Finite Elements for the Navier-Stokes Problem with Outflow Condition

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Abstract This work is devoted to the Directional Do-Nothing (DDN) condition as an outflow boundary condition for the incompressible Navier-Stokes equation. In contrast to the Classical Do-Nothing (CDN) condition, we have stability, existence of weak solutions and, in the case of small data, also uniqueness. We derive an a priori error estimate for this outflow condition for finite element discretizations with inf-sup stable pairs. Stabilization terms account for dominant convection and the divergence free constraint. Numerical examples demonstrate the stability of the method.

1 Introduction

The classical do-nothing condition is very often prescribed at outflow boundaries for fluid dynamical problems. However, in the case of the Navier-Stokes equations in a domain $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, not even existence of weak solutions can be shown, see [10]. The reason is that this boundary condition does not exhibit any control about inflow across such boundaries, see [4]. This has also severe impact onto the stability of numerical algorithms for flows at higher Reynolds numbers. Denoting the velocity field by $\boldsymbol{u}$ and the pressure by p, the directional do-nothing (DDN) boundary condition

$$\nu \nabla \boldsymbol{u} \cdot \boldsymbol{n} - p\boldsymbol{n} - \beta(\boldsymbol{u} \cdot \boldsymbol{n})_{-}\boldsymbol{u} = 0 \quad \text{at } S_1 \tag{1}$$

on $S_1 \subseteq \partial\Omega$ with normal vector $\boldsymbol{n}$ and a parameter $\beta \geq 0$ is one possibility to circumvent this disadvantage. Here, $(\boldsymbol{u} \cdot \boldsymbol{n})_{-}(x) = \min(0, \boldsymbol{u}(x) \cdot \boldsymbol{n}(x))$ denotes the negative part of the flux across the boundary at $x \in \partial\Omega$. In particular, existence of

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weak solutions is proved in [4], and in several applications the stability is enhanced compared to the classical do-nothing condition ($\beta = 0$), see e.g. [2, 12].

In the case of pure outflow, i.e. if $\boldsymbol{u} \cdot \boldsymbol{n} \geq 0$ on S_1, this condition is identical to the classical do-nothing condition (CDN). In particular, it reproduces Poiseuille flow for a laminar flow in a tube with parabolic inflow.

The outflow condition (1) has similarities with the *convective boundary condition* in [5, 6], but no reference solution nor Stokes solution on a larger domain is needed. We also like to refer to the recent work [8] where a different open boundary condition is proposed which makes use of a smoothed step function and overcomes backflow instabilities as well.

2 Variational Formulation

The variational spaces for velocity and pressure are given by

$$\begin{aligned} \boldsymbol{V} &:= \{\boldsymbol{u} \in H^1(\Omega)^d \,|\, \boldsymbol{u} = 0 \text{ a.e. on } S_0\}, \\ Q &:= L^2(\Omega), \end{aligned}$$

respectively. The norm in $L^2(\omega)$ (and in $L^2(\omega)^d$) for $\omega \subseteq \Omega$ is denoted by $\|\cdot\|_\omega$. For $\omega = \Omega$ we surpress the index. The $H^{-1}(\Omega)$-norm is denoted by $\|\cdot\|_{-1}$. In order to formulate the Navier-Stokes system in variational form we consider the decomposition

$$((\boldsymbol{w}\cdot\nabla)\boldsymbol{u}, \boldsymbol{\phi}) = \frac{1}{2}\left(((\boldsymbol{w}\cdot\nabla)\boldsymbol{u}, \boldsymbol{\phi}) - (\boldsymbol{u}, (\boldsymbol{w}\cdot\nabla)\boldsymbol{\phi})\right) + \frac{1}{2}\int_{\partial\Omega} (\boldsymbol{w}\cdot\boldsymbol{n})\boldsymbol{u}\cdot\boldsymbol{\phi}\, ds,$$

of the convective term for divergence free vector fields $\boldsymbol{w}$ and use the notation

