Stability Analysis of the ALE-STDGM for Linear Convection-Diffusion-Reaction Problems in Time-Dependent Domains

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Abstract In this paper we investigate the stability of the space-time discontinuous Galerkin method (STDGM) for the solution of nonstationary, linear convection-diffusion-reaction problem in time-dependent domains formulated with the aid of the arbitrary Lagrangian-Eulerian (ALE) method. At first we define the continuous problem and reformulate it using the ALE method, which replaces the classical partial time derivative with the so called ALE-derivative and an additional convective term. In the second part of the paper we discretize our problem using the space-time discontinuous Galerkin method. The space discretization uses piecewise polynomial approximations of degree $p \geq 1$, in time we use only piecewise linear discretization. Finally in the third part of the paper we present our results concerning the unconditional stability of the method.

1 Formulation of the Continuous Problem

We consider an initial-boundary value nonstationary, linear convection-diffusion-reaction problem in a time-dependent bounded domain:

Find a function $u = u(x,t)$ with $x \in \Omega_t,\ t \in (0,T)$ such that

$$\frac{\partial u}{\partial t} + \boldsymbol{v} \cdot \nabla u - \epsilon \triangle u + cu = g \quad \text{in} \quad \Omega_t,\ t \in (0,T), \tag{1}$$

$$u = u_D \quad \text{on} \quad \partial\Omega_t,\ t \in (0,T), \tag{2}$$

$$u(x,0) = u^0(x), \quad x \in \Omega_0. \tag{3}$$

We assume that $\boldsymbol{v} = (v_1, v_2),\ c,\ g,\ u_D,\ u^0$ are given functions and $\epsilon > 0$ is a given constant. Moreover let $Q_T = \{(x,t);\ t \in (0,T),\ x \in \Omega_t\}$, and let us assume

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that there exist constants $c_v,\ c_c > 0$, such that

$$\begin{aligned} &\boldsymbol{v} \in C([0,T];\ W^{1,\infty}(\Omega_t)),\ |\nabla \boldsymbol{v}| \le c_v,\ |\boldsymbol{v}| \le c_v \quad \text{in} \quad Q_T, \\ &c \in C([0,T], L^{\infty}(\Omega_t)),\ |c(x,t)| \le c_c \quad \text{in} \quad Q_T. \end{aligned}$$

Problem (1)–(3) will be reformulated using the so called arbitrary Lagrangian-Eulerian (ALE) method. It is based on a regular one-to-one ALE mapping of the reference domain Ω_0 onto the current configuration Ω_t:

$$\begin{aligned} &\mathscr{A}_t : \overline{\Omega}_0 \to \overline{\Omega}_t, \\ &X \in \overline{\Omega}_0 \to x = x(X,t) = \mathscr{A}_t(X) \in \overline{\Omega}_t, \quad t \in [0,T]. \end{aligned}$$

We assume that $\mathscr{A}_t \in C^1([0,T]; W^{1,\infty}(\Omega_t))$, i.e. the mapping $\mathscr{A}_t$ belongs to the Bochner space of continuously differentiable functions in $[0,T]$ with values in the Sobolev space $W^{1,\infty}(\Omega_t)$. We define the ALE velocity by

$$\begin{aligned} &\tilde{z}(X,t) = \frac{\partial}{\partial t}\mathscr{A}_t(X), \quad t \in [0,T],\ X \in \Omega_0, \\ &z(x,t) = \tilde{z}(\mathscr{A}_t^{-1}(x),t), \quad t \in [0,T],\ x \in \Omega_t. \end{aligned}$$

Let $|z(x,t)|,\ |\mathrm{div}\, z(x,t)| \le c_z$ for $x \in \Omega_t,\ t \in (0,T)$. Further, we define the ALE derivative $D_t f = Df/Dt$ of a function $f = f(x,t)$ for $x \in \Omega_t$ and $t \in [0,T]$ as

$$D_t f(x,t) = \frac{D}{Dt} f(x,t) = \frac{\partial \tilde{f}}{\partial t}(X,t),$$

where $\tilde{f}(X,t) = f(\mathscr{A}_t(X),t)$, $X \in \Omega_0$, and $x = \mathscr{A}_t(X) \in \Omega_t$. The use of the chain rule yields the relation

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + z \cdot \nabla f, \tag{4}$$

which allows us to reformulate problem (1)–(3) in the ALE form:

Find $u = u(x,t)$ with $x \in \Omega_t,\ t \in (0,T)$ such that

$$D_t u + (\boldsymbol{v} - z) \cdot \nabla u - \epsilon \triangle u + cu = g \quad \text{in} \quad \Omega_t,\ t \in (0,T), \tag{5}$$

$$u = u_D \quad \text{on} \quad \partial\Omega_t, \tag{6}$$

$$u(x,0) = u^0(x), \quad x \in \Omega_0. \tag{7}$$

In what follows, we shall use the notation $\boldsymbol{w} = \boldsymbol{v} - z$ for the ALE transport velocity.

