Dynamic Modeling and Econometrics in Economics and Finance 22

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Dynamic Perspectives on Managerial Decision Making

Essays in Honor of Richard F. Hartl

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Introduction

Herbert Dawid, Karl F. Doerner, Gustav Feichtinger, Peter M. Kort, and Andrea Seidl

In the Introduction to their seminal 1986 book on Optimal Control Feichtinger and Hartl paraphrase Joseph Schumpeter's famous quote that dealing with capitalist economies without taking into account dynamics is like Hamlet without the prince of Denmark. A similar statement certainly applies also to the analysis of firm behavior. Dynamic aspects are of crucial importance for almost any area of managerial decision making. Firms are exposed to changing market environments and technological landscapes and can actively influence these developments among others through marketing activities or by investing in the development of product innovations or of new technologies. On an operational level firms face challenging dynamic planning problems when designing their production processes, supply chains and distribution logistics. All these different dynamic challenges gave rise to highly active areas of research, but there are very few scholars who have been

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able to make contributions to research on managerial decision making in several of these operational and strategic domains. Richard F. Hartl is one of them. His highly influential research covers domains ranging from daily operational challenges like Vehicle Routing and Scheduling (e.g. Bullnheimer et al. 1999a; Kovacs et al. 2015) to dynamic advertising strategies (e.g. Feichtinger et al. 1994) and strategic decisions of firms like those concerning capital investments in different technologies (e.g. Feichtinger et al. 2006).

However, not only the breadth of the areas of research to which Richard Hartl has contributed is impressive, but also the methodological diversity of his work. Most notably, his work bridges two methodological camps which in Economics and Management typically are strongly separated: those working with dynamic optimization methods, like optimal control theory, mainly relying on analytical findings and those employing meta-heuristics, mainly relying on computational analysis. Richard Hartl has not only heavily used both approaches in his work, but has also made important methodological contributions. With respect to optimal control theory he has for example provided an influential result on the monotonicity of optimal paths in problems with one-dimensional state-space (Hartl 1987) and has been among the pioneers systematically studying optimal solutions in higher-dimensional control problems with multiple stable steady states (e.g. Haunschmied et al. 2003). Furthermore, books and surveys co-authored by Richard Hartl (e.g. Feichtinger and Hartl 1986; Hartl et al. 1995) have contributed strongly to the diffusion of advanced optimal control techniques in Economics and Management. In the area of meta-heuristics he has contributed to the development of improved versions of algorithms, in particular ant systems (e.g. Bullnheimer et al. 1999b), variable neighbourhood search (e.g. Polacek et al. 2004) and adaptive large neighbourhood search (e.g. Kovacs et al. 2012) for rich vehicle routing problems.

Published research is only one channel through which Richard F. Hartl had impact on the profession. The second important channel is personal interaction as supervisor, co-author and colleague. The size of the network of co-authors (to which also all five Editors of this book belong) and the impressive number of former students or assistants of Richard Hartl who have launched a successful academic career clearly illustrates that also in this respect he has been highly effective.

The aim of this volume is to pay tribute to these achievements of Richard F. Hartl. It collects contributions from co-authors, (present or former) colleagues and friends which are as versatile as the work of Richard Hartl both with respect to topics and methodology. The common denominator of the papers is that (almost) all of them deal with dynamic problems in Economics and Management. The volume is divided in the two parts,

- Economic Dynamics
- Firm Management

which are distinguished according to the level of aggregation of the phenomenon under consideration, covering the aggregate level and the firm level.

1 Economic Dynamics

Caulkins and *Tragler* investigate the optimal mix of enforcement, treatment, and prevention over the course of a drug epidemic. Depending on epidemic parameters which are based on empirical data, it may be optimal to either eradicate the epidemic, to "accommodate" it by letting it grow, or to eradicate if control begins before drug use passes a Skiba threshold but accommodate if control begins later. They show that relatively modest changes in parameters can push the model from one solution regime to another, perhaps explaining why opinions concerning proper drug policy diverge so sharply.

History-dependency also plays an important role in *Yegorov et al.* The authors show how the success of an individual's career depends on the initial stock of human capital as well as on the market access. They analyze the impact of talent and discuss the reinforcing or deterring effect of market access on individual investment decisions and the payoff of talents.

Dawid et al. study the strategic interactions between an incumbent in a market and a potential competitor, which tries to enter the market through product innovation. It is shown that in the presence of upper bounds on investment activities of both firms a Markov-Perfect-Equilibrium exists under which, depending on the initial conditions, the knowledge stock converges either to a positive steady state, thereby inducing an entry probability of one, or to a steady state with zero knowledge of the potential entrant. It is the first paper to present a Markov-Perfect-Equilibrium of an asymmetric differential game which gives rise to a Skiba phenomenon with two co-existing locally stable steady states.

Jorgensen and *Sigue* extend the Lanchester advertising model by considering three types of advertising as separate decision variables, namely offensive, defensive, and generic advertising. The paper provides closed-form expressions for equilibrium advertising strategies.

Lambertini studies a differential oligopoly game of resource extraction in which firms delegate production decisions to managers who are remunerated based on profit and output. It is shown that in Markov-perfect Equilibrium with linear feedback strategies an increase in the weight put on output in the remuneration scheme implies an increase in the long-run level of the resource. Furthermore, an increase in this weight also enlarges the set of stable non-linear feedback solutions.

Boucekkine and *El Ouardighi* build on Stokey (1998) by studying a model of capital accumulation and growth in the presence of a negative environmental externality. They extend this framework by including the option of recycling that both reduces the stock of waste and generates income.

Bednar-Friedl et al. consider residential density and air pollution in terms of a predator–prey model. They analyse the impact of pollution control on the level of the long-run steady state and the transition path towards it. Furthermore, they discuss the results of an experimental study in which participants seek to maximise revenues from simultaneously harvesting a predator and a prey species while avoiding their overexploitation.

Belyakov and *Veliov* take the age-structure of the fish population into account when determining the optimal harvesting strategy. They show that the optimal efforts may be periodic in case of selective harvesting (i.e. only fish of certain sizes are harvested).

Moser et al. study in an optimal control framework the age specific labor force participation rate of females which maximizes total female labor force participation taking into account that fertility rates of working women differ from that of non-working females. It is shown that participation rates should be higher at lower and higher ages where fertility is relatively low. Furthermore, the paper explores the impact of family policy on the optimal female participation rate profile.

Sorger shows that a neoclassical one-sector growth model in continuous time with elastic labor supply and a learning-by-doing externality can have a continuum of balanced growth paths. The paper also provides a condition guaranteeing that a balanced growth path with constant labor supply is a locally unique equilibrium.

Blueschke and *Neck* study the interactions between fiscal (governments) and monetary (common central bank) policy makers by means of a small stylized nonlinear two-country macroeconomic model. They find that the cooperative solution always dominates over the non-cooperative solution, providing incentives for decision makers to coordinate their policies.

Novak et al. consider a government that fights against insurgents. To develop optimal government behavior they take the Lanchester model as a basis. Stable limit cycles are obtained in a simplified version of the model where intelligence is the only control variable.

The paper by *Luptáčik et al.* is devoted to the measuring of the economic performance of national economies taking simultaneously into account three dimensions of social welfare: economic, environmental, and social. The intertemporal analysis reveals the prevailing role of technology in improving overall social welfare as well as its three constituent dimensions.

2 Firm Management

Dogramaci and *Sethi* focus on organizational nimbleness, by which is meant the speed with which an organization responds to a failure requiring scrapping of capital equipment. The paper builds on Kamien and Schwartz (1971), where in the latter a model of a non-nimble organization is analysed.

In the study of *Inderfurth*, a two-stage manufacturer-retailer supply chain with stochastic production yield and deterministic customer demand is considered in a single-period context that allows for an analytical study of the impact of yield randomness on safety stock determination. It is shown that different yield types, like stochastically proportional, binomial, and interrupted geometric yield, can be associated with completely different properties concerning how safety stocks react on various price and cost parameters in supply chain planning. In a model-based analysis it is demonstrated that the safety stock properties not only differ between the respective yield types, but also between systems of central and decentralized supply chain decision making.

In the study of *Minner* dynamic pricing is analyzed. Dynamic pricing is an instrument to achieve operational efficiency in inventory management. We consider an economic ordering context with strategic customers who forward-buy or postpone their purchases in anticipation of price changes. By charging a lower price when inventories are high, significant profit improvements can be achieved. The paper analyzes a simple EOQ-type model with a single price change within each inventory cycle. Numerical examples illustrate its impact on profits, order cycle duration and optimal price discounts. In particular for slow moving and low margin products, improvements in retail-type environments are substantial.

Reimann studies a joint manufacturing-remanufacturing problem of a manufacturer under demand uncertainty. Supply of remanufacturable units is constrained by the availability of used and returned cores, which depends on previous supply of new units. Potential cost savings due to remanufacturing in later periods may induce the manufacturer to reduce its short-term profits by artificially increasing its supply in earlier periods. For dealing with this trade-off they formulate an intertemporal optimization problem in a classical two-period newsvendor setting. Exploiting first period information when taking second period supply decisions they provide analytical insights into the optimal strategy and compare this optimal strategy with a previously proposed heuristic.

Almeder compares different approaches for integrating the lot-sizing and the scheduling decisions in multi-stage systems. They show their abilities and limitations in describing relevant aspects of a production environment. By applying the models to benchmark instances they analyze their computational behavior. The structural and numerical comparisons show that there are considerable differences between the approaches although all models aim to describe the same planning problem.

In the paper of *Felberbauer et al.* a recently developed model for project scheduling and staffing by addressing two practically important features, namely the possibility of interruptions between the execution periods of a project on the one hand, and decisions between different types of labor contracts on the other hand is considered. A hybrid metaheuristic employs a decomposition of the problem into a project scheduling problem and a personnel planning problem.

Vetschera study methods to transform probabilistic information into a strict preference relation among alternatives, as such strict preferences are needed to actually make a decision. Decision makers are often not able to provide precise preference information, which is required to solve a multicriteria decision problem. Thus, many methods of decision making under incomplete information have been developed to ease the cognitive burden on decision makers in providing preference information. One popular class of such methods, exemplified by the uses a volume-based approach in parameter space and generates probabilistic statements about relations between alternatives.

Wagner et al. study investigates the effect of fear-inducing public service announcements on evaluations of subsequent commercials in a commercial break.

In two laboratory experiments, the authors measured participants' evaluations of advertisements using a program analyzer. In line with affective priming theory, the results showed that fear-inducing public service announcements can negatively affect evaluations of subsequent commercials.

Rauner et al. develop a Strategic Disaster Management wiki to address challenges associated with a multi-agency response in emergency situations (e.g., lack of coordination, information, and interoperability). The wiki provides main emergency glossary terms, definitions, and standards to improve decision making.

In the paper of *Brandstätter et al.* the main optimization problems arising in Ecar-sharing systems at strategic, tactical and operational levels are reviewed and the existing approaches often developed for similar problems, for example in car-sharing systems with traditional vehicles are discussed. We also outline open problems and fruitful research directions. The design and management of Ecar-sharing systems poses several additional challenges with respect to those based on traditional combustion vehicles, mainly related with the limited autonomy allowed by current battery technology.

Tricoire et al. introduce the golf tourist problem (GTP) as a generalization of the bi-objective orienteering problem, with specific constraints related to different regions. We develop a biobjective branch-and-bound algorithm as well as a standard epsilon-constraint algorithm for the GTP. Experiments demonstrate the superiority of the biobjective branch-and-bound approach, which provides exact solution sets for real-world instances in reasonable run times.

In the paper of *Doerner et al.* the contributions in the field of vehicle routing of by Richard F. Hartl and his coworkers is surveyed. The vehicle routing problem was formulated more than 50 years ago and has attracted great attention since then, not least due to its high practical relevance and its computational complexity. Throughout the years, various generalizations and solution techniques were proposed. Richard Hartl worked more than 20 years on different variants of vehicle routing problems especially with metaheuristics.

3 Appendix

Mehlmann eloquently discusses the game theoretic problem of Achilles, disguised as a daughter of king Lykomedes, and Ulysses, who tries to expose his opponent.

We wish to thank all the contributors and anonymous reviewers who made this volume possible. We are grateful to Willi Semmler and to Martina Bihn and Yuliya Zeh from Springer for their support in publishing this book.

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Part I
Economic Dynamics

Dynamic Drug Policy: Optimally Varying the Mix of Treatment, Price-Raising Enforcement, and Primary Prevention Over Time

Jonathan P. Caulkins and Gernot Tragler

Abstract A central question in drug policy is how control efforts should be divided among enforcement, treatment, and prevention. Of particular interest is how the mix should vary dynamically over the course of an epidemic. Recent work considered how various pairs of these interventions interact. This paper considers all three simultaneously in a dynamic optimal control framework, yielding some surprising results. Depending on epidemic parameters, one of three situations pertains. It may be optimal to eradicate the epidemic, to "accommodate" it by letting it grow, or to eradicate if control begins before drug use passes a DNSS threshold but accommodate if control begins later. Relatively modest changes in parameters such as the perceived social cost per unit of drug use can push the model from one regime to another, perhaps explaining why opinions concerning proper policy diverge so sharply. If eradication is pursued, then treatment and enforcement should be funded very aggressively to reduce use as quickly as possible. If accommodation is pursued then spending on all three controls should increase roughly linearly but less than proportionally with the size of the epidemic. With the current parameterization, optimal spending on prevention varies the least among the three types of control interventions.

1 Introduction

Illicit drugs impose enormous costs on society (Harwood et al. 1998; United Nations Office on Drugs and Crime (UNODC) 2004), and there is considerable debate over how policy makers should respond. A central question concerns the relative roles of three broad strategies: enforcement, treatment, and prevention.

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Drug use varies dramatically over time in ways that can fairly be described as epidemics even though there is no literal pathogen (Golub and Johnson 1996; Ferrence 2001; Caulkins 2001, 2005). For example, cocaine initiation in the US increased roughly four-fold in the 1970s, then the "infectivity" (number of new initiates recruited per current user) subsequently fell over time (Caulkins et al. 2004).

Traditionally drug control effectiveness has been evaluated in a static framework (e.g., Rydell and Everingham 1994), but intuitively the relative roles of enforcement, treatment, and prevention should vary over the course of an epidemic. Indeed, this has been argued for various pairs of interventions (Behrens et al. 2000; Caulkins et al. 2000; Tragler et al. 2001). The present paper yields substantial new insights by simultaneously considering key elements of all three principal classes of drug control interventions in a dynamic model parameterized for the most problematic drug (cocaine) for the country with the most dependent users (the US).

Enforcement, treatment, and prevention are broad classes of interventions, not single programs, so it is important to clarify what specifically is modeled. Enforcement here refers to actions taken against the drug supply chain that raise the cost of producing and distributing drugs and thereby increase retail prices (cf., Reuter and Kleiman 1986). Such actions account for the majority of US enforcement spending. For enforcement within US borders the largest cost driver is incarceration. Simply put, prison (at $25–30,000 per cell-year) costs more than arrest or adjudication (Greenwood et al. 1994). More people are arrested for possession than sale, but on the order of 90+ % of those imprisoned for drug-law violations in the US were involved in drug distribution (Sevigny and Caulkins 2004).[1]

A smaller share of enforcement dollars are spent outside US borders on interdiction in source countries and the "transit zone". There is debate concerning whether these activities are best thought of as driving up equilibrium prices or as creating spot shortages (Rydell and Everingham 1994; Crane et al. 1997; Manski et al. 1999; Caulkins et al. 2000). Modeling price raising enforcement is of interest even if enforcement outside the US has no impact on equilibrium prices, but we suspect that it does have at least some such effects.

Enforcement has been hypothesized to work through other mechanisms as well. Moore (1973) and Kleiman (1988) suggest it might increase non-monetary "search costs" that users incur to obtain drugs. These costs are non-negligible, even for experienced users (Rocheleau and Boyum 1994), but since regular users often have 10–20 alternative suppliers (Riley 1997) enforcement's effects through increasing search time are second-order for established markets (Caulkins 1998a) such as

[1] Possesion arrests include "possession with intent to distribute", which is essentially a distribution charge, but offenders arrested for simple possession are less likely to be incarcerated and when they are, they serve shorter sentences. Note that many of those involved in distribution also use drugs, but generally it is not the use per se that leads to their incarceration.

those for cocaine in the US today.[2] Likewise, enforcement against suppliers of mass market drugs does not work primarily through incapacitation; there are few barriers to entry, so incarcerated sellers are rapidly replaced (Kleiman 1993).

Prevention is similarly multi-faceted. Unfortunately there is little scientific evidence concerning the effectiveness of most forms of prevention other than school-based prevention (Cuijpers 2003), so we focus on school-based programs and adapt parameter estimates from Caulkins et al. (1999, 2002).

Caulkins et al.'s estimates are based on lifetime projections of results for "best practice" programs evaluated in randomized control trials run through the end of high school. This has two implications. First, since data are only available on impacts through the end of high school, there is unavoidable uncertainty about prevention's effectiveness over a lifetime. Second, the estimates pertain to model programs. Historically most school districts have not implemented research-based programs with high fidelity (Hallfors and Godette 2002). By using Caulkins et al.'s data, were are examining what the optimal level of spending on school-based prevention would be if the best currently available prevention technologies were employed.

There are many kinds of treatment, and they are of varying quality (Institute of Medicine (IOM) 1990, 1996). Effectiveness data from randomized-controlled trials for cocaine treatment is lacking (Manski et al. 2001). Hence, we model treatment somewhat abstractly as simply increasing the net quit rate and ignore the possibility that it might reduce the social damage per unit of consumption. For consistency we use the same basecase assumptions about treatment's average cost and effectiveness as did Rydell and Everingham (1994), Tragler et al. (2001), but in light of Manski et al. we do sensitivity analysis with respect to those assumptions.

Note that our goal is not to anoint any one of these classes of interventions as the "winner" in some cost-effectiveness horse race. Rather, the goal is to understand better how their relative roles might vary over the course of an epidemic.

2 The Model

2.1 Clarifying Some Common Misconceptions

Before proceeding it is important to dispel some common misconceptions about drug markets. First, most new users are introduced to drugs by other users, typically friends or siblings. This is the sense in which drug use is "contagious". Dealers rarely "push" drugs on unwitting innocents (Kaplan 1983). Furthermore, drug supply is characterized by intense and atomistic competition (Reuter 1983), not monopolistic control. Hence, drug suppliers do not act strategically. There are

[2] Infrequent or "light" users may have fewer alternative suppliers, but they account for a modest share of all consumption because they use so much less, per capita, than do heavier users.

simply too many of them; well over a million Americans sold cocaine within only 12 months (Caulkins 2004).[3] Hence, one can develop sensible models of drug markets without explicitly modeling strategic behavior by suppliers. Instead, one can simply abstract the drug supply sector by what amounts to a supply curve (albeit one whose position depends on enforcement).

Second, drug initiation and use are affected by prices. There was once a lore that drug addicts "had to have their drug" regardless of the price, but a considerable literature has clearly established that cocaine use responds to price changes (Grossman and Chaloupka 1998; Chaloupka et al. 1999; Chaloupka and Pacula 2000; Rhodes et al. 2001; DeSimone 2001; DeSimone and Farrelly 2003; Dave 2004). Gallet (2014) provides a nice, new literature review and synthesis. This should not be surprising. Merely consuming less when prices rise in no way implies or requires perfect foresight or full rationality. What is somewhat surprising is the magnitude of the response. Best estimates for the elasticity of demand for cocaine are in the neighborhood of -1 (Caulkins and Reuter 1998), implying that a one percent increase in price is associated with a one percent reduction in use. This substantial responsiveness may stem from the fact that the vast majority of cocaine is consumed by dependent users who spend a large share of their disposable income on their drug of choice. All other things being equal, price elasticities tend to be larger for things that are important budget items (e.g., housing) than for incidentals (e.g., toothpaste).

Unfortunately, there is much less information concerning what proportion of the overall elasticity stems from reduced per capita consumption by existing users vs. reduced initiation or increased quitting changing the number of users. In the absence of better information, we follow Rydell and Everingham (1994) and Tragler et al. (2001) in assuming an equal division between these categories and likewise divide the latter (price elasticity of prevalence) equally between effects on initiation and quitting.

2.2 Model Structure

The present model extends that of Tragler et al. (2001). It tracks the number of users ($A(t)$) over time t. Initiation is modeled as an increasing (but concave) function of the current number of users that is modified by price, through a constant price elasticity of initiation, and by prevention.

Primary prevention is typically modeled as reducing initiation by a certain percentage, where the percentage depends on program intensity. Diminishing

[3]Market power is most concentrated at the export level in Colombia, and never more so than in the heyday of the Medellin "cartel". Yet this supposed "cartel" was not able to stave off a precipitous decline in prices. In reality, the cartel was formed more for protection against kidnapping than to strategically manipulate prices. Today there are several hundred operators even at that market level.

returns are presumed through an exponential decay as in Behrens et al. (2000). As mentioned, effectiveness estimates are based on Caulkins et al.'s (1999, 2002) analysis of "model" or "best practice" programs. Note: even "model" prevention is no panacea. As Caulkins et al. observe, prevention tends to be cost-effective primarily because it is so cheap, not because it is extremely effective. If kids who were going to initiate drug use in the absence of a prevention intervention are given cutting edge school-based drug prevention, most (though not all) would still initiate drug use. That does not necessarily mean prevention programs are poorly designed. It may simply indicate that there is little one can possibly do in 30 or so school contact-hours to counteract the influence of many thousands of hours of television, peers, etc.

The background quitting rate is assumed to be a simple constant per capita rate. (Even such simple modeling can fit historical data surprisingly well; cf., Caulkins et al. 2004.) Like initiation, this flow is affected by price through a constant elasticity and by an intervention, in this case treatment. As in Rydell and Everingham (1994) and Tragler et al. (2001), treatment is assumed to exhibit diminishing returns because some users are more likely to relapse than others, and the treatment system has some capacity to target interventions first on those for whom the prognosis is most favorable.

Price is a function of enforcement intensity. The underlying theoretical paradigm is Reuter and Kleiman's (1986) "risks and prices" model, operationalized as in Caulkins et al. (1997). The key insight is that some component of price (the intercept) is due to the "structural consequences of product illegality" (Reuter 1983; Caulkins and Reuter 2010) accompanied by some minimal enforcement. The increment in price above that intercept is driven by the *intensity*, not the *level*, of enforcement because of "enforcement swamping" (Kleiman 1993). Sellers do not care per se about the level of enforcement, e.g., the number of arrests. They care about their individual arrest risk, which is essentially the total number of arrests divided by the number of sellers subject to those arrests. Hence, for any given level of enforcement, the *intensity* is inversely related to the number of sellers. Since we do not model sellers explicitly, we divide by the number of users, implicitly assuming that the number of sellers is proportional to the number of users.

We assume that the social planner wishes to minimize the discounted weighted sum of drug use and of drug control spending. The cost coefficient on consumption is simply the average social cost per unit of cocaine use. Clearly marginal costs would be more relevant, but we have no way to estimate them.

The quantity of cocaine consumed is simply the number of users times the baseline consumption per user, adjusted for the short-term price elasticity of consumption per capita. Consumption per capita varies across users and the mix of light and heavy users varies over the course of an epidemic. Our consumption per capita is calibrated to our base year (1992), a time when roughly one-third of all users were heavy users (weekly or more often).

2.3 Mathematical Formulation

If we let $u(t)$, $v(t)$, and $w(t)$ denote treatment, enforcement, and prevention spending, respectively, then the discussion above suggests the following formulation:

$$\min_{\{u(t),v(t),w(t)\}} J = \int_0^\infty e^{-rt} \left(\kappa\theta A(t)p(A(t),v(t))^{-\omega} + u(t) + v(t) + w(t)\right) dt$$

subject to

$$\dot{A}(t) = kA(t)^\alpha p(A(t),v(t))^{-a}\Psi(w(t)) - c\beta(A(t),u(t))A(t) - \\ -\mu p(A(t),v(t))^b A(t)$$

and the non-negativity constraints

$$u(t) \geq 0, v(t) \geq 0, w(t) \geq 0,$$

where

J = discounted weighted sum of the costs of drug use and control,
r = time discount rate,
κ = social cost per unit of consumption,
θ = per capita rate of consumption at baseline prices,
$A(t)$ = number of users at time t,
$p(A(t),v(t))$ = retail price,
ω = absolute value of the short-run price elasticity of demand,
k = constant governing the rate of initiation,
α = exponent governing concavity of contagious aspect of initiation,
a = absolute value of the elasticity of initiation with respect to price,
$\Psi(w(t))$ = proportion of initiation remaining after prevention,
c = treatment efficiency proportionality constant,
$\beta(A(t),u(t))$ = outflow rate due to treatment,
μ = baseline per capita rate at which users quit without treatment, and
b = elasticity of desistance with respect to price.

As in Tragler et al. (2001), treatment's increment to the per capita outflow rate is assumed to be proportional to treatment spending per capita raised to an exponent (z) that reflects diminishing returns, with a small constant in the denominator (δ) to prevent division by zero:

$$\beta(A(t),u(t)) = \left(\frac{u(t)}{A(t)+\delta}\right)^z.$$

We model enforcement's effect on price as in Caulkins et al. (1997) and Tragler et al. (2001):

$$p(A(t), v(t)) = d + e\frac{v(t)}{A(t) + \epsilon},$$

where d describes the price with minimal enforcement, e is the enforcement efficiency proportionality constant, and ϵ is an arbitrarily small constant that avoids division by zero.

Following Behrens et al. (2000), we model prevention as reducing initiation by a certain proportion. That proportion increases with prevention spending but at a decreasing rate because of diminishing returns. Specifically, we model

$$\Psi(w(t)) = h + (1 - h)e^{-mw(t)}$$

for positive constants h and m.

2.4 Parameters

Tragler et al. (1997) describe in detail how parameters are derived from the literature. Briefly, the price elasticity parameters (a, b, and ω) collectively generate a long term price elasticity of demand of -1 (Caulkins et al. 1997; Caulkins and Reuter 1998), half coming from reduced consumption by current users (ω) and half from changes in the number of users, with the latter divided equally between impacts on initiation (a) and quitting (b).

For consistency with Rydell and Everingham (1994) and Tragler et al. (2001), we take the baseline price to be \$106.73 per pure gram and choose as initiation parameters $\alpha = 0.3$ and $k = 5167$ to make initiation 1,000,000 per year when the number of users $A = 6{,}500{,}000$ in base conditions. They estimate total baseline consumption as 291 (pure) metric tons, so we set $\theta = 14.6259$ (since $14.6259 \times 0.10673^{-0.5} = 291{,}000{,}000/6{,}500{,}000$ and price is expressed in thousands of dollars).

Rydell and Everingham (1994, p. 38) report cocaine-related health and productivity costs of \$19.68B for cocaine in 1992, dividing by 291 metric tons of consumption implies an average social cost per gram of \$67.6/g (in 1992 dollars). These figures do not include crime-related costs, so in light of Miller et al. (1996), we take \$100/g as our base value ($\kappa = 0.1$ since dollars are measured in thousands). In view of Caulkins et al. (2002) we also consider larger values in the sensitivity analysis.

The price function parameters ($d = 0.06792$ and $e = 0.02655$) reflect a price of \$106.73 per gram under base case enforcement spending and an elasticity of price with respect to enforcement spending of 0.3636 as in Caulkins et al. (1997).

As in Tragler et al. (2001) we assume $c = 0.04323$ and $z = 0.6$. These values reflect Rydell and Everingham's (1994) estimates that spending an average of \$1700–\$2000 per admission to treatment provides a 13 % chance of ending heavy use, over and above baseline exit rates.

We adopt Behrens et al.'s (2000) value of $h = 0.84$, but modify their value of m slightly (1.93×10^{-6} vs. 2.37×10^{-6}) to reflect better the size of the birth cohorts on whom prevention is targetted.

The outflow parameter $\mu = 0.18841$ was selected to make the outflow be 700,000 users per year at base case prices, which reflects the observed population change (Office of National Drug Control Policy (ONDCP) 1996) net of initiation and treatment during the recent years of relative stability. The discount rate is set at $r = 0.04$ as in Rydell et al. (1996) and Caulkins et al. (1997).

These values are summarized in Table 1. Two values are given for parameters d, e, k, κ, μ, and θ. The values in brackets are the ones just described. For analytical convenience, we adjust d, e, k, and μ so that $\kappa = 1$ and $\theta = 1$, yielding the second set of values for those parameters.

Table 1 Base case parameter values

Parameter	Value	Description
a	0.25	Absolute value of the elasticity of initiation with respect to price
α	0.3	Exponent reflecting contagiousness of initiation
b	0.25	Elasticity of desistance with respect to price
c	0.04323	Treatment efficiency proportionality constant
d	0.03175 [0.06792]	Price with minimal enforcement (in thousands of \$)
δ	0.001	Constant to avoid division by zero
e	0.01241 [0.02655]	Enforcement efficiency proportionality constant
ϵ	0.001	Constant to avoid division by zero
h	0.84	One minus maximum proportion of baseline initiation prevention can avert with full implementation
k	$4,272$ $[5,167]$	Initiation constant
κ	1 [0.1]	Social cost per gram consumed (in thousands of \$)
m	1.93×10^{-6}	Prevention efficiency proportionality constant
μ	0.22786 [0.18841]	Natural outflow rate from use
ω	0.5	Absolute value of the short run elasticity of demand
θ	1 [14.6259]	Per capita consumption constant
r	0.04	Annual discount rate (time preference rate)
z	0.6	$1 - z$ reflects treatment's diminishing returns

3 Base Case Analysis

Note that for simplicity, the time argument t will mostly be omitted from now on. The model cannot be solved analytically, but the Appendix describes the derivation of the necessary optimality conditions according to Pontryagin's maximum principle (cf. Feichtinger and Hartl 1986; Grass et al. 2008; Léonard and Long 1992). Due to the concavity of the Hamiltonian with respect to all three controls (u, v, w), setting the first-order partial derivatives equal to zero leads to the unrestricted extremum. These equations allow one to describe u and w as functions of v and A, so the solutions are described in terms of phase portraits in the A-v plane.

Steady state values are given by intersections of the isoclines obtained by setting to zero the derivatives of the state (A) and control (v) variables (dark gray and black curves, respectively, in Fig. 1). With parameter values from Table 1, there are two intersections, a left-hand (lower A) intersection $\left(\hat{A}^{(l)} = 0.2 \times 10^6,\ \hat{v}^{(l)} = 1.04 \times 10^7\right)$ that is an unstable focus and a right-hand (larger A) intersection that is a saddle point $\left(\hat{A}^{(h)} = 3.24 \times 10^6,\ \hat{v}^{(h)} = 1.14 \times 10^7\right)$. Every saddle point equilibrium in a two-dimensional phase portrait has a stable manifold which consists of two branches. Locally, these branches are determined by the eigenvector associated with the negative eigenvalue of the Jacobian evaluated at the steady state. This is used to numerically compute the complete stable manifolds (light gray curves in Fig. 1) which, in optimal control theory, are known to be candidates for the optimal trajectories.

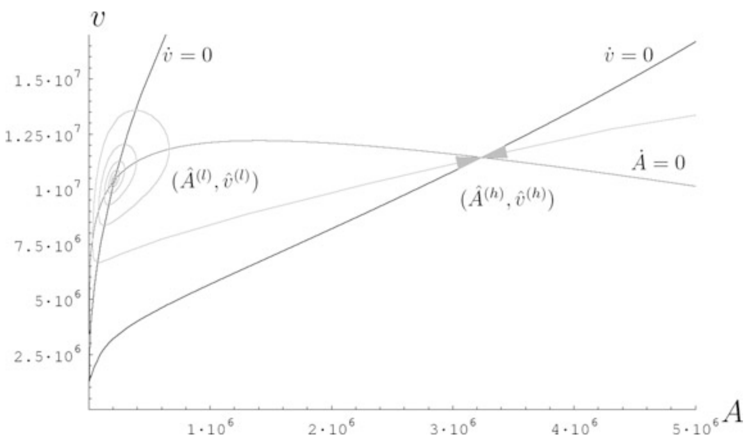

Fig. 1 Phase portrait with base case parameter values. The intersections of the isoclines $\dot{A} = 0$ and $\dot{v} = 0$ give the two steady-state solutions. The *light gray curves* represent the stable manifolds of the saddle point

The stable manifold from the right describes directly what trajectory one should follow to drive the number of users down to the saddle point equilibrium if the initial conditions have $A(0) > \hat{A}^{(h)}$. The stable manifold from the left emanates from the unstable focus, so it is not immediately obvious what the optimal policy should be when starting to the left of that focus. If control begins when the number of users is below its steady state value but still above a certain threshold A_{DNSS} to be described shortly, then the optimal treatment, prevention, and enforcement rates gradually increase while $A(t)$ converges to the equilibrium $\hat{A}^{(h)}$. (The opposite holds for initial states above the steady state value, but we presume that control begins with $A(0) < \hat{A}^{(h)}$.) Note this means that even if the optimal policy is pursued, the number of users will increase over time toward the equilibrium $(\hat{A}^{(h)})$.

Figure 2 shows the optimal amounts of treatment, prevention, and enforcement spending as functions of the number of users. When $A(0) > A_{DNSS}$, the optimal levels of control spending (u, v, and w) are each approximately linear in the size of the epidemic (A). The treatment (u) and enforcement (v) lines are almost parallel, implying that as time goes by, increments in the treatment and enforcement budgets should be approximately equal. Since with these parameter values the enforcement spending trajectory has a higher "intercept", for $A(0) > A_{DNSS}$ it is always optimal to spend more on enforcement than on treatment, but enforcement's share of the total control budget shrinks as time goes on.

According to Fig. 2, spending on prevention should also increase as the epidemic grows but not by much for the simple reason that prevention should already be

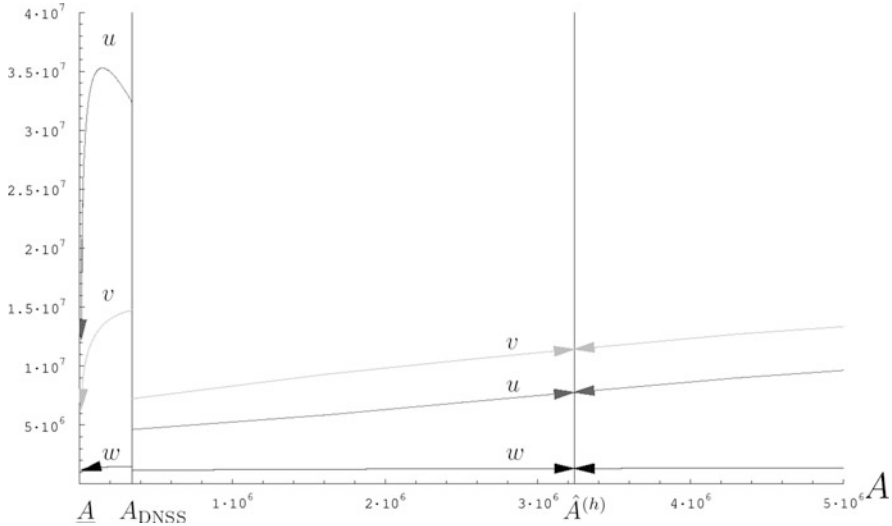

Fig. 2 Treatment (*dark gray*), enforcement (*light gray*), and prevention (*black*) as functions of A along the optimal paths. The *left and right vertical lines* represent the DNSS threshold and the saddle point at $\hat{A}^{(h)}$, respectively

almost "maximally funded" even when the epidemic is small. "Maximally funded" is in quotes because there is no literal bound on prevention spending, but the least it is ever optimal to spend on prevention is about \$1B per year. A cutting edge junior-high school-based prevention program costs about \$150 per youth, even including "booster sessions" in the two subsequent years (Caulkins et al. 2002), so \$1B per year would be enough to give six million youth per year an excellent prevention program. Since there are only about four million children in a birth cohort in the US, that \$1B would be enough to cover every seventh grader and also half of all fourth graders with a curriculum designed for younger children.

The great advantage of prevention is that it is so inexpensive compared to treatment or incarceration. It is not extremely powerful, at least with current technology, but it is powerful enough to make it optimal to "fully fund" prevention for almost any level of the epidemic. Still, even when fully funded, prevention does not absorb a large proportion ($< 10\,\%$) of drug control spending.

The total optimal level of spending in equilibrium, summing across the three programs, is about \$20B per year. That is probably roughly comparable to what the US has spent historically. More precise statements are difficult to make because data are not available for *national* drug control spending *by drug*. Figures are published annually for *federal* spending to control *all drugs*. Rydell and Everingham (1994) estimated that in the early 1990s, national cocaine control spending was roughly equal to federal spending on all drugs, and the federal drug control budget was \$18.8B for FY2002 (Office of National Drug Control Policy (ONDCP) 2002), which is quite close to the prescribed \$20B per year.[4]

Returning to Fig. 1, in addition to the "high volume" saddle point equilibrium, there is a second "low volume" equilibrium that is an unstable focus, so the optimal policy is more complicated when control starts when the epidemic is still small. For initial numbers of users below some critical level the solution is qualitatively different than a slow approach to the high volume saddle point equilibrium.

In particular, for smaller initial numbers of users ($A(0)$) it is not possible to jump onto the stable manifold that leads to the saddle point equilibrium. If we assume there is some lower limit, $\underline{A}$, on the number of users (e.g., $\underline{A} = 10{,}000$) below which control efforts cannot drive the problem (e.g., because these residual users cannot be detected), then the point $(\underline{A}, \underline{v})$ becomes another equilibrium, where $\underline{v}$ is given by the intersection of $A = \underline{A}$ and the isocline $\dot{A} = 0$. This steady state is approached along a trajectory which spirals out of the low volume equilibrium.

For low enough initial numbers of users it is only possible to jump on the stable manifold that approaches the lower limit equilibrium. For high enough values, it is clear one should approach the high volume equilibrium. For intermediate values,

[4]National budgets after 2003 have reported in a substantially different and non-comparable format. Walsh (2004) gives a quick, readable account of some of the changes in budgeting procedures and definitions.

there is a so-called Dechert-Nishimura-Sethi-Skiba (DNSS) point (Dechert and Nishimura 1983; Sethi 1977, 1979; Skiba 1978; cf. Grass et al. 2008) that defines two basins of attraction according to whether the optimal policy is to effectively eradicate drug use (push it to the lower limit equilibrium) or to just moderate its approach to the high volume saddle point equilibrium, as above. For the base case parameter values, that point is $A_{DNSS} = 344{,}339$ users.

Figure 2 shows that if the initial number of users is to the left of the DNSS point, treatment and enforcement spending are very high in absolute terms and, thus, truly enormous per user. Prevention spending is also higher than it is immediately to the right of the DNSS point, but less dramatically so. If it is optimal to eradicate the drug epidemic, then apparently it is optimal to do so aggressively and quickly (cf. Baveja et al. 1997). By spending enormous amounts on control in the early years, one avoids getting stuck at the high volume equilibrium.

Price is approximately a linear function of enforcement spending relative to market size (i.e., linear in v/A). It turns out to be a decreasing function of A for all A, with a sharp downward discontinuity at the DNSS point (since v^* is much higher just to the left of A_{DNSS} then it is just to the right of A_{DNSS}) (Fig. 3). Since when one starts to the right of A_{DNSS} one moves to the right (still assuming $A(0) < \hat{A}^{(h)}$), and when one starts to the left of A_{DNSS} one moves to the left, that means that the optimal price trajectory is very different depending on whether the optimal strategy is to eradicate or accommodate the epidemic. In particular, if the optimal

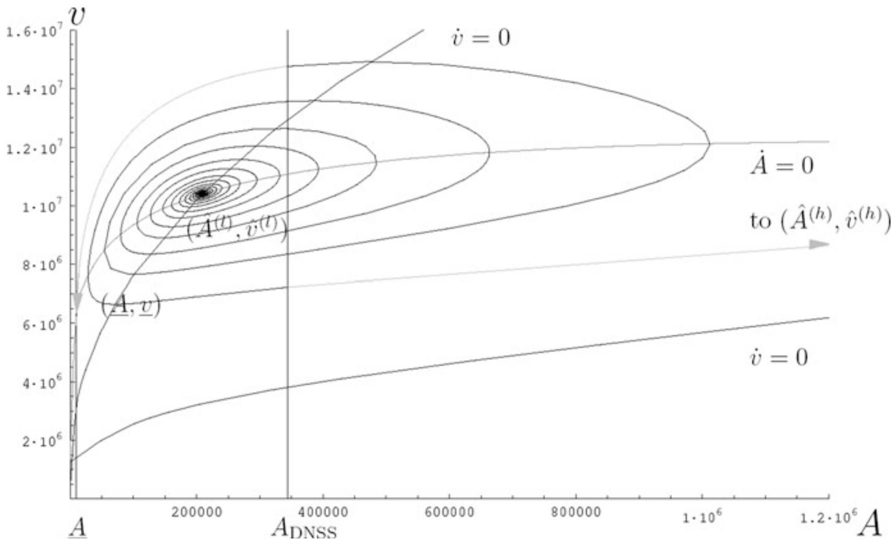

Fig. 3 The DNSS threshold A_{DNSS}. The two *gray curves* represent the optimal policy. On the *left side* of the DNSS threshold, the optimal policy leads to $(\underline{A}, \underline{v})$, on the *right side* optimal convergence is towards $\left(\hat{A}^{(h)}, \hat{v}^{(h)}\right)$

strategy is to accommodate, then it is optimal to allow the price to decline over time. Enforcement spending increases, but less than proportionally in A, so v/A and, hence, p^* decreases as one approaches the high-volume saddle point equilibrium. Conversely, if the optimal strategy is to eradicate the market, then it is optimal to start with a high price and keep driving it higher and higher until A reaches its lower limit. Even though enforcement spending declines over time with the eradication strategy, A declines faster so v/A and, hence, p^* increase over time when one opts for eradication.

To summarize, at the strategic level the policy prescription is simple. When control starts, one must judge whether the current epidemic size ($A(0)$) is greater or less than the critical DNSS threshold (A_{DNSS}). If it is greater than the threshold, then the optimal strategy is to grudgingly "accommodate" the epidemic, allowing it to grow to its high-volume equilibrium ($\hat{A}^{(h)}$). Spending on all controls should increase, but less than proportionately in A so control levels increase, but control intensity decreases. If, on the other hand, the initial epidemic size is below that critical threshold, then it is optimal to "eradicate" the epidemic in the sense of pursuing all controls extremely aggressively, quickly driving the epidemic down to its minimum size ($\underline{A}$).

Note: if spending were constrained to be proportional to the current size of the problem for some sufficiently small proportionality constant, e.g., because it is hard for politicians to muster support for massive spending on a problem that is currently small, then eradication might not be feasible and approaching the high-volume saddle point equilibrium might be the only alternative (cf. Tragler et al. 2001).

One final observation. The total discounted cost of the epidemic under optimal control, counting both the social costs of use and the costs of control, are monotonically increasing in the initial number of users. That is not surprising. What is surprising, is that the relationship is almost linear with a kink at the DNSS threshold. (Figure not shown.) Roughly speaking, for initial numbers of users below 1,000,000, total discounted costs increase by about $200,000 per person increase in $A(0)$ for $A(0) < A_{DNSS}$, and by about $80,000 per person for $A(0) > A_{DNSS}$. Those are astoundingly large numbers with a dramatic policy implication. In the absence of controls, for A near A_{DNSS}, modeled initiation is on the order of 1000 people per day, so the cost of delaying onset of control by even a day is very large. The actual values per day of delay are not simply 1000 times the figures above because one must account for what happens during the day of waiting. Doing so, it turns out that when the number of users is near the DNSS threshold ($A_{DNSS}/2 < A(0) < 2A_{DNSS}$), a one-day delay (or interruption) in control costs approximately $240 million per day to the left of the DNSS threshold and $60 million per day to its right. A corollary is that very significant investments in data collection systems may be justified if those systems can help detect future epidemics in their nascent stages.

4 Sensitivity Analysis

4.1 Sensitivity Analysis Concerning the Strength of Prevention

There is a reasonably strong basis for believing that current, model primary prevention technologies can reduce initiation by about $1 - h = 16\,\%$, but sensitivity analysis with respect to parameter h is still of interest for three reasons. First, many programs that are actually being used are not model programs, so the effectiveness of prevention today may be smaller (higher h). Second, better prevention technologies may be available in the future. For example, immunotherapies being developed to treat cocaine addiction might conceivably be used for primary prevention (Harwood and Myers 2004). There are plausible circumstances under which such vaccinations could be highly cost-ineffective for prophylactic purposes, but the very existence of such research suggests that prevention technology is not static. Finally, a fundamental contribution of this paper is adding prevention to the mix of interventions considered, so sensitivity with respect to its performance is of particular interest.

It turns out that if more effective types of prevention were available, that could quite dramatically affect what is the optimal policy and the resulting magnitude of drug problems. Figure 4 illustrates this with regard to optimal spending on the three types of control at the lower limit (quantities with an under-bar) and the right-hand saddle equilibrium (quantities with a hat).

Moving from right to left corresponds to prevention becoming more powerful (reducing h). Not surprisingly, spending on prevention increases as it becomes more effective (until the far left when it becomes so effective that slightly reduced levels of spending are sufficient). What is striking is the extent to which spending on

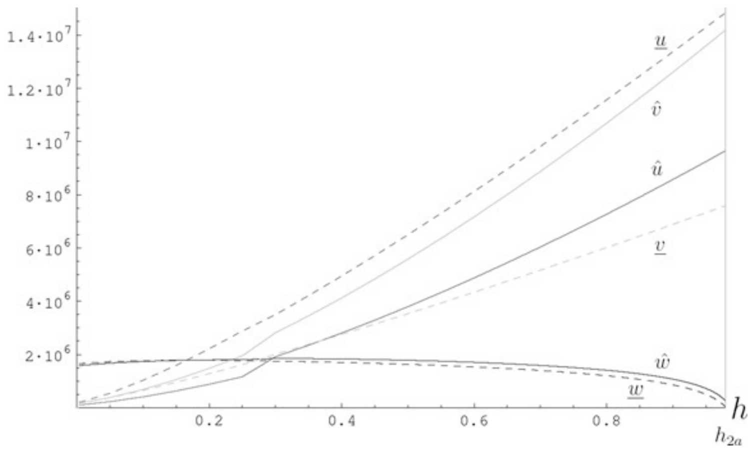

Fig. 4 The levels of optimal control spending as functions of h at $\hat{A}^{(h)}$ (*continuous*) and at $\underline{A}$ (*dashed*)

enforcement and treatment decline as prevention becomes more effective. Better prevention substitutes for these costly interventions. Furthermore, since prevention spending saturates at between $1B and $2B per year, total drug control spending declines as that particular drug control technology improves. Despite the declines in total control spending, with more effective prevention the right-hand saddle moves steadily to the left, roughly linearly in h, indicating fewer users in the steady state reached when accommodating the epidemic. Reduced control spending and reduced use both translate into lower social costs. Indeed, the present value of all social costs declines almost linearly by over 50 % as prevention effectiveness increases enough to reduce h from 1.0 to about 0.6. That potential may justify continued investment in prevention research even though the progress to date has been more incremental than dramatic. One initially counter-intuitive result is that as prevention's effectiveness increases, the DNSS point shifts to the left, not the right. One might have expected that as the tools of drug control improved, it would be not only feasible but also desirable to eradicate epidemics even if the initial number of users were somewhat larger. However, recall that a given level of prevention spending reduces initiation by a given percentage, regardless of what that level of initiation would have been, and that initiation is increasing in the number of users. Hence, increments in prevention's effectiveness are relatively more valuable when the number of users A is large, not when it is small. Hence, while increased prevention effectiveness reduces the cost of eradicating the epidemic, it reduces the social cost from accommodating that epidemic even more, shifting to the left the DNSS point, where one is indifferent between the strategies of eradication and accommodation.

4.2 Sensitivity Analysis with Respect to Treatment Effectiveness

As mentioned, a parameter about whose value there is considerable uncertainty is the treatment effectiveness coefficient c. Our basecase value is derived from Rydell and Everingham's (1994) analysis of data from the Treatment Outcomes Prospective Study, and treatment experts generally believe a 13 % probability of quitting per episode of treatment is conservative. Indeed, at several points in Rydell and Everingham's analysis, they erred on the side of conservativism. Nevertheless, Manski et al. (2001) note that selection effects could have introduced an upward bias and, more generally, there is next to no definitive data from randomized controlled trials concerning the effectiveness of cocaine treatment. Hence, this parameter is an appropriate object of sensitivity analysis.

Varying this parameter affects the saddle-point equilibrium in predictable ways. The more effective treatment is, the greater its share of control spending in steady state, and the fewer users there are in steady state. In particular, if treatment were 1 % more effective, it would be optimal in steady state to spend about 1 % more on treatment and almost 1 % less on enforcement (+0.97 % and −0.86 %, respectively, to be precise). Even though enforcement spending declines, enforcement intensity increases because the decline in the number of users is even greater (−1.65 %), inflating the ratio of v over A. Prevention spending also declines but less dramatically (by 0.22 %), which is consistent with the general finding that the optimal level of prevention spending is stable in multiple respects. Overall, improved treatment technology acts as a substitute for enforcement and prevention. Indeed, because with base case parameter values more is spent on enforcement ($11.4B) than treatment ($7.8B), the increase in treatment effectiveness actually leads to a reduction in total steady-state control spending.

4.3 Sensitivity Analysis with Respect to Initiation Exponent α

It is generally presumed that initiation is an increasing but concave function of the current number of users, modeled here as initiation being proportional to the current number of users A raised to an exponent α, with $\alpha = 0.3$ in the base case. Sensitivity analysis with respect to α is of interest because prevention is related to initiation and because it turns out that the location of the DNSS point is greatly affected by the value of α.

When α is varied, we vary k as well to keep the rate of initiation under base case conditions constant at 1,000,000 per year. That means that as α is reduced, the leading coefficient k is increased, and rather dramatically. By definition the reduction in α exactly offsets the increase in k when the number of users is 6.5 million, but for smaller numbers of users typical of earlier stages of the epidemic, the increase in k dominates. So in these sensitivity analyses, reducing α implies increasing rather substantially the force or "power" of initiation early in the epidemic.

Predictably, then, reducing α moves the DNSS point to the left, implying that eradication is the optimal strategy only under narrower circumstances. That makes sense. The appeal of eradication is that one drives use down to such a low level that initiation is also modest. When α is smaller, initiation with small A is much greater, so the benefit from reduced initiation achieved by driving A down to $\underline{A}$ is much smaller.

Still, the extent to which this turns out to be the case is striking. If α drops merely to 0.28, the DNSS point disappears and accommodation is always optimal. On the other hand, if α increases to 0.3415, the DNSS point moves so far to the right that it reaches the high-value saddle equilibrium (which also has been moving left), implying that eradication is always the optimal strategy.

4.4 Sensitivity Analysis Concerning the Social Cost per Gram Consumed

We observed that the optimal total level of spending at the saddle point equilibrium may be roughly comparable to what the US has spent historically on cocaine control. However, what level is optimal depends substantially on the presumed social cost per gram of cocaine consumed, and there is considerable uncertainty as to whether the base case value ($\kappa = \$100/\text{g}$) is "correct", both because of data limitations and because there can be genuine disagreement concerning what categories of costs should be included as social costs.[5] Generally, the greater the perceived social cost per unit of consumption, the more it is optimal to spend at the saddle point equilibrium and, hence, the lower the level of use in that equilibrium. In particular, if the social cost per gram were believed to be 20 % higher, then the optimal level of drug control spending at the saddle equilibrium would be 11 % higher. Likewise, if κ were 20 % lower, the optimal steady state spending would be 17 % lower, with the changes being most dramatic for treatment and least dramatic for prevention.

In contrast, the level of control spending at the lower limit $\underline{A}$ is almost unaffected by κ, presumably because the value of that spending is whatever it takes to prevent an epidemic from exploding, not an amount that is determined by balancing current control costs with current social costs of use.

Sensitivity of the optimal policy to variation in the assumed social cost per gram of use is even more pronounced for larger variations from the base case. In particular, reducing κ affects the DNSS threshold in qualitatively the same way as reducing α does, as is illustrated in Fig. 5, albeit for quite different reasons. As κ declines, the DNSS threshold shifts to the left, disappearing when κ drops to 0.7. Similarly, the DNSS threshold shifts to the right as κ increases, merging with the saddle point equilibrium when $\kappa = 1.474$.

Hence, someone who thinks the social costs per gram of cocaine use are less than \$70 per gram ought always to favor accommodation, whereas someone who thinks they are over \$147 per gram ought always to strive for eradication, even if the epidemic is already established. That is striking sensitivity inasmuch as it is easy for two reasonable people to disagree by a factor of 2 or more concerning the social cost per gram of cocaine.

An obvious implication is a plausible explanation for the persistent heated disagreements between drug policy "hawks" who favor having the goal be a "drug-free America" and "doves" who think the social costs of eradication exceed its benefits.

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[5] Notable examples include social costs borne by family members, any benefits of use of an illicit substance, valuation of a human life beyond that person's labor market earnings, and valuation of pain and suffering associated with crime and with addiction itself.

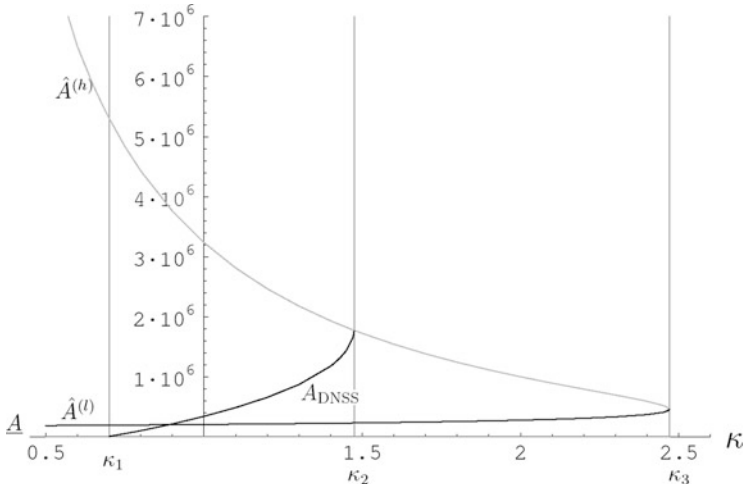

Fig. 5 The influence of κ on the equilibrium values and the DNSS threshold. The relation between κ and the high equilibrium $\hat{A}^{(h)}$ is displayed in the *upper gray* branch, while the *black lower* branch shows the relation between κ and the unstable focus at $\hat{A}^{(l)}$. The *black curve* between κ_1 and κ_2 bending upwards represents the level of the DNSS threshold, and the *horizontal gray line* at the very *bottom stands* for the lower limit at $\underline{A}$

A more subtle point emerges from the observation that the social cost per gram consumed is not an immutable physical constant like π or the speed of light. There are a whole set of policies not modeled here but popular in countries such as Australia and the Netherlands that go under the banner of "harm reduction". That term is highly controversial and widely misused and misunderstood. For the moment, let it mean simply and literally programs that reduce the social harm per unit of drug used, i.e., that reduce κ. The paradigmatic harm reduction policy, distributing clean syringes to injection drug users, is largely irrelevant for cocaine in the US, which is not primarily injected. Another favorite of harm reduction advocates is increasing treatment availability, which is already included in the model and is not actually likely to have as its primary outcome reductions in κ. Still, one can imagine other harm reduction tactics that would be relevant for cocaine in the US, including offering various forms of social support to the families of cocaine abusers, particularly their children; developing immunotherapies that treat cocaine overdose more effectively (Harwood and Myers 2004); and pursuing different types of law enforcement that push street markets into forms of distribution that generate less violence per kilogram sold and used, rather than seeking to reduce use by driving up prices.[6] Whatever the specifics, according to this model there can be a strong interaction between the presence of effective harm reduction and

[6]One partial explanation for why homicides have fallen so dramatically in New York City may be that much retail drug distribution has shifted from anonymous street markets where controlling "turf" produces profits to instances in which seller-user dyads arrange private meetings in covert locations, often using cell phones.

whether the optimal policy is eradication or accommodation. If one can design harm reduction strategies that reduce the average social cost per gram consumed, then accommodation might be the better alternative, even if eradication would be preferred in their absence.

4.5 Sensitivity Analysis Concerning the Lower Limit on the Number of Users

The larger the lower limit, $\underline{A}$, below which control cannot drive the number of users, the smaller the DNSS point. For example, doubling $\underline{A}$ from 10,000 to 20,000 roughly reduces the DNSS point by two thirds (reduces it from 334,339 to 128,268). This seemingly counter-intuitive result has a simple explanation. The smaller the lower limit on A, the more appealing that low-volume steady state is and, hence, the more the decision maker would be willing to invest in order to drive the epidemic to that lower steady state. Willingness to invest more means being willing to pursue the "eradication" strategy even if the initial number of users is somewhat larger.

If the minimum number of users is interpreted as the number below which users are essentially invisible, this has an interesting implication. Policy makers would like to push that lower limit down as far as possible. Doing so raises the DNSS point and, thus, increases the time it takes an epidemic to reach the "point of no return", beyond which the best that policy can do is moderate expansion to the high volume equilibrium.

As noted above, similar logic explains the otherwise surprising result that the more effective prevention is (i.e., the lower h is) the lower is the DNSS threshold.

5 Discussion

The analysis here confirms the observation of Behrens et al. (2000) and Tragler et al. (2001) that it can be misleading to discuss the merits of different drug control interventions in static terms (e.g., asserting that prevention is better than enforcement or vice versa without reference to the stage of the epidemic). Even this simple model of drug use and drug control can yield optimal solutions that involve substantially varying the mix of interventions over time.

Furthermore, the broad outlines of the policy recommendations are similar to those in Tragler et al. (2001). When a new drug problem emerges, policy makers must choose whether to essentially eradicate use or to accommodate the drug by grudgingly allowing it to grow toward a high-volume equilibrium. If the decision is to eradicate, then control should be very aggressive, using truly massive levels of both enforcement and treatment relative to the number of users to drive prevalence down as quickly as possible. If accommodation is pursued, levels of spending on price-raising enforcement, treatment, and primary prevention should increase linearly but less than proportionally with the number of users (i.e., linearly with a positive intercept). So the total level of drug control spending should grow as the epidemic matures, but spending per user would decline.

Of all the interventions, optimal spending on primary prevention is least dependent on the stage of the epidemic. To a first-order approximation, prevention spending should be about the same throughout. With our particular parameterization, that level is roughly enough to offer a good school-based program to every child in a birth cohort, but not dramatically more than that. That relative independence on the state of the epidemic is fortuitous inasmuch as there are built in lags to primary prevention, at least for school-based programs. Such programs are usually run with youth in junior high, but the median age of cocaine initiation in the US is 21 (Caulkins 1998b).

However, these observations do not in any way imply that adding prevention to this dynamic model does not alter the results. Prevention is a strong substitute for price-raising enforcement and treatment. The more effective prevention is, the less that should be spent on those other interventions. Furthermore, a truly effective prevention program would be such a strong substitute that both the amount of drug use and the combined optimal levels of drug control spending would decline, leading of course to a substantial reduction in the total social costs associated with the drug epidemic.

The catch is that to date even the better primary prevention programs seem to be only moderately effective (Caulkins et al. 1999, 2002), and the programs actually implemented are often not the best available (Hallfors and Godette 2002). Hence, with respect to the wisdom of further investments in improving the "technology" of primary prevention, one can see the glass as half full or half empty. The pessimists would point to limited progress to date and suggest focusing elsewhere. The optimists would see the tremendous benefits that a truly effective primary prevention program would bring and redouble their efforts.

The second broad policy contribution of this paper relative to the prior literature is the sensitivity analysis with respect to the location of the DNSS threshold and, hence, of when each broad strategy (eradication or accommodation) is preferred. In short, the finding is that the location of the DNSS threshold is highly sensitive to three quantities that are difficult to pin down for various reasons: the social cost per gram of cocaine consumed, the exponent in the initiation function governing how contagious the spread of drug use is, and the lower limit on prevalence below which it is assumed that control cannot drive the epidemic.

A depressing implication is that it will generally be exceedingly difficult to make an informed decision concerning the strategic direction for policy concerning a newly emergent drug. More is known and more data are available about the current cocaine epidemic in the US than about any other epidemic of illicit drug use, yet these parameters still cannot be pinned down even for cocaine in the US. It is hard to imagine that when a new drug epidemic emerges, we will have better information about it, at least at that early stage, and one of the results above was a startlingly high increase in social cost for each *day* that initiation of control is delayed. So a "wait and study" option may not be constructive.

Another depressing implication concerns the result for the lower limit on prevalence and its interpretation in a world of polydrug use. The model considered explicitly just one drug, cocaine. If there were just one illicit drug entering a "virgin" population, it might be somewhat plausible to drive use of that drug down to very low levels. However, the US already has several million dependent drug users who tend to use a wide variety of drugs, including new ones that come along. So if the US now faced a new epidemic, it might be that the only way it could drive use of that drug down to levels such as the lower limit considered here, would be to also eliminate use of the existing established drugs such as cocaine, heroin, and methamphetamine. That may be impossible or at least, according to this model, likely not optimal. Inasmuch as higher lower limits on prevalence make eradication strategies less appealing, accommodation may be the best option for future epidemics, even if eradication would have been the better course if we could turn back the clock to 1965.

