Projective Transverse Structures for Some Foliations

Paulo Sad

A Pepe, com admiração e amizade

Abstract. We construct examples of regular foliations of holomorphic surfaces which are generically transverse to a compact curve and have a projective transverse structure.

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1. Introduction

Let us consider a codimension 1 holomorphic foliation $\mathcal{F}$ of a complex manifold M. A (singular) projective transverse structure for $\mathcal{F}$ is defined by the following data:

1. a covering of the complement M^* of a finite set of embedded leaves by open sets $\{U_i\}$; in each U_i there is a trivialization of the foliation.
2. a collection $\{f_i\}$ of holomorphic functions $f_i : U_i \to \bar{\mathbb{C}}$ which are first integrals of $\mathcal{F}$ in each U_i and a collection of Moebius transformations $\{\phi_{ij}\}$ such that $f_i = \phi_{ij} \circ f_j$ whenever $U_i \cap U_j \neq \emptyset$.

In general, the map ϕ_{ij} which relates f_i to f_j is simply a diffeomorphism between open sets of $\bar{\mathbb{C}}$.

The presence of projective transverse structure leads to the existence of a multivalued first integral of $\mathcal{F}$ in M^*; we start with some open set, say U_1, and extend f_1 along paths starting at U_1 just by composing it with a convenient choice of functions ϕ_{ij}.

Let us present some examples.

Example 1.1. A classical example comes from projective structures on Riemann surfaces. Let C be a compact Riemann surface with an atlas of coordinate charts

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such that all changes of coordinates are given by elements of $\mathrm{PSL}(2,\bar{\mathbb{C}})$ (Moebius transformations of $\bar{\mathbb{C}}$). After applying the process mentioned above we get a developing map $\mathcal{D} : \tilde{C} \to \bar{\mathbb{C}}$ where $\tilde{C}$ is the universal covering of C ($\tilde{C} = \mathbb{D}$ when the genus of C is greater than 1 and $\tilde{C} = \mathbb{C}$ when the genus of C is 1) and a monodromy representation $\rho : \pi_1(C) \to \mathrm{PSL}(2,\bar{\mathbb{C}})$ which are related by the equality $\mathcal{D}(\gamma(p)) = \rho(\gamma)\mathcal{D}(p)$, for $p \in \tilde{C}$ and $\gamma \in \pi_1(C)$ view as a deck transformation.

Then $\pi_1(C)$ acts on $\tilde{C} \times \bar{\mathbb{C}}$: to each $\gamma \in \pi_1(C)$ we associate the map $(p,z) \mapsto (\gamma(p), \rho(\gamma)(z))$. The action preserves the horizontal and vertical fibrations of $\tilde{C} \times \bar{\mathbb{C}}$; therefore, the space of orbits of the action is a compact surface that has a rational fibration over C (coming from the vertical fibration) and a transversely projective foliation $\mathcal{F}$ (coming from the horizontal fibration). The action preserves also the graph $\{(p,\mathcal{D}(p)); p \in \tilde{C}\}$ of $\mathcal{D}$ (because $(p,\mathcal{D}(p)) \mapsto (\gamma(p), \rho(\gamma)\mathcal{D}(p)) = (\gamma(p), \mathcal{D}(\gamma(p)))$, which becames in the quotient a section of the rational fibration, generically transverse to $\mathcal{F}$. The foliation induces in this section the same projective structure of C.

We may adapt this construction to some cases where the projective structure of C has a number of singularities (for example, multivalued maps of the form $z \mapsto z^\alpha$). The local monodromy around the singularity has to be realized as the monodromy of a foliation of $\mathbb{D} \times \bar{\mathbb{C}}$ which is transverse to the fibers $\{z\} \times \bar{\mathbb{C}}$ for $z \neq 0$ and has $\{0\} \times \bar{\mathbb{C}}$ as a leaf; then we glue this foliation with the one constructed as before outside the singularities.