Numerical methods for linear convection-diffusion-reaction equations in a domain Ω independent of time were analyzed e.g. in [5]. In the case, when problem (1)–(3) is considered in a fixed domain, error estimates for the space-time discontinuous Galerkin discretization were derived in [4]. These results were generalized to the case of nonlinear convection and diffusion (cf. [3]). The paper [1] is devoted to the proof of unconditional stability of the space-time discontinuous Galerkin method (STDGM) applied to nonlinear convection-diffusion problems. The STDGM was used with success for the numerical solution of compressible flow in time-dependent domains and also for the dynamical linear and nonlinear elasticity (see [3]). In [2], the stability of the time discontinuous Galerkin semi-discretization of problem (5)–(7) was analyzed. Here we are concerned with the investigation of the stability of the complete STDGM applied to problem (5)–(7) in a time-dependent domain.

2 Space-Time Semidiscretization

In the time interval $[0, T]$ we construct a partition formed by time instants $0 = t_0 < t_1 < \ldots < t_M = T$ and set $I_m = (t_{m-1}, t_m)$ and $\tau_m = t_m - t_{m-1}$ for $m = 1, \ldots, M$. Further we set $\tau = \max_{m=1,\cdots,M} \tau_m$. For a function φ defined in $\bigcup_{m=1}^{M} I_m$ we denote one-sided limits at t_m as $\varphi_m^{\pm} = \varphi(t_m\pm) = \lim_{t\to t_m\pm} \varphi(t)$ and the jump as $\{\varphi\}_m = \varphi(t_m+) - \varphi(t_m-)$.

For any $t \in [0, T]$ we denote by $\mathscr{T}_{h,t}$ a partition of the closure $\overline{\Omega}_t$ into a finite number of closed triangles with mutually disjoint interiors. We set $h_K = \mathrm{diam}(K)$ for $K \in \mathscr{T}_{h,t}$. The boundary of the domain will be divided into two parts: $\partial\Omega_t = \partial\Omega_t^- \cup \partial\Omega_t^+$:

$$\boldsymbol{w}(x,t)\cdot\boldsymbol{n}(x) < 0 \text{ on } \partial\Omega_t^-, \forall t \in [0,T] \text{ (inflow boundary)}$$
$$\boldsymbol{w}(x,t)\cdot\boldsymbol{n}(x) \geq 0 \text{ on } \partial\Omega_t^+, \forall t \in [0,T] \text{ (outflow boundary)},$$

where $\boldsymbol{n}$ denotes the unit outer normal to ∂K. Similarly for each $K \in \mathscr{T}_{h,t}$ we set

$$\partial K^-(t) = \{x \in \partial K;\ \boldsymbol{w}(x,t)\cdot\boldsymbol{n}(x) < 0\},$$
$$\partial K^+(t) = \{x \in \partial K;\ \boldsymbol{w}(x,t)\cdot\boldsymbol{n}(x) \geq 0\}.$$

By $\mathscr{F}_{h,t}$ we denote the system of all faces of all elements $K \in \mathscr{T}_{h,t}$. It consists of the set of all inner faces $\mathscr{F}_{h,t}^I$ and the set of all boundary faces $\mathscr{F}_{h,t}^B$: $\mathscr{F}_{h,t} = \mathscr{F}_{h,t}^I \cup \mathscr{F}_{h,t}^B$. Each $\Gamma \in \mathscr{F}_{h,t}$ will be associated with a unit normal vector $\boldsymbol{n}_\Gamma$. By $K_\Gamma^{(L)}$ and $K_\Gamma^{(R)} \in \mathscr{T}_{h,t}$ we denote the elements adjacent to the face $\Gamma \in \mathscr{F}_{h,t}$. We shall use the convention that $\boldsymbol{n}_\Gamma$ is the outer normal to $\partial K_\Gamma^{(L)}$. Over a triangulation $\mathscr{T}_{h,t}$, for each positive integer k, we define the broken Sobolev space $H^k(\Omega_t, \mathscr{T}_{h,t}) = \{\varphi; \varphi|_K \in H^k(K) \quad \forall K \in \mathscr{T}_{h,t}\}$.

