Quasi-Optimality Constants for Parabolic Galerkin Approximation in Space

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Abstract We consider Galerkin approximation in space of linear parabolic initial-boundary value problems where the elliptic operator is symmetric and thus induces an energy norm. For two related variational settings, we show that the quasi-optimality constant equals the stability constant of the L^2-projection with respect to that energy norm.

1 Introduction

A Galerkin method S for a variational problem is quasi-optimal in a norm $\|\cdot\|$ if there exists a constant q such that

$$\|u - U_S\| \leq q \inf_v \|u - v\| , \tag{1}$$

where u is any variational solution, U_S its associated Galerkin approximation and v varies in the discrete trial space. The quasi-optimality constant q_S is the best constant q in (1), and thus measures how well the Galerkin method S exploits the approximation potential offered by the discrete trial space. The determination or estimation of q_S is therefore the ideal first step in an a priori error analysis.

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B. Karasözen et al. (eds.), *Numerical Mathematics and Advanced Applications ENUMATH 2015*, Lecture Notes in Computational Science and Engineering 112, DOI 10.1007/978-3-319-39929-4_11

Here we are interested in Galerkin approximation in space for linear parabolic initial-boundary value problems like

$$\begin{gathered} \partial_t u - \Delta u = f \text{ in } \Omega \times (0,T), \\ u = 0 \text{ on } \partial\Omega \times (0,T), \quad u(\cdot,0) = w \text{ in } \Omega. \end{gathered} \tag{2}$$

Whereas for the stationary case, i.e. elliptic problems, quasi-optimality results like Céa's lemma are very common, such results have been less explored for parabolic problems. A common assumption of such results is that the L^2-projection onto the underlying discrete space is H^1-stable; see, e.g., [4, 5, 7], where the norm in (1) is either the one of $H^1(H^{-1}) \cap L^2(H^1)$ or the one of $L^2(H^1)$. Recently, the authors [8] have clarified the role of this assumption by showing that it is also necessary. This follows by applying the inf-sup theory [2, 3] to two weak, essentially dual formulations: the standard weak formulation with trial space $H^1(H^{-1}) \cap L^2(H^1)$ and the ultra-weak formulation with trial space $L^2(H^1)$.

This short note underlines the close relationship between parabolic quasi-optimality and the H^1-stability of the L^2-projection. It improves the results of [8] in the special case of a time-independent symmetric elliptic operator. For the model problem (2) and both variational formulations, this improvement reads as follows: the quasi-optimality constant of a Galerkin approximation with values in a discrete subspace S of H^1_0 is given by the operator norm in H^1_0 of the L^2-projection onto S:

$$q_{\text{std};S} = \|P_S\|_{\mathscr{L}(H^1_0)} = q_{\text{ult};S}. \tag{3}$$

2 Petrov-Galerkin Framework and Quasi-Optimality

This section, which is taken from [8], provides the general framework for the derivation of our quasi-optimality results. Let $(H_1, \|\cdot\|_1)$ and $(H_2, \|\cdot\|_2)$ be two real Hilbert spaces. The dual space H_2^* of H_2 is equipped with the usual dual norm $\|\ell\|_{H_2^*} = \sup_{\|\varphi\|_2=1} \ell(\varphi)$ for $\ell \in H_2^*$. Moreover, let b be a real-valued bounded bilinear form on $H_1 \times H_2$ and set $C_b := \sup_{\|v\|_1=\|\varphi\|_2=1} |b(v,\varphi)|$. We consider the problem

$$\text{given } \ell \in H_2^*, \text{ find } u \in H_1 \text{ such that } \forall \varphi \in H_2 \quad b(u,\varphi) = \ell(\varphi) \tag{4}$$

and say that it is well-posed if, for any $\ell \in H_2^*$, there exists a unique solution that continuously depends on ℓ. This holds if and only if there hold the following two conditions involving the so-called inf-sup constant c_b, cf. [3]:

$$c_b := \inf_{\|v\|_1=1} \sup_{\|\varphi\|_2=1} b(v,\varphi) > 0 \qquad \text{(uniqueness)}, \tag{5a}$$

$$\forall \varphi \in H_2 \setminus \{0\}\ \exists v \in H_1 \quad b(v,\varphi) > 0 \qquad \text{(existence)}. \tag{5b}$$

If (5) is satisfied, we have the duality

$$\inf_{\|v\|_1=1} \sup_{\|\varphi\|_2=1} b(v,\varphi) = \inf_{\|\varphi\|_2=1} \sup_{\|v\|_1=1} b(v,\varphi). \tag{6}$$