Example 1.2. We mentioned in the last example the difficulties that arise in the presence of singularities of the projective structure. There is a related construction that avoids this problem by using "pre-integration" data. We take a line bundle L over a compact, holomorphic curve C. In some covering $\mathcal{U} = \{U_i\}$ of C we write $\{\lambda_{ij}\}$ for the transition functions of L and take trivializations (x_i, z_i) of L with $z_i = \lambda_{ij} z_j$. Let now $\{\omega_i\}$, $\{\eta_i\}$ and $\{\xi_i\}$ be meromorphic 1-forms defined in the open sets U_i satisfying $\omega_i = \lambda_{ij}\omega_j$, $\eta_i - \eta_j = d\log\lambda_{ij}$ and $\xi_j = \lambda_{ij}\xi_i$. We notice that the equations $dz_i = z_i\eta_i + \frac{z_i^2}{2}\xi_i + \omega_i$ define a foliation $\mathcal{G}$ in L. In fact, we may compactify L as a $\bar{\mathbb{C}}$ over C and extend the foliation to the closure of L. Except by the fibers over some pole (which are leaves), the other fibers are transverse to the leaves. This implies the existence of a transverse projective structure for $\mathcal{G}$ outside the vertical leaves. These are called Ricatti foliations (see [6]).

We may think of course that the pre-integration data produce a (singular) projective structure for the curve C; we explain in Section 2 how this works.

In this paper we combine in Theorem 3.1 features of Examples 1.1 and 1.2. We start with pre-integration data in a curve and obtain a (singular) projective structure. Then we embed the curve into a surface which comes with a regular foliation generically transverse to it, with the aditional property that the structure of the curve can be extended to the surface along the leaves of the foliation. We get many more examples, at the price of not having a compact surface and no fibration over the curve. It is worth noticing that we also describe the singularities of the projective structure.

Example 1.3. Let us consider a codimension 1 foliation defined by some integrable holomorphic 1-form ω. We suppose that this 1-form is completed to a triplet of holomorphic 1-forms with η and ξ such that $d\omega = \eta \wedge \omega$, $d\eta = \omega \wedge \xi$ and $d\xi = \xi \wedge \eta$. According to Darboux (see [5]), we may write locally triplets of holomorphic functions (f, g, h) such that

$$\omega = -g df, \ \ \eta = \frac{dg}{g} + h\,\omega, \ \ \xi = dh + h\,\eta + \frac{h^2}{2}\omega.$$

Furthermore, if $(\bar{f}, \bar{g}, \bar{h})$ is another triplet of functions satisfying the same relations then $f = \phi \circ \bar{f}$ for some Moebius transformation ϕ. This implies that the foliation has a projective transverse structure.

In general, we may work with a triplet of meromorphic 1-forms, and the projective structure is singular. Many authors take the existence of such a triplet as the definition of a transversely projective foliation (see for example [6]). Of course a true transverse projective structure appears for the foliation outside the poles of the 1-forms; but it is not clear if we can produce a triplet out of a transverse projective structure defined outside a divisor. We treat this question for the examples constructed in Theorem 1.3, where we know the nature of the singularities. We give in Section 4 an answer adding a negativity hypothesis for the embedding of the curve. We do not know if this is true for the other cases.

Example 1.4. This is an example of a different nature, but it shows how foliations with transverse projective structures appear quite naturally. Let K be a radial type Kupka component of a codimension 1 foliation in some complex manifold of dimension $n \geq 3$. This means that K is covered by open sets $\{U_i\}$ such that in each U_i the foliation is conjugated by a diffeomorphism Θ_i to the foliation $x_i\, dy_i - y_i d\, x_i = 0$ of $\mathbb{D}^{n-2} \times \mathbb{D}^2$ and $\Theta_i(K \cap U_i) = \mathbb{D}^{n-2} \times \{(0,0)\}$. Now we take the function $f_i = \dfrac{y_i}{x_i} \circ \Theta_i$ defined in $U_i \setminus K$ as a first integral for the foliation in U_i. If $U_i \cap U_j \neq \emptyset$ and $f_i = \phi_{ij} \circ f_j$, we see that ϕ_{ij} is a Moebius transformation because f_i and f_j are surjective onto $\bar{\mathbb{C}}$ (see [4]).

There is a lot of important work done the subject of transversely projective foliations; let us mention [1], [2] and [3] where the structure of these foliations is discussed.

We follow all the time part of the presentation of [4], which relies upon the paper [5]. We are grateful to J.V. Pereira for helping to establish the right setting of our construction. We are also grateful to the referee for valuable comments.

2. Projective Structures for Curves

Let us consider a line bundle L over a compact Riemann surface C; we cover C by a family $\mathcal{U} = \{U_i\}$ of open sets, and write $\{\lambda_{ij}\} \in H^1(\mathcal{U}, \mathcal{O}_C^*)$ for the transition functions of L; the notation $\mathcal{O}_C^*$ stands for the sheaf of non vanishing germs of holomorphic functions of C. We will also write $\mathcal{M}_C$ for the sheaf of germs of meromorphic functions of C and $\mathcal{M}_C^1$ for the sheaf of germs of meromorphic 1-forms of C.