If $\varphi \in H^1(\Omega_t, \mathscr{T}_{h,t})$ and $\Gamma \in \mathscr{F}_{h,t}$, then $\varphi|_\Gamma^{(L)}, \varphi|_\Gamma^{(R)}$ will denote the traces of φ on Γ from the side of elements $K_\Gamma^{(L)}, K_\Gamma^{(R)}$ adjacent to Γ. For $\Gamma \in \mathscr{F}_{h,t}^I$ we set

$$\langle\varphi\rangle_\Gamma = \frac{1}{2}\left(\varphi|_\Gamma^{(L)} + \varphi|_\Gamma^{(R)}\right), \quad [\varphi]_\Gamma = \varphi|_\Gamma^{(L)} - \varphi|_\Gamma^{(R)},$$
$$h(\Gamma) = \frac{h_{K_\Gamma^{(L)}} + h_{K_\Gamma^{(R)}}}{2} \quad \text{for } \Gamma \in \mathscr{F}_{h,t}^I, \quad h(\Gamma) = h_{K_\Gamma^{(L)}} \quad \text{for } \Gamma \in \mathscr{F}_{h,t}^B.$$

If $u, \varphi \in H^2(\Omega_t, \mathscr{T}_{h,t})$, $\theta \in \mathbb{R}$ and $c_W > 0$, we introduce the following forms.

Convection form:

$$b_h(u,\varphi,t) = \sum_{K\in\mathscr{T}_{h,t}} \int_K \boldsymbol{w}\cdot\nabla u\,\varphi\,dx - \sum_{K\in\mathscr{T}_{h,t}} \int_{\partial K^- \cap \partial\Omega_t} \boldsymbol{w}\cdot\boldsymbol{n}u\varphi\,dS - \sum_{K\in\mathscr{T}_{h,t}} \int_{\partial K^- \setminus \partial\Omega_t} \boldsymbol{w}\cdot\boldsymbol{n}[u]\varphi\,dS,$$

Diffusion form:

$$a_h(u,\varphi,t) = \sum_{K\in\mathscr{T}_{h,t}} \int_K \nabla u\cdot\nabla\varphi\,dx - \sum_{\Gamma\in\mathscr{F}_{h,t}^I} \int_\Gamma \left(\langle\nabla u\rangle\cdot\boldsymbol{n}_\Gamma\,[\varphi] + \theta\,\langle\nabla\varphi\rangle\cdot\boldsymbol{n}_\Gamma\,[u]\right)\,dS - \sum_{K\in\mathscr{T}_{h,t}} \int_{\partial K^- \cap \partial\Omega_t} \left(\nabla u\cdot\boldsymbol{n}_\Gamma\,\varphi + \theta\nabla\varphi\cdot\boldsymbol{n}_\Gamma\,u - \theta\nabla\varphi\cdot\boldsymbol{n}_\Gamma\,u_D\right)\,dS,$$

Interior and boundary penalty:

$$J_h(u,\varphi,t) = c_W \sum_{\Gamma\in\mathscr{F}_{h,t}^I} h(\Gamma)^{-1} \int_\Gamma [u]\,[\varphi]\,dS + c_W \sum_{K\in\mathscr{T}_{h,t}} h(\Gamma)^{-1} \int_{\partial K^- \cap \partial\Omega_t} u\,\varphi\,dS,$$

$$A_h(u,\varphi,t) = \epsilon a_h(u,\varphi,t) + \epsilon\,J_h(u,\varphi,t),$$

Reaction form:

$$c_h(u,\varphi,t) = \sum_{K\in\mathscr{T}_{h,t}} \int_K cu\varphi\,dx,$$

Right-hand side form:

$$l_h(\varphi, t) = \sum_{K\in\mathscr{T}_{h,t}} \int_K g\varphi\, dx + \epsilon\, c_W \sum_{\Gamma\in\mathscr{F}^B_{h,t}} h(\Gamma)^{-1} \int_\Gamma u_D\, \varphi\, dS.$$

Let us note that in integrals over faces we omit the subscript Γ. We consider $\theta = 1$, $\theta = 0$ and $\theta = -1$ and get the symmetric (SIPG), incomplete (IIPG) and nonsymmetric (NIPG) variants of the approximation of the diffusion terms, respectively.