For notational simplicity, we take the viewpoint that a Petrov-Galerkin method for problem (4) is characterized by one pair of subspaces, instead of a family of pairs. Let $M_i \subset H_i$, $i = 1, 2$, be nontrivial and proper subspaces. The Petrov-Galerkin method $M = (M_1, M_2)$ for (4) reads

$$\text{given } \ell \in H_2^*, \text{ find } U_M \in M_1 \text{ such that } \forall \varphi \in M_2 \quad b(U_M,\varphi) = \ell(\varphi). \tag{7}$$

Problem (7) is well-posed if and only if there hold the semidiscrete counterparts of (5), involving the semidiscrete inf-sup constant c_M:

$$c_M := \inf_{v\in M_1:\|v\|_1=1} \sup_{\varphi\in M_2:\|\varphi\|_2=1} b(v,\varphi) > 0,$$

$$\forall \varphi \in M_2 \setminus \{0\}\ \exists v \in M_1 \quad b(v,\varphi) > 0.$$

A method M is quasi-optimal if there exists a constant $q \geq 1$ such that, for any $\ell \in H_2^*$, there holds

$$\|u - U_M\|_1 \leq q \inf_{v\in M_1} \|u - v\|_1 . \tag{8}$$

The quasi-optimality constant q_M of the method M is the smallest constant verifying (8). The formula for q_M in [8, Theorem 2.1] or combining [2, 3] with [9] imply

$$\frac{c_b}{c_M} \leq q_M \leq \frac{C_b}{c_M}. \tag{9}$$

3 Two Weak Formulations of Linear Parabolic Problems

In order to cast parabolic initial-boundary value problems in the form (4), we briefly recall two suitable weak formulations thereof.

Let V and W be two separable Hilbert spaces such that $V \subset W \subset V^*$ forms a Hilbert triplet. The scalar product in W as well as the duality pairing of $V^* \times V$ is denoted by $\langle \cdot, \cdot \rangle$. The norms are indicated by $\|\cdot\|_V$, $\|\cdot\|_W$, and $\|\cdot\|_{V^*} = \sup_{\|v\|_V=1} \langle \cdot, v \rangle$.

Let $A \in \mathscr{L}(V, V^*)$ be a linear and continuous operator arising from a symmetric bilinear form a via $\langle Av, \varphi \rangle = a(v, \varphi)$. We assume that a is bounded and coercive, i.e.

$$\nu_a := \inf_{\|v\|_V=1} a(v,v) > 0, \qquad C_a := \sup_{\|v\|_V=\|\varphi\|_V=1} a(v,\varphi) < \infty. \tag{10}$$

In view of (10) and the symmetry of a, the energy norm $\|\cdot\|_a = \langle A\cdot, \cdot\rangle^{1/2}$ and the dual energy norm $\|\cdot\|_{a;*} := \sup_{\|\varphi\|_a=1} \langle \cdot, \varphi \rangle$ are equivalent to $\|\cdot\|_V$ and $\|\cdot\|_{V^*}$, respectively. Moreover, for every $\ell \in V^*$ we have

$$\|\ell\|_{a;*} = \sup_{\|\varphi\|_a=1} \left\langle A\varphi, A^{-1}\ell \right\rangle = \left\|A^{-1}\ell\right\|_a = \sqrt{\langle \ell, A^{-1}\ell \rangle}. \tag{11}$$

Finally, given a final time $T > 0$ and a Hilbert space X, we set $I := (0, T)$ and denote with $L^2(X) := L^2(I; X)$ the space of all Lebesgue-measurable and square-integrable functions of the form $I \to X$. In addition, if Y is another Hilbert space, we set $H^1(X, Y) := \{v \in L^2(X) \mid v' \in L^2(Y)\}$ and write $H^1(X)$ for $H^1(X, X)$.

3.1 Standard Weak Formulation

The standard weak formulation is very common, also for some nonlinear parabolic problems. In the above setting, it reads

$$\begin{gathered} \text{given } f \in L^2(V^*) \text{ and } w \in W, \text{ find } u \in H^1(V, V^*) \text{ such that} \\ u' + Au = f \text{ in } I, \quad u(0) = w \end{gathered} \tag{12}$$

and can be cast in the form (4) by choosing $H_1 = H^1(V, V^*)$ and $H_2 = \{\varphi = (\varphi_0, \varphi_1) \mid \varphi_0 \in W, \varphi_1 \in L^2(V)\}$ with norms

$$\|v\|_1^2 = \|v(T)\|_W^2 + \int_I \|v\|_a^2 + \left\|v'\right\|_{a;*}^2, \quad \|\varphi\|_2^2 = \|\varphi_0\|_W^2 + \int_I \|\varphi_1\|_a^2. \tag{13}$$

Bilinear form and right-hand side are given, respectively, by

$$b(v, \varphi) = b_{\text{std}}(A; v, \varphi) := \langle v(0), \varphi_0 \rangle + \int_I \left\langle v', \varphi_1 \right\rangle + \langle Av, \varphi_1 \rangle \tag{14}$$

and $\ell(\varphi) = \langle w, \varphi_0 \rangle + \int_I \langle f, \varphi_1 \rangle$. We denote the constants of b_{std} by C_{std} etc.