We study now (singular) projective structures of C. We take meromorphic 1-forms $\omega = \{\omega_i\} \in H^0(\mathcal{U}, \mathcal{M}_C^1 \otimes L)$ (that is, $\omega_i = \lambda_{ij}\,\omega_j$ whenever $U_i \cap U_j \neq \emptyset$) and meromorphic 1-forms $\eta = \{\eta_i\} \in C^0(\mathcal{U}, \mathcal{M}_C^1)$ such that $\eta_i - \eta_j = d\log\lambda_{ij}$. We also select meromorphic functions $h = \{h_i\} \in C^0(\mathcal{U}, \mathcal{M})$. Proceeding formally, we define $f = \{f_i\}$ as the solution of the system

$$d\log g_i = \eta_i + h_i\omega_i \tag{2.1}$$

$$\omega_i = -g_i\, df_i \tag{2.2}$$

The elements of f will be taken as coordinate charts for C deprived of a finite set of singularities. We will impose later conditions on ω, η and h in order to assure the existence of f and be able to describe its singularities. For the moment we will compare the elements of f assuming that they are difeomorphisms over open sets of $\bar{\mathbb{C}}$; let $f_i = \phi_{ij}(f_j)$, the functions ϕ_{ij} being difeomorphisms between open sets of $\bar{\mathbb{C}}$. Let us write $\psi_{ij}(t) = d\log\phi'_{ij}(t)$.

Lemma 2.1. $\psi_{ij}\, df_j = h_i\,\omega_i - h_j\,\omega_j$ *in* $U_i \cap U_j$.

Proof. Since $f_i = \phi_{ij}(f_j)$, we have $df_i = \phi'_{ij}(f_j)\, df_j$. From equation (2.2) it follows that $\omega_i/g_i = \phi'_{ij}(f_j)\,\omega_j/g_j$ and therefore $\lambda_{ij}g_j/g_i = \phi'_{ij}(f_j)$. We apply then $d\log$ to both sides and use equation (2.2).

We remark that if $h = 0$, that is, $h_i = 0$ for every i, then the coordinate charts give a (singular) affine structure for C.

Now we will make another choice for h; we consider a collection $\xi = \{\xi_i\} \in C^0(\mathcal{U}, \mathcal{M}_C^1)$ of meromorphic 1-forms and define

$$dh_i + h_i\,\eta_i + \frac{h_i^2}{2}\omega_i = \xi_i. \tag{2.3}$$

Remind that the Schwarz derivative of a holomorphic function l is given by

$$S(l)(t) = \psi'(t) - \frac{\psi(t)^2}{2}$$

where $\psi(t) = d\log l'$.

Lemma 2.2. $\dfrac{1}{g_j}S(\phi_{ij})(f_j)df_j = \lambda_{ij}\xi_i - \xi_j$ *in* $U_i \cap U_j$.

Proof. We have from Lemma 2.1:

$$\phi_{ij}(f_j) = -g_j(h_j - \lambda_{ij}\,h_i).$$

It follows that

$$\frac{1}{g_j}\psi'_{ij}(f_j)df_j = -\xi_j - \frac{h_j^2}{2}\omega_j + \lambda_{ij}(dh_i + h_i\eta_i) + \lambda_{ij}h_ih_j\omega_j$$

and

$$\frac{1}{g_j}\frac{\psi(f_j)^2}{2}\,df_j = -\frac{h_j^2}{2}\omega_j - \lambda_{ij}\frac{h_i^2}{2}\omega_i + \lambda_{ij}h_i\,h_j\omega_j.$$

Consequently the Lemma follows. □

The case which will be of interest for us is when $\xi_j = \lambda_{ij}\xi_i$ whenever $U_i \cap U_j \neq \emptyset$, that is, $\xi \in H^0(\mathcal{U}, \mathcal{M}^1_C \otimes L^*)$. Lemma 2.2 implies that all ϕ_i are Moebius transformations of $\bar{\mathbb{C}}$, so that the collection $f = \{f_i\}$ provides a (singular) projective structure for C.