$$\begin{aligned} c(\boldsymbol{w};\boldsymbol{u},\boldsymbol{\phi}) := {}& \tfrac{1}{2}((\boldsymbol{w}\cdot\nabla)\boldsymbol{u}, \boldsymbol{\phi}) - \tfrac{1}{2}(\boldsymbol{u}, (\boldsymbol{w}\cdot\nabla)\boldsymbol{\phi}) \\ & + \int_{S_1} \left(\tfrac{1}{2}(\boldsymbol{w}\cdot\boldsymbol{n}) - \beta(\boldsymbol{w}\cdot\boldsymbol{n})_-\right) \boldsymbol{u}\cdot\boldsymbol{\phi}\, ds. \end{aligned}$$

Lemma 1 *The nonlinear convective term can be expressed by*

$$c(\boldsymbol{w};\boldsymbol{u},\boldsymbol{\phi}) = ((\boldsymbol{w}\cdot\nabla)\boldsymbol{u}, \boldsymbol{\phi}) + \tfrac{1}{2}(div\, \boldsymbol{w}\, \boldsymbol{u}, \boldsymbol{\phi}) - \beta \int_{S_1} (\boldsymbol{w}\cdot\boldsymbol{n})_- \boldsymbol{u}\cdot\boldsymbol{\phi}\, ds$$

for all $\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{\phi} \in \boldsymbol{V}$.

Proof This identity follows easily by integration by parts.

The semi-linear form for the Navier-Stokes system with DDN condition reads

$$A(\boldsymbol{w};\boldsymbol{u},p;\boldsymbol{\phi},\chi) := c(\boldsymbol{w};\boldsymbol{u},\boldsymbol{\phi}) + (\nu\nabla\boldsymbol{u}, \nabla\boldsymbol{\phi}) - (p, \operatorname{div}\boldsymbol{\phi}) + (\operatorname{div}\boldsymbol{u}, \chi).$$

We seek $(\boldsymbol{u}, p) \in \boldsymbol{V} \times Q$ s.t.

$$(\partial_t \boldsymbol{u}, \boldsymbol{\phi}) + A(\boldsymbol{u}; \boldsymbol{u}, p; \boldsymbol{\phi}, \chi) = \langle \boldsymbol{f}, \boldsymbol{\phi} \rangle \qquad \forall \boldsymbol{\phi} \in \boldsymbol{V}, \forall \chi \in Q. \tag{2}$$

Diagonal testing with the solution, $\boldsymbol{\phi} := \boldsymbol{u}$, $\chi := p$, results in

$$\frac{1}{2}\partial_t \|\boldsymbol{u}\|^2 + \nu \|\nabla \boldsymbol{u}\|^2 + \int_{S_1} \left(\frac{1}{2}(\boldsymbol{u} \cdot \boldsymbol{n})_+ + \left(\frac{1}{2} - \beta \right) (\boldsymbol{u} \cdot \boldsymbol{n})_- \right) |\boldsymbol{u}|^2 ds$$
$$\leq \|\boldsymbol{f}\|_{-1} \|\nabla \boldsymbol{u}\|.$$

For $\beta \geq 1/2$ the arising boundary integral is non-negative. This property is needed to show existence of weak solutions, see techniques in [4]. Moreover, the solution is unique in the case of small data, see [3].

3 Finite Element Discretization

For the discretization of (2) in space we use inf-sup stable finite elements of order k for $\boldsymbol{u}_h$, for instance the classical Taylor-Hood element Q_2/Q_1 on quadrilaterals (for $d = 2$). Due to the fact that we use a divergence-free projection in the analysis below, we require for $d = 3$ on hexahedrons Q_3/Q_2 elements, see [9]. It is well-known that the convective terms and the divergence-free constraint should be stabilized in order to obtain more accurate discrete solutions with enhanced divergence properties and less over- and undershoots. We use a combination of div-div stabilization and local projection (LPS) of the convective terms