The norm $\|\cdot\|_1$ in (13) slightly differs from the corresponding definition in [8] because it involves $v(T)$ instead of $v(0)$. This modification offers the following advantage, which was already observed in [1]: the norms in (13) mimic the energy norm for a linear elliptic problem in that the operator $v \mapsto b(v, \cdot)$ is an isometry. We provide a proof because its arguments will be used in what follows.

Proposition 1 (Isometry) *For every $v \in H_1$, we have $\|b(v,\cdot)\|_{H_2^*} = \|v\|_1$.*

Proof In view of $\int_I \langle v', v\rangle = \|v(T)\|_W^2 - \|v(0)\|_W^2$, the symmetry of A and (11), we have the identity

$$\|v(0)\|_W^2 + \int_I \left\|A^{-1}v' + v\right\|_a^2 = \|v(T)\|_W^2 + \int_I \left\|A^{-1}v'\right\|_a^2 + \|v\|_a^2 = \|v\|_1^2 \tag{15}$$

for every $v \in H_1$. On the one hand, this gives, for every $v \in H_1$, $\varphi \in H_2$,

$$\begin{aligned} b(v,\varphi) &= \langle v(0), \varphi_0\rangle + \int_I \langle v', \varphi_1\rangle + \langle Av, \varphi_1\rangle \\ &\le \left(\|v(0)\|_W^2 + \int_I \left\|A^{-1}v' + v\right\|_a^2\right)^{1/2} \|\varphi\|_2 = \|v\|_1 \|\varphi\|_2, \end{aligned}$$

which implies $\|b(v,\cdot)\|_{H_2^*} \le \|v\|_1$. On the other hand, choosing

$$\varphi_0 = v(0), \qquad \varphi_1 = v' + A^{-1}v \tag{16}$$

and using again (15), we get $\|\varphi\|_2 = \|v\|_1$ and

$$b(v,\varphi) = \|v(0)\|_W^2 + \int_I \langle v', v\rangle + \langle v', A^{-1}v\rangle + \langle Av, v\rangle = \|v\|_1^2.$$

Hence, $\|b(v,\cdot)\|_{H_2^*} \ge \|v\|_1$. □

Corollary 2 (Standard bilinear form) *The bilinear form b in (14) is continuous and satisfies the inf-sup condition with $C_{\mathrm{std}} = c_{\mathrm{std}} = 1$.*

Proof The equalities follow readily from Proposition 1. The proof of the non-degeneracy condition (5b) can be found in [8, Prop. 3.1]. □

3.2 Ultra-Weak Formulation

Discontinuous Galerkin methods, applications in optimization and stochastic PDEs motivate to consider solution notions with less regularity in time. In order to obtain such a solution notion for (12), one may multiply the differential equation with a test function

$$\varphi \in H_T^1(V, V^*) := \{\varphi \in L^2(I; V) \mid \varphi' \in L^2(I, V^*), \varphi(T) = 0\},$$

integrate in time and by parts. This results in the ultra-weak formulation, which can be cast in the form (4) by choosing $H_1 = L^2(V)$, and $H_2 = H^1_T(V, V^*)$, with norms

$$\|v\|_1^2 = \int_I \|v\|_V^2, \qquad \|\varphi\|_2^2 = \int_I \|\varphi\|_a^2 + \|\varphi'\|_{a;*}^2.$$

Here, bilinear form and right-hand side are given, respectively, by

$$b(v,\varphi) = b_{\text{ult}}(A; v, \varphi) := \int_I -\langle \varphi', v\rangle + \langle Av, \varphi\rangle \tag{17}$$

and $\ell(\varphi) = \langle w, \varphi(0)\rangle + \int_I \langle f, \varphi\rangle + \langle \varphi', f_1\rangle$, with $f \in L^2(V^*)$, $f_1 \in L^2(V)$ and $w \in W$. We denote the constants of b_{ult} by C_{ult} etc. Every solution of the standard weak formulation is one of the ultra-weak formulation.

