Super Pairwise Comparison Matrix in the Dominant AHP

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Abstract We have proposed an SPCM (Super Pairwise Comparison Matrix) to express all pairwise comparisons in the evaluation process of D-AHP (the dominant analytic hierarchy process) or the multiple dominant AHP as a single pairwise comparison matrix. This paper shows that the evaluation value resulting from the application of LLSM (the logarithmic least-squares method) to an SPCM matches the evaluation value determined by the application of D-AHP to the evaluation values obtained from each pairwise comparison matrix by using the geometric mean.

Keywords Super pairwise comparison matrix • The dominant AHP • Logarithmic least-squares method

1 Introduction

AHP (the Analytic Hierarchy Process) proposed by Saaty [1] enables objective decision making by top-down evaluation based on an overall aim.

In actual decision making, a decision maker often has a specific alternative (regulating alternative) in mind and makes an evaluation on the basis of the alternative. This was modeled in D-AHP(the dominant AHP), proposed by Kinoshita and Nakanishi [2].

If there are more than one regulating alternatives and the importance of each criterion is inconsistent, the overall evaluation value may differ for each regulating alternative. As a method of integrating the importance in such cases, CCM (the

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concurrent convergence method) was proposed. Kinoshita and Sekitani [3] showed the convergence of CCM.

Ohya and Kinoshita [4] proposed an SPCM (Super Pairwise Comparison Matrix) to express all pairwise comparisons in the evaluation process of the dominant analytic hierarchy process (AHP) or the multiple dominant AHP (MDAHP) as a single pairwise comparison matrix.

Ohya and Kinoshita [5] showed, by means of a numerical counterexample, that in MDAHP an evaluation value resulting from the application of the logarithmic least-squares method (LLSM) to an SPCM does not necessarily coincide with that of the evaluation value resulting from the application of the geometric mean multiple dominant AHP (GMMDAHP) to the evaluation value obtained from each pairwise comparison matrix by using the geometric mean method.

Ohya and Kinoshita [6] showed, using the error models, that in D-AHP an evaluation value resulting from the application of the logarithmic least squares method (LLSM) to an SPCM necessarily coincide with that of the evaluation value resulting obtained by using the geometric mean method to each pairwise comparison matrix.

Ohya and Kinoshita [7] showed the treatment of hierarchical criteria in D-AHP with super pairwise comparison matrix.

Ohya and Kinoshita [8] showed the example of using SPCM with the application of LLSM for calculation of MDAHP.

This paper shows that the evaluation value resulting from the application of LLSM to an SPCM agrees with the evaluation value determined by the application of D-AHP to the evaluation value obtained from each pairwise comparison matrix by using the geometric mean.

2 D-AHP and SPCM

This section explains D-AHP and an SPCM to express the pairwise comparisons appearing in the evaluation processes of D-AHP and MDAHP as a single pairwise comparison matrix. Section 2.1 outlines D-AHP procedure and explicitly states pairwise comparisons, and Sect. 2.2 explains an SPCM that expresses these pairwise comparisons as a single pairwise comparison matrix.

2.1 Evaluation in D-AHP

The true absolute importance of alternative $a(a=1,\ldots,A)$ at criterion $c(c=1,\ldots,C)$ is v_{ca}. The final purpose of the AHP is to obtain the relative value (between alternatives) of the overall evaluation value $v_a=\sum_{c=1}^{C} v_{ca}$ of alternative a. The procedure of D-AHP for obtaining an overall evaluation value is as follows:

D-AHP

Step 1: The relative importance $u_{ca} = \alpha_c v_{ca}$ (where α_c is a constant) of alternative a at criterion c is obtained by some kind of methods. In this paper, u_{ca} is obtained by applying the pairwise comparison method to alternatives at criterion c.

Step2: Alternative d is the regulating alternative. The importance u_{ca} of alternative a at criterion c is normalized by the importance u_{cd} of the regulating alternative d, and $u_{ca}^{d}\ (= u_{ca}/u_{cd})$ is calculated.

Step3: With the regulating alternative d as a representative alternative, the importance w_c^d of criterion c is obtained by applying the pairwise comparison method to criteria, where, w_c^d is normalized by $\sum_{c=1}^{C} w_c^d = 1$.

Step4: From u_{ca}^{d}, w_c^d obtained at Steps 2 and 3, the overall evaluation value $t_a = \sum_{c=1}^{C} w_c^d u_{ca}^d$ of alternative a is obtained. By normalization at Steps 2 and 3, $u_d = 1$. Therefore, the overall evaluation value of regulating alternative d is normalized to 1.