Further, we set

$$(\varphi, \psi)_\omega = \int_\omega \varphi\psi\, dx, \quad \|\varphi\|_\omega = \left(\int_\omega |\varphi|^2\, dx\right)^{1/2},$$
$$\|\eta\|_{\boldsymbol{w},\sigma} = \left\| \sqrt{|\boldsymbol{w}\cdot\boldsymbol{n}|}\, \eta \right\|_{L^2(\sigma)},$$

where $\omega \subset \mathbb{R}^2$, σ is either a subset of $\partial\Omega$ or ∂K and $\boldsymbol{n}$ denotes the corresponding outer unit normal to $\partial\Omega$ or ∂K, provided the integrals make sense.

Let $p,\ q \geq 1$ be integers. For any $m = 1, \ldots, M$ and $t \in [0, T]$ we define the finite-dimensional spaces

$$S^p_{h,t} = \left\{\varphi \in L^2(\Omega_t);\ \varphi|_K \in P^p(K),\ K \in \mathscr{T}_{h,t},\ t \in [0, T]\right\},$$
$$S^{p,q}_{h,\tau} = \Big\{\varphi \in L^2(Q_T);\ \varphi = \varphi(x, t),\ \text{for each } X \in \Omega_0$$
$$\text{the function } \varphi(\mathscr{A}_t(X), t) \text{ is a polynomial}$$
$$\text{of degree } \leq q \text{ in } t,\ \varphi(\cdot, t) \in S^p_{h,t} \text{ for every } t \in I_m,\ m = 1, \ldots, M\Big\}.$$

Definition 1 We say that function U is an approximate solution of problem (5)–(7), if $U \in S^{p,q}_{h,\tau}$ and

$$\int_{I_m} \big((D_t U, \varphi)_{\Omega_t} + A_h(U, \varphi, t) + b_h(U, \varphi, t) + c_h(U, \varphi, t)\big)\, dt \tag{8}$$
$$+(\{U\}_{m-1}, \varphi^+_{m-1})_{\Omega_{t_{m-1}}} = \int_{I_m} l_h(\varphi, t)\, dt \quad \forall\varphi \in S^{p,q}_{h,\tau}, \quad m = 1, \ldots, M,$$
$$U_0^- \in S^p_{h,0}, \quad (U_0^- - u^0, v_h) = 0 \quad \forall v_h \in S^p_{h,0}. \tag{9}$$

3 Analysis of the Stability

In our further considerations for each $t \in [0, T]$ we introduce a system of conforming triangulations $\{\mathscr{T}_{h,t}\}_{h\in(0,h_0)}$, where $h_0 > 0$. We assume that it is shape regular and locally quasiuniform. Under these assumptions, the multiplicative trace inequality and the inverse inequality hold.

Moreover, we assume that $\mathscr{T}_{h,t} = \{K_t = \mathscr{A}_t(K_0); K_0 \in \mathscr{T}_{h,0}\}$. This assumption is usually satisfied in practical computations, when the ALE mapping $\mathscr{A}_t$ is a continuous, piecewise affine mapping in $\overline{\Omega}_0$ for each $t \in [0, T]$.

In the space $H^1(\Omega, \mathscr{T}_{h,t})$ we define the norm

$$\|\varphi\|_{DG,t} = \left(\sum_{K \in \mathscr{T}_{h,t}} |\varphi|^2_{H^1(K)} + J_h(\varphi, \varphi, t) \right)^{1/2}.$$

Moreover, over $\partial\Omega$ we define the norm

$$\|u_D\|_{DGB,t} = \left(c_W \sum_{K \in \mathscr{T}_{h,t}} h^{-1}(\Gamma) \int_{\partial K^- \cap \partial\Omega_t} |u_D|^2 \, dS \right)^{1/2}.$$

If we use $\varphi := U$ as a test function in (8), we get the basic identity

$$\int_{I_m} \big((D_t U, U)_{\Omega_t} + A_h(U, U, t) + b_h(U, U, t) + c_h(U, U, t) \big) \, dt \tag{10}$$

$$+ (\{U\}_{m-1}, U^+_{m-1})_{\Omega_{t_{m-1}}} = \int_{I_m} l_h(U, t) \, dt.$$

Let us denote

$$\sigma(U) = \frac{1}{2} \sum_{K \in \mathscr{T}_{h,t}} \left(\|U\|^2_{\boldsymbol{w}, \partial K \cap \partial\Omega} + \|[U]\|^2_{\boldsymbol{w}, \partial K^- \setminus \partial\Omega} \right). \tag{11}$$