We proceed now to introduce conditions on $\omega \in H^0(\mathcal{U}, \mathcal{M}^1_C \otimes L)$, $\eta = \{\eta_i\} \in C^0(\mathcal{U}, \mathcal{M}^1_C)$ such that $\eta_i - \eta_j = d\log\lambda_{ij}$ and $\xi \in H^0(\mathcal{U}, \mathcal{M}^1_C \otimes L^*)$ which allow us to analyse more carefully the family f.

We will use $Z(\cdot)$ and $P(\cdot)$ for the set of zeros or poles of a 1-form.

Lemma 2.3. *There exist* 1*-forms as above such that all poles are simple, there are no common poles for any pair of* 1*-forms and*

- $Z(\omega) \neq \emptyset$,
- $Z(\omega) \cap (P(\eta) \cup P(\xi)) = \emptyset$.

Proof. We will use here classical results of Complex Analysis (in the case of curves) such as Riemann-Roch's Theorem and the existence of meromorphic 1-forms with pre-assigned polar parts.

1) Let us start with some $\bar{\omega} \in H^0(\mathcal{U}, \mathcal{M}^1_C \otimes L)$. We denote by $q_1, \ldots, q_s$ the poles of $\bar{\omega}$, $m_1, \ldots, m_s$ being their polar orders. We select a disjoint set of points $p_1, \ldots, p_r, p_{r+1}$ where $\bar{\omega}$ is regular and non vanishing and look for a meromorphic function l of C such that

$$(l) \geq p_{r+1} - \sum p_j + \sum m_i q_i =: -D.$$

The vector space $\mathcal{L}(D)$ of meromorphic functions of C whose divisor is greater or equal to $-D$ has dimension $l(D) \geq \deg(D) - g + 1$, where g is the genus of C. Therefore $l(D) \geq r - \sum m_i - g$ is positive for large r; we take $l \in \mathcal{L}(D)$. Then $\omega =: l\bar{\omega} \in H^0(\mathcal{U}, \mathcal{M}^1_C)$ has certainly a zero at the point p_{r+1} and all possible poles are at the points $p_1, \ldots p_r$ (with order at most 1).

2) A 1-form $\bar{\eta} = \{\bar{\eta}_i\}$ with the property $\bar{\eta}_i - \bar{\eta}_j = d\log\lambda_{ij}$ whenever $U_i \cap U_j \neq \emptyset$ can be obtained as $\bar{\eta}_i = \dfrac{du_i}{u_i}$ where $u = \{u_i\}$ is a meromorphic section of L. We can add to $\bar{\eta}$ a meromorphic 1-form θ of C, which is interesting if we are willing to move poles of $\bar{\eta}$. Notice firstly that $\bar{\eta}$ has simple poles, with residues α_i; of course $\sum \alpha_i = c(L)$, where $c(L)$ is the Chern class of L. We select θ with simple poles at the same points but with residues $-\alpha_i$, besides other poles which can be taken as simple poles with residues μ_k satisfying $-\sum \alpha_i + \sum \mu_k = 0$, or $\sum \mu_k = c(L)$. We ask also $P(\theta) \cap (Z(\omega) \cup P(\omega)) = \emptyset$; finally we take $\eta = \bar{\eta} + \theta$.

3) We apply to some $\bar{\xi} \in H^0(\mathcal{U}, \mathcal{M}^1_C \otimes L^*)$ the same technique as in step 1) of this proof. □

We remark that the only restriction to the residues μ_k in the choice of θ is that $\sum \mu_k = c(L)$. We will demand that $Im\mu_k \neq 0$ for any k.

Lemma 2.4. *Equation* (2.3) *has always meromorphic solutions.*

Proof. First of all we remark that if ω_i, η_i and ξ_i are holomorphic then there exists a (unique) holomorphic solution for a given initial condition $h_i(0)$.We have to deal with the case where some (simple) pole appears.

