continuous approximation in time for cG–FEM or piecewise constant approximation in time for dG–FEM, is equivalent (up to suitable quadrature of the right–hand side) to Crank–Nicolson, resp. backward Euler method, in time and FEM in space.

## 4 A Posteriori Error Analysis

In this section we shall propose suitable error measure and derive a posteriori error estimate of this measure.

### *4.1 Error Measure*

Let $d_{K,m} > 0$ be an arbitrary parameter associated with space-time element $K \times I_m$, e.g. $d_{K,m}^2 = h_K^2 + \tau_m^2$ or $d_{K,m} = 1$ or $d_{K,m} = h_K$ or $d_{K,m}^2 = (h_K^{-2} + \tau_m^{-2})^{-1}$. Let us define the space

$$Y^\tau = \{v \in X : v'|_{I_m} \in L^2(I_m, L^2(\Omega))\} \qquad (11)$$

of piecewise continuous functions with respect to time. We define the norm

$$\|v\|_{Z,K,m}^2 = \frac{h_K^2 \|\nabla v\|_{K,m}^2 + \tau_m^2 \|v'\|_{K,m}^2}{d_{K,m}^2}, \quad \|v\|_Z^2 = \sum_{K,m} \|v\|_{Z,K,m}^2. \qquad (12)$$

Since $Y^\tau \subset X$, we gain from (3) that the exact solution $u \in Y$ satisfies

$$\int_{I_m} (f, v) - (u', v) - (\nabla u, \nabla v) dt - (\{u\}_{m-1}, v_+^{m-1}) = 0 \tag{13}$$
$$\forall m = 1, \ldots, r, \ \forall v \in Y^\tau.$$

The existence of the solution $u$ of problem (13) comes clearly from the existence of the solution of problem (3). We shall focus on uniqueness of the solution of problem (13). Let us assume that there exists another solution $u_1 \in Y^\tau$ of problem (13). After subtracting the equation for $u$ from the equation for $u_1$ and setting $v = 2(u - u_1)$ we gain

$$\begin{aligned} 0 &= \int_{I_m} 2(u' - u_1', u - u_1) + 2\|\nabla(u - u_1)\|^2 dt \\ &\quad + 2(\{u - u_1\}_{m-1}, (u - u_1)_+^{m-1}) \\ &= \|(u - u_1)_-^m\|^2 - \|(u - u_1)_-^{m-1}\|^2 + \|\{u - u_1\}_{m-1}\|^2 \\ &\quad + 2\int_{I_m} \|\nabla(u - u_1)\|^2 dt \end{aligned} \tag{14}$$

Summing this relation over $m = 1, \ldots, r$ and using the fact $u_-^0 = u^0 = u_{1-}^0$ we gain

$$\|(u - u_1)_-^r\|^2 + \sum_{m=1}^{r} \|\{u - u_1\}_{m-1}\|^2 + 2\int_0^T \|\nabla(u - u_1)\|^2 dt = 0, \tag{15}$$

which implies $u = u_1$.

It is natural to define error measure EST for both variants of discretization as residual of (13)

$$\begin{aligned} \mathrm{EST}(w) = \sup_{0 \neq v \in Y^\tau} \frac{1}{\|v\|_Z} \Bigg( & \sum_{K,m} (f, v)_{K,m} - (w', v)_{K,m} \\ & - (\nabla w, \nabla v)_{K,m} - (\{w\}_{m-1}, v_+^{m-1})_K \Bigg) \end{aligned} \tag{16}$$

for $w \in X_h^\tau$.

It is possible to show that the uniqueness of the solution of problem (13) implies that $\mathrm{EST}(u_h^\tau) = 0$, if and only if $u_h^\tau$ is equal to the exact solution $u$.

## 4.2 Reconstruction of the Solution with Respect to Time

Since the exact solution $u \in Y_0 \subset C([0,T], L^2(\Omega))$, i.e. $u$ is continuous in time and $u(0) = u^0$, we will reconstruct the discrete solution $u_h^\tau$ in such a way, that the reconstruction satisfies these properties too. For conforming variant of discretization (cG-FEM) this task is easier, since the solution is already continuous in time, but the initial condition can still be violated. For nonconforming version we need to reconstruct for both reasons.