2.2 SPCM

The relative comparison values $r_{c'a'}^{ca}$ of importance v_{ca} of alternative a at criteria c as compared with the importance $v_{c'a'}$ of alternative a' in criterion c', are arranged in a (CA × CA) or (AC × AC) matrix. This is proposed as an SPCM $R = (r_{c'a'}^{ca})\ or\ (r_{a'c'}^{ac})$.

In a (CA × CA) matrix, index of alternative changes first. In a (CA × CA) matrix, SPCM's $(A(c-1)+a,\ A(c'-1)+a')$ th element is $r_{c'a'}^{ca}$.

In a (AC × AC) matrix, index of criteria changes first. In a (AC × AC) matrix, SPCM's $(C(a-1)+c,\ C(a'-1)+c')$ th element is $r_{a'c'}^{ac}$.

In an SPCM, symmetric components have a reciprocal relationship as in pairwise comparison matrices. Diagonal elements are 1 and the following relationships are true:

If $r_{c'a'}^{ca}$ exists, then $r_{ca}^{c'a'}$ exists and

$$r_{ca}^{c'a'} = 1/r_{c'a'}^{ca}, \tag{1}$$

$$r_{ca}^{ca} = 1. \tag{2}$$

Pairwise comparison at Step 1 of D-AHP consists of the relative comparison value $r_{ca'}^{ca}$ of importance v_{ca} of alternative a, compared with the importance $v_{ca'}$ of alternative a' at criterion c.

Pairwise comparison at Step 3 of D-AHP consists of the relative comparison value $r_{c'd}^{cd}$ of importance v_{cd} of alternative d at criterion c, compared with the importance $v_{c'd}$ of alternative d at criterion c', where the regulating alternative is d.

SPCM of D-AHP or MDAHP is an incomplete pairwise comparison matrix. Therefore, the LLSM based on an error model or an eigenvalue method such as the Harker method [9] or two-stage method is applicable to the calculation of evaluation values from an SPCM.

3 SPCM + LLSM in the Dominant AHP

This section shows that an evaluation value resulting from the application of the LLSM to an SPCM agrees with the overall evaluation value resulting from the application of D-AHP to the evaluation value obtained by application of the geometric mean method to each pair-wise comparison matrix.

In Sect. 3.1, an overall evaluation value is obtained by applying D-AHP to evaluation values that are obtained by applying the geometric mean method to each pairwise comparison matrix. In Sect. 3.2, an overall evaluation value is obtained by applying the LLSM to an SPCM to show that it agrees with the overall evaluation value obtained in Sect. 3.1.

Hereinafter, the regulating alternative in D-AHP is assumed to be alternative 1. This assumption can generally be satisfied by renumbering alternatives.

3.1 D-AHP and Geometric Mean Method

Pairwise comparison at Step 1 of D-AHP consists of the relative comparison value $r_{ca'}^{ca}$ of importance v_{ca} of alternative a as compared with the importance $v_{ca'}$ of alternative a' from the view point of criterion c. R_c^A is the pairwise comparison matrix between alternatives from the view point of criterion c, whose (a, a') th element is $r_{ca'}^{ca}$. Therefore, the relative importance u_{ca} of alternative a at criterion c resulting from the application of the geometric mean method to the pairwise comparison matrix R_c^A becomes the geometric mean of the values in row a of R_c^A. In other words, u_{ca} is calculated with following equation.

$$u_{ca} = \left(\prod_{a'=1}^{A} r_{ca'}^{ca} \right)^{1/A}, \quad c = 1, \ldots, C, \; a = 1, \ldots, A. \tag{3}$$

At Step 2, this value is normalized by u_{c1} and we obtain the following equation:

$$u_{ca}^{1} = u_{ca}/u_{c1} = \left(\prod_{a'=1}^{A} r_{ca'}^{ca} \right)^{1/A} / \left(\prod_{a'=1}^{A} r_{ca'}^{c1} \right)^{1/A}, \quad c = 1, \ldots, C, a = 1, \ldots, A. \tag{4}$$

Pairwise comparison at Step 3 of D-AHP consists of the relative comparison value $r_{c'1}^{c1}$ of importance v_{c1} of alternative 1 at criterion c, compared with the importance v_{c1} of alternative 1 at criterion c', R_1^C is the pairwise comparison matrix between criteria of the dominant alternative 1 whose (c, c') th element is $r_{c'1}^{c1}$. Therefore, the relative importance w_{c1} of alternative 1 at criterion c resulting from the application of the geometric mean method to the pairwise comparison matrix R_1^C becomes the geometric mean of the values in row c of R_1^C, where w_{c1} is normalized to $\sum_{c=1}^{C} w_c^1 = 1$ as shown in the following equation:

$$w_c^1 = \left(\prod_{c'=1}^{C} r_{c'1}^{c1}\right)^{1/C} / \sum_{c''=1}^{C} \left(\prod_{c'=1}^{C} r_{c'1}^{c''1}\right)^{1/C}, \quad c = 1, \ldots, C. \tag{5}$$