1. ω_i has a pole. Let us write (2.3) as $y' + b(x)y + a(x)y^2 = c(x)$, with $a(x) = \frac{A(x)}{x})$; $A(x)$ is a holomorphic function and $A(0) \neq 0$. This equation can be written as the system
$$\frac{dx}{dt} = x, \ \frac{dy}{dt} = xc(x) - (xb(x) + A(x)y)y.$$
Therefore $(0,0)$ is a saddle-node with a strong separatrix $x \to (x, y_0(x))$ transverse to the vertical line $x = 0$ (which is the weak separatrix). We take then $x \to y_0(x)$ as our solution.
2. η_i has a pole. We write (2.3) as $y' + (\frac{\mu}{x} + B(x))y + a(x)y^2 = c(x)$, where $B(x)$ is a holomorphic function and $\mathrm{Im}(\mu) \neq 0$. The corresponding system is
$$\frac{dx}{dt} = x, \ \frac{dy}{dt} = xc(x) - (\mu + xB(x))y - xa(x)y^2$$
which has a (unique) non-vertical separatrix $x \to (x, y_1(x))$ at $(0,0)$. We select $x \to y_1(x)$ as the solution for (2.3).
3. ξ_i has a pole. We write (2.3) as $y' + b(x)y + a(x)y^2 = \frac{C(x)}{x}$; $C(x)$ is a holomorphic function and $C(0) \neq 0$. The corresponding system is
$$\frac{dx}{dt} = x, \ \frac{dy}{dt} = C(x) - (b(x)x + xa(x))y.$$
We look to this system in a neighborhood of $(0, \infty)$; using $u = y^{-1}$ we get
$$\frac{dx}{dt} = x, \ \frac{du}{dt} = -uxb(x) + xa(x) + C(x)u^2.$$
This is a saddle-node at $(x, u) = (0, 0)$, with a non-vertical strong separatrix $x \to (x, u_0(x))$ in the coordinates (x, u). We take then as solution $x \to u_0^{-1}(x)$; it is meromorphic with a simple pole at $0 \in \mathbb{C}$.

This ends the proof of the Lemma. □

Lemma 2.4 allows us to describe the singularities of the functions in $f = \{f_i\}$. We have the following cases:

- at a zero of ω_i: let us look to the equation (2.1); since $\eta_i + h_i\omega_i$ is holomorphic, we can say the same for $g_i(x_i) = g_i(0)e^{\int_0^{x_i} \eta_i + h_i\omega_i}$ (x_i is a coordinate in U_i and $g_i(0) \neq 0$). From equation (2.2) it follows that f_i is a holomorphic function of the type $f_i(x_i) = a + x_i^m v(x_i)$ where $v(x_i)$ is a non-vanishing holomorphic function and $m \in \mathbb{N}$ is greater or equal to 2.

- at a pole of ω_i: by our construction $h_i\omega_i$ is holomorhic, and so is $g_i(x_i) = g_i(0)e^{\int_0^{x_i}\eta_i + h_i\omega_i}$. Equation (2.2) gives $f_i(x_i) = a\log x_i + \beta(x_i)$, for $a \in \mathbb{C}$ and $\beta(x_i)$ holomorphic.
- at a pole of η_i: Equation (2.1) gives $log\, g_i = \mu \log x_i + \gamma(x_i)$, so $g_i(x_i) = Cx^{\mu}e^{\int_0^{x_i}}$ and $f_i'(x_i) = x_i^{-\mu}\bar{\gamma}(x_i)$.
- at a pole of ξ_i: now $h_i\omega_i$ has a simple pole. From (2.2) we have $g_i(x_i) = x_i^r\bar{\delta}(x_i)$ for $r \in \mathbb{C}$ and $\bar{\delta}(x_i)$ holomorphic. Consequently $f_i'(x_i) = x_i^{-r}\delta(x_i)$, for $\delta(x_i)$ holomorphic.

We remark that at the singularities of f we should take convenient sectors in order to have well defined branches for each f_i. Another remark that we shall use later is: suppose $c_1(L) \le g - 1$, where g is the genus of C; then the 1-form ξ can be chosen as a holomorphic 1-form. This is another simple consequence of Riemann-Roch's theorem.

3. Constructing Foliations

We give in Theorem 3.1 below a simple construction of foliations which have a projective transverse structure.

Before doing this, let us remark the following facts. Consider some ω_i in the collection ω with a zero of order k_i, and take the Equation (2.2) $\omega_i = -g_i\, df_i$. The 1-form $-\dfrac{\omega_i}{g_i}$ can be written as $x_i^{k_i}\, a(x_i)dx_i$ where $a(x_i)$ is a nonvanishing holomorphic function. It can be easily shown that there exists a holomorphic local diffeomorphism s_i such that $s_i(0) = 0, s_i^{k_i+1} = s_i \circ s_i \circ \cdots \circ s_i = \mathrm{Id}$ (the composition is taken k_i+1 times) and $f_i \circ s_i = f_i$ (or $s_i^*(df_i) = df_i$). The periodic diffeomorphism s_i is conjugated to the linear rotation $l_i(x_i) = s_i'(0)x_i$ via a diffeomorphism $\phi_i : (\mathbb{D}, 0) \to (\mathbb{D}, 0)$: $\phi_i^*\, s_i = l_i \;\; \phi_i'(0) = 1$; furthermore, $(\phi_i^* f_i) \circ l_i = \phi^* f_i$.