Corollary 3 (Ultra-weak bilinear form) *The bilinear form b in (17) is continuous and satisfies the inf-sup condition with* $C_{\text{ult}} = c_{\text{ult}} = 1$.

Proof We exploit the duality with the standard weak formulation. Setting $\iota v(t) := v(T-t)$, $t \in I = (0, T)$ and using the symmetry of A, we have

$$\forall v_1 \in L^2(V), v_2 \in H^1_T(V, V^*) \quad b_{\text{ult}}(A; v_1, v_2) = b_{\text{std}}(A; \iota v_2, \iota v_1); \tag{18}$$

see [8, Lemma 4.1]. Since Proposition 1 holds also with $H^1_0(V, V^*) := \{v \in H^1(V, V^*) \mid v(0) = 0\}$ in place of $H^1(V, V^*)$, we thus deduce $C_{\text{ult}} = C_{\text{std}} = 1$ and $c_{\text{ult}} = c_{\text{std}} = 1$ with the help of (6). □

4 Galerkin Approximation in Space and Quasi-Optimality Constants

We review Galerkin approximation in space for the standard and the ultra-weak formulation and then derive identities for the corresponding quasi-optimality constants.

Let S be a finite-dimensional, nontrivial, and proper subspace of V. Observe that S is also a subspace of W and, with the identification $S^* = S$, also of V^*. As a subspace of V^*, we can equip $S = S^*$ with

$$\|\ell\|_{a;*} = \sup_{\varphi \in V} \frac{\langle \ell, \varphi\rangle}{\|\varphi\|_a} \quad \text{as well as} \quad \|\ell\|_{a;S^*} := \sup_{\varphi \in S} \frac{\langle \ell, \varphi\rangle}{\|\varphi\|_a}.$$

The following relationship, which can be found, e.g., in [8, Proposition 2.5], will be crucial:

$$\sup_{\ell \in S} \frac{\|\ell\|_{a;*}}{\|\ell\|_{a;S^*}} = \|P_S\|_{\mathscr{L}(V, \|\cdot\|_a)} := \sup_{\|w\|_a = 1} \|P_S w\|_a, \tag{19}$$

where P_S is the W-orthogonal projection onto S satisfying $\langle P_S w, \varphi \rangle = \langle w, \varphi \rangle$ for all $\varphi \in S$ and every $w \in W$.

4.1 Standard Weak Formulation

We first consider the standard weak formulation and define the spaces H_1, H_2, their norms and the bilinear form b as in Sect. 3.1. The Galerkin approximation with values in S is characterized by (7) with $M = (M_1, M_2)$ where

$$M_1 = H^1(S) \subset H_1, \qquad M_2 = S \times L^2(S) \subset H_2. \tag{20}$$

In order to determine the associated inf-sup constant $c_{\mathrm{std};S}$ in (5a), we first derive a discrete counterpart of Proposition 1. To this end, we define on M_1 the following S-dependent variant of $\|\cdot\|_1$:

$$\|v\|_{1;S}^2 := \|v(T)\|_W^2 + \int_I \|v\|_a^2 + \left\|v'\right\|_{a;S^*}^2,$$

where we replaced the dual norm $\|\cdot\|_{a;*}$ of the time derivative with the discrete dual norm $\|\cdot\|_{a;S^*}$. This gives rise to

$$\tilde{c}_{\mathrm{std};S} := \inf_{v \in M_1} \sup_{\varphi \in M_2} \frac{b(v,\varphi)}{\|v\|_{1;S} \|\varphi\|_2}, \qquad \tilde{C}_{\mathrm{std};S} := \sup_{v \in M_1} \sup_{\varphi \in M_2} \frac{b(v,\varphi)}{\|v\|_{1;S} \|\varphi\|_2}$$

and

$$\inf_{v \in M_1} \frac{\|v\|_{1;S}}{\|v\|_1} \tilde{c}_{\mathrm{std};S} \le c_{\mathrm{std};S} \le \inf_{v \in M_1} \frac{\|v\|_{1;S}}{\|v\|_1} \tilde{C}_{\mathrm{std};S}. \tag{21}$$

Proposition 4 (Discrete isometry) *For every $v \in M_1$, we have*

$$\|b(v,\cdot)\|_{M_2^*} := \sup_{\varphi \in M_2} \frac{b(v,\varphi)}{\|\varphi\|_2} = \|v\|_{1;S}.$$