$$S_h(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{\phi}) := \sum_{M \in \mathcal{M}_h} \gamma_M (\operatorname{div} \boldsymbol{u}, \operatorname{div} \boldsymbol{\phi})_M + \alpha_M (\kappa_M[(\boldsymbol{w}_M \cdot \nabla)\boldsymbol{u}], \kappa_M[(\boldsymbol{w}_M \cdot \nabla)\boldsymbol{\phi}])_M$$

with local fluctuation operator $\kappa_M : L^2(M) \to L^2(M)$ on patches M, and piecewise constant approximation $\boldsymbol{w}_M$ of $\boldsymbol{w}$. We allow for the one-level ($M \in \mathcal{T}_h$) or the two-level ($M \in \mathcal{T}_{2h}$) variant, but the common requirements according to [11] should be satisfied. The stabilization parameter γ_M for the divergence stabilization is patch-wise constant in the following range:

$$0 < \gamma_0 h_{max} \leq \gamma_M \leq \gamma_{max}, \tag{3}$$

with positive constants $\gamma_0, \gamma_{max} > 0$ and the maximal mesh size $h_{max} = \max\{h_T : T \in \mathcal{T}_h\}$. The LPS parameter α_M must be non-negative (may vanish) and may depend on $\boldsymbol{u}_M$ but is bounded ($\alpha_0 \geq 0$):

$$0 \leq \alpha_M \leq \alpha_0 |\boldsymbol{u}_M|^{-2}. \tag{4}$$

Similar to the work [1] we assume for the a priori estimate the following local approximation property:

$$\|\boldsymbol{u} - \boldsymbol{u}_M\|_{L^\infty(M)} \leq C\|\boldsymbol{u}\|_{L^\infty(M)}. \tag{5}$$

The semi-discrete system consists in seeking $\boldsymbol{u}_h \in \boldsymbol{V}_h, p_h \in Q_h$ s.t.

$$(\partial_t \boldsymbol{u}_h, \boldsymbol{\phi}) + A(\boldsymbol{u}_h; \boldsymbol{u}_h, p_h; \boldsymbol{\phi}, \chi) + S_h(\boldsymbol{u}_h; \boldsymbol{u}_h, \boldsymbol{\phi}) = \langle \boldsymbol{f}, \boldsymbol{\phi} \rangle \tag{6}$$

for all $\boldsymbol{\phi} \in \boldsymbol{V}_h$ and all $\chi \in Q_h$.

4 A Priori Estimate

For the a priori estimate we split the error $\boldsymbol{e}_u := \boldsymbol{u} - \boldsymbol{u}_h$ into interpolation error $\boldsymbol{\eta}_u := \boldsymbol{u} - \boldsymbol{i}_h\boldsymbol{u}$ and projection error $\boldsymbol{\xi}_u := \boldsymbol{i}_h\boldsymbol{u} - \boldsymbol{u}_h$. Here, $\boldsymbol{i}_h : \boldsymbol{V} \to \boldsymbol{V}_h$ is a divergence-free projection. We use the following norm in $\boldsymbol{V}$:

$$\begin{aligned} |||\boldsymbol{u}|||_{\boldsymbol{u}_h}^2 = \nu\|\nabla \boldsymbol{u}\|^2 + \int_{S_1} \left(\tfrac{1}{2}(\boldsymbol{u}_h \cdot \boldsymbol{n})_+ + (\tfrac{1}{2} - \beta)(\boldsymbol{u}_h \cdot \boldsymbol{n})_-\right) |\boldsymbol{u}|^2 d\sigma \\ + S_h(\boldsymbol{u}_h; \boldsymbol{u}, \boldsymbol{u}). \end{aligned}$$

A bound on the interpolation error $\boldsymbol{\eta}_u$ is well-known, see [9]. Therefore we focus on the projection error.