Theorem 3.1. *Let C be a smooth, compact, holomorphic curve and $n \in \mathbb{Z}$. Let L be a line bundle over C such that $c_1(L) = n$ and $\omega = \{\omega_i\} \in H^0(C, \mathcal{M}_C^1 \otimes L)$ with $Z(\omega) \neq \emptyset$ and simple poles. There exists a surface S and a embedding of C in S such that:*

1) *C has n as its self-intersection number in S;*
2) *C is generically transverse to a foliation of S which has a (singular) projective transverse structure;*
3) *C is tangent to this foliation only at the points of $Z(\omega)$.*

Proof. The total space of L total space is foliated by the fibers. We will modify this foliation in order to make it tangent to C at the points of $Z(\omega)$. From the analysis we did in Section 1, we have a (singular) projective structure for C given by a collection $f = \{f_i\}$. Let us take one of these elements, say f_1, defined in a neighborhood U_1 of a zero p_1 of order k_1 ($x_1(p_1) = 0$). Consider in a neighborhood of $(0, 0) \in \mathbb{C}^2$ the foliation $\mathcal{I}$ defined by the level curves of $(x, t) \to t - x^{k_1+1}$; we replace near p_1 the foliation by fibers of L by $\mathcal{I}$. This can be done as follows:

- we take an annulus $\mathbb{A} \subset \mathbb{C} \times \{0\}$ with center at $(0,0)$ and a small neighborhood V of this annulus in $\mathbb{C}^2$ saturated by the leaves of $\mathcal{I}$. Let R_V (respectively R) be a holomorphic vector field transverse to A (respect. transverse to $\phi_1(A)$) and tangent to the leaves of $\mathcal{I}$ (respec. tangent to the fibers of L); we denote by $R_V(.,.)$ and $R(.,.)$ their flows. Then V is diffeomorphic to a neighborhood W of $\phi_1(\mathbb{A})$ via a diffeomorphism Φ_1 such that $\Phi_1(R_V((x,0),T) = R((\phi_1(x),0),T)$, where $(x,0) \in \mathbb{A}$ and $T \in \mathbb{C}$ is small; clearly Φ extends ϕ.
- W is a subset of some neighborhood P of p_1 which is saturated by the fibers of L and is diffeomorphic to a polydisc (W itself is diffeomorphic to the product of an annulus by a disc). Similarly, V is a subset of a neighborhood of $(0,0)$ which is diffeomorphic to a polydisc Q to which we may restrict $\mathcal{I}$. Finally we remove all fibers of L which pass through $P \setminus W$ and add Q, using Φ_1 as the gluing map between V and W.

We see that the new surface is foliated with the fibers of L except at some neighborhood of p_1, where now there is a leaf tangent to this point with order k_1. The definition of Φ_1 guarantees that C has $c_1(L)$ as self-intersection number in this surface. Furthermore, f_1 has the same value at each point (close to p_1) of intersection of a leaf with C, which allows us to extend f_1 along the nearby leaves. We can repeat the same construction for any subset of $Z(\omega)$, getting a foliation $\mathcal{F}$. At the other points of C where no modification is made we simply extend the elements of f along the fibers of L. □

4. Projective transverse structures and 1-forms

As we have seen in the Introduction, a projective transverse structure for a codimension 1 foliation $\mathcal{G}$ of a manifold M may be obtained from the folowing data:

1. a covering $\bar{\mathcal{U}} = \{\bar{U}_i\}$ of M;
2. a line bundle $\bar{L} = \{\Lambda_{ij}\} \in H^1(\bar{\mathcal{U}}, \mathcal{O}_M^*)$;
3. a triplet of meromorphic 1-forms $\bar{\omega} = \{\bar{\omega}_i\}$, $\bar{\eta} = \{\bar{\eta}_i\}$ and $\bar{\xi} = \{\bar{\xi}_i\}$ defined in the open sets of $\bar{\mathcal{U}}$ such that $\bar{\omega}_i = 0$ defines $\mathcal{G}$ in each $\bar{U}_i$ and satisfy:

$$d\bar{\omega}_i = \bar{\eta}_i \wedge \bar{\omega}_i \ \ , d\bar{\eta}_i = \bar{\omega}_i \wedge \bar{\xi}_i \ \ , d\bar{\xi}_i = \bar{\xi}_i \wedge \bar{\eta}_i$$

and

$$\bar{\omega}_i = \Lambda_{ij}\bar{\omega}_j \ \ , \bar{\eta}_i - \bar{\eta}_j = d\log\Lambda_{ij} \ \ , \bar{\xi}_j = \Lambda_{ij}\bar{\xi}_i$$

in each $\bar{U}_i$ and $\bar{U}_i \cap \bar{U}_j$, respectively.

A similar procedure as in Section 1 yields a transverse projective structure for the foliation, at least outside zeroes and poles of the 1-forms. The question we address now is whether the transverse projective structure of the foliations constructed in Theorem 1 may be defined from a triplet of 1-forms as above.

Let us use the same construction and notation of Sections 1 and 2, but with a modification. We will deal here with the case $c_1(L) \leq g-1$, so we can choose $\xi \in H^0(\mathcal{U}, \Omega_C^1 \otimes L^*)$ (that is, a holomorphic 1-form); in particular, all the functions of $h = \{h_i\}$ are holomorphic and vanish at the poles of ω and η. We introduce the following notation: if in the open set U_i there is a point of tangency of the foliation we put $\tilde{g}_i = g_i$; otherwise we put $\tilde{g}_i \equiv 1$. We replace then each ω_i by $\dfrac{\omega_i}{\tilde{g}_i}$; consequently we have to replace ξ_j by $\tilde{g}_j \xi_j$ and η_i by $\eta_i - d\log \tilde{g}_i$. We remark that all $\tilde{g}_i$ are non-vanishing holomorphic functions. The corresponding functions h_i and g_i are affected by these changes: they become respectively $\tilde{g}_i\, h_i$ and $\frac{g_i}{\tilde{g}_i}$, but the functions in the collection f remain the same. The transition functions for the line bundle L become $\frac{\tilde{g}_j}{\tilde{g}_i}\,\lambda_{ij}$. The implication is that in the setting of Theorem 1, we may assume that all 1-forms ω_i and all functions g_i and f_i can be extended to a neighborhood $\bar{U}_i$ of U_i in S along the leaves of $\mathcal{F}$ to $\bar{\omega}_i$, G_i and F_i. Our interest relies in fact in $\bar{\omega}_i$, $d\,G_i$ and $d\,F_i$.

Let us write $\bar{\omega}_i = \Lambda_{ij}\,\bar{\omega}_j$, $\bar{\omega}_i = -G_i\,dF_i$ and $\tilde{\eta}_i = \dfrac{d\,G_i}{G_i}$; it follows that $\tilde{\eta}_i - \tilde{\eta}_j = d\log\dfrac{G_i}{G_j} = d\log\Lambda_{ij} - d\log\dfrac{dF_i}{dF_j}$. this last quotient makes sense since the functions F_i are constant along the leaves of $\mathcal{F}$. As $F_i = \phi_{ij}(F_j)$, it follows that $\dfrac{dF_i}{dF_j} = {\phi_{ij}}'(F_j)$ and $\tilde{\eta}_i - \tilde{\eta}_j = d\log\Lambda_{ij} - \dfrac{\phi''_{ij}(F_j)}{\phi'_{ij}(F_j)}\,dF_j$, so finally

$$\tilde{\eta}_i - \tilde{\eta}_j = d\log\Lambda_{ij} - \psi_{ij}(F_j)dF_j.$$

Theorem 4.1. *Let $g \geq 1$ and suppose that the selfintersection number $C.C$ of C inside S satisfies $C.C < 2-2g$. Then there exists a meromorphic 1-form $\bar{\eta} = \{\bar{\eta}_i\}$ such that $d\bar{\omega}_i = \bar{\eta}_i \wedge \bar{\omega}_i$ for all $\bar{U}_i$ and $\bar{\eta}_i - \bar{\eta}_j = d\log\Lambda_{ij}$ whenever $\bar{U}_i \cap \bar{U}_j \neq \emptyset$.*

Proof. We start noticing that $c_1(L) < g-1$, so that we are in the setting above. We intend to write $\{\psi_{ij}dF_j\}$ as a special coboundary. We notice also that the 1-forms $h_i\omega_i$ defined in the open sets U_i are all holomorphic.