Let $r_m \in P^{q+1}(I_m)$ be the right Radau polynomial on $I_m$, i.e. $r_m(t_{m-1}) = 1$, $r_m(t_m) = 0$ and $r_m$ is orthogonal to $P^{q-1}$. Then there exists a polynomial reconstruction $R_h^\tau = R_h^\tau(u_h^\tau)$ for both variants of discretization such that

$$R_h^\tau(t) = u_h^\tau(t) - \{u_h^\tau\}_{m-1} r_m(t), \quad \forall t \in I_m. \tag{17}$$

Since the cG-FEM solution $u_h^\tau$ is continuous in time, the reconstruction $R_h^\tau$ is equal to $u_h^\tau$ except $I_1$. It is still necessary to reconstruct the discrete initial condition $\Pi_\Omega^p u^0$ on $I_1$, see (8).

The resulting function $R_h^\tau$ is continuous in time and satisfies the initial condition, i.e. $R_h^\tau \in Y_0$. Moreover,

$$\begin{aligned} \int_{I_m} ((R_h^\tau)', v)dt &= \int_{I_m} ((u_h^\tau)', v) - r_m'(\{u_h^\tau\}_{m-1}, v)dt \\ &= \int_{I_m} ((u_h^\tau)', v)dt + \int_{I_m} r_m(\{u_h^\tau\}_{m-1}, v')dt \\ &\quad - r_m(t_m)(\{u_h^\tau\}_{m-1}, v_-^m) + r_m(t_{m-1})(\{u_h^\tau\}_{m-1}, v_+^{m-1}) \\ &= \int_{I_m} ((u_h^\tau)', v)dt + (\{u_h^\tau\}_{m-1}, v_+^{m-1}), \quad \forall v \in P^q(I_m, L^2(\Omega)). \end{aligned} \tag{18}$$

Such a reconstruction is used to show the equivalence among Radau IIA Runge–Kutta method, Radau collocation method and discontinuous Galerkin method. For the details see, e.g. [6] and [7]. Such a reconstruction is also used for proving a posteriori nodal superconvergence in [2].

## 4.3 Reconstruction of the Solution with Respect to Space

It is possible to show that the exact solution satisfies $\nabla u \in L^2(0,T,H(\text{div}))$. Since $\nabla u_h^\tau \notin L^2(0,T,H(\text{div}))$ in general, we reconstruct also $\nabla u_h^\tau$. Let $RTN_p(K)$ be the Raviart-Thomas-Nedelec space of order $p$, i.e. $RTN_p(K) = P_p(K)^d + xP_p(K)$. Let us denote the patch $\mathscr{T}_a = \bigcup_{a \in K} K$ of vertex $a$. Then we can define $RTN$ spaces on $\mathscr{T}_a$

$$RTN_p^{N,0}(\mathscr{T}_a) = \{v \in RTN_p(\mathscr{T}_a) : v \cdot n = 0 \ \forall e \subset \partial \mathscr{T}_a\}, \quad a \notin \partial\Omega, \tag{19}$$
$$RTN_p^{N,0}(\mathscr{T}_a) = \{v \in RTN_p(\mathscr{T}_a) : v \cdot n = 0 \ \forall e \subset \partial \mathscr{T}_a \setminus \partial\Omega\}, \quad a \in \partial\Omega.$$

Let us denote by $P_*^p(\mathscr{T}_a)$ piecewise polynomials of order $p$ for $a \in \partial\Omega$. Moreover, $P_*^p(\mathscr{T}_a)$ consists of functions with zero mean value for $a \notin \partial\Omega$. Let us denote $\psi_a$ piecewise linear "hat" function associated with vertex $a$ with $\psi(a) = 1$, $\psi = 0$ on $\partial\mathscr{T}_a$.

We formulate space–time version of patch–wise reconstruction from [3]. We seek $\sigma_a^\tau|_{\mathscr{T}_a \times I_m} \in P^q(I_m, RTN_p^{N,0}(\mathscr{T}_a))$ and $r_a^\tau \in P^q(I_m, P_*^p(\mathscr{T}_a))$ such that

$$\begin{aligned}(\sigma_a^\tau, v)_{\mathscr{T}_a,m} - (r_a^\tau, \nabla \cdot v)_{\mathscr{T}_a,m} &= (\psi_a \nabla u_h^\tau, v)_{\mathscr{T}_a,m}, \\ &\qquad \forall v \in P^q(I_m, RTN_p^{N,0}(\mathscr{T}_a)), \\ (\nabla \cdot \sigma_a^\tau, q)_{\mathscr{T}_a,m} &= (\psi_a(f - (R_h^\tau)'), q)_{\mathscr{T}_a,m} + (\nabla\psi_a \cdot \nabla u_h^\tau, q)_{\mathscr{T}_a,m}, \\ &\qquad \forall q \in P^q(I_m, P_*^p(\mathscr{T}_a)).\end{aligned} \tag{20}$$