For a sufficiently large constant c_W, whose lower bound is determined by the constants from the multiplicative trace inequality, inverse inequality and local quasiuniformity of the meshes, we can prove the coercivity of the diffusion and penalty terms:

$$\int_{I_m} A_h(U, U, t) \, dt \geq \frac{\epsilon}{2} \int_{I_m} \|U\|^2_{DG,t} \, dt - \frac{\epsilon}{2} \int_{I_m} \|u_D\|^2_{DGB,t} \, dt. \tag{12}$$

Furthermore, if $k_1 > 0$, then the following inequalities for the convective term, reaction term and for the right-hand side form hold:

$$b_h(U, U, t) = \sigma(U) - \frac{1}{2} \int_{\Omega_t} U^2 \nabla \cdot \boldsymbol{w} \, dx, \tag{13}$$

$$\int_{I_m} |c_h(U, U, t)| \, dt \leq c_c \int_{I_m} \|U\|^2_{\Omega_t} \, dt, \tag{14}$$

$$\int_{I_m} |l_h(U,t)|\, dt \le \frac{1}{2}\int_{I_m} \left(\|g\|^2_{\Omega_t} + \|U\|^2_{\Omega_t}\right)\, dt \qquad (15)$$
$$+\epsilon k_1 \int_{I_m} \|u_D\|^2_{DGB,t}\, dt + \frac{\epsilon}{k_1}\int_{I_m} \|U\|^2_{DG,t}\, dt.$$

In what follows, we are concerned with the derivation of inequalities based on estimating the expression $\int_{I_m} (D_t U, U)_{\Omega_t}\, dt$. By some manipulation we find that

$$\int_{I_m} (D_t U, U)_{\Omega_t}\, dt + \left(\{U\}_{m-1}, U^+_{m-1}\right)_{\Omega_{t_{m-1}}} \qquad (16)$$
$$\ge \frac{1}{2}\left(\|U^-_m\|^2_{\Omega_{t_m}} - \|U^-_{m-1}\|^2_{\Omega_{t_{m-1}}} + \|\{U\}_{m-1}\|^2_{\Omega_{t_{m-1}}}\right)$$
$$-\frac{1}{2}\int_{I_m} (U^2, \nabla\cdot \boldsymbol{z})_{\Omega_t}\, dt,$$

and

$$\int_{I_m} (D_t U, U)_{\Omega_t}\, dt + (\{U\}_{m-1}, U^+_{m-1})_{\Omega_{t_{m-1}}} \qquad (17)$$
$$\ge \frac{1}{2}(\|U^-_m\|^2_{\Omega_{t_m}} + \frac{1}{2}\|U^+_{m-1}\|^2_{\Omega_{t_{m-1}}}) - (U^-_{m-1}, U^+_{m-1})_{\Omega_{t_{m-1}}}$$
$$-\frac{1}{2}\int_{I_m} (U^2, \nabla\cdot \boldsymbol{z})_{\Omega_t}\, dt.$$

Taking into account that $\sigma(U) \ge 0$ and $\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{z}$, from (10), (14) and (12)–(16) and putting $k_1 = 4$, we get the relation

$$\|U^-_m\|^2_{\Omega_{t_m}} - \|U^-_{m-1}\|^2_{\Omega_{t_{m-1}}} - \int_{I_m} (U^2, \nabla\cdot \boldsymbol{v})_{\Omega_t}\, dt \qquad (18)$$
$$+ \int_{I_m} (2c-1, U^2)_{\Omega_t} + \frac{\epsilon}{2}\int_{I_m} \|U\|^2_{DG,t}\, dt$$
$$\le c_1 \int_{I_m} \left(\|g\|^2_{\Omega_t} + \|u_D\|^2_{DGB,t}\right)\, dt$$

with a constant c_1 independent of data, h and τ.

First, let us assume that

$$2c - \nabla\cdot \boldsymbol{v} \ge 1. \qquad (19)$$

Then the summation of (18) over $m = 1, \ldots, k \ \leq M$ yields the estimate

$$\|U_k^-\|_{\Omega_{t_k}} + \frac{\epsilon}{2} \sum_{m-1}^{k} \int_{I_m} \|U\|_{DG,t}^2 \, dt \tag{20}$$

$$\leq \|U_0^-\|_{\Omega_0}^2 + c_1 \sum_{m-1}^{k} \int_{I_m} \left(\|g\|_{\Omega_t}^2 + \|u_D\|_{DGB,t}^2\right) \, dt,$$

which proves the stability.