Let us look at the conormal bundle $N^*_{\mathcal{F}}$ of the foliation $\mathcal{F}$ and at the short exact sequence

$$0 \to \mathcal{I}_C.N^*_{\mathcal{F}} \to N^*_{\mathcal{F}} \to N^*_{\mathcal{F}}/\mathcal{I}_C.N^*_{\mathcal{F}} \to 0$$

in some neighborhood $\bar{U} \subset \cup\bar{U}_i$ of C; $\mathcal{I}_C$ is the ideal sheaf of C. We have the associated long exact sequence

$$\cdots \to H^1(\bar{U},\mathcal{I}_C.N^*_{\mathcal{F}}) \to H^1(\bar{U},N^*_{\mathcal{F}}) \to H^1(\bar{U},N^*_{\mathcal{F}}/\mathcal{I}_C.N^*_{\mathcal{F}}) \to \dots$$

By a theorem of Grauert, $\bar{U}$ may be chosen as a Levi strongly pseudoconvex neighborhood of C (since $C.C < 0$), and $H^1(\bar{U},\mathcal{I}_C.N^*_{\mathcal{F}}) = 0$ (because we have $C.C < 2-2g$; more generally, $H^1(\bar{U},\mathcal{I}_C.\mathcal{A}) = 0$ for any coherent sheaf $\mathcal{A}$ defined in $\bar{U}$). Consequently the map

$$H^1(\bar{U},N^*_{\mathcal{F}}) \to H^1(\bar{U},N^*_{\mathcal{F}}/\mathcal{I}_C.N^*_{\mathcal{F}})$$

is injective. Let us consider some holomorphic extension $\bar{H}_i\bar{\omega}_i$ of $h_i\omega_i$ to $\bar{U}_i$. Now the cocycle $\{\psi_{ij}(F_j)dF_j\}$ restricted to C coincides with $\{\psi_{ij}(f_j)df_j = h_i\omega_i - h_j\omega_i\}$; by injectivity, $\{\psi_{ij}dF_j\} = \{\bar{H}_i\bar{\omega}_i - \bar{H}_j\bar{\omega}_j\}$ in $H^1(\bar{U},N^*_{\mathcal{F}})$ since they have the same image in $H^1(\bar{U}, N^*_{\mathcal{F}}/\mathcal{I}_C.N^*_{\mathcal{F}})$. Consequently $\{\psi_{ij}dF_j\}$ is a 1-cobord: $\{\psi_{ij}dF_j\} = \{\tilde{\omega}_i - \tilde{\omega}_j\}$ for a collection $\{\tilde{\omega}_i\} \in C^0(\bar{U}, N^*_{\mathcal{F}})$. We define then $\bar{\eta}_i = \tilde{\eta}_i - \tilde{\omega}_i$. We have $d\bar{\omega}_i = \bar{\eta}_i \wedge \bar{\omega}_i$ since both sides vanish. □

Let us make some remarks:

- in the case we have an affine transverse structure, $\psi_{ij} = 0\ \forall i, j$ such that $U_i \cap U_j \neq \emptyset$. I follows that $\tilde{\eta}_i - \tilde{\eta}_j = d\log \Lambda_{ij}$, and there is no need of the negativity hypothesis on $C.C$.
- take now $g = 0$. Then the same proof applies when $C.C < -1$. If $C.C = -1$, then a neighborhood of C can be blown down to a neighborhood of $(0, 0)$ in $\mathbb{C}^2$. We find directly the 1-forms $\bar{\omega}$ and $\bar{\eta}$ such that $\bar{\omega} = 0$ defines the foliation and $d\bar{\omega} = \bar{\eta} \wedge \bar{\omega}$.
- It can be readily seen that the third 1-form of the triplet can be taken as $\bar{\xi}_i = dH_i + H_i\bar{\eta}_i + \frac{H_i^2}{2}\bar{\omega}_i$ where H_i comes from $\tilde{\omega}_i = H_i\bar{\omega}_i\ \forall i$. This illustrates an interesting property of foliations with a projective transverse structure: the existence of the two first 1-forms of the triplet implies the existence of the third 1-form (see [4]).

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Paulo Sad
Instituto de Matemática Pura e Aplicada (IMPA)
Estrada Dona Castorina 110, Jardim Botânico
22460-320 Rio de Janeiro
Brazil
e-mail: sad@impa.br