Theorem 2 *Under the previous assumptions* (3), (4) *and* (5)*, enough regularity of the continuous solution* $\boldsymbol{u}$*,* p*, and* $\beta > 1/2$ *it holds for the projection error:*

$$\|\boldsymbol{\xi}_u\|^2_{L^\infty(0,T;L^2(\Omega))} + \int_0^T |||\boldsymbol{\xi}_u(t)|||^2_{\boldsymbol{u}_h} dt \leq C \int_0^T e^{C_G(t-\tau)} \sum_M \phi_M(\tau) d\tau$$

with the Gronwall constant

$$C_G := c(1 + |\boldsymbol{u}|_{L^\infty(0,T;W^{1,\infty}(\Omega))} + h\|\boldsymbol{u}\|^2_{L^\infty(0,T;W^{1,\infty}(\Omega))} + (1+\nu^{-1})\|\boldsymbol{u}\|_{L^\infty(S_1)}),$$

and the quantity ϕ_M *depending on* $\boldsymbol{u}$ *and on the interpolation errors* $\boldsymbol{\eta}_u$*,* η_p*:*

$$\phi_M := \|\partial_t \boldsymbol{\eta}_u\|_M^2 + (c_1 + c_3)\|\nabla \boldsymbol{\eta}_u\|_M^2 + c_2\|\boldsymbol{\eta}_u\|_M^2 + c_3\|\kappa_M(\nabla \boldsymbol{u})\|_M^2 + c_4\|\eta_p\|_M^2,$$

and coefficients $c_1, \ldots, c_4$*:*

$$c_1 = \nu + \gamma_M,\ c_2 = h_M^{-2} + \nu^{-1}\|\boldsymbol{u}\|^2_{L^\infty(M)},\ c_3 = \alpha_M|\boldsymbol{u}|_M^2,\ c_4 = (\nu + \gamma_M)^{-1}.$$

This bound is similar to the one published in [1] (for $S_1 = \emptyset$). The difference is the additional term $\nu^{-1}\|\boldsymbol{u}\|_{L^\infty(S_1)}$ in the Gronwall constant.

Proof In the first step we subtract (2) and (6) and perform diagonal testing. Due to the additive splitting of the error, $\boldsymbol{e}_u = \boldsymbol{\eta}_u + \boldsymbol{\xi}_u$, and after reordering terms we arrive at

$$\frac{1}{2}\partial_t\|\boldsymbol{\xi}_u\|^2 + |||\boldsymbol{\xi}_u|||^2_{\boldsymbol{u}_h} = -(\partial_t\boldsymbol{\eta}_u, \boldsymbol{\xi}_u) - \nu(\nabla\boldsymbol{\eta}_u, \nabla\boldsymbol{\xi}_u) + (\operatorname{div}\boldsymbol{\xi}_u, \eta_p) - (\operatorname{div}\boldsymbol{\eta}_u, \xi_p)$$
$$-c(\boldsymbol{u};\boldsymbol{u},\boldsymbol{\xi}_u) + c(\boldsymbol{u}_h;\boldsymbol{i}_h\boldsymbol{u},\boldsymbol{\xi}_u) + S_h(\boldsymbol{u}_h;\boldsymbol{i}_h\boldsymbol{u},\boldsymbol{\xi}_u)$$