At step 4, with u_{ca}^1, w_{c1} shown in Eqs. (4) and (5), the overall evaluation value $u_a = \sum_{c=1}^{C} w_c^1 u_{ca}^1$ of alternative a is

$$v_{ca} = w_c^1 u_{ca}^1 = \frac{\left(\prod_{c'=1}^{C} r_{c'1}^{c1}\right)^{1/C} \left(\prod_{a'=1}^{A} r_{ca'}^{ca}\right)^{1/A}}{\sum_{c=1}^{C} \left(\prod_{c'=1}^{C} r_{c'1}^{c1}\right)^{1/C} \cdot \left(\prod_{a'=1}^{A} r_{ca'}^{c1}\right)^{1/A}} \tag{6}$$

3.2 LLSM Application to SPCM in the Dominant AHP

For existing pairwise comparison values $r_{c'a'}^{ca}$ $(c < c', a < a')$ in an SPCM, an error model is assumed as follows:

$$r_{c'a'}^{ca} = \varepsilon_{c'a'}^{ca} \frac{v_{ca}}{v_{c'a'}}. \tag{7}$$

where $\varepsilon_{c'a'}^{ca}$ is error term, v_{ca} is non-random but unobservable parameters. Taking the logarithms (base e) of both sides gives

$$\ln r_{c'a'}^{ca} = \ln v_{ca} - \ln v_{c'a'} + \ln \varepsilon_{c'a'}^{ca} \tag{8}$$

To simplify the equation, logarithms will be represented by over dots as $\dot{r}_{c'a'}^{ca} = \log r_{c'a'}^{ca}$, $\dot{v}_{ca} = \log v_{ca}$, $\dot{\varepsilon}_{c'a'}^{ca} = \log \varepsilon_{c'a'}^{ca}$. Using this notation, Eq. (8) becomes

$$\dot{r}_{c'a'}^{ca} = \dot{v}_{ca} - \dot{v}_{c'a'} + \dot{\varepsilon}_{c'a'}^{ca}, c, c' = 1, \ldots, C, \ a, a' = 1, \ldots, A \tag{9}$$

From Eqs. (1) and (2), we have followings.

If $\dot{r}^{ca}_{c'a'}$ exists, then $\dot{r}^{c'a'}_{ca}$ exists and

$$\dot{r}^{c'a'}_{ca} = -\dot{r}^{ca}_{c'a'} \tag{10}$$

$$\dot{r}^{ca}_{ca} = 0. \tag{11}$$

There are two types of pairwise comparison in the dominant AHP: $r^{ca}_{ca'}$ at Step 1 and $r^{c1}_{c'1}$ at Step 3. In the least-squares method, therefore, $\widehat{v}_{ca}$ is obtained from the pairwise comparison $\dot{r}^{ca}_{ca'}(a = 1, \ldots, A-1, a' = a+1, \ldots, A, c = 1, \ldots, C)$ and $\dot{r}^{c1}_{c'1}(c = 1, \ldots, C-1, c' = c+1, \ldots, C)$ to minimize

$$S = \sum_{c=1}^{C} \sum_{a=1}^{A-1} \sum_{a'=a+1}^{A} (\dot{r}^{ca}_{ca'} - \hat{v}_{ca} + \hat{v}_{ca'})^2 + \sum_{c=1}^{C-1} \sum_{c'=c+1}^{C} (\dot{r}^{c1}_{c'1} - \hat{v}_{c1} + \hat{v}_{c'1})^2. \tag{12}$$

In Eq. (12), the first term $\sum_{c=1}^{C} \sum_{a=1}^{A-1} \sum_{a'=a+1}^{A} (\dot{r}^{ca}_{ca'} - \hat{v}_{ca} + \hat{v}_{ca'})^2$ associates the relative comparison value $r^{ca}_{ca'}$ of importance v_{ca} of alternative a, compared with the importance $v_{ca'}$ of alternative a' from the view point of criterion c, and the second term $\sum_{c=1}^{C-1} \sum_{c'=c+1}^{C} (\dot{r}^{c1}_{c'1} - \hat{v}_{c1} + \hat{v}_{c'1})^2$ associates the relative comparison value $r^{c1}_{c'1}$ of importance v_{c1} of alternative 1 at criterion c, compared with the importance $v_{c'1}$ of alternative 1 at criterion c'.

As Eq. (7) shows, only the ratio is important with regard to $\hat{v}_{ca}$ and the constant multiple is arbitrary, becoming an arbitrary additive constant in the logarithm $\hat{v}_{ca}$ form.

From $\frac{\partial S}{\partial \hat{v}_{ca}} = 0$, we have

$$\begin{aligned} \frac{1}{2}\frac{\partial S}{\partial \hat{v}_{c1}} &= \sum_{a=1}^{A} (\hat{v}_{c1} - \hat{v}_{ca} - \dot{r}^{c1}_{ca}) + \sum_{c'=1}^{C} (\hat{v}_{c1} - \hat{v}_{c'1} - \dot{r}^{c1}_{c'1}) \\ &= (A\hat{v}_{c1} - \sum_{a=1}^{A} \hat{v}_{ca} - \sum_{a=1}^{A} \dot{r}^{c1}_{ca}) + (C\hat{v}_{c1} - \sum_{c'=1}^{C} \hat{v}_{c'1} - \sum_{c'=1}^{C} \dot{r}^{c1}_{c'1}) = 0, \\ & \qquad c = 2, \ldots, C \end{aligned} \tag{13}$$

$$\begin{aligned} \frac{1}{2}\frac{\partial S}{\partial \hat{v}_{ca}} = \sum_{a'=1}^{A} (\hat{v}_{ca} - \hat{v}_{ca'} - \dot{r}^{ca}_{ca'}) = (A\hat{v}_{ca} - \sum_{a'=1}^{A} \hat{v}_{ca'} - \sum_{a'=1}^{A} \dot{r}^{ca}_{ca'}), \\ a = 2, \ldots, A, \ c = 2, \ldots, C \end{aligned} \tag{14}$$

Second term $\sum_{c'=1}^{C} (\hat{v}_{c1} - \dot{r}^{c1}_{c'1} - \hat{v}_{c'1})$ of Eqs. (13) is obtained from $\frac{\partial}{\partial \hat{v}_{c1}} \sum_{c=1}^{C-1} \sum_{c'=c+1}^{C} (\dot{r}^{c1}_{c'1} - (\hat{v}_{c1} - \hat{v}_{c'1}))^2$.

The fact that

$$\hat{v}_{ca} = \frac{1}{C}\sum_{c'=1}^{C} \dot{r}_{c'1}^{c1} + \frac{1}{A}\sum_{a'=1}^{A} \dot{r}_{ca'}^{ca} - \frac{1}{A}\sum_{a'=1}^{A} \dot{r}_{ca'}^{c1} + Const.,\ c=1,\ \ldots,C,\ a=1,\ \ldots,A \tag{15}$$

satisfies Eqs. (13) and (14) is easy to confirm using Eqs. (10) and (11). In Eq. (15), Const. is arbitrary constant.

From Eq. (15),

$$\hat{v}_{ca} = Const.\left(\prod_{c'=1}^{C} r_{c'1}^{c1}\right)^{1/C} \frac{\left(\prod_{a'=1}^{A} r_{ca'}^{ca}\right)^{1/A}}{\left(\prod_{a'=1}^{A} r_{ca'}^{c1}\right)^{1/A}},\ c=1,\ \ldots,C,\ a=1,\ \ldots,A \tag{16}$$

In accordance with the normalization at Step 3 of the dominant AHP, the normalized equation $\hat{v}_{ca}(a=1,\ \ldots,A, c=1,\ \ldots,C)$ is such that the overall evaluation value of the regulating alternative (alternative 1) will be 1. In other words, $\sum_{c=1}^{C}\hat{v}_{c1}=1$. Therefore,

$$\hat{v}_{ca} = \frac{\left(\prod_{c'=1}^{C} r_{c'1}^{c1}\right)^{1/C}}{\sum_{c''=1}^{C}\left(\prod_{c'=1}^{C} r_{c'1}^{c''1}\right)^{1/C}} \frac{\left(\prod_{a'=1}^{A} r_{ca'}^{ca}\right)^{1/A}}{\left(\prod_{a'=1}^{A} r_{ca'}^{c1}\right)^{1/A}}, c=1,\ \ldots,C,\ a=1,\ \ldots,A \tag{17}$$

From Eq. (17), we see that the overall evaluation value $\hat{v}_a = \sum_{c=1}^{C}\hat{v}_{ca}$ of alternative a agrees with Eq. (6).

As shown above, an evaluation value resulting from the application of the LLSM to an SPCM agrees with the overall evaluation value resulting from the application of the dominant AHP to evaluation values that are obtained by applying the geometric mean method to each pairwise comparison matrix.

4 Conclusion

It is well known that in complete pairwise comparison matrix, the evaluation values applying the geometric mean method agree with the evaluation values resulting from the application of the LLSM. This paper shows that the evaluation values resulting from the application of the LLSM to an SPCM agree with the evaluation values resulting from the application of the dominant AHP to evaluation values that are obtained by applying the geometric mean method to each pairwise comparison matrix.

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