Proof In order to proceed as in the proof of Proposition 1, we introduce the discrete counterpart of A, namely the operator $A_S : S \to S^*$ given by $\langle A_S v, \varphi \rangle = a(v, \varphi)$, for every $v, \varphi \in S$. In analogy to (11), we have $\left\langle \ell, A_S^{-1} \ell \right\rangle = \|A_S^{-1}\ell\|_a^2 = \|\ell\|_{a;S^*}^2$. We thus conclude as in the proof of Proposition 1, upon replacing $\varphi = (\varphi_0, \varphi_1)$ in (16) with $\varphi_0 = v(0) \in S$, $\varphi_1 = v + A_S^{-1} v' \in L^2(S)$. □

Consequently, the counterparts of the identities in Corollary 2 are

$$\tilde{c}_{\mathrm{std};S} = \tilde{C}_{\mathrm{std};S} = 1, \tag{22}$$

which imply a symmetric error estimate for $\|\cdot\|_{1;S}$, similar to the one in [6]. For $\|\cdot\|_1$ instead, we have:

Theorem 5 (Quasi-optimality in $H^1(V, \|\cdot\|_a; V^*, \|\cdot\|_{a;*})$) *The quasi-optimality constant of the Galerkin method (20) is given in terms of the W-projection onto S by*

$$q_{\mathrm{std};S} = \|P_S\|_{\mathscr{L}(V,\|\cdot\|_a)} .$$

Proof Identity (19) entails that the ratio of the two norms in the trial space is

$$\sup_{v \in M_1} \frac{\|v\|_1}{\|v\|_{1;S}} = \|P_S\|_{\mathscr{L}(V,\|\cdot\|_a)} , \tag{23}$$

see [8, Proposition 2.5 and (3.14)]. We thus deduce

$$q_{\mathrm{std};S} = c_{\mathrm{std};S}^{-1} = \sup_{v \in M_1} \frac{\|v\|_1}{\|v\|_{1;S}} = \|P_S\|_{\mathscr{L}(V,\|\cdot\|_a)} . \tag{24}$$

by using Corollary 2 in (9) and (22) in (21). □

Remark 6 (Non-symmetric case) If a is not symmetric, Theorem 5 can be generalized to

$$\kappa_a^{-1} \|P_S\|_{\mathscr{L}(V,\|\cdot\|_a)} \le q_{\mathrm{std};S} \le \kappa_a \|P_S\|_{\mathscr{L}(V,\|\cdot\|_a)} ,$$

where $\|\cdot\|_a$ is given by the symmetric part of a and κ_a depends on C_a and ν_a, with $\kappa_a = 1$ whenever a is symmetric. To this end, the bilinear form is split into its symmetric and skew-symmetric part, where the latter part is treated as a perturbation. An alternative and more general approach is offered by [8]. That analysis appears to be simpler but we only have $\kappa_a = \sqrt{2}$ if a is symmetric and one adopts the above energy-norm setting.

4.2 Ultra-Weak Formulation

We turn to Galerkin approximation based upon the ultra-weak formulation. Let the spaces H_1, H_2, their norms and the bilinear form b be given as in Sect. 3.2. The corresponding Galerkin approximation with values in S is characterized by (7) with $M = (M_1, M_2)$ where

$$M_1 = L^2(S) \subset H_1, \quad M_2 = H_T^1(S) := H^1(S) \cap H_T^1(V, V^*) \subset H_2. \tag{25}$$

Also, the Galerkin approximation of the ultra-weak formulation generalizes the Galerkin approximation of the standard weak formulation. Moreover:

Theorem 7 (Quasi-optimality in $L^2(V, \|\cdot\|_a)$) *The quasi-optimality constant of the ultra-weak Galerkin method (25) is determined in terms of the W-projection onto S by*

$$q_{\mathrm{ult};S} = \|P_S\|_{\mathscr{L}(V,\|\cdot\|_a)} .$$

Proof We exploit again duality. To this end, notice first that Proposition 4 and (23) hold also if $H^1(S)$ is replaced by $H^1_0(S) := \{v \in H^1(S) \mid v(0) = 0\}$. Hence, the discrete inf-sup constant does not change under this replacement and (18) yields $c_{\mathrm{ult};S} = c_{\mathrm{std};S}$. We thus obtain

$$q_{\mathrm{ult};S} = c^{-1}_{\mathrm{ult};S} = c^{-1}_{\mathrm{std};S} = \|P_S\|_{\mathscr{L}(V,\|\cdot\|_a)}$$

by using Corollary 3 in (9) and (24). □

Theorems 5 and 7 with $W = L^2(\Omega)$, $V = H^1_0(\Omega)$ and $A = -\Delta$ yield (3).

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