Using Lemma 1 we obtain

$$\begin{aligned}
&-c(\boldsymbol{u};\boldsymbol{u},\boldsymbol{\xi}_u) + c(\boldsymbol{u}_h;\boldsymbol{i}_h\boldsymbol{u},\boldsymbol{\xi}_u)\\
&= -(\boldsymbol{u}\cdot\nabla\boldsymbol{u},\boldsymbol{\xi}_u) + (\boldsymbol{u}_h\cdot\nabla\boldsymbol{i}_h\boldsymbol{u},\boldsymbol{\xi}_u) + \tfrac{1}{2}(\operatorname{div}\boldsymbol{u}_h\,\boldsymbol{i}_h\boldsymbol{u},\boldsymbol{\xi}_u)\\
&\quad+\beta\int_{S_1}\left\{(\boldsymbol{u}\cdot\boldsymbol{n})_-\boldsymbol{u} - (\boldsymbol{u}_h\cdot\boldsymbol{n})_-\boldsymbol{i}_h\boldsymbol{u}\right\}\cdot\boldsymbol{\xi}_u ds\\
&= -(\boldsymbol{e}_u\cdot\nabla\boldsymbol{u},\boldsymbol{\xi}_u) - (\boldsymbol{u}_h\cdot\nabla\boldsymbol{\eta}_u,\boldsymbol{\xi}_u) + \tfrac{1}{2}(\operatorname{div}\boldsymbol{u}_h\,\boldsymbol{i}_h\boldsymbol{u},\boldsymbol{\xi}_u)\\
&\quad+\beta\int_{S_1}\left\{(\boldsymbol{u}\cdot\boldsymbol{n})_-\boldsymbol{u} - (\boldsymbol{u}_h\cdot\boldsymbol{n})_-\boldsymbol{i}_h\boldsymbol{u}\right\}\cdot\boldsymbol{\xi}_u ds.
\end{aligned}$$

Integration by parts a second time yields

$$(\boldsymbol{u}_h\cdot\nabla\boldsymbol{\eta}_u,\boldsymbol{\xi}_u) = -(\boldsymbol{\eta}_u,\boldsymbol{u}_h\cdot\nabla\boldsymbol{\xi}_u) - (\operatorname{div}\boldsymbol{u}_h\,\boldsymbol{\eta}_u,\boldsymbol{\xi}_u) + \int_{S_1}(\boldsymbol{u}_h\cdot\boldsymbol{n})\boldsymbol{\eta}_u\cdot\boldsymbol{\xi}_u\,ds.$$

We obtain the identity

$$\frac{1}{2}\partial_t\|\boldsymbol{\xi}_u\|^2 + |||\boldsymbol{\xi}_u|||^2_{\boldsymbol{u}_h} = R + T,$$

with volume integrals

$$\begin{aligned}
R := &-(\partial_t\boldsymbol{\eta}_u,\boldsymbol{\xi}_u) - \nu(\nabla\boldsymbol{\eta}_u,\nabla\boldsymbol{\xi}_u) + (\operatorname{div}\boldsymbol{\xi}_u,\eta_p) - (\operatorname{div}\boldsymbol{\eta}_u,\xi_p)\\
&-(\boldsymbol{e}_u\cdot\nabla\boldsymbol{u},\boldsymbol{\xi}_u) + \tfrac{1}{2}(\operatorname{div}\boldsymbol{u}_h\,\boldsymbol{i}_h\boldsymbol{u},\boldsymbol{\xi}_u) + (\boldsymbol{\eta}_u,\boldsymbol{u}_h\cdot\nabla\boldsymbol{\xi}_u)\\
&+(\operatorname{div}\boldsymbol{u}_h\,\boldsymbol{\eta}_u,\boldsymbol{\xi}_u) + S_h(\boldsymbol{u}_h;\boldsymbol{i}_h\boldsymbol{u},\boldsymbol{\xi}_u),
\end{aligned}$$

and boundary integrals

$$T := \int_{S_1}\left\{\beta(\boldsymbol{u}\cdot\boldsymbol{n})_-\boldsymbol{u} - \beta(\boldsymbol{u}_h\cdot\boldsymbol{n})_-\boldsymbol{i}_h\boldsymbol{u} - (\boldsymbol{u}_h\cdot\boldsymbol{n})\boldsymbol{\eta}_u\right\}\cdot\boldsymbol{\xi}_u ds.$$

For R we may use the result of [1]:

$$R \le \frac{1}{2}\|\partial_t \boldsymbol{\eta}_u\|^2 + C_G\|\boldsymbol{\xi}_u\|^2 + \frac{1}{4}|||\boldsymbol{\xi}_u|||^2_{\boldsymbol{u}_h} + 2\sum_M \phi_M.$$