The one positive observation, though, is that there exist, at least in theory, another set of drug control interventions, not modeled here, that would target not drug use but the objective function coefficient associated with that use. Introducing interventions of that sort into this framework would be one of many productive avenues for further research.

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Appendix: Optimality Conditions

The current value Hamiltonian H is given by

$$H = -(\kappa\theta A p^{-\omega} + u + v + w) + \lambda(kA^{\alpha}p^{-a}\Psi - c\beta A - \mu p^{b}A),$$

where λ describes the current-value costate variable.

Note that it is not necessary to formulate the maximum principle for the Lagrangian, which incorporates the non-negativity constraints for the controls, since u, v, and w all turn out to be positive in the analysis described in this paper.

According to Pontryagin's maximum principle we have the following three necessary optimality conditions:

$$u = \arg\max_{u} H,$$

$$v = \arg\max_{v} H,$$

and

$$w = \arg\max_{w} H.$$

Due to the concavity of the Hamiltonian H with respect to (u, v, w), setting the first order partial derivatives equal to zero leads to the unrestricted extremum, and we get the following expressions for the costate λ:

$$H_u = 0 \Rightarrow \qquad \lambda = \tfrac{-1}{c\beta_u A}, \tag{1}$$

$$H_w = 0 \Rightarrow \qquad \lambda = \tfrac{1}{kp^{-a}A^{\alpha}\Psi_w}, \tag{2}$$

$$H_v = 0 \Rightarrow \lambda = \tfrac{1-\kappa\theta\omega p^{-\omega-1}p_v A}{-akp^{-a-1}p_v A^{\alpha}\Psi - \mu b p^{b-1} p_v A}, \tag{3}$$

where subscripts denote derivatives w.r.t. the corresponding variables.

The concavity of the maximized Hamiltonian with respect to the state variable, however, cannot be guaranteed, so the usual sufficiency conditions are not satisfied.

With Eqs. (1)–(3) we can describe u, w, and λ as functions of A and v as follows:

$$\lambda(A, v) := \frac{\frac{p_v}{p}\left(\frac{a}{m} + \kappa\theta\omega p^{-\omega}A\right) - 1}{ahkp^{-a-1}p_v A^{\alpha} + \mu b p^{b-1} p_v A}, \tag{4}$$

$$u(A, v) := \left(\frac{-(A+\delta)^z}{czA\lambda(A, v)}\right)^{\frac{1}{z-1}},$$

$$w(A, v) := \frac{1}{m}\ln\left((h-1)kmp^{-a}A^{\alpha}\lambda(A, v)\right).$$

Due to this simplification we can concentrate on the two variables A and v.

To gain an equation for $\dot{v}$ we differentiate $\lambda(A, v)$ with respect to time:

$$\dot{\lambda} = \lambda_A \dot{A} + \lambda_v \dot{v}. \tag{5}$$

Setting (5) equal to the costate equation

$$\dot{\lambda} = r\lambda - H_A,$$

yields:

$$\dot{v} = \frac{r\lambda - H_A - \lambda_A \dot{A}}{\lambda_v},$$

where we insert $\lambda(A, v)$ from (4) and the corresponding derivatives λ_A and λ_v as well as H_A given by

$$H_A = -\kappa\theta p^{-\omega-1}(p - \omega p_A A) + \lambda[kp^{-a-1}\Psi(\alpha A^{\alpha-1}p - aA^{\alpha}p_A) - \\ -c(\beta_A A + \beta) - \mu p^{b-1}(bp_A A + p)].$$

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Economics of Talent: Dynamics and Multiplicity of Equilibria

Yuri Yegorov, Franz Wirl, Dieter Grass, and Andrea Seidl

Abstract The economics of art and science differs from other branches by the small role of material inputs and the large role of given talent and access to markets. E.g., an African violinist lacks the audience (=market) to appreciate her talent unless it is so large that it transgresses regional constraints; conversely, a European violinist of equal talent may be happy to end up as a member of one of the regional orchestras. This paper draws attention to this second aspect and models dynamic interactions between investments into two stocks, productive capital and access (or bargaining power). It is shown that there exists multiple equilibria. The separation between pursuing an artistic career or quitting depends on both idiosyncracies, individual talent and individual market access (including or depending on market size), which explains the large international variation in the number of people choosing a career in arts as market access is affected by geographic, linguistic, and aesthetic dimensions.

JEL Classification: D90, L11, L82, Z11.

1 Introduction

An important part of skilled labour—artists, writers, scientists, etc.—are creators. They produce specific output requiring talent. When we study an interaction

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between creators and the market, we have to take into account not only the process of individual human capital formation (a necessary input that allows talent to create an output) but also the possibility of market access required to sell this output and get some benefit. This second aspect is stressed in a recent article in *The Economist* (2015), namely that authors must invest time and resources in promoting themselves.

Economics literature suggests that equilibrium return to labour depends on its marginal productivity. There is also literature where this return depends on bargaining power. Both approaches explain part of the evidence. In the case of art there are two additional complexities: individual heterogeneity with respect to (non-observable) talent and the peculiarities of the market for art with network externalities and other scale economies that amplify heterogeneity of return to labour.

Becker (1976) has formulated a framework for labour skills, their market pricing and individual investment in time allocation. Acemoglu and Autor (2012) review the book of Goldin and Katz and suggest a richer set of interactions between skills and technologies. Hence, it is important to study not only individual accumulation of human capital, but also its interaction with the market.

A model of individual career evolution requires the investigation of the dynamics of human capital accumulation. There are many works in this field. For example, O'Brien and Hapgood (2012) use an evolutionary model based on a logistic equation and its modification in order to explain disadvantage of females in scientific career. Their basic assumption is the necessity for woman to work part-time (especially in the period of child bearing), and this prolongs the moment of reaching critical scientific mass of individual research output described by the logistic curve. While this model explains the disparity between males and females in their research careers, this is not the only heterogeneity that should be addressed. The second heterogeneity is of initial talent. Moreover, it is also important to complement such analysis by the third heterogeneity, in market access for different individuals. We will show how important it is to consider these asymmetries by more detailed analysis of the market structure for authors.

1.1 Importance of the Topic

In this article we propose a dynamic model capturing the interactions between a creator and her market. A talented individual accumulates her human capital, while market accessibility is not given a priori but has to be conquered using complementary capital. In this sense, we explore another application of economic growth model related to capital accumulations. Market access in our model is not costless, because there might be: (a) intermediaries separating direct access of producer to consumers, (b) asymmetric cost of market access (e.g., if firms have different transport costs to the market). The second feature is a topic of regional science, while the first is covered in literature on market barriers (see, for example, Schmalensee 1981; Weizsäcker 1980).

If we consider a creator (or talent) instead of a usual producer (e.g. of agricultural goods, steel), the analysis also becomes intrinsically dynamic. We want to trace the dynamics from a perspective of an individual talent, who has to invest into his or her human capital and his or her access to the market. How market barriers due to language, distance and taste interact with investment in human capital is an interesting question. The social dimension, i.e. the dynamics of a population of talents with different initial endowments, is a related topic. How will market accessibility influence the outcome? Potential applications include influence on policy, or the probability to 'foster' sport champions.

1.2 Goals

The goal of this article is to investigate the dynamic interactions between a creator and the market. Creators are producers of individually differentiated output that differ by type (different kinds of art, science, etc.) and quality determined by the level of individual talent. This suggests a market structure different from the standard labour market where workers receive the marginal value of their product. On the one hand, there is some kind of monopolistic competition between talents. On the other hand, consumers are not informed about each creator since there are too many varieties (in the extreme case coinciding with the number of creators). Therefore, the role of intermediaries (like publishers and promoters) is important. Their risk aversion and the possibility of at least local monopoly power reduces an author's return with the consequence that the distribution of returns to talents is more skewed than the distribution of talents.

The analysis starts from a creator's perspective whose returns depend on talent, human capital (education and training), and on market access. While talent is given, human capital and market access are stocks that can be expanded by investments. We find that multiple equilibria exist over a broad range of parameter values. Therefore, not only talent but initial circumstances like education and access to markets determine whether one should pursue a career in creative industries or not.[1]

1.3 Literature Survey

Art is an interesting and important case of talent performance, and thus we consider this market in more detail. The economics of art differs from other sectors in several

[1] This outcome explains the choice of this paper for Richard's Festschrift. If there is a regret then the one that I (= Franz Wirl) did not manage to write papers jointly with Richard except for the two emerged from extremely fruitful discussions and work at a conference in Alghero that turned down our submissions (i.e. of Richard, Gustav and myself; about Peter I do not recall for sure).

aspects. In a seminal paper Baumol and Bowen (1966) introduced the term "cost disease" for relative cost growth of live performances, which explains the increasing dependency of this kind of art (e.g., orchestras, since it makes no sense to play Mozart's clarinet concert faster) on state subsidies. Rosen (1981) stresses another particularity of the economics of art by focusing on the very skewed distributions of returns to talent. More precisely, little addition to talent can yield very high extra return while a slightly lower talent may have to work for very small wage. In order to explain this effect, Rosen (1981) introduced fixed cost for each consumption of art (e.g., cost in terms of leisure to watch art performance) that pushes consumer choices towards more talented artists. These (abnormal) returns to talent have increased over the recent decades partially due to globalization, which takes place not only in economics, but in art, music, film and TV. Our paper is complementary to this literature by emphasizing dynamics and a crucial second (but much less explored) dimension of market access.

Creative industries refer to a range of economic activities that generate or exploit knowledge and information and include publishing, film, music of all varieties, TV, design, fashion, art (painting and sculpture), advertising, computer games, etc. Howkins (2001) also includes here research and development in science and technology. They all depend on individual creativity, skill and talent. Hesmondhalgh (2002) reduces the list to core cultural industries that engage in some form of industrial reproduction. According to Caves (2000), creative industries are characterized by several economic principles, the first of which is demand uncertainty ("nobody knows principle"). Others are product differentiation by quality, uniqueness, and vertical differentiation of skills with small differences leading to huge differences in (not only financial) success.

Cultural and creative industries do not fit easily into the standard economic framework. This is due to their origin in cults (and thus outside of markets) and due to the existence of network effects. The paper of Potts et al. (2008) argues that "the central economic concern [for creative industries] is not with the nature of inputs or outputs in production per se, or even with competitive structures, but the nature of the market that coordinates this industry". This suggests a crucial role for potentially complex social networks. There exist two extremes of consumers' choice. At the one extreme, consumers act as autonomous rational agents. At the other extreme, consumers rely purely on the decision of others (due to peer groups, as sociologists stress this point). Potts et al. (2008) argue that in mature markets preferences are fixed (consumers already learned them), while in creative industries the network effect dominates.[2] While that article it non-mathematical, it opens a wide possibility for mathematical modelling. First, the authors divide all markets into mature markets where the neoclassical economic approach is valid (agricultural, mining, industrial, etc.) and those driven mostly by economic of networks (art,

[2]If one looks at the dynamics of top-ten songs in the world, it is easy to discover high variability over time, and complete independence on agents with fixed tastes, i.e. those who like music of, let say, Mozart or "Beatles".

science, etc.). The latter behave as open-complex-adaptive systems. New cultural industries as well as science are heavily influenced by social networks, that in turn influence consumer's choice and participate in value creation. In our paper we will model the interaction between creators and such network in a highly stylized but simple way.

Social externalities have been modelled in Schelling (1973), Cooper and John (1988) and generalized in Glaeser and Scheinkman (2003), all within a static context. Krugman (1991) accounts for dynamics within social networks (actually, the labour market), see Fukao and Benabou (1993) for a correction, and Wirl and Feictinger (2006) and Wirl (2007) for recent generalizations. However, this paper does not model complex networks (see e.g., Barabasi 2002) but takes a network of consumers as a given to which producers can connect.

There exist a few dynamic models that describe career building in art. Adler (1985) argues that the emergence of superstars has its origin not in the difference of talents but in a random factor. He correctly argues that the preferences of consumers towards art are not self-regarding but also depend on discussions with friends and articles in newspapers. Thus, we have network externalities in this market, correctly mentioned also by other authors. In the model of Adler it is chance that picks up randomly one artist among equally talented artists, thus bringing fame and a lot of money to one and frustration and poorness to the rest. His basic argument is that due to these specific preferences of consumers there is simply no place for many superstars at one moment. We agree with Adler that not only talent plays its role but are willing to decompose "chance" by a more deterministic structure in our model (that might be poorly observable in practice). Indeed, can we measure talent exactly? Why talents should be absolutely equal? There might be small difference between them and also in other factors that determine success. We also can observe the market outcome when an artist with visibly lower talent has higher success, and this cannot be explained by the model of Adler but can be explained by our model.

Chan et al. (2008) model (classic) musicians who can choose between playing mainstream and fringe. They take into account both the preferences of consumers and their own preferences for style. The model considers two time periods, two types of musicians' quality but a continuum of style types. One of the major results is self-selection. Musicians with extreme preferences for mainstream will choose to play such music in both periods, while fringe extremists will play fringe music in both periods sacrificing profits in favour of their preferred style. What is more interesting is that playing mainstream in the first, and fringe in the second period becomes the dominant choice for intermediate types, because they first gain reputation and then shift to their preferred style. This model shows the importance of the interplay of several factors for an artist's career choice, a point which our model addresses in a different way.

One goal of our paper is to develop a model that can explain some of these stylized facts, in particular, to demonstrate the role of market accessibility for the decisions of an individual creator within a dynamic set up. The standard economic assumption is that market entry and access is free so that a producer has to care only about investment into productive capital. However, some markets are characterized by barriers for entrants; Weizsäcker (1980) is a classic treatment, Brock (1983) considers lobbying capital that can erect barriers with threshold effects similar to our analysis below. Creative industry is one example in which substantial costs are required to connect poets, writers, composers, singers, artists or even scientists to their customers. This may require promotion, e.g., by advertising in TV and newspapers, or it may involve investments in establishing the necessary personal links, or even bribes, e.g., in order to pass some bureaucratic procedures. In addition access is linked in many cases to geographic, linguistic and cultural[3] idiosyncrasies. The number of readers that an author of a book can expect depends on whether the book is written originally in English (or Chinese) or in a language less spoken, and/or with many illiterates, or whether it is a novel or poetry. The importance of this effect, which plays also a crucial role in determining thresholds for investments into an artistic career, can be seen by the dominance of books and songs originating from areas with a large (linguistic and cultural) home basis such as the Anglo-Saxon. Everything else equal, a larger market size encourages more people to pursue a career as an artist. Indeed, there are more songwriters and pop musicians with Anglo-Saxon roots (e.g., compare U.K. versus Germany), more football players in the US but more people investing into their soccer skills in the U.K. and Brazil.

Section 2 presents the model of talent and market access as a dynamic optimization model. Section 3 presents numerical examples and classifies the types of equilibria. Section 4 deals with basic properties of economics of art, describes its complexity and discusses the possibility to apply this model there. Section 5 concludes.

2 A Model of Talent and Market Access

We start from basic description of economics of art with the focus on economics of a writer. In order to develop her talent, there is a necessity to invest not only in human capital (that gives higher quality of writing) but also in bargaining power with the editors that will determine the author's share in payoff. After definition of all the variables and parameters of the model, it is formulated as a dynamic optimization problem with two states and two controls. Optimality conditions bring us to the system of 4 differential equations and 4-dimensional polynomial for the steady states which is further investigated numerically.

[3]The arts are and were always full of fashions. A composer with a talent for melody will not make in today's atonal serious music. Similarly even a painter with talents equal to those of Rembrandt with respect to realistic paintings will not make it with his peculiar talent in today's market dominated by abstract paintings.

2.1 Economics of Art

Some properties of creative business are crucial even for a simple mathematical framework. Let us start with examples from the age prior to internet, TV and even radio. Jack London in his “Martin Eden” has described the difficult process for a talented writer to achieve popularity and prosperity. Fjodor I. Dostoevsky spent his life in poverty and debts, although he became one of the most popular authors after his death and remains so until today. The paintings of impressionists had very low prices during their first creative decade. One objective is to explain why the path to popularity and market recognition may be so difficult for individuals.

Until the recent past authors were separated from the network of readers by editors. Therefore, any beginner faced the difficulty of getting her work published and even after first publication had a difficulty to live on the proceeds. Suppose that talent did not change much between his first and last publication. Why was the return on talent growing over time? Can we explain this effect only by investment in talent? Clearly not. We have to introduce a new variable that will measure the bargaining power of author that describes the splitting surplus of writing activity with the publishers.

Nowadays after the emergence of the internet the situation seems different: a book can be put online and an author can become popular even without editor. However, the problem of rent collection remains or is even aggravated for internet publications. Furthermore, such an author is required to promote the work in order to catch attention among all the material available online. And this skill is complementary to the talent of an author. Therefore in both cases, traditional writing and modern creative industries, a talent has to invest in some skills complementary to making art, and we propose a dynamic model that covers both cases.

Suppose that a particular creator has an output potential (aK). If he is not or only weakly connected to the market, e.g., at the beginning of her creative activity, there is no way for an outsider to assess her potential. However, the (potential) author knows this value (perfectly). This individual potential can increase over time due to investment, otherwise it depreciates, i.e., even to keep productive talent constant one needs to work with it.[4] Therefore, the dynamics of the human capital stock K (education, fitness, research) requires investments in the form of learning and/or practicing. For simplicity it is assumed that all creators are equal with respect to the costs of acquiring this human capital stock K (which can be measured by years of schooling relevant for a particular creative activity). However, talents differ across agents and this is measured by the parameter a, which is not observable to outsiders, at least at the beginning.

[4] A postcard from Oxford read: “The more you study, the more you know. The more you know, the more you forget. The more you forget, the less you know. So why study?”.

2.2 Economics of a Writer

We consider in the following a writer in order to outline the need for a second state. Suppose, a writer writes the first book and brings it to an editor. Since the editor may be risk averse (after all he has very little idea about this author, here the parameter a), the offer might be only a small fraction (say, 10 %) of the expected profits as equilibrium honorarium. Let $0 < G < +\infty$ be the power of this author to bargain, which is another skill, complementary to talent. Then we postulate that his equilibrium share (of the pie) is given by $s(G) = G/(1 + G)$, where $0 < s < 1$.[5] When our author has full and costless access to market, writer's payoff is exactly equal to aK, and the bargaining share is one. In this case, one may think about author-publisher in one person and indeed currently all famous authors are able to retain the copyrights over their works.

Publishing a book involves high fixed cost (F) and low variable cost (negligible for simplicity). Assume, that the ex ante book price is p and given[6] and that the number of potential readers N is proportional to the artistic quality of the author (= product of his talent and human capital): $N = \gamma aK$. Then publication is profitable if $Np > F$. The publisher's expected profit is positive given the distribution of quality and his acceptance rule for manuscripts because otherwise he cannot survive. Risk averse publishers can impose external costs on potential writers. One can argue that free entry drives profits towards zero such that only risk-neutral editors will survive. However in reality even the business of publishing may affect market access. For example, in small markets local monopoly may exist, and it may exist even in large markets due to regulation, just as only national TV-channels were allowed in most European countries until rather recently. As a consequence, the market access for a potential TV-script writer was much more limited in Europe compared with the U.S. and its many channels. This may explain the dominance of US TV-series.

But now readers can go online. Thanks to social media, word of mouth spreads faster than ever before, giving unknown writers a better chance. Today, a bestseller must usually appeal either to young people (who use social media a lot) or women (who dominate reading groups).[7] Will this change a lot in our arguments of market access? Before a writer had to fight with a publisher for his share, but today he has to access a complex network of readers with preferences that depend not only on product quality. In both cases our author has to develop some skills complementary to those needed for a writer. An example of a network in the market is considered in the Appendix.

[5]This is the simplest mapping from real half-line $[0, \infty)$ to a unit interval $[0, 1]$.

[6]Although we can consider that playing in price can influence demand, this is not true. If a reader dislikes certain book, he might not buy it even with 80 % discount. At the same time, he is not likely to pay price above average book price for unknown author.

[7]The Economist, May 5th 2012, p.65.

2.3 Model Formulation

Here we develop a dynamic model for a talent from the perspective of an individual. Some literature considers "talent" only in its developed form, when it can be measured by public success, for example. We think that in fact such approach does not measure individual abilities and skills, but the product of their interaction with learning process and the market. Since we want to focus on those interactions, by "talent" we will understand an individual heterogeneity in particular skills (related to some arts) that can create some art product after the process of skill development, or learning. By this assumption we depart from the literature on human capital that typically assumes all agents to be identical a priori and to differ only by years of schooling. It is well known that the same education can result in quite different quantity (and quality) of output. This is especially important when such output has some unique characteristics which are typical for the area of the arts.

An individual creator is endowed with talent a, an initial stock of human capital K_0 (e.g., education before starting to develop his special talent) and market access G_0 and can invest in each period t into both $(I_K(t), I_G(t))$ at linear-quadratic costs. Both stocks depreciate at the same common factor δ for reasons of simplicity (although market access depreciates presumably faster than educational attainments) and which does not lead to more complex dynamics but only to more parameters. The same input can yield widely different outputs. This variation is due to differences in individual talent that is proxied by the coefficient a. Assuming that the payoff to a creator depends (linearly) on her or his talent a (unobservable to all outsiders), acquired human capital K (e.g., given by educational degrees), and the share of the cake, $G/\,(1+G)$, the creator has to solve the following dynamic optimization problem ($r > 0$ denotes a discount factor), where the argument, time t, is omitted:

$$\max_{I_K \geq 0,\, I_G \geq 0} \int_0^{\infty} e^{-rt} \left[\frac{G}{1+G} aK - b_1 I_K - b_2 I_K^2 - c_1 I_G - c_2 I_G^2 \right] dt, \tag{1}$$

$$\text{s.t.} \qquad \dot{K} = I_K - \delta K,\; K(0) = K_0,\; K \geq 0, \tag{2}$$

$$\dot{G} = I_G - \delta G,\; G(0) = G_0,\; G \geq 0. \tag{3}$$

The two idiosyncrasies—individual talent and market access given the artist's origin (geography, language, and type of art[8] affect the returns)—are reflected in the parameter $a = \alpha\beta$, where α refers to talent and β to the size of the potential market for say a novel written in Albanian about topics peculiar to Albania. Alternatively, playing with the cost coefficients (c_1, c_2) allows capturing how much easier it is for English-writing authors to access the global market compared with, say an Albanian, writer. The returns to each investment satisfy the law of diminishing

[8]Similarly for sports: playing golf, tennis, professional boxing delivers much higher returns to such a talent than biking, badminton, and chess.

returns (only weakly with respect to K but this is not crucial and a power relation (K^{κ}) allows for similar conclusions); yet increasing returns apply to both inputs (K, G), which capture the abnormal returns to the right people in the right place, i.e., to high talents (as already noted in Rosen 1981) and to market access, which is specific in our case.

2.4 Optimality Conditions

Maximizing the current value Hamiltonian of the optimal control problem (1)–(3),

$$H \equiv [\frac{G}{1+G}aK - b_1 I_K - b_2 I_K^2 - c_1 I_G - c_2 I_G^2 + \lambda[I_K - \delta K] + \mu[I_G - \delta G], \quad (4)$$

with respect to the controls yields (from the familiar condition that marginal costs equal their intertemporal shadow prices λ and μ),

$$I_K = \frac{\lambda - b_1}{2b_2}, \qquad I_G = \frac{\mu - c_1}{2c_2}, \quad (5)$$

such that the following system of canonical equations results:

$$\dot{K} = \frac{\lambda - b_1}{2b_2} - \delta K, \quad (6)$$

$$\dot{G} = \frac{\mu - c_1}{2c_2} - \delta G, \quad (7)$$

$$\dot{\lambda} = \lambda(r+\delta) - a\frac{G}{1+G}, \quad (8)$$

$$\dot{\mu} = \mu(r+\delta) - a\frac{K}{(1+G)^2}. \quad (9)$$

The steady states of the system (6)–(9) can be found by solving $\dot{\lambda} = 0, \dot{\mu} = 0, \dot{K} = 0, \dot{G} = 0$, which can be reduced into a single equation,

$$(r+\delta)(1+G)\left[b_1 + \delta(c_1 + \delta G)(1+G)^2\frac{r+\delta}{a}\right] = aG. \quad (10)$$

This is a polynomial of the fourth order, that has four roots. We are interested only in real and positive roots, $G \in (0, \infty)$. In addition, we have the corner solution,

$$K = 0, G = 0, I_K = 0, I_G = 0, \quad (11)$$

as another steady state of the system.

The possibility of multiple long run outcomes separated by a threshold is a consequence from the increasing returns in the joint inputs (K, G), which capture inter alia the aspects addressed in Rosen (1981). Hence, given low (initial) values and/or little talent it does not payoff to invest until the increasing returns can be enjoyed.

A threshold (curve) in the state space, if existing, separates interior and positive outcomes from the corner solution $(K \to 0, G \to 0)$ such that an individual's choice depends not only on the payoffs but also on the initial conditions. An unstable steady state is necessary (but not sufficient) for the existence of different long run outcomes. Such an unstable steady state is characterized by a single negative eigenvalue (the others being positive or have positive real parts) of the Jacobian (J), which is equivalent to a negative determinant, $\det J < 0$; for details see Appendix.

3 Numerical Examples and Equilibria Types

3.1 The Case of Unique Equilibrium

Given the nonlinear two states optimal control problem (thus a four-dimensional phase space of states and co-states) one has to recourse to numerical methods. The following examples focus on multiple equilibria. Indeed, the standard case of a unique equilibrium is of little interest, because it is then (generically) the trivial equilibrium at the origin of the state space (although it is the outcome for the vast majority of the population), e.g., for $a = 0.5, b_1 = c_1 = 1, r = 0.2, \delta = 0.3$ there are no positive roots of (10), and $K = 0, G = 0$ is therefore the only equilibrium.

In the case $a = 1, b_1 = c_1 = 1, r = 0.05, \delta = 0.25$ three (non-negative) steady states exist with saddle paths converging either to an interior equilibrium $(K = 5.35, G = 2.35)$ or to the corner equilibrium, $(K = 0 = G)$. The second interior steady state $(K = 0.84, G = 0.57)$ is unstable $(\det J < 0)$. This raises the question for the optimal choice between the two feasible long run outcomes. However it is not optimal to go to the corner equilibrium in this example (Fig. 1). The reason is that the saddle point branch towards the interior steady state dominates its counterpart towards the origin globally, i.e., given high talent $(a = 1)$, it is optimal to pursue the career of an artist and to invest into skills (K) and market access (G) irrespective of initial conditions. This possibility of a unique outcome in spite of multiple steady states is unfortunately ignored in many of the related papers, which take the existence of an unstable steady state as sufficient for multiple long run outcomes; one exception is the thorough bifurcation analysis of the shallow lake model in Wagener (2003) another is Hartl et al. (2004). Figure 1 shows four trajectories converging to the stable interior steady state starting from different initial conditions.

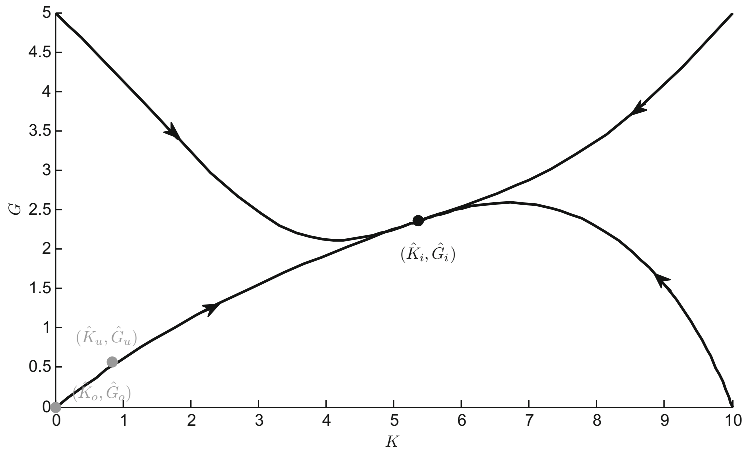

Fig. 1 Case 2 Phase portrait with unique equilibrium (in spite of three steady states): $a = 1, b_1 = c_1 = 1, r = 0.05, \delta = 0.25, b_2 = c_2 = 0.5$. Zero and unstable equilibria are not attractors

3.2 Skiba Threshold

In the case $a = 1, b_1 = c_1 = 1, r = 0.2, \delta = 0.3$ the set of steady states and their characteristics are as above, yet both stable steady states will be attained depending on initial conditions, and the basins of attraction are separated by a negatively sloped threshold in the state space, $G = G_s(K)$, see Fig. 2. This is clearly the most interesting outcome. The threshold line in the $K - G$ space is often called the Skiba curve but in Grass et al. (2008) DNSS-curve in reference to Sethi (1977), Skiba (1978) and Dechert and Nishimura (1983). If we start above this threshold, e.g., $G_0 > G_s(K)$, then it is optimal to converge to the higher of the interior steady states, $K = 1.81,\ G = 1.094$. For $G < G_s(K)$, it is optimal to converge to the corner equilibrium $K = 0,\ G = 0$. Therefore, given only modest access to market (a low G_0), a large initial talent (i.e., the value of $K(0)$ for a given parameter a) is needed to justify investments in human capital and market access to approach the higher equilibrium.

For $\delta = 0.19$ we have another Skiba point (Fig. 3), and likely to have it in some neighborhood of those parameter values. For this Skiba we also have negatively sloped threshold in $K - G$ space, separating the space of initial $(K(0), G(0))$ into two attractors—towards the origin and larger node. When an agents stays on the threshold, he is indifferent between two paths going to different equilibria. In this case the threshold is located further from the higher node.

Alternatively, good market access can substitute for weak talent, also ending up in the high equilibrium. There are many possibilities and corresponding real world examples for this: a publisher is a relative or friend of the family of writer, a potential author is famous due to successes in sport, business or film, or is a prominent politician, or a journalist with experience in a topical issue (e.g., the revolution

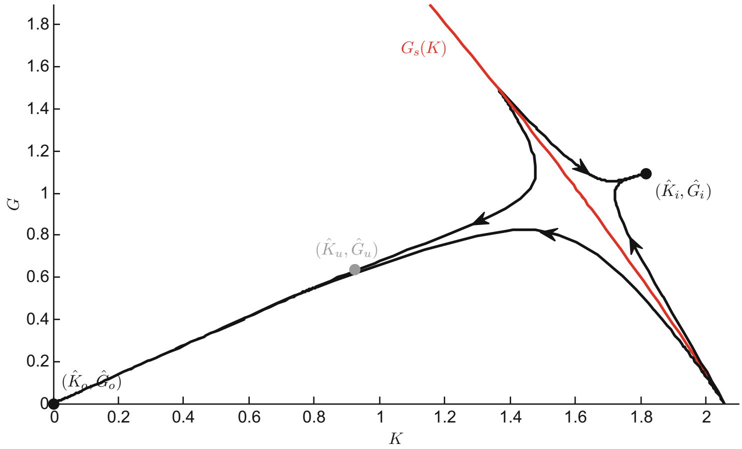

Fig. 2 Case 1—Threshold: $a = 1, b_1 = c_1 = 0.5, r = 0.2, \delta = 0.3, b_2 = c_2 = 0.5$. The threshold, which separates the basins of attractions of the origin $(\hat{K}_0, \hat{G}_0)$, or the interior, stable node $(\hat{K}_i = 1.82, \hat{G}_i = 1.094)$, has negative slope. The unstable node $(\hat{K}_u = 0.92, \hat{G}_u = 0.63)$ is depicted in gray and is off the threshold or Skiba curve $G_s(K)$. (This is the usual interpretation in the literature, namely, that this steady state is not part of the policy function. This is true if the steady state is a focus, but need not be true if it is a node (as in our case) even if the objective is non-concave, Hartl et al. (2004). If this point is on the policy function then the threshold is identical to the one-dimensional manifold in the state space that allows convergence to this state.)

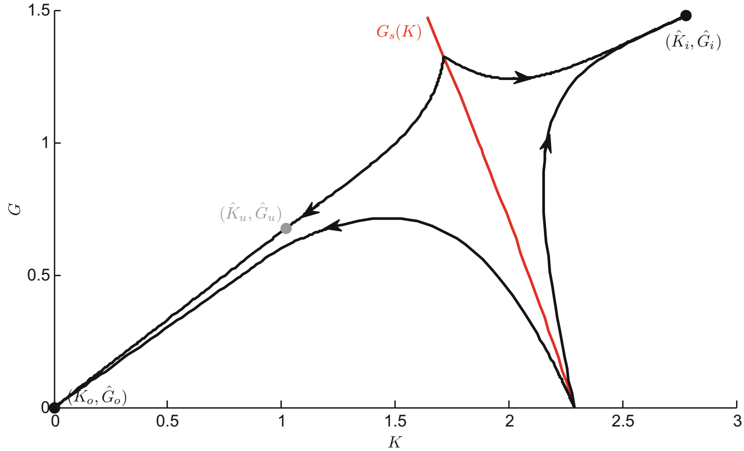

Fig. 3 Skiba curve. Case 2: $a = 0.5, b_1 = c_1 = 0.5, r = 0.1, \delta = 0.19, b_2 = c_2 = 0.5$. We have again negatively sloped threshold in $K - G$ space. The stable node has $\hat{G}_i = 1.47$, unstable node has $\hat{G}_u = 0.67$

in Arab countries). Thus, the set of successful artists consists not only of young geniuses but also includes people with good connections or luck at the start. At the same time, we observe many unrealized talents among those who converge to the corner equilibrium. Again different stories can be told. Those with low initial values of both stocks may probably never attempt seriously to become an artist. Those with higher levels of $K(0)$ may choose to "drink" their talent,[9] since their returns are too low factoring in the costs for market access if departing from too small $G(0)$. This can include talents with few connections, e.g. a talented pop musician, who lived in a remote country/area, say in Maoist Albania during the days of Enver Hoxha.

3.3 Bifurcation Analysis

Departing from a unique positive equilibrium, varying parameters allows to trace the implications on the number and properties of emerging or vanishing equilibria. Numerical experiments show that increasing the parameters r and δ has a negative effect on the interior and stable steady state, while higher talent a leads to higher steady state not only in K but also in G.

So far, we have bifurcation diagrams for a and δ. Figure 4 shows that for $a = 0.5, b_1 = c_1 = 0.5, r = 0.1$ we have two positive roots for G for $\delta < 0.198$,

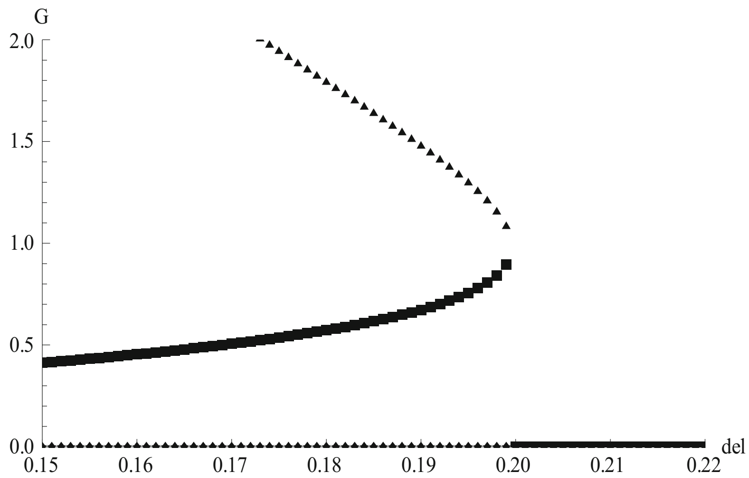

Fig. 4 Bifurcation diagram with respect to $\delta(=$ del) for $a = 0.5, b_1 = 0.5, c_1 = 0.5, r = 0.1$. After $\delta = 0.198$ all positive roots disappear (bifurcation point). Below this critical level, e.g. at $\delta = 0.19$, thresholds exist that separate the basins of attractions

[9]Here we mean that a potentially high talent may choose rationally not to invest in it but to enjoy life (drink instead).

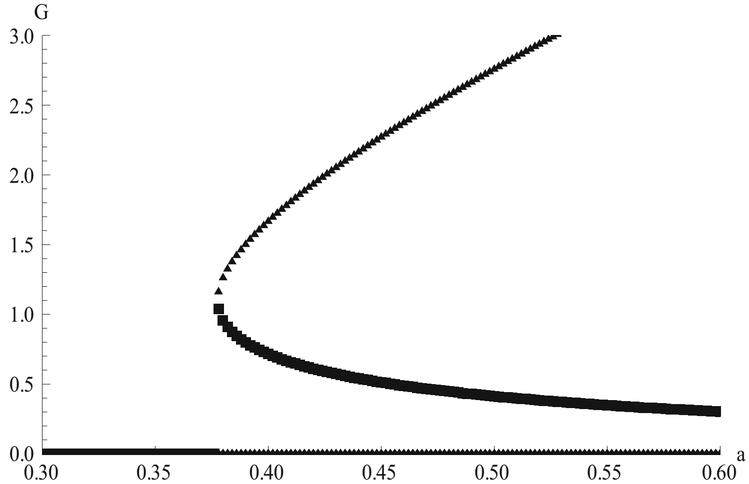

Fig. 5 Bifurcation diagram with respect to parameter a for $b_1 = 0.5, c_1 = 0.5, r = 0.1, \delta = 0.15$

and for larger depreciation only zero equilibrium remains. We can call $\delta = 0.198$ the threshold value of bifurcation parameter.

Concerning this bifurcation, three possibilities emerge even restricted to two interior steady states (without determining the threshold, if existing, at each parameter constellation):

(a) although two nodes emerge (close to threshold in the value of bifurcation parameter), the origin remains the unique equilibrium; for example, for $a = 0.5, b_1 = c_1 = 0.5, r = 0.13, \delta = 0.18$, we have two close nodes, $G = 0.93$ (unstable) and $G = 1.12$ (stable), but none of them is an attractor (all trajectories go to zero equilibrium);
(b) Skiba phenomenon for a subset of the cases with a threshold line in (G, K) space starting close to upper (stable) node and moving down;
(c) further increase of the bifurcation parameter leads to a disappearance of Skiba phenomenon: only the upper and stable as the global attractor.

Figure 5 shows the bifurcation diagram with respect to parameter a (talent). As it is natural to expect, low values of talent (below the threshold of about $a = 0.38$) produce only trivial equilibrium, while for larger values we get two nodes. Typically, the lower node is unstable (having one positive, one negative eigenvalue and two complex eigenvalues with positive real parts). The upper node is typically stable (has two positive and two negative eigenvalues).

The diagram for r is likely to be similar one for δ. This also can be seen from the similar role of r and δ in the expression for the Jacobian.

3.4 Policy Implications

The threshold phenomenon, which is characteristic for our model, has important policy implications. First, it describes some stylized facts from creative industries. Here the small differences in talents can produce not only much larger differences in payoffs, but can also induce creators to select very different career paths, either developing her or his talent or forgetting about it and choosing another activity (moving to zero equilibrium).

When we consider writers with different talents (a), human capital (K_0) and market access (G_0) within a normal society, their earnings depend on those parameters. However, environment also matters. If an economy is corrupt, this imposes a negative externality on at least a subset of artists. Higher bribes for market access cut off some artists from the market. In a dictatorship the attitude can be selective and manipulative. Those artists who are loyal to the regime, get easier market access, while those who oppose it have little or no access. As a result, the set of published authors is not equal to the set of most talented authors. A famous example is Solzhenitsyn from the former USSR, who was unable to publish more than one book at home, but was awarded a Nobel prize after moving to the West.

4 Additional Complexity of Art

This section addresses the complexity of market interactions in the economics of art. Our model was formulated initially for the case of writer, but it has a wider range of applications to other kinds of art and sciences that are briefly discussed below.

4.1 Market Interactions

The complexity of market structure in arts (especially audio-visual) is much higher than for normal goods. What characteristics are crucial for a market in creative industries? Firstly, it is a market offering a huge number of varieties. Secondly, monopolistic competition prevails between the contributors to all of these different varieties. Although free entry drives profits of the marginal entrant to zero, others can earn quasi-rents on their talent and these rents can be huge.[10] Thirdly, physical inputs are not crucial. Hence, it is reasonable to treat creators as small firms, producing their own variety (paintings, songs all with an individual style or touch), having some initially given access to market (direct or indirect), competing with similar creators for consumers and maximizing the net present value obtainable

[10]Damien Hirst sale breaks art auction record, raking in £70.5 million in its first day, Daily Telegraph, Tuesday 29 March 2011.

from their talent over time. Fourthly, preferences of consumers are subject to strong social influences (e.g. by a peer group or the entire society). It might be difficult for some consumers to defend a movie or book that only they like. These social effects often increase non-predictability of success, since a small random effect can either increase or decrease popularity.[11]

Fifthly, there is an additional layer of decision makers (intermediaries) such as producers and promoters. Producers (of a movie, for a example) select a group of talents. Their skills influence the product and in the end the payoff of each artist (return on individual talent). Promoters can sometimes influence the formation of public opinion about a particular art product, and thus they also influence the payoff of creators. Summarizing, the complexity of this market and the role of intermediaries in shaping this market is high. It is also difficult to track productivity of each worker in the chain of bringing art from creator to the public.

Network effects also influence traditional art, like writing. An even more pronounced justification for risk aversion of publishers is possibly due to network effects. Suppose that the developed talents $A \equiv aK$ of all writers visiting a publisher are uniformly distributed over the interval $[0, 1]$. The demand is determined by a distribution how readers value works of a particular author, which is (strongly) influenced by "word-of-mouth" and thus by network effects. The key assumption of economies with social networks (compare Shy 2011) is that individual preferences are socially influenced (e.g., Glaeser and Scheinkman 2003). It is then possible to show (see the Appendix) that social influence on individual preferences leads to excessive skewness in distribution of returns to talents. On the one hand, this can make editors even more risk averse, on the other hand, authors above a critical threshold benefit from higher sales due to this network externality.

4.2 Market Access

Of the many possibilities to access a market, direct public performance is the simplest one. Consider a musician standing in some public place and collecting money for his or her performance. However, creator may get a higher payoff to the talent if some publisher will put this music on CD and sell it through a variety of channels. In general, there exists a spectrum of possibilities for creators to access the market, both directly and indirectly. Recently, the costs for access have been reduced considerably, e.g., music (producing a CD or an online song is very cheap), small videos can be displayed in YouTube with the potential of reaching wide audiences, which may have far fetching consequences on people pursuing a corresponding career.

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[11] Santa Fe Institute (US) communicates an example of two restaurants, with similar quality. The choice of the first person becomes crucial, since the others will follow him. In certain cases even the worse restaurant can win popularity.

Easier market access does not mean higher payoff, because more talents have to compete in a pool of internet. Those having higher skills to advertise their work on internet will be more successful. However, creation of a good personal webpage to make additional publicity again requires a skill complementary to what we consider here a talent.

There exist different technical possibilities to create a piece of art. As an extreme example of direct marketing consider a painter in a tourist location, where he or she paints (landscapes or portraits) and sells directly. In this case of direct access to market, there are no intermediaries, and the creator receives full value for this art product.[12] Direct access to market (if we understand it as standing on a market square are selling one's art) may not work for talents above some threshold, because their potential buyers visit only auctions. And in the last case intermediary is required.

4.3 Economics of Theater

Now let us consider a theater performance, where several actors work in a team together with people of other professions (designers, managers, electricians, etc.) to create a joint product (performance) and to sell it to the public. Moore (1968) analyzes production and operating costs on Broadway in 1960–1961 (presented also in Heilbrun and Gray 1993). Pre-opening production costs (a kind of fixed cost per play) are dominated by cost of scenery and designers ($45,135 out of total $111,422). On the other hand, weekly operating costs are $27,309 (hence, performances in a month are equal to fixed costs, and a play running for 1 year, has only small fraction of fixed costs in its total costs), and its major components are theater rent ($7639, or 29 %) and salary of actors ($7297, or 28 %). The royalty of author(s) is also important: $2334, or 8 %. Thus, if we consider only artists and authors as creators, their payoff is only about 36 %. This can give us a proxy for estimation of creator's share, $s(G) = G/(1 + G)$.

In the case of theater (or other products of art that include substantial material costs) the share complementary to that of the creator, although substantial, goes mostly to cover the production cost. If intermediaries (agents, producers, financiers) are involved, they may accrue a substantial share of total profits.

[12] In fact, we have to add material costs, but usually they represent a small fraction in the price of the final product, especially for art of high value.

4.4 Applications and Extensions

When we consider an access of talent to the market in general framework, we have to take into account different forms of interactions. In the case of old-type art (like writing books) this interaction comes via an intermediary (publisher, for example). In this case the crucial aspect for us is in the interaction between writer and publisher, and G corresponds to the bargaining power of our writer, who shares the profit with publisher. The more a writer is famous, the greater is the competition between publishers, and the higher this share will be. However, it may be naive to expect that fame will come in one day, if the writer invests only in his talent, but not in his bargaining power. Only in some cases is bargaining share a direct function of talent, $B = B(K)$, but we will not consider those cases here.

In the case of modern art (that includes access to TV, publishing discs, giving concerts) we have the case of a network industry. Contrary to classical art (where public opinion on a book was formed as a sum of impressions of different readers not interacting between themselves), public opinion in modern art formed via a complex network (see Potts et al. 2008). It has complex topology with an asymmetry in different nodes (valuators of art) and the possibility to transmit information between nodes. Finally, the valuation of an artist comes not as a sum of independent opinions, but as a complex product of interactions in this network. The size of this network depends partially on advertisement (like TV broadcasting), but there might be a "killer", when authors' public reputation can first rise, but then is lost after a scandal article about him in press. The complexity of this network that forms the values in modern art is little known even to producers and there are few (if any) mathematical models of it. But one thing is clear: a modern talent who wants success should invest not only in the development of own talent, but also in this network. On a practical level, this may include putting videoclips on internet (YouTube, for example), having a personal page on Facebook and similar activities. Given the complexity of this network, we never know exact effect of this, but we assume that the artist's share is again a positive function of this complementary capital G, now in the sense of creator's human capital, associated with network access and reputation there. Further we proceed with a formal model of optimal accumulation of both types of capital.

For both cases of classical and modern art a creator can engage in activities that lead to an increase of her or his share at the expense of intermediary agents; e.g., famous authors get a much higher slice from book sales and often retain the copyright. This is formally done by investing in G that is per se unrelated to talent. An example may be as follows. A talented and already popular (high aK and G) pop-star asks, let say, \$1 million for her concert and will get it, since there is competition among producers for her. The winning producer will receive (almost) zero profit. Another example is about an equally talented, but not yet popular star (low G) who has to accept much lower payoff and moreover a lower share that allows the producer to earn profit (but at a considerable risk given the lack of popularity).

The model can be also applied to science. Although scientific productivity depends not only on the years of schooling, but also includes heterogeneity in talents, this case is different. Indeed, a scientist might need some popularity to get promotion but typically the salary is not defined by the pubic vote but by few experts in the field (who decide about job and grant applications). Nowadays, a researcher can go to special site, like Research Gate, and put there his or her works. It makes access to (scientific) public easier. However, the obtained scores have no direct effect on the promotion of scientists. Summarizing, we see that market access plays lower role in science compared to arts.

In sports we also have huge heterogeneity of abilities. Even an individual might not know if he or she has better skills for soccer of skiing. Like in science, market access is easier, especially for sports that do not require huge investment in equipment. Payoff depends on the results in competition and thus evaluation is more objective than in arts. Still, the common feature is the low observability of the talent parameter, a, both by coaches and sportsmen himself. The measurement of talent requires some minimal initial investment in training, and this is a costly procedure. Many potential champions can be cut off at the initial stage. That is why a country with more efficient screening and cheap access for initial training can raise more Olympic champions.

The model can be extended in various directions. In particular, one can allow for a more general specifications of the payoff, e.g. $\pi(K, G)$ and also of the dynamic relations. This in turn allows for alternative extensions and outcomes. Concerning the more general payoff, the amount of already proven performance (i.e. of K) can raise the share that the artist can ask for and this is actually the case with famous authors who are often able to retain their copyright. The lack of joint concavity of the objective combined with nonlinear dynamics may lead to cycles. For example, let G measure fame so that its dynamic follows a growth pattern, e.g., a logistic one. This can lead to cycles, exploiting one's own fame and then rebuilding it. This includes possibility of an unstable cycle rendering a cycle as the threshold curve in the (K, G)-plane. Another extension is to move from a single decision maker to the associated market equilibrium either of the competitive type or of monopolistic competition. Last but not least, one may test empirically the implication of this paper that the pursue of an artistic career depends not only on talent and education but also on market accessibility.

5 Conclusions and Policy Implications

It is observed that payoff to talent in art is quite heterogeneous. Individuals have quite heterogeneous abilities, but this difference cannot explain the phenomenon of higher payoffs for less talented individuals that happen sometimes in our society. It is important to understand the market structure in arts. In classical markets (like one for books) there is an intermediary between an author and readers; market access depends on intermediary, and asymmetric information about the talent and market power of a publisher can reduce the author's payoff. In the market for modern arts (pop music, TV shows, etc.) the crucial role belongs to complex network structure emerging from non-self-regarding preferences of the audience. The success in this market depends not so much on a talent per se but on the activities of producers and promoters. In both cases a talent willing to get higher share of payoff for his output has to invest in the skills complementary to his talent.

We have proposed a stylized model in order to explain the reinforcing or deterring effect of market access on the individual investment decisions and the payoffs of talents. Talent per se is an important but not the only relevant ingredient for success in the market. Apart from initial differences in talents, creators differ also with respect to initial access to market (or share of payoff to the talent, that they can get). Given very favorable parameters, in particular, sufficiently high talent and low discount and depreciation rates then it is optimal to start a career and to converge to a unique steady state irrespective of initial conditions. However, a creator should invest not only in his or her human capital but also into market access. In the case of less ideal circumstances (little talent, high discount rate, high investment costs) the corner solution of unrealized talent is the optimal long run outcome.

Most interesting are intermediate cases where the above two sketched outcomes depend on initial conditions. More precisely a threshold line in $K - G$ space characterizes the locus where an agent is indifferent between realizing his talent (going to the high interior steady state) or not (going to the origin of the state space). Along this threshold lower talent can be compensated by higher market access, $dG/dK < 0$. This explains asymmetry of talents, of not only with respect to returns but also how many will pursue a career depending on an individual's market accessibility.

Typically, for low initial talent a we have no positive roots, i.e. only non-realization of the talent is optimal. At a certain critical level of a an interior equilibrium emerges, that splits immediately into two. In a certain region of parameter a we get the Skiba phenomenon (multiplicity), while at even higher a only the high equilibrium survives, and it grows non-linearly with the respect to a. Mathematically similar bifurcations take place for the depreciation parameter δ (but reverse compared to a).

This model can be applied not only to arts, but to related fields that depend on talent, like sports and possibly science. Moreover and as sketched, our results could be transferred to other fields, like marketing, or even to markets for physical goods and services with costly access to market, e.g., export markets.

Appendix

Vector Field Structure and Jacobian

The vector field $\vec{F}(\vec{y})$ associated with the dynamic system (2) is smooth and regular in all the points except for the discrete set of critical points, where $\vec{F} = \vec{0}$, for which we get the polynomial equation. Local behavior in these critical points (also named as nodes, or steady states) is determined by the set of eigenvalues. If there are two negative and two positive eigenvalues, the node is "stable" in the sense that all flows within the associated stable 2-dimensional manifold converge to this steady state; similar stability properties apply for a pair of complex eigenvalues with negative real parts except that the steady state is a focus and attained by transient oscillations. The crucial case for a threshold is that where only one eigenvalue is negative.

The dynamic system (6)–(9) has the following matrix of partial derivatives J:

$$J = \begin{pmatrix} -\delta & 0 & 1/2b_2 & 0 \\ 0 & -\delta & 0 & 1/2c_2 \\ 0 & -a/(1+G)^2 & r+\delta & 0 \\ -a/(1+G)^2 & 2aK/(1+G)^3 & 0 & r+\delta \end{pmatrix}. \tag{12}$$

The determinant of the Jacobian (J) is then given by:

$$\det J = (r+\delta)^2\delta^2 + \frac{a(r+\delta)K}{c_2(1+G)^3} - \frac{a^2}{4b_2c_2(1+G)^4}. \tag{13}$$

The theorem of Dockner (1985) says that $\det J < 0$ is equivalent to one negative eigenvalue of J, such that stability is restricted to a one-dimensional manifold in the state space (and thus of zero probability). Since the sum of principal minors of dimension 2 in (12) is negative (more precisely, $-2(r+\delta)\delta$), $\det J > 0$ implies saddlepoint stability, i.e., two eigenvalues are either negative or have negative real parts. Therefore, a higher b_2 and c_2 reduce the negative term in $\det J$ and thus the chances for an unstable steady state.

Lemma 1 *The trivial equilibrium $K = 0, G = 0$ is stable if $a^2 < 2b_2c_2\delta^2(r+\delta)^2$. Thus, for sufficiently low talent and sufficiently high values of discount and depreciation there is convergence to zero equilibrium for at least a subset of initial values $(K(0), G(0))$.*

The proof is trivial, since $K = 0$ implies zero in the second term of expression (3). In the opposite case of inequality this equilibrium is unstable,[13] and thus sufficiently high talent gives an escape from such an outcome.

[13] Although we have to consider the boundary problem separately.

The condition for an unstable steady state ($\det J < 0$) is:

$$a^2 > (r + \delta)^2\delta^2(1 + G)^4 + 2a\delta(r + \delta)K(1 + G). \qquad (14)$$

Since $G > 0, K > 0$ at an interior steady state, a sufficiently high talent is necessary to unsettle the origin as the only equilibrium. Two positive steady states, one stable the other unstable, is then typical.

Network Effect on Market: Social Influence Versus Normal Preferences

We consider the case of heterogeneous consumers, who define the market for each author, that can be also of many types, A. We want to show that market structure (with or without network influence on preferences) has a pronounced effect on the return to talents. In particular, network effect raises the threshold for market entry, but also makes returns for more talented authors higher.

For the sake of mathematical transparency, assume a continuum of individuals (consumers) of measure one, uniformly distributed over interval [0, 1] and indexed by i. We start from the model of "normal market", with individual valuation of a book without social network effect given by $V_i = Ai$. Let the price of a book, p, be given exogenously. Each individual of type i decides whether to buy this book or not, comparing his valuation with the price, i.e. one book is purchased by those who have $Ai > p$. It is easy to find that the threshold agent is characterized by $i^* = p/A$, and that the measure of such agents is $1 - i^*$. Hence, in the normal case the demand is given by the formula $d1 = 1 - p/A$. Normalizing price to one, we get the dependence of market size on talent. For $A < p$, nobody will buy a book from this author, while for $A > p$ the demand grows as $d1 = 1 - 1/A$, approaching in the limit $A \to \infty$ the maximal demand, equal to one in our case.

There exist many possibilities to model social influence. We assume that social influence transforms the individual preferences into $W_i = V_i f(M)$. Let us consider mathematically simple and transparent case $W_i = 2MV_i$, where M denotes the measure of consumers who like this book. Again, there is a uniform distribution of individual valuations $V_i = Ai$. Since the total measure of consumers is one, the average measure is 1/2 (when only half like), and factor 2 is taken for comparative transparency: when half of agents like an author, the social effect does not change the valuation of the mean individual. When we account for social influence and look for a threshold consumer, $\bar{\imath}$, we need to equate his valuation, $W(\bar{\imath})$ with price. This leads to a quadratic equation $2A\bar{\imath}(1 - \bar{\imath}) = p$ with two solutions: $\bar{\imath}_{1,2} = 1/2(1 \pm \sqrt{1 - 2p/A})$. Note that the function W has its maximum for a medium consumer: as soon as he buys, a cascade of purchases will occur, and a finite measure will buy. Now the threshold becomes different: for $A < 2p$ nobody will buy a book, and the demand is thus zero. The we have a discontinuity: demand

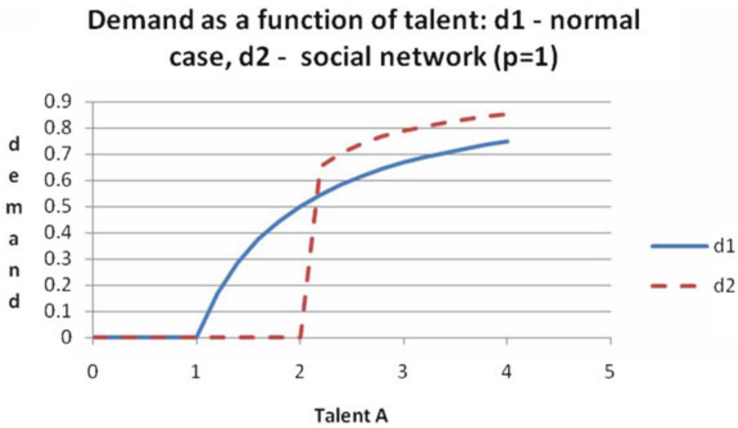

Fig. 6 Demand for creative work as a function of talent for unit price. Normal case corresponds to uniform distribution of valuations by consumers. In the case of social network the threshold grows, but later demand rises sharply

jumps immediately to 1/2, and after increases slower: $d2 = 1/2(1 + \sqrt{1 - 2p/A})$. Figure 6 presents both curves for comparison.

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Skiba Phenomena in Markov Perfect Equilibria of Asymmetric Differential Games

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Abstract This paper examines the existence of Markov-Perfect-Equilibria that give rise to coexisting locally stable steady states in asymmetric differential games. The strategic interactions between an incumbent in a market and a potential competitor, which tries to enter the market through product innovation, are considered. Whereas the potential entrant invests in the build-up of a knowledge stock, which is essential for product innovation, the incumbent tries to reduce this stock through interference activities. It is shown that in the presence of upper bounds on investment activities of both firms a Markov-Perfect-Equilibrium exists under which, depending on the initial conditions, the knowledge stock converges either to a positive steady state, thereby inducing an entry probability of one, or to a steady state with zero knowledge of the potential entrant. In the later case the entry probability is close to zero. It is shown that this Markov-Perfect-Equilibrium is characterized by a discontinuous value function for the incumbent and it is discussed that this feature is closely related to the existence of upper bounds on the investments of the players. Removing these constraints in general jeopardizes the existence of a Markov-Perfect-Equilibrium with multiple locally stable steady states.

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1 Introduction

Professor Richard F. Hartl is not a fan of differential games. The reason is that he dislikes the multiplicity of equilibria. For instance, there are quite a number of papers deriving Markov-Perfect-Equilibria where the controls linearly depend on the states. It makes him wonder how many more equilibria would there be if the assumption of linear strategies would be relaxed. In particular, how can coordination between the players about the use of linear strategies be accomplished ("shall we play the game admitting linear strategies, dear competitor?")?

Although the authors of this contribution and Richard are very good friends for decades, still we decided to publish a differential games paper in the book written in Richard's honor. We have two good reasons to believe that this action will not do any harm to our friendly relationship with Richard. The first is that the Markov-Perfect-Equilibrium we derive is not based on any ex-ante assumptions about the functional form of equilibrium strategies. Although uniqueness of these equilibria cannot be guaranteed, at least they seem to have focal point properties since they are robustly obtained by applying standard numerical dynamic programing methods. Second, the main contribution of this work is that this is a differential game the Markov-Perfect-Equilibrium of which exhibits a Skiba point, where the latter is a topic that appears in more than twenty publications (co-)authored by Richard F. Hartl. Moreover, his masterpiece written together with Feichtinger and Hartl (1986) shows a Skiba threshold on the cover.

Consider first an optimal control model exhibiting Skiba behavior. A Skiba point is located at the boundary between the basins of attraction of two (locally) stable fixed points under the equilibrium dynamics. At this point it holds that the decision maker is indifferent between converging to either of the equilibria. This can, for instance, be a high capital stock equilibrium where the firm has scale enough for abatement efforts to be efficient, and a low capital stock equilibrium with no abatement efforts (see, e.g., Hartl and Kort 1996). The firm has to invest more to reach the high capital stock equilibrium, which results in a discontinuity of the policy function at the Skiba point, k^S : in this sense we write the investments, I, as a function of capital stock, k, and find that the investment cost function $I(k)$ has a jump at k^S. Immediately to the right of k^S the firm invests considerably more than to the left of k^S, which is needed in order to converge to the equilibrium with the large capital stock. Note that the discontinuity of the policy function $I(k)$ at k^S goes along with a continuous value function because by definition right at the Skiba point k^S the values of converging to the low and the high capital stock equilibrium are equal.

Later on, Wirl and Feichtinger (2005) have shown that existence of a Skiba point also admits a continuous policy function. In particular this requires the unstable steady state to be a node. The ideas of this study are applied to capital accumulation models in Hartl et al. (2004) and Hartl and Kort (2004), where the main contribution of these works is that a complete representation is given with regard to what kind of unstable steady state occurs for which parameter values. Concerning the unstable steady state a distinction is made between a focus, a continuous node, or a discontinuous node.