If condition (19) is not valid, then the stability analysis is more complicated. In this case, instead of (18) we get the inequality

$$\|U_m^-\|_{\Omega_{t_m}}^2 - \|U_{m-1}^-\|_{\Omega_{t_{m-1}}}^2 + \frac{\epsilon}{2} \int_{I_m} \|U\|_{DG,t}^2 \, dt \tag{21}$$

$$\leq c_1 \sum_{m-1}^{k} \int_{I_m} \left(\|g\|_{\Omega_t}^2 + \|u_D\|_{DGB,t}^2\right) \, dt + c_2 \int_{I_m} \|U\|_{\Omega_t}^2 \, dt.$$

It is necessary to estimate the term $\int_{I_m} \|U\|_{\Omega_t}^2 \, dt$. It is rather technical and the proof has been carried out for $q = 1$, i.e., for piecewise linear time discretization. Then it is possible to show that there exist constants L_1 and M_1 such that

$$\|U_{m-1}^+\|_{\Omega_{t_{m-1}}}^2 + \|U_m^-\|_{\Omega_{t_m}}^2 \geq \frac{L_1}{\tau_m} \int_{I_m} \|U\|_{\Omega_t}^2 \, dt, \tag{22}$$

$$\|U_{m-1}^+\|_{\Omega_{t_{m-1}}}^2 \leq \frac{M_1}{\tau_m} \int_{I_m} \|U\|_{\Omega_t}^2 \, dt.$$

This allows to prove that there exists a constant $c^* > 0$ depending on c_2 and L_1 such that

$$\int_{I_m} \|U\|_{\Omega_t}^2 dt \leq \frac{2c_1}{L_1} \tau_m \int_{I_m} \left(\|g\|_{\Omega_t}^2 + \|u_D\|_{DGB,t}^2\right) \, dt + \frac{8M_1}{L_1^2} \tau_m \|U_{m-1}^-\|_{\Omega_{t_{m-1}}}^2 \tag{23}$$

holds, if $0 < \tau_m \leq c^*$.

Now, by virtue of (21) and (23), the summation over $m = 1, \ldots, k \ \leq M$ and the application of the discrete Gronwall lemma we get the following result.

Theorem 2 *Let* $q = 1$ *and* $0 < \tau_m \le c^*$. *Then there exists a constant* $c_3 > 0$ *such that*

$$\|U_m^-\|^2_{\Omega_{t_m}} + \sum_{j=1}^{m} \|\{U_{j-1}\}\|^2_{\Omega_{t_{j-1}}} + \frac{\beta_0}{2} \sum_{j=1}^{m} \int_{I_j} \|U\|^2_{DG,j}\, dt \tag{24}$$

$$\le c_3 \left(\|U_0^-\|^2_{\Omega_{t_0}} + \sum_{j=1}^{m} \int_{I_j} R_j\, dt \right),\ m = 1, \ldots, M,\ h \in (0, h_0),$$

where

$$R_j = c_1 \left(1 + \frac{2c_2}{L_1} \tau_j\right) \left(\|g\|^2_{\Omega_j} + \|u_D\|^2_{DGB,t} \right).$$

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References

1. M. Balázsová, M. Feistauer, M. Hadrava, A. Kosík, On the stability of the space-time discontinuous Galerkin method for the numerical solution of nonstationary nonlinear convection-diffusion problems. J. Numer. Math. **23**(3), 211–233 (2015)
2. A. Bonito, I. Kyza, R.H. Nochetto, Time-discrete higher-order ALE formulations: stability. SIAM J. Numer. Anal. **51**(1), 577–604 (2013)
3. V. Dolejší, M. Feistauer, *Discontinuous Galerkin Method – Analysis and Applications to Compressible Flow* (Springer, Cham, 2015)
4. M. Feistauer, J. Hájek, K. Švadlenka, Space-time discontinuous Galerkin method for solving nonstationary linear convection-diffusion-reaction problems. Appl. Math. **52**, 197–233 (2007)
5. H.-G. Roos, M. Stynes, L. Tobiska, *Robust Numerical Methods for Singularly Perturbed Differential Equations* (Springer, Berlin, 2008)