The techniques in [1] do not require any further integration by parts. Therefore, the approach for the Dirichlet case without any outflow condition also applies to bound R in our case. The remaining terms of T will be bounded in the sequel. A basic calculus yield

$$T = T_1 + T_2$$

with

$$T_1 := -\int_{S_1} \{(\boldsymbol{u}_h \cdot \boldsymbol{n})_+ + (1-\beta)(\boldsymbol{u}_h \cdot \boldsymbol{n})_-\}\boldsymbol{\eta}_u \cdot \boldsymbol{\xi}_u \, ds,$$

$$T_2 := \beta \int_{S_1} \{(\boldsymbol{u} \cdot \boldsymbol{n})_- - (\boldsymbol{u}_h \cdot \boldsymbol{n})_-\}\boldsymbol{u} \cdot \boldsymbol{\xi}_u \, ds.$$

Since the triple-norm includes control on the boundary fluxes, T_1 is bounded by $|||\cdot|||_{\boldsymbol{u}_h}$ provided $\beta > \frac{1}{2}$:

$$\begin{aligned} T_1 &= -\int_{S_1} \{(\boldsymbol{u}_h \cdot \boldsymbol{n})_+ + (1-\beta)(\boldsymbol{u}_h \cdot \boldsymbol{n})_-\}\boldsymbol{\eta}_u \cdot \boldsymbol{\xi}_u \, ds \\ &\le \max\left(2, \frac{|\beta - 1|}{\beta - \frac{1}{2}}\right) |||\boldsymbol{\eta}_u|||_{\boldsymbol{u}_h} |||\boldsymbol{\xi}_u|||_{\boldsymbol{u}_h}. \end{aligned}$$

T_2 can be bounded by the trace theorem in L^1-norm and the product rule with arbitrary $\epsilon > 0$:

$$\begin{aligned} T_2 &= \beta \int_{S_1} \{(\boldsymbol{u} \cdot \boldsymbol{n})_- - (\boldsymbol{u}_h \cdot \boldsymbol{n})_-\}\boldsymbol{u} \cdot \boldsymbol{\xi}_u \, ds \\ &\le c_S \|\boldsymbol{u}\|_{L^\infty(S_1)} \| |\boldsymbol{e}_u||\boldsymbol{\xi}_u| \|_{W^{1,1}(\Omega)} \\ &\le c_S \|\boldsymbol{u}\|_{L^\infty(S_1)} (\| |\boldsymbol{e}_u||\boldsymbol{\xi}_u| \|_{L^1(\Omega)} + \|\nabla(|\boldsymbol{e}_u||\boldsymbol{\xi}_u|)\|_{L^1(\Omega)}) \\ &\le c_S \|\boldsymbol{u}\|_{L^\infty(S_1)} (1 + \nu^{-1})(\|\boldsymbol{\xi}_u\|^2 + \|\boldsymbol{\eta}_u\|^2) + \epsilon(|||\boldsymbol{\xi}_u|||^2_{\boldsymbol{u}_h} + |||\boldsymbol{\eta}_u|||^2_{\boldsymbol{u}_h}). \end{aligned}$$

Hence, the sum of the two terms T_1 and T_2 can now be bounded by

$$T \le C_G(\|\boldsymbol{\xi}_u\|^2 + \|\boldsymbol{\eta}_u\|^2) + (\epsilon + c_\beta \epsilon^{-1})|||\boldsymbol{\eta}_u|||^2_{\boldsymbol{u}_h} + \epsilon|||\boldsymbol{\xi}_u|||^2_{\boldsymbol{u}_h},$$

with a still arbitrary parameter $\epsilon > 0$. In combination with the upper bound for R we arrive for $\epsilon = \frac{1}{4}$ at

$$\partial_t \|\boldsymbol{\xi}_u\|^2 + |||\boldsymbol{\xi}_u|||^2_{\boldsymbol{u}_h} \leq \|\partial_t \boldsymbol{\eta}_u\|^2 + 4\sum_M \phi_M + C_G(\|\boldsymbol{\xi}_u\|^2 + \|\boldsymbol{\eta}_u\|^2) + c'_\beta |||\boldsymbol{\eta}_u|||^2_{\boldsymbol{u}_h}.$$