Now consider a differential game with two firms, say firm 1 and 2, where both firms have a capital stock as state variable. Both firms can invest to increase their capital stock. The capital stock is used to produce goods that are sold on the market. Assume the market is homogenous, i.e. the firms produce identical goods and the firms compete in quantities (Cournot). In such a case a firm likes its competitor to invest less because this means this firm attracts a low capital stock and this in turn implies that the competitor's output is low so that the output price will be high. Exactly this property that the opponent profits from a decrease of investment of firm 1 gives rise to conceptual problems if we want to consider a Skiba scenario in an (asymmetric) differential game.

To illustrate this point let us consider a situation in which a Skiba curve exists for firm 1 (given the strategy of firm 2). Since investments of firm 2 reduce investments incentives of firm 1 this curve is upwards sloping in the (k_1, k_2)-plane. We denote firm 1's Skiba point (for a given value of k_2) by k_1^S (subscript 1 points to firm 1), where it holds that for $k_1(0) > k_1^S$ firm 1 converges to a high long run capital stock equilibrium, while for $k_1(0) < k_1^S$ firm 1 chooses the trajectory ending at a low long run capital stock equilibrium. Since firm 1 produces more when capital stock is higher, firm 2 prefers that firm 1 chooses to converge to the low capital stock equilibrium. This is reflected by firm 2's value function jumping down at the moment k_1 reaches k_1^S from below. It follows that firm 2 wants to avoid that firm 1 chooses the path ending at the high long run capital stock equilibrium, which can be accomplished by overinvesting, since an increase of k_2 induces an increase of k_1^S. The result is that under optimal investment behavior of the opponent firm 1 also for $k_1(0) > k_1^S$, chooses the trajectory ending up at the low long run capital stock equilibrium implying that k_1^S is not a Skiba point.

This overinvesting by firm 2 with the aim to push firm 1 into the basin of attraction of the low long run capital stock equilibrium, is not possible if firm 2's investment is sufficiently restricted from above. Then, as a result, there can be an equilibrium where firm 1 has a Skiba point. This is essentially what we show in this paper: existence of a Markov-Perfect Equilibrium in an asymmetric differential game, where due to the fact that investment of the other firm is sufficiently bounded above, a Skiba point exists separating two locally stable steady states. In order to make the argument as transparent as possible and to restrict the number of complex calculations, we illustrate this idea by employing a differential game model with only one state variable. In this model firm 1 is the incumbent and firm 2 the potential entrant.[1] Firm 2 needs a high knowledge stock to increase the probability of entry.

[1]There is an abundant literature on dynamic entry deterrence. One stream of contributions based on limit pricing on the product market has been pioneered by Gaskins (1971). The model consists of a low-cost dominant firm and fringe firms holding a share of the market that evolves continuously at a rate proportional to expected profits. It is an optimal control problem where the incumbent firm sets the pricing strategy. Judd and Petersen (1986) give a game-theoretic extension of this work by allowing fringe firms to react by controlling the retention rate of their earnings. Our paper is

Firm 2 can invest to increase this knowledge stock, while firm 1 can invest in interference activities to reduce this stock of knowledge. A convex form of the hazard rate makes that in principle a Skiba point related to the knowledge level exists above which firm 2 keeps on investing to enter, but below which firm 2's activities will eventually die out. We formally show that without an upperbound on investment firm 1 will jeopardize this Skiba behavior while having the upperbound keeps the Skiba point valid.

In spite of the substantial literature on the existence and properties of Skiba points in dynamic optimization problems, such threshold phenomena have so far been hardly addressed in the framework of Markov-Perfect-Equilibria (MPE) of differential games. Dockner and Wagener (2014) present an MPE exhibiting a Skiba point in a symmetric differential game, but to our knowledge this contribution is the first to present and discuss an MPE giving rise to multiple locally stable steady states in the framework of an asymmetric differential game.

The paper is organized as follows. Section 2 presents the model, whereas Sect. 3 contains the equilibrium analysis. The paper ends with conclusions and a general discussion.

2 The Model

We consider a dynamic incumbent entrant problem, where firm 1 is the incumbent in a market and firm 2 tries to enter the market through product innovation. Entry time of firm 2 is stochastic and the hazard rate is given by

$$h(k) = \alpha k^2,$$

where k is the knowledge stock of firm 2. Knowledge evolves over time based on *R&D* investments of firm 2, denoted by $I_2(t)$ as well as interference activities of firm 1, denoted by $I_1(t)$. Examples for interference activities are the strategic spreading of misleading information or the headhunting of key employees of the competitor. Additionally, we assume that due to technological progress there is some obsolescence effect of existing knowledge. Formally, we have

$$\dot{k} = I_2 - \lambda I_1 - \delta k, \tag{1}$$

where $\lambda > 0$ measures the efficiency of firm 1's interference activities and δ is the technical obsolescence rate. The activities of both firms are associated with costs of

closer to another stream of contributions that features strategic investment as a deterrent. One of the best known works in that stream is Fudenberg and Tirole (1983). Building on the static model of Spence (1979), it investigates the investment behavior of two firms in the steady-state post-entry game.

the form

$$c_i(I_i) = \beta_i I_i + \frac{\gamma_i}{2} I_i^2, \quad i = 1, 2$$

with $\beta_i, \gamma_i \geq 0$ and we assume that investments of both firms are non-negative and bounded above by $\bar{I}$. Furthermore, the knowledge stock has to be non-negative and due to (1) it is bounded above by $\bar{k} = \frac{\bar{I}}{\delta}$. The instantaneous profit of the incumbent is π_1^m as long as it is monopolist on the market, whereas after the entry of firm 2 the profits of the two firms read $0 < \pi_1^d < \pi_1^m$ and $0 < \pi_2^d$. Both firms choose their activities in order to maximize expected discounted payoffs.

Formally, the game is interpreted as a multi-mode game with the two modes: m_1 prior to entry of firm 2, and m_2 after the entry. Using this notation we write the firm profits as functions of the mode:

$$\pi_1(m_1) = \pi_1^m,\ \pi_1(m_2) = \pi_1^d,\ \pi_2(m_1) = 0, \pi_2(m_2) = \pi_2^d.$$

The objective function of the players are given by

$$J_i = \max_{I_i} \mathbb{E}\left[\int_0^\infty e^{-rt}\left(\pi_i(m(t)) - c_i(I_i(t))\right) dt\right]$$

subject to the state dynamics (1) and the mode process

$$\lim_{\Delta \to 0} \frac{1}{\Delta} \mathbb{P}\left[m(t+\Delta) = m_2 \mid m(t) = m_1\right] = h(k).$$

Initial conditions are $m(0) = m_1$ and $k(0) = k^{ini}$.

3 Equilibrium Analysis

In what follows we consider stationary Markov Perfect Equilibria (MPEs) of the game described in the previous section. Hence, we assume that investment by the two firms is given by Markovian feedback strategies of the form $\phi_i(k)$. These strategies are only relevant in mode m_1 of the game, but since no further actions are taken in mode m_2, we abstain from explicitly including the mode as an argument of the strategies. Furthermore, we assume that the equilibrium strategies are almost everywhere continuous on $[0, \bar{k}]$.

The value functions of the players in mode m_2 are independent of the state k and given by

$$V_i(m_2) = \frac{\pi_i^d}{r}.$$

In an MPE with strategy profile (ϕ_1, ϕ_2) the value functions in mode m_1 have to satisfy the Hamilton-Jacobi-Bellman (HJB) equations

$$rV_1(k;m_1) = \max_{I_1}\left[\pi_1^m - c_1(I_1) + \frac{\partial V_1(k;m_1)}{\partial k}(\phi_2(k) - \lambda I_1 - \delta k) \right.$$
$$\left. + h(k)(V_1(m_2) - V_1(k;m_1))\right] \tag{2}$$

$$rV_2(k;m_1) = \max_{I_2}\left[-c_2(I_2) + \frac{\partial V_2(k;m_1)}{\partial k}(I_2(k) - \lambda\phi_1(k) - \delta k) \right.$$
$$\left. + h(k)(V_2(m_2) - V_2(k;m_1))\right]. \tag{3}$$

It is easy to see that the maximization of the right hand side of these equations gives

$$\phi_i(k) = \begin{cases} \frac{1}{\gamma_1}\left(-\beta_1 - \lambda\frac{\partial V_1(k;m_1)}{\partial k}\right), \; i = 1 \\ \frac{1}{\gamma_2}\left(-\beta_2 + \frac{\partial V_2(k;m_1)}{\partial k}\right). \;\; i = 2 \end{cases} \tag{4}$$

Inserting these strategies into (2) and (3) results in a set of nonlinear ordinary differential equations for $V_1(.;m_1)$, $V_2(.;m_1)$, and in general no closed form solution is available. Therefore, we rely on numerical methods to determine the value functions and feedback strategies in equilibrium. In particular, we employ a homotopy method, where the value functions is approximated by a polynomial generated from a basis of Chebychev functions and the coefficients of the basis functions are determined such that the Hamilton-Jacobi Bellman equations hold on a set of Chebychev nodes (see Vedenov and Miranda 2001 or Dawid et al. 2015 for more details). Since the polynomial approximation of the value function is by construction continuous and smooth it can not directly capture potential discontinuities or kinks in the value functions, which might arise in scenarios where there are several coexisting locally stable fixed points with basins of attraction of positive size. In order to analyze such scenarios we calculate 'local value functions' around the stable steady states by numerically solving the HJB equations on different overlapping parts of the state space, where each of them contains only one steady state. As will be discussed below, a combination of these local value functions can then be used to characterize MPEs of the game which give rise to coexisting stable steady states. It should be pointed out that no ex-ante knowledge about the exact size and shape of the basins of attraction of the two stable steady-states is needed for the application of the described method. The two overlapping parts of the state space used to determine the local value functions just have to be chosen sufficiently large such that they contain the basin of attraction of the corresponding steady states. Also, the local value functions in this approach are determined as solutions of the Hamilton-Jacobi-Bellman equation on the entire considered part of the state space, rather than by local approximation around the steady state. In this sense, the proposed method is a global procedure. An alternative global procedure for determining basins of attraction in optimal control problems, relying on Nonlinear Model Predictive Control, was recently introduced in Grüne et al. (2015).

The value functions of the two players determined by our method are almost everywhere continuous and differentiable. They satisfy the HJB equations in all points of the state space where they have these properties, because in each basin of attraction value functions are continuous and differentiable. Skiba points are such that value functions attached to different basins of attraction have equal value for one player. There we can have kinks in this player's value function and/or a discontinuity in the value of the other player. attraction.

In the model considered here an increase of the knowledge stock in mode m_1 has two qualitatively different and counteracting effects on the payoffs of the two players. First, due to the convex shape of the hazard rate, the marginal effect of a change in the knowledge stock on the hazard rate is more pronounced if k is large. Second, a large k induces a large hazard rate and therefore the expected time before the jump to mode m_2 occurs is small. Since the size of k becomes irrelevant once mode m_2 is reached, a shorter expected stay in mode m_1 reduces the impact of an increase (decrease) of k on the expected future payoff stream of both players.[2] Since both players' incentives to invest are driven by the marginal impact of a change in k on their value functions, these considerations suggest that equilibrium steady states with high and low investments of both players might coexist. To illustrate this point and the numerical approach sketched above we calculate local value functions of the two players on the subsets $k \in [0, 0.45]$ and $k \in [0.25, 1]$ of the state space for an appropriate parametrization of our model.[3] We denote the two local value functions by $V_i^l(k)$ and V_i^h, $i = 1, 2$ respectively and Fig. 1 shows these two functions of each player. Figure 2 depicts the feedback functions resulting from inserting the local value functions into (4), which are denoted by ϕ_i^l and ϕ_i^h.

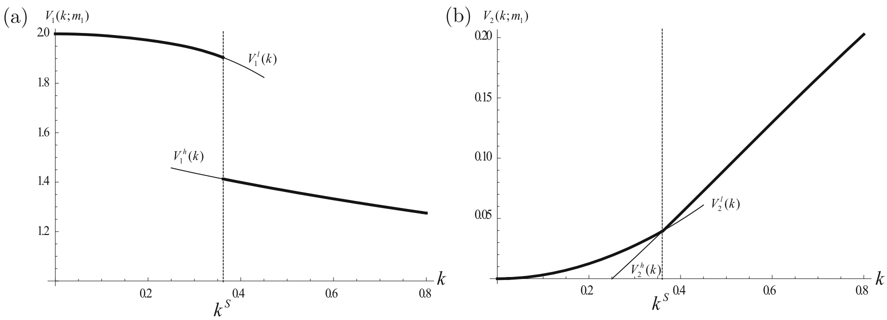

Fig. 1 Local value function of the incumbent (**a**) and the potential entrant (**b**) in mode m_1

[2]Doraszelski (2003) refers to the second effect as the 'pure knowledge effect'.

[3]The default parameter setting considered in the following analysis is: $r = 0.05, \alpha = 0.3, \beta_1 = 0, \beta_2 = 0.15, \gamma_1 = 1, \gamma_2 = 2, \delta = 0.2, \lambda = 0.1, \pi_1^m = 0.1, \pi_1^d = 0.05, \pi_2^d = 0.025, \bar{I} = 0.2$. The upper bound of the state space for this parametrization is given by $\bar{k} = 1$.

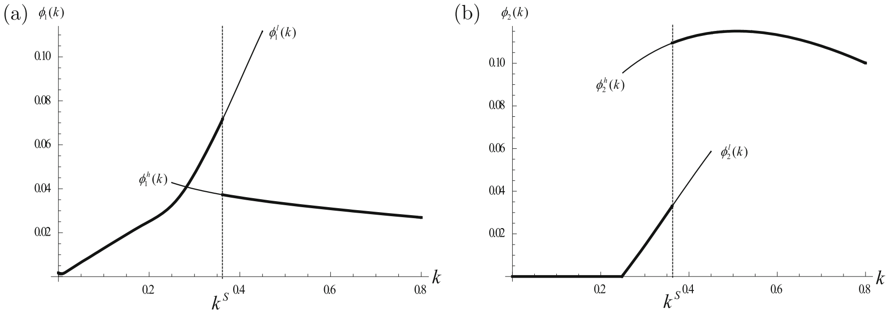

Fig. 2 Feedback function of the incumbent (**a**) and the potential entrant (**b**) based on the local value functions in mode m_1

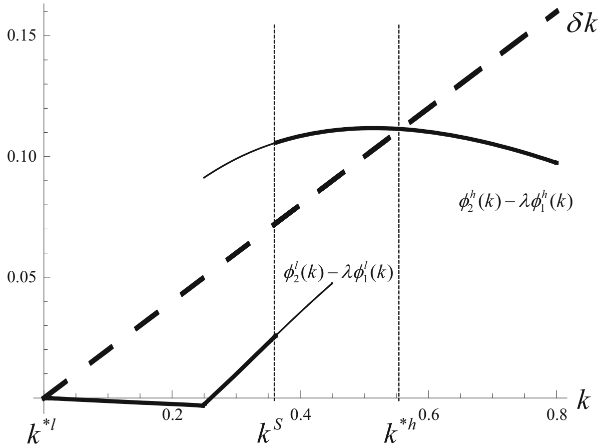

Fig. 3 Net investment and knowledge depreciation

Care should be taken in interpreting these local value functions. Consider for example the local value functions calculated on the upper segment of the state space, i.e. on [0.25, 1]. First, considering the induced investment functions depicted in Fig. 2, it is easy to see that the interval [0.25, 1] is invariant under the state dynamics resulting from these investment functions (see also Fig. 3 below). Taking into account that V_i^h and ϕ_i^h satisfy the HJB equations on [0.25, 1] this implies that it is optimal for player i to follow the feedback function $\phi_i^h(k)$ given that the other player sticks to ϕ_{-i}^h and considering only investment functions that keep the state in the considered interval [0.25, 1]. A priori it is however not clear that it is not optimal for player i to choose a strategy deviating from ϕ_i^h that would lead the state outside the interval [0.25, 1]. This aspect will turn out to be crucial for verifying that a considered profile is indeed an MPE of the game.

Figure 3 shows the net investment $\phi_2(k) - \lambda\phi_1(k)$ under the two local feedback profiles as well as knowledge depreciation. Keeping in mind that the dynamics of the knowledge stock is determined by the difference between net investment

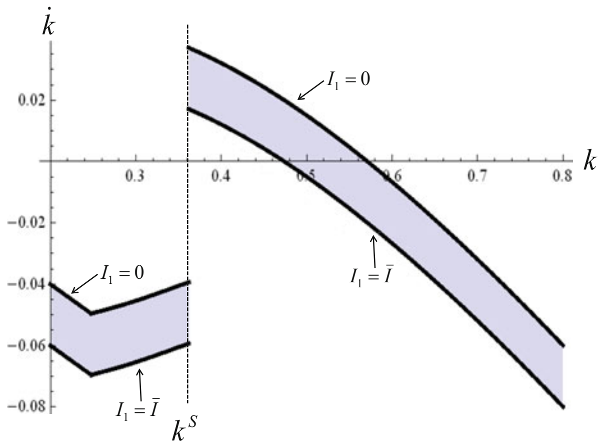

Fig. 4 Range of possible values of $\dot{k}$ for values of $I_1 \in [0, \bar{I}]$ and $I_2 = \phi_2^*(k)$

and depreciation, this figure first confirms that the two considered segments of the state-space are invariant under the dynamics induced by the local feedback functions and, second, shows that in each of the two segments there is a unique locally asymptotically stable steady state, namely $k^{*l} = 0$ in the lower segment and $k^{*h} = 0.556$ in the upper segment. Considering Fig. 1 it can be seen that the two local value functions of player 2 intersect, whereas for player 1 $V_1^l > V_1^h$ holds on the entire overlap of the two considered parts of the state space. This is quite intuitive, since under the investment functions (ϕ_1^l, ϕ_2^l) the knowledge stock and therefore also the hazard rate converges monotonically towards zero, which means that firm 1 will with a large probability stay a monopolist in the market, whereas under (ϕ_1^h, ϕ_2^h) the hazard rate converges towards the positive value $h(k^{*h})$ and therefore with probability one eventually firm 2 will enter the market. The discontinuity in the value function at k^S can thus be understood by noting that firm 1 prefers firm 2 not to enter. As shown in Fig. 4 later on, the boundedness of firm 1's investment prevents that for k slightly above k^S firm 1 can reduce firm 2's knowledge stock sufficiently such that the trajectory corresponding to the long run equilibrium $k = 0$ will prevail.

Firm 2 faces a more involved trade-off between moving towards the steady-state with positive respectively zero knowledge. If the current knowledge stock is small, then immediate returns from investment are small and the firm has to compare the accumulated costs of bringing the knowledge stock up to a level where a positive long run hazard rate is generated to the expected profits from entering the market. For very small levels of k this results in zero investment by the firm, whereas for an intermediate range positive investment is made, which is however not sufficient to outweigh the investments of the opponent and to induce an increasing knowledge stock. Considering now the intersection point $k^S = 0.36$ between V_2^l and V_2^h, firm 2 is indifferent between pushing the knowledge stock upwards to k^{*h}, thereby ensuring market entry, and letting the knowledge stock decrease towards $k^{*l} = 0$, which implies that with positive probability the firm will not enter the market. This indifference is conditional on the assumption that the opponent firm 1 switches at $k = k^S$ from $\phi_1^l(k)$ to $\phi^h(k)$. Considering Fig. 3 it becomes clear that under a strategy profile where both firms switch their strategies at $k = k^S$, the state converges to k^{*l} whenever initial knowledge is below k^S, whereas it converges to k^{*h} if $k^{ini} > k^S$. Hence, the threshold k^S is a Skiba point of this game, which separates the basins of attraction of the two locally stable fixed points. The following Proposition shows that in our setting a profile where both firms switch their feedback functions at k^S is indeed an MPE of the game.

Proposition 1 *Under the considered default parametrization of the model the strategy profile* $(\phi_1^*(k), \phi_2^*(k))$ *with*

$$\phi_i^*(k) = \begin{cases} \phi_i^l(k) & 0 \leq k \leq k^S, \\ \phi_i^h(k) & k^S < k \leq \bar{k} \end{cases}$$

for $i = 1, 2$ *constitutes a Markov-Perfect Equilibrium of the game. The corresponding value functions are given by*

$$V_i(k; m_1) = \begin{cases} V_i^l(k) & 0 \leq k \leq k^S, \\ V_i^h(k) & k^S < k \leq \bar{k}. \end{cases}$$

for $i = 1, 2$.

Proof Given that the profiles (ϕ_1^l, ϕ_2^l) and (ϕ_1^h, ϕ_2^h) satisfy the HJB equations on $k \in [0, k^S]$ and $k \in [k^S, \bar{k}]$ respectively, all we have to prove is that none of the firms can increase its discounted payoff stream by pushing the knowledge stock k across the Skiba point k^S. Consider first player 2 and assume that for some $\tilde{k}^{ini} > k^S$ there exists an investment path for firm 2, which, given that firm 1 uses $\phi_1^*(k)$, leads into the interval $[0, k^S)$ and generates a discounted payoff $\tilde{V}_2 > V_2^h(\tilde{k}^{ini})$. Denote by $\tilde{k}(t)$ the corresponding state dynamics (assuming the mode stays in m_1). Given that this is a problem with a one-dimensional state space it follows from Hartl (1987) that any optimal path has to be monotonous and therefore we restrict attention to such paths. Define $\tau = \min[t : \tilde{k}(t) \leq k^S]$ as the time the path enters $[0, k^S]$. Then, we have

$$\begin{aligned} \tilde{V}_2 &= \int_0^\tau e^{-rt}(-c_2(\phi_2^h(\tilde{k})) + h(\tilde{k})V_2(m_2))dt + e^{-r\tau}V_2^l(k^S) \\ &= \int_0^\tau e^{-rt}(-c_2(\phi_2^h(\tilde{k})) + h(\tilde{k})V_2(m_2))dt + e^{-r\tau}V_2^h(k^S). \end{aligned}$$

The second line implies that there must exist an investment path inducing a state trajectory in the interval $[k^S, \bar{k}]$ yielding an (expected) value of $\tilde{V}_2 > V_2^h(\tilde{k}^{ini})$ which is a contradiction to the fact that V_2^h is the value function of firm 2 for the game played with the state space $[k^S, \bar{k}]$. Hence, no profitable deviation from ϕ_2^h exists on $[k^S, \bar{k}]$. The same argument shows that no profitable deviation from ϕ_2^l exists on $[0, k^S]$.

Considering firm 1, the grey area in Fig. 4 shows the range of potential values of $\dot{k}$ that firm 1 can induce by choosing $I_1 \in [0, \bar{I}]$ if firm 2 determines its investment according to its equilibrium feedback strategy ϕ_2^*. It can be clearly seen that for values of $k = k^S$ and in the neighborhood above this threshold the whole range of possible investments for firm 1 leads to a positive value of $\dot{k}$. Put differently, the minimal investment needed to induce a decrease in knowledge is above the upper bound on investment $\bar{I}$. Hence, given the investment strategy ϕ_2^* of firm 2, it is not possible for firm 1 to push the knowledge stock k below the Skiba point k^S once it has exceeded that level. Hence, for $k \geq k^S$ the strategy ϕ_1^h is clearly optimal for firm 1. Assume now that for some $\tilde{k}^{ini} < k^S$ there exists an investment path for firm 1, which, given that firm 2 uses $\phi_2^*(k)$, leads into the interval $(k^S, \bar{k}]$ and generates a discounted payoff $\tilde{V}_1 > V_1^l(\tilde{k}^{ini})$. Denote again by $\tilde{k}(t)$ the corresponding state dynamics (assuming the mode stays in m_1). Define $\tau = \min[t : \tilde{k}(t) \geq k^S]$ as the time the path enters $[k^S, \bar{k}]$. Then, we have

$$\begin{aligned}
\tilde{V}_1 &= \int_0^\tau e^{-rt}(\pi_1^m - c_1(\phi_1^l(\tilde{k})) + h(\tilde{k})V_1(m_2))dt + e^{-r\tau}V_1^h(k^S) \\
&< \int_0^\tau e^{-rt}(\pi_1^m - c_1(\phi_1^l(\tilde{k})) + h(\tilde{k})V_1(m_2))dt + e^{-r\tau}V_1^l(k^S),
\end{aligned}$$

where the inequality follows from the fact $V_1^l(k^S) > V_1^h(k^S)$. Since an optimal path is monotonous (Hartl 1987), the second line implies that there must exist an investment path inducing a state trajectory in the interval $[0, k^S]$ yielding an (expected) value of $\tilde{V}_1 > V_1^l(\tilde{k}^{ini})$ which is contradiction to the fact that V_1^l is the value function of firm 1 for the game played with the state space $[0, k^S]$. Hence, no profitable deviation from ϕ_1^l exists on $[0, k^S]$ and we have shown that (ϕ_1^*, ϕ_2^*) is an equilibrium profile.

In Figs. 1 and 2 the value and investment functions under this equilibrium are indicated by bold lines. At first sight it might be surprising that the value function of firm 1 in equilibrium exhibits a downward jump as k crosses the Skiba point k^S from below. Intuitively, for a current knowledge stock slightly above k^S the incumbent could gain substantially by pushing the knowledge stock below k^S, since this would result in a sharp decrease of R&D activities of firm 2 and the probability of firm 2 eventually entering the market would jump downwards. However, close to k^S the R&D investments of firm 2 are so large that firm 1 cannot prevent an increase in the knowledge stock, even if it chooses its investment level at the upper bound. As becomes also clear from the proof of the proposition, a jump in the value function of one of the players can only occur in equilibrium if this player has no possibility to choose own controls such that the state moves from the lower branch of the value function to the upper one. Quite naturally this raises the question how an MPE of the game could look like if the upper bound on investment would be sufficiently

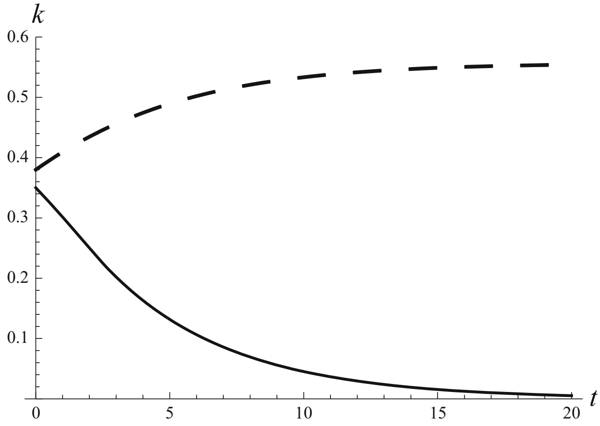

Fig. 5 Dynamics of the knowledge stock for $k^{ini} = 0.35$ (*solid line*) and $k^{ini} = 0.38$ (*dashed line*)

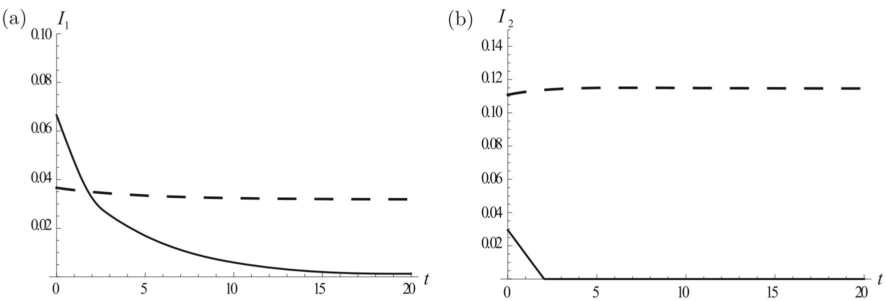

Fig. 6 Investment of the incumbent (**a**) and the potential entrant (**b**) for $k^{ini} = 0.35$ (*solid line*) and $k^{ini} = 0.38$ (*dashed line*)

large such that firm 1 could induce a decrease of the knowledge stock at k^S. We will come back to this issue in the next section.

To further examine the properties of the equilibrium characterized in Proposition 1, we depict the equilibrium dynamics of the knowledge stock and of investment by both firms for initial knowledge stocks slightly above and slightly below the Skiba point k^S in Figs. 5 and 6. The Skiba phenomenon can be clearly seen in that the knowledge stocks converge to the two different steady states depending on whether k^{ini} is above or below k^S. Concerning the investment paths it is interesting to observe that during the initial period firm 1 invests more in the scenario with a lower knowledge stock. At first sight this might seem at odds with the convex form of the hazard rate, which implies that the marginal effect of a reduction of knowledge stock on the hazard rate is smaller for a lower knowledge stock. However, the driving force of this behavior is a strategic consideration of firm 1. As can be seen in Fig. 2, the feedback function of firm 2 has a steep positive slope in the interval just below k^S. This means that by accelerating the decrease of the knowledge stock in this part of the state space, the incumbent can strongly decrease future R&D investments of the opponent, which provides additional investment incentives for firm 1.

At approximately $k = 0.25$ firm 2 stops any R&D investments and these additional strategic investment incentives for firm 1 vanish. In the depicted trajectories this value of k is reached at about $t = 2$ in the scenario with low initial knowledge stock and it can be seen in Fig. 6 that indeed from this time on the investments of firm 1 in the case scenario with low knowledge stock are always below that in the scenario with high knowledge stock. Intuitively, in case where initial knowledge is close to the Skiba threshold, but too low to yield a positive long run knowledge stock, the incumbent initially overinvests in order to make the opponent earlier accept that expected returns from R&D are below the costs and to abolish any R&D activities.

In this respect our result is related to the rich literature discussing strategic reduction of firm entry by incumbent firms. Overinvestment by the incumbent has been identified as a main instrument of entry deterrence in the seminal contributions by Spence (1977) and Dixit (1979, 1980). In these papers a one-time overinvestment is enough to prevent entry forever. However, our setting is dynamic and there is uncertainty. The implication is that the incumbent (firm 1) has to keep on overinvesting during a time interval with positive length. And still entry is not prevented for sure since the hazard rate is positive although declining, because also the knowledge stock is decreasing over time.

4 General Discussion and Conclusions

To our knowledge this paper is the first to present a Markov-Perfect-Equilibrium of an asymmetric differential game which gives rise to a Skiba phenomenon with two co-existing locally stable steady states. A particular feature of our example is that the value function of one player exhibits a jump at the Skiba point and, as discussed above, this feature can only occur in equilibrium because the combination of the structure of the state dynamics and control constraints make it impossible for this player to move the state from the lower branch of his value function to the upper branch. It is easy to see that the strategy profile considered here would no longer be an equilibrium profile if the control constraint would be removed. In such a scenario, for values of the knowledge stock just slightly above k^S, firm 1 could increase his discounted payoff by investing heavily for a short time-period thereby moving the knowledge stock below k^S, which would imply that firm 1 would reach the upper branch of his value function. More generally, it seems evident that in games where at every point of the state space both players can move the state in both directions (if we restrict attention to one-dimensional problems) the value functions of both players have to be continuous under the equilibrium profile. Since generically it will be impossible in an asymmetric game to have a single point where the local value functions of both players intersect, the issue arises whether in such games an MPE exhibiting Skiba points can exist. Exploring this issue in more detail is a challenge for future research.

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A Dynamic Advertising Game with Market Growth

Steffen Jørgensen and Simon Pierre Sigué

Abstract The paper considers a market in which two firms compete on advertising over time. Each firm can use three types of advertising: offensive advertising which attempts to attract customers from the rival firm, defensive advertising which aims at protecting a firm's customer base from the rival's attacks, and generic advertising to make industry sales grow. We address questions like: How should a strategy for the simultaneous use of the three types of advertising be designed? How would the resulting time paths of sales look like? The paper studies a differential game played over an infinite time horizon and provides closed-form expressions for equilibrium advertising strategies and sales rate trajectories.

1 Introduction

Business firms competing in growing markets have the choice of targeting their marketing activities at three different segments: rival firms' customers, own customers and potential customers. Marketing instruments used to influence the three segments include advertising, pricing, personal selling, and customer service. Such activities often have longer-term effects and should be studied in an intertemporal setting. This paper focuses on advertising and investigates how three types of advertising can be used in a dynamic and competitive market.

Offensive advertising aims at attracting rival firms' customers and its purpose is to stimulate brand switching. *Defensive advertising* is mainly targeted at a firm's own customers and its goal is to avoid that current customers switch to competing brands. In recent years there has been an increasing interest in defensive strategies as an efficient way to reduce a firm's marketing expenditures, based on an observation that in many industries it is considerably cheaper to retain a current customer

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than to attract a new one. The main goal of *generic advertising* is to expand demand for the category, mainly by inducing potential customers to start buying. Generic advertising does not promote a particular brand and has been used to create awareness in the early stages of the life cycle of a new product or in mature markets to promote new uses or increased consumption. Generic advertising has been used for a variety of product categories, notably agricultural products such as bacon and milk.

The concurrent use of all three types of advertising gives a firm a wider range of strategic marketing options. Previous research has not addressed the strategic interactions between all three types of advertising. In a mature (i.e., saturated) market, Erickson (1993) and Martín-Herrán et al. (2012) studied the design of offensive and defensive strategies. Bass et al. (2005a,b) designed offensive and generic advertising strategies in a growing market.

The current research constructs and analyzes a dynamic advertising game model to provide recommendations of how firms should design offensive, defensive, and generic advertising strategies in a growing market. Market contraction has been studied, although in a rather stylized fashion in, e.g., Wang and Wu (2001), Espinosa and Mariel (2001), but is disregarded in the current research.

For reasons of tractability we confine our interest to firms that are symmetric, with the exception of their initial sales rates. We follow the main trend in the literature and consider a duopolistic market. The aims of the paper are as follows. We wish to

- characterize the conditions, in terms of the model parameters, under which various combinations of offensive, defensive, and generic advertising emerge as equilibrium outcomes
- design feedback strategies for each type of advertising
- determine the time paths of sales that result from using equilibrium advertising strategies.

For this purpose we propose an extension of the Lanchester advertising model which includes the three types of advertising as separate decision variables. The remainder of the paper is organized as follows. Section 2 provides a literature review and Sect. 3 presents the advertising differential game model. Section 4 identifies equilibria of the game in Sects. 3 and 5 concludes.

2 Literature Review

The Lanchester model (see, e.g., Jørgensen and Zaccour 2004 for a survey) has often been used as a stylized representation of advertising competition. The standard Lanchester advertising model assumes a market of fixed size and describes how firms 'battle for market share', using offensive advertising to increase their individual sales to the detriment of their rivals.

A number of modifications of the standard Lanchester model have been proposed (see Jørgensen and Zaccour 2004, Bass et al. 2005a, and Huang et al. 2012 for reviews). Fruchter (1999), Piga (1998), Espinosa and Mariel (2001), among others, relaxed the assumption of a fixed market size and consider an expanding market. Market growth is attributed to advertising that plays the role of both offensive and generic efforts. Bass et al. (2005a,b) model generic advertising as a separate decision variable. Bass et al. (2005a) assume an infinite planning horizon while Bass et al. (2005b) use a finite horizon setting.

Erickson (1993) and Martín-Herrán et al. (2012) consider a market of fixed size and study defensive as well as offensive advertising. The latter research incorporates interaction between offensive and defensive marketing. Bass et al. (2005a,b) also consider defensive advertising, although not as a separate decision variable.

The current research builds on the above extensions of the Lanchester model, in particular Erickson (1993) and Bass et al. (2005a,b), and suggests a new model to study offensive, defensive, and generic advertising, all being treated as separate decision variables. This is a proper generalization of the work by Erickson (1993), Bass et al. (2005a,b), and Martín-Herrán et al. (2012). We assume an infinite time horizon. The finite horizon case is dealt with in Jørgensen and Sigué (2015).

3 A Differential Advertising Game

Let $a_i(t), d_i(t)$, and $g_i(t)$ be the rates of offensive, defensive, and generic advertising efforts, respectively, of firm $i = 1, 2$ at time $t \in [0, \infty)$. Time t is continuous. Firms are symmetric which means that any model parameter has the same (unspecified) value for both firms, with the exception that initial sales rates are unequal. The case of fully asymmetric firms is analytically intractable. (Because defensive advertising is absent in their model, Bass et al. (2005a,b) were able to find analytical solutions for an asymmetric case).

Denote by $S_i(t)$ the rate of sales of firm i at time t. The evolution of sales over time is modelled by two differential equations that extend the Lanchester advertising model to include three types of advertising effort and market expansion:

$$\dot{S}_i(t) = f_i(a_i(t), d_j(t))\sqrt{S_j(t)} - f_j(a_j(t), d_i(t))\sqrt{S_i(t)} + h_i(g_1(t), g_2(t))$$

in which $i, j \in \{1, 2\}, i \neq j$. Function f_i is called an attraction rate and, following Erickson (1993), we let it depend on a firm's offensive advertising effort and the defensive effort of the rival firm. Attraction rates must be nonnegative and their partial derivatives satisfy the obvious inequalities $\partial f_i / \partial a_i > 0$ and $\partial f_i / \partial d_j < 0$. Sales rates $S_1(t)$ and $S_2(t)$ enter on the right-hand side of the sales dynamics in a concave fashion which implies that effects of advertising are subject to diminishing marginal returns (see, e.g., Sorger 1989; Bass et al. 2005a,b).

Function h_i represent a firm's share of industry sales growth and depends on generic advertising efforts of both firms (Dearden and Lilien 2001; Bass et al. 2005a,b). The growth function h_i takes nonnegative values, increases in its arguments, and satisfies $h_i(0,0) = 0$. The latter is a simplification because there clearly are instances where a market grows even if no generic advertising is done.

Remark 1 The sales dynamics proposed in this research generalize those in Erickson (1993) to include market growth and generic advertising. If there is no defensive advertising, our sales dynamics are the same as in the symmetric case of Bass et al. (2005a,b).

Following the major part of the literature in the area, we employ specific functional forms for attraction and market growth rates. Our choices are simple[1]:

$$f_i(a_i, d_j) = \beta a_i - \lambda d_j; \;\; h_i(g_1, g_2) = \frac{k\,(g_1 + g_2)}{2}.$$

in which β and λ are positive constants reflecting the efficiency of offensive and defensive advertising, respectively. We adopted the market growth component from Bass et al. (2005a,b) and—due to symmetry—allocate to each firm one half of the increase in industry sales. The parameter $k > 0$ measures the efficiency of total generic advertising efforts.

The unit profit margin of a firm equals $m > 0$ and is constant. Costs of advertising efforts are quadratic, a functional form that has often been used in the literature. It reflects a hypothesis that advertising effort is subject to decreasing marginal returns. Advertising cost functions are as follows:

$$C_a(a_i) = \frac{c_a}{2}a_i^2, \;\; C_d(d_i) = \frac{c_d}{2}d_i^2; \; C_g(g_i) = \frac{c_g}{2}g_i^2$$

in which $c_a > 0, c_d > 0, c_g > 0$ are constants. (Bass et al. (2005a,b) assumed $c_a = c_g$.) The objective function of firm i then is

$$J_i(a_i, d_i, g_i) = \int_0^\infty e^{-\rho t}\left[mS_i(t) - \frac{c_a}{2}a_i^2(t) - \frac{c_d}{2}d_i^2(t) - \frac{c_g}{2}g_i^2(t)\right]dt$$

where $\rho > 0$ is a constant discount rate.

Advertising and sales rates must satisfy the nonnegativity constraints

$$a_i(t) \geq 0, \;\; d_i(t) \geq 0, \;\; g_i(t) \geq 0, \;\; S_i(t) \geq 0 \text{ for all } t \text{ and } i \tag{1}$$

and attraction rates must take nonnegative values:

$$f_i(a_i, d_j) = \beta a_i - \lambda d_j \geq 0 \text{ for all } t \text{ and } i. \tag{2}$$

[1]Martín-Herrán et al. (2012) also use a linear formulation of attraction rates. Moreover, they include multiplicative interaction between offensive and defensive advertising.

Rather than imposing (1) and (2) as constraints on advertising and sales rates, we address the nonnegativity issue when equilibrium values of these rates have been determined.

4 Nash Equilibrium Advertising Strategies

An autonomous problem is one (like ours) in which the model parameters are time-independent. If such a problem has an infinite planning horizon, the usual approach in the literature has been to focus on equilibria with stationary strategies. This paper assumes that firms use stationary Markovian strategies, denoted $a_i(S_1,S_2), d_i(S_1,S_2)$, and $g_i = g_j(S_1,S_2)$.

Let $V^1(S_1,S_2)$ and $V^2(S_1,S_2)$ be the value functions of the firms. A value function represents the optimal profit to be earned by a firm during the time interval $[t,\infty)$ when the dynamic system is in state (S_1,S_2) at time t. Equilibria are identified by verifying the existence of continuously differentiable functions $V^i(S_1,S_2)$ that solve the following Hamilton-Jacobi-Bellman (HJB) equations for all $t \geq 0, S_i \geq 0, i=1,2$:

$$\rho V^i(S_1,S_2) = \max_{a_i\geq 0,\ d_i\geq 0,\ g_i\geq 0} \Big\{ mS_i - \frac{c_a}{2}a_i^2 - \frac{c_d}{2}d_i^2 - \frac{c_g}{2}g_i^2 + \tag{3}$$

$$\frac{\partial V^i}{\partial S_i}\left[(\beta a_i - \lambda d_j)\sqrt{S_j} - \left(\beta a_j - \lambda d_i\right)\sqrt{S_i} + \frac{k\left(g_i+g_j\right)}{2}\right] +$$

$$\frac{\partial V^i}{\partial S_j}\left[(\beta a_j - \lambda d_i)\sqrt{S_i} - \left(\beta a_i - \lambda d_j\right)\sqrt{S_j} + \frac{k\left(g_i+g_j\right)}{2}\right]\Big\}$$

in which a_j, d_j, and g_j are stationary Markovian strategies. Transversality conditions associated with the HJB equations are

$$\lim_{t\to\infty} e^{-\rho t} V^i(S_1(t),S_2(t)) = 0; \quad \lim_{t\to\infty} e^{-\rho t}\frac{\partial V^i}{\partial S_i}(S_1(t),S_2(t)) = 0. \tag{4}$$

Performing the maximizations indicated on the right-hand side of the HJB equation yields

$$-c_a a_i + \beta\sqrt{S_j}\left(\frac{\partial V^i}{\partial S_i} - \frac{\partial V^i}{\partial S_j}\right)\begin{cases} \leq 0 \text{ if } a_i = 0 \\ = 0 \text{ if } a_i > 0 \end{cases}$$

$$-c_d d_i + \lambda\sqrt{S_i}\left(\frac{\partial V^i}{\partial S_i} - \frac{\partial V^i}{\partial S_j}\right)\begin{cases} \leq 0 \text{ if } d_i = 0 \\ = 0 \text{ if } d_i > 0 \end{cases}$$

$$-c_g g_i + \frac{k}{2}\left(\frac{\partial V^i}{\partial S_i} + \frac{\partial V^i}{\partial S_j}\right)\begin{cases} \leq 0 \text{ if } g_i = 0 \\ = 0 \text{ if } g_i > 0. \end{cases} \tag{5}$$

The first expression in (5) shows that offensive advertising of firm i increases if the sales rate of the rival firm j increases. The reason is that a firm exploits that its rival has a larger sales rate that can be attacked. Defensive advertising of a firm increases as its own sales rate increases because the firm has more to protect. It holds that $a_i > 0 \Leftrightarrow d_j > 0$ and $a_i = 0 \Leftrightarrow d_j = 0$ which is intuitive: If firm i attacks [does not attack] sales of firm j, then firm j will [will not] defend its sales.

The partial derivatives $\partial V^i/\partial S_i$ and $\partial V^i/\partial S_j$ can be interpreted as shadow prices of sales rates S_i and S_j, respectively. When offensive and defensive advertising rates a_i and d_i are positive, the shadow prices must satisfy $\partial V^i/\partial S_i > \partial V^i/\partial S_j$. This inequality means that a firm will use these two types of advertising if it yields a higher marginal profit to have a marginal increase in its own sales rather than in rival's sales. This is intuitive. If the generic advertising rate g_i is positive, the inequality $\partial V^i/\partial S_i + \partial V^i/\partial S_j > 0$ must hold. The aggregate effect of a marginal increase in the two sales rates being positive justifies the use of generic advertising.

A consequence of the stationary, infinite horizon setup is that the shadow prices and model parameters that determine the strategies will not change over time. The implication is that the decision to use a particular type of advertising is irreversible: The choice is made at the start of the game and will not be changed.

The following assumption is needed for reasons of tractability:

$$\frac{\beta^2}{c_a} > \frac{\lambda^2}{c_d} \tag{6}$$

and means that offensive advertising is the more cost-effective type of advertising. Whether this is true or not depends on the institutional characteristics of the market to which the model is applied.

Suppose that all equilibrium advertising rates are positive. Substituting the strategies from (5) on the right-hand sides of the HJB equations yields

$$\begin{aligned} \rho V^i &= mS_i + \frac{k^2}{8c_g}\left(\frac{\partial V^i}{\partial S_i} + \frac{\partial V^i}{\partial S_j}\right)^2 + \frac{k^2}{4c_g}\left(\frac{\partial V^i}{\partial S_i} + \frac{\partial V^i}{\partial S_j}\right)\left(\frac{\partial V^j}{\partial S_i} + \frac{\partial V^j}{\partial S_j}\right) \\ &\quad + \frac{1}{2}\left(\frac{\partial V^i}{\partial S_i} - \frac{\partial V^i}{\partial S_j}\right)^2\left[\frac{\beta^2}{c_a}S_j + \frac{\lambda^2}{c_d}S_i\right] \\ &\quad + \left(\frac{\partial V^i}{\partial S_i} - \frac{\partial V^i}{\partial S_j}\right)\left(\frac{\partial V^j}{\partial S_i} - \frac{\partial V^j}{\partial S_j}\right)\left[\frac{\beta^2}{c_a}S_i + \frac{\lambda^2}{c_d}S_j\right]. \end{aligned} \tag{7}$$

Bass et al. (2005a) showed (having $\lambda = 0$) that linear value functions solve the equations in (7). Guided by this result, we guess that value functions are

$$V^i(S_1, S_2) = \alpha + \varphi S_i + \psi S_j. \tag{8}$$

Value function coefficients α, φ, and ψ are the same for both firms (due to symmetry) and time-invariant (due to stationarity). Equation (8) shows that $\varphi = \partial V^i/\partial S_i$ and $\psi = \partial V^i/\partial S_j$.

It is straightforward to show that for the conjectured value functions to satisfy the HJB equations, α, φ, and ψ must be roots of the algebraic equations

$$\begin{aligned}\alpha &= \frac{3k^2}{8\rho c_g}(\varphi+\psi)^2 \\ \varphi &= \frac{m}{\rho} + \frac{1}{\rho}\left(\frac{\lambda^2}{2c_d} - \frac{\beta^2}{c_a}\right)(\varphi-\psi)^2 \\ \psi &= \frac{1}{\rho}\left(\frac{\beta^2}{2c_a} - \frac{\lambda^2}{c_d}\right)(\varphi-\psi)^2 .\end{aligned} \tag{9}$$

Remark 2 If $\lambda = 0$, these equations appear in Bass et al. (2005a) who showed that $\varphi > 0, \psi > 0$, and $\varphi > \psi$ for all feasible values of the model parameters. In our model we shall see that—due to defensive advertising—it may happen that $\psi < 0$.

The expressions in (5) show that to determine equilibrium advertising strategies we only need the sum and the difference of shadow prices φ and ψ. For this purpose it is convenient in (9) to define three constants

$$C_1 = \frac{1}{\rho}\left(\frac{\lambda^2}{2c_d} - \frac{\beta^2}{c_a}\right); \;\; C_2 = \frac{1}{\rho}\left(\frac{\beta^2}{2c_a} - \frac{\lambda^2}{c_d}\right); \;\; C = \frac{m}{\rho} > 0$$

where we note that (6) implies $C_1 < 0, C_1 - C_2 < 0, C_1 + C_2 < 0$. In (9) we now have

$$\begin{aligned}\varphi &= C + C_1(\varphi-\psi)^2 \\ \psi &= C_2(\varphi-\psi)^2 .\end{aligned} \tag{10}$$

The equations in (10) admit two pairs of roots.[2] To select one of these we use a test proposed in Bass et al. (2005a). If firms have zero profit margins, i.e., $m = 0$, value functions should vanish because firms have no revenues and do not advertise. This happens if $\alpha = \varphi = \psi = 0$. It is readily verified that the following roots satisfy the test:

$$\begin{aligned}\varphi &= \frac{C_1 - C_1\sqrt{4C(C_2-C_1)+1} + 2CC_2(C_2-C_1)}{2(C_1-C_2)^2} \\ \psi &= -\frac{C_2}{C_1}\left(C - \frac{C_1 - C_1\sqrt{4C(C_2-C_1)+1} + 2CC_2(C_2-C_1)}{2(C_1-C_2)^2}\right).\end{aligned} \tag{11}$$

[2] It holds that $C_1 \neq 0, C_1 \neq C_2$ because $C_1 < 0$ and $C_1 < C_2$. Roots are real since $4C(C_2 - C_1) + 1 > 0$.

The following lemma is important for the characterization of offensive and defensive advertising strategies.

Lemma 1 *It holds that* $\varphi - \psi > 0$.

Proof All proofs are in the Appendix ■

As a technicality, we have shown in the Appendix that attraction rates are positive, given (6) and the above lemma. The lemma implies that it always pays to use offensive and defensive advertising. For the generic advertising rate, there will be two cases to consider, either positive or zero. Our task, therefore, is to find the signs of shadow prices φ and ψ (to determine the value functions) and their sum $\varphi + \psi$ (to characterize the generic advertising strategies). We consider two cases: (1) $C_2 > 0$ and (2) $C_2 < 0$.

Remark 3 Using (10) yields

$$\varphi - C = \frac{C_1}{C_2}\psi$$

where the right-hand side is negative. (In Case 1 we have $C_1 < 0, C_2 > 0, \psi > 0$ while $C_1 < 0, C_2 < 0, \psi < 0$ in Case 2). Then $\varphi - C < 0$, i.e., the shadow price φ of a firm's sales has an upper bound, C. Recalling that $C = m/\rho$, this is intuitive. The shadow price φ represents the marginal change in a firm's optimal profit, from time t to infinity, produced by a marginal increase in the firm's own sales rate at time t. The number C is the present value of a perpetuity which earns the amount m per unit of sales and when future earnings are discounted at rate ρ.

The following proposition summarizes our results for the shadow prices of individual and industry sales.

Proposition 1 *In Case 1 it holds that* $\psi > 0, \varphi > 0$, *implying* $\varphi + \psi > 0$ *and the generic advertising rates* $g_i(t)$ *are positive. In Case 2 it holds that* $\psi < 0$ *and*

$$\phi \begin{pmatrix} > \\ < \end{pmatrix} 0 \text{ if } C \begin{pmatrix} > \\ < \end{pmatrix} - \frac{C_1}{C_2^2} \triangleq b$$

where $b > 0$. *Moreover*

$$\varphi + \psi \begin{pmatrix} > \\ \leq \end{pmatrix} 0 \text{ if } C \begin{pmatrix} > \\ \leq \end{pmatrix} - \frac{C_1 + C_2}{4C_2^2} \triangleq B \tag{12}$$

where $B > b$. *In Case 2, the generic advertising rate* g_i *is positive if* $C > B$. *If* $C \leq B$, g_i *is zero.*

The results in Proposition 1 have the following interpretation. The results for Case 1 are similar to those obtained in Bass et al. (2005a) for $\lambda = 0$. This case is characterized by $c_d\beta^2 > 2c_a\lambda^2$, i.e., offensive advertising is the dominant type of advertising. Then it certainly pays to attack rival sales and therefore the shadow price of these sales is positive. Both shadow prices are positive which makes it worthwhile to use generic advertising.

In Case 2, the shadow price ψ is negative which occurs if $c_d\beta^2 < 2c_a\lambda^2$. In this situation defensive advertising is dominant. The proposition shows that φ is positive for $C = m/\rho$ larger than B. This inequality is valid if the margin m is sufficiently high and/or the discount rate ρ is sufficiently small. Thus, a firm which is sufficiently far-sighted and/or has a sufficiently high margin, will have a positive shadow price of its own sales rate. This is quite intuitive. If φ is negative, both shadow prices are negative and the firms should not spend money on generic advertising. The results for Case 2 are summarized as follows:

$C < b$	$\phi < 0$ and $\varphi + \psi < 0$	$g = 0$
$C \in (b, B)$	$\phi > 0$ and $\varphi + \psi < 0$	$g = 0$
$C > B$	$\phi > 0$ and $\varphi + \psi > 0$	$g > 0.$

It remains to determine the equilibrium sales rate dynamics. Inserting the equilibrium advertising strategies into the dynamics yields

$$\dot{S}_1(t) = K_1 S_2(t) - K_1 S_1(t) + K_2; \quad S_1(0) = s_{10} > 0 \tag{13}$$
$$\dot{S}_2(t) = K_1 S_1(t) - K_1 S_2(t) + K_2; \quad S_2(0) = s_{20} > 0$$

where

$$K_1 = \left(\frac{\beta^2}{c_a} - \frac{\lambda^2}{c_d}\right)(\varphi - \psi), \quad K_2 = \frac{k^2}{2c_g}(\varphi + \psi)$$

are constants. K_1 is positive, cf. Lemma 1. K_2 is positive if the generic advertising rate g is positive, nonpositive if $g = 0$. Solving the differential equations in (13) yields

$$S_1(t) = \frac{s_{10} + s_{20}}{2} + \frac{s_{10} - s_{20}}{2} e^{-2K_1 t} + K_2 t \tag{14}$$
$$S_2(t) = \frac{s_{10} + s_{20}}{2} - \frac{s_{10} - s_{20}}{2} e^{-2K_1 t} + K_2 t$$

from which we obtain

$$S_1(t) - S_2(t) = e^{-2K_1 t}(s_{10} - s_{20}). \tag{15}$$

If $s_{10} > s_{20}$ we conclude from (15) that $S_1(t) - S_2(t) > 0$ for any finite t. Therefore, the initial sales advantage of firm 1 will persist for any practical purpose. Firm 2, however, gradually catches up with firm 1 and in the limit $t \to +\infty$, sales rates are equal. If $s_{20} > s_{10}$ the opposite happens.

Remark 4 Using (14) one can determine the value functions. In a scenario where firms use generic advertising, the value function of firm 1 is

$$V^1(t) = \frac{(s_{10}+s_{20})(\psi+\varphi)}{2} + \frac{(s_{10}-s_{20})(\varphi-\psi)}{2} e^{-2K_1 t} + \frac{k^2(\psi+\varphi)^2}{2c_g}\left(\frac{3}{4\rho}+t\right) \tag{16}$$

which is positive if $s_{10} > s_{20}$. In this case the sign of $V^2(t)$ cannot be ascertained. The value function $V^2(t)$ is positive if $s_{20} > s_{10.}$. In this case the sign of $V^1(t)$ cannot be determined.

We can now prove that the conditions in (4) are satisfied. Since the shadow price φ is constant, the condition $\lim_{t\to\infty} e^{-\rho t}\varphi = 0$ is satisfied and the condition $\lim_{t\to\infty} e^{-\rho t}V^i(t) = 0$ holds because $\lim_{t\to\infty} te^{-\rho t} = 0$.

Given the equilibrium sales rate trajectories, one can investigate how advertising rates evolve over time. Since advertising strategies depend on sales we first characterize the sales trajectories. Differentiation with respect to time in (14) provides

$$\dot{S}_1(t) = K_2 + K_1(s_{20}-s_{10})e^{-2K_1 t}; \quad \dot{S}_2(t) = K_2 + K_1(s_{10}-s_{20})e^{-2K_1 t}$$

and therefore

$$\begin{aligned} \dot{S}_1(t) \begin{pmatrix} > \\ < \end{pmatrix} 0 \text{ if } (s_{20}-s_{10})e^{-2K_1 t} \begin{pmatrix} > \\ < \end{pmatrix} -\frac{K_2}{K_1} \\ \dot{S}_2(t) \begin{pmatrix} > \\ < \end{pmatrix} 0 \text{ if } (s_{10}-s_{20})e^{-2K_1 t} \begin{pmatrix} > \\ < \end{pmatrix} -\frac{K_2}{K_1}. \end{aligned} \tag{17}$$

Define a constant

$$\chi = -\frac{K_2}{K_1(s_{20}-s_{10})} = \frac{kg}{\left(\frac{\beta^2}{c_a}-\frac{\lambda^2}{c_d}\right)(\varphi-\psi)(s_{10}-s_{20})} \geq 0$$

and a time instance $\tilde{t}$ which is the unique root of the equation $\exp\{-2K_1\tilde{t}\} = \chi$ if $\chi < 1$. There will be two cases to consider: Case A where $g > 0 \Rightarrow \chi > 0$ and Case B where $g = 0 \Rightarrow \chi = 0$. Proposition 2 follows readily from (17) and its proof is omitted.

Proposition 2 *Given $s_{10} > s_{20}$, the time paths of sales rates are as follows. The trajectory $S_1(t)$ is convex and $S_2(t)$ is concave.*

***Case A**: $S_2(t)$ increases for all t. **(A1)**: If $\chi \in (0,1]$, $S_1(t)$ decreases for $t \in [0,\tilde{t})$ and increases for $t > \tilde{t}$. **(A2)**: If $\chi > 1$, $S_1(t)$ increases for all t.*

***Case B**: $S_1(t)$ decreases and $S_2(t)$ increases for all t.*

The time paths of sales rates in Proposition 2 for Cases A1, A2, and B are illustrated in Figs. 1–3 appearing at the end of the Appendix. Sales rates $S_1(t)$ and $S_2(t)$ are respectively given by the upper and bottom lines.

From (5) we have

$$a_i = \frac{\beta}{c_a}\sqrt{S_j}\,(\varphi - \psi) > 0;\;\; d_i = \frac{\lambda}{c_d}\sqrt{S_i}\,(\varphi - \psi) > 0$$

which shows, for example, that the time paths of offensive advertising a_1 of firm 1 and defensive advertising d_2 of firm 2 mimic the time path of $S_2(t)$. It holds that

$$a_2(t) - d_2(t) = \left(\frac{\beta}{c_a}\sqrt{S_1(t)} - \frac{\lambda}{c_d}\sqrt{S_2(t)}\right)(\varphi - \psi) > 0 \text{ for all } t$$

which shows that firm 2 which has the smaller sales rate spends more on offensive than on defensive advertising. A similar result is reported in Erickson (1993). Moreover

$$a_2(t) - d_1(t) = \left(\frac{\beta}{c_a} - \frac{\lambda}{c_d}\right)\sqrt{S_1(t)}\,(\varphi - \psi) > 0$$
$$a_1(t) - d_2(t) = \left(\frac{\beta}{c_a} - \frac{\lambda}{c_d}\right)\sqrt{S_2(t)}\,(\varphi - \psi) > 0$$

which means that a firm uses more offensive effort to attack rival sales than the latter uses to defend its sales. This result is driven by the higher effort-to-cost effectiveness of offensive advertising.

The difference between Cases A and B in Proposition 2 lies in the use of generic advertising. In Case B there is no market growth and if the sales rate of a firm increases, the sales rate of the rival must decrease. The sales rate of firm 2 increases because (1) this firm uses more effort to attack sales of firm 1 than firm 1 uses to defend its sales ($a_2(t) > d_1(t)$), (2) offensive advertising is more cost-effective than defensive advertising, and (3) the sales rate of firm 1 is the largest.

In Case A1, there is an initial interval of time, ending at $\tilde{t}$, during which the sales rate of firm 1 is diminishing. As plausible responses, firm 2 reduces its offensive advertising effort while firm 1 reduces its defensive effort. After time $\tilde{t}$, both sales rates increase which means that offensive and defensive advertising rates of both firms increase. In Case A2, both sales rates are increasing over time.

To finish our analysis of the game we wish to see what happens in the "very long run". If there is no generic advertising, sales trajectories approach monotonically the limit $(s_{10} + s_{20})/2$ as $t \to \infty$. This is not surprising. When generic advertising is used, sales rates do not converge to a steady state. However, considering market shares $M_i(t) = S_i(t)/(S_1(t) + S_2(t))$ we can state Proposition 3. The result is intuitive and is a special case of one derived in Bass et al. (2005a) for the case of symmetric firms. Proposition 3 follows directly from (14) and the result is expected.

Proposition 3 *In Case A of Proposition 2, market shares converge to a steady state:*

$$\lim_{t\to\infty} M_i(t) = \frac{1}{2}.$$

Sensitivity analyses of equilibrium strategies, sales rates, and value function parameters with respect to model parameters are complicated and lead to quite many instances in which conclusions are ambiguous. Bass et al. (2005a), on the other hand, were able to obtain many definitive sensitivity results for their model, most likely due to the absence of defensive advertising.

5 Conclusions

We have suggested a modification of the Lanchester advertising model with the purpose of studying a duopolistic market in which firms have the option to use generic, offensive, and defensive advertising efforts. Previous works have studied offensive advertising alone, offensive and defensive advertising, and offensive and generic advertising. Our model offers a wider range of strategic advertising possibilities to an advertising manager and generalizes previous works.

The paper has identified Markovian equilibrium advertising strategies, determined the associated equilibrium sales rate trajectories, and illustrated how advertising and sales paths evolve over time. If firms do not use generic advertising, offensive and defensive advertising strategies are qualitatively similar to those reported in Erickson (1993). On the other hand, in a growing market, advertising strategies are different from those reported in previous works (Erickson 1993; Bass et al. 2005a,b).

Two important assumptions of the model are that firms are symmetric (except for their initial sales rates) and offensive advertising is more cost-effective than defensive advertising. The assumption of symmetry has been used in previous studies but clearly is a simplification. Whether offensive advertising is the more cost-effective kind of advertising is an empirical matter.

The assumption of a duopolistic market has been used in the majority of related work. Future research might explore the extension to an oligopolistic industry with asymmetric firms. Generalizing to an oligopoly is feasible but would probably not provide much more than a possibility to study the impact on equilibrium outcomes of the number of firms in the industry. An asymmetric model will most likely be analytically intractable.

Finally, a topic for further research would be how firms can agree on investing in generic advertising, given that they compete on offensive and defensive efforts. Here some kind of cooperation, perhaps in the spirit of “coopetition”, seems to be required.

Appendix

Nonnegativity of Attraction Rates If $a_i > 0, d_j > 0$, attraction rates are given by

$$f_i\left(a_i, d_j\right) = \beta a_i - \lambda d_j = \left(\frac{\beta^2}{c_a} - \frac{\lambda^2}{c_d}\right)\sqrt{S_j}\,(\varphi - \psi)$$

and invoking (6) we have $f_i\left(a_i, d_j\right) > 0$ if $\varphi - \psi > 0$. Lemma 1 shows that given (6), the difference $\varphi - \psi$ is positive and hence attraction rates are positive. Q.E.D.

Proof of Lemma 1 Using (11) provides

$$\varphi - \psi = -\frac{\sqrt{4C\,(C_2 - C_1) + 1} - 1}{2\,(C_1 - C_2)}.$$

Our assumption $\beta^2/c_a > \lambda^2/c_d$ implies $C_1 - C_2 < 0$ and hence $\varphi - \psi > 0$. Q.E.D.

Proof of Proposition 1 In Case 1, $\psi > 0$ follows from (10). Using $\psi > 0$ and $\varphi - \psi > 0$ shows that $\varphi > 0$ and then $\varphi + \psi > 0$. In Case 2, $\psi < 0$ follows from (10). The result in (12) is established by using (10) and the fact that $B > b$. Offensive and defensive advertising rates are positive in both cases since $\varphi > \psi$. The generic advertising rate g is positive in Case 1. In Case 2, use (11) to see that g is positive if $C > B$, zero if $C \leq B$. Q.E.D.

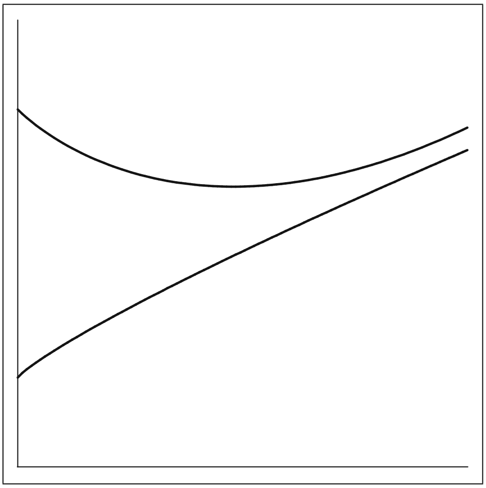


Fig. 1 Case A1: equilibrium sales paths

Fig. 2 Case A2: equilibrium sales paths

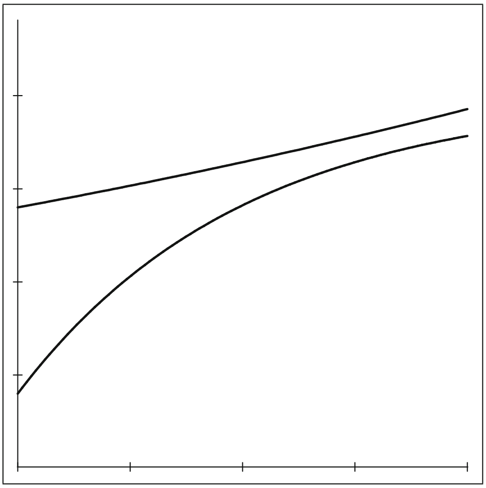
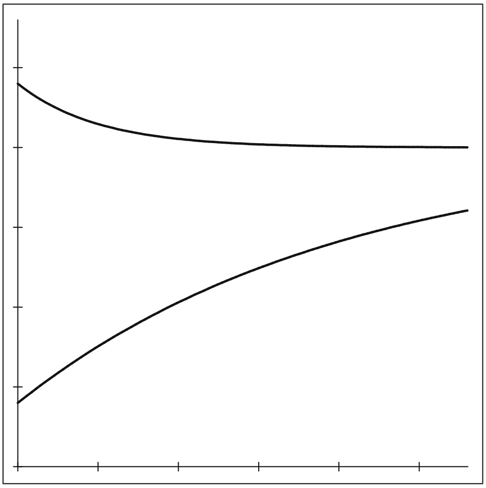


Fig. 3 Case B: equilibrium sales paths



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Managerial Delegation in a Dynamic Renewable Resource Oligopoly

Luca Lambertini

Abstract I propose a differential oligopoly game of resource extraction under linear and nonlinear feedback strategies, where firms are managerial and delegation contract are based on output levels. The model shows that delegation expands the set of stable nonlinear feedback equilibria as well as the residual steady state resource stock. Additionally, the separation between ownership and control mitigates the voracity effect associated with high values of the reproduction rate of the resource.

JEL Codes: C73, L13, Q2

1 Introduction

One of the most debated issue in environmental and resource economics is the joint exploitation of common pool resources and the related tragedy of commons.[1] Some recent extensions of the literature on this matter examines the exploitation of a renewable resource in differential oligopoly games with profit-seeking firms, i.e., pure entrepreneurial units (Benchekroun 2003, 2008; Fujiwara 2008, 2011; Colombo and Labrecciosa 2013, 2015; Lambertini and Mantovani 2014). However, casual observation reveals that in many industries, most of the firms (in particular large ones) are indeed managerial entities in which control is separate from

[1]See the seminal contributions by Gordon (1954) and Hardin (1968). The subsequent literature includes, among many others, Levhari and Mirman (1980), Clemhout and Wan (1985), Clark (1990), Benhabib and Radner (1992), Dockner and Sorger (1996), Dawid and Kopel (1997), Sorger (1998), and Benchekroun and Long (2002), among many others. Advanced overviews of the early stages of this debate are in Dasgupta and Heal (1979) and Clark (1990). A recent survey of differential games of resource extraction is in Lambertini (2013, Chap. 9).

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ownership and managers in charge of determining their respective firms' strategies receive remunerations based on combinations of profits and some other magnitudes, such as sales, revenues or market shares.

Most of the extant literature on strategic delegation (based on Vickers 1985, Fershtman and Judd 1987, Sklivas 1987, and Fershtman et al. 1991) neatly points out that delegation relying on contracts based on sales or revenues accounting for a managerial preference towards output expansion makes firms more aggressive as compared to what they would be if directly led by shareholders. This, in turn, produces a more competitive outcome with a lower price and a higher industry output. These results, attained in static models, are confirmed by the dynamic version of such games, as in Cellini and Lambertini (2008). A priori, operating the separation between ownership and control via such contracts may be expected to jeopardise the survival of the resource.

To examine this issue, I am proposing an extension of the differential oligopoly game investigated by Benchekroun (2003, 2008), Fujiwara (2008), Lambertini and Mantovani (2014), and Colombo and Labrecciosa (2015) to account for the generalised presence of managerial contracts based on sales. After outlining the linear feedback solution, which, as a by-product, also yield the open-loop equilibrium, I will set out to consider nonlinear feedback information.

The main results of the analysis can be summarised in the following terms. At the proper linear feedback equilibrium, delegation enhances the residual resource stock. This happens because the managerial incentive *à la* Vickers is observationally equivalent to expanding market size. This also applies, and to a higher degree, under nonlinear feedback rules, to the extent that delegation (1) expands the set of stable nonlinear feedback equilibria and, in particular, (2) raises its upper bound in the state-control space.

Combining delegation with feedback rules allows one to revisit the so-called voracity effect (Lane and Tornell 1996; Tornell and Lane 1999) operating for sufficiently high levels of the resource growth rate. This effect emerges whenever higher growth rates lead to lower steady state resource stocks, because of the resulting hastening of the extraction activity on the part of firms, combined with the usual pre-emption effect associated with feedback rules, whose presence has been abundantly stressed in previous research on differential oligopoly games.[2] While a priori one could expect delegation to exert a flywheel effect in combination with the voracity effect in compromising resource preservation, indeed the opposite applies, the reason being again the fact that sales incentive modify the managers' perception of market size and compensate the voracity generated by the rate of reproduction of the resource.

[2]This idea is pervasive in the literature treating differential oligopoly games with quantity competition (Driskill and McCafferty 1989); capacity accumulation (Reynolds 1987, 1991); sticky prices (Fershtman and Kamien 1987; Cellini and Lambertini 2004).

The remainder of the paper is organised as follows. Section 2 illustrates the setup. The linear feedback solution is illustrated in Sect. 3. Nonlinear feedback strategies are dealt with in Sect. 4. Concluding remarks are in Sect. 5.

2 The Model

The setup is an extension of Lambertini and Mantovani (2014) and Benchekroun (2008), where a common property productive asset oligopoly is considered, and encompasses the duopoly model used in Benchekroun (2003) and Fujiwara (2008). The model illustrates a differential oligopoly game of resource extraction unravelling over continuous time $t \in [0, \infty)$. The market is supplied by $n \geq 2$ firms[3] producing a homogeneous good, whose inverse demand function is $p = a - Q$ at any time t, with $Q = \sum_{i=1}^{n} q_i$. Firms share the same technology, characterised by marginal cost $c \in (0, a)$, constant over time. Firms operate without any fixed costs. During production, each firm exploits a renewable natural resource, whose accumulation is governed by the following dynamics:

$$\dot{S} = F(S) - Q \tag{1}$$

with

$$F(S) = \begin{cases} \delta S \ \forall \, S \in \left(0, S_y\right] \\ \delta S_y \left(\dfrac{S_{\max} - S}{S_{\max} - S_y} \right) \ \forall \, S \in \left(S_y, S_{\max}\right] \end{cases} \tag{2}$$

where S is the resource stock, $\delta > 0$ is its *implicit* growth rate when the stock is at most equal to S_y and δS_y is the maximum sustainable yield. Taken together, (1)–(2) imply that *(1)* if the resource stock is sufficiently small the population grows at an exponential rate; and *(2)* beyond S_y, the asset grows at a decreasing rate. Moreover, $S_{\max}$ is the *carrying capacity* of the habitat, beyond which the growth rate of the resource is negative, being limited by available amounts of food and space. In the remainder, we will confine our attention to the case in which $F(S) = \delta S$.[4]

Firms play noncooperatively and choose their respective outputs simultaneously at every instant. At $t = 0$, each firm hires a manager whose contract specifies the instantaneous objective which the manager has to maximise. Delegation contracts are observable. As in Vickers (1985), the delegation contract establishes that the

[3]Under monopoly the delegation to managers would not be operated by stockholders, so I'm assuming this case away.

[4]As in Benchekroun and Long (2002), Fujiwara (2008) and Tornell and Velasco (1992), among several others.

instantaneous objective function of manager i is a linear combination of profits and output[5]:

$$M_i = \pi_i + \theta q_i \, , \; \theta > 0 \tag{3}$$

In the remainder, I will treat θ as a constant for the sake of simplicity. It is worth noting that this type of contract implies that, through delegation, the owners intend to affect the manager's perception of marginal cost, inducing the latter to act as if his firm's marginal cost were indeed lower than c. To appreciate how this sort of technological illusion operates, it suffices to observe that (3) can be rewritten as follows:

$$M_i = \pi_i + \theta q_i = (a - Q - c + \theta)\, q_i = (a - Q - \hat{c})\, q_i \tag{4}$$

where $\hat{c} = c - \theta$. Hence, one could say that the manager behave as if he were maximising the profit function $\pi_i = (a - Q - \hat{c})\, q_i$, associated to a firm endowed with a more efficient technology. In a similar way, one could say that any $\theta > 0$ induces the manager to behave as if the reservation price were higher than it actually is, say, $A = a + \theta$. Be that as it may, this contract induces the sales expansion constituting the core of Vickers's model of strategic delegation. In the remainder, in order to make notation a bit more parsimonious, I will pose $\sigma \equiv a - \hat{c} > 0$.

The ith manager maximises the following discounted payoff flow

$$\Pi_i = \int_0^\infty M_i e^{-\rho t} dt, \tag{5}$$

under the constraint posed by the state equation

$$\dot{S} = \delta S - Q \tag{6}$$

Parameter $\rho > 0$ is the discount rate, common to all managers and constant over time. Obviously, if $\theta = 0$, firms behave as pure profit-seeking entrepreneurial units.

The analysis will be carried out under the following assumption:

Assumption 1 $\delta > \rho \left(n^2 + 1\right)/2$.

This guarantees the positivity of the residual resource stock at the steady state under linear feedback rules. That is, in the remainder I will leave the possibility of

[5]This contract is equivalent to that considered in Fershtman and Judd (1987), where the maximand is a weighted average of profits and revenues, $M_i = \alpha \pi_i + (1 - \alpha) R_i$, $R_i = p q_i$. A proof of the equivalence is in Lambertini and Trombetta (2002).

resource exhaustion due to an excessively large number of firms out of the picture, in order to focus solely on the effects of delegation.[6]

If firms don't internalise the consequences of their behaviour at any time and play the individual (static) Cournot-Nash output $q^{CN} = \sigma/(n+1)$ at all times, then the residual amount of the natural resource in steady state is $S^{CN} = n\sigma/[\delta(n+1)] = Q^{CN}/\delta$. As the remainder of the analysis is about to show, it is worth noting that the static solution corresponds to the open-loop steady state one, which in this game is unstable (see below). Let the initial condition be $S(0) = S_0 > 0$. The relevance of the size of S_0 on the final resource stock as well as on the stability of solutions will be discussed in the ensuing analysis.

3 The Linear Feedback Solution

The game can be solved under feedback rules conjecturing a linear-quadratic value function with unknown coefficients to be determined solving the system of Riccati equations. The Hamilton-Jacobi-Bellman (HJB) equation writes as:

$$\rho V_i(S) = \max_{q_i} \left[(\sigma - Q) q_i + V_i'(S)(\delta S - Q) \right] \tag{7}$$

where $V_i(S)$ is the firm i's value function; and $V_i'(S) = \partial V_i(S)/\partial S$. The first order condition (FOC) on q_i is

$$\sigma - 2q_i - \sum_{j \neq i} q_j - V_i'(S) = 0 \tag{8}$$

In view of the *ex ante* symmetry across firms, one can impose the symmetry conditions $q_i = q(S)$ and $V_i(S) = V(S)$ for all i and solve FOC (8) to obtain

$$q^F(S) = \max\left\{0, \frac{\sigma - V'(S)}{n+1}\right\} \tag{9}$$

where superscript F stands for *feedback*. Consider the case where $\sigma - V'(S) > 0$. Substituting $q^F(S) = (\sigma - V'(S))/(n+1)$ into (7), the latter can be rewritten as follows:

$$\frac{\sigma\left[\left(n^2+1\right)V'(S) - \sigma\right] + (n+1)^2\left[\rho V(S) - \delta S V'(S)\right] - n^2 V'(S)^2}{(n+1)^2} = 0 \tag{10}$$

[6]The analysis of this specific aspect in the corresponding model without any form of delegation can be found in Lambertini and Mantovani (2014, pp. 119–21).

Conjecturing $V(S) = \varphi_1 S^2 + \varphi_2 S + \varphi_3$, so that $V'(S) = 2\varphi_1 S + \varphi_2$, (10) implies the following system of Riccati equations:

$$\varphi_1 \left[(n+1)^2 (\rho - 2\delta) - 4n^2 \varphi_1 \right] = 0 \tag{11}$$

$$2\varphi_1 \left(n^2+1\right) \sigma + \varphi_2 \left[(n+1)^2 (\rho - \delta) - 4n^2 \varphi_1 \right] = 0 \tag{12}$$

$$\varphi_3 (n+1)^2 \rho + \varphi_2 \left[\left(n^2+1\right) \sigma - n^2 \varphi_2 \right] - \sigma^2 = 0 \tag{13}$$

Equations (12)–(13) are solved by

$$\begin{aligned} \varphi_3 &= \frac{\sigma^2 - \varphi_2 \left[\left(n^2+1\right)\sigma - n^2 \varphi_2 \right]}{(n+1)^2 \rho} \\ \varphi_2 &= \frac{\varphi_3 (n+1)^2 \rho}{(n+1)^2 (\delta - \rho) + 4n^2 \varphi_1} \end{aligned} \tag{14}$$

while the roots of (11) are

$$\varphi_{11} = 0; \; \varphi_{12} = \frac{(n+1)^2 (\rho - 2\delta)}{4n^2} \tag{15}$$

whereby, if $\varphi_1 = \varphi_{11}$, the individual equilibrium output is $q^{F1}(S) = q^{OL}$,[7] while if $\varphi_1 = \varphi_{12}$, the individual equilibrium output is

$$q^{F2}(S) = \frac{\sigma \left[\rho \left(n^2+1\right) - 2\delta \right] + (n+1)^2 (2\delta - \rho) \delta S}{2\delta (n+1) n^2} \tag{16}$$

which, if $\theta = 0$ so that $\sigma = a - c$, coincides with expression (11) in Lambertini and Mantovani (2014, p. 118). The expression on the r.h.s. of (16) belongs to $[0, \sigma/(n+1)]$ for all

$$S \in \left[\frac{\sigma \left(n^2+1\right)}{\delta (n+1)^2}, \frac{\sigma \left[2\delta - \rho \left(n^2+1\right) \right]}{(n+1)^2 (2\delta - \rho) \delta} \right] \tag{17}$$

[7] That is, here the open-loop solution is a degenerate feedback one. For more on games with this feature, see Fershtman (1987), Mehlmann (1988, Chap. 4), Dockner et al. (2000, Chap. 7), and Cellini et al. (2005), inter alia.

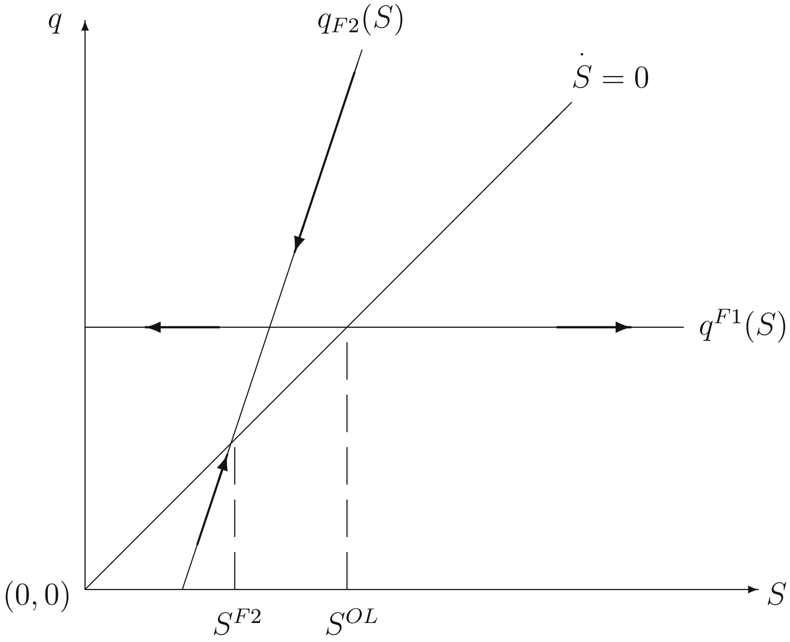

Fig. 1 Open-loop and linear feedback solutions in the (S, q) space

If $q = q^{F2}(S)$, the steady state level of the natural resource stock is

$$S^{F2} = \frac{\sigma\left[2\delta - \rho\left(n^2+1\right)\right]}{\delta(n+1)\left[2\delta - \rho(n+1)\right]} > 0 \tag{18}$$

for all values of δ satisfying Assumption 1. It is evident that $\partial S^{F2}/\partial\theta > 0$, which entails a property which will become useful in the remainder:

Lemma 1 *At the linear feedback equilibrium, any increase in the extent of delegation increases the residual stock of resources in steady state.*

Solutions $q^{F1}(S) = q^{OL}$ and $q^{F2}(S)$, together with the locus $\dot{S} = 0$, are represented in the space (q, S) in Fig. 1, where arrows illustrate the dynamics of variables and the stability of $q^{F2}(S)$, as opposed to the instability of the alternative solution $q^{F1}(S)$, and therefore also of the open-loop solution. As for its stability properties, for all $S_0 < S^{OL}$, the resource stock is bound to shrink to zero; otherwise, for all $S_0 > S^{OL}$, the stock will grow beyond S_y, not represented along the horizontal axis of Fig. 1. Hence, under open-loop rules, the ultimate destiny of the natural resource depends on initial conditions. It is also worth stressing that $\partial S^{OL}/\partial\theta > 0$, which implies that the interval of initial conditions leading to resource extinction under open-loop (or quasi-static) strategies expands in the extent of managerial delegation.

4 Nonlinear Feedback Equilibria

Here, I will rely on the procedure illustrated in Rowat (2007),[8] and confine my attention to symmetric equilibria arising under nonlinear feedback rules. Imposing the symmetry condition $q_i = q(S)$ for all i and solving (8), one obtains $V'(S) = \sigma - (n+1)q(S)$. Substituting this into (7) yields an identity in S. Differentiating both sides with respect to S and rearranging terms, any feedback strategy is implicitly given by the following differential equation:

$$q'(S) = \frac{(\delta - \rho)[\sigma - (n+1)q(S)]}{\sigma(n-1) + \delta(n+1)S - 2n^2 q(S)}, \tag{19}$$

which must hold together with terminal condition $\lim_{t\to\infty} e^{-\rho t} V(s) = 0$. Examining expression (19) reveals that

$$q'(S) = 0 \Leftrightarrow q_0(S) = \frac{\sigma}{n+1} = q^{F1}(S) = q^{OL} \tag{20}$$

$$q'(S) \to \pm\infty \Leftrightarrow q_\infty(S) = \frac{\sigma(n-1) + (n+1)\delta S}{2n^2} \tag{21}$$

Now, considering that the feedback control $q^F(S)$ is defined as in (9), the HJB equation is

$$\rho V(S) - \delta S V'(S) = 0 \tag{22}$$

if $\sigma \leq V'(S)$. Otherwise, it is (10), so that

$$\rho V(S) = \frac{n^2 V'(S)^2 + \sigma^2 + V'(S)\left[\delta S(n+1)^2 - \sigma\left(n^2+1\right)\right]}{(n+1)^2} \tag{23}$$

Consider first the case $\sigma \leq V'(S)$. Equation (22) is easily solved, yielding

$$V(S) = S^{\rho/\delta} \boldsymbol{C} \tag{24}$$

and the related condition on $V'(S)$ allows (24) to hold for

$$\boldsymbol{C} > \frac{\delta \sigma S^{\frac{\delta - \rho}{\delta}}}{\rho} \tag{25}$$

[8] Nonlinear feedback solutions have been investigated in oligopoly theory, environmental and resource economics and other fields. See Tsutsui and Mino (1990), Shimomura (1991), Dockner and Sorger (1996), Itaya and Shimomura (2001), Rubio and Casino (2002), and Colombo and Labrecciosa (2015), inter alia.

For all $\sigma > V'(S)$, differentiating (23) one obtains

$$w'(S) = \frac{(\delta - \rho)(n+1)^2 w(S)}{\sigma(n^2+1) - \delta S(n+1)^2 - 2nw(S)} \tag{26}$$

where $w(S) \equiv V'(S)$ and $\sigma\left(n^2+1\right) - \delta S(n+1)^2 - 2nw(S) \neq 0$. Equation (26) can be transformed using $w = \varkappa + A$ and $S = \psi + B$, with

$$A = 0;\ B = \frac{\left(n^2+1\right)\sigma}{\delta(n+1)^2} \tag{27}$$

into one which is homogeneous of degree zero in its variables:

$$\frac{d\varkappa}{d\psi} = \frac{\left(n^2+1\right)(\rho - \delta)\varkappa}{\delta(n+1)^2\psi + 2n^2\varkappa} \tag{28}$$

Now, defining $z \equiv \varkappa/\psi$, whereby $\varkappa = \psi z$ and $\partial\varkappa/\partial\psi = z + \psi \cdot \partial z/\partial\psi$, one may rewrite the above equation as

$$z + \psi\frac{\partial z}{\partial\psi} = \frac{\left(n^2+1\right)(\rho - \delta)z}{\delta(n+1)^2 + 2n^2 z} \tag{29}$$

which has two constant solutions:

$$z_{\mathbf{a}} = 0 \text{ and } z_{\mathbf{b}} = \frac{(n+1)^2(\rho - 2\delta)}{2n^2} \tag{30}$$

Going back to the original control variables, we have

$$q_{\mathbf{a}} = \frac{\sigma - A - z_{\mathbf{a}}(S-B)}{n+1};\ q_{\mathbf{b}} = \frac{\sigma - A - z_{\mathbf{b}}(S-B)}{n+1} \tag{31}$$

i.e.,

$$q_{\mathbf{a}} = \frac{\sigma}{n+1};\ q_{\mathbf{b}} = \frac{\sigma}{n+1} + \frac{(n+1)(2\delta - \rho)\left[\delta(n+1)^2 S - \sigma\left(n^2+1\right)\right]}{2\delta n^2(n+1)^2} \tag{32}$$

The levels of S solving $\dot{S} = 0$ at $q = q_{\mathbf{a}}$ and $q = q_{\mathbf{b}}$ are, respectively,

$$S_{\mathbf{a}} = \frac{\sigma n}{\delta(n+1)};\ S_{\mathbf{b}} = \frac{\sigma\left(n^2+1\right)}{\delta(n+1)} \tag{33}$$

with $S_{\mathbf{b}} < B$ since $\delta > (2n+1)\rho/2 > (n+1)\rho/2$. Then, for $z \neq \{z_{\mathbf{a}}, z_{\mathbf{b}}\}$, one can solve

$$\frac{d\psi}{\psi} = \frac{\left[\delta (n+1)^2 / \left(2n^2\right) + z\right] dz}{(z - z_{\mathbf{a}})(z - z_{\mathbf{b}})} = \frac{\gamma_{\mathbf{a}} dz}{(z - z_{\mathbf{a}})} + \frac{\gamma_{\mathbf{b}} dz}{(z - z_{\mathbf{b}})} \tag{34}$$

where coefficients

$$\gamma_{\mathbf{a}} = \frac{\delta}{2\delta - \rho}\,;\; \gamma_{\mathbf{b}} = \frac{\delta - \rho}{2\delta - \rho}\,;\; \gamma_{\mathbf{a}} + \gamma_{\mathbf{b}} = 1 \tag{35}$$

are determined via the method of partial fractions. Then, integrating (34), one obtains

$$\ln|\psi| = \widetilde{\boldsymbol{K}} + \gamma_{\mathbf{a}} \ln|z - z_{\mathbf{a}}| + \gamma_{\mathbf{b}} \ln|z - z_{\mathbf{b}}| \tag{36}$$

in which $\widetilde{\boldsymbol{K}}$ is an integration constant. Exponentiation yields

$$|\psi| = \frac{1}{\boldsymbol{K}} |z - z_{\mathbf{a}}|^{\gamma_{\mathbf{a}}} |z - z_{\mathbf{b}}|^{\gamma_{\mathbf{b}}} \tag{37}$$

with $\boldsymbol{K} = e^{-\widetilde{\boldsymbol{K}}}$. The latter can be rewritten in terms of $w(S)$ and S as follows:

$$\boldsymbol{K} = \left[w(S) - A - S_{\mathbf{a}}(S - B)\right]^{\gamma_{\mathbf{a}}} \cdot \left[w(S) - A - S_{\mathbf{b}}(S - B)\right]^{\gamma_{\mathbf{b}}}. \tag{38}$$

Accordingly,

Lemma 2 *For all $\sigma > V'(S)$, the solution to (23) is*

$$\boldsymbol{K} = \left[w(S) - A - S_{\mathbf{a}}(S - B)\right]^{\gamma_{\mathbf{a}}} \cdot \left[w(S) - A - S_{\mathbf{b}}(S - B)\right]^{\gamma_{\mathbf{b}}} =$$

$$\left[w(S) - \frac{\sigma n}{\delta(n+1)}\left(S - \frac{\left(n^2+1\right)\sigma}{\delta(n+1)^2}\right)\right]^{\frac{\delta}{2\delta-\rho}} \cdot \left[w(S) - \frac{\sigma\left(n^2+1\right)}{\delta(n+1)}\left(S - \frac{\left(n^2+1\right)\sigma}{\delta(n+1)^2}\right)\right]^{\frac{\delta-\rho}{2\delta-\rho}}$$

where $\boldsymbol{K}$ is a real constant of integration.

Changing the arbitrary value of $\boldsymbol{K}$ generates infinitely many nonlinear solutions. Figure 2 describes the evolution of state and control variables over time, enabling one to single out the properties of any nonlinear feedback solutions, including the very specific one generated by the tangency point with the locus $\dot{S} = 0$ (point T in the figure). Figure 2 (which is nothing but a more detailed version of Fig. 1) also portrays the loci $q'(S) = 0$ (along which $q_0(S) = q^{F1}(S) = q^{OL}$) and $q'(S) \to \infty$.

The arrows along the curve tangent to the locus $\dot{S} = 0$ in point T shows that the tangency solution is indeed unstable. However, there exist infinitely many solutions identified by the intersections along the segment delimited by points T and LF. This

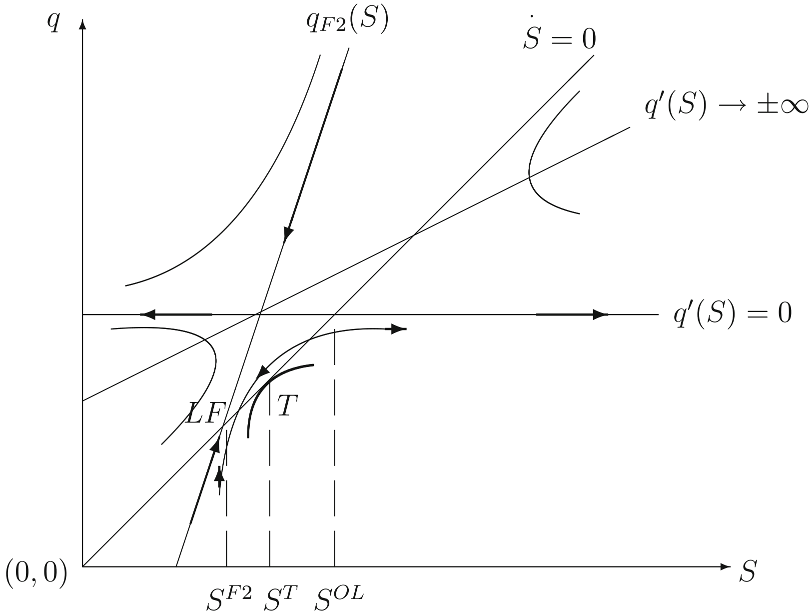

Fig. 2 Linear and nonlinear feedback solutions in the (S, q) space

set of *stable nonlinear solutions*, which can be labelled as *SNS*, is sensitive to the extent of delegation θ, which affects the loci $\dot{S} = 0$ and $q^{F2}(S)$, and therefore also the position of the tangency point T. The set *SNS* has the size of such a segment:

$$SNS = \sqrt{(S^T - S^{LF})^2 + (q^T - q^{LF})^2} \tag{39}$$

where $S^{LF} = S^{F2}$, the latter being defined in (18),

$$S^T = \frac{\sigma(\delta - \rho n)}{\delta[2\delta - n(1+\rho)]} = \frac{nq^T}{\delta} \tag{40}$$

and $q^{LF} = q^{F2}\left(S = S^{LF}\right)$, i.e.,

$$q^{LF} = \frac{\sigma\left[2\delta - \rho\left(n^2 + 1\right)\right]}{n(n+1)[2\delta - \rho(n+1)]} \tag{41}$$

Using these expressions, one obtains $SNS = \sigma\sqrt{\Phi(n, \delta, \rho)}$, with $\Phi(\cdot) > 0$. Consequently,

$$\frac{\partial SNS}{\partial \theta} = \sqrt{\Phi(n, \delta, \rho)} > 0 \tag{42}$$

by the definition of σ. This boils down to the following:

Proposition 3 *The separation between ownership and control via delegation contracts based on output expansion enlarges the set of stable nonlinear feedback solutions.*

In particular, since $\partial S^T / \partial \theta > 0$, the above proposition is accompanied by a relevant corollary:

Corollary 4 *The adoption of managerial incentives based on output expansion increases the upper bound of the SNS set.*

This result can be rephrased to say that these particular type of managerial incentives allows for a larger stock of the resource surviving in correspondence of a nonlinear feedback solution, and prompts for the analysis of the so-called *voracity effect* (Lane and Tornell 1996; Tornell and Lane 1999), which can be briefly summarised as follows. In line of principle, one would expect that the higher the resource growth rate is, the higher should be the volume of that resource in steady state. However, this may not hold true as firms respond to any increase in the growth rate by hastening resource extraction, whereby one observes that $\partial S/\partial \delta < 0$ in steady state, at least for sufficiently high levels of δ. The arising of such voracity effect has been highlighted, with pure profit-seeking units, in Benchekroun (2008) and Lambertini and Mantovani (2014). In particular, Lambertini and Mantovani (2014, p. 121) show that under linear feedback information the voracity effect operates, i.e., $\partial S^{LF}/\partial \delta < 0$, for all

$$\delta > \tilde{\delta} = \frac{r}{2}\left[n^2 + 1 + \sqrt{\left(n^2+1\right)\left(2+n\left(n+1\right)\right)}\right] \tag{43}$$

while under nonlinear feedback information the stock corresponding to the tangency solution exhibits the following property:

$$\frac{\partial S^T}{\partial \delta} < 0 \Leftrightarrow \delta > \hat{\delta} = r\left[n + \sqrt{\frac{n\left(3n+1\right)}{2}}\right] \tag{44}$$

with $\tilde{\delta} > \hat{\delta}$. These properties, combined with Lemma 1, entail

Proposition 5 *If delegation contracts allow for output expansion and $\delta > \hat{\delta}$, then managerial incentives soften the voracity effect over the entire interval of nonlinear feedback solutions SNS.*

It would be tempting to interpret this conclusion as implying a beneficial effect of managerialization on resource preservation. However, this would be hazardous as the same issue should indeed be reassessed in presence of alternative incentive schemes, based for instance on market shares (Jansen et al. 2007; Ritz 2008) or comparative performance evaluation (Salas Fumas 1992; Miller and Pazgal 2001). Yet, the possibility that delegating control to agents interested in expanding production might ultimately mitigate the pressure on the resource is a striking and unexpected feature of the present model. This fact finds its explanation in the multiplicative effect of this form of delegation on equilibrium outputs and the resource stock, as the delegation parameter θ appears in market size σ and makes it larger as seen from the managers' standpoint. Since σ is a at the same time a measure of profitability or demand level, this type of delegation (1) increases the maximum mark-up from $a - c$ to $a - c + \theta$ or equivalently (2) shifts the demand upwards by θ. Consequently, the managerial inclination to expanding output is routed in the direction of affecting the mark-up level and this mechanism operates as a partial remedy to voracity in the range where the latter takes place. Therefore, albeit with some caution, this design of delegation contracts—admittedly, far from being general—is of public interest because it couples the usual elements connected with consumer surplus and profits with additional motives (perhaps more far-reaching) dealing with the impact of the separation between ownership and control on resource (and species) preservation.

5 Concluding Remarks

I have investigated the implications of the separation between ownership and control in a differential oligopoly game where firms exploit a renewable resource over an infinite horizon. The analysis have been carried out assuming that delegation contracts are based on sales expansion *à la* Vickers (1985).

The foregoing analysis has shown that, while under open-loop strategies (yielding a degenerate feedback equilibrium) the final outcome depends on initial conditions, under feedback rules delegation favours the preservation of the resource stock. More specifically, the size of the set of stable nonlinear feedback solution is increasing in the extent of delegation. Moreover, the use of incentives explicitly allowing for the presence of output in the instantaneous managerial objective function mitigates the voracity effect appearing whenever the reproduction rate of the resource is high enough.

The robustness of these results to a change in managerial incentive schemes, which could instead rely on market shares or comparative profit performance, is left for future research.

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Optimal Growth with Polluting Waste and Recycling

Raouf Boucekkine and Fouad El Ouardighi

Abstract We study an optimal AK-like model of capital accumulation and growth in the presence of a negative environmental externality in the tradition of Stokey (Int Econ Rev 39(1):1–31, 1998). Both production and consumption activities generate polluting waste. The economy exerts a recycling effort to reduce the stock of waste. Recycling also generates income, which is fully devoted to capital accumulation. The whole problem amounts to choosing the optimal control paths for consumption and recycling to maximize a social welfare function that notably includes the waste stock and disutility from the recycling effort. We provide a mathematical analysis of both the asymptotic behavior of the optimal trajectories and the shape of transition dynamics. Numerical exercises are performed to illustrate the analysis and to highlight some of the economic implications of the model. The results suggest that when recycling acts as an income generator, (1) a contraction of both the consumption and capital stock is observed in the long run after an expansion phase; (2) whether polluting waste is predominantly due to production or consumption, greater consumption and lower capital stock are obtained in the long run compared with the situation when recycling does not create additional income; (3) greater recycling effort and lower stock of waste are resulted in the long run.

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1 Introduction

Since the Meadows report (1972), economic research on the limits of the contemporaneous growth regime and the design of optimal sustainable policies has been central to many agendas. While the very first attempts along these investigation lines have been directed at the issue of non-renewable resources (see for example, Stiglitz 1974, or Hartwick 1977), substantial efforts have been devoted to studying the impact of pollution on growth and social welfare, especially since the '90s. A fundamental theoretical contribution to this topic is that of Stokey (1998). Within various optimal growth settings, Stokey studied the implications of pollution externalities for optimal capital accumulation (and therefore for optimal growth). A remarkable outcome of this study is the analysis of the AK economy with and without pollution externalities. In the latter case, the economy optimally starts on exponentially growing balanced growth paths with strictly positive growth rates. When pollution produces negative welfare losses, the economy no longer follows these virtuous paths; rather, it converges to optimal steady states (therefore, with zero growth). In other words, pollution drastically limits (optimal) growth. Several authors have extended Stokey (1998)'s framework to account for more technological or ecological ingredients; Boucekkine et al. (2013) is one of the most recent extensions. When pollution is irreversible (that is, when the environmental absorption capacity irreversibly declines above a certain pollution stock level), these authors show that the optimal relationship between income and pollution can take a much richer set of forms, although optimal exponential growth invariably vanishes under pollution externalities.

This paper incorporates polluting waste and recycling into this type of optimal relationship analysis. More precisely, we are interested in polluting waste for which natural absorption takes extremely long. A characteristic example of such pollution is that of plastic waste, that is, bags, bottles, etc., which can be absorbed by the environment only after four to six centuries. Because of this extremely long time, no natural amenities can be reasonably assumed, and recycling efforts are required to avoid massive accumulation with harmful consequences in the long run. In the case of plastic waste, it seems that insufficient efforts have been deployed to prevent such a scenario, as evidenced by the emergence of several plastic vortexes in the oceans (Kaiser 2010). Jambeck et al. (2015) calculated that 275 million metric tons of plastic was generated in 192 coastal countries in 2010, with 4.8–12.7 million metric tons entering the ocean. In spite of the significant development of recycling and energy recovery activities, post-consumer plastic waste predominantly goes to landfill (PlasticsEurope 2015). Jambeck et al. (2015) consider that without waste management infrastructure improvements, the cumulative quantity of plastic waste available to enter the ocean from land is predicted to reach 80 million metric tons by 2025.

Several authors have already modeled waste generation and recycling activities within economic frameworks. Of course, these models are largely found in the industrial organization literature (see for example, Martin 1982, or Grant 1999).

Nonetheless, waste and recycling are increasingly being examined from a more macroeconomic perspective. The recent macroeconomic literature contains studies of the impact of recycling on aggregate fluctuations (see De Beir et al. 2010, for example). There have also been several attempts to incorporate waste and recycling in a sustainability analysis, as we are doing in our paper, most often in decentralized equilibrium frameworks with technological progress (see for example, the recent paper by Fagnart and Germain 2011).

Below we provide a first-best central planner analysis in line with Stokey's seminal paper incorporating waste (both from consumption and production) and recycling. Perhaps the work that is most closely related to ours is that of Lusky (1976). Lusky (1976) also solved a central planner problem regarding waste and recycling. However, our problem differs from his along three essential dimensions. First, Lusky (1976) has strictly concave production functions with labor as a unique input, whereas capital accumulation is an essential feature of our model and we use an AK production technology as in Stokey (1998). Second, recycling is modeled differently in Lusky (1976): recycling produces a consumable good (so it increases consumption possibilities), while in our model recycling output goes to capital accumulation given that our focus is sustainable growth via capital accumulation, to which recycling contributes. Third, the social welfare functions are not the same. In particular, while in both papers waste produces negative externalities, recycling increases instantaneous utility in Lusky (1976) via the recycled consumption good whereas the recycling effort supposes a strictly concave welfare loss in our set-up.

The problem we consider is an infinite time horizon problem with two control variables (consumption and recycling effort) and two states (capital and stock of waste). The state equations are linear mainly due to the linearity of the production function and of waste generation processes. Using a version of the maximum principle, we can extract the (necessary and sufficient) optimality conditions, and study the asymptotic properties of the optimal paths. The economic implications of this analysis are then evaluated in light of the sustainability literature à la Stokey.

Our results suggest that when recycling acts as an income generator, (1) a contraction of both the consumption and capital stock is observed in the long run after an expansion phase; (2) whether polluting waste is predominantly due to production or consumption, greater consumption and lower capital stock are obtained in the long run compared with the situation when recycling does not create additional income; (3) greater recycling effort and lower stock of waste are resulted in the long run.

The paper is organized as follows. Section 2 gives the specifications of our central planner problem with waste and recycling. Section 3 analyzes the mathematical properties of the model. In Section 4, we perform some quantitative exercises and we bring out the main lessons we can draw from the point of view of sustainability. Section 5 concludes the paper.

2 Model

Assume that the social planner owns a stock of productive capital that provides a continuous flow of revenue, $aK(t)$, where $K(t) \geq 0$ denotes the capital stock at time t, and $a > 0$, the constant marginal unit of revenue generated by the productive capital stock. The production function is linear in the stock of capital, as in Stokey (1998) and Boucekkine et al. (2013). The flow of revenue allows for a certain consumption level, $c(t) > 0$. The revenue generation process and the consumption decisions are both assumed to generate polluting waste, $w(t) \geq 0$, where $w(t) = \alpha a K(t) + \beta c(t)$, $\alpha, \beta > 0$ being the marginal wasting impact of the revenue generation process and the current consumption, respectively.

Though most of the polluting waste is generally related to productive processes (Klassen 2001), it is not clear whether the consumption process generates more waste than the revenue generation process. In the case of the plastics industry, polluting waste is predominant in either process depending on the market segment and the polymer type (PlasticsEurope 2015). In this regard, we assume that $\alpha \gtreqless \beta$. By construction (no capital accumulation), waste is generated only by consumption in Lusky (1976). The same is assumed in the macroeconomic literature in line with De Beir et al. (2010), which relies heavily on the related industrial organization (e.g., Martin 1982): waste produced from consumption is used one period ahead as an input in the recycling sector. In this paper, waste can come from both consumption and production (or capital utilization), and some of the essential properties of optimal paths may depend on whether $\alpha \gtreqless \beta$.

To reduce the stock of polluting waste, denoted by $W(t) \geq 0$, the social planner may invest in recycling efforts $v(t) \geq 0$ over time. We assume that the waste generating processes and recycling operations are mutually independent so that the recycling efforts are non-proportional to the waste emissions (e.g., El Ouardighi et al. 2015). This assumption allows for unbounded recycling efforts, i.e., $v(t) \lesseqgtr w(t)$, to account for the possibility of reduction of past waste emissions. The environmental absorption capacity of polluting waste is approximated by zero. This approximation is consistent with the extremely long time needed for natural absorption of plastic waste. In economic terms, it implies that the social planner cannot benefit from any natural abatement of pollution waste.[1]

Finally, a fixed proportion of recycled waste is supposed to generate additional revenues, $\varphi v(t)$, and therefore to positively influence the capital accumulation process, $1 \geq \varphi \geq 0$ being the marginal proportion of the recycled waste that adds to capital accumulation. An illustration of this assumption is related to the plastics industry, where 60 million tons of plastics diverted from landfills are equivalent to over 60 billion euros (PlasticsEurope 2015).

[1]This assumption is optimistic because accumulation of polluting waste can lead to negative environmental absorption capacity that might create additional negative externalities (see El Ouardighi et al. 2014).

In more elaborated industrial organization models of recycling, the recycling sector may produce profits (as in Martin 1982), which are later redistributed to the owners. In our central planner setting, the idea is pretty much the same: recycling not only decreases the level of polluting waste, but it also generates an income, which contributes to capital accumulation. Both functions of recycling help alleviate the sustainability problem faced by the economy. Only abatement plays this role in Stokey (1998) for example. If $\varphi = 0$, the income generation channel of recycling is shut down, and we are closer to Stokey's framework regarding the impact of pollution control instruments.

Based on these assumptions, the endogenous capital accumulation process is described as follows:

$$\dot{K} = aK(t) - c(t) + \varphi v(t), \quad K(0) = K_0 > 0$$

where a positive difference between the total revenues from capital and recycled waste, and current consumption results in investment in productive capital, while a negative difference leads to disinvestment. The initial endowment in productive capital is given by $K_0 > 0$.

The dynamics of polluting waste are given by:

$$\dot{W} = \alpha a K(t) + \beta c(t) - v(t), \quad W(0) = W_0 > 0$$

where the recycling efforts are such that $\dot{W} \gtreqless 0$, $\forall t > 0$, for a given initial stock of waste, $W_0 > 0$.

Regarding the objective function of the social planner, we make the following assumptions. At each period, the instantaneous social utility is given as the difference between the utility drawn from current consumption and the costs incurred from the stock of waste and the recycling efforts, respectively. The instantaneous utility from current consumption is a concave function, that is, $\ln c(t)$. In addition, the stock of waste entails negative externalities such as environmental pollution and destruction of the biomass (e.g., Barnes 2002). These negative externalities are valued as an increasing convex function of the stock of waste, that is, $eW(t)^2/2$, $e > 0$. Lastly, the recycling effort generates an increasing quadratic cost, denoted by $fv(t)^2/2, f > 0$. Without loss of generality, we set $f = 1$.

Denoting the discounting rate by $r > 0$, and assuming an infinite planning horizon, the social planner's optimal control problem is:

$$U = \int_0^\infty e^{-rt} \left(\ln c(t) - \frac{eW(t)^2}{2} - \frac{v(t)^2}{2} \right) \mathrm{d}t \quad (1)$$

subject to:

$$\dot{K} = aK(t) - c(t) + \varphi v(t), \quad K(0) = K_0 > 0 \tag{2}$$

$$\dot{W} = \alpha aK(t) + \beta c(t) - v(t), \quad W(0) = W_0 > 0 \tag{3}$$

As mentioned in the introduction, our social welfare function differs from Lusky (1976)'s in that it includes a welfare loss due to the recycling effort and has no additional (recycled) consumption term. Note that we consider a pollution (via aggregate waste) negative externality while the macroeconomic literature of recycling and fluctuations does not (see for example De Beir et al. 2010). The same comparison holds with the ecological sustainability literature (see Fagnart and Germain 2011).

We now come to the mathematical resolution of the optimal control problem considered. We limit our presentation to the case of an interior solution.

3 Analysis

Skipping the time index for convenience, the current-value Hamiltonian is:

$$H = \ln c - \frac{eW^2}{2} - \frac{v^2}{2} + \lambda\,(aK - c + \varphi v) + \mu\,(\alpha aK + \beta c - v) \tag{4}$$

where $\lambda \equiv \lambda(t)$ and $\mu \equiv \mu(t)$ are costate variables, $j = 1, 2$, that evolve according to:

$$\dot{\lambda} = (r - a)\,\lambda - \mu\alpha a \tag{5}$$

$$\dot{\mu} = r\mu + eW \tag{6}$$

Necessary conditions for optimality are:

$$H_c = \frac{1}{c} - \lambda + \mu\beta = 0 \Rightarrow c = \frac{1}{\lambda - \mu\beta} \tag{7}$$

$$H_v = -v + \lambda\varphi - \mu = 0 \Rightarrow v = \lambda\varphi - \mu \tag{8}$$

Because the stock of polluting waste has a negative marginal influence on the social planner's objective function, its implicit price should be non-positive, i.e., $\mu \leq 0$. Along with $\lambda > 0$, this should result in strictly positive consumption and recycling effort respectively in (7) and (8).

The Legendre-Clebsch condition of concavity of the Hamiltonian with respect to the control variables is satisfied, as the Hessian:

$$\begin{bmatrix} H_{cc} & H_{cv} \\ H_{vc} & H_{vv} \end{bmatrix} = \begin{bmatrix} -c^{-2} & 0 \\ 0 & -1 \end{bmatrix}$$

is negative definite. This guarantees a maximum of the Hamiltonian.

Lemma 1 *The necessary conditions are sufficient for optimality.*

Plugging the respective expressions of c *and* v *from* (7) *and* (8) *in* (4) *results in the maximized Hamiltonian:*

$$H^0 = \ln\left(\frac{1}{\lambda - \mu\beta}\right) - 1 + \frac{(\lambda\varphi - \mu)^2}{2} + (\lambda + \mu\alpha)\, aK - \frac{eW^2}{2}$$

from which the Hessian matrix:

$$\begin{bmatrix} H_{KK} & H_{KW} \\ H_{WK} & H_{WW} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -e \end{bmatrix}$$

is negative semi-definite. This ensures that the necessary conditions are also sufficient for optimality. □

Plugging the value of c and v from (7) and (8) in (2) and (3), respectively, the equations:

$$\dot{K} = aK - \frac{1}{\lambda - \mu\beta} + \varphi(\lambda\varphi - \mu) \tag{9}$$

$$\dot{W} = \alpha aK + \frac{\beta}{\lambda - \mu\beta} - \lambda\varphi + \mu \tag{10}$$

along with (5) and (6), form the canonical system in the state-costate space.

We now prove that as in Stokey (1998), the dynamic system will not converge to a balanced growth path despite the assumed impacts of recycling on the stock of pollution and on income generation. We first show the existence of steady states and then assess stability.

Proposition 1 *In the case of a patient social planner, i.e., $r < a$, and $\beta\varphi < 1$, the steady state is unique and given by:*

$$
\begin{aligned}
\left(K^S \quad W^S \quad c^S \quad v^S\right)^T = \Bigg(& \frac{(1-\beta\varphi)\sqrt{(1+\varphi\alpha)\,a-r}}{a\sqrt{(1+\alpha\varphi)\,(\alpha+\beta)\,\Phi}} \\
& \times \frac{r\,(a-r)\sqrt{\alpha+\beta}}{e\sqrt{(1+\varphi\alpha)\,[(1+\varphi\alpha)\,a-r]\,\Phi}} \\
& \times \sqrt{\frac{(1+\varphi\alpha)\,[(1+\varphi\alpha)\,a-r]}{(\alpha+\beta)\,\Phi}} \\
& \times \sqrt{\frac{(\alpha+\beta)\,[(1+\varphi\alpha)\,a-r]}{(1+\varphi\alpha)\,\Phi}} \Bigg)
\end{aligned} \tag{11}
$$

where $\Phi = [a\,(\alpha+\beta) - r\beta]$, and the superscript '$S$' stands for steady state.

Proof Equating the RHS of (9)-(10)-(5)-(6) to 0 and solving by identification and substitution, we get:

$$
\lambda^S = \frac{\alpha a\sqrt{\alpha+\beta}}{\sqrt{(1+\varphi\alpha)\,[(1+\varphi\alpha)\,a-r]\,\Phi}}
$$

$$
\mu^S = -\frac{(a-r)\sqrt{\alpha+\beta}}{\sqrt{(1+\varphi\alpha)\,[(1+\varphi\alpha)\,a-r]\,\Phi}}
$$

and K^S and W^S as given in (11). Note that $r < a$ implies that $r < (1+\varphi\alpha)\,a < (\alpha+\beta)\,a/\beta$, which allows for a feasible steady state stock of waste. Conversely, for any $r > a$, the steady state stock of waste is not feasible. Plugging the above expressions in (7) and (8), respectively, and simplifying, yields c^S and v^S in (11).

From (11), it can be shown that the limiting transversality conditions are satisfied for the saddle-paths because:

$$
\lim_{t\mapsto+\infty} e^{-rt}\lambda(t)K(t) = \lim_{t\mapsto+\infty}\left[\frac{(1-\beta\varphi)\,\alpha e^{-rt}}{(1+\varphi\alpha)\,\Phi}\right] = 0
$$

$$
\lim_{t\mapsto+\infty} e^{-rt}\mu(t)W(t) = \lim_{t\mapsto+\infty}\left\{-\frac{r\,(\alpha+\beta)\,(a-r)^2 e^{-rt}}{e\,(1+\varphi\alpha)\,[(1+\varphi\alpha)\,a-r]\,\Phi}\right\} = 0
$$

This ensures the uniqueness of the globally optimal solution. □

In the case where $r < a$, and $\beta\varphi < 1$, a zero steady state is obtained which is not optimal. The reason is the following. To be steady, zero waste and capital require zero recycling and consumption, which with respect to costate Eqs. (5) and (6), implies that the corresponding costate variables must be zero as well. Substituting zero costates into the optimality condition (7), we find that the optimal consumption tends to infinity rather than to zero. Consequently, a zero waste and capital is steady but not optimal. This indeed does not affect the results as they are derived for a non-zero steady state.

From the canonical system (5)-(6)-(9)-(10), the isoclines of $\dot{K} = 0$ and $\dot{W} = 0$ are given by:

$$K^S_{|\dot{K}=0} = \frac{r(a-r)}{a\Phi eW^S} - \frac{\varphi\left[(1+\alpha\varphi)a - r\right]eW^S}{ra(a-r)} \tag{12}$$

$$K^S_{|\dot{W}=0} = \frac{1}{\alpha a}\left[\frac{\left[(1+\alpha\varphi)a - r\right]eW^S}{r(a-r)} - \frac{r\beta(a-r)}{\Phi eW^S}\right] \tag{13}$$

Using (12)–(13) and (11), Fig. 1 describes the sensitivity of the state variables at the steady state to the parameters.

From Fig. 1, the marginal wasting impact of both the revenue generation process (α) and the consumption process (β) on the capital stock is negative. In contrast, the influence of these processes on the stock of waste is different because an increase in the marginal wasting impact of the revenue generation (consumption) process results in less (more) waste stock. In contrast, an increase in the marginal impact of the recycling effort on the capital accumulation process (φ) reduces both the capital stock and the waste stock. Note that the influence of the discounting rate and the marginal revenue coefficient on the stock of capital and of waste is similar to that of the marginal polluting impact of revenue generation (α) and the marginal impact of the recycling effort on the capital accumulation process (φ). The marginal revenue coefficient (a) has a similar impact on the stock of capital and the stock of waste as the marginal wasting impact of the consumption process (β). Finally, a greater

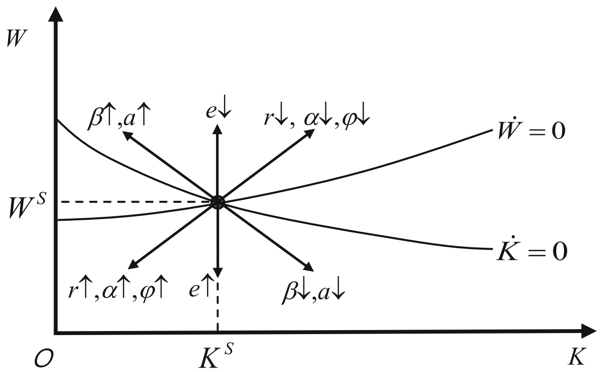

Fig. 1 Sensitivity of the steady state to the parameter values

cost coefficient of the waste stock (e) lowers the waste stock and does not affect the capital stock, and *vice-versa*.

We now move to the stability analysis and investigate the structure of the associated stable manifolds.

Proposition 2 *The steady state exhibits a (local) two-dimensional stable manifold if the social planner is moderately patient (i.e.,* $r < (1+\varphi\alpha)\,a$*), and a one-dimensional stable manifold otherwise.*

Proof To analyze the stability of the steady state, we compute the Jacobian matrix of the canonical system (9)-(10)-(5)-(6), that is:

$$J = \begin{bmatrix} a & 0 & \frac{1}{(\lambda-\beta\mu)^2} + \varphi^2 & -\frac{\beta}{(\lambda-\beta\mu)^2} - \varphi \\ \alpha a & 0 & -\frac{\beta}{(\lambda-\beta\mu)^2} - \varphi & \frac{\beta^2}{(\lambda-\beta\mu)^2} + 1 \\ 0 & 0 & r-a & -\alpha a \\ 0 & e & 0 & r \end{bmatrix}$$

Given that λ and μ are evaluated at their steady state value, we compute the determinant:

$$\begin{aligned} |J| &= ae\left\{(1+\varphi\alpha)\left[(1+\varphi\alpha)\,a - r\right] + \frac{(\alpha+\beta)\,\Phi}{(\lambda-\beta\mu)^2}\right\} \\ &= 2ae\,(1+\varphi\alpha)\left[(1+\varphi\alpha)\,a - r\right] \end{aligned}$$

which has a positive value for a moderately patient social planner ($r < (1+\varphi\alpha)\,a$), and negative otherwise. As shown in Dockner and Feichtinger (1991), a negative determinant is a necessary and sufficient condition for the Jacobian matrix to have one negative eigenvalue and either three positive eigenvalues or one positive eigenvalue and two with positive real parts. In terms of dynamic behavior, this corresponds to the case of a one-dimensional stable manifold.

In the case of a moderately patient social planner ($r < (1+\varphi\alpha)\,a$), we use Dockner's formula (Dockner 1985) to determine the sum of the principal minors of J of order 2 minus the squared discounting rate, that is:

$$\begin{aligned} \Psi &= -a\,(a-r) - e\left[\frac{\beta^2}{(\lambda-\beta\mu)^2} + 1\right] \\ &= -a\,(a-r) - e\left[\frac{\beta^2\,(1+\varphi\alpha)\left[(1+\varphi\alpha)\,a - r\right]}{(\alpha+\beta)\,\Phi} + 1\right] \end{aligned}$$

The necessary and sufficient conditions that ensure that two eigenvalues have negative real parts and two have positive real parts, which corresponds to the case of a two-dimensional stable manifold, are $|J| > 0$ and $\Psi < 0$. The sign of Ψ is negative, which implies that a two-dimensional stable manifold (saddle-point) exists in the case of a moderately patient social planner. □

According to Proposition 2, if the social planner is relatively impatient, the zero steady state cannot be reached from some or all initial states, which confirms that it is not optimal. Conversely, if the social planner is moderately patient, the positive saddle-point exists that can be reached. We may dig deeper in the analysis and draw further properties of the optimal paths, in particular about the existence of oscillatory transitions to the steady states. This gives a rough idea of the ability of the model to generate non-monotonic optimal trajectories and fluctuations, a central issue in the recent recycling-related macroeconomic literature (De Beir et al. 2010).

Proposition 3 *Assuming a patient social planner (i.e., $r < a$), for any given a, e, and $\beta\varphi < 1$, there exists a threshold $\tilde{\alpha} > 0$ such that for any $\alpha > \tilde{\alpha}$, the convergence to the steady state is oscillatory.*

Proof To determine whether the optimal path is monotonic or follows cyclical motions, we compute the expression (Dockner 1985):

$$\Omega = \Psi^2 - 4\,|J| = \left\{a\,(a-r) + e\left[\frac{\beta^2\,(1+\varphi\alpha)\,[(1+\varphi\alpha)\,a-r]}{(\alpha+\beta)\,\Phi} + 1\right]\right\}^2 -8ae\,(1+\varphi\alpha)\,[(1+\varphi\alpha)\,a-r]$$

A positive (negative) sign of Ω indicates that convergence to the saddle-point is monotonic (spiraling) near the steady state. Because the sign of Ω is ambiguous, a limit value analysis highlights the role played by a, e, φ, α and β in the sign of Ω for a given $r < a$ (Table 1).

The results suggest that Ω is generally positive, which implies that convergence to the saddle-point is monotonic near the steady state in general. However, given that $\lim_{\alpha\mapsto 0^+} \Omega > 0$ and $\lim_{\alpha\mapsto\infty} \Omega = -\infty$, it can be shown that $\partial\Omega/\partial\alpha < 0$.

Therefore, we conclude that there exists a threshold $\tilde{\alpha} > 0$ such that for any $\alpha > \tilde{\alpha}$, we have $\Omega < 0$. □

According to Proposition 3, if the social planner is patient, convergence to the locally stable steady state is either monotonic or oscillatory, depending on the magnitude of the wasting impact of the revenue generation process. That is, if the wasting impact of the revenue generation process is excessively high, the optimal

Table 1 Limit value analysis

$\lim_{a\mapsto 0^+} \Omega = \frac{e^2[\alpha(1+\beta\varphi)+2\beta]^2}{(\alpha+\beta)^2}$	$\lim_{a\mapsto\infty} \Omega = \infty$
$\lim_{e\mapsto 0^+} \Omega = a^2(a-r)^2$	$\lim_{e\mapsto\infty} \Omega = \infty$
$\lim_{\varphi\mapsto 0^+} \Omega = a^2(a-r)^2$	$\lim_{\varphi\mapsto(1/\beta)^-} \Omega = \infty$
$\lim_{\alpha\mapsto 0^+} \Omega = [a\,(a-r)-2e]^2$	$\lim_{\alpha\mapsto\infty} \Omega = -\infty$
$\lim_{\beta\mapsto 0^+} \Omega = \lim_{\beta\mapsto(1/\varphi)^-} \Omega = a^2(a-r)^2 - 8\alpha\varphi ae\,[(2+\alpha\varphi)\,a-r] - e\,[6a\,(a-r)-e]$	

policy follows a spiralling path which has the effect of reducing both the long run stock of capital and stock of waste, as suggested in Fig. 1. Therefore, a monotonic convergence is less detrimental to the capital stock but also more detrimental to the environment than a spiralling convergence.

Proposition 4 *The saddle paths of the control and state variables in the neighborhood of the steady state are:*

$$c(t) = \frac{1}{\lambda^S - \beta\mu^S + (B_1 - \beta B_3)\, e^{\chi_1 t} + (B_2 - \beta B_4)\, e^{\chi_3 t}} \tag{14}$$

$$v(t) = \varphi\lambda^S - \mu^S + (\varphi B_1 - B_3)\, e^{\chi_1 t} + (\varphi B_2 - B_4)\, e^{\chi_3 t} \tag{15}$$

$$K(t) = K^S + B_5 e^{\chi_1 t} + B_6 e^{\chi_3 t} \tag{16}$$

$$W(t) = W^S + B_7 e^{\chi_1 t} + B_8 e^{\chi_3 t} \tag{17}$$

where $B_1, \ldots, B_8$ *are constants of integration and* $\chi_1, \chi_3 < 0$.

Proof The linear approximation of the system (5)-(6)-(9)-(10) around the steady state is:

$$\dot{\lambda} = (r - a)\,\lambda - \mu\alpha a$$

$$\dot{\mu} = r\mu + eW$$

$$\dot{K} = aK + \frac{\lambda - \beta\mu - 2\,(\lambda - \beta\mu)}{(\lambda - \beta\mu)^2} + \varphi(\lambda\varphi - \mu)$$

$$\dot{W} = \alpha a K - \frac{\beta\,[\lambda - \beta\mu - 2\,(\lambda - \beta\mu)]}{(\lambda - \beta\mu)^2} - (\lambda\varphi - \mu)$$

Using Dockner's formula (1985), the four eigenvalues associated with the Jacobian matrix of the canonical system are:

$$\begin{aligned} {}_1^{\,3}\chi_2^4 &= \frac{r}{2} \pm \sqrt{\frac{r^2}{4} - \frac{\Psi}{2} \pm \frac{1}{2}\sqrt{\Psi^2 - 4\,|J|}} \\ &= \frac{r}{2} \pm \sqrt{\frac{r^2}{4} + \frac{1}{2}\left\{a\,(a - r) + e\left[\frac{\beta^2\,(1 + \varphi\alpha)\,[(1 + \varphi\alpha)\,a - r]}{(\alpha + \beta)\,\Phi} + 1\right]\right\} \pm \frac{\sqrt{\Omega}}{2}} \end{aligned}$$

and λ^S and μ^S are given in the proof of Proposition 1. As expected, two eigenvalues, χ_1 and χ_3, have a negative sign and two have a positive sign, χ_2 and χ_4. Choosing the negative roots for convergence, the time paths of the costate and state variables are written as:

$$\lambda(t) = \lambda^S + B_1 e^{\chi_1 t} + B_2 e^{\chi_3 t}$$

$$\mu(t) = \mu^S + B_3 e^{\chi_1 t} + B_4 e^{\chi_3 t}$$

$$K(t) = K^S + B_5 e^{\chi_1 t} + B_6 e^{\chi_3 t}$$

$$W(t) = W^S + B_7 e^{\chi_1 t} + B_8 e^{\chi_3 t}$$

These equations involve 10 unknowns (i.e., $B_1, \ldots, B_8, \lambda(0), \mu(0)$) that can be solved with the 10 following equations, which are drawn from the above expressions and the linearized versions of (5)-(6)-(9)-(10):

$$\lambda(0) = \lambda^S + B_1 + B_2$$

$$\mu(0) = \mu^S + B_3 + B_4$$

$$K_0 = K^S + B_5 + B_6$$

$$W_0 = W^S + B_7 + B_8$$

$$(r-a)\,\lambda^S + (r-a-\chi_1)\,B_1 + (r-a-\chi_3)\,B_2 - \alpha a\left(\mu^S + B_3 + B_4\right) = 0$$

$$r\mu^S + (r-\chi_1)\,B_3 + (r-\chi_3)\,B_4 + e\left(W^S + B_7 + B_8\right) = 0$$

$$aK^S - \frac{\lambda^S - \beta\mu^S - B_1 - B_2 + \beta\,(B_3 + B_4)}{(\lambda^S - \beta\mu^S)^2} + (a-\chi_1)\,B_5 + (a-\chi_3)\,B_6$$
$$+\varphi\left[\varphi\left(\lambda^S + B_1 + B_2\right) - \mu^S - B_3 - B_4\right] = 0$$

$$\alpha a\left(K^S + B_5 + B_6\right) + \frac{\beta\left[\lambda^S - \beta\mu^S - B_1 - B_2 + \beta\,(B_3 + B_4)\right]}{(\lambda^S - \beta\mu^S)^2}$$
$$-\varphi\left(\lambda^S + B_1 + B_2\right) + \mu^S + B_3 + B_4 - \chi_1 B_7 - \chi_3 B_8 = 0$$

$$(r-a)^2\lambda^S + \left[(r-a)^2 - \chi_1^2\right]B_1 + \left[(r-a)^2 - \chi_3^2\right]B_2$$
$$+\alpha a\left[(a-2r)\left(\mu^S + B_3 + B_4\right) - e\left(W^S + B_7 + B_8\right)\right] = 0$$

$$r^2\mu^S + \left(r^2 - {\chi_1}^2\right)B_3 + \left(r^2 - {\chi_3}^2\right)B_4 + e\left[rW^S + (r+\chi_1)\,B_7 + (r+\chi_3)\,B_8\right] = 0$$

This system of equations can be solved numerically. Finally, using (7), and (8) yields (14), (15), (16), and (17).

4 Numerical Example

We now give a numerical example to suggest the economic insight that could be gained from our model. To this end, we use the following parameter values (Table 2).

The parameter values in Table 2 reflect a configuration characterized by players' relative patience with a discounting rate, r, similar to the market interest in normal time (that is, 5 %), and lower than the marginal revenue from the productive capital stock, a, which is set at an intermediate value (that is, 10 %). By varying parameters α, β and φ, we show that various structurally different solutions are possible. The numerical solutions were computed with *Maple* 18.0.

The induced steady state values are reported in the following table (Table 3).

The main relationship to be investigated is that between the optimal stocks of capital and pollution to uncover a possible environmental Kuznets curve as in Stokey (1998) or Boucekkine et al. (2013). Regarding this relationship, the results of our experiments are reported in Fig. 2 below. Several remarkable features can be deduced from these figures.

The first result is that the relationship between capital (or income) and pollution is non-monotonic, if recycling generates additional income (that is $\varphi > 0$). This is true when production is more wasteful than consumption (Fig. 2a) and in the opposite case (Fig. 2). In both cases, we observe that the stock of pollution decreases while capital rises initially, but in the last stage of convergence to the steady state both stocks go down. In contrast, also in both cases, the relationship is permanently monotonic if recycling does not generate income (that is, $\varphi = 0$): the pollution stock decreases to its steady state value whereas capital increases to its corresponding stationary value. No turning point is observed in such cases.

Table 2 Parameter values

r	a	α	β	φ	e	K_0	W_0
0.05	0.1	(0.5, 0.8)	(0.5, 0.8)	(0, 0.1)	1	5	5

Table 3 State and control steady state values under various configurations

	Production more wasteful than consumption $\alpha = 0.8, \beta = 0.5$		Consumption more wasteful than production $\alpha = 0.5, \beta = 0.8$	
	Capital-improving recycling $\varphi = 0.1$	Capital-neutral recycling $\varphi = 0$	Capital-improving recycling $\varphi = 0.1$	Capital-neutral recycling $\varphi = 0$
K^S	5.95880809	6.052275326	6.155758759	6.537204503
W^S	0.03514723465	0.039333978962	0.0395379762	0.04249182928
c^S	0.6774223934	0.6052275326	0.7025594235	0.8698354766
v^S	0.815415844	0.7867957925	0.8698354766	0.8498365856

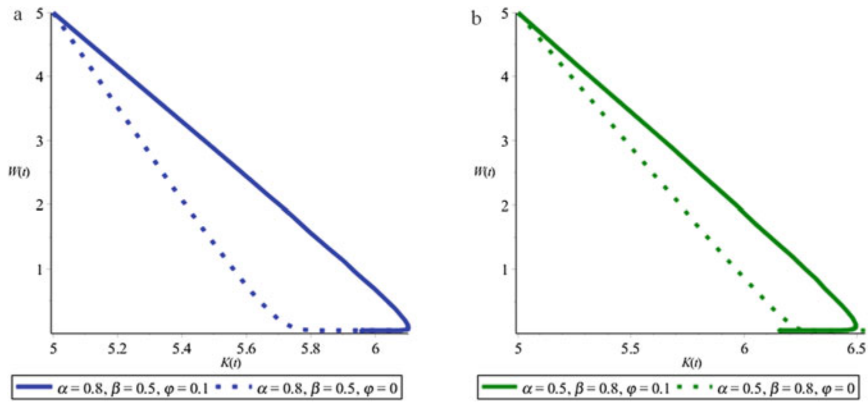

Fig. 2 Phase-planes of stock of capital and stock of waste. (**a**) Production more wasteful than consumption; (**b**) Consumption more wasteful than production

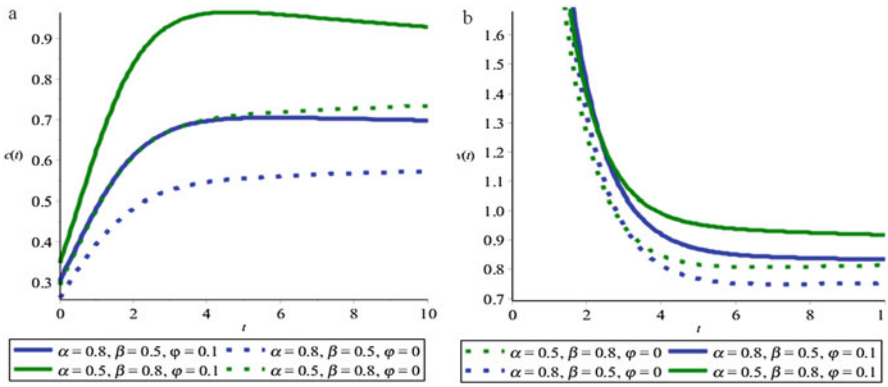

Fig. 3 Time paths of control variables (logarithmic time scale). (**a**) Consumption; (**b**) Recycling effort

It is worth pointing out two interesting connected outcomes at this stage. First, the kind of non-monotonicity we get is not of the environmental Kuznets curve kind, which is therefore not the rule in our optimal recycling model. Second, this non-monotonicity only arises when recycling generates additional income. To understand these remarkable properties, it is useful to have a look at the optimal control time paths. They are reported in Fig. 3.

In all the parametric configurations, optimal consumption starts at a relatively low level, in contrast to recycling. We are therefore in a situation where both controls act as substitutes. In our calibrated economy, priority is given to pollution control in the short run to decrease the stock of polluting waste as quickly as possible. In the transition to the steady states, consumption increases steadily to its corresponding stationary value when recycling has no additional income ($\varphi = 0.1$) or rises then decreases to the steady state value when recycling does generate income ($\varphi > 0$).

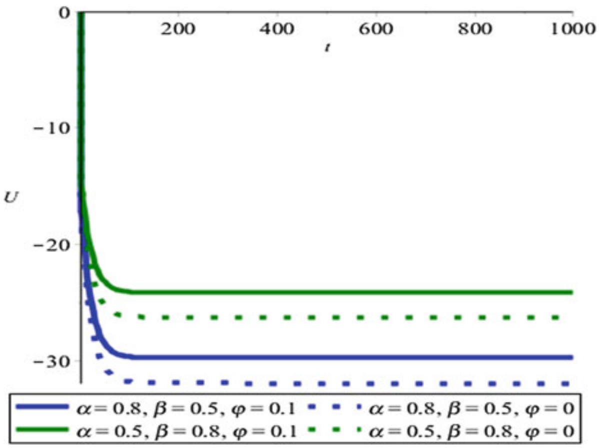

Fig. 4 Overall social utility

Whatever the parameterization, recycling effort always starts at a high value and follows an unambiguous monotonically decreasing path to its steady state value.

With these elements in hand, one can rationalize the optimal relationship between income and pollution we have uncovered. When recycling generates income, capital increases markedly in the initial stage of transitional dynamics because recycling is highest at this stage, and income from recycling goes entirely to capital accumulation. This significant increment in the stock of capital ends up boosting production and therefore consumption (notably with respect to the case where recycling does not generate income). In the medium term, consumption is quite high while recycling drops sharply. In the last stage of the transitional dynamics, the consumption level is still markedly higher than in the case $\varphi = 0$, while income from recycling is much lower than in the initial stage of the transition dynamics (because the amount recycled in also much lower). The conjunction of the two latter forces pushes capital down in the ultimate adjustment stage given the law of motion of capital (2), generating the turning point observed and described above.

Therefore, the non-monotonicity observed is mainly due to the timing of optimal recycling, which massively takes place in the initial periods. The welfare implications of such timing can be seen in Fig. 4, where the social welfare function is computed for different time horizons. The initial sharp drop in social welfare is due to the initial intense recycling period.

5 Conclusion

In this paper, we have studied the sustainability of a Stokey-inspired AK model, one that considers a negative environmental externality that arises both from production and consumption. Instead of the typical abatement technologies, we present novel recycling modelling where recycling also generates income that is fully devoted to capital accumulation.

We have studied the qualitative properties of the resulting optimal control problem, notably in terms of optimal asymptotic states, stability and transition. We have also worked out a numerical example and got some highly interesting economic results in comparison with the seminal framework of Stokey, both in terms of the optimal pace of recycling and the relationship between income and pollution.

In particular, the role played by recycling as an income generator is crucial in the sense that it gives rise to a contraction of both the consumption and capital stock in the long run after an expansion phase. Whether polluting waste is predominantly due to production or consumption, when recycling generates additional income, greater consumption and lower capital stock are obtained in the long run compared with the situation when recycling does not create additional income. In parallel, when recycling generates additional income, greater recycling effort and lower stock of waste are resulted in the long run than when recycling has no additional income.

Of course, this is just a preliminary investigation; further analyses involving control-state constraints and alternative specifications within the same class of models along with alternative calibrations are needed to corroborate and complement this study.

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Handling the Complexity of Predator-Prey Systems: Managerial Decision Making in Urban Economic Development and Sustainable Harvesting

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Abstract In this paper we deal with complex systems and how to handle them. We focus on a well-known class of dynamical systems, namely predator-prey models, firstly by applying this type of model to urban economic development and secondly by testing models in an experimental setting in order to ascertain how successful human decision makers are in managing such a system. Regarding urban economic development, we illustrate that residential density and air pollution can be understood in terms of a predator-prey model and show how pollution control affects the level of the long-run equilibrium and the transition path towards it. We do this to support the understanding of how urban economic development works today and how it can be managed by decision makers via different interventions to improve the quality of living in urban areas. Regarding the task of handling predator-prey

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systems, we analyse the results of an experimental study in which participants take the role of a decision maker who seeks to maximise revenues from simultaneously harvesting a prey and a predator species while avoiding their overexploitation. We find that participants fall significantly shorter of the optimal strategy when the price assigned to the predator species is very high, compared to the price assigned to the prey species, in contrast to the case where the price difference is smaller. We offer several explanations for this observation that shed light on the human capability to handle predator-prey systems in general.

1 Introduction

Complex systems are omnipresent. They can be found in chemistry (e.g. Lehn 2002; Ludlow and Otto 2008) and biology (e.g. Kauffman 1993) and they can also be found in economics (e.g. Hayek 1967, 1937; Goodwin 1951) and economic policy (Hager et al. 2001; Neck and Behrens 2004, 2009), psychology (e.g. Hayek 1992), medicine (e.g. Coffey 1998), communications engineering (e.g. Prehofer and Bettstetter 2005), transportation (e.g. Pavone et al. 2011; Grippa et al. 2014) and complex systems optimisation (e.g. Dörner et al. 2007), management (e.g. Rivkin and Siggelkow 2007) and capital budgeting (e.g. Leitner and Behrens 2015a, 2015b), organisational design (e.g. Siggelkow and Levinthal 2003; Siggelkow and Rivkin 2005), health care planning (e.g. Plsek and Greenhalgh 2001) and even arms race modelling (e.g. Richardson et al. 1960; Behrens et al. 1997), to mention but a few.

It was essentially classical economic thought that, more than two centuries ago, prepared the ground for modern complex systems theory in the social sciences. This way of thinking was carried on by the Austrian school of economics which focused on what we today call complex phenomena. For example, these researchers enquired into the idea that order in market systems emerges spontaneously, in that it is '*the result of human action but not of human design*' (Hayek 1967) and skipped the central intelligence assumption in favour of the concept of dispersed knowledge (Hayek 1937). The types of systems under the 'complexity' umbrella embrace adaptive complex systems, but usually also chaotic systems and nonlinear dynamical systems. In this paper we focus on the latter, and present two case studies, both utilising a predator-prey system.

Predator-prey models which have been successfully applied to the understanding of ecological systems (e.g. Begon et al. 1996), as well as economic problems such as the business cycle (Goodwin 1967), consist of no more than two interconnected components. Still, we regard the science of predator-prey models, being representative of the simplest form of an interdependent system, as an important part of the study of complex systems. We, therefore, briefly outline them in Sect. 3. Following the multiple lines of evidence approach, we present the results of two case studies to illustrate how predator-prey systems can be applied to support environmental decision making. The first case study, presented in Sect. 4, illustrates how environmental decision problems can be conceptualised in a predator-prey system by expanding the standard predator-prey model to replicate interactions between local air pollution and population growth in a city (i.e. Graz, Austria). Moreover, we discuss options for policy interventions, how they can be integrated in such a framework, and what the impacts are for the transition path to the system's long-run equilibrium.

The second case study addresses the next step in the decision process: how successful are decision makers in steering a predator prey system into a desired direction? To answer this question, we use the results of an experimental study in which participants are asked to harvest two fish populations in order to maximise revenues over a certain period of time, while avoiding the extinction of both species (cf. Grabner et al. 2009). We then compare the outcomes of the experimental study to the optimal solution and test how strongly participants deviate from the optimal solution, depending on the context of the decision making problem. The specific context provided is to consider different prices for the two species harvested.

2 Predator-Prey Models and Their Applications

Independently of each other, Lotka (1925) and Volterra (1928) pioneered in modelling the dynamics of a complex biological system within which two species interact, one as a predator and the other as a prey. The corresponding mathematical formulation is given by two interrelated, nonlinear, first-order differential equations:

$$dx(t)/dt = (a - by(t))\,x(t), \quad x(0) > 0, \tag{1a}$$

$$dy(t)/dt = (cbx(t) - d)\,y(t), \quad y(0) \geq 0, \tag{1b}$$

with $a, b, c, d > 0$.

The variable $x(t)$ represents a prey population at time t that would grow exponentially in absence of a predator population; indicated by the term $ax(t)$, with reproduction rate $a > 0$. The prey's mortality rate depends on the current number of predators, represented by the variable $y(t)$, with the prey as the exclusive source of food. The associated parameter $b > 0$, therefore, denotes both the prey population's mortality rate per predator and the predator population's uptake rate of nutrition per prey. The parameter $c > 0$ measures the predator population's efficiency of converting prey eaten into offspring (i.e. the predator's metabolic efficiency) and the parameter $d > 0$ represents the predator's natural mortality rate.

An extension of the Lotka-Volterra model (1a, 1b) includes intraspecific competition among the prey species, upper boundaries on consumption and options for colonising ecological niches (Holling 1959). In its most general form, predator-prey dynamics are thus given by:

$$dx(t)/dt = g(x(t)) - f(x(t))y(t), \quad x(0) > 0, \tag{2a}$$

$$dy(t)/dt = (cf(x(t)) - d)y(t), \quad y(0) \geq 0, \tag{2b}$$

with $g(x(t)) \geq 0$ denoting the natural growth function of the prey population, and $f(x(t)) \geq 0$ standing for the functional response of the predator population to prey supply (e.g. Begon et al. 1996). The parameters $c > 0$ and $d > 0$ have the same meaning as in (1b).

The choice of different functional forms for g and f leads to different types of short-run and long-run system behaviour. For the basic Lotka-Volterra model (1a, 1b) with $g(x(t)) = ax(t)$, the phase space is filled with infinitely many closed orbits, i.e. wherever one starts (corresponding to initial conditions expressed in terms of $x(0)$ and $y(0)$) one undergoes, also in the long run, recurrent behaviour for both species, except if located exactly at the non-hyperbolic (non-trivial) equilibrium.

In contrast, augmenting the Lotka-Volterra model with $g(x(t)) = ax(t)(1 - x(t)/K)$, i.e. a logistic growth function with carrying capacity K, yields a richer and considerably different system behaviour: while the trivial and fatal equilibrium at the origin remains a saddle point, as also observed for (1a, 1b), we find that in the long run the extinction of the predator species can happen if its metabolic efficiency is too low relative to its mortality rate, i.e. $d \geq cbK$, while for $d < cbK$ the system is stable, i.e. non-recurrent, and coexistence of both populations can be guaranteed. This result is generated by the constrained environment and the resulting intraspecific competition for the prey population. If the carrying capacity for the prey population is sufficiently high, i.e. $K > d/cb$, the ecosystem thus remains intact and stabilises without any further intervention necessary. The more efficient the predator can convert food into offspring, reflected by an increasing parameter c, the higher the degree of variation towards the stable state, i.e. the higher the amplitude of the damped oscillation during the transient phase.

While predator-prey models were initially developed to describe ecological problems at the population level, namely the evolution of fish populations in the Mediterranean Sea (Volterra 1928), they have since been also proven applicable to other fields, including economics (e.g. the theory of the business cycle by Goodwin (1967)). In the environmental context, the Lotka-Volterra model has been, next to harvesting problems, employed in problems such as pest control in agriculture (Christiaans et al. 2007), nature conservation in protected areas (Friedl and Behrens 2007), or the fall of the Easter Islands (Brander and Taylor 1998). In the urban context, the Lotka-Volterra model has been used to describe the dynamic interaction of residential growth and urban rents or income (Capello and Faggian 2002; Dendrinos and Mullally 1983; Orishimo 1987; Camagni 1992). As diverse as these fields of application may be, the specific characterisation of the problems is similar: a system of two dynamic variables which interact and influence each other over time in a non-trivial way, i.e. a predator species which is not viable without the prey species.

In the next two sections, we use two case studies to illustrate how predator-prey systems can be used in different stages of a decision making process, first in framing and understanding the decision making problem and second in steering the decision making problem in practice. While the two case studies draw on different fields of application (the first on urban environmental problems, the second on harvesting a renewable resource), due to the universality of the underlying principle(s) of a generic predator-prey system the findings can be generalised to other contexts, as will be discussed in the concluding section.

3 A Predator-Prey System Approach for Urban Environmental Problems

In the urban context, the Lotka-Volterra model has been used to describe the dynamic interaction of residential growth and urban rents or income (Capello and Faggian 2002).[1] Dendrinos and Mullally (1983), for example, investigate the interaction between urban population and per-capita income. Orishimo (1987), by contrast, focus on population and land rent, with the latter as a proxy for the intensity of land use. The two main variables in the model of Camagni (1992) have been income and urban rent. But, while in Orishimo's model, urban rent was an indicator of cumulated capital stock, it was a negative localisation factor in Camagni's model. Capello and Faggian (2002) tested the empirical validity of different functional specifications for several Italian cities.

We seek to complement this stream of literature by using a predator-prey system with residential density and the quality of living as variables to epitomise the essential features of urban economic development (UED). A predator-prey approach is considered as appropriate because it allows to go beyond comparative statics and shifts the research focus from the analysis of the long-run equilibrium state towards the analysis of the system's transients. In the following subsection, we select suitable indicators (i.e. time series) for the two variables in order to describe the dynamics of the urban predator-prey system.

3.1 UED Case Study for Graz, Austria: Empirical Data and Selection of Suitable Indicators

With 276,526 residents in 2014, Graz is Austria's second largest city.[2] For more than a decade, Graz has been Austria's most rapidly growing residential urban area, and it has increased by 14 % between 2002 and 2013. The growing city size today poses a number of challenges to urban planners. Since the permanent settlement area has stayed almost constant since 1995 (being equal to approximately 97 km^2 out of a total available area of 127 km^2), these challenges come in the form of limited land resources, but also in the form of increasing environmental pressure. Specifically, air pollution is a serious problem in Graz, which strongly affects the well-being

[1]The dominant modelling approach for UED management problems has focused on the spatial distribution of population and economic activity between the centre and the surroundings of a city (as e.g. in the New Economic Geography, cf. Fujita et al. 1999; Brakman et al. 2001). For a New Economic Geography based model of urban settlement structure, commuting and local environmental quality see Koland (2010) and Bednar-Friedl et al. (2011). We abstain here from the regional aspect in favour of understanding the intertemporal evolution of urban economic development and investigate the associated problem as one that progresses over time (but not in space).

[2]Data source: Präsidialabteilung of the City of Graz.

of its residents, i.e. Graz suffers from high concentrations of particulate matter (PM10) which, especially in winter during stagnating weather conditions, multiply in excess of the European Union (EU) pollution limits. This situation is exacerbated by its geographical location south of the Alpine bow and the associated frequent weather inversions. While targeted policy measures have helped to substantially reduce emissions over the last two decades, air pollution is likely to remain an issue in the future, due to increasing traffic volumes and growing population numbers.[3]

A city like Graz clearly pools the advantages of proximity, such as the supply of a variety of goods and services (e.g. healthcare and education) and a sound public transport infrastructure, and it offers economic opportunities such as jobs, consumers and suppliers. Yet, along with the benefits of urbanisation come environmental, social and health problems. The quality of living in Graz results from a combination of centripetal and centrifugal forces which co-determine, in conjunction with institutional and political conditions, residential location choice. When a city comes closer to its carrying capacity, the negative effects of overpopulation tend to outweigh the benefits of urban life, and maintaining urban areas as liveable places becomes a substantial challenge for local planners. The following variables may be used to describe the UED problem for Graz and similar cities:

Residential density is regarded to be a key driver of UED dynamics. It can, however, be described by a spectrum of empirical measurements; apart from obvious ones such as population density, land use measurements may also be used to quantify residential density. To cover most of the spectrum, for the study focus of Graz, the following groups of time series data were collected and/or calculated: (1) urban population and population density (population per unit area), (2) construction of dwellings, building surface and building density (share of building surface in total permanent settlement area) and (3) sealed area and sealing ratio (share of sealed area in settlement area).[4] For Graz, we find a high positive and significant correlation of the urban population data for the period 2002 to 2013 with all other residential density indicators (see Table 1, left column) and infer that the measurements for the number of residents can be used as a satisfactory proxy for residential density.

Quality of living is perceived to be the second key driver for UED. This broad qualitative concept *inter alia* includes environmental conditions (e.g. air pollution, noise, accessibility of parks), security conditions and the traffic situation (cf. Magistrat 2009, 2013) alongside the supply of goods and services (e.g. healthcare, education, childcare) and the cost of living (e.g. housing prices). To capture the quality of living and its development over time for the city of Graz, the following data were collected: (1) air pollution (particulate matter concentration as averages per day, month and year as well as the number of exceedance days with respect

[3]Traffic volume causes half of the particulate matter emissions in Graz (39 % in winter season), followed by the emissions from industry and commerce (27 %; 22 % in winter) and domestic heating (23 %; 39 % in winter) (Heiden et al. 2008).

[4]Data sources: own calculations based on data supplied by Statistics Austria, the Federal Office for Metrology and Surveying (BEV) and the Environment Agency Austria (UBA). See Table 4 in the Appendix for more details.

Table 1 Correlation (Spearman-Rho) of various indicators for Graz

	Urban population (measured on 31st Dec of each year)		PM10 concentration (yearly average)
Building surface	0.988**	PM10 exceedance days (per year)	0.963**
Building density	0.970**	Green space (recreational area and gardens)	−0.731*
Sealed area	1.000**	Price of owner-occupied flats	−0.782**
Sealing ratio	1.000**	Property price	−0.763**

Data sources: Statistics Austria; Regional Statistics Styria; Federal Office for Metrology and Surveying; Regional Government of Styria, Department of Air Monitoring; Austrian Federal Economic Chamber, Association of Real Estate and Asset Trustees; 1995–2013 own calculations
Significance levels of correlations: ** 0.01, * 0.05

to EU-limits), (2) green space (parks and recreational areas), (3) wage income and (4) housing prices (prices of owner-occupied flats, of property, and of single-family houses).[5] We acknowledge that environmental quality is an important ingredient for the quality of living and a key determining factor in people's settlement decisions. Since a region's environmental quality suffers from the emission and the re-circulation of air pollutants, and since it is affected by the level of sealed soil surface (buildings, road infrastructure), reducing exposure to particulate matter in urban areas is important for assure public health. A large number of epidemiological studies find that the exposure to nanoparticles such as PM10 leads to adverse health effects (e.g. Murr and Garza 2009; McConnell et al. 2006; Samet 2007; Samal et al. 2008). In this regard, daily peaks and exceedance days, which mark EU limits for critical PM10 thresholds, matter the most, because PM10 concentration is harmful to the residents as soon as the threshold is reached.

Part of Graz's permanent local air quality monitoring network is the *Graz Don Bosco* station that was already implemented in 2001. For example, in 2001 it counted 158 exceedance days of PM10 emissions while the number declined to 46 in 2013 (see Fig. 1).[6] Looking at the Graz data, the yearly PM10 averages follow the same dynamics as the number of exceedance days per year, as can be seen in Fig. 1 (there is a significant correlation of +0.963 between the two time series, see Table 1). Therefore, we regard the yearly PM10 concentration as reasonable proxy for local air pollution, and we epitomise 'quality of living' by 'air pollution' (i.e.

[5]Data sources: Regional Government of Styria, Department of Air Monitoring; the Austrian Federal Economic Chamber, Association of Real Estate and Asset Trustees; Statistics Austria; Regional Statistics Styria; own calculations based on data supplied by the Federal Office for Metrology and Surveying (BEV) and the Environment Agency Austria (UBA). See also Table 5 in the Appendix for more details.

[6]For PM10, the EU's clean air policy imposes a limit of 50 $\mu g/m^3$ of air for the daily average value. This limit may be exceeded by up to 35 days per year (25 days according to IG-L, the Pollution Protection Act/Air for Austria).

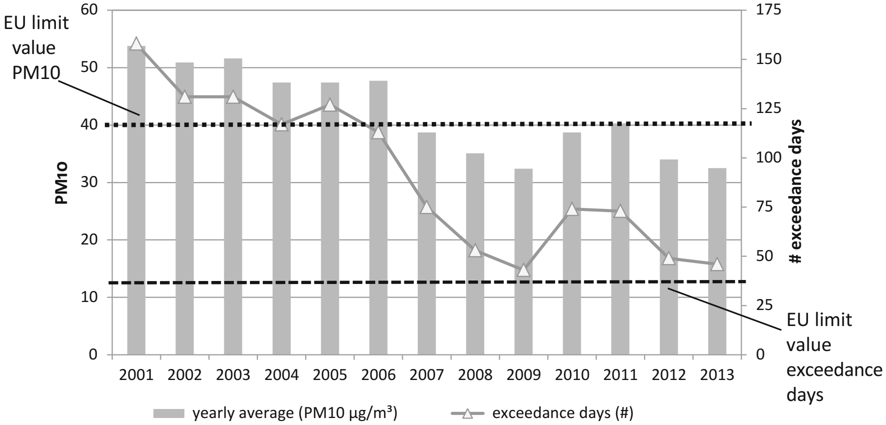


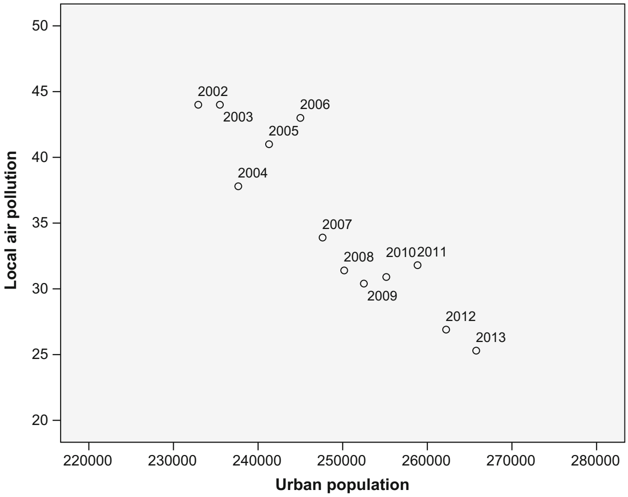
Fig. 1 Yearly PM10 averages and number of exceedance days for the period 2001–2013 in Graz Don Bosco including limit values (*dashed lines*) for PM10 (40 μg/m^3 yearly limit) and exceedance days (35 days per year). Data source: Regional Government of Styria, Division 15 of Energy, Housing & Technology, Department of Air Monitoring, own illustration

inverse environmental quality). Furthermore, there are also strong and significant correlations of PM10 concentration with the other indicators for quality of living such as housing prices (Table 1, right column). On the one hand, the positive correlation between green space and PM10 concentration indicates that more green space leads to less PM10. On the other hand, the negative correlation between PM10 and, for example, property prices indicates that housing prices reflect environmental attributes and that environmental quality is a non-market good that people value.

Figure 2 illustrates the relationship of the key variables in the UED system: residential density (measured by urban population) and quality of living (measured by local air pollution).

3.2 The Predator-Prey Model for Urban Economic Development

High environmental quality naturally attracts residents, while high levels of pollution are likely to push residents to the less-polluted periphery of an urban centre, comparable to the effect of a massive increase in the cost of living in central locations, i.e. rising housing prices (cf. Breheny 1992; Glaeser and Kahn 2003; Song and Zenou 2009). Recurrent combinations of environmental pollution and population density may, therefore, emerge due to mutually lagged reactions to the abundance of the other factor. In terms of a predator-prey model, this can be



	Urban population [# residents]	Air pollution [PM10 µg/m³]
2002	232,930	50.9
2003	235,485	51.6
2004	237,658	47.4
2005	241,298	47.4
2006	244,997	47.7
2007	247,624	38.7
2008	250,177	35.1
2009	252,502	32.4
2010	255,152	38.7
2011	258,847	40.3
2012	262,223	34
2013	265,778	32.5

Fig. 2 Local air pollution (in μg/m^3) vs. urban population (number of residents) for the period 2002–2013 in Graz

described as follows: Pollution (predator) increases, when population density (prey) is high. This leads to an increased predator pressure on prey, i.e. residents move out of the region where pollution is high. This in turn causes a recovery of environmental quality, i.e. a decrease in pollution levels (less predator abundance). Residents are in turn attracted by a higher quality of living (cf. e.g. Gyourko et al. 1999) and resettle the region (increase in prey abundance), where the cycle starts again (with subsequent rise in predator or pollution abundance). However, if residents respond to congestion by increasingly used public transport and bicycles, the quality of living would improve again. Particulate matter (e.g. PM10) is a stock variable and accumulates (degrades) over time with activity (inactivity) of the residents, but in addition every city has to cope with a certain background level of emissions because of pollution inflows from other locations.

In agreement with most of the previous literature, we conceptualise residents ($R(t)$) as the prey variable, since even at zero pollution the population would not exhibit more than logistic growth behaviour, i.e.

$$\dot{R}(t) = a\, R(t) \left(1 - \frac{R(t)}{R_{\max}}\right) - b\ R(t)\ P(t), \qquad R(0) = R_0 > 0, \tag{3}$$

where a denotes the annual net population growth rate, $R_{\max}$ denotes the city's carrying capacity and b denotes the annual dose response relationship between mortality and air pollution (PM10).

Pollution is regarded as the predator variable, since it would disappear in the absence of people driving cars and/or needing housing facilities. Empirical evidence suggests, however, that a certain share of local air pollution is caused by economic activity outside the city. Thus, even removing a city entirely would leave the geographical region where it is situated with a positive level of emissions. We take account of this important fact and introduce into the predator-prey model the parameter τ, which indicates the level of background emissions that prevail in the absence of residents. Consequently, pollution dynamics are described by:

$$\dot{P}(t) = \tau + c \ b \ R(t) \ P(t) - d \ P(t), \qquad P(0) = P_0 \geq 0. \tag{4}$$

Parameter c in (4) captures how the interaction of population and pollution $(b\,R(t)P(t))$ contributes to the accumulation of the pollutant. For instance, this parameter reflects environmental awareness and behaviour of residents, e.g. the distribution across transport modes (car vs. public transport vs. walking or cycling) or the distribution across heating systems (central heating vs. oil vs. biomass). Parameter d stands for the level of pollution control measures implemented by the mayor or the urban planner, such as driving restrictions.[7]

In other words, (4) captures the annual change of the average PM10 level, not its day-to-day changes responding to wind, rain, fog, traffic, heating, etc. As opposed to the behaviour of emission levels on a shorter time scale, the annual average PM10 level does not decay unless the city shrinks, residents change habits or environmental policies successfully ban (or substitute) individual traffic or support environmentally friendly heating facilities, *ceteris paribus*.

This model has three steady states, determined by the intersection of the isoclines of (3) and (4), i.e.

$$\widehat{R}_1 = 0, \ \ \widehat{P}_1 = \frac{\tau}{d}; \qquad \widehat{R}_{2,3} = \frac{a \ (d + bcR_{\max}) \pm A}{2abc}, \ \ \widehat{P}_{2,3} = -\frac{a \ (d - bcR_{\max}) \pm A}{2b^2cR_{\max}},$$

where $A := \sqrt{a \left(a \ (d - bcR_{\max})^2 + 4b^2cR_{\max}\tau\right)}$. The first steady state $E_1 = \left(\widehat{R}_1, \ \widehat{P}_1\right)$ is characterised by zero residents but a positive level of pollution which depends on the background emission level τ and is inversely related to the control-induced decay rate d. Inspection of $\widehat{R}_2$ reveals that $\widehat{R}_2 > R_{\max}$. Moreover, $A > a\,(d - bcR_{\max})$ implies that steady state pollution is negative, i.e. $\widehat{P}_2 < 0$. Hence, $E_2 = \left(\widehat{R}_2, \ \widehat{P}_2\right)$ is not economically feasible. Finally, when both the numerator in

[7] It is plausible to assume that these two parameters are exogenous: According to the environmental psychological literature, both heating and mobility behaviour show high habitual continuity, resulting in temporal stability and resistance to change (Klöckner and Matthies 2004; Verplanken 2006). Often, it takes a biographical event to revise habitual mobility patterns, such as moving to a new house or city (Bamberg et al. 2003; Scheiner 2007). Regarding the pollution control variable, it is also a well-founded assumption that the urban planner takes his or her decision based on observing the urban system, yet in a discrete way because e.g. laws regulating speed limits or driving bans need to be passed by the city authorities.

$\widehat{R}_3$ and $(d - bcR_{\max})$ are positive (which is ensured by a sufficiently large value of d), an interior solution exists, i.e. $\widehat{R}_3 \in (0, R_{\max})$ and $\widehat{P}_3 > 0$.

To calibrate the system to the data of the city of Graz, we use forecasts of population growth up to the 2030s (Stadt Graz 2012), yielding $a = 0.108909$ and $R_{\max} = 295,163$. The dose response relationship is based on WHO estimates for pre-natural mortality due to higher exposure to PM10 in Austria (Künzli et al. 2000), yielding $b = 3.7 \times 10^5$. According to expert opinions, the level of Graz's PM10 background emissions from external sources approximately amounts to $\tau = 16.25$ $\mu g/m^3$. In order to calibrate parameters c and d, we divide the pollution time series into two parts, the first part representing the pre-policy phase (up to 2006) and the second part representing the policy phase (2007 onwards). This allows estimating parameter c for the pre-policy phase and using that result in estimating parameter d. We thus obtain $c = 0.097267$ and $d = 1.44831$. However, as d captures the effect of specific control measures which have been implemented in the recent past, this does not necessarily imply that new measures will lead to similar effects; on the contrary, diminishing efficiency seems a more plausible assumption. We, therefore, set d arbitrarily equal to 0.5 and explore the sensitivity of results to higher or lower values of d.

Figure 3 illustrates system behaviour based on these estimated parameter values in the phase plane. The positively-sloped dashed line represents the $\dot{P}(t) = 0$ isocline and the negatively-sloped dashed line represents the $\dot{R}(t) = 0$ isocline. The

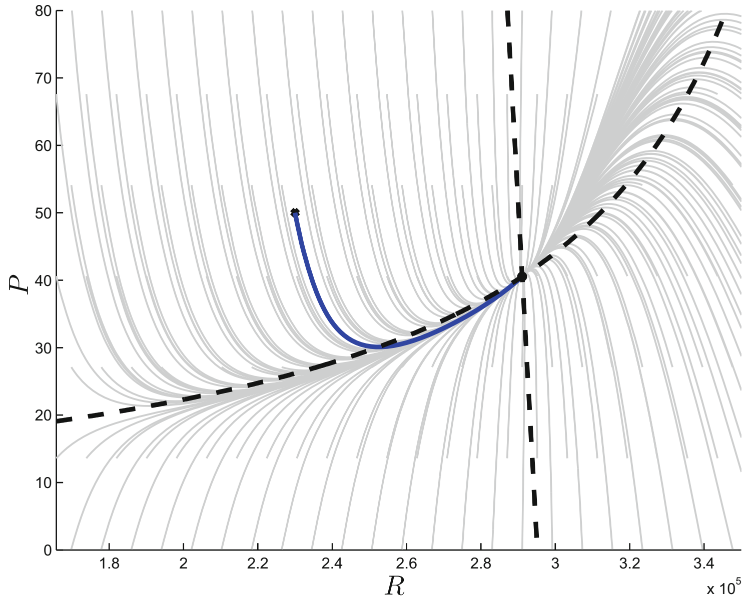


Fig. 3 The state space for residents and population in Graz around $E_3 > 0$; for $a = 0.108909$, $b = 3.7 \times 10^5$, $c = 0.097267$, $d = 0.5$, $R_{\max} = 295,163$, $\tau = 16.25$

latter is rather steep, as residential numbers hardly respond to changes in pollution, whereas pollution responds strongly to changes in residents and thus the $\dot{P}(t) = 0$ isocline is flat. At the intersection of the isoclines, we find the third steady state ($\widehat{R}_3 = 291,040; \widehat{P}_3 = 41.1$), which is a stable node as indicated by the trajectories. The black trajectory illustrates the transition from PM10 levels in the 1990s to the steady state and shows that for $d = 0.5$ residents will increase and pollution will be slightly higher than at the starting point. The second steady state is economically non-admissible, since $\widehat{P}_2 < 0$, and the first steady state (that would lead to the extinction of residents, i.e. $\widehat{R}_1 = 0$), is unstable.

When cities are about to reach their carrying capacities K, the residents' disutilities from overpopulation exceed the benefits they perceive from urban life. The main question then is how such a complex system represented by two interrelated UED-indicators can (should) be handled by an urban planner and how UED can (should) affect people and environment in the future. The EU's clean air policy imposes strict limits for particulate matter concentrations, PM10 and PM2.5, which Austria and especially Graz struggle to meet. Basically, an urban planner has different options for handling the urban system, namely (1) market-based measures, (2) command-and-control measures (standards, regulations) and (3) measures of governance (changes in institutions, information campaigns). Utilising these measures mutually affects each other via the system's dynamics. Therefore, it is important to analyse the capability of humans to understand a predator-prey-type system and to deploy interventions to (possibly optimally) steer it into a desired direction.

To illustrate the effect of such a policy intervention, we conduct a sensitivity analysis with regard to d. An increase in d by 1 % leads to a decline in steady state pollution, computed according to $E_3 = \left(\widehat{R}_3, \ \widehat{P}_3\right)$, by 11.20 % and to an increase in residents by 0.16 %. Thus, doubling d implies a steady state average pollution level of 18.2 $\mu g/m^3$ of PM10, compared to 41.1 $\mu g/m^3$ in the base case scenario and to a steady state population size equal to 293,949 residents, instead of 291,040.

To do so, the handling of predator-prey models has not only been scrutinised analytically or numerically; they have also been investigated in economic experiments. For example, Dörner (1989, 1996) started already very early with a new theory about the structure of the human mind, by using experimental studies to find out regularities for human behaviour.

4 Experimental Case Study on Optimal Harvesting

Regarding the task of handling or steering predator-prey systems, we analyse now the results of an experimental study of human behaviour in a simulation environment built upon a predator-prey ecology (cf. Grabner et al. 2009). The main motivation for experimentally studying managerial decision making in a predator-prey ecology is the attempt to corroborate analytically-derived optimal strategies by empirically observed behaviour. The approach of presenting problems to human decision makers as computer simulated scenarios, which they have to explore and control, follows the ideas of Reichert and Dörner (1988).

The starting point of this analysis is that previous economic experiments have shown that in certain settings human decision makers fail to achieve the optimal solution. Fehr and Zych (1998) report, for example, that addicts consume systematically more than what would be rationally optimal. Another non-linear dynamic system successfully used in experimental economics is presented by Becker and Murphy (1988), who show that addictive consumer behaviour is consistent with intertemporal utility maximisation. In a follow-up paper, Dockner and Feichtinger (1993) demonstrate that addiction may lead to persistent oscillations of consumption rates. This is where behavioural economic research, and in particular experiments, can contribute to a better understanding of the manageability of a complex system and thus to an improved way of actually managing it.

In this section, we investigate how successful decision makers are in steering a predator-prey system into a desired direction. We use the results of an experimental study in which participants are asked to harvest two fish populations in order to maximise the revenues. By comparing the outcomes of the experimental study, we test how strongly participants deviate from the optimal solution depending on the context of the decision making problem, i.e. the influence of different price levels for the two species.

4.1 The Setup of the Experiments

The main idea of this experiment, which studies the behaviour of human decision makers in a simulation environment based on a Lotka-Volterra ecology (see 1a, 1b), dates back to Becker and Leopold-Wildburger (1995), Becker et al. (2004) and Reichert and Dörner (1988). To conduct the experiment, we reuse the EXPOSIM simulation software designed by Fritsch and Siegelmann (2006) and which was applied by Grabner et al. (2009) in a related context.

The simulated predator and prey populations are denoted by $y = y(t)$ and $x=x(t)$, respectively. By assumption, they form a biotope that consists of populations of small fish (x) and large fish (y).[8] Figure 4 depicts the prototypical behaviour of small (x) and large fish (y) over time in the absence of intraspecific competition among the prey and in the absence of interventions. We observe that, due to the absent competition within the prey population, the number of small fish and the number of large fish (which feed upon the small fish) fluctuate periodically, but out of phase. That is, after the predators have become too many in number and feed upon too many small fish, the large fish population begins to shrink due to a shortage of food.

[8] An alternative interpretation of y and x would be an ecosystem that consists of populations of foxes and rabbits.

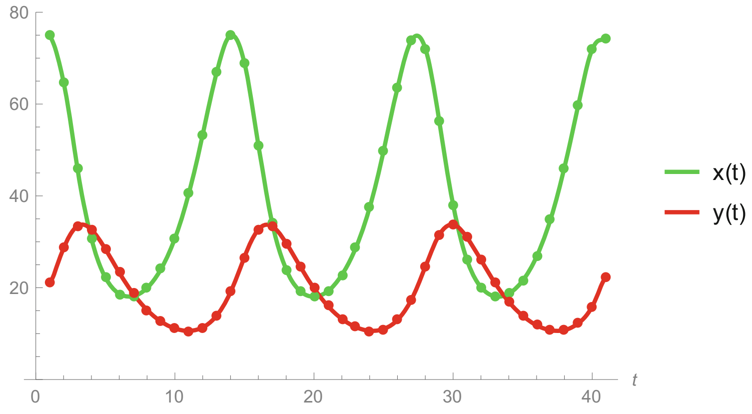


Fig. 4 Development of predator and prey populations over time following the Lotka-Volterra dynamics (1a,1b) (for $x_0 = 70$, $y_0 = 15$, $a = 0.60$, $b = 0.03$, $c = 0.01$, $d = 0.4$)

This gives the small fish a chance to recover. This, in turn, allows for the large fish population to gradually recover as well and then to prosper because of an abundance of food, which starts the process all over again.

The participants of the experiment were put into the position of a decision maker and requested to simultaneously harvest two fish species for $T = 42$ periods, such that the revenues from catching and selling $h_x(1) + \cdots + h_x(T)$ small fishes and $h_y(1) + \cdots + h_y(T)$ large fishes are maximised, while keeping the respective stocks of fish sufficiently large so that no species is driven to extinction. As is a standard assumption in the resource economics literature (see e.g. Clark 2010; Conrad 1999; Perman et al. 2011), fishermen are assumed to operate on competitive markets and, therefore, the price of fish is exogenous to each fisherman and not influenced by fluctuations within their own fishing ground. For our experiment, we follow this strand of the literature by assuming that we look into the management problem of an individual fisherman who owns a fish pond.

After each period, the remaining stocks of small and large fish were disclosed to the participants. To emphasise the importance of maintaining the ecosystem in the long run and to make the objective more tangible, we integrated the market value of the stocks of small and large fish in the terminal period, i.e. $p_x x(T) + p_y y(T)$, into the current value of the manager's revenue function, where prices for small and large fishes are denoted by p_x and p_y, respectively. For example, the revenue function to be maximised by the participants, which was displayed on the screen in the beginning of the experiment (and remained displayed throughout), was given by

$$\Pi = \sum_{t=1}^{T} p_x h_x(t) + p_y h_y(t) + p_x x(T) + p_y y(T), \qquad T = 42, \tag{5}$$

with $x(t)$ and $y(t)$ determined by

$$\dot{x}(t) = (a - b\,y(t))\,x(t) - h_x(t), \tag{6a}$$

$$\dot{y}(t) = (c\,b\,x(t) - d)\,y(t) - h_y(t). \tag{6b}$$

The complexity to be handled by the participants was not only to manage a simulated environment with increasing and decreasing population numbers. It was that the two harvesting instruments, $h_x(t)$ and $h_y(t)$ for $t = 1, \ldots, T$, were interdependent and each intervention did not only have an impact on the targeted population but, via the interrelated evolution of the ecosystem, also on the other population and *vice versa*.

4.2 Results and Discussion

By construction, the optimal harvesting trajectory, which results from solving the finite maximisation problem (5) subject to (6a and 6b), depends on the market prices paid for the two fish species. Therefore, we varied the price for each species, resulting in four different types of treatment characterised by p_y/p_x (see Table 2). We observed that the goal to harvest alongside keeping the final stocks as high as possible was fulfilled by nearly all participants for all types of treatment. In particular, it turned out that for all treatments the participants of the experiment behaved in a reasonable way by actively searching for the optimal orbit.

For treatment $p_y/p_x = 4/2$ and 5/2 the optimal behaviour was, however, significantly more often reached than for the other treatments (cf. Table 2): participants reached on average 71 % and 70 % of the optimal revenues for treatments 5/2 and 4/2 as compared to 66 % and 65 % for treatments 5/1 and 4/1, respectively. This difference becomes even more pronounced when comparing the results for the best 50 % to best 5 % of respondents, as shown in Table 3.

According to Table 3, the difference between pooled treatments 4/1 & 5/1 and pooled treatments 4/2 & 5/2 is significant according to a t-test on mean

Table 2 Optimal revenues Π^*, mean revenues $\overline{\Pi}$ and relative mean revenues $\overline{\Pi}/\Pi^*$ for the different treatments

Treatment p_y/p_x	Number of participants	Optimal revenue Π^*	Participants' mean revenue		Participants' mean revenue ratio	
			Value	Stdev	Value	Stdev
4/1	205	1.636	1.078	369	0.659	0.225
4/2	117	1.990	1.392	364	0.699	0.176
5/1	476	2.017	1.314	360	0.651	0.191
5/2	247	2.253	1.594	357	0.708	0.212

Table 3 Difference in mean ratios between pooled treatments 4/1 & 5/1 compared to 4/2 & 5/2 for best 50 % to best 5 % of participants

		4/1 & 5/1		4/2 & 5/2		t Test	
		Mean ratio	Stdev	Mean ratio	Stdev	S.E.	t
Best	50 %	0.809	0.079	0.857	0.062	0.074	−7.018*
	45 %	0.821	0.076	0.866	0.058	0.070	−6.735*
	40 %	0.833	0.071	0.875	0.055	0.066	−6.228*
	35 %	0.846	0.067	0.884	0.053	0.062	−5.634*
	30 %	0.861	0.061	0.895	0.050	0.057	−5.033*
	25 %	0.877	0.052	0.907	0.045	0.050	−4.674*
	20 %	0.896	0.040	0.920	0.040	0.040	−4.223*
	15 %	0.913	0.029	0.935	0.035	0.031	−4.147*
	10 %	0.929	0.020	0.953	0.029	0.024	−4.975*
	5 %	0.945	0.018	0.977	0.013	0.016	−6.833*

*Significance level $p < 0.01$

equivalence. In particular, for 4/2 & 5/2, participants achieved significantly better results than participants did for treatments 4/1 & 5/1. This result emerges no matter whether one looks at the best-performing 50 %, 45 %, ..., 10 % or 5 % of the participants.

Inspecting the data reveals that the significantly lower outcome for pooled treatments 4/1 & 5/1 results from a selection of sub-optimal interventions: in the experimental setting, harvesting prey (fishing small fish) was significantly more often used than optimal, even when the price of this species was very low (as in treatments 4/1 & 5/1) in relation to the price of the predator (large fish).

Several *ex post* justifications for why the prey population was systematically overharvested, while predators were under-harvested, seem plausible:

- The participants' decisions could have been guided by bounded rationality, or phrased in terms of Dörner (1989), by a reduction to simple decisions.
- As the future evolution of an ecosystem is difficult to predict (in spite of 'knowing' its dynamics) a misperception of feedback could have occurred.
- Participants could have had an inclination to harvest too much of the prey population, due to an overestimation of both its available amount (falsely associating 'small' with 'many') or its accumulated value (falsely associating 'quantity' with 'revenue').

Generally speaking, the results of the empirical analysis show that the behaviour of the participants is significantly dependent on the amount of available objects, independent of the revenue. Subjects perform efficiently according to the rules when valuable objects are prey (i.e. small fish, population density), however behave significantly inefficient when high price objects are predator (i.e., large fish, pollution). This is remarkable as it underlines that individuals underestimate the threat for the existence of the prey population via removing prey and overestimate the threat conveyed by harvesting.

5 Discussion and Conclusions

The science of predator-prey models is an important aspect of the study of complex systems, despite the interaction of only two variables. In this paper, we have used two case studies to illustrate how a predator-prey model can aid environmental decision making. In the first case study on urban economic development, we have shown how such a model can be adapted to reflect residential development as prey population which is limited by settlement area and negatively affected by air pollution. Air pollution, as a proxy for (inverse) quality of living in a city, was regarded as predator population. While the main objective was to conceptualise the model dynamics, we briefly discussed how decision makers could intervene into the system. One possible policy option is pollution control which reduces air pollution by speed limits or driving bans, but additional policies which target the feedback of population growth on pollution could reduce pollution as well. Examples of this latter approach are provision of information and behavioural measures, which induce residents to switch their mobility behaviour towards less environmentally harmful modes of transport.

Regarding the task of handling or steering predator-prey systems for optimally supporting some overarching objectives, in Sect. 4 we have analysed the results of an experimental study of human decision making behaviour in a simulation environment built upon a predator-prey ecology (cf. Grabner et al. 2009). The participants resumed the role of a manager who seeks to maximise revenues from simultaneously harvesting a prey and a predator species (while avoiding their extinction) over a certain time period. We find that participants fall significantly short of the optimal solution when the price paid for the predator species is very high, as compared to the price paid for the prey species, in contrast to the case where the price difference is smaller. We provide some explanations for why a moderate price difference is more adequately incorporated into the participants' managerial decisions than a high price difference, i.e. why predators are systematically under-harvested in spite of the growing intensity of the price signal provided. The experimental setting in the predator-prey study on fishery provides insights for the management and the manageability of similar systems. The lesson learned is that carrying out management tasks in a predator-prey ecology and similarly behaving systems is inherently difficult for human decision makers and can be flawed in spite of perfect information, especially if two interdependent quantities have to be managed at the same time.

This contribution, therefore, demonstrates that predator-prey systems are applicable to a broad class of environmental decision making problems; but it also demonstrates that these systems require substantial problem understanding on the part of decision makers. It is, therefore, a task for future research to support decision making, such that desired and effective control strategies can also be achieved in the real world.

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Appendix

See Tables 4 and 5.

Table 4 Empirical data for residential density

Indicators for residential density		
Variable	Description	Source
BEV	Residential population per region[a] [number]	Statistics Austria (STAT.AT), 2002–2013
BEVDI	Residential population per unit of total permanent settlement area *DSR* per region[a] [number/1000 m^2]	Own calculation based on STAT.AT and Federal Office for Metrology and Surveying (BEV), 2002–2011
DSR	Permanent settlement area [1000 m^2]	Own calculation based on BEV (data) and Environment Agency Austria (EEA), 1995–2011
BAUFL	Building surface and hard-surfaced area per region [1000 m^2]	Own calculation based on BEV (data) and EEA, 1995–2011
BAUDI	Share of *BAUFL* in total permanent settlement area DSR per region [share] (building density)	Own calculation based on BEV (data) and EEA, 1995–2011
DWE	Construction of dwellings [number and floor space in 1000 m^2]	STAT.AT, 1980–2002
VSIEFL	Sealed area per region [1000 m^2]	Own calculation based on BEV (data) and EEA, 1995–2011
VSIEGR	Share of *VSIEFL* in total permanent settlement area *DSR* per region (sealing rate) [share]	Own calculation based on BEV (data) and EEA, 1995–2011

[a]Regions are the political districts of Graz and Graz surroundings

Table 5 Empirical data for the quality of living

Indicators for the quality of living		
FEIN	Yearly, monthly and daily average of particulate matter (PM10) concentration per region [$\mu g/m^3$]	Regional Government of Styria, Department of Air Monitoring, 2001–2013
FEIN_d	PM10 exceedance days [number]	Regional Government of Styria, Department of Air Monitoring, 2001–2013
GRÜFL	Recreational area and gardens per region [1000 m^2]	Own calculation based on STAT.AT and BEV, 2002–2011
GRÜAN	Share of *GRÜFL* in total permanent settlement area *DSR* per region [share]	Own calculation based on BEV (data) and EEA, 1995–2011
IMMO_eigw	Average price for owner-occupied flats per region [EUR/m^2]	Austrian Federal Economic Chamber, 2000–2013
IMMO_grund	Average property price per region [EUR/m^2]	Austrian Federal Economic Chamber, 2000–2013
IMMO_ehaus	Average price for single-family houses per region [EUR/m^2]	Austrian Federal Economic Chamber, 2000-2013
EK	Wage income of employees per region [EUR]	Regional Statistics Styria, 2002–2012
BWS	Gross value added for the NUTS-3 region Graz [million EUR]	Statistics Austria, 2002–2011

[a]Regions are the political districts of Graz and Graz surroundings

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On Optimal Harvesting in Age-Structured Populations

Anton O. Belyakov and Vladimir M. Veliov

Abstract The problem of optimal harvesting (in a fish population as a benchmark) is stated within a model that takes into account the age-structure of the population. In contrast to models disregarding the age structure, it is shown that in case of selective harvesting mode (where only fish of certain sizes are harvested) the optimal harvesting effort may be periodic. It is also proved that the periodicity is caused by the selectivity of the harvesting. Mathematically, the model comprises an optimal control problem on infinite horizon for a McKendrick-type PDE with endogenous and non-local dynamics and boundary conditions.

1 Introduction

This paper contributes to the understanding of the controversial issue of the pattern of exploitation of renewable resources: is it optimal to extract a renewable resource with constant (in the long run) intensity or it is better to implement a periodic extraction pattern in which periods of intense extraction are followed by recovery periods of no extraction. This issue arises in several contexts, including agricultural land-use and fishing. The later case is brought to our attention by Ulf Dieckmann, who suggested that a selective fishing mode (harvesting only sufficiently large fishes) may lead to detrimental evolutionary changes, due to which a periodic harvesting may be advantageous, since it gives time for the fish population to recover

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(see the more sophisticated analysis in Landi et al. (2015) and the bibliography therein).

In the recent paper (Belyakov and Veliov 2014) we investigated the problem of optimal industrial fishing in a closed basin (in the sense that migration of the considered fish species in and out of the basin is negligible). It was shown there that in the problem of maximization of the averaged net revenue from the fishing activity on the infinite time-horizon $[0, \infty)$ a periodic fishing effort may give a better objective value than every constant one. It is also shown that the selective fishing mode plays a key role for this result. Namely, we proved that in case of non-selective fishing and stationary vital rates the optimal fishing effort is constant.

The averaged objective functional takes into account only the asymptotics of the net revenue. In contrast, in the present paper we consider the total discounted net revenue as an objective of maximization. This is an essentially different problem, the solution of which depends on the discount rate, and may have a qualitatively different asymptotic behavior than the solution of the problem with averaged objective functional. However, in the analysis and in the numerical investigation in the present paper we substantially use the results obtained in Belyakov and Veliov (2014). Namely, on certain assumptions we show that for sufficiently small discount rates the asymptotic behavior of the solution of the discounted problem inherits the qualitative properties of the solution of the averaged problem. As a benchmark case for the numerical study we use the same data specifications for which we obtain in Belyakov and Veliov (2014) superiority of a proper periodic fishing effort (compared with any constant ones). In this case study we establish periodicity of the optimal fishing effort in the discounted problem for sufficiently small discount rates.

In short, in the present paper we show that a selective fishing mode may lead to an optimal fishing effort which is asymptotically proper periodic. This may happen at least for sufficiently small discount rates. On the other hand, we show that if the fishing mode is not selective, then the optimal fishing effort is asymptotically constant. That is, the possible periodicity of the optimal fishing effort is caused by the selectivity of fishing.

In order to enable involvement of selective fishing modes in the mathematical analysis, we need to consider a distributed optimal control model, that takes into account the size- (or age-) structure of the fish population. The model we use employs a standard age-structured controlled differential equation (see e.g. Aniţa 2000; Feichtinger et al. 2003; Iannelli 1995; Webb 1985). The main difficulty is, that the theory of establishing (asymptotically) periodic behavior of the optimal control for problems involving such equations is not developed (in contrast to the ODE case). For this reason we make use of the results in Belyakov and Veliov (2014), as explained above.

The plan of the paper is as follows. In the next section we formulate the model and the main assumptions. In Sect. 3 we present a result suggesting that a periodic fishing can be superior compared with any (asymptotically) constant one. In Sect. 4 we give numerical results that exhibit periodicity of the optimal fishing effort in a case study with a selective fishing mode. In Sect. 5 we show that in case of non-selective fishing the optimal fishing effort is asymptotically constant. Some concluding remarks are given in Sect. 6.

2 Model Formulation and Preliminaries

The dynamics of the age-structured fish population is described by the same model as in Belyakov and Veliov (2014), which we reproduce for readers' convenience.

Everywhere below $t \geq 0$ is the time, a is the fish age, assumed to be restricted in a finite interval $[0, A]$, where A is a proxy for the maximal age that a fish may achieve (meaning, that fishes of age higher than A can be disregarded, since they are not fertile and their total biomass is negligible). By $n(t, \cdot)$ we denote the (non-normalized) age-density of fish at time $t \in [0, +\infty)$. Informally, one says that $n(t, a)$ is the number of fish of age a at time t. Moreover, $z(t)$ will be the total biomass of fish, and $u(t)$ will be the harvesting intensity (effort) at time t. The latter can be interpreted as the number of ships or nets involved in harvesting at t.

The model involves the following age-specific data:

$\mu(a)$—natural (without competition and harvesting) mortality rate at age a;
$\beta(a)$—fertility rate;
$\gamma(a)$—(average) biomass of a single fish of age a;
$\chi(a)$—age-profile of the selective harvesting.

The last function needs an explanation. Essentially, the function $\chi(a)$ indicates at which ages (hence, indirectly, at which size) a fish is a subject to harvesting. Typically, $\chi(a)$ is zero for small ages and equals 1 after a certain age. Generally, $\chi(a)u(t)n(t, a)$ is the harvested flow of fish of age a resulting from fishing effort $u(t)$ in a population with density $n(t, \cdot)$.

Moreover, there is an additional mortality rate $M(z)$ depending on the total biomass, z. Such additional mortality due to overpopulation may be explained by competition for food and space, and also by higher mortality due to predation.

The dynamics of the fish population is described by the age-structured system

$$\mathscr{D}n(t,a) = -\,(\mu(a) + M(z(t)) + u(t)\,\chi(a))\,n(t,a), \quad (t,a) \in [0,\infty) \times [0,A], \tag{1}$$

$$n(0,a) = n^0(a), \tag{2}$$

$$n(t,0) = \int_0^A \beta(a)\,n(t,a)\,\mathrm{d}a, \tag{3}$$

$$z(t) = \int_0^A \gamma(a)\,n(t,a)\,\mathrm{d}a, \tag{4}$$

where $n^0(a)$ is a given initial density, and

$$\mathscr{D}n(t,a) := \lim_{\varepsilon \to 0+} \frac{n(t+\varepsilon, a+\varepsilon) - n(t,a)}{\varepsilon}$$

is the directional derivative of n in the direction $(1,1)$. (Traditionally, the partial differentiation $\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right)$ is used instead of $\mathscr{D}$, meaning the same.) Since n is assumed absolutely continuous on almost every characteristic line $t-a = \text{const.}$, the traces $n(t,0)$ and $n(0,a)$ make sense [see e.g. Aniţa 2000; Feichtinger et al. 2003; Iannelli 1995; Webb 1985 for the precise meaning of a solution of (1)–(4)].

The following assumptions are standing all over the paper.

Assumption (A1) The functions μ, β, γ, $\chi : [0,A] \to [0,\infty)$ are measurable and bounded, $\gamma(a) \geq \gamma_0 > 0$ for every $a \in [0,A]$.

Assumption (A2) The following inequality holds:

$$\int_0^A \beta(a)\, e^{-\int_0^a \mu(\eta)\,\mathrm{d}\eta}\, \mathrm{d}a > 1.$$

Assumption (A3) $M : [0,+\infty) \to [0,+\infty)$ is continuously differentiable, $M(0) = 0$, $\lim_{z\to+\infty} M(z) = +\infty$, and $M'(z) > 0$ for $z > 0$.

Assumption (A4) The set $\{a \in [0,A] :\ n^0(a)\beta(a) > 0\}$ has a positive measure.

Assumption (A2) is known as *above replacement fertility* (if the initial n^0 contains fertile fishes and there is no harvesting, then the population grows as long as its density $M(z(t))$ is sufficiently small. Assumption (A3) ensures that the population cannot grow infinitely even without harvesting. Assumption (A4) ensures that the population does not extinct in a finite time.

It is well known that on the above assumptions for every measurable and bounded function u a solution of system (1)–(4) exists on $[0,+\infty)$ and is unique (see e.g. Aniţa 2000, Theorem 2.2.3). Moreover (A3) implies that the solution n is bounded.

Within the above model the harvested biomass, $x(t)$, per unit of harvesting effort $u(t)$ is

$$x(t) = \int_0^A \gamma(a)\, \chi(a)\, n(t,a)\, \mathrm{d}a. \tag{5}$$

If $c \geq 0$ is the cost of a unit of harvesting effort, then a reasonable optimization problem is to maximize the aggregated net discounted revenue:

$$J^\delta(u) := \int_0^\infty e^{-\delta t}\, (x(t) - c)\, u(t)\, \mathrm{d}t \ \longrightarrow \max, \tag{6}$$

with respect to the control function (harvesting effort) $u(\cdot)$. Admissible control functions are all measurable functions $u : [0,\infty) \to [0,U]$, where U is the maximal feasible harvesting effort. Denote this set of admissible controls by $\mathscr{U}$. The discount rate δ is assumed strictly positive.

Since $\delta > 0$, the integral in (6) is finite for every $u \in \mathscr{U}$ due to the boundedness of x and assumption (A1). Then optimality has the classical meaning, namely, $\hat{u} \in \mathscr{U}$ is optimal if $J^\delta(\hat{u}) \geq J^\delta(u)$ for every $u \in \mathscr{U}$.

As explained in the introduction, the main goal of this paper is to show that an optimal fishing control $u(t)$ is not necessarily tending to a constant in the long run. In order to formalize this claim we introduce the following notations:

$$\mathscr{U}_c^\infty := \{u \in \mathscr{U} : \exists\, u^\infty \in [0, U] \text{ such that } \lim_{\tau \to +\infty} \|u(\cdot) - u^\infty\|_{L_\infty(\tau,+\infty)} = 0\}, \quad (7)$$

$$\mathscr{U}_p := \{u \in \mathscr{U} : u \text{ is a proper periodic function}\}.$$

The first set consists of all asymptotically constant admissible controls, while $\mathscr{U}_p$ consists of all (essentially) periodic admissible controls for which a minimal positive period exists. In these notations, our goal is to show that for appropriate configurations of data $(\mu, M, \chi, \beta, \gamma, c, \delta > 0)$ there exists $\hat{u}_p \in \mathscr{U}_p$ such that

$$\sup_{u \in \mathscr{U}_c^\infty} J^\delta(u) < J^\delta(\hat{u}_p). \quad (8)$$

In other words, there exists a proper periodic admissible control $\hat{u}_p$ which gives a strictly better performance value of J^δ than any asymptotically constant admissible control.

3 An Analytic Approach

This section deals with the case of positive discounting, $\delta > 0$. A direct formal verification of the key inequality (8) for a given data configuration seems to be difficult. Therefore, we propose an indirect way that uses the results in the recent paper (Belyakov and Veliov 2014) where a relation similar to (8) for the long run averaged maximization problem is obtained. Below we extract the needed here information from Belyakov and Veliov (2014).

For a given $u \in \mathscr{U}$ define

$$J_\tau(u) := \frac{1}{\tau} \int_0^\tau (x[u](t) - c)\, u(t)\, \mathrm{d}t$$

and

$$J(u) := \liminf_{\tau \to +\infty} J_\tau(u), \quad (9)$$

where $x[u]$ is defined as in (5) for the solution n of (1)–(4) corresponding to u. The existence of the above limit follows from the boundedness of $x[u]$.

In Belyakov and Veliov (2014) we have shown that for appropriate configurations of data $(\mu, M, \chi, \beta, \gamma, c)$ there exists a proper periodic control $\hat{u}_p \in \mathscr{U}_p$ and a number $\alpha > 0$ such that

$$J(u) \le J(\hat{u}_p) - \alpha \quad \text{for every constant control } u \in [0, U]. \tag{10}$$

This is done by using a combination of analytic and reliable numerical arguments (the latter resulting from an enhancement of the so-called *properness test* originally developed in Colonius and Kliemann (1989)), and under certain additional conditions that we do not mention here. Instead, we merely assume that for a given particular configuration of data $(\mu, M, \chi, \beta, \gamma, c)$ inequality (10) is fulfilled. The existence of such data is shown in Belyakov and Veliov (2014) and the used arguments clearly indicate that such data configurations are not exceptional.

Our goal in this and in the next section is to show that (10) implies (8) for all sufficiently small discount rates $\delta > 0$. In this section we propose an analytic approach for achieving this goal, which is based on results in Aniţa et al. (2008) and Grüne (1998), which we briefly present for readers' convenience. We begin with the following consequence of Aniţa et al. (2008, Theorem 2.2).

Corollary 1 (of Theorem 2.2 in Aniţa et al. 2008) *For every $u \in \mathscr{U}_p$ with a period $\omega > 0$ there exists a measurable and bounded function $\bar{n} : [0, +\infty) \times [0, A] \to \mathbf{R}$ such that the mapping $t \mapsto \bar{n}(t, \cdot)$ is ω-periodic and the corresponding solution n of system (1)–(4) satisfies*

$$\lim_{t \to +\infty} \|n(t, \cdot) - \bar{n}(t, \cdot)\|_{L_\infty(0,A)} = 0.$$

We stress that Theorem 2.2 in Aniţa et al. (2008) is proved under conditions, denoted there by (H1)–(H5). These conditions follow from (A1)–(A3), together with the following additional one: there is an interval $[a_0, a_0 + \omega] \subset (0, A)$ such that $\beta(a) \ge \beta_0 > 0$ on $[a_0, a_0 + \omega]$. Since this additional condition might be redundant for our model, we do not pose it explicitly, merely assuming that the claim of Corollary 1 holds true.

Lemma 1 *Let $u \in \mathscr{U}_p$. Then $\lim_{\tau \to +\infty} J_\tau(u)$ does exist.*

Proof With the notations from Corollary 1 we can easily show that due to the convergence of $n(t, \cdot)$ we have

$$\lim_{\tau \to +\infty} \left| J_\tau(u) - \frac{1}{\tau} \int_0^\tau (\bar{x}(t) - c) u(t) \, \mathrm{d}t \right| = 0,$$

where

$$\bar{x}(t) = \int_0^A \gamma(a) \, \chi(a) \, \bar{n}(t, a) \, \mathrm{d}a.$$

Since $f(t) := (\bar{x}(t) - c)u(t)$ is ω-periodic, we have

$$\frac{1}{\tau}\int_0^\tau f(t)\,\mathrm{d}t = \frac{1}{\tau}\left(\sum_{i=0}^{k-1}\int_{i\omega}^{(i+1)\omega} f(t)\,\mathrm{d}t + \int_{k\omega}^{\tau} f(t)\,\mathrm{d}t\right),$$

where k is the maximal natural number for which $k\omega \le \tau$, so that $\alpha_\tau := \tau - k\omega \in [0, \omega)$. The above exposed expression can be written as

$$\frac{1}{k\omega + \alpha_\tau}\left(k\int_0^\omega f(t)\,\mathrm{d}t + \int_{k\omega}^{\tau} f(t)\,\mathrm{d}t\right) \longrightarrow \frac{1}{\omega}\int_0^\omega f(t)\,\mathrm{d}t,$$

where the convergence is with respect to $\tau \to +\infty$, that is, $k \to +\infty$. □

The next result that we use below is taken from Grüne (1998).

Lemma 2 *Let $q : \mathbf{R} \to \mathbf{R}$ be a measurable function satisfying $|q(t)| \le N$ for almost every $t \in [0, \infty)$. Assume that there exist $T > 0$ and $\sigma \in \mathbf{R}$ such that*

$$\frac{1}{\tau}\int_0^\tau q(t)\,\mathrm{d}t < \sigma \quad \textit{for all } \tau > T.$$

Then for any $\varepsilon > 0$ and all $\delta \in (0, \varepsilon/T(N + |\sigma + \varepsilon|))$ the following inequality holds:

$$\delta\int_0^\infty e^{-\delta t} q(t)\,\mathrm{d}t \le \sigma + \varepsilon.$$

Remember that for $u \in \mathscr{U}_c$ we denoted by $u^\infty \in [0, U]$ the limit value of $u(t)$ at infinity [see (7)]. Thus for $u_\delta \in \mathscr{U}_c$ (that appears below), u_δ^∞ will denote the corresponding limit value.

Proposition 1 *Assume that (10) is fulfilled. Let $u_\delta \in \mathscr{U}_c^\infty$ be any function such that*

$$J^\delta(u_\delta) \ge \sup_{u\in\mathscr{U}_c^\infty} J^\delta(u) - \frac{\alpha}{16\delta}. \tag{11}$$

Assume also that there exist $T_0 > 0$ and $\delta_0 > 0$ such that

$$J_\tau(u_\delta) < J(u_\delta^\infty) + \frac{\alpha}{16} \quad \text{for all } \delta \in (0, \delta_0) \text{ and all } \tau \ge T_0. \tag{12}$$

Then there exists $\bar{\delta} > 0$ such that

$$\sup_{u\in\mathscr{U}_c^\infty} J^\delta(u) < J^\delta(\hat{u}_p) - \frac{\alpha}{2\delta} \quad \text{for all } \delta \in (0, \bar{\delta}).$$

Proof By the definition of $J(\hat{u}_p)$ and Lemma 1 there exists $\bar{T} \geq T_0$ such that

$$\left|J_\tau(\hat{u}_p) - J(\hat{u}_p)\right| < \frac{\alpha}{16} \quad \text{for all } \tau \geq \bar{T}.$$

Hence,

$$-\frac{1}{\tau}\int_0^\tau (x[\hat{u}_p](t) - c)\hat{u}_p(t)\,\mathrm{d}t < -J(\hat{u}_p) + \frac{\alpha}{16}.$$

Then we apply Lemma 2 with $q(t) = -(x[\hat{u}_p](t) - c)\hat{u}_p(t)$, $\sigma = -J(\hat{u}_p) + \alpha/16$, and $\varepsilon = \alpha/16$. It gives that

$$-\delta J^\delta(\hat{u}_p) = \delta\int_0^\infty -e^{-\delta t}(x[\hat{u}_p](t) - c)\hat{u}_p(t)\,\mathrm{d}t \leq \sigma + \varepsilon = -J(\hat{u}_p) + \frac{\alpha}{8},$$

for every

$$\delta \in \left(0, \frac{\varepsilon}{(N + |\sigma + \varepsilon|)\bar{T}}\right) = \left(0, \frac{\alpha}{16(N + |-J(\hat{u}_p) + \alpha/8|)\bar{T}}\right),$$

where N is an upper bound for $|(x[\hat{u}_p](t) - c)\hat{u}_p(t)|$, $t \in [0, \infty)$. Since obviously also $J(\hat{u}_p) \leq N$, we obtain that

$$\delta J^\delta(\hat{u}_p) \geq J(\hat{u}_p) - \frac{\alpha}{8} \quad \text{for all } \ \delta \in \left(0, \frac{\alpha}{2(16N + \alpha)\bar{T}}\right).$$

On the other hand, from (12) we have for every $\delta \in (0, \delta_0)$ that

$$\frac{1}{\tau}\int_0^\tau (x[u_\delta](t) - c)u_\delta(t)\,\mathrm{d}t < J(u_\delta) + \frac{\alpha}{16}. \tag{13}$$

Then we can apply Lemma 2 with $q(t) = (x[u_\delta](t) - c)u_\delta(t)\,\mathrm{d}t$, $\sigma = -J(u_\delta) + \alpha/16$, and $\varepsilon = \alpha/16$, and obtain in the same way as above that

$$\delta J^\delta(u_\delta) \leq J(u_\delta) + \frac{\alpha}{8} \quad \text{provided that } \ \delta \in \left(0, \frac{\alpha}{2(16N + \alpha)\bar{T}}\right).$$

Denoting $\bar{\delta} = \min\{\delta_0, \alpha/(2(16N + \alpha))\bar{T})\}$ and combining the last inequality with (11) and (13) we obtain that for every $\delta \in (0, \bar{\delta})$ and $\tau \geq \bar{T}$

$$\delta \sup_{u \in \mathscr{U}_c^\infty} J^\delta(u) \leq \delta J^\delta(u_\delta) + \frac{\alpha}{16} \leq J(u_\delta) + \frac{\alpha}{8} + \frac{\alpha}{16} \leq J(\hat{u}_p) - \frac{13\alpha}{16}$$
$$\leq \delta J^\delta(\hat{v}_p) - \frac{13\alpha}{16} + \frac{\alpha}{8} < \delta J^\delta(\hat{v}_p) - \frac{\alpha}{2}.$$

□

Clearly, the claim of the above proposition implies the desired inequality (8), since $\hat{u}_p \in \mathscr{U}_p$. However, the assumptions of the proposition need a discussion. The existence of the controls u_δ in (11) is evident. However, (12) is problematic. On one hand, since $u_\delta(t) \to u_\delta^\infty$, it is possible to prove that the "liminf" in (9) is in fact "lim", and then $J_\tau(u_\delta^\infty) \to J(u_\delta^\infty)$ by definition. Thus the convergence $J_\tau(u_\delta) \to J(u_\delta^\infty)$ can be verified. However, the assumption in Proposition 1 requires this convergence to be uniform with respect to δ in a sufficiently small interval $(0, \delta_0)$. One can further elaborate this assumption, but it still requires a number of qualitative properties of system (1)–(4) which are not available in the literature and are not simple to obtain in general. Therefore, we combine the argument in the above proposition with a numerical study done in the next section.

4 Periodic Optimal Fishing: Numerical Results

In this section we present a numerical results for a particular problem of the type (1)–(6), where the optimal fishing control $u(t)$ is clearly periodic. The specifications for this case study are taken from Belyakov and Veliov (2014). Namely, here $A = 30$, $\mu(a) = 0.005$, $M(z) = 0.001z$, the fertility function $\beta(a)$ equals 0.16 for ages $a \in [10, 20]$ and is zero for all other ages, the biomass of a fish of age a is $\gamma(a) = \frac{2a}{10+a}$. Moreover, the fishing is selective with $\chi(a) = 0$ for $a \leq 10$ and $\chi(a) = 1$ for $a > 10$. The cost of a unit of fishing effort is $c = 2$.

The choice of the above specifications has a good reason. It was shown in Belyakov and Veliov (2014) that there exists a proper periodic admissible control $\hat{u}_p(t)$ that gives averaged objective value $J(\hat{u}_p)$ which is larger than the averaged objective value $J(u)$ resulting from any constant admissible control u. Then Proposition 1 suggests that $J^\delta(\hat{u}_p) > \sup_{u \in \mathscr{U}_c^\infty} J^\delta(u)$ for all sufficiently small $\delta > 0$. This turns out to be the case. The numerical results below are obtained for discount rate $\delta = 0.01$ and maximal fishing effort $U = 0.5$. The solution of problem (1)–(6) in the class $\mathscr{U}$ of admissible controls turns out to be periodic in the long run. It is obtained by using the optimality conditions in Feichtinger et al. (2003) (applied for a finite, but sufficiently large horizon) and utilization of a gradient projection method. A detailed description of the numerical approximation scheme, including error analysis, is presented in the forthcoming paper (Veliov 2015).

Figure 1 present the optimal control, $u(t)$, together with the average age of fish and the number of newborns, all restricted to the time-interval $[100, 140]$. After an initial transitional time-interval the optimal control stabilizes to a periodic pattern, which is of bang-bang type: either the harvesting has maximal intensity ($u(t) = U$), or the fish stock is left to recover ($u(t) = 0$).

Figure 1 helps to understand the reason for which selective fishing may lead to a periodic optimal control. We see that during an intense fishing period both the average age of fish and the inflow of newborns steeply decrease. The former, due to the fact that young fish is not harvested, the later, due to the harvesting of fish at fertile ages. A stock of young fish with maturation still to come stays

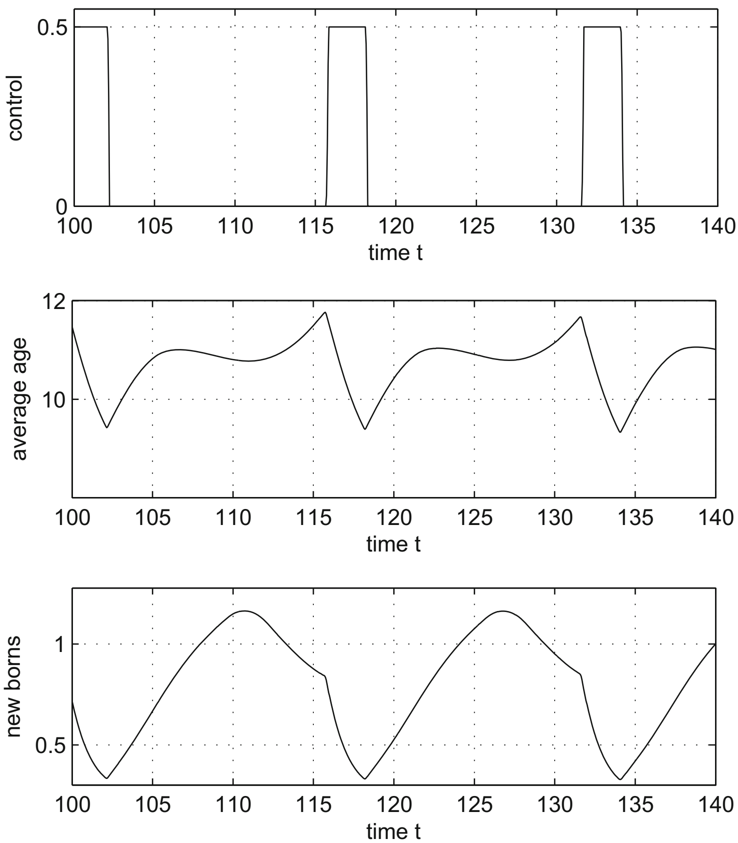

Fig. 1 Optimal harvesting (*top*), average population age (*middle*), and newborn flow (*bottom*) for $\delta = 0.01$ and $U = 0.5$

unaffected. If the harvesting continued longer, then a part of this stock would have failed to give offspring. A period of no-harvesting enables the still unaffected wave of young fish to reproduce. In our case study this period takes about 14 units of time (call them years). As the third plot in Fig. 1 shows, in this period the number of newborns steeply increases for a while (some 10 years), then decreases again since the preserved stock of young fish becomes infertile (this also explains the increase of the average age some 10 years after the harvesting period).

We should mention that the above explanations are conditional: as suggested by Proposition 1, they apply to sufficiently small discount rates. If the discount rate δ is too large, then the concern about the regeneration potential is small. Indeed, further numerical analysis of the case study shows that for $\delta = 0.1$ the optimal harvesting effort converges to a constant level, that is, it is not asymptotically proper periodic.

On the other hand, the periodicity of the optimal control is robust with respect to presence of a quadratic term in the cost function. Of course, in this case the optimal control will be continuously oscillating instead of jumping.

5 The Case of Non-selective Fishing

In this section we show that the selectivity of fishing (defined by the age-dependent fishing pattern $\chi(a)$) is responsible for the asymptotically non-constant/periodic behavior of the optimal fishing. To do this we consider the case of non-selective fishing: $\chi(a) \equiv \chi \in (0, \infty)$. Without any restriction we assume that $\chi = 1$.

Let us fix an arbitrary admissible control $u \in \mathscr{U}$. Following Sect. 3.2, Chap. 3 in Aniţa (2000) (see also Iannelli 1995; Thieme 2003) and making use of the stationarity of the vital rates in our model, we represent the unique solution $n(t, a)$ of system (1)–(4) in the form

$$n(t, a) = y(t)\, \tilde{n}(t, a),$$

where y is absolutely continuous and $\tilde{n}$ is measurable and absolutely continuous along the characteristic lines of $\mathscr{D}$. To do this, for an arbitrary real number α (viewed as a parameter) we consider the following two systems:

$$\mathscr{D}\tilde{n}(t, a) = -\,(\mu(a) + \alpha)\, \tilde{n}(t, a), \tag{14}$$

$$\tilde{n}(0, a) = n^0(a), \tag{15}$$

$$\tilde{n}(t, 0) = \int_0^A \beta(a)\, \tilde{n}(t, a)\, \mathrm{d}a, \tag{16}$$

and

$$\dot{y}(t) = -\,(M(\tilde{z}(t)\, y(t)) - \alpha + u(t))\, y(t), \quad y(0) = 1, \tag{17}$$

where $\tilde{z}(t)$ is defined as

$$\tilde{z}(t) = \int_0^A \gamma(a)\, \tilde{n}(t, a)\, \mathrm{d}a. \tag{18}$$

We chose α to be the *Malthusian parameter*, that is, the value for which the unique solution of system (14)–(16), $\tilde{n}(t, \cdot)$, converges to a steady state $\bar{n}$ in the space $L_\infty(0, A)$. The Malthusian parameter α is determined from the condition that the *net reproduction rate* is equal to 1:

$$\int_0^A \beta(a)\, e^{-\int_0^a \mu(\eta)\, \mathrm{d}\eta - \alpha a}\, \mathrm{d}a = 1.$$

Due to Assumption (A2), for $\alpha = 0$ the left-hand side is strictly bigger than 1. Since the left-hand side is strictly decreasing and continuous in α, and converges to zero when $\alpha \to +\infty$, the above equation has a unique solution $\alpha > 0$.

It is a matter of direct substitution to check that $n = y\tilde{n}$ is the (unique) solution of system (1)–(4). Notice that $\tilde{n}$, thus also $\tilde{z}$, is independent of the control u. Then the optimal control problem (1)–(6) can be equivalently reformulated as

$$\int_0^\infty e^{-\delta t} \left(\tilde{z}(t)\, y(t) - c\right) u(t)\, \mathrm{d}t \to \max_{u(\cdot)}, \qquad u(t) \in [0, U], \tag{19}$$

subject to (17).

Due to Assumption (A3) for M it is straightforward to prove that there is a constant K such that the interval $[0, K]$ is invariant with respect to (17) with any admissible control u, and any trajectory enters $[0, K]$ in finite time.

We mention that by a standard argument problem (17), (19) has a solution due to the linearity with respect to u, the compactness of the admissible control values, $[0, U]$, the above boundedness property, and the inequality $\delta > 0$ (take an L_2-weakly convergent maximizing sequence and pass to the limit).

Our aim in the rest of the section is to prove (on minor additional assumptions) that any solution of problem (17), (19) is asymptotically constant. Then the same will be true for the optimal fishing problem (1)–(6).

We begin with some preliminary considerations and notations. The steady state $\bar{n}$ is nontrivial due to assumption (A4). This implies that $\tilde{n}(t, \cdot)$ is nontrivial for every $t \geq 0$. From (18) and $\gamma(a) \geq \gamma_0 > 0$ we have that $\tilde{z}(t) > 0$. Passing to the limit in (18) with $t \to +\infty$ and using the non-triviality of $\bar{n}(\cdot)$ we obtain that $\tilde{z}(t)$ converges to $\bar{z} := \int_0^A \gamma(a)\bar{n}(a)\, \mathrm{d}a > 0$. Since there exists $\bar{N}$ such that $\tilde{n}(t, a) \leq \bar{N}$ for all (t, a) and is Lipschitz continuous along the characteristic lines of (14), it is easy to prove that $\tilde{z}(\cdot)$ is (Lipschitz) continuous. Together with the already established properties of $\tilde{z}$, this implies that there exist $z_0 > 0$ and $z_1 > z_0$ such that $z(t) \in [z_0, z_1]$ for all $t \geq 0$.

Denote by y^* and y^0 the unique numbers for which

$$\bar{z}y^* = c \quad \text{and} \quad M(\bar{z}y^0) = \alpha.$$

Proposition 2 *Let $(\tilde{y}, \tilde{u})$ be a solution of problem (17), (19). Then the following claims hold:*

(i) If $M(c) = \alpha$ then $\tilde{y}(t)$ converges to y^0 when $t \to +\infty$;
(ii) if $M(c) > \alpha$ then $\tilde{u}(t) = 0$ for all sufficiently large t and $\tilde{y}(t)$ converges to y^0 when $t \to +\infty$;
(iii) if $M(c) < \alpha$ and if, additionally, $c > 0$, μ is continuous, β is continuously differentiable, M is twice continuously differentiable, $M(c) - \alpha + U \geq 0$, and

$$2M'(\xi) + (\xi - c)M''(\xi) > 0 \quad \text{for every} \quad \xi \in [c, M^{-1}(\alpha)], \tag{20}$$

then $\tilde{y}(t)$ and $\tilde{u}(t)$ converge when $t \to +\infty$.

Before proving the proposition we make some comments. First, in case (i) we prove only convergence of $\tilde{y}(t)$. It might be true that also $\tilde{u}(t)$ converges (to zero), however, this requires further analysis. We pay no attention to this case, since it is non-generic: notice that α is determined solely by the population parameters while c has purely economic meaning, thus the two are not correlated. Second, obviously assumption (20) is fulfilled if M is convex (a plausible assumption), but is even weaker than that. The other additional assumptions in (iii) are also not restrictive, except $M(c) - \alpha + U \geq 0$. The latter requires that the upper bound U is sufficiently large. We believe that the last assumption can be substantially relaxed, but the proof would require some additional work.

Proof (of Proposition 2) We introduce the notations $\tilde{y}^*(t) = c/\tilde{z}(t)$ and $\tilde{y}^0(t) = M^{-1}(\alpha)/\tilde{z}(t)$. Observe that the optimal solution $(\tilde{y}, \tilde{u})$ has the following property: if $\tilde{y}(t) < \tilde{y}^*(t)$ then $\tilde{u}(t) = 0$. (Since harvesting decreases the stock of fish, the above statement has the obvious meaning that fish is harvested only if the net revenue is positive. The formal proof is easy.)

Part 1. Let $M(c) \geq \alpha$. Due to the monotonicity of M this implies that $y^0 \leq y^*$. Fix an arbitrary positive number $\rho < y^0$. Due to the convergence of $\tilde{z}(t)$ to $\bar{z}$, we have that $\tilde{y}^*(t) \to y^*$ and $\tilde{y}^0(t) \to y^*$. Thus there exists a number $\tau(\rho)$ such that $|\tilde{y}^0(t) - y^0| \leq \rho/2$ and $|\tilde{y}^*(t) - y^*| \leq \rho/2$ for every $t \geq \tau(\rho)$. Since $M'(\xi) > 0$ for $\xi > 0$, there exists $\varepsilon > 0$ such that $M(\tilde{z}(t)y) \geq \alpha + \varepsilon$ for $y \geq y^0 + \rho$, $M(\tilde{z}(t)y) \leq \alpha - \varepsilon$ for $y \leq y^0 - \rho$, and $\tilde{z}(t)y < c$ for $y \leq y^* - \rho$.

Assume that for some $t \geq \tau(\rho)$ it holds that $\tilde{y}(t) \geq y^0 + \rho$. Then

$$\dot{\tilde{y}}(t) \leq -(M(\tilde{z}(t)\tilde{y}(t)) - \alpha)\tilde{y}(t) \leq -\varepsilon\tilde{y}(t).$$

This implies that $\tilde{y}(t) \leq y^0 + \rho$ for all sufficiently large t.

Now assume that for some $t \geq \tau(\rho)$ it holds that $\tilde{y}(t) \leq y^0 - \rho$. Then

$$\dot{\tilde{y}}(t) = -(M(\tilde{z}(t)\tilde{y}(t)) - \alpha)\tilde{y}(t) \geq \varepsilon\tilde{y}(t),$$

where we use that $\tilde{z}(t)\tilde{y}(t) < c$, hence $\tilde{u}(t) = 0$. This implies that $\tilde{y}(t) \geq y^0 - \rho$ for all sufficiently large t. Since $\rho > 0$ was arbitrarily chosen (sufficiently small), we obtain that $\tilde{y}(t)$ converges to y^0, thus claim (i).

Part 2. Under the condition in claim (ii) we have, in addition, that $y^* > y^0$, thus $\tilde{y}(t) < y^* - \rho$ for some $\rho > 0$ and all sufficiently large t. Then $\tilde{z}(t)\tilde{y}(t) < c$ for all sufficiently large t, hence $\tilde{u}(t) = 0$. Claim (ii) is proved.

Part 3.1. The non-trivial case is $M(c) < \alpha$, which implies $y^* < y^0$. First, we shall establish some smoothness and convergence properties of $\tilde{z}$. Consider the function

$$B(t) := \int_0^A \beta(a)\tilde{n}(t,a)\,\mathrm{d}a.$$

Similarly as for $\tilde{z}$, one can argue that B is (Lipschitz) continuous. Moreover, from (14) $\tilde{n}$ can be expressed as

$$\tilde{n}(t,a) = e^{-\int_0^a (\mu(\theta)+\alpha)\,\mathrm{d}\theta} B(t-a), \qquad \text{for} \quad t > A. \tag{21}$$

Hence, B satisfies for $t > A$ the equation

$$B(t) = \int_0^A k(a)B(t-a)\,\mathrm{d}a, \qquad k(a) := \beta(a)e^{-\int_0^a (\mu(\theta)+\alpha)\,\mathrm{d}\theta}.$$

Using the continuity of B and the above equality, it is a routine task to prove that B is continuously differentiable and

$$\dot{B}(t) = \int_0^A k'(a)B(t-a)\,\mathrm{d}a + \beta(0)B(t) - k(A)B(t-A). \tag{22}$$

Due to the convergence of $\tilde{n}(t,\cdot)$ we have that $B(t)$ also converges when $t \to +\infty$ to the limit $\bar{B} := \int_0^A \beta(a)\bar{n}(a)\,\mathrm{d}a$. Then the right-hand side in (22) converges to

$$\int_0^A k'(a)\bar{B}\,\mathrm{d}a + \beta(0)\bar{B} - k(A)\bar{B} = 0.$$

Thus $\dot{B}(t)$ converges to zero. Then from (22) we recursively conclude that B is two-times differentiable for all $t > 2A$, three-times differentiable for all $t > 3A$, and so on. Each derivative converges to zero. Then from (21) it follows, in particular, that $\tilde{n}$ is two-times differentiable with respect to t for all $t > 3A$ and each partial derivative in t converges to zero uniformly in a. From (18) we obtain that $\dot{\tilde{z}}(t)$ and $\ddot{\tilde{z}}(t)$ exist for all $t > 3A$ and both converge to zero when $t \to +\infty$.

Part 3.2. Inequality (20) is equivalent to

$$2M'(\bar{z}y) + (\bar{z}y - c)M''(\bar{z}y) > 0 \quad \text{for every} \quad y \in [y^*, y^0]. \tag{23}$$

Then there exist positive numbers σ and $\rho < y^*$ such that

$$\bar{z}^2[2M'(\bar{z}y) + (\bar{z}y - c)M''(\bar{z}y)] > 2\sigma \quad \text{for every} \quad y \in [y^* - \rho, y^0 + \rho]. \tag{24}$$

Since $\tilde{z}(t)$, $\tilde{y}^*(t)$, $\tilde{y}^0(t)$ converge to $\bar{z}$, y^*, y^0, respectively, there exist $\tau > 0$ and $\nu > 0$ such that[1] for $t \geq \tau$ we have $\tilde{y}^*(t) > y^* - \rho$ and

$$M(\tilde{z}(t)y) - \alpha < -\nu \quad \text{for } y \leq y^* - \rho, \quad M(\tilde{z}(t)y) - \alpha > \nu \quad \text{for } y \geq y^0 + \rho.$$

[1] The number ρ may be eventually decreased, and τ increased, later on in a correct way, that is, this can be made right here, but we postpone it to a later point for more clarity.

This allows to prove that the interval $[y^0 - \rho, y^* + \rho]$ is invariant with respect to Eq. (17) with $u = \tilde{u}$, and $\tilde{y}(t)$ reaches it in a finite time. Indeed, if for some $t \geq \tau$ it holds that $\tilde{y}(t) > y^0 + \rho$ then

$$\dot{\tilde{y}}(t) = -(M(\tilde{z}(t)\tilde{y}(t)) - \alpha + \tilde{u}(t))\tilde{y}(t) \leq -\nu\tilde{y}(t).$$

If it holds that $\tilde{y}(t) < y^* - \rho$, then $\tilde{y}(t) < \tilde{y}^*(t)$, thus $\tilde{u}(t) = 0$ and

$$\dot{\tilde{y}}(t) = -(M(\tilde{z}(t)\tilde{y}(t)) - \alpha)\tilde{y}(t) \geq \nu\tilde{y}(t).$$

These two inequalities prove both the invariance of $[y^0 - \rho, y^* + \rho]$ and the fact that it is reached in a finite time. Thanks to this and the dynamic programming principle, for all sufficiently large τ the restriction of $(\tilde{y}, \tilde{u})$ to $[\tau, \infty)$ is a solution of the problem

$$\int_\tau^\infty e^{-\delta t}\,(\tilde{z}(t)\,y(t) - c)\,u(t)\,\mathrm{d}t \to \max_{u(\cdot)}, \qquad u(t) \in [0, U], \tag{25}$$

$$\dot{y}(t) = -\,(M(\tilde{z}(t)\,y(t)) - \alpha + u(t))\,y(t), \quad y(\tau) = \tilde{y}(\tau), \tag{26}$$

$$y(t) \in [y^0 - \rho, y^* + \rho]. \tag{27}$$

Part 3.3. In order to investigate the solution $(\tilde{y}, \tilde{u})$ for $t > \tau$ we apply to problem (25)–(27) the most rapid approach path theorem (Hartl and Feichtinger 1987, Theorem 3.1), which deals with non-stationary problems.

Solving (26) for u and substituting u in (25) we come up with the problem

$$\max_y \int_{\tau_0}^\infty e^{-\delta t}[P(t, y(t)) + Q(t, y(t))\dot{y}(t)]\,\mathrm{d}t, \tag{28}$$

where,

$$P(t, y) = -(\tilde{z}(t)y - c)(M(\tilde{z}(t)y) - \alpha), \quad Q(t, y) = -\frac{\tilde{z}(t)y - c}{y}.$$

The initial condition is $y(\tau) = \tilde{y}(\tau) > 0$ and due to the constraint $u \in [0, U]$ the function y has to satisfy the inclusion

$$\dot{y}(t) \in \Omega(t, y(t)) \quad \text{with} \quad \Omega(t, y) := [-(M(\tilde{z}(t)y) - \alpha + U)y, -(M(\tilde{z}(t)y) - \alpha)y], \tag{29}$$

along with the state constraints (27). The restriction of $(\tilde{y}, \tilde{u})$ to $[\tau, \infty)$ is a solution of this problem.

In order to apply (Hartl and Feichtinger 1987, Theorem 3.1) we have to investigate for $y \in [y^* - \rho, y^0 + \rho]$ the equation $I(t, y) = 0$ with

$$I(t, y) := -\delta Q(t, y) + Q_t(t, y) - P_y(t, y)$$
$$= -\dot{\tilde{z}}(t) + \delta \left(\tilde{z}(t) - \frac{c}{y} \right) + \tilde{z}(t)[M(\tilde{z}(t)y) - \alpha + (\tilde{z}(t)y - c)M'(\tilde{z}(t)y)].$$

Observe that

$$I(t, y) = \delta \left(\bar{z} - \frac{c}{y} \right) + \bar{z}[M(\bar{z}y) - \alpha + (\bar{z}y - c)M'(\bar{z}y)] + \varepsilon(t, y) =: \bar{I}(y) + \varepsilon(t, y),$$

where due to the convergence of $\tilde{z}(t)$ to $\bar{z}$ and of the first two derivatives of $\tilde{z}$ to zero, also the function $\varepsilon(t, y)$ as well as the derivatives $\varepsilon_t(t, y)$ and $\varepsilon_y(t, y)$ converge to zero when $t \to +\infty$, uniformly in $y \in [y^*, y^0]$. Since

$$\bar{I}(y^*) = \bar{z}[M(\bar{z}y^*) - \alpha] < 0, \qquad \bar{I}(y^0) = \delta \frac{\bar{z}y^0 - c}{y^0} + \bar{z}(\bar{z}y^0 - c)M'(\bar{z}y^0) > 0. \tag{30}$$

and [see (24)]

$$\bar{I}'(y) = \delta \frac{c}{y^2} + \bar{z}^2[2M(\bar{z}y) + M''(\bar{z}y)(\bar{z}y - c)] \geq 2\sigma,$$

the function $\bar{I}$ has a unique zero $\bar{y} \in (y^*, y^0)$, which is the unique zero also in $[y^* - \rho, y^0 + \rho]$ if $\rho > 0$ is fixed as sufficiently small. Then the inequalities

$$I(t, y^* - \rho) < 0, \quad I(t, y^0 + \rho) > 0, \quad I_y(t, y) \geq \sigma$$

hold for every $y \in [y^* - \rho, y^0 + \rho]$ and $t \geq \tau$, provided that $\rho > 0$ and τ are appropriately fixed (see Footnote 1). Consequently, $I(\cdot, t)$ has a unique single zero $\hat{y}(t) \in ([y^* - \rho, y^0 + \rho)$. The implicit function theorem also claims that $\hat{y}$ is differentiable and

$$\dot{\hat{y}}(t) = \frac{I_t(t, \hat{y}(t))}{I_y(t, \hat{y}(t))} = \frac{\varepsilon_t(t, \hat{y}(t))}{\bar{I}'(\hat{y}(t)) + \varepsilon_y(t, \hat{y}(t))} \longrightarrow 0.$$

Since every condensation point of $\hat{y}(t)$ at $+\infty$ satisfies $\bar{I}(y) = 0$ and belongs to $[y^* - \rho, y^0 + \rho]$, it must coincide with $\bar{y}$. Thus $\lim_{t \to +\infty} \hat{y}(t) = \bar{y}$.

As a recapitulation, in the paragraph above we show in particular that $I_y(t, y) > 0$, in $[y^* - \rho, y^* + \rho]$ and the equation $I(t, y) = 0$ has a unique solution, $\hat{y}(t)$, in this interval. This is one of the suppositions in Hartl and Feichtinger (1987, Theorem 3.1).

A second assumption in Hartl and Feichtinger (1987, Theorem 3.1) is that

$$\lim_{t \to +\infty} e^{-\delta t} \int_{y(t)}^{\hat{y}(t)} Q(t, \xi)\, \mathrm{d}\xi \geq 0$$

for every admissible $y(\cdot)$ in problem (28), (29), (27). Obviously the above limit equals zero due to $y(t),\ \hat{y}(t) \in [y^* - \rho, y^0 + \rho]$, where $Q(t, \cdot)$ is bounded uniformly in t.

The third assumption in Hartl and Feichtinger (1987, Theorem 3.1) is that $\hat{y}(t)$ satisfies the inclusion in (29), which reads as

$$\dot{\hat{y}}(t) \in [-(M(\tilde{z}(t)\hat{y}(t)) - \alpha + U)\hat{y}(t),\ -(M(\tilde{z}(t)\hat{y}(t)) - \alpha)\hat{y}(t)].$$

Since we proved that $\hat{y}(t)$ converges to $\bar{y}$ and $\dot{\hat{y}}(t)$ converges to zero, and since $-(M(\bar{z}\bar{y}) - \alpha)\bar{y} > 0$ (due to $\bar{y} < y^0$) the upper constraint is satisfied by $\dot{\hat{y}}(t)$ for $t \geq \tau$, provided that τ is fixed sufficiently large. If we assume that the lower constraint is not satisfied for arbitrarily large t, then $\dot{\hat{y}}(t_k) < -(M(\tilde{z}(t_k)\hat{y}(t_k)) - \alpha + U)\hat{y}(t_k)$ for some sequence $t_k \to +\infty$. Passing to the limit we obtain

$$0 \leq -(M(\bar{z}\bar{y}) - \alpha + U)\bar{y}, \quad \text{hence,} \quad 0 \geq M(\bar{z}\bar{y}) - \alpha + U > M(c) - \alpha + U,$$

which contradicts the assumption in part (iii) of the theorem.

Now, we can apply Theorem 3.1 in Hartl and Feichtinger (1987), which claims that $\tilde{y}(t)$ reaches $\hat{y}(t)$ in a finite time and then coincides with it, hence it converges to $\bar{y}$. This completes the proof of the proposition. □

6 Final Discussion

Below we indicate some open questions related to subject of the present paper.

1. The proof of superiority of proper periodic controls given in Sect. 3 is based on the assumption (12) that is not directly checkable. Proving that it is fulfilled would make the conclusion for non-optimality of asymptotically constant controls ultimate (at least for small discount rates).
2. The harvesting age-profile $\chi(a)$ is assumed fixed in this paper. However, it makes sense to consider it also as a decision function, as in Aniţa (2000, Chap. 3, Sect. 3.3) or Park et al. (1998) for example, where control depends also on age, $u(t, a)$. It is an open question if the optimal harvesting effort will still be asymptotically non-constant in this case.

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Maximizing Female Labor Force Participation in a Stationary Population

Elke Moser, Alexia Prskawetz, Gustav Feichtinger, and Bilal Barakat

Abstract As many European countries have to cope with a shrinking and aging labor force, one important goal of redistributing work is to increase female labor force participation. In some countries, however, this increase could come at the cost of a reduced fertility rate as childcare facilities might be rare or institutional settings and social support are not sufficiently family friendly. In this paper we investigate how and especially at which ages female labor force participation could be increased in a country such as Austria, with an apparently strong negative correlation between childbearing and labor force participation, without reducing fertility even further. Our results indicate that an increase in female labor force participation is indeed possible if the participation rate remains low in the most fertile ages. It turns out, however, that the optimal labor force participation for females strongly depends on the initial fertility pattern of the female population.

1 Introduction

Faced with an aging and shrinking labor force, the redistribution of work is a major challenge in Europe (c.f. Vaupel and Loichinger 2006). Most European countries are aiming to integrate older workers, the unemployed and migrants into their labor force and to increase the labor force participation of women. With respect to the latter group, an obvious trade off exists between increasing female labor force and keeping fertility from falling even further. In countries such as Sweden or France,

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female labor force participation has a long tradition. Institutional settings, social support and family friendly norms and values foster the combination of childbearing and employment. In contrast, in countries such as Austria or Italy, the negative correlation between female labor force participation and fertility is persistent and partly explained by the difficulty to combine active employment with childbearing. So far, simulation models of future developments of the labor force, that assume an increase in female labor force participation, have not yet taken into account the feedback mechanisms that higher labor force participation of women may exert on fertility. The aim of our study is to present a formal model that allows to derive the optimal age-specific female labor force participation in a stationary population where the objective is to maximize overall employment of women but taking into account the repercussion of female labor force participation on fertility and hence on future developments of the population size. We calibrate our model to the Austrian age specific fertility pattern and female labor force participation rate. Our results indicate that the optimal age specific labor force participation to obtain a stationary population size should closely follow the inverse of the age specific fertility pattern. Depending on the strength of the negative repercussion of labor force on fertility, we can analytically derive the ages where labor force should be reduced (increased) above the Austrian labor force participation rates.

Figure 1a shows the average age-specific female labor force participation rates over the time period 2000–2009 for Austria, Sweden and Italy while Fig. 1b depicts the age-specific fertility rates for these countries over the same time period. Sweden stands out with the highest female labor force participation rate combined with the highest fertility rates among the set of three countries we consider. In contrast, Austria and particularly Italy have much lower female labor force participation rates combined with lower fertility rates. Obviously the correlation between female labor force participation and fertility depends on country specific institutional settings that ease or hamper the combination of child-rearing and employment. Sweden is well known for its support towards families with children in terms of child care facilities but also for its more generous and flexible maternal/paternal leave regulations. On the contrary in Austria and Italy the availability of child care facilities is much more

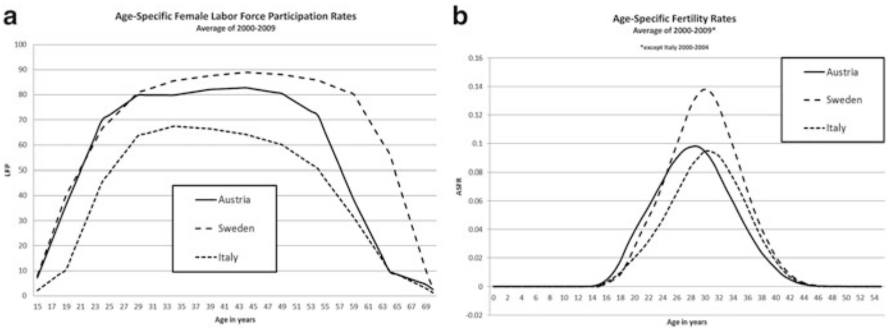


Fig. 1 (**a**) Age-specific female labor force participation rates. (**b**) Age-specific fertility rates

restricted and the labor market regulations are often less flexible. Moreover, norms and values towards combining childbearing and work, are only changing slowly in Austria and Italy.

2 The Model

To investigate whether and at which ages the Austrian female labor force participation rate could be increased without further reducing overall fertility, we consider an optimal control model. The objective is to determine the optimal age specific female labor participation rates that would maximize overall female labor force participation taking into account the negative feedback of female labor force participation rates on fertility rates and restricting our population to a stationary population. We furthermore assume that the population is closed, and unisex with only females. We constrain the age specific labor force participation assuming the participation rates are bounded from below by the Italian labor force participation rates and from above by the Swedish labor force participation rates.[1] If the optimal age specific labor force participation rate exceeds the corresponding Austrian value, we assume that the corresponding age specific fertility rate declines. Our argument is that an increase in the labor force participation rates is incompatible with the same fertility rate since less time is left for child-rearing and child-care facilities are supposed to be scarce and therefore only a limited substitute for own time spent on childbearing. Similarly, we argue that an optimal labor force participation rate below the Austrian one will increase the corresponding age specific fertility level as more time can be devoted to childrearing. We are aware of the fact that other correlates like income, norms, values, etc. are important determinants of fertility. For analytical convenience we neglect those determinants to focus solely on the relation between female labor force participation and fertility.

With our model we aim to highlight the possible optimal tradeoff between fertility and labor force participation without arguing that the implementation of such policies may indeed be feasible for a social planner. Our framework is intended to indicate during which life cycle stages of females family and labor market policies may be in highest demand if we would aim for an overall increase in total female labor force. In an ideal world, labor market and family policies should be coordinated and related to each other. However, in reality these two policies are often independently designed and therefore we ignore these interactions in our model set up as well. We are also aware of the fact that a reduction/increase in the labor force participation rate may not necessarily result in an increase/decrease of

[1]Note, that there are small intervals around the age of 20 and 65, where the Austrian labor force participation rate is slightly higher/lower than the Swedish/Italian rate for this short period. In these cases we use the Austrian rate as upper/lower bound for the control.

fertility. Various other socio-economic and institutional factors as well as norms and values will influence these decisions. However, the aim of our model is to demonstrate how far we could theoretically increase female labor force if fertility and labor force participation would be negatively related and assuming a stationary population. The fertility pattern is assumed to follow the observed age specific fertility rates for Austria, while the labor force participation is determined to maximize the overall labor force but assumed to be bounded from below and above. We are neither aiming to describe or predict female labor force participation behavior in our model, but our aim is to discuss how labor force would have to be redistributed in a society to maximize the overall labor force without further reducing population growth.

In the following we present the various analytical expressions of our model.[2] Assuming that females in the labor force have a different age specific fertility rate $x_W(a)$ as those out of the labor force (whose fertility rate is $x_N(a)$),[3] and denoting by $p^A(a)$ the age-specific labor force participation rate in Austria of the considered time period 2000–2009, then the Austrian fertility rate at each age a can be described as a linear combination of the form

$$f^A(a) = x_W(a)p^A(a) + x_N(a)(1 - p^A(a)). \tag{1}$$

We assume that the fertility of a working woman, $x_W(a)$ is proportional to the one of a non-working woman, $x_N(a)$, with the proportionality factor $c + \tau$ being the same for each age group. The parameter $0 < c < 1$ denotes the ratio between the fertility rate of a working and a non-working women in case that the government does not offer any social support. By offering child care facilities or other institutional settings, the fertility rate of working mothers may be higher. In this case we assume that the proportionality factor c increases by the value τ, i.e. the sum $0 < c + \tau < 1$ indicates the ratio between working and non working females fertility . Hence, this relation reads as

$$x_W(a) = (c + \tau)\, x_N(a) \text{ with } 0 < c + \tau < 1. \tag{2}$$

The specific relation between fertility rates of working and non working mothers as introduced above constitutes a rather specific case and has been chosen for simplification in a first set up of the model. The size of this proportionality factor will determine the difference between fertility of working and non working females assuming that the total fertility is given by the age-specific Austrian fertility rate. The calibration of the fertility of non working and working mothers given

[2]Note that our model is based on aggregate period indicators, i.e. we consider age specific aggregate fertility and labor force participation of the whole Austrian population at a specific time period. Our model does not represent a life cycle model which would aim to replicate the life cycle decisions of individual.

[3]Note that we only consider full time employment.

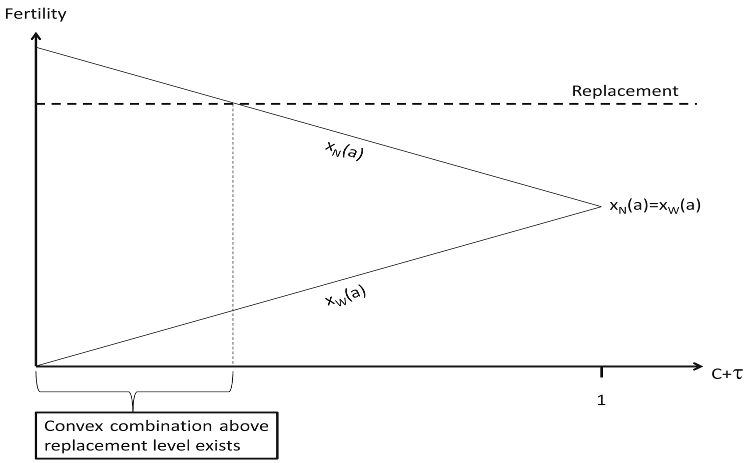


Fig. 2 Estimation of $x_N(a)$ and $x_W(a)$ with respect to $c + \tau$ and given an age specific schedule of Austrian fertility

the observed overall fertility determines the initial fertility pattern of our model population. We fix $c = 0.01$ which represents a high differential between fertility rates of working and non working females in case that no social support is offered by the government. The parameter τ will be changed in the course of the analysis to investigate various combinations of working versus non working female‘s fertility. Figure 2 illustrates the relation between working and non-working mother's fertility if we vary the proportionality factor $c + \tau$. For low values of $c + \tau$, we obtain a population with a very pronounced difference between working and non-working mother's fertility and the overall fertility is mainly determined by the fertility level of the non-working mothers. On the other hand , high values of $c + \tau$ represent a population where child care facilities enable a high fertility even for working women and both populations (the working and non-working females) contribute almost equally to the population's fertility. If $c + \tau = 1$, both fertility rates are the same. Figure 2 also shows that a convex combination of $x_N(a)$ and $x_W(a)$ will exceed the replacement level fertility only for low values of $c + \tau$ where the fertility rate of non working females x_N exceeds replacement level fertility. Hence, for our calibration that is based on the Austrian age specific fertility rates, any change in the labor force participation rate will have only a pronounced effect on the overall fertility if the fertility levels between working and non-working females are sufficiently distinct. Otherwise, any change of the female labor force would be too low to have a pronounced effect on overall fertility.

Combining Eqs. (1) and (2) yields

$$f^A(a) = (c+\tau)\, x_N(a)p^A(a) + x_N(a)(1-p^A(a)) \tag{3}$$

$$\Leftrightarrow f^A(a) = x_N(a)\left((c+\tau)p^A(a) + (1-p^A(a))\right) \tag{4}$$

$$\Leftrightarrow x_N(a) = \frac{f^A(a)}{1-p^A(a)(1-(c+\tau))}. \tag{5}$$

Equation (5) states the postulated relation between the age-specific Austrian fertility rate $f^A(a)$, the Austrian specific labor force participation rate $p^A(a)$, and the age specific fertility rate of non working females for a given set of parameters c, τ.

Starting from this initial fertility pattern, a change in the female labor force participation rate would affect the overall fertility by changing the linear combination of working and non working women's fertility rates. Further on, however, we assume that changing the labor force participation rate also directly affects the fertility of working females. In contrast, the fertility of women outside the labor force is assumed to remain unchanged. This assumption is quite realistic for Austria since an increase in the labor force participation of women has often not been accompanied with the required extension of childcare facilities. Therefore, an increase in the female labor force participation may imply a decrease in fertility for women in the labor force due to the excess demand for childcare facilities. Hence, we set the age specific fertility of working females, in the following denoted as $x_W^*(a)$, as

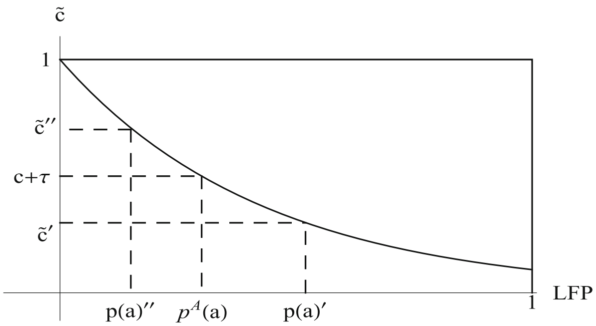
$$x_W^*(a) = \tilde{c}(p(a))x_N(a), \tag{6}$$

where $\tilde{c}(p(a))$ depends on the age specific labor force participation rate. We choose the following analytical function of $\tilde{c}(p(a))$

$$\tilde{c}(p(a)) = (c+\tau)^{\frac{p(a)}{p^A(a)}}. \tag{7}$$

For a given set of parameters c and τ (that represent the prevailing family policies), $\tilde{c}(p(a))$ represents the change in the fertility rate when the labor force participation rate is modified compared to the prevailing levels for Austria in the considered time period 2000–2009. Note, that this function is convex in the labor force participation rate $p(a)$ as illustrated in Fig. 3. If $p(a) = p^A(a)$, the relation between working and non working female's fertility remains the same. If, however, $p(a) > p^A(a)$ holds, then the higher labor force participation rate comes along with scarcer childcare facilities. Therefore, $x_W^*(a)$ declines below the level of the Austrian fertility rate of working females and becomes more distinct to the fertility rate of non working females. In case that $p(a) < p^A(a)$ holds, the scarcity of childcare facilities is reduced and the fertility of working females $x_W^*(a)$ increases. In this case the difference between working and non working female's fertility decreases. Intuitively, the convexity of the function represents the fact that the reduction in fertility due to an increase in the labor force participation rate is less than the

Fig. 3 Fertility share for working women under a changing labor force participation rate



increase of fertility rates as a consequence of lower labor force participation rates. Put differently, a movement from $p^A(a)$ to $p(a)'$ in Fig. 3 will imply that child care facilities are short in supply. However, one may argue that not all of the women who are joining the labor market will reduce their fertility, and henceforth the decline in age specific fertility rate is convex. On the other hand, a movement from $p^A(a)$ to $p(a)''$ in Fig. 3 implies that less women are in the labor force and hence the demand of childcare facilities decreases. As a consequence more females who have previously decided to stay childless once they enter the labor force may now change their mind and decide for children.[4] For this reasons $\tilde{c}(p(a))$ increases at an increasing rate if labor force participation drops below $p^A(a)$.

Applying the new definition of the fertility rate of working females, the overall age specific fertility can again be written as linear combination as follows:

$$f(a) = x_W^*(a)p(a) + x_N(a)(1 - p(a)). \tag{8}$$

Using Eqs. (5) and (6) yields

$$f(a) = \frac{(c+\tau)^{\frac{p(a)}{p^A(a)}} f^A(a)}{1 - p^A(a)(1-(c+\tau))} p(a) + \frac{f^A(a)}{1 - p^A(a)(1-(c+\tau))}(1 - p(a)) \tag{9}$$

$$\Leftrightarrow f(a) = \frac{f^A(a)}{1 - p^A(a)(1-(c+\tau))}\left((c+\tau)^{\frac{p(a)}{p^A(a)}} p(a) + (1 - p(a))\right) \tag{10}$$

$$\Leftrightarrow f(a) = f^A(a) \underbrace{\frac{1 - p(a)(1-(c+\tau)^{\frac{p(a)}{p^A(a)}})}{1 - p^A(a)(1-(c+\tau))}}_{=:D(p(a))} =: f(p(a), a). \tag{11}$$

[4]Note that in the arguments about the role of female labor force participation on fertility we assumed that only working females demand childcare facilities. Indeed our results would change once we introduce demand for childcare services by non working females as well.

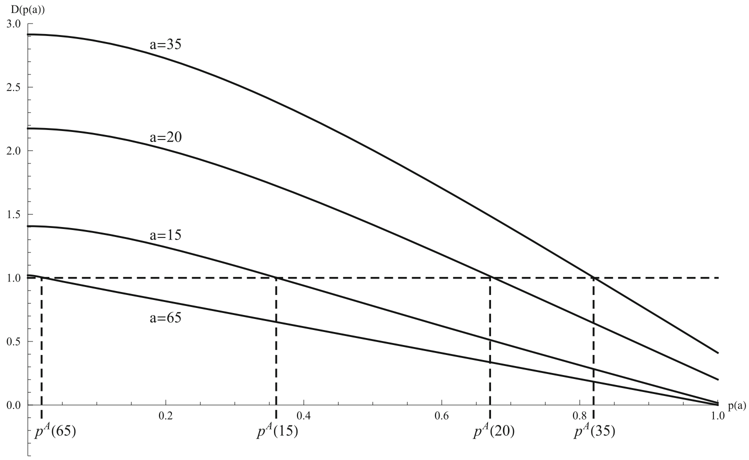


Fig. 4 Effect $D(p(a))$ of the chosen labor force participation rate $p(a)$ on fertility for different age groups

The term $D(p(a))$ represents the feedback of changes in the labor force participation rate on overall age specific fertility rates $f^A(a)$ given an initial relation of working- and non working fertility in the population as represented by the parameter $(c + \tau)$ [cf. Eq. (5)]. Figure 4 shows the relation between $D(p(a))$ and $p(a)$ for different ages. If $p(a) = p^A$, then $D(p^A) = 1$ and hence the Austrian fertility rate does not change. If the participation rate exceeds the currently prevailing Austrian participation rate of the considered time period 2000–2009, $D(p(a))$ is below one and fertility is reduced. Otherwise, if the participation rate is lower, $D(p(a))$ is higher than one and fertility is boosted. Note, that the function $D(p(a))$ is concave in $p(a)$. As mentioned above, the fertility patterns of working and non working women are more similar the lower the labor force participation rate. This implies that the marginal gain in fertility due to a further reduction of the labor force participation rate below the Austrian one decreases because in this region the fertility of working women already is high as well. In contrast, the higher the labor force participation is, the higher is the difference between the two fertility patterns and hence the higher is the marginal loss in fertility if the labor force participation rate is even further increased beyond the Austrian one as the fertility of the working women here is already very low.

The objective of the social planner is now to maximize the Austrian female labor force participation by optimally choosing the female labor force participation rates at each age. Denoting the female population at age a by $N(a)$, and recalling that $p(a)$ denotes the age specific female labor force participation rate, the objective is to maximize at each age:

$$\max_{p(a)} p(a)N(a) \tag{12}$$

The choice of $p(a)$ is subject to some restrictions, of course. As mentioned before, we use the Swedish participation rate as upper, and the Italian one as lower bound, in order to restrict $p(a)$ to reasonable values, hence

$$p^I(a) \leq p(a) \leq p^S(a), \forall a \in [0, \omega]. \tag{13}$$

We further postulate that the population is stationary, implying the following population dynamics:

$$N'(a) = -\mu(a)N(a), \tag{14}$$

$$N(0) = \int_0^\omega f(a)N(a)\,da, \tag{15}$$

$$N = \int_0^\omega N(a)\,da, \tag{16}$$

where $\mu(a)$ denotes the age specific mortality rate, $f(a)$ denotes the age specific fertility rate and the population size N is assumed to stay constant over time. Hence, the population at age a is given by the stationary population equation:

$$N(a) = N(0) \underbrace{\exp\left(-\int_0^a \mu(\tau)\,d\tau\right)}_{=:l(a)} = N(0)l(a), \tag{17}$$

For the total population size we obtain the fundamental equation of a stationary population:

$$1 = \int_0^\omega f(p(a), a)l(a)\,da, \tag{18}$$

$$N = N(0) \underbrace{\int_0^\omega l(a)da}_{=:e_0} = N(0)e_0, \tag{19}$$

where e_0 defines the life expectancy at age 0. Using this transformation as well as the defined control dependent fertility rate of Eq. (11), the optimal control model can be written as

$$\max_{p(a)} \int_0^\omega p(a)l(a)\frac{N}{e_0}\,da \tag{20}$$

$$\text{s.t.:} \quad 1 = \int_0^\omega f(p(a), a)l(a)\,da, \tag{20a}$$

$$p(a) \leq p^S(a), \quad \forall a \in [0, 110], \tag{20b}$$

$$-p(a) \leq -p^I(a), \quad \forall a \in [0, 110]. \tag{20c}$$

In order to model the labor force participation rates in Austria, Italy and Sweden with a deterministic function, it turns out that a combination of *Sigmoid* functions yields a satisfying fit,

$$p^A(a) = b^A \left(\frac{1}{1 + \exp\left(-d_1^A(a - a_1^A)\right)} - \frac{1}{1 + \exp\left(-d_2^A(a - a_2^A)\right)} \right), \tag{21}$$

$$p^I(a) = b^I \left(\frac{1}{1 + \exp\left(-d_1^I(a - a_1^I)\right)} - \frac{1}{1 + \exp\left(-d_2^I(a - a_2^I)\right)} \right), \tag{22}$$

$$p^S(a) = b^S \left(\frac{1}{1 + \exp\left(-d_1^S(a - a_1^S)\right)} - \frac{1}{1 + \exp\left(-d_2^S(a - a_2^S)\right)} \right). \tag{23}$$

To model the mortality rate we choose the so-called *Gompertz* function (cf. Gompertz 1825; Preston et al. 2001; Sundt 2004) which is a two-parameter formula and has proven to be a remarkably good model in different populations and epochs. Gompertz noticed that, especially in past middle adult ages, the mortality curve displays a nearly exponential increase in age. This geometric progression, however, was meant to describe especially underlying mortality, i.e. mortality purged of infectious and accidental reasons. The latter often occurs in early ages and is known as the *accident-hump*, for example due to cars and motorcycles. Moreover, it has been observed that at old ages (over 80), death rates often increase at a diminishing rate. However, according to the scope of this analysis, the Gompertz function is accurate enough to guarantee satisfying results.

$$\mu^A(a) = \phi \exp(\psi a). \tag{24}$$

Fertility is modeled with the *Hadwiger* function (cf. Hadwiger 1940; Gibe and Yntema 1971) because it contains only a few parameters with clear meanings, see Eq. (25). The parameter R represents the gross reproduction rate and T is the mean age at childbearing. For Austria, the values of these two parameters are at $R = 0.71$ and $T = 29.14$, as the subsequent parameter estimation shows.

$$f^A(a) = \frac{RH}{T\pi^{\frac{1}{2}}} \left(\frac{T}{a}\right)^{\frac{3}{2}} \exp\left(-H^2\left(\frac{T}{a} - \frac{a}{T} - 2\right)\right) \tag{25}$$

The deterministic functions in Eqs. (21)–(25) should reflect the observed behavior of the available data (cf. Statistik Austria 2011a,b; OECD.StatExtracts 2011), which we obtained by taking the average rates of the series from 2000–2009. In order to minimize the residuals, the parameter values accordingly are estimated by

Table 1 Estimated parameter values

Function	Parameter	Value
$p^A(a)$	a_1^A	15.6934
	a_2^A	54.7576
	b^A	0.822994
	d_1^A	0.352882
	d_2^A	0.340225
$p^I(a)$	a_1^I	18.3908
	a_2^I	54.227
	b^I	0.664392
	d_1^I	0.422108
	d_2^I	0.256721
$p^S(a)$	a_1^S	15.7371
	a_2^S	61.0396
	b^S	0.87767
	d_1^S	0.253495
	d_2^S	0.389959
$\mu^A(a)$	ϕ	0.00003418
	ψ	0.0936099
$f^A(a)$	R	0.705845
	T	29.1404
	H	3.42672

using the method of least squares[5] and can be seen in Table 1. Figures 5 and 6 show these deterministic functions (dashed line) with the empirical data[6] (solid line).

[5]To estimate the parameters for the Gompertz function, we used data from Statistik Austria (2011b) that includes annual mortality rates from age 10 to 95. To get rates for even higher ages up to 110, we continued this trend.

[6]One may expect that the observed data exhibits the typical M-shaped form, which occurs due to the decline of female labor force participation in fertile ages. However, because women in paid maternity leave are counted as employed, this decline is less visible in countries with high social support. Austria, Sweden and Italy are examples for such countries with a comparatively long or well paid maternity leave period (see Fagan et al. 2007). Besides the fact that Sweden has an above average and Italy a below average labor force participation rate, the similarity in the shape with the one of Austrian data is another reason why we have chosen these two countries as boundaries. Further on note that due to the smooth shape of the data the estimation functions in Eqs. (21)–(23) yield the best fit. If, however, the data would exhibit a M-shaped from, other estimation functions would have to be used.

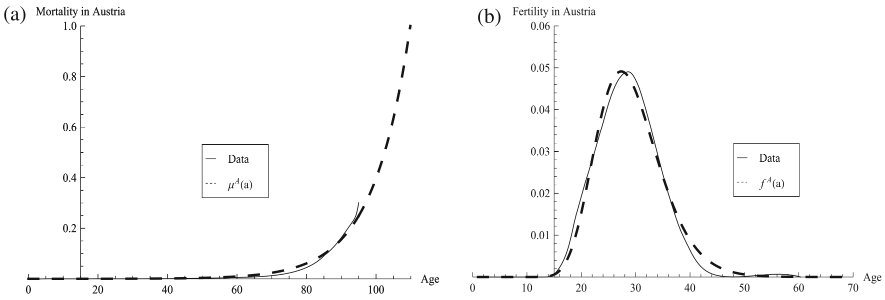


Fig. 5 Approximating functions (*dashed line*) for given data (*solid line*) for Austrian mortality (**a**) and fertility (**b**)

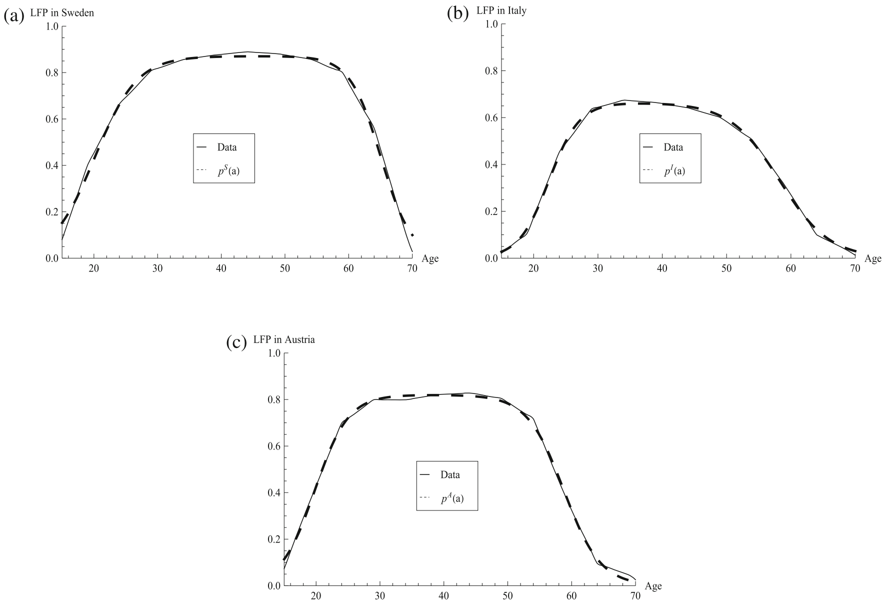


Fig. 6 Approximating functions (*dashed line*) for given data (*solid line)* for female labor force participation rates in Sweden (**a**), Italy (**b**) and Austria (**c**)

3 Analytical Solution

We have to solve the maximization problem with an integral equality constraint, given in Eq. (20). While in Bryson and Ho (1975) such models with infinite horizon are solved by replacing the integral condition with a new state variable, we are faced with a finite horizon, which complicates matters. We therefore follow Alekseev et al. (1987) and apply the *Lagrange Principle* for equality and inequality conditions. If $p^*(a)$ is a local maximum of problem Eqs. (20)–(20c), similar to the standard dynamic optimization problems with equality constraints (cf. Sage and White 1977) there exist *Lagrange Multipliers* $\lambda = (\lambda_0, \lambda_1, \lambda_2(a), \lambda_3(a))$, not all zero, such that the *Lagrangian* reads as

$$\mathscr{L}(p(a), \lambda) = \int_0^\omega \lambda_0 p(a) l(a) \frac{N}{e_0} + \lambda_1 f^A(a) D(p(a)) l(a) da$$
$$-\lambda_1 + \lambda_2(a)(p(a) - p^S(a)) + \lambda_3(a)(p^I(a) - p(a)), \qquad (26)$$

and in the local maximum

$$\frac{\partial \mathscr{L}}{\partial p(a)}(p^*(a), \lambda^*) = 0 \qquad (27)$$

holds, with $\lambda_2^*(a), \lambda_3^*(a) \geq 0$.

Further on, the *complementary slackness conditions* (for more detail see also Grass et al. 2008),

$$\lambda_2(a)(p(a) - p^S(a)) = 0, \quad \forall a \in [0, \omega], \quad \lambda_2(a) \geq 0,$$
$$\lambda_3(a)(p^I(a) - p(a)) = 0, \quad \forall a \in [0, \omega], \quad \lambda_3(a) \geq 0,$$

are fulfilled.

Without loss of generality we set $\lambda_0 = 1$ for the following analysis.

According to the Lagrange Principle, maximizing the Largangian in Eq. (26) is equivalent to maximizing

$$\tilde{\mathscr{L}}(p(a), \lambda) = p(a) l(a) \frac{N}{e_0} + \lambda_1 f^A(a) D(p(a)) l(a) - \lambda_1 + \lambda_2(a)(p(a)$$
$$-p^S(a)) + \lambda_3(a)(p^I(a) - p(a)). \qquad (28)$$

Then, the *First-Order Condition* reads as

$$\frac{\partial \tilde{\mathscr{L}}}{\partial p(a)} = l(a) \frac{N}{e_0} + \lambda_1 f^A(a) l(a) \frac{\partial D(p(a))}{\partial p(a)} + \lambda_2(a) - \lambda_3(a) = 0. \qquad (29)$$

If $p^*(\tilde{a})$ lies within the admissible region $p^I(\tilde{a}) < p^*(\tilde{a}) < p^S(\tilde{a})$ for some $\tilde{a} \in [0, \omega]$, which implies that $\lambda_2(\tilde{a}) = \lambda_3(\tilde{a}) = 0$, the *second order condition*,

$$\frac{\partial^2 \tilde{\mathscr{L}}}{\partial p(a)^2} = \lambda_1 f^A(a) l(a) \frac{\partial^2 D(p(a))}{\partial p(a)^2} \leq 0, \qquad (30)$$

guarantees, that the optimal solution is indeed a local maximum of the objective function. If $p^*(\tilde{a})$ lies at one of the admissible boundaries, Eq. (30) does not have to be satisfied. Further on, $\lambda_2(\tilde{a}) \neq 0$ or $\lambda_3(\tilde{a}) \neq 0$ holds in this case.

Hence, in order to find the optimal labor force participation rate $p^*(a)$ at each age, we have to solve

$$\frac{\partial \tilde{\mathscr{L}}}{\partial p(a)}(p^*(a)) = l(a)\frac{N}{e_0} - \lambda_1 f^A(a)l(a)\frac{(c+\tau)^{\frac{p(a)}{p^A(a)}}(p^A(a) + p(a)\log(c+\tau)) - p^A(a)}{p^A(a) + (c+\tau-1)p^A(a)^2} = 0, \tag{31}$$

$$\frac{\partial^2 \tilde{\mathscr{L}}}{\partial p(a)^2}(p^*(a)) = \lambda_1 f^A(a)l(a)\frac{(c+\tau)^{\frac{p(a)}{p^A(a)}}\log(c)(2p^A(a) + p(a)\log(c+\tau))}{p^A(a)^2(1 + (c+\tau-1)p^A(a))} \leq 0, \tag{32}$$

for the inner case ($\lambda_2(a) = \lambda_3(a) = 0$), and

$$\begin{aligned}\frac{\partial \tilde{\mathscr{L}}}{\partial p(a)}(p^S(a)) = -\lambda_1 f^A(a)l(a)&\frac{(c+\tau)^{\frac{p^S(a)}{p^A(a)}}(p^A(a) + p^S(a)\log(c+\tau)) - p^A(a)}{p^A(a) + (c+\tau-1)p^A(a)^2} \\ &+ l(a)\frac{N}{e_0} + \lambda_2(a) = 0,\end{aligned} \tag{33}$$

$$\begin{aligned}\frac{\partial \tilde{\mathscr{L}}}{\partial p(a)}(p^I(a)) = -\lambda_1 f^A(a)l(a)&\frac{(c+\tau)^{\frac{p^I(a)}{p^A(a)}}(p^A(a) + p^I(a)\log(c+\tau)) - p^A(a)}{p^A(a) + (c+\tau-1)p^A(a)^2} \\ &+ l(a)\frac{N}{e_0} - \lambda_3(a) = 0,\end{aligned} \tag{34}$$

for the boundary cases $p^*(a) = p^S(a)$ and $p^*(a) = p^I(a)$, respectively.

4 Numerical Solution for $\tau = 0.19$

The numerical analysis for this paper was done with *Wolfram Mathematica 7.0*. We start our numerical investigations by assuming that the fertility of women in the labor market amounts to 20 % of the fertility of women outside the labor market, hence $c + \tau = 0.2$. The numerical solution for these parameter values yields the optimal labor force participation rate, which is given as

$$p^*(a) = \begin{cases} p_{int}(a) & \text{if } a \in [21.34, 21.85) \text{ or } a \in [37.69, 37.89), \\ p^I(a) & \text{if } a \in [21.85, 37.69), \\ p^S(a) & \text{else,} \end{cases} \tag{35}$$

with $p_{int}(a)$ defining the interior solution according to Eqs. (31) and (32). The optimal age specific labor force participation $p^*(a)$ is shown in Fig. 7 as black dashed line. The gray lines are the two boundaries, the Swedish participation rate $p^S(a)$ as upper, and the Italian participation rate $p^I(a)$ as lower one. For young ages, the optimal participation rate is along the upper boundary. Comparing this solution with the Austrian participation rate $p^A(a)$ (gray dotted line) shows, that the optimal solution obviously lies above this rate and hence is supposed to have a

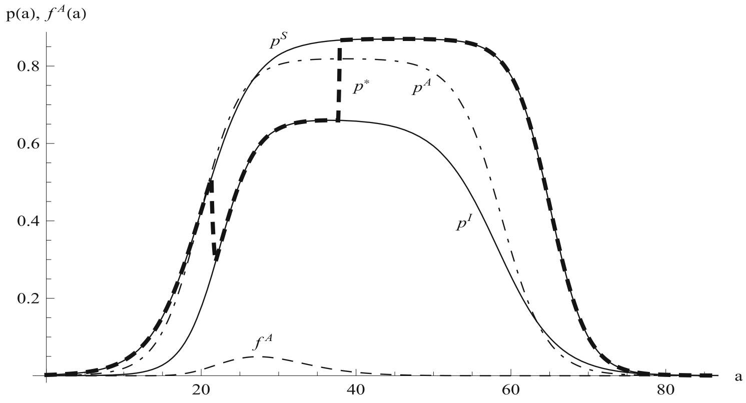


Fig. 7 Optimal labor force participation for $\tau = 0.19$

negative impact on the fertility of women in the labor market. This loss in fertility, however, is minor because the fertility rate (gray dashed line) at these ages is very low and hence, the highest possible participation rate is optimal in this period. However, the higher the fertility rate, the more labor force participation will depress fertility. Consequently, when the fertility rate starts to increase at around $a = 15$, it is optimal to reduce the labor force participation rate. In order to maintain a stationary population, the participation rate has to decline during the most fertile ages, which explains the sudden decrease at $a = 20.34$. A lower participation rate is advantageous as it increases fertility. The optimal solution finally follows the lower boundary in this interval. For less fertile ages, an increase in labor force participation rates above the corresponding Austrian values becomes optimal again. Hence, the optimal age specific labor force participation rate increases from the lower boundary at $a = 37.69$ until it reaches the upper boundary at $a = 37.89$ and remains there for the rest of the working age period. Note that a strong but reasonable restriction in this approach is that the solution should not fall below the Italian participation rate in order to have a realistic outcome. However, interesting insights can be given by releasing this constraint and considering the optimal solution when the lower boundary is zero. The optimal labor participation rate (again for $c + \tau = 0.2$) is now given as

$$p^*(a) = \begin{cases} p_{int}(a) & \text{if } a \in [24.82, 32.25), \\ p^S(a) & \text{else,} \end{cases} \tag{36}$$

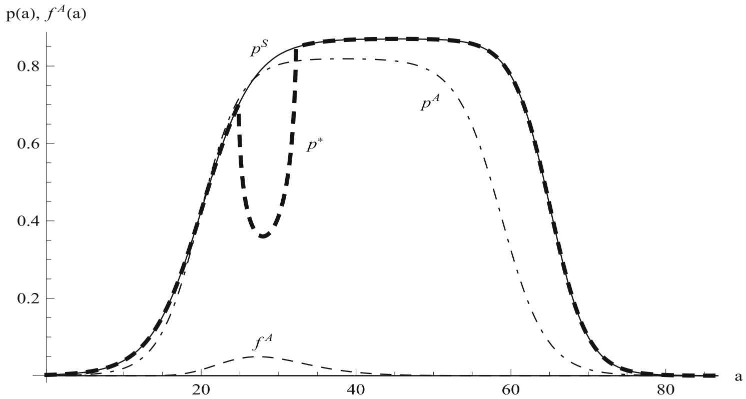


Fig. 8 Optimal labor force participation for $\tau = 0.19$ without lower boundary

and is shown in Fig. 8. Comparing this outcome with the previous results shows that relaxing the lower boundary enables a high participation rate over a longer period. Only during the most fertile ages the participation rates are reduced. This time, however, the decline in the labor force participation is even more pronounced to guarantee high enough fertility to obtain a stationary population.

5 Sensitivity Analysis

The results presented in the previous section referred to a parameter setting of $\tau = 0.19, c = 0.01$. In this section we investigate how the optimal age specific labor force participation rate will change when we increase the value of τ, i.e. when we assume that family policies help to combine working and childrearing. We recall the key expression that reflects the feedback from the labor force participation rate on fertility as given by

$$D(p(a)) = \frac{1 - p(a)(1 - \tilde{c}(c + \tau, P))}{1 - p^A(a)(1 - (c + \tau))}. \tag{37}$$

Note that in the following we omit the age argument a in case of no ambiguity. The derivative of $D(p)$ with respect to p is given as

$$\frac{\partial D}{\partial p} = \frac{(\tilde{c}(c + \tau, p) - 1) + p\frac{\partial \tilde{c}(c+\tau,p)}{\partial p}}{p^A(c + \tau - 1) + 1} < 0, \tag{38}$$

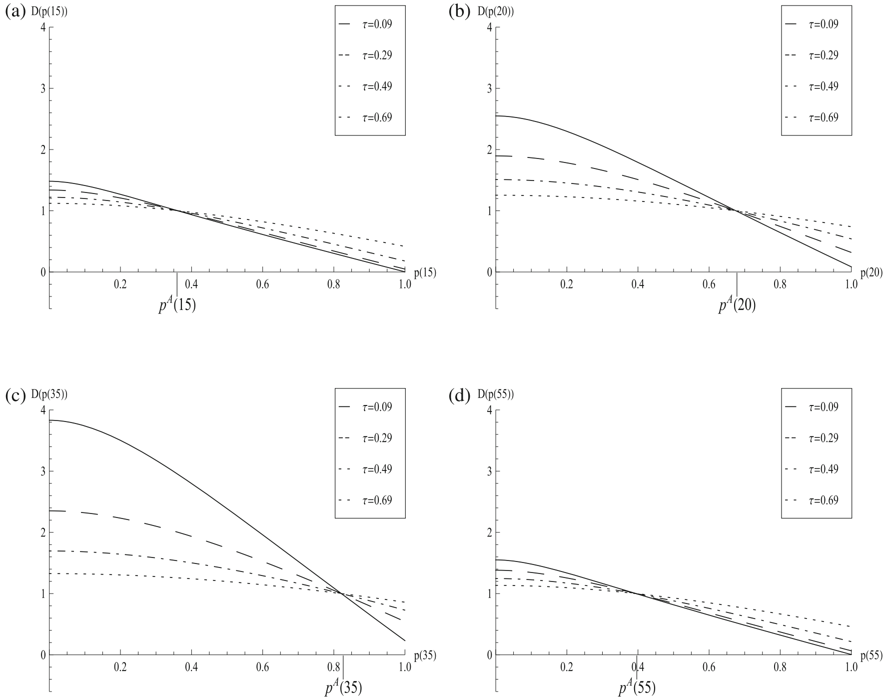


Fig. 9 $D(p(a))$ at different ages for $\tau = 0.09$ (*Solid line*), $\tau = 0.29$ (*Dashed line*), $\tau = 0.49$ (*Dotdashed line*) and $\tau = 0.69$ (*Dotted line*). (**a**) $a = 15$. (**b**) $a = 20$. (**c**) $a = 35$. (**d**) $a = 55$

and shows, according to our assumption, that an increase in the labor force participation rate p reduces the value of $D(.)$. For the second derivative,

$$\frac{\partial}{\partial \tau}\left(\frac{\partial D}{\partial p}\right) = \frac{1}{(p^A(c+\tau-1)+1)^2}\Bigg(-p^A\left((\tilde{c}(c+\tau,p)-1) + p\frac{\partial \tilde{c}(c+\tau,p)}{\partial p}\right) + \left(\frac{\partial \tilde{c}(c+\tau,p)}{\partial \tau} + p\frac{\partial}{\partial \tau}\frac{\partial \tilde{c}(c+\tau,p)}{\partial p}\right)\left(p^A(c+\tau-1)+1\right)\Bigg), \quad (39)$$

one can show that in case of a concave function $\tilde{c} = (c+\tau)^{p/p^A}$, it is always positive. The latter argument implies that an increase of the labor force participation reduces the value of $D(p)$ to a lower extend the higher the value of τ. As Fig. 9 shows, the smaller the value of τ, the higher is the difference between the fertility of working and non working women ($x_N(a)$ and $x_W(a)$) and hence the higher is the impact of a change in the labor force participation rate on the fertility rates of working and non working women.

To investigate how the optimal age specific labor force participation changes with varying values of τ we consider various scenarios, shown in Fig. 10.

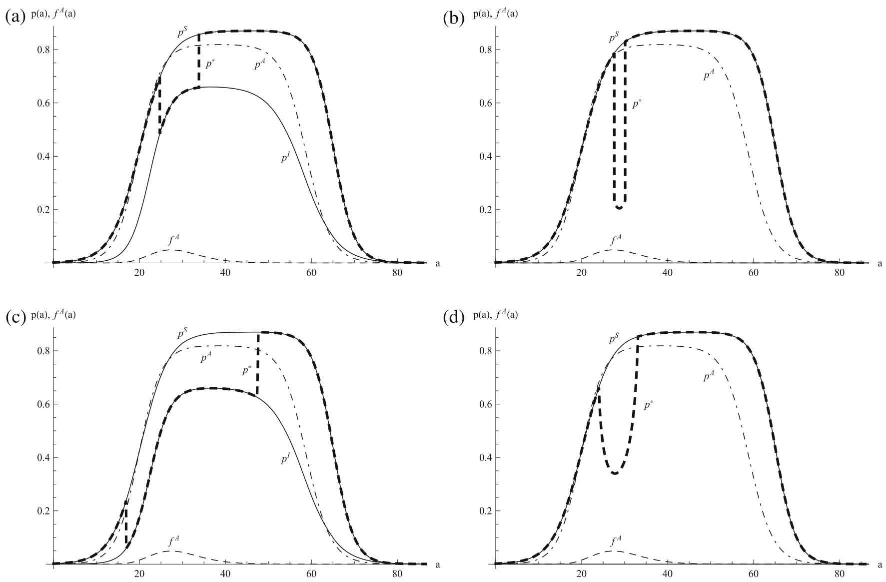


Fig. 10 Optimal labor force participation rate $p^*(a)$ for varying values of τ. (**a**) $\tau = 0$. (**b**) $\tau = 0$ without lower boundary. (**c**) $\tau = 0.26$. (**d**) $\tau = 0.26$ without lower boundary

Figure 10a shows the optimal labor force participation rate for $\tau = 0$, i.e. in case that family policies are not implemented and therefore the parameter $c = 0.01$ determines the relation between a working and non working woman. Recalling our discussion in Sect. 2 a lower value of τ will imply a smaller reduction in overall fertility when the labor force participation rate is increased. Put differently, since $c = 0.01$ is fixed, a lower value of τ implies a lower value of x_W and a higher value of x_N. Consequently, the interval in which the participation rate has to decrease in order to satisfy the condition for a stationary population, which without exceptions always happens during the most fertile ages, decreases with lower values of τ. Figure 10b then shows the same scenario without the Italian participation rate as lower boundary. Assuming that the labor participation rates may decline even below the Italian ones, the length of the interval where labor force participation is decreased becomes smaller. In contrast, Fig. 10c shows the results for a higher value of $\tau = 0.26$. Since a higher value of τ also indirectly reduces the fertility of woman outside the labor market, see Eq. (5), the participation rate has to be reduced over a longer interval in order to yield a stationary population. If the value of τ is even further increased, it turns out that an optimal labor force participation no longer exists, because the fertility rate $x_N(a)$ of non working women here is so close to the replacement level that even a reduction of the labor force participation rate to $p(a) = p^I(a)$ for all ages is not enough to enable a stationary population level. Figure 10d correspondingly shows the results without the lower boundary where we

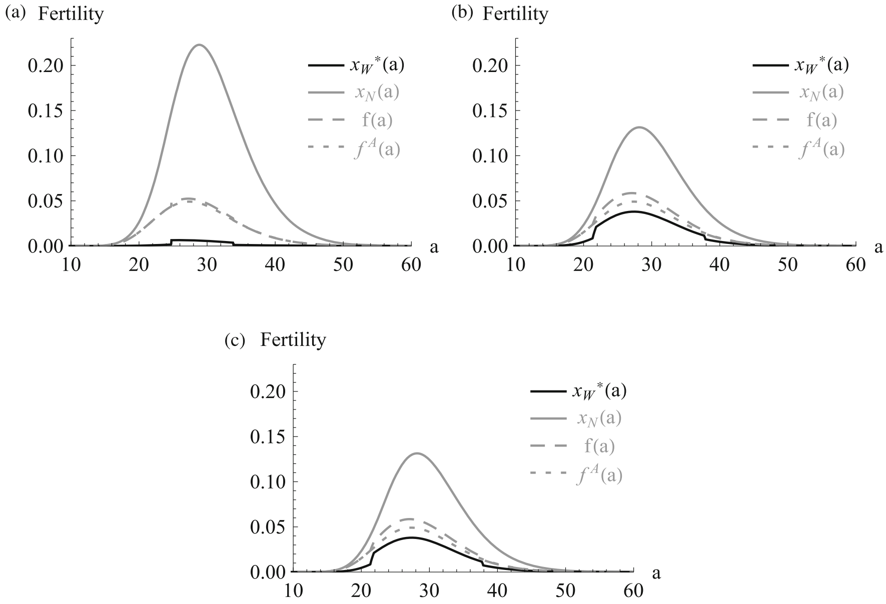


Fig. 11 Fertility of working women, $x_W^*(a)$, and non working women, $x_N(a)$, for different values of τ. (**a**) $\tau = 0$. (**b**) $\tau = 0.19$. (**c**) $\tau = 0.26$

can see again that in this unrestricted case the optimal solution is to have a stronger decline of the labor force participation but over a smaller interval.

In Fig. 11 we have plotted for various values of τ the optimal age specific fertility of working and non working women together with the age specific fertility of all women for the initial Austrian population and the optimal stationary population. A higher value of τ implies a lower difference between the age specific fertility of working and non-working women and at the same time the difference between the initial Austrian population and the optimal stationary population increases.

To give at least some idea about what happens for higher values of τ, we consider this case without including the lower boundary $p^I(a)$. Then a solution exists, which is shown in Fig. 12 for $\tau = 0.49$ which yields a proportionality factor of 50 %. Here, one can see that the interval in which a inner solution is optimal as well as the extend of the decline in this interval have become very large, compared to the previous results (Fig. 10).

6 Conclusion

While the redistribution of work across life is increasingly discussed as a way to sustaining a specific level of the labor force in aging and shrinking populations (see for example Vaupel and Loichinger 2006), research so far has mainly focused on

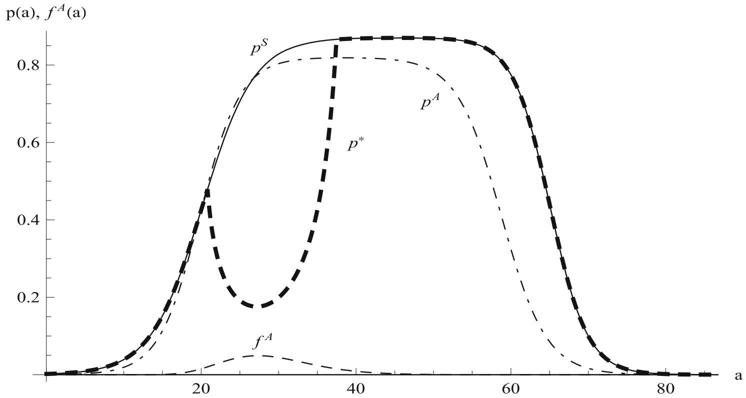


Fig. 12 Optimal labor force participation for $\tau = 0.49$ without lower boundary

simulation models and ignored the feedback effects of the redistribution of work on other demographic events. In this paper we focus on a specific group of the labor force, female employees, and ask how their labor force participation can be redistributed across life to maximize the overall female labor force. Moreover, we take into account that an increase in the female labor force participation rate may reduce fertility and thereby even further reduce the total labor force. We therefore introduce the constraint of a stationary population, i.e. any redistribution of the female labor force across age has to take into consideration that fertility is still sufficiently high to sustain a stationary population level. Applying optimal control theory we derive the optimal age specific labor force participation of females and the resulting fertility rates where we differentiate between the fertility levels of working and non-working females. We assume that at each age the fertility rate of working females is proportional to the fertility rate of non-working women with the proportionality factor being determined by the age specific labor force participation rate together with exogenous parameters that represent family policies. The closer both fertility rates are, the more effective family policies are in combining childrearing and work. We initialize our model with Austrian age-specific data on mortality, fertility and female labor force participation. We furthermore assume a lower and upper bound of the female labor force participation represented by the Italian and Swedish labor force rates respectively.

Our results demonstrate that a redistribution of the female labor force participation may indeed be optimal in terms of maximizing the overall female labor force. More specifically, it is optimal to redistribute work to lower and higher ages where fertility is lower, but at the same time to reduce labor force participation during the most fertile ages. Moreover our results indicate that a higher compatibility of work and childbearing and hence a lower differential between working and non-working females' fertility implies two counteracting policies. On the one hand it is easier to increase the labor force participation of females without depressing fertility too much. On the other hand, however, the more equal the fertility rate of the working and non-working women are, the more difficult it will be that an increase in the labor force will help to sustain a stationary population. The latter result is driven by the fact that we initialize our simulations with the Austrian age specific fertility rate. I.e. if we assume the fertility rate of working and non-working females to be more similar at each age, we may end up in a situation where both fertility rates are below replacement level fertility. Indeed, an extension of our model could be to take into consideration that an increase in the compatibility of childrearing and work will also alter the Austrian age specific fertility.

Our paper constitutes a first attempt to formalize the redistribution of work across life and to take repercussions of the labor force participation on demographic events into account. We foresee several extensions. With respect to the objective function, the redistribution of the overall labor force over the life cycle may be considered. A possible feedback could be that lowering the intensity of the labor force participation at each age, but extending the working life, may indeed have positive repercussions on productivity and health over the life cycle and possibly also on the family life cycle. In further applications we also aim to relax the assumption of a stationary and closed population. More generally, our framework may be applied to other age specific characteristics that may be optimally redistributed over the life cycle, such as for example education, consumption, savings, etc. These life cycle events will in turn produce feedback effects on the timing and sequencing of demographic events and thereby determine the evolution of aggregate levels of those variables, i.e. aggregate human capital, aggregate consumption and aggregate savings.

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Multiplicity of Balanced Growth Paths in an Endogenous Growth Model with Elastic Labor Supply

Gerhard Sorger

Abstract We consider the neoclassical one-sector growth model in continuous time with elastic labor supply and a learning-by-doing externality. It is shown that this model can have a continuum of balanced growth paths. Some of these balanced growth paths can be locally unique (determinate) whereas others can be indeterminate.

Journal of Economic Literature **Classification Codes:** C61, E13, O41

1 Introduction

The neoclassical one-sector growth model with infinitely-lived households and endogenous labor supply combines two of the most fundamental macroeconomic tradeoffs in a simple dynamic general equilibrium setting: the division of output between consumption and investment and the division of time between productive activities and leisure. It is therefore not surprising that this model forms the non-stochastic backbone of real business cycle theories, which have been developed to simulate the reaction of output and employment to various types of exogenous shocks.[1]

[1] In growth theory, however, the labor-leisure trade-off is surprisingly often disregarded. Eriksson (1996) writes that "The choice between work and leisure has been remarkably neglected in the theory of economic growth" (Eriksson 1996, p. 533) and even the very comprehensive and more recent survey of economic growth theory provided by Acemoglu (2009) does not discuss the case of elastic labor supply except for briefly mentioning real business cycle models in Section 17.3.

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The present note seeks to contribute to the understanding of the complexities that can arise in this rather simple model. More specifically, we augment the basic framework by a mechanism that generates endogenous growth and prove that the resulting model can have infinitely many different balanced growth paths. In order to see how this finding adds to existing knowledge, we start by providing a brief and selective survey of the literature. In doing this, it is useful to distinguish between results that have been proved for the discrete-time version of the model and those that hold in its continuous-time counterpart.

Using a discrete-time framework and considering the model without a growth-generating mechanism, De Hek (1998) shows by means of numerical examples that there can be multiple (i.e., finitely many) steady states as well as stable period-2 cycles. Kamihigashi (2015) addresses the multiplicity of steady states in a more systematic way and proves that the model can have any finite number of steady states or even a continuum of steady states. Moreover, multiplicity of steady states can occur for all values of the time-preference factor between 0 and 1 and for all production functions satisfying standard assumptions. Sorger (2015) proves a similar result for period-2 cycles by showing that such periodic solutions can occur for all values of the time-preference factor between 0 and 1 even if one restricts the technology to be of the Cobb-Douglas type. He also shows that the model can generate period-3 cycles and topological chaos if the time-preference factor is sufficiently small.

Whereas the findings of Kamihigashi (2015) can easily be transferred to the continuous-time setting, this is not the case for those of Sorger (2015). This follows from a result by Hartl (1987) according to which all optimal solutions to continuous-time dynamic optimization problems with a single state variable must be monotonic. Sorger (2000a) studies the model in continuous time without endogenous growth as a decentralized market economy and allows that the households differ from each other with respect to their initial capital holdings (they are assumed to be identical in all other respects). He finds that even with the standard parameterizations used in real business cycle models, there exists a continuum of steady states that differ from each other not only with respect to the distribution of capital among households but also with respect to the level of aggregate output. Sorger (2000b) transfers these results into a model with endogenous growth that is driven by a learning-by-doing externality à la Arrow (1962) and Romer (1986). He shows that such an endogenous growth model admits a continuum of balanced growth paths that differ from each other with respect to the income distribution and the long-run growth rate of the economy. It is important to emphasize, however, that the possibility of infinitely many steady states or balanced growth paths in Sorger (2000a,b), respectively, is solely due to the heterogeneous initial endowments of the households. In both papers, the steady state or balanced growth path, respectively, would be unique if a representative household assumption would be imposed. Benhabib and Farmer (1994) already note that the model with a representative household can have two interior balanced growth paths if one allows for production externalities that generate increasing returns to scale on the aggregate level. Eriksson (1996) uses a continuous-time model and standard parameterizations of preferences and technology to show that the long-run growth rate typically depends on the preference parameters both when growth is due to exogenous technological progress and when it is generated endogenously.

In view of the above mentioned results, the present note can be interpreted in two different ways: either as a translation of the results about the multiplicity of steady states from Kamihigashi (2015) into a setting with endogenous growth, or as a complement to Sorger (2000b) which shows the possibility of a continuum of balanced growth paths without resorting to heterogeneity of households. In this respect, it has to be noted that the translation of the results by Kamihigashi (2015) to a framework with endogenous growth is by no means trivial. This is so because the elasticity of marginal utility of consumption must be constant along every balanced growth path, which considerably restricts the preferences for which balanced growth paths can occur. Therefore one cannot proceed as in Kamihigashi (2015), namely by fixing the technology (in an arbitrary way) and choosing preferences such that multiple steady states exist. Instead we start from a fixed preference specification that is consistent with balanced growth and choose the production function so as to allow for a continuum of balanced growth paths.

The paper is organized as follows. Section 2 formulates the model and defines equilibria and balanced growth paths and Sect. 3 presents the results.

2 Model Formulation

In this section we formulate the one-sector growth model with elastic labor supply. The basic structure of the model is identical to the deterministic version of the standard real business cycle model in continuous time. In order to allow for endogenous growth, we include a learning-by-doing externality à la Arrow (1962) and Romer (1986). Similar formulations have been studied for example by Benhabib and Farmer (1994), Eriksson (1996), and Sorger (2000b).

Consider an economy that evolves continuously over the infinite time-horizon $\mathbb{R}_+$. There exists a unit interval of identical and infinitely-lived households. The representative household is endowed with $k(0) > 0$ units of capital at time 0 as well as with a constant flow (normalized to 1) of time that can be used either for work or for leisure. Let us denote the rate at which the household consumes output at time t by $c(t) \geq 0$, the rate at which it supplies labor to the firms by $\ell(t) \in [0, 1]$, and the capital holdings (wealth) of the household at time t by $k(t)$. This implies that, at time t, leisure is consumed at rate $1 - \ell(t)$. The flow budget constraint of the representative household can be expressed as

$$\dot{k}(t) = [q(t) - \delta]k(t) + w(t)\ell(t) - c(t), \tag{1}$$

where $\delta > 0$ is the rate at which capital depreciates and where $q(t)$ and $w(t)$ are the factor prices of capital and labor, respectively, at time t. The household maximizes the objective functional

$$\int_0^{+\infty} e^{-\rho t} u(c(t), 1 - \ell(t))\, \mathrm{d}t, \tag{2}$$

subject to the flow budget constraint (1) and the no-Ponzi game condition

$$\lim_{t \to +\infty} e^{-\int_0^t [q(\tau)-\delta]\, \mathrm{d}\tau} k(t) \geq 0, \tag{3}$$

where $\rho > 0$ is the time-preference rate and where u is the instantaneous utility function.

Assumption 1 The function $u : \mathbb{R}_+ \times [0, 1] \mapsto \mathbb{R}$ is continuous, non-decreasing, and concave. On the interior of its domain it is twice continuously differentiable as well as strictly increasing and strictly concave in each of its arguments.

There exists a unit interval of identical firms $i \in [0, 1]$, which rent the factor inputs from the households and maximize their profits. Denoting the capital input and the labor input of firm i at time t by $K_i(t)$ and $L_i(t)$, respectively, firm i produces output at time t at the rate $F(K_i(t), A(t)L_i(t))$, where $A(t)$ is the efficiency of labor at time t and where F is the production function. Every firm i takes the factor prices as well as the efficiency of labor as given and maximizes the profit rate, which is given by

$$F(K_i(t), A(t)L_i(t)) - q(t)K_i(t) - w(t)L_i(t) \tag{4}$$

at each time t subject to non-negativity constraints on both factor inputs.

Assumption 2 The function $F : \mathbb{R}_+^2 \mapsto \mathbb{R}_+$ is continuous, non-decreasing, concave, and homogeneous of degree 1. On the interior of its domain it is twice continuously differentiable as well as strictly increasing and strictly concave in each of its arguments. Furthermore, it holds that $F(0, 1) = 0$.

Since all firms are identical it holds that $K_i(t) = K(t)$ and $L_i(t) = L(t)$, where $K(t) = \int_0^1 K_i(t)\, \mathrm{d}i$ and $L(t) = \int_0^1 L_i(t)\, \mathrm{d}i$ are the aggregate factor inputs at time t. We follow Arrow (1962) and Romer (1986) and assume that the efficiency of labor is positively related to the aggregate capital stock. More specifically, we assume that $A(t)$ is proportional to $K(t)$ and we normalize the factor of proportionality without loss of generality by 1, that is,

$$A(t) = K(t). \tag{5}$$

An *economy* is a quadruple (u, ρ, F, δ) consisting of a utility function u satisfying Assumption 1, a time-preference rate $\rho > 0$, a production function F satisfying Assumption 2, and a capital depreciation rate $\delta > 0$.

An *equilibrium* of the economy (u, ρ, F, δ) is a family of functions $\{k, \ell, c, (K_i, L_i)_{i\in[0,1]}, K, L, A, q, w\}$, all defined on the common domain $\mathbb{R}_+$, such that the following conditions hold:

1. Given q and w, the triple (k, ℓ, c) maximizes (2) subject to (1) and (3).
2. For all $i \in [0, 1]$, all $t \in \mathbb{R}_+$, and given $(q(t), w(t), A(t))$ it holds that the pair $(K_i(t), L_i(t))$ maximizes (4).
3. The factor markets clear at all times, that is, $L_i(t) = L(t) = \ell(t)$ and $K_i(t) = K(t) = k(t)$ hold for all $t \in \mathbb{R}_+$.[2]
4. The externality condition (5) holds for all $t \in \mathbb{R}_+$.

An equilibrium is said to be *interior* if none of the non-negativity constraints on the functions ℓ, $1 - \ell$, c, $(K_i, L_i)_{i\in[0,1]}$ binds at any time t.

A *balanced growth path* (BGP) is an equilibrium such that all functions k, ℓ, c, $(K_i, L_i)_{i\in[0,1]}$, K, L, A, q, and w take strictly positive values and have constant growth rates.[3]

3 Results

Let us define the intensive production function $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$ by $f(x) = F(x, 1)$. Under Assumption 2 it follows that the intensive production function is continuous, strictly increasing, and concave, and that it is twice continuously differentiable on the interior of its domain. Moreover, it holds that $f(0) = 0$. The following lemma states necessary and sufficient equilibrium conditions.

Lemma 1 *Consider an economy* (u, ρ, F, δ) *and let* f *be the intensive production function.*

(a) *If* $\{k, \ell, c, (K_i, L_i)_{i\in[0,1]}, K, L, A, q, w\}$ *is an interior equilibrium of the economy* (u, ρ, F, δ)*, then it follows that the functions* k, ℓ, c *satisfy the conditions*[4]

$$\dot{k}(t)/k(t) = \ell(t)f(1/\ell(t)) - \delta - c(t)/k(t), \tag{6}$$

$$u_2(c(t), 1 - \ell(t)) = k(t)[f(1/\ell(t)) - f'(1/\ell(t))/\ell(t)]u_1(c(t), 1 - \ell(t)), \tag{7}$$

$$\begin{aligned} &u_{11}(c(t), 1 - \ell(t))\dot{c}(t) - u_{12}(c(t), 1 - \ell(t))\dot{\ell}(t) \\ &\quad = [\rho + \delta - f'(1/\ell(t))]u_1(c(t), 1 - \ell(t)), \end{aligned} \tag{8}$$

$$\lim_{t\to+\infty} e^{-\rho t}u_1(c(t), 1 - \ell(t))k(t) = 0. \tag{9}$$

[2]It follows from Walras' law that the output market clears as well.

[3]The growth rate of a function $z : \mathbb{R}_+ \mapsto \mathbb{R}_{++}$ at time t is given by $\dot{z}(t)/z(t)$.

[4]Partial derivatives are denoted by subscripts. For example, $u_1(c, 1 - \ell)$ denotes the partial derivative of the function u with respect to its first argument evaluated at the point $(c, 1 - \ell)$.

(b) *Conversely, if there exist strictly positive functions k, ℓ, c such that $\ell(t) < 1$ and conditions (6)–(9) hold, then one can find functions $(K_i, L_i)_{i\in[0,1]}$, K, L, A, q, and w such that $\{k, \ell, c, (K_i, L_i)_{i\in[0,1]}, K, L, A, q, w\}$ is an equilibrium of the economy (u, ρ, F, δ).*

Proof The necessary and sufficient conditions for $(K_i(t), L_i(t))$ to be an interior maximum in (4) are $q(t) = F_1(K_i(t), A(t)L_i(t))$ and $w(t) = A(t)F_2(K_i(t), A(t)L_i(t))$. Since equilibrium requires $K_i(t) = K(t) = k(t)$, $L_i(t) = L(t) = \ell(t)$, and (5) and since F is homogeneous of degree 1, this implies that

$$q(t) = f'(1/\ell(t)) \qquad \text{and} \qquad w(t) = k(t)[f(1/\ell(t)) - f'(1/\ell(t))/\ell(t)]. \tag{10}$$

The Hamiltonian of the representative household's optimization problem is

$$H(k, c, \ell, \lambda, t) = u(c, 1 - \ell) + \lambda\{[q(t) - \delta]k + w(t)\ell - c\},$$

where λ denotes the adjoint variable. The necessary and sufficient first-order conditions for an interior maximum are[5]

$$u_1(c(t), 1 - \ell(t)) - \lambda(t) = 0,$$
$$-u_2(c(t), 1 - \ell(t)) + \lambda(t)w(t) = 0,$$
$$\dot{\lambda}(t) = [\rho + \delta - q(t)]\lambda(t),$$
$$\lim_{t\to+\infty} e^{-\rho t}\lambda(t)k(t) = 0.$$

Eliminating the adjoint variable from these conditions and using the expressions for the factor prices from (10), we obtain the conditions stated in part (a). It is obvious that these steps can be reversed such that part (b) is proven as well. □

Let us now focus on BGP. Since $\ell(t) = L(t) = L_i(t) \in [0, 1]$ must hold for all $i \in [0, 1]$ and all $t \in \mathbb{R}_+$, the growth rate of $\ell(t)$ must be non-positive. Eriksson (1996) focuses on the case where $\ell(t)$ has a negative growth rate because he wants to match the stylized fact of falling labor supply. BGP with declining labor supply, however, are only possible if the production function is of the Cobb-Douglas form (at least for sufficiently large capital-labor ratios), as Eriksson (1996) indeed assumes. Furthermore, capital and consumption must also be declining. This is shown in our next lemma.

Lemma 2 *Consider an economy (u, ρ, F, δ) and assume that there exists a BGP along which labor supply declines. Then there exist numbers A, α, and $\underline{\kappa}$ such that the intensive production function f satisfies $f(x) = Ax^{\alpha}$ for all $x \geq \underline{\kappa}$. Furthermore, the growth rates of capital and consumption along the BGP must be negative as well.*

[5] See Feichtinger and Hartl (1986).

Proof Suppose that there exists a BGP along which capital, consumption, and labor grow at the rates γ_k, γ_c, and γ_ℓ. Suppose furthermore that $\gamma_\ell < 0$. From (6) it follows that

$$\gamma_k + \delta = \ell(0)e^{\gamma_\ell t}f(\ell(0)^{-1}e^{-\gamma_\ell t}) - \frac{c(0)}{k(0)}e^{(\gamma_c-\gamma_k)t}. \tag{11}$$

Suppose first that $\gamma_c = \gamma_k$. In this case it is obvious that $\ell(0)e^{\gamma_\ell t}f(\ell(0)^{-1}e^{-\gamma_\ell t})$ must be independent of t, that is, there exists $m \in \mathbb{R}_+$ such that $\ell(0)e^{\gamma_\ell t}f(\ell(0)^{-1}e^{-\gamma_\ell t}) = m$ holds for all $t \in \mathbb{R}_+$. This implies that

$$f(1/\ell(t)) = f(\ell(0)^{-1}e^{-\gamma_\ell t}) = \frac{m}{\ell(0)}e^{-\gamma_\ell t}.$$

Differentiating this equation with respect to t we obtain $f'(1/\ell(t)) = f'(\ell(0)^{-1}e^{-\gamma_\ell t}) = m$ for all $t \in \mathbb{R}_+$. Obviously, this is a contradiction to the strict concavity of f and it follows that $\gamma_c \neq \gamma_k$.

If $\gamma_c \neq \gamma_k$, then Eq. (11) can only hold for all $t \in \mathbb{R}_+$ if $\gamma_k = -\delta$ and if

$$f(1/\ell(t)) = f(\ell(0)^{-1}e^{-\gamma_\ell t}) = me^{\mu t}$$

holds for all $t \in \mathbb{R}_+$, where $m = c(0)/[k(0)\ell(0)]$ and $\mu = \gamma_c - \gamma_k - \gamma_\ell$. Differentiation with respect to t yields

$$f'(1/\ell(t)) = f'(\ell(0)^{-1}e^{-\gamma_\ell t}) = -\frac{\ell(0)m\mu}{\gamma_\ell}e^{(\gamma_\ell+\mu)t} = -\frac{\mu}{\gamma_\ell}\frac{f(1/\ell(t))}{1/\ell(t)}.$$

Obviously, this implies that $f(x) = Ax^\alpha$ for some $A \in \mathbb{R}$, $\alpha = -\mu/\gamma_\ell$, and all $x \geq \underline{\kappa} = 1/\ell(0)$. Since $\gamma_\ell < 0$ and $\alpha = -\mu/\gamma_\ell = (\gamma_k - \gamma_c)/\gamma_\ell + 1 \in (0, 1)$ must hold, it follows that $\gamma_c < \gamma_k = -\delta < 0$. This completes the proof of the lemma. □

Note that the proof of the above lemma uses only the equilibrium condition (6), which is the capital accumulation equation. In other words, the lemma holds independently of the specification of the preferences. We consider the case of a BGP along which capital, consumption, and labor supply decline as rather uninteresting and will therefore from now on focus on those BGP, along which the labor supply remains constant. Furthermore, in what follows, we assume that the utility function is specified by

$$u(c, 1-\ell) = c^{1-\sigma}(1-\ell)^\sigma, \tag{12}$$

where $\sigma \in (0, 1)$ is the elasticity of marginal utility of consumption. It is easy to see that this function satisfies Assumption 1. The following lemma characterizes those BGP along which labor supply remains constant.

Lemma 3 *Consider an economy* (u, ρ, F, δ) *in which the utility function u is given by (12) and let f be the intensive production function. There exists an interior BGP along which labor supply is constant and equal to* $\hat{\ell}$ *if and only if the conditions*

$$\left(1-\frac{1-\sigma}{\hat{\ell}}\right)\hat{\ell}f(1/\hat{\ell})+\rho+(1-\sigma)\delta=\left(2-\sigma-\frac{1-\sigma}{\hat{\ell}}\right)f'(1/\hat{\ell}), \quad (13)$$

$$f'(1/\hat{\ell})<\frac{\rho+(1-\sigma)\delta}{1-\sigma} \quad (14)$$

hold.

Proof Suppose that $\ell(t)=\hat{\ell}$ holds for all $t\in\mathbb{R}_+$. It follows immediately from (6) that k and c must grow at the same rate, such that $k(t)/c(t)=k(0)/c(0)$ holds for all $t\in\mathbb{R}_+$. Let us denote the common growth rate of capital and consumption by γ. With this notation, we can rewrite the equilibrium conditions for a BGP (with constant labor supply $\hat{\ell}$) from (6)–(9) as

$$\gamma=\hat{\ell}f(1/\hat{\ell})-\delta-\frac{c(0)}{k(0)},$$

$$\frac{\sigma c(0)}{(1-\sigma)(1-\hat{\ell})k(0)}=f(1/\hat{\ell})-\frac{f'(1/\hat{\ell})}{\hat{\ell}},$$

$$-\sigma\gamma=\rho+\delta-f'(1/\hat{\ell}),$$

$$-\rho+\gamma(1-\sigma)<0.$$

Solving the first and the third of these conditions for γ and $c(0)/k(0)$ and substituting the results into the other two conditions, we obtain (13)–(14). □

We are now ready to state the main result of the paper.

Theorem 1 *Fix any strictly positive values* ρ *and* δ*. There exist functions u and F satisfying Assumptions 1 and 2, respectively, such that the economy* (u, ρ, F, δ) *admits a continuum of mutually different BGP.*

Proof Because of Lemma 3 it is sufficient to show that there exists a strictly increasing, strictly concave, and smooth function $f:\mathbb{R}_+\mapsto\mathbb{R}_+$ with $f(0)=0$ such that conditions (13)–(14) hold for a continuum of values $\hat{\ell}\in(0,1)$. We proceed in four steps.

STEP 1: Let Q be an arbitrary positive number and define the function $\phi:[0,(2-\sigma)/(1-\sigma)]\mapsto\mathbb{R}_+$ by

$$\phi(x)=Q[2-\sigma-(1-\sigma)x]^{(1-\sigma)/(2-\sigma)}x^{1/(2-\sigma)}+\frac{\rho+(1-\sigma)\delta}{1-\sigma}x.$$

Obviously, ϕ is continuous, satisfies $\phi(0) = 0$, and is twice continuously differentiable on the interior of its domain. The first- and second-order derivatives of ϕ are

$$\phi'(x) = Q[1-(1-\sigma)x][2-\sigma-(1-\sigma)x]^{-1/(2-\sigma)}x^{-(1-\sigma)/(2-\sigma)} + \frac{\rho+(1-\sigma)\delta}{1-\sigma}$$

and

$$\phi''(x) = -Q(1-\sigma)[2-\sigma-(1-\sigma)x]^{-(3-\sigma)/(2-\sigma)}x^{-(3-2\sigma)/(2-\sigma)}.$$

Because of $Q > 0$ and $\sigma \in (0,1)$ we have $\phi''(x) < 0$ for all $x \in (0, (2-\sigma)/(1-\sigma))$ and it follows that ϕ is strictly concave.

STEP 2: Define $\underline{x} = 1/(1-\sigma)$. It is easy to see that $\lim_{x\to 0}\phi'(x) = +\infty$, $\phi'(\underline{x}) = [\rho+(1-\sigma)\delta]/(1-\sigma) > 0$, and $\lim_{x\to(2-\sigma)/(1-\sigma)}\phi'(x) = -\infty$. Consequently, there exists a unique value $x_0 \in (\underline{x}, (2-\sigma)/(1-\sigma))$ for which $\phi'(x_0) = 0$ holds. We choose an arbitrary number $\bar{x} \in (\underline{x}, x_0)$ and note that $\phi'(\bar{x}) > 0$ holds.

STEP 3: Let $\tilde{\phi} : [\bar{x}, +\infty) \mapsto \mathbb{R}_+$ be an arbitrary smooth, strictly increasing, and strictly concave function that satisfies $\tilde{\phi}(\bar{x}) = \phi(\bar{x})$, $\tilde{\phi}'(\bar{x}) = \phi'(\bar{x})$, and $\tilde{\phi}''(\bar{x}) = \phi''(\bar{x})$. We define the intensive production function $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$ by

$$f(x) = \begin{cases} \phi(x) & \text{if } 0 \le x \le \bar{x}, \\ \tilde{\phi}(x) & \text{otherwise} \end{cases}$$

and the production function $F : \mathbb{R}^2_+ \mapsto \mathbb{R}_+$ by $F(K, AL) = ALf(K/(AL))$. From steps 1 and 2 and the construction of f it follows that f is continuous, strictly increasing, strictly concave, and twice continuously differentiable on the interior of its domain. Moreover, it holds that $f(0) = 0$. It is easy to see that these properties imply that F satisfies Assumption 2.

STEP 4: Finally consider an arbitrary $\hat{\ell} \in [1/\bar{x}, 1/\underline{x})$ and define $\hat{x} = 1/\hat{\ell}$. Note that this implies $\hat{x} \in (\underline{x}, \bar{x}]$ and, consequently, that $f(\hat{x}) = \phi(\hat{x})$ and $f'(\hat{x}) = \phi'(\hat{x})$ hold. It follows therefore that conditions (13)–(14) can be written as

$$[1-(1-\sigma)\hat{x}]\frac{\phi(\hat{x})}{\hat{x}} + \rho + (1-\sigma)\delta = [2-\sigma-(1-\sigma)\hat{x}]\phi'(\hat{x})$$

and

$$\phi'(\hat{x}) < \frac{\rho+(1-\sigma)\delta}{1-\sigma}.$$

Using the expressions for $\phi(x)$ and $\phi'(x)$ stated in step 1, it is straightforward to verify that the first condition holds. The second one follows from $\hat{x} > \underline{x}$ and the definition of $\underline{x}$. The proof of Theorem 1 is now complete.

□

The above theorem demonstrates that there exists an economy with a continuum of interior BGP. Using arguments similar to those employed in the proof of Proposition 3 in Kamihigashi (2015) one can also show that, for every integer n, there exists an economy with exactly n different BGP. We will not provide the details of such a proof here. Instead, we want to illustrate a different way of proving the possibility of multiple BGP. This alternative proof has the advantage that it allows us also to verify local uniqueness or indeterminacy, respectively, of the BGP.

Lemma 4 *Consider an economy* (u, ρ, F, δ) *in which the utility function is given by (12). Let* f *be the intensive production function and define* $W : \mathbb{R}_+ \mapsto \mathbb{R}_+$ *by* $W(x) = f(x) - xf'(x)$.

(a) *If* $\{k, \ell, c, (K_i, L_i)_{i\in[0,1]}, K, L, A, q, w\}$ *is an interior equilibrium of the economy* (u, ρ, F, δ)*, then it follows that the function* ℓ *satisfies the differential equation*

$$\frac{\sigma W'(1/\ell(t))\dot{\ell}(t)}{\ell(t)^2 W(1/\ell(t))} = \left[1 - \frac{1-\sigma}{\ell(t)}\right]\ell(t)f(1/\ell(t)) + \rho + (1-\sigma)\delta - \left[2 - \sigma - \frac{1-\sigma}{\ell(t)}\right]f'(1/\ell(t)). \tag{15}$$

(b) *Conversely, if there exists a function* ℓ *such that* $\ell(t) \in (0, 1)$ *and Eq. (15) hold, then one can find functions* k, c, $(K_i, L_i)_{i\in[0,1]}$, K, L, A, q, *and* w *such that the family of functions* $\{k, \ell, c, (K_i, L_i)_{i\in[0,1]}, K, L, A, q, w\}$ *forms an interior equilibrium of* (u, ρ, F, δ).

Proof If the utility function is given by (12), then conditions (7)–(8) simplify to

$$\frac{\sigma c(t)}{(1-\sigma)[1-\ell(t)]} = k(t)W(1/\ell(t)), \tag{16}$$

$$-\sigma\frac{\dot{c}(t)}{c(t)} - \sigma\frac{\dot{\ell}(t)}{1-\ell(t)} = \rho + \delta - f'(1/\ell(t)). \tag{17}$$

From (16) it follows that

$$c(t) = \frac{1-\sigma}{\sigma}k(t)W(1/\ell(t))[1-\ell(t)] \tag{18}$$

which, in turn, implies

$$\frac{\dot{c}(t)}{c(t)} = \frac{\dot{k}(t)}{k(t)} - \frac{W'(1/\ell(t))\dot{\ell}(t)}{W(1/\ell(t))\ell(t)^2} - \frac{\dot{\ell}(t)}{1-\ell(t)}. \tag{19}$$

Combining (6) with (18) one obtains

$$\frac{\dot{k}(t)}{k(t)} = \ell(t)f(1/\ell(t)) - \delta - \frac{1-\sigma}{\sigma}W(1/\ell(t))[1-\ell(t)].$$

Using this equation to eliminate $\dot{k}(t)/k(t)$ from (19) and substituting the resulting expression for $\dot{c}(t)/c(t)$ into (17) we obtain after some algebraic manipulations equation (15). Conversely, if a solution ℓ of Eq. (15) is given such that $\ell(t) \in (0, 1)$ holds for all $t \in \mathbb{R}_+$, then one can compute $c(t)/k(t)$ from (18). Substituting this result into (6) one obtains a differential equation that can be used to compute k. □

Note that Eq. (15) is a differential equation for the labor supply, which is a jump variable. Hence, no initial value $\ell(0)$ is given. This observation allows us to derive the following characterization of the determinacy of BGP.

Theorem 2 *Consider an economy* (u, ρ, F, δ) *in which the utility function is given by (12) and let* f *be the intensive production function. Consider an interior BGP with constant labor supply equal to* $\hat{\ell}$*. This BGP is a locally unique equilibrium if*

$$\hat{\ell}^2 f(1/\hat{\ell}) - \hat{\ell} f'(1/\hat{\ell}) + \left(2 - \sigma - \frac{1-\sigma}{\hat{\ell}}\right) f''(1/\hat{\ell}) > 0$$

holds, and it is indeterminate if the above inequality holds with the reversed (strict) inequality sign.

Proof The labor supply $\ell(t)$ is a jump variable for which no initial value is given. A BGP with constant labor supply $\hat{\ell}$ is therefore locally unique if $\hat{\ell}$ is an unstable fixed point of Eq. (15) and it is indeterminate if $\hat{\ell}$ is a stable fixed point of (15). Denoting the right-hand side of Eq. (15) by $g(\ell(t))$ it is easy to see that

$$g'(\ell) = \frac{1}{\ell^2}\left[\ell^2 f(1/\ell) - \ell f'(1/\ell) + \left(2 - \sigma - \frac{1-\sigma}{\ell}\right) f''(1/\ell)\right].$$

Since $W'(x) > 0$ holds for all $x \in \mathbb{R}_+$, it is clear that $\hat{\ell}$ is an unstable fixed point of (15) if $g'(\hat{\ell}) > 0$ holds and that it is locally asymptotically stable if $g'(\hat{\ell}) < 0$ is satisfied. This completes the proof of the theorem. □

It is well known that the one-sector growth model with elastic labor supply and production externalities can have indeterminate equilibria; see Benhabib and Rustichini (1994) or Benhabib and Farmer (1994). The reason why we include the above theorem in this paper is that our setting allows for multiple BGP, whereas the existing literature typically makes assumptions under which there exists exactly one BGP. As will be illustrated in the following example, the model can simultaneously have determinate and indeterminate BGP.

Example 1 Suppose that the intensive production function f satisfies

$$f(0) = 0\,,\ f(5/2) = 15/2\,,\ f(8/3) = 38/5\,,\ f'(5/2) = 1\,,\ f'(8/3) = 3/10.$$

It is not difficult to see that a smooth, strictly increasing, and strictly concave function f with these properties exists. Now consider the utility function from (12) with parameter $\sigma = 1/2$. Furthermore, choose the unit of time in such a way that

$\rho + \delta/2 = 1$. With these specifications it follows that conditions (13)–(14) hold for both $\hat{\ell} = 3/8$ and $\hat{\ell} = 2/5$. Hence, the economy (u, ρ, F, δ) has (at least) two interior BGP with labor supplies 3/8 and 2/5, respectively. Substituting the value $\hat{\ell} = 3/8$ into the condition stated in Theorem 2, it can be seen that the BGP with $\hat{\ell} = 3/8$ is indeterminate if $f''(3/8) < -459/80$ whereas it is locally unique if $f''(3/8) > -459/80$. Analogously, the BGP with $\hat{\ell} = 2/5$ is indeterminate if $f''(2/5) < -16/5$ and it is locally unique if $f''(2/5) > -16/5$.

Finally, let us define the elasticities $\varepsilon_0(x) = f'(x)x/f(x)$ and $\varepsilon_1(x) = |f''(x)x/f'(x)|$. Using this notation we can write the inequality stated in Theorem 2 as

$$\frac{1}{\varepsilon_0(1/\hat{\ell})} - \left(2 - \sigma - \frac{1-\sigma}{\hat{\ell}}\right)\varepsilon_1(1/\hat{\ell}) > 1.$$

This shows that the BGP with constant labor supply equal to $\hat{\ell}$ is locally unique if both elasticities $\varepsilon_0(1/\hat{\ell})$ and $\varepsilon_1(1/\hat{\ell})$ are small, whereas it is indeterminate if at least one of these elasticities is sufficiently large.

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Fiscal and Monetary Policies in a Monetary Union: Conflict or Cooperation?

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Abstract In this paper we present an application of the dynamic tracking games framework to a monetary union. We use a small stylized nonlinear two-country macroeconomic model (MUMOD1) of a monetary union to analyse the interactions between fiscal (governments) and monetary (common central bank) policy makers, assuming different objective functions of these decision makers. Using the OPTGAME algorithm we calculate solutions for two game strategies: a cooperative solution (Pareto optimal) and a non-cooperative equilibrium solution (the Nash game for the feedback information pattern). We show how the policy makers react to demand shocks under non-cooperation and cooperation scenarios. The cooperative solution dominates the non-cooperative solution in all scenarios, which may be interpreted as an argument for coordinating fiscal and monetary policies in a monetary union in a situation of high public debt such as in the recent sovereign debt crisis in Europe.

1 Introduction

Economic decisions usually aim to achieve goals as successfully as possible according to some system of preferences, both at the level of an individual firm or household and at the level of an entire economy. As economic systems usually involve relations over time, both microeconomic and macroeconomic models are mostly dynamic. Therefore dynamic optimization or optimum control techniques are appropriate instruments for determining optimal decisions in economics. This has been recognized for quite some time now; see e.g. Feichtinger and Hartl (1986). However, there is a broad class of problems in economics where decisions also have to take other rational decision makers into account. In a dynamic context, this opens the way for the application of dynamic game theory, whose development started even before most current techniques for dynamic optimization were known.

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Nowadays, dynamic game theory is fairly well developed and several applications to economics have shown its potential for providing a framework for interactive economic strategic decision making [see e.g. Acocella et al. (2013), Basar and Olsder (1999), Petit (1990), van Aarle et al. (2002)].

In this paper we present an application of the dynamic tracking game framework to a macroeconomic model of a monetary union. In such a union a supranational central bank interacts strategically with sovereign governments as national fiscal policy makers in the member states. Such conflicts can be analysed using either large empirical macroeconomic models or small stylized models. We follow the second line of research and use a small stylized nonlinear two-country macroeconomic model of a monetary union (called MUMOD1) for analysing the interactions between fiscal (governments) and monetary (common central bank) policy makers, assuming different objective functions for these decision makers. Using the OPTGAME algorithm we calculate equilibrium solutions for two game strategies: one cooperative (Pareto optimal) and one non-cooperative game (the Nash game for the feedback information pattern). Applying the OPTGAME algorithm to the MUMOD1 model we show how the policy makers react optimally to symmetric and asymmetric demand shocks. Some comments are given about possible applications to the recent sovereign debt crisis in Europe.

2 Nonlinear Dynamic Tracking Games

The nonlinear dynamic game-theoretic problem which we consider in this paper is given in tracking form. The players are assumed to aim at minimizing quadratic deviations of the equilibrium or optimal values (according to the respective solution concept) of state and control variables over time from given desired values. Thus each player minimizes an objective function J^i given by:

$$\min_{u_1^i,\ldots,u_T^i} J^i = \sum_{t=1}^{T} L_t^i(x_t, u_t^1, \ldots, u_t^N), \qquad i = 1, \ldots, N, \tag{1}$$

with

$$L_t^i(x_t, u_t^1, \ldots, u_t^N) = \frac{1}{2}[X_t - \tilde{X}_t^i]' \Omega_t^i [X_t - \tilde{X}_t^i], \qquad i = 1, \ldots, N. \tag{2}$$

The parameter N denotes the number of players (decision makers). T is the terminal period of the finite planning horizon, i.e. the duration of the game. X_t is an aggregated vector

$$X_t := [x_t \;\; u_t^1 \;\; u_t^2 \;\; \ldots \;\; u_t^N]', \tag{3}$$

which consists of an $(n_x \times 1)$ vector of state variables and N $(n_i \times 1)$ vectors of control variables determined by the players $i = 1, \ldots, N$. Thus X_t (for all $t = 1, \ldots, T$) is an r-dimensional vector where

$$r := n_x + n_1 + n_2 + \cdots + n_N. \tag{4}$$

The desired levels of the state variables and the control variables of each player enter the quadratic objective functions [as given by Eqs. (1) and (2)] via the terms

$$\tilde{X}_t^i := [\tilde{x}_t^i \;\; \tilde{u}_t^{i1} \;\; \tilde{u}_t^{i2} \;\; \ldots \;\; \tilde{u}_t^{iN}]'. \tag{5}$$

Equation (2) contains an $(r \times r)$ penalty matrix Ω_t^i, weighting the deviations of states and controls from their desired levels in any time period t. Thus the matrices

$$\Omega_t^i = \begin{bmatrix} Q_t^i & 0 & \cdots & 0 \\ 0 & R_t^{i1} & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & R_t^{iN} \end{bmatrix}, \; i = 1, \ldots, N, \; t = 1, \ldots, T, \tag{6}$$

are in block-diagonal form, where the blocks Q_t^i and R_t^{ij} $(i, j = 1, \ldots, N)$ are symmetric and the R^{ii} are positive definite. Blocks Q_t^i and R_t^{ij} correspond to penalty matrices for the states and the controls respectively.

In a frequent special case, a discount factor α is used to calculate the penalty matrix Ω_t^i in time period t:

$$\Omega_t^i = \alpha^{t-1} \Omega_0^i, \tag{7}$$

where the initial penalty matrix Ω_0^i of player i is given.

The dynamic system, which constrains the choices of the decision makers, is given in state-space form by a first-order system of nonlinear difference equations:

$$x_t = f(x_{t-1}, x_t, u_t^1, \ldots, u_t^N, z_t), \quad x_0 = \bar{x}_0. \tag{8}$$

$\bar{x}_0$ contains the initial values of the state variables. Vector z_t contains non-controlled exogenous variables. For the algorithm, we require that the first and second derivatives of the system function f with respect to x_t, x_{t-1} and $u_t^1, \ldots, u_t^N$ exist and are continuous.

Equations (1), (2) and (8) define a nonlinear dynamic tracking game problem. For each solution concept, the task is to find N trajectories of control variables u_t^i which minimize the postulated objective functions subject to the dynamic system.

In order to solve the stated game we use the OPTGAME algorithm as described in Behrens and Neck (2015) and Blueschke et al. (2013). The OPTGAME algorithm allows us to approximate game solutions for different game strategies. In this paper we consider two solution concepts: a cooperative game (Pareto optimal) and a non-cooperative Nash game.

3 The MUMOD1 Model

We use a dynamic macroeconomic model of a monetary union consisting of two countries (or two blocs of countries) with a common central bank. This model is called MUMOD1 and slightly improves on the one introduced in Blueschke and Neck (2011) and Neck and Blueschke (2014). For a similar framework in continuous time, see van Aarle et al. (2002). The model is calibrated so as to deal with the problem of public debt targeting in a situation that resembles the one currently prevailing in the Eurozone.

The model is formulated in terms of deviations from a long-run growth path and includes three decision makers. The common central bank decides on the prime rate R_{Et}, a nominal rate of interest under its direct control. The national governments decide on the fiscal policy instruments, where g_{it} denotes country i's ($i = 1, 2$) real fiscal surplus (or, if negative, its fiscal deficit), measured in relation to real GDP.

The model consists of the following equations:

$$y_{it} = \delta_i(\pi_{jt} - \pi_{it}) - \gamma(r_{it} - \theta) + \rho_i y_{jt} - \beta_i \pi_{it} + \kappa_i y_{i,t-1} - \eta_i g_{it} + zd_{it}, \tag{9}$$

$$r_{it} = I_{it} - \pi^e_{it}, \tag{10}$$

$$I_{it} = R_{Et} - \lambda_i g_{it} + \chi_i D_{it}, \tag{11}$$

$$\pi_{it} = \pi^e_{it} + \xi_i y_{it}, \tag{12}$$

$$\pi^e_{it} = \varepsilon_i \pi_{i,t-1} + (1 - \varepsilon_i)\pi^e_{i,t-1}, \quad \varepsilon \in [0, 1], \tag{13}$$

$$y_{Et} = \omega y_{1t} + (1 - \omega) y_{2t}, \quad \omega \in [0, 1], \tag{14}$$

$$\pi_{Et} = \omega \pi_{1t} + (1 - \omega)\pi_{2t}, \quad \omega \in [0, 1], \tag{15}$$

$$D_{it} = (1 + BI_{i,t-1} - \pi^e_{i,t-1}) D_{i,t-1} - g_{it}, \tag{16}$$

$$BI_{it} = \frac{1}{6} \sum_{\tau=t-5}^{t} I_{it}. \tag{17}$$

List of variables:

y_{it}	real output (deviation from natural output)
π_{it}	inflation rate
r_{it}	real interest rate
g_{it}	real fiscal surplus
I_{it}	nominal interest rate
π_{it}^{e}	expected inflation rate
R_{Et}	prime rate
D_{it}	real government debt
B_{it}	interest rate on public debt

The goods markets are modelled for each country i by the short-run income-expenditure equilibrium relation (IS curve) (9) for real output y_{it} at time t ($t = 1, \ldots, T$). The natural real rate of output growth, $\theta \in [0, 1]$, is assumed to be equal to the natural real rate of interest.

The current real rate of interest r_{it} is given by Eq. (10). The nominal rate of interest I_{it} is given by Eq. (11), where $-\lambda_i$ and χ_i (assumed to be positive) are risk premiums for country i's fiscal deficit and public debt level respectively. This allows for different nominal rates of interest in the union in spite of a common monetary policy.

The inflation rates for each country π_{it} are determined in Eq. (12) according to an expectations-augmented Phillips curve. π_{it}^{e} denotes the rate of inflation expected to prevail during time period t, which is formed according to the hypothesis of adaptive expectations at (the end of) time period $t-1$ [Eq. (13)]. $\varepsilon_i \in [0, 1]$ are positive parameters determining the speed of adjustment of expected to actual inflation.

The average values of output and inflation in the monetary union are given by Eqs. (14) and (15), where parameter ω expresses the weight of country 1 in the economy of the whole monetary union as defined by its output level. The same weight ω is used for calculating union-wide inflation.

The government budget constraint is given as an equation for real government debt D_{it} (measured in relation to GDP) and is shown in Eq. (16). The interest rate on public debt (on government bonds) is denoted by BI_{it}, which assumes an average government bond maturity of 6 years, as estimated in Krause and Moyen (2013).

The parameters of the model are specified for a slightly asymmetric monetary union. Here an attempt has been made to calibrate the model parameters so as to fit the Euro Area (EA). The data used for calibration include average economic indicators for the 19 EA countries from EUROSTAT up to the year 2014. Mainly based on the public finance situation, the EA is divided into two blocs: a "core" (country or bloc 1) and a "periphery" (country or bloc 2). The first bloc includes twelve EA countries (Austria, Belgium, Estonia, Finland, France, Germany, Latvia, Lithuania, Luxembourg, Malta, the Netherlands, and Slovakia) with a more solid fiscal situation and inflation performance. This bloc has a weight of 67 % in the entire economy of the monetary union. The second bloc has a weight of 33 % in the economy of the union; it consists of seven countries in the EA with higher public debt and/or deficits and higher interest and inflation rates on average (Cyprus, Greece, Ireland, Italy, Portugal, Slovenia, and Spain). These weights correspond to the blocs' shares in EA real GDP.

The initial values of all players are calibrated based on the data from 2014 and presented in Table 1. For the other parameters of the model, we use values in accordance with econometric studies and plausibility considerations (see Table 2).

For the intertemporal nonlinear policy game, the individual objective functions of the national governments ($i = 1, 2$) and of the common central bank (E) are given by

$$J_i = \frac{1}{2}\sum_{t=1}^{T}\left(\frac{1}{1+\frac{\theta}{100}}\right)^t\{\alpha_{\pi i}(\pi_{it}-\tilde{\pi}_{it})^2+\alpha_{yi}(y_{it}-\tilde{y}_{it})^2+\alpha_{Di}(D_{it}-\tilde{D}_{it})^2+\alpha_{gi}g_{it}^2\} \quad (18)$$

$$J_E = \frac{1}{2}\sum_{t=1}^{T}\left(\frac{1}{1+\frac{\theta}{100}}\right)^t\{\alpha_{\pi E}(\pi_{Et}-\tilde{\pi}_{Et})^2+\alpha_{yE}(y_{Et}-\tilde{y}_{Et})^2+\alpha_{E}(R_{Et}-\tilde{R}_{Et})^2\} \quad (19)$$

where all α are the weights of state variables representing their relative importance to the policy maker in question. A tilde denotes the desired ("ideal") values of the variable. Note that we assume different weights between the core and periphery for the state variable public debt: the core is assumed to care much more about its budgetary situation compared to the periphery (Tables 3 and 4).

Using a finite planning horizon T seems adequate for short-run problems of stabilization policy but has the consequence of neglecting developments at $t > T$. This leads to well-known end-of-planning period effects unless one introduces a

Table 1 Initial values ($t = 0$) of the two-country monetary union

y_1	y_2	π_1	π_2	I_1	I_2	D_1	D_2	g_1	g_2
−1	−1.3	0.7	0	1.4	3.1	81	121	−1.6	−4.2

Table 2 Parameter values for an asymmetric monetary union, $i = 1, 2$

T	θ	ω	$\delta_i, \eta_i, \varepsilon_i$	$\beta_i, \gamma_i, \rho_i, \kappa_i$	λ_i	ξ_i	χ_i	μ_i, μ_E
30	2	0.67	0.5	0.25	0.125	0.1	0.00625	0.333

Table 3 Weights of the variables in the objective functions

α_{yi}, α_{gi}	$\alpha_{\pi E}$	$\alpha_{yE}, \alpha_{\pi i}$	α_{D1}	α_{D2}	α_{RE}
1	2	0.5	0.01	0.0001	6

Table 4 Target values for the asymmetric monetary union

$\tilde{D}_{1t}$	$\tilde{D}_{2t}$	$\tilde{\pi}_{it}$	$\tilde{\pi}_{Et}$	$\tilde{y}_{it}$	$\tilde{y}_{Et}$	$\tilde{g}_{it}$	$\tilde{R}_{Et}$
60	80↘60	2	2	0	0	0	3

scrap value for the last period in the objective function. The present version of OPTGAME does not allow for this; hence results for the last few periods should be neglected when interpreting the trajectories of the state and control variables.

The joint objective function for calculating the cooperative Pareto-optimal solution is given by the weighted sum of the three objective functions:

$$J = \mu_1 J_1 + \mu_2 J_2 + \mu_E J_E, \qquad (\mu_1 + \mu_2 + \mu_3 = 1). \tag{20}$$

Here we assume equal weights for the three players ($\mu_i = 1/3, i = 1, 2, E$).

The dynamic system, which constrains the choices of the decision makers, is given in state-space form by the MUMOD1 model as presented in Eqs. (9)–(17). Equations (18), (19) and the dynamic system (9)–(17) define a nonlinear dynamic tracking game problem which can be solved for different solution concepts using the OPTGAME3 algorithm (see Blueschke et al. 2013).

4 Simulation Results

The MUMOD1 model can be used to simulate the effects of different shocks acting on the monetary union which are reflected in the paths of the exogenous non-controlled variables, and the effects of policy reactions towards these shocks. In this study we consider demand-side shocks in the goods markets as represented by the variables $zd_{it} (i = 1, 2)$. First, we assume a negative symmetric demand shock as given in Table 5. After that, we analyze the effects of asymmetric shocks affecting the core ($zd_{2t} = 0$) or the periphery ($zd_{1t} = 0$) only.

4.1 Effects of a Negative Symmetric Demand-Side Shock

In this section we investigate how the dynamics of the model and the results of the policy game (9)–(17) depend on the strategy choice of the decision makers in the case of a symmetric demand-side shock. We calculate three different solutions: a non-controlled simulation (no-policy scenario, keeping control variables at their desired levels), a non-cooperative feedback Nash equilibrium solution (which is strongly time-consistent) and one cooperative Pareto game solution. The non-controlled scenario does not include any policy intervention and describes a simple simulation of the dynamic system.

Table 5 Negative demand shocks in the asymmetric monetary union

t	1	2	3	4	5	6	$\cdots$	30
zd_1	−2	−4	−2	0	0	0	$\cdots$	0
zd_2	−2	−4	−2	0	0	0	$\cdots$	0

Figures 1, 2, 3, 4, and 5 show the results of these simulations. Figures 1 and 2 show the results for the control variables of the players while Figures 3, 4, and 5 show the results of selected state variables: namely output, inflation, and public debt.

Without policy intervention (the scenario denoted by 'simulation'), both countries suffer from the economic downturn caused by the demand-side shock during the first three periods. The output in the core drops by about 5 % points and the output in the periphery drops by about 3 % points. The development of public debt is even more dramatic. Without policy intervention it increases during the whole planning horizon and arrives at levels of 190 % of GDP for country 1 (or the core bloc) and 470 % for country 2 (or the periphery bloc). We are aware that this is not a realistic scenario as both countries (or at least the periphery) would go bankrupt

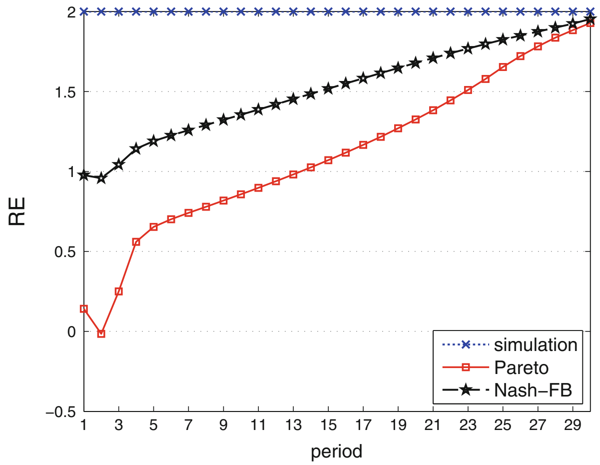

Fig. 1 Prime rate R_{Et} controlled by the central bank

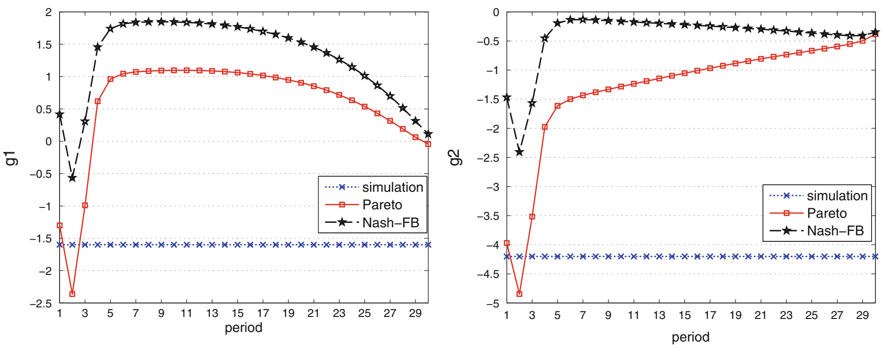

Fig. 2 Country i's fiscal surplus g_{it} (control variable) for $i = 1$ (core; *left*) and $i = 2$ (periphery; *right*)

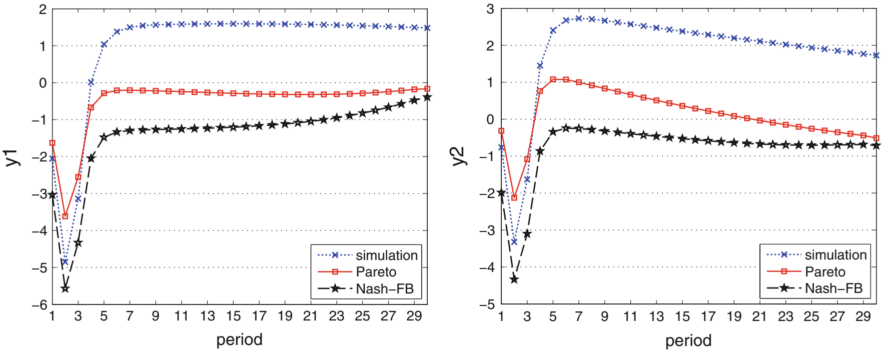

Fig. 3 Country i's output y_{it} for $i = 1$ (core; *left*) and $i = 2$ (periphery; *right*)

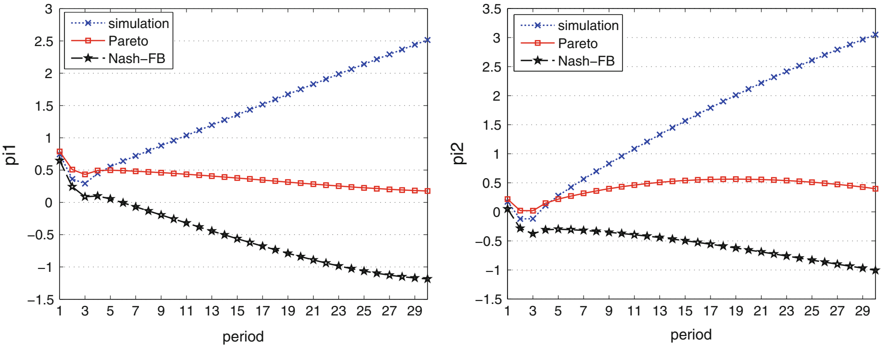

Fig. 4 Country i's inflation rate π_{it} for $i = 1$ (core; *left*) and $i = 2$ (periphery; *right*)

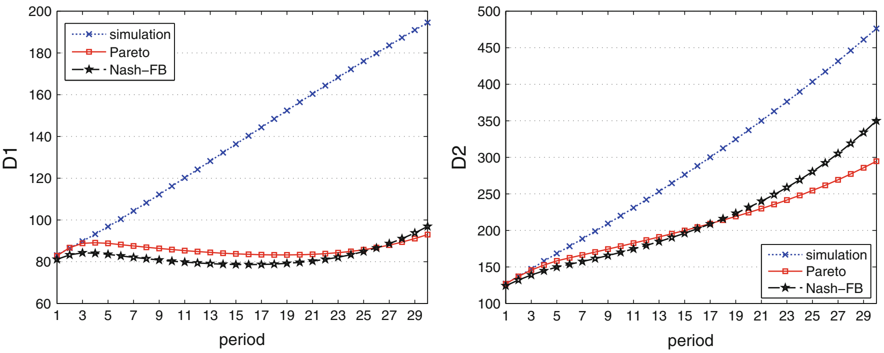

Fig. 5 Country i's debt level D_{it} for $i = 1$ (core; *left*) and $i = 2$ (periphery; *right*)

long before the end of the planning horizon. Instead, the non-controlled simulation scenario points toward a need for policy actions to preserve the solvency of the governments in the monetary union.

An optimal reaction of the players (in terms of the defined objective function) depends on the presence or absence of cooperation. For example, optimal monetary policy has to be expansionary (lowering the prime rate) in both solution concepts considered, but in the cooperative Pareto solution it is more active when compared to the non-cooperative feedback Nash equilibrium solution.

With respect to fiscal policy, even stronger differences can be seen between optimal policies for the core and the periphery. The periphery is required to set expansionary actions and to create deficits in the first three periods in order to help absorb the demand-side shock. This expansionary fiscal policy is much more active in the case of the cooperative solution compared to the feedback Nash equilibrium solution. Such fiscal policies help reduce the effects of the demand-side shock on output but result in relatively small improvements in the public debt situation. Government debt still goes up to very high values of around 300 % of GDP in the Pareto solution and 350 % of GDP in the Nash solution. Compared to 470 % in the non-controlled simulation this is a significant improvement but these levels of public debt are still unsustainable.

The core bloc in the Pareto solution also creates deficits during the presence of the demand-side shock but switches to a restrictive fiscal policy directly afterwards. In the case of the feedback Nash equilibrium solution, fiscal policy is even more restrictive and allows for a small deficit only at the peak of the negative shock in period 2. The effects of this more restrictive fiscal policy on economic performance are relatively small except for public debt. Although it does not allow the bloc to fulfill the Maastricht criteria it nevertheless leads to a significant decrease in public debt, which stays below 100 % of GDP.

One major reason for the need for a more restrictive (less expansionary) fiscal policy in the non-cooperative than in the cooperative solution is the less expansionary monetary policy in the former. This leads to higher nominal interest rates in both countries. The more restrictive overall policy stance causes (mild) deflation in the non-cooperative solution which, in combination with higher nominal interest rates, leads to high and increasing real interest rates, which contribute to strongly increasing public debt in spite of lower budget deficits than in the cooperative solution.

Finally Table 6 summarizes the objective function values as calculated by Eqs. (18) and (19), showing the advantages of cooperation (lower values of the objective functions) for all three players.

Table 6 Values of the objective functions (loss functions, to be minimized)

Strategy	J_E	J_1('core')	J_2('periphery')	$J_E + J_1 + J_2$
Simulation	45.95	686.88	374.03	1106.86
Pareto	146.36	69.28	93.93	309.57
Nash-FB	185.35	125.91	102.99	414.25

4.2 *Effects of a Negative Demand-Side Shock in the Core*

In this section we analyze a negative demand-side shock which occurs in the core bloc, with the same values for zd_1 as in Table 5 and $zd_2 = 0$ for all t. Figures 6, 7, 8, and 9 show the results of this experiment.

As the shock influences the core bloc only, its effects are smaller than under a symmetric shock in both countries. Optimal monetary policy remains expansionary but significantly less so than in the previous scenario. The fiscal policy of the core also remains expansionary during the shock, becoming restrictive immediately after it disappears. A comparison between the cooperative and the non-cooperative solution for the periphery shows the effects of cooperation. Even without being

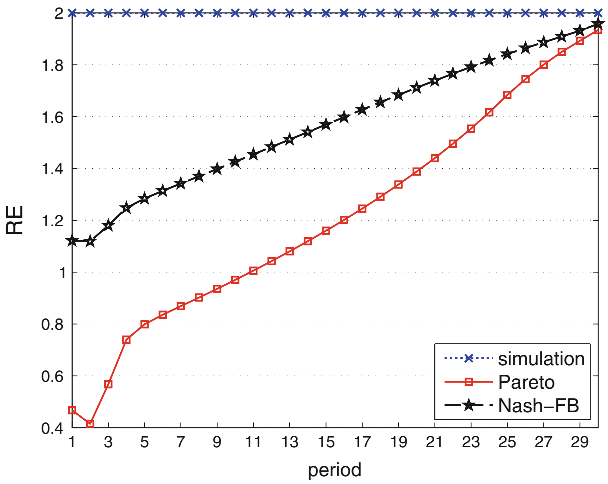

Fig. 6 Prime rate R_{Et} controlled by the central bank

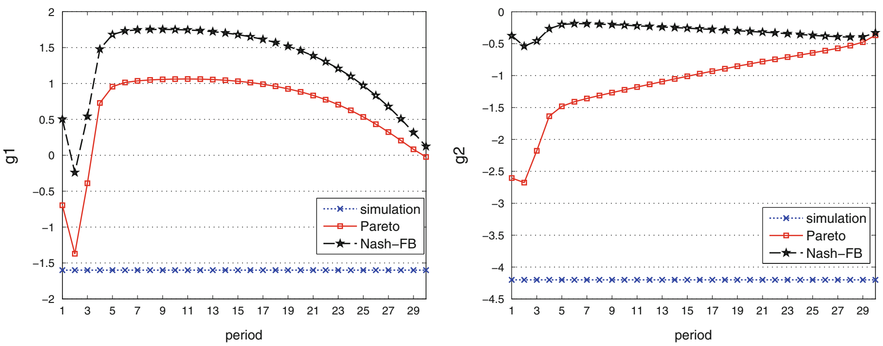

Fig. 7 Country i's fiscal surplus g_{it} (control variable) for $i = 1$ (core; *left*) and $i = 2$ (periphery; *right*)

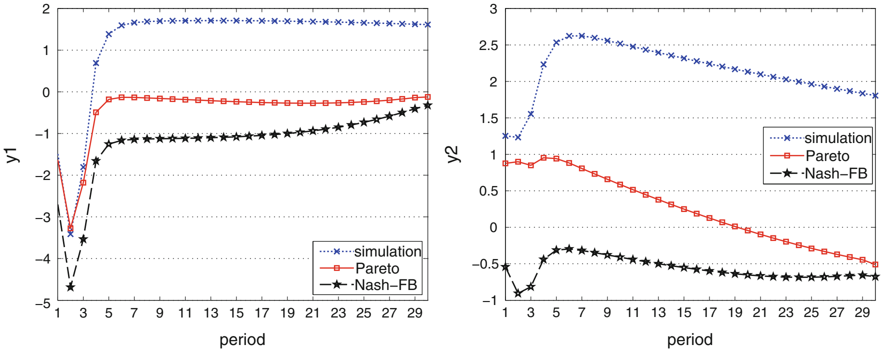

Fig. 8 Country i's output y_{it} for $i = 1$ (core; *left*) and $i = 2$ (periphery; *right*)

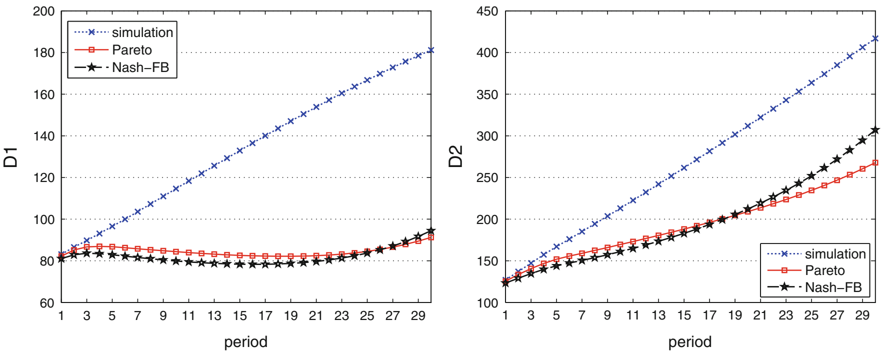

Fig. 9 Country i's debt level D_{it} for $i = 1$ (core; *left*) and $i = 2$ (periphery; *right*)

directly affected by the shock the periphery bloc nevertheless runs an expansionary fiscal policy in order to improve upon the joint objective function, which causes only small additional costs due to its weaker preference for fiscal prudence. In contrast, in the non-cooperative Nash solution we see nearly no reaction in the periphery's fiscal policy to the shock.

The asymmetric shock also produces asymmetric results for the state variables. While there is nearly no effect on the output of the second bloc, the first bloc experiences a significant decrease in output. However this small spillover of the negative shock does not help the second bloc to get its public debt situation under control. Its public debt grows further and arrives at a value of 270 % of GDP in the Pareto solution and 310 % of GDP in the feedback Nash equilibrium solution. This result indicates that the problem of the periphery's high public debt cannot be solved without deeper changes in the affected economies, which can be modelled in our framework by giving a higher weight to this state variable in the objective function of the periphery. In contrast, the core bloc can hold its public debt on a relatively constant level despite the occurrence of the negative shock. The effects on inflation are similar to those in the previous subsection.

Table 7 summarizes the objective function values as calculated by Eqs. (18) and (19). Note that the periphery does not suffer from cooperating with the core even though it is only minimally affected directly by the core-specific shock.

4.3 *Effects of a Negative Demand-Side Shock in the Periphery*

In this section we analyze a negative demand-side shock of the same size as before which occurs in the periphery bloc only. Figures 10, 11, 12, and 13 and Table 8 show the results.

Reversing the shock to affect the periphery only also turns around the results. The periphery is now affected negatively and is required to run a stronger expansionary fiscal policy in order to mitigate the effects of the shock. The core bloc instead can concentrate more on its public debt, which stays below 85 % of GDP most of the time in this scenario. This is achieved by creating nearly constant budget surpluses. The only exceptions are the first two periods in the cooperative Pareto solution, where small deficits occur. The optimal monetary policy is still expansionary but slightly less active than in the other two scenarios.

Table 7 Values of the objective functions (loss functions, to be minimized)

Strategy	J_E	J_1('core')	J_2('periphery')	$J_E + J_1 + J_2$
Simulation	34.90	592.47	345.38	972.75
Pareto	113.77	57.78	62.21	233.75
Nash-FB	144.64	104.10	62.93	311.67

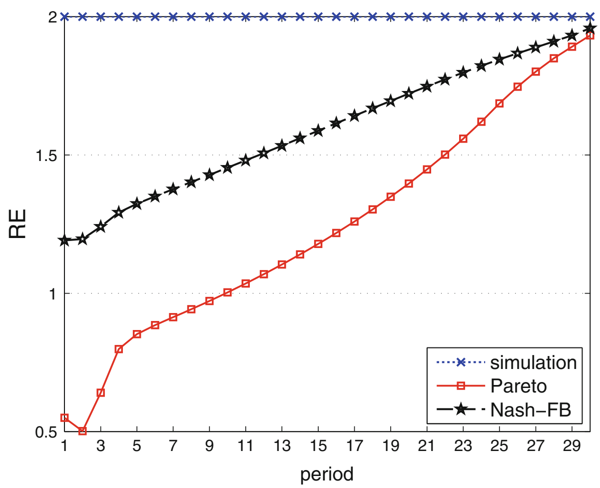

Fig. 10 Prime rate R_{Et} controlled by the central bank

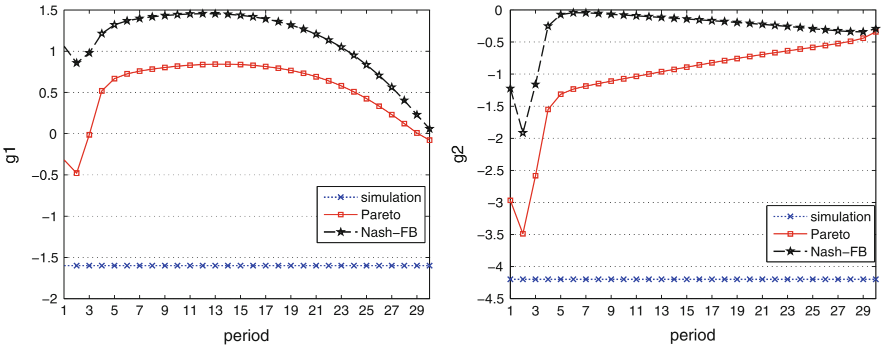

Fig. 11 Country i's fiscal surplus g_{it} (control variable) for $i = 1$ (core; *left*) and $i = 2$ (periphery; *right*)

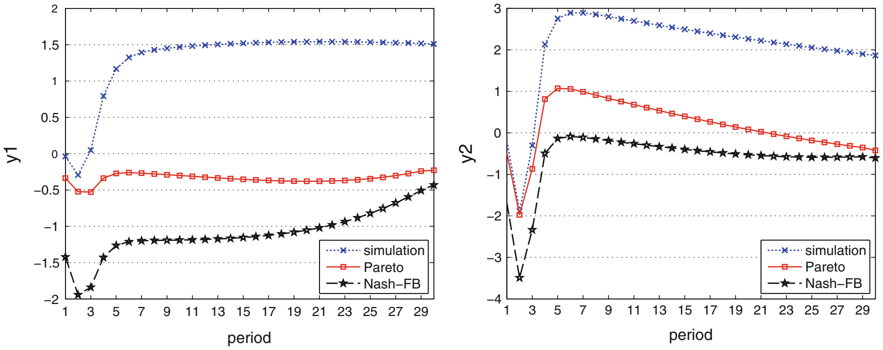

Fig. 12 Country i's output y_{it} for $i = 1$ (core; *left*) and $i = 2$ (periphery; *right*)

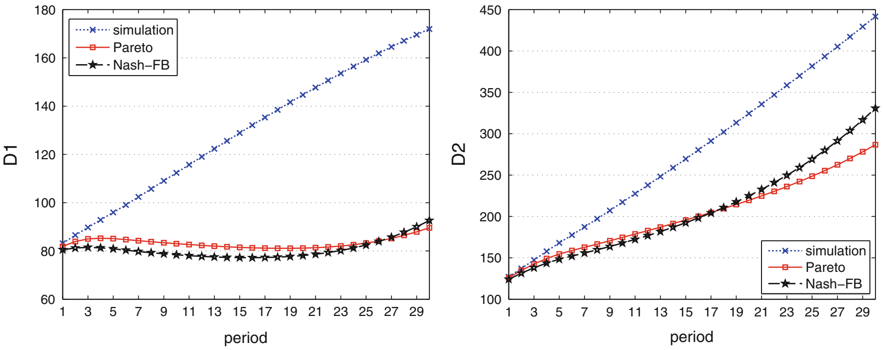

Fig. 13 Country i's debt level D_{it} for $i = 1$ (core; *left*) and $i = 2$ (periphery; *right*)

Finally Table 8 summarizes the objective function values as calculated by Eqs. (18) and (19).

Table 8 Values of the objective functions (loss functions, to be minimized)

Strategy	J_E	J_1(‘core’)	J_2(‘periphery’)	$J_E + J_1 + J_2$
Simulation	30.91	506.93	363.30	901.14
Pareto	102.88	38.10	74.95	215.92
Nash-FB	128.29	74.19	84.13	286.62

5 Concluding Remarks

In this paper we analysed the interactions between fiscal (governments) and monetary (common central bank) policy makers by applying a dynamic game approach to a simple macroeconomic model of a two-country monetary union in a situation of high public debt. Using the OPTGAME3 algorithm, which allows us to find approximate solutions for nonlinear-quadratic dynamic tracking games, we obtained some insights into the design of economic policies when facing negative shocks on the demand side. To this end we introduced three different shocks on the monetary union: a negative symmetric and two negative asymmetric demand-side shocks. The monetary union was assumed to be asymmetric in the sense of consisting of a core with smaller initial public debt and a periphery with higher initial public debt, which was meant to reflect the current situation in the Eurozone.

Our results show that there is a trade-off between sovereign debt sustainability and output (and hence employment) stability when the monetary union is confronted with a negative demand-side shock. Fiscal policy decisions in the periphery tend towards prioritizing the output target at the expense of its budgetary targets while the core, with its higher preference for fiscal prudence, acts in a more restrictive way without much of a negative side effect on its output. The cooperative solution in all cases gives better results for all decision makers, even in the case of an asymmetric shock. An expansionary (low interest rate) monetary policy contributes to accommodating fiscal policy in combating the shocks, especially in the cooperative solution. If the cooperative solution is interpreted as a form of fiscal and monetary union or pact, this can provide an argument for greater coordination among fiscal policies and between monetary and fiscal policies. However, this presumes a binding and permanent agreement among the policy makers, which is notoriously difficult to achieve on a supranational level.

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On the Optimal Trade-Off Between Fire Power and Intelligence in a Lanchester Model

A.J. Novák, G. Feichtinger, and G. Leitmann

Abstract Combat between governmental forces and insurgents is modelled in an asymmetric Lanchester-type setting. Since the authorities often have little and unreliable information about the insurgents, 'shots in the dark' have undesirable side-effects, and the governmental forces have to identify the location and the strength of the insurgents. In a simplified version in which the effort to gather intelligence is the only control variable and its interaction with the insurgents based on information is modelled in a non-linear way, it can be shown that persistent oscillations (stable limit cycles) may be an optimal solution. We also present a more general model in which, additionally, the recruitment of governmental troops as well as the attrition rate of the insurgents caused by the regime's forces, i.e. the 'fist', are considered as control variables.

1 Introduction

In a remarkably early paper Lanchester (1916) describes the course of a battle between two adversaries by two ordinary differential equations. By modelling the attrition of the forces as result of the attacks by their opponent, the author is able to predict the outcome of the combat. There are dozens, maybe even hundreds of papers dealing with variants of Lanchester model and extending it. There is, however, a remarkable fact.

Together with A.K. Erlang's idea to model telephone engineering problems by queueing methods, Lanchester's approach may be seen as one of the forerunners of Operations Research. And OR is—at least to a certain extent—the science of

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optimization. But, strangely enough, virtually all existing Lanchester models neglect optimization aspects.

Lanchester's model has been modified and extended in various directions. Deitchman (1962) and later Schaffer (1968) study **asymmetric** combat between a government and a group of insurgents. Since the regime's forces have only limited situational awareness of their enemies, they 'shoot in the dark'. But such tactics lead to collateral damage. Innocent civilians are killed and infrastructure is destroyed. Moreover, insurgents may escape unharmed and will continue their terroristic actions.

Collateral damage will trigger support for the insurgency and new potential terrorists will join the insurgents, as modelled, e.g., in Caulkins et al. (2009) and Kress and Szechtmann (2009); see also Caulkins et al. (2008). For a more detailed discussion on such 'double-edged sword' effects see also Kaplan et al. (2005), Jacobson and Kaplan (2007), and Washburn and Kress (2009, Sect. 10.4). In a recent paper Kress and MacKay (2014) generalize Deitchman's guerrilla warfare model to account for trade-off between intelligence and firepower.

To avoid responses with these undesirable side-effects improved intelligence is required. As stressed by Kress and Szechtmann (2009): '*Efficient counter-insurgency operations require good intelligence*'. These authors were first to include intelligence in a dynamic combat setting. While their model was descriptive, Kaplan et al. (2010) presented a first attempt to apply optimization methods to intelligence improvement.

The present paper uses dynamic optimization to determine an optimal intelligence gathering rate. Acting as decision maker, the government tries to minimize the damage caused by the insurgents, the cost of gathering intelligence (the 'eye') and militarily attacking the insurgents (the 'fist') as well as the costs for maintaining its forces, and, possibly, recruitment costs.

The continuous-time version in which we will formulate the model below (see Sect. 2) and the use of optimal control theory allows one to derive interesting insights into the course of the various variables of the model especially concerning the occurrence of persistent oscillations. Using Hopf bifurcation theory we will demonstrate (at least numerically) the existence of stable limit cycles for the variables of the model.

The paper is organized as follows. The model is presented in Sect. 2. In Sect. 3 a simplified model is presented and analysed leading to two results. On one hand it can be shown analytically that interior steady states can only exist iff the marginal effect of casualties on the increase of insurgents is smaller than the average effect; on the other hand the existence of periodic solutions is shown numerically by applying Hopf bifurcation theory. Moreover the qualitative structure of periodic solution paths is discussed. Finally, in Sect. 4 some conclusions are drawn and suggestions for possible extensions are given.

2 The Model

Following the model of Kress and Szechtmann (2009) we describe the interaction between governmental forces and insurgents by a variant of Lanchester's model by

$$\dot{G}(t) = -\alpha I(t) + \beta(t) - \delta G(t) \tag{1}$$

$$\begin{aligned}\dot{I}(t) &= -\gamma(t)G(t)\,[\mu(\epsilon(t), I(t)) + (1 - \mu(\epsilon(t), I(t)))I(t)] + \theta(C(t)) \\ &= -\gamma(t)G(t) + C(t) + \theta(C(t))\end{aligned} \tag{2}$$

$$\text{with } C(t) = \gamma(t)G(t)[1 - \mu(\epsilon(t), I(t))](1 - I(t)), \tag{3}$$

with given initial values $G(0)$ and $I(0)$. The variable $t \in [0, \infty)$ denotes time,[1] $G(t)$ and $I(t)$ describe the size of governmental forces, and the fraction of insurgents in the population at time t, respectively. Population size is constant over time and normalized to 1 for simplicity. $C(t)$ describes the collateral damages which may result in the increase of insurgents $\theta(C)$. [2] It is reasonable to assume that $\theta(C)$ is a positive, strictly monotonically increasing continuous function of collateral damages C. α and δ are non-negative constants; α is the attrition rate of the government force due to insurgents' actions, whereas δ is the natural decay rate of soldiers due to natural attrition and defection. The reinforcement rate $\beta(t)$ as well as the attrition rate $\gamma(t)$ may be seen either as time dependent decision variables of the government, or as constant fixed parameters in a simplified version of the model.[3]

Key in the interaction between soldiers and insurgents is the level of intelligence $\mu(t)$,($0 \leq \mu(t) \leq 1$). Without knowledge about the location of insurgents (i.e. $\mu(t) = 0$) the government is 'shooting in the dark', where the probability of reducing insurgents is proportional to the size of the insurgency. On the other hand insurgents can be combated precisely if intelligence is perfect, that is $\mu(t) = 1$. This effect can also be seen by the number of collateral casualties $C(t)$ which is zero under perfect information $\mu(t) = 1$.

Contrary to the model of Feichtinger et al. (2012) it is assumed that $\mu(t)$ cannot be directly chosen by the government, rather it depends on the effort $\epsilon(t) \geq 0$ of intelligence gathering (e.g., number of informants) but also on the level of the insurgency, i.e. $\mu(t) = \mu(\epsilon(t), I(t))$. We assume that $\mu(0, I) = 0$, the partial derivative with respect to ϵ being positive, $\mu_\epsilon > 0$, and that $\mu_{\epsilon\epsilon} < 0$ which means that μ is concave w.r.t. ϵ. Additionally a saturation effect in the sense that $\lim_{\epsilon\to\infty} \mu(\epsilon, I) \leq 1$ should hold. With respect to I the level of intelligence should be hump-shaped. This accounts for the fact that it may be harder to gather information

[1] Note that the time argument t will be dropped to simplify notation whenever appropriate.

[2] This term models the double-edged sword effect as described in Sect. 1.

[3] The attrition rate γ is assumed to be non-negative, but there are no restrictions on β because it can be seen as hiring/firing rate and negative values correspond to dismissing soldiers.

for low fractions of insurgents (as the cooperating population may not be aware of the location of terrorists) as well as for high levels of insurgencies (as nobody would dare to give corresponding information to the government). For the mixed second order partial derivative $\mu_{\epsilon,I}$ we assume that it is positive for small and negative for large values of I. For a possible specification see Eq. (14) below.

In our model the government, as a decision maker, has to decide on the effort $\epsilon(t)$ invested in intelligence as well as on recruitment $\beta(t)$ and attrition $\gamma(t)$ to minimize the damage caused by insurgents together with the costs of counter-measures. To keep the analysis simple we assume an additive structure. Therefore we consider the following discounted stream of instantaneous costs over an infinite time horizon

$$\int_0^\infty e^{-\rho t} \left[D(I(t)) + K_0(G(t)) + K_1(\epsilon(t)) + K_2(\gamma(t)) + K_3(\beta(t))\right] dt \tag{4}$$

as objective value of our intertemporal optimization problem which has to be minimized.

The first term $D(I)$ describes the (monetary) value of damages created by the insurgents which is an important but also problematic issue. Insurgencies always lead to human casualities, and it is almost cynical to measure these losses in monetary terms. Nevertheless, efforts have been made to determine the economic value of a human life (see, e.g., Viscusi and Aldy 2003). Additionally there are also financial damages, such as destroyed infrastructure, which can be evaluated straightforwardly. It is assumed that $D(0) = 0$ and that damages are an increasing and convex function depending on the size of the insurgency.

The second term $K_0(G)$ captures the costs of keeping an army of size G. The remaining terms K_i denote the costs of the control variables ϵ, γ, β, respectively. These cost functions $K_j, j = 0, \cdots, 3$ are assumed to be increasing and convex. The positive time preference rate (discount rate) of the government is denoted as ρ.

Summarizing the control problem and modelling it as an maximization problem leads to

$$\max_{\epsilon,\gamma,\beta} \int_0^\infty e^{-\rho t} \left[-D(I) - K_0(G) - K_1(\epsilon) - K_2(\gamma) - K_3(\beta)\right] dt \tag{5}$$

subject to

$$\dot{G} = -\alpha I + \beta - \delta G \tag{6}$$

$$\dot{I} = -\gamma G + C + \theta(C) \tag{7}$$

$$\text{where } C = \gamma G[1 - \mu(\epsilon, I)](1 - I) \tag{8}$$

with given initial conditions $G(0)$ and $I(0)$ under the constraints

$$0 \le \epsilon, \quad 0 \le \gamma, \quad 0 \le G, \quad 0 \le I \le 1. \tag{9}$$

Note that our model does not address any operational requirements and it stipulates that the decision variables are optimized via a cost function. In a more realistic context of the problem a government might have the objective that the number of attacks by insurgents or the fraction of insurgents within the population must never exceed a certain threshold. Also, it might be a requirement that the size of the governmental troops never falls below a certain threshold as this might be seen as a sign of weakness to third parties trying to exploit the conflict. By reinforcement of the troops, one can keep the size of the governmental troops above a threshold. Additionally, more intense attacks by the government's forces mean a stronger decline of the insurgents leading to less attacks. In order to keep the fraction of the insurgents below a certain threshold, one can attack the insurgents more intensely and put more efforts into intelligence to make counter-insurgency operations more effective. However, a higher recruitment of the government leads to higher capabilities related to counter-insurgency actions. A future task is to investigate how the introduction of such political objectives affects the optimal application of the available control instruments. One could compare the costs of a strict enforcement of such objectives opposed to a policy where the decision maker does not care about the size of the two groups on the short run as long as the long run outcome is favorable. The inclusion of such state constraints will most likely lead to the occurrence of additional steady states, history-dependence and areas in the state space where no solution is feasible.

3 Simplified Model

To show that periodic solutions may be optimal we consider a simplified version of the above model. As the main goal of our analysis is to study the effect of intelligence, the effort ϵ in increasing this level of intelligence is the only control variable. Recruitment β as well as the strength of fighting insurgents γ are assumed to be constants and do not enter the objective functional. Moreover $\delta = 0$ is chosen and the damage and cost functions are linear; i.e. we assume $D(I) = fI, K_0(G) = gG, K_1(\epsilon) = \epsilon$ with positive parameters f, g.

Therefore this simplified version can be written as

$$\max_{\epsilon} \int_0^{\infty} e^{-\rho t} \left[-fI - gG - \epsilon)\right] dt \tag{10}$$

subject to the system dynamics

$$\dot{G} = -\alpha I + \beta \tag{11}$$

$$\dot{I} = -\gamma G + C + \theta(C) \tag{12}$$

$$\text{with } C = \gamma G[1 - \mu(\epsilon, I)](1 - I) \tag{13}$$

where initial values $G(0)$ and $I(0)$ are given.

For simplicity the dependence of the level of intelligence on the effort and on the size of the insurgency is modelled as

$$\mu(\epsilon, I) = A\left(1 - \frac{1}{1+\epsilon}\right) I(1-I), \tag{14}$$

a function which shows all features like saturation effect w.r.t. ϵ and unimodality w.r.t. I required in the general model setup. With this specification the intelligence measures act most efficiently in case that the level of insurgency is about 50 %. Note that by introducing a further parameter this hump of $I(1-I)$ at $I = 0.5$ could be shifted to any arbitrary value within the interval [0, 1].

Additionally the constraints

$$0 \leq \epsilon, \quad 0 < A \leq 4, \quad 0 \leq G, \quad 0 \leq I \leq 1 \tag{15}$$

have to hold.[4]

3.1 Analysis

We analyse the above optimal control problem by applying Pontryagin's maximum principle and derive the canonical system of differential equations in the usual way. Note that in our analysis we could not verify any sufficiency conditions and therefore the canonical system only leads to extremals which are only candidates for optimal solutions. First the current value Hamiltonian is built and given by

$$H = \lambda_0 [-fI - gG - \epsilon] + \lambda_1\{-\alpha I + \beta\} + \lambda_2\{-\gamma G + C(G, I, \epsilon) + \theta(C(G, I, \epsilon