Then

$$\sigma_h^\tau = \sum_a \sigma_a^\tau. \tag{21}$$

The reconstructions $\sigma_a^\tau$ and $\sigma_h^\tau$, exist and satisfy

$$\begin{aligned}0 &= (f - (R_h^\tau)' + \nabla \cdot \sigma_h^\tau, v)_{K,m} \\ &= (f - (u_h^\tau)' + \nabla \cdot \sigma_h^\tau, v)_{K,m} - (\{u_h^\tau\}_{m-1}, v_+^{m-1})_K, \\ &\qquad \forall v \in P^q(I_m, P^p(K)).\end{aligned} \tag{22}$$

## 4.4 Upper Error Bound

In this section we will present a posteriori upper bound for $\mathrm{EST}(u_h^\tau)$, i.e. we will present the estimate of $\mathrm{EST}(u_h^\tau)$ in terms of data $f$ and $u^0$, discrete solution $u_h^\tau$ (both versions of time discretizations are covered) and functions $R_h^\tau$ and $\sigma_h^\tau$ that are derived and easily computable from the discrete solution $u_h^\tau$.

**Theorem 4 (Upper error bound)** *Let $u \in Y_0$ be the solution of (3) and $u_h^\tau \in X_h^\tau$ be arbitrary. Let $R_h^\tau$ be the reconstructions obtained from $u_h^\tau$ by (17) and $\sigma_h^\tau$ be the* *reconstruction obtained from* $u_h^\tau$ *by* (20) *and* (21)*. Then*

$$EST(u_h^\tau) \le \left( \sum_{K,m} \left( \frac{d_{K,m}}{\pi} \| f - (R_h^\tau)' + \nabla \cdot \sigma_h^\tau \|_{K,m} + \frac{d_{K,m}}{h_K} \| \sigma_h^\tau - \nabla u_h^\tau \|_{K,m} + \frac{d_{K,m}}{\tau_m} \| (R_h^\tau - u_h^\tau)' \|_{K,m} \right)^2 \right)^{1/2} \tag{23}$$

The proof of Theorem 4 is a straightforward application of (17) and (22), but it is quite long. For this reason we omit it.

### *4.5 Asymptotic Lower Error Bound*

The goal of this section is to show that the local individual terms from a posteriori estimate (23) are locally effective, i.e. provide a local lower bound to $\mathrm{EST}(u_h^\tau)$, at least in asymptotic sense.

To be able to apply the result in a local way, we need following notation. Let $\mathscr{T}_K$ be a patch consisting of elements surrounding $K$ and $K$ itself. Let $M \subset \overline{\Omega}$, e.g. $M = K$ or $M = \mathscr{T}_K$. We define local version of space $Y^\tau$

$$Y_{M,m}^\tau = \{ v \in Y^\tau : \mathrm{supp}(v) \subset \overline{M \times I_m} \}, \tag{24}$$

and local version of $\mathrm{EST}(w)$

$$\mathrm{EST}_{M,m}(w) = \sup_{0 \neq v \in Y_{M,m}^\tau} \frac{1}{\|v\|_Z} \left( \sum_{K,m} (f, v)_{K,m} - (w', v)_{K,m} - (\nabla w, \nabla v)_{K,m} - (\{w\}_{m-1}, v_+^{m-1})_K \right). \tag{25}$$

For the purpose of the effectivity analysis let us assume that $f$ is a space–time polynomial. Otherwise, it is necessary to deal with the classical oscillation term.

**Theorem 5 (Local effectivity estimate)** *Let* $u \in Y_0$ *be the solution of* (3) *and* $u_h^\tau \in X_h^\tau$ *be arbitrary. Let* $R_h^\tau$ *be the reconstructions obtained from* $u_h^\tau$ *by* (17) *and* $\sigma_h^\tau$ *be the reconstruction obtained from* $u_h^\tau$ *by* (20) *and* (21)*. Let* $f$ *be a space–time* *polynomial. Then there exists a constant* $C > 0$ *such that*

$$d_{K,m}^2 \|f - (R_h^\tau)' - \nabla \cdot \sigma_h^\tau\|_{K,m}^2 + \frac{d_{K,m}^2}{\tau_m^2} \|R_h^\tau - u_h^\tau\|_{K,m}^2 \tag{26}$$
$$+ \frac{d_{K,m}^2}{h_K^2} \|\sigma_h^\tau - \nabla u_h^\tau\|_{K,m}^2 \leq C\, EST_{\mathscr{T}_K,m}(u_h^\tau)^2.$$

The proof of Theorem 5 is very technical and quite long. For these reasons we shall skip it in this paper.

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