Application of the Gronwall lemma yields the assertion. □

Remark 3 If the solution $\boldsymbol{u}$ is sufficiently smooth, the previous Theorem can be used to derive an upper bound of the projection error in terms of powers of h_M by using

$$\phi_M(\tau) \leq Ch_M^{2k}\left(\|\partial_t \boldsymbol{u}(\tau)\|^2_{H^2(M)} + (c_5 + c_6)\|\boldsymbol{u}(\tau)\|^2_{H^{k+1}(M)} + c_4\|p(\tau)\|^2_{H^k(M)}\right)$$

with $c_4 = (\nu + \gamma_M)^{-1}$, $c_5 = 1 + \nu + \gamma_{max} + \alpha_0$, and $c_6 = h_M^2 \nu^{-1} \|\boldsymbol{u}\|^2_{L^\infty(M)}$.

Remark 4 The term c_6 in the previous Remark can be avoided by using a different bound in the proof of Theorem 2, see Dallmann [7] (page 44). However, this leads to a larger Gronwall constant $\widetilde{C}_G = C_G + \|\boldsymbol{u}_h\|^2_{L^\infty(\Omega)}$.

5 Numerical Results

We want to support the above analysis by numerical examples that show the desired convergence results in space. In particular we like to see that the error does not blow up in time or space, even if there is inflow at the boundary S_1.

The considered domain is given by $\Omega := (0, 2\pi) \times (-\pi, \pi)$. We use the directional do-nothing (DDN) at $S_1 := \{(2\pi, y) : -\pi \leq y \leq \pi\}$ with the parameter $\beta = 1$, and Dirichlet boundary at $S_0 := \partial\Omega \setminus S_1$. Let $\chi : \mathbb{R} \to [0, 1]$ defined as $\chi(y) = 1$ if $y < 0$, and $\chi(y) = 0$ for $y \geq 0$. The exact solution in analytical form and the corresponding right hand side are given by

$$\begin{aligned}
\boldsymbol{u}(x, y) &= (\sin(y)\cos(t)^2, 0)^T, \\
p(x, y) &= -\tfrac{1}{2}\chi(y)\sin(y)^2\cos(t)^4, \\
\boldsymbol{f}(x, y) &= (-\sin(2t)\sin(y) + \cos(t)^2\sin(y)\nu, -\chi(y)\cos(y)\sin(y)\cos(t)^4)^T.
\end{aligned}$$

We investigate the convergence behavior for the classical Taylor-Hood pair Q_2/Q_1. Since we are not interested in the error due to time discretization we set $\Delta t = 10^{-4}$ and evaluate the error at $T = 10^{-2}$.

In Fig. 1 we depict the L^2-errors with respect to velocity and pressure, $\|\boldsymbol{u} - \boldsymbol{u}_h\|$ and $\|p - p_h\|$, in dependence of a uniform mesh size h for various viscosities ν. We compare with ($\gamma = 1$) and without ($\gamma = 0$) div-div stabilization. For the velocity error in L^2 we observe convergence of third order in the case $\gamma = 1$. Without div-div stabilization the convergence order of $\|\boldsymbol{u} - \boldsymbol{u}_h\|$ is reduced. For the pressure, second

