Improvement of the Weights Due to Inconsistent Pairwise Comparisons in the AHP

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Abstract One of the most important problems in the Analytic Hierarchy Process (AHP) is consistency of pairwise comparisons by the decision maker. This study focuses on the comparison methods to be used when the weights of the alternatives and criteria in AHP are inconsistent. In general, the weights in AHP use the principal eigenvector of the pairwise comparison matrix. However, for example, due to the decision maker's misunderstandings, inconsistencies in pairwise comparisons sometimes arise. The consistency of the pairwise comparison matrix is usually determined using Consistency Index (*CI*) values. In the traditional AHP, when judged inconsistent, repeating the pairwise comparison is usually recommended. However, if the repeated comparison is arbitrarily performed, the results will not be optimal. In fact, to obtain the overall evaluation of alternatives, we often use inconsistent weights, even given the inconsistencies in the latter. Another method for judging the consistency of the pairwise comparison is to use a directed graph. Cycles in a directed graph represent comparison inconsistencies. Therefore in this paper, based on the principal eigenvalue and cycles in the directed graph of the pairwise comparison matrix, a method of correcting the principal eigenvector taking into consideration consistency is proposed.

Keywords AHP · Pairwise comparison · Consistency Index · Directed graph · Cycles

1 Introduction

In the Analytic Hierarchy Process (AHP) [9] and the Analytic Network Process (ANP) [10], some problems, for example rank reversal and weight normalization, were pointed out and were improved [1–4, 6–8, 11].

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I. Czarnowski et al. (eds.), *Intelligent Decision Technologies 2016*,
Smart Innovation, Systems and Technologies 57,
DOI 10.1007/978-3-319-39627-9_35

In the AHP, the pairwise comparison matrix $\boldsymbol{A}$, which consists of n alternatives, is constructed by the decision maker. In this paper, we assume that $\boldsymbol{A}$ consists of complete comparisons. Through AHP, we calculate the principal eigenvalue and corresponding eigenvector of $\boldsymbol{A}$ using the power method. The weights of alternatives in the AHP usually use the principal eigenvector.

In this study, the weights of alternatives taking into account the consistency of comparisons are considered.

In the AHP, we usually evaluate comparisons as "consistent" or "inconsistent" based on Consistency Index (CI) values. CI is calculated using Eq. (1) based on the principal eigenvalue.

$$CI = (\lambda_{max} - n)/(n - 1), \tag{1}$$

where λ_{max} is the principal eigenvalue of $\boldsymbol{A}$. In general, if $CI < 0.1$ then we consider $\boldsymbol{A}$ to be consistent and if $CI > 0.1$ then it is considered inconsistent.

Another method of determining consistency involves using the directed graph of $\boldsymbol{A}$ [5]. In the pairwise comparison, if alternative "i" is better than alternative "j", then we indicate " i → j". If alternatives "i" and "j" are equally important, then we indicate " i — j". In directed graph of $\boldsymbol{A}$, there is either no cycle or there are some cycles of various length. If directed graph of $\boldsymbol{A}$ has no cycle, and thus is compliant with the transitive law, then $\boldsymbol{A}$ is considered to be consistent. If some cycles of length three are observed in the directed graph of $\boldsymbol{A}$, thus being compliant with the circulation law, then we consider it as inconsistent. If there is a cycle of length m ($m > 3$) in the directed graph of the complete comparisons, such cycle always includes some cycles of length three. Therefore, it should be considered only the cycles of length three.

For example, two kinds of pairwise comparison are illustrated. In these cases, the simplest pairwise comparisons are carried out. These are called binary comparisons. Using parameter θ, we can construct comparison matrix $\boldsymbol{A}$. If alternative "i" is better than alternative "j", then the element of $\boldsymbol{A}$, that is $\boldsymbol{a}_{ij} = \theta$, and $\boldsymbol{a}_{ji} = 1/\theta$. In these examples, assume that three alternatives a1, a2 and a3 are being compared.

First, the consistent comparison matrix is shown in Eq. (2).

$$\boldsymbol{A} = \begin{bmatrix} 1 & \theta & \theta \\ 1/\theta & 1 & \theta \\ 1/\theta & 1/\theta & 1 \end{bmatrix} \tag{2}$$

Calculating as $\theta = 2$, we get the principal eigenvalue $\lambda_{max} = 3.0536$ and we get corresponding eigenvector in Eq. (3). In this study, $\boldsymbol{w}$ is not normalized.

$$\boldsymbol{w} = \begin{bmatrix} 1.000000 \\ 0.629961 \\ 0.396850 \end{bmatrix} \tag{3}$$

From Eq. (1), we obtain $CI = 0.0268$. Thus $\boldsymbol{A}$ is considered consistent.

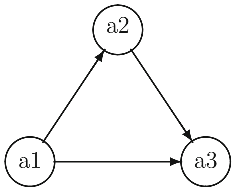


Fig. 1 Directed graph of no cycle

Another method to determine consistency is through use of the directed graph of comparison matrix. The directed graph of $\boldsymbol{A}$ is drawn in Fig. 1 based on Eq. (2).

From Fig. 1, $\boldsymbol{A}$ has no cycle, and thus complies with the transitive law. Thus $\boldsymbol{A}$ is considered consistent.

The next example is of inconsistency. The comparison matrix is shown in Eq. (4).

$$\boldsymbol{A} = \begin{bmatrix} 1 & \theta & 1/\theta \\ 1/\theta & 1 & \theta \\ \theta & 1/\theta & 1 \end{bmatrix} \tag{4}$$

Calculating as $\theta = 2$, we get the principal eigenvalue $\lambda_{max} = 3.5000$ and we get the corresponding eigenvector in Eq. (5).

$$\boldsymbol{w} = \begin{bmatrix} 1.000000 \\ 1.000000 \\ 1.000000 \end{bmatrix} \tag{5}$$

From Eq. (1), we obtain $CI = 0.2500$. Thus $\boldsymbol{A}$ is considered inconsistent.

Figure 2 shows the directed graph of $\boldsymbol{A}$ based on Eq. (4).

In Fig. 2, there is one cycle of length three. Thus the directed graph of $\boldsymbol{A}$ complies with the circulation law, $\boldsymbol{A}$ is considered inconsistent.

In the traditional AHP, for comparisons judged inconsistent, repeat of pairwise comparisons is usually recommended. However, a repeat of pairwise comparisons usually results in arbitrary evaluations, so it is not a good method. Therefore in this paper, for the case of inconsistent comparisons, an improvement method is proposed.

This paper consists of following the sections. Section 2 describes the proposed method. Examples of the proposed method are illustrated in Sect. 3. And finally, in Sect. 4, the results obtained through the proposed method are discussed and the study concluded.

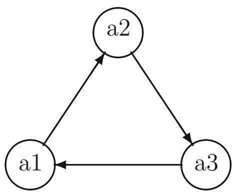


Fig. 2 Directed graph with one cycle

2 The Proposed Method

This section describes the proposed method. In the case of perfect consistency of the pairwise comparison, the element of comparison matrix $\boldsymbol{A}$, that is a_{ij} supports Eq. (6). Where w_i is the weight of alternative ai.

$$a_{ij} = w_i/w_j \tag{6}$$

In the perfectly consistent $\boldsymbol{A}$, as shown, we get $\lambda_{max} = n$. On the other hand, in the inconsistent case, we get $\lambda_{max} > n$.

To correct the weights of alternatives for consistency, in this study, the corrective parameter α is defined in Eq. (7). α means rough consistency.

$$\alpha = n/\lambda_{max} \tag{7}$$

Based on α, the corrected weight w_i' is calculated using Eq. (8). Where k_i is the number of cycles which are related to alternative ai on the directed graph of $\boldsymbol{A}$.

$$w_i' = \alpha^{k_i+1} w_i \tag{8}$$

In the case of perfect consistency, that is $\lambda_{max} = n$, we get $\alpha = 1$ from Eq. (7). In this case $\boldsymbol{w}' = \boldsymbol{w}$. If there is no cycle in the directed graph, that is, $k_i = 0$ but $\lambda_{max} > n$, we get $\boldsymbol{w}' = \alpha \boldsymbol{w}$ from Eq. (8).

The procedure of the proposed method is as follows.

- P1: Calculate the principal eigenvalue and corresponding eigenvector of the pairwise comparison matrix $\boldsymbol{A}$ using the power method.
- P2: Calculate the corrective parameter α using Eq. (7).
- P3: Find cycles of length three on the directed graph of $\boldsymbol{A}$.
- P4: Count the number of cycles k_i which are related to the alternative ai.
- P5: Correct the weights using Eq. (8).

If the weights of alternatives are calculated from $\boldsymbol{A}$ using the geometric mean, unfortunately we do not have λ_{max}. Therefore we need to calculate the approximate eigenvalue $\bar{\lambda}$ using the following well known procedure.

Through the geometric mean, the weights of alternatives w_i is obtained using Eq. (9).

$$w_i = \sqrt[n]{\prod_{j=1}^{n} a_{ij}} \tag{9}$$

To obtain the approximate eigenvalue, we perform calculations only once for the iteration in the power method. From Eq. (10), the approximate vector $\boldsymbol{x}$ is obtained. Where initial vector $\boldsymbol{w}$ is calculated using Eq. (9).

$$\boldsymbol{A}\boldsymbol{w} = \boldsymbol{x} \tag{10}$$

If the power method is carried completion, we get $\boldsymbol{Aw} = \lambda_{max}\boldsymbol{w}$. However when once iteration, $\boldsymbol{x}$ in Eq. (10) is the approximate vector, then different eigenvalues λ_i are obtained using Eq. (11).

$$\lambda_i = x_i / w_i \tag{11}$$

The approximate eigenvalue $\bar{\lambda}$ is obtained by calculating the average of λ_i from Eq. (12).

$$\bar{\lambda} = \frac{1}{n} \sum_{i=1}^{n} \lambda_i \tag{12}$$

Using $\bar{\lambda}$ instead of λ_{max} in Eq. (7), α is determined.

3 Examples

In this section, using the proposed method, three examples are illustrated. The directed graph in Example 1 shows no cycles, Example 2 has one cycle and Example 3 has four cycles. Applying the proposed method to these examples, each principal eigenvector is corrected.

3.1 Example 1

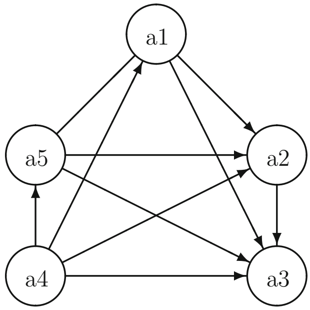
The first example consists of five alternatives, a1 to a5. Comparison matrix $\boldsymbol{A_1}$ is shown in Eq. (13).

$$\boldsymbol{A_1} = \begin{bmatrix} 1 & 3 & 2 & 1/2 & 1 \\ 1/3 & 1 & 5 & 1/4 & 1/2 \\ 1/2 & 1/5 & 1 & 1/4 & 1/3 \\ 2 & 4 & 4 & 1 & 2 \\ 1 & 2 & 3 & 1/2 & 1 \end{bmatrix} \tag{13}$$

Through the proposed procedure "P1", we get $\lambda_{max} = 5.3813$ and we get corresponding eigenvector in Eq. (14). However $\boldsymbol{w_1}$ is not normalized.

$$\boldsymbol{w_1} = \begin{bmatrix} 0.567933 \\ 0.365566 \\ 0.179137 \\ 1.000000 \\ 0.533289 \end{bmatrix} \tag{14}$$

Fig. 3 Directed graph of $\boldsymbol{A}_1$



In this example, we get $CI = 0.0953$ from Eq. (1), however we dither over consistency because $CI \fallingdotseq 0.1$.

Next, Fig. 3 shows the directed graph of $\boldsymbol{A}_1$.

Since no cycle is observed, it appears to be consistent.

Through the proposed procedure "P2", we get $\alpha = 0.929151$ using Eq. (7). Because no cycle is observed in Fig. 3, we get $k_i = 0$ through the proposed procedures "P3" and "P4". Through the proposed procedure "P5", we get corrected vector $\boldsymbol{w_1}'$ in Eq. (15) using Eq.(8).

$$\boldsymbol{w_1}' = \alpha \boldsymbol{w_1} = \begin{bmatrix} 0.527696 \\ 0.339666 \\ 0.166445 \\ 0.929151 \\ 0.495506 \end{bmatrix} \tag{15}$$

Through the proposed method, in this example, there is no change in the ordering of alternatives.

Based on the weights obtained through geometric mean using Eq. (9), we get $\bar{\lambda} = 5.368394$ using Eq. (12) and we get $\alpha = 0.931377$.

3.2 Example 2

The next example consists of five alternatives, a1 to a5. Comparison matrix $\boldsymbol{A_2}$ is shown in Eq. (16).

$$\boldsymbol{A_2} = \begin{bmatrix} 1 & 1/4 & 1/6 & 1/8 & 1/7 \\ 4 & 1 & 1/2 & 1/3 & 1/3 \\ 6 & 2 & 1 & 1/2 & 2 \\ 8 & 3 & 2 & 1 & 1/2 \\ 7 & 3 & 1/2 & 2 & 1 \end{bmatrix} \tag{16}$$

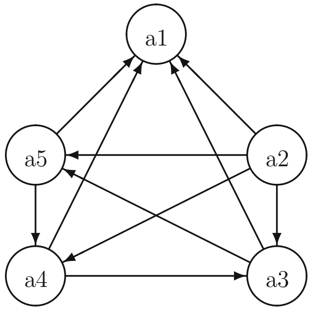


Fig. 4 Directed graph of $\boldsymbol{A_2}$

Table 1 Cycles—alternatives in $\boldsymbol{A_2}$

Cycles\Alternatives	a1	a2	a3	a4	a5
(a3-a5-a4)	0	0	1	1	1
The number of related cycles (k_i)	0	0	1	1	1

Using P1, we get $\lambda_{max} = 5.3518$ and we get corresponding eigenvector $\boldsymbol{w_2}$ in Eq. (17).

$$\boldsymbol{w_2} = \begin{bmatrix} 0.116882 \\ 0.363940 \\ 0.902340 \\ 0.995339 \\ 1.000000 \end{bmatrix} \tag{17}$$

From Eq. (1), we get $CI = 0.0880$ in this example. As in Example 1, we dither over consistency because $CI \fallingdotseq 0.1$. Using P2, we get $\alpha = 0.934258$.

Next, Fig. 4 shows the directed graph of $\boldsymbol{A_2}$.

In Fig. 4, there is one cycle of length three, (a3-a5-a4). So it is seems to be inconsistent. Table 1 is obtained from Fig. 4.

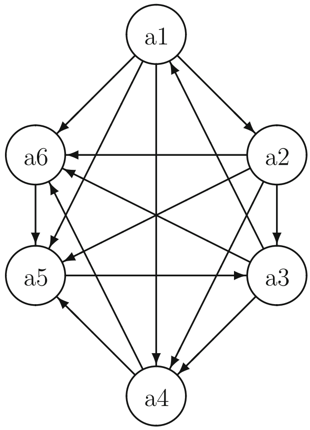
Through P3 and P4, we get k_i from Table 1 and through P5, we get corrected vector $\boldsymbol{w_2}'$ in Eq. (18) using Eq. (8).

$$\boldsymbol{w_2}' = \begin{bmatrix} \alpha^1 \times 0.116882 \\ \alpha^1 \times 0.363940 \\ \alpha^2 \times 0.902340 \\ \alpha^2 \times 0.995339 \\ \alpha^2 \times 1.000000 \end{bmatrix} = \begin{bmatrix} 0.109198 \\ 0.340014 \\ 0.787597 \\ 0.868770 \\ 0.872839 \end{bmatrix} \tag{18}$$

The results using the proposed method, in this example, show that there is no change in the ordering of alternatives.

We get $\bar{\lambda} = 5.341311$ and $\alpha = 0.936100$.

Fig. 5 Directed graph of $\boldsymbol{A_3}$



3.3 Example 3

In the next example, the binary comparisons consists of six alternatives, a1 to a6. The comparison matrix is shown in Eq. (19).

$$\boldsymbol{A_3} = \begin{bmatrix} 1 & \theta & 1/\theta & \theta & \theta & \theta \\ 1/\theta & 1 & \theta & \theta & \theta & \theta \\ \theta & 1/\theta & 1 & \theta & 1/\theta & \theta \\ 1/\theta & 1/\theta & 1/\theta & 1 & \theta & \theta \\ 1/\theta & 1/\theta & \theta & 1/\theta & 1 & 1/\theta \\ 1/\theta & 1/\theta & 1/\theta & 1/\theta & \theta & 1 \end{bmatrix} \tag{19}$$

Using P1, calculating as $\theta = 2$, we get $\lambda_{max} = 6.7587$ and we get corresponding eigenvector $\boldsymbol{w_3}$ in Eq. (20). Then $CI = 0.1517$ is obtained and we consider $\boldsymbol{A_3}$ as inconsistent.

$$\boldsymbol{w_3} = \begin{bmatrix} 1.000000 \\ 0.974743 \\ 0.869358 \\ 0.618222 \\ 0.570360 \\ 0.498701 \end{bmatrix} \tag{20}$$

Using P2, we get $\alpha = 0.887741$.

Next, Fig. 5 shows the directed graph of $\boldsymbol{A_3}$.

In Fig. 5, there are four cycles of length three, (a1-a2-a3), (a1-a5-a3), (a3-a4-a5) and (a3-a6-a5), so as a result $\boldsymbol{A_3}$ is considered inconsistent.

Table 2 is obtained based on Fig. 5.

Using P3 and P4, we get k_i from Table 2, and through P5, we get corrected vector $\boldsymbol{w_3}'$ in Eq. (21) using Eq. (8).

Table 2 Cycles—alternatives in A_3

Cycles\Alternatives	a1	a2	a3	a4	a5	a6
(a1-a2-a3)	1	1	1	0	0	0
(a1-a5-a3)	1	0	1	0	1	0
(a3-a4-a5)	0	0	1	1	1	0
(a3-a6-a5)	0	0	1	0	1	1
The number of related cycles (k_i)	2	1	4	1	3	1

$$\boldsymbol{w_3}' = \begin{bmatrix} \alpha^3 \times 1.000000 \\ \alpha^2 \times 0.974743 \\ \alpha^5 \times 0.869358 \\ \alpha^2 \times 0.618222 \\ \alpha^4 \times 0.570360 \\ \alpha^2 \times 0.498701 \end{bmatrix} = \begin{bmatrix} 0.699614 \\ 0.768179 \\ 0.479325 \\ 0.487211 \\ 0.354237 \\ 0.393018 \end{bmatrix} \tag{21}$$

The results obtained through the proposed method demonstrate, in this example, that the order of alternatives of $\boldsymbol{w_3}'$ are different from $\boldsymbol{w_3}$. The order in $\boldsymbol{w_3}$ is a1 > a2> a3> a4> a5> a6, however, in $\boldsymbol{w_3}'$, it is a2> a1 >a4 >a3> a6 >a5.

We get $\bar{\lambda} = 6.739464$ and $\alpha = 0.890278$.

4 Conclusion

In this study, a method for correcting inconsistent pairwise comparisons in AHP was proposed. The proposed method is as follows.

1. Using the principal eigenvalue of pairwise comparison matrix $\boldsymbol{A}$, the corrective parameter α was defined.
2. Using α and the number of cycles of length three in the directed graph of $\boldsymbol{A}$, a corrective procedure was proposed.

Applying the proposed method to the three examples, the following results could be obtained.

1. In the case of inconsistencies, i.e. those represented by cycles of length three in directed graph of $\boldsymbol{A}$, the order of alternatives in terms of priority were changed.
2. The proposed method offers promising results for determining the overall evaluation of alternatives demonstrating inconsistency.

Topics for further study are as follows.

1. Appropriateness of the corrective parameter α.
2. The evaluation of the proposed method that includes application to other examples demonstrating inconsistency is required.

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