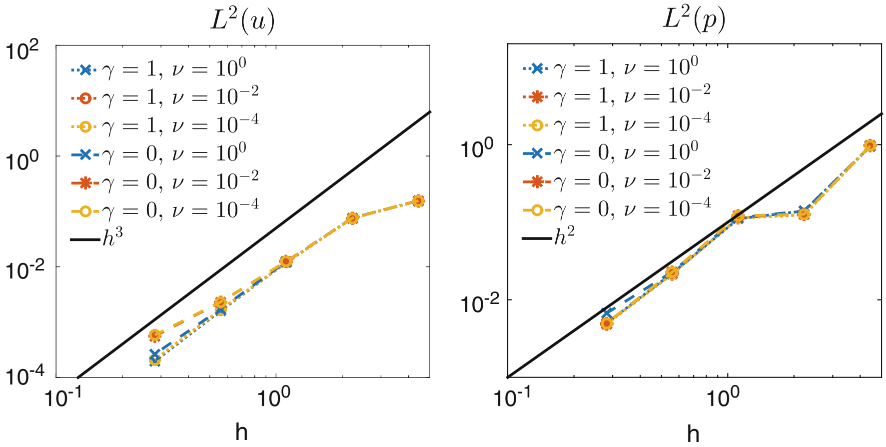

Fig. 1 L^2-errors of $\boldsymbol{u}$ and p for Taylor-Hood ($\mathscr{Q}_2/\mathscr{Q}_1$) elements

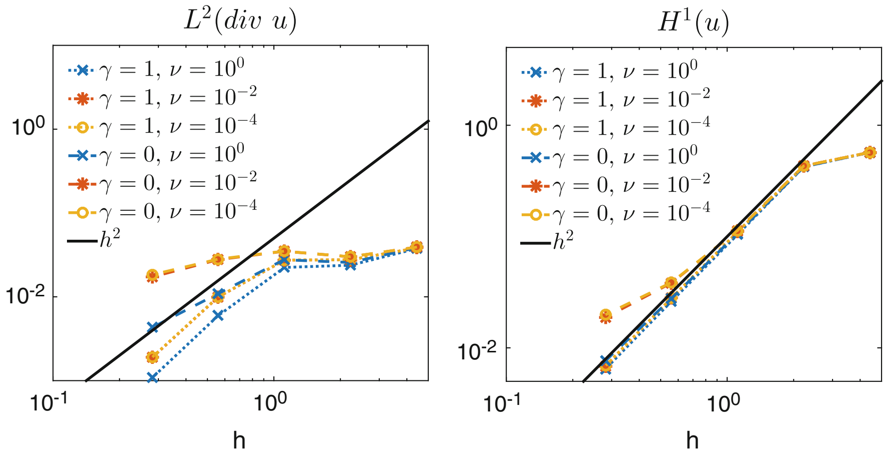

Fig. 2 Errors $\|\text{div}\,(\boldsymbol{u} - \boldsymbol{u}_h)\|$ and $|\boldsymbol{u} - \boldsymbol{u}_h|_1$ for Taylor-Hood ($\mathscr{Q}_2/\mathscr{Q}_1$) elements

order convergence can be observed which is in line with our analysis. The pressure error does essentially not depend on any of the parameters.

In Fig. 2 we show the errors $|\boldsymbol{u} - \boldsymbol{u}_h|_1$, and $\|\text{div}\,(\boldsymbol{u} - \boldsymbol{u}_h)\|$. Both quantities show quadratic convergence, i.e. at optimal rate, if div-div stabilization is used. For the velocity energy error $\|\boldsymbol{u} - \boldsymbol{u}_h\|$ and the $H^1(\Omega)$ error the results deviate from the optimal rate of convergence (h^3 resp. h^2) if no div-div stabilization is used. However, the biggest impact of the stabilization can be seen for the divergence error $\|\text{div}\,(\boldsymbol{u} - \boldsymbol{u}_h)\|$. For sufficiently small viscosity the error stays nearly constant if no div-div stabilization is used. Optimal convergence rates can be recovered if div-div stabilization is used. With respect to the LPS stabilization we did not observe any significant influence for the considered norms. Compared to results in [1] for Dirichlet boundary conditions the div-div stabilization seems to play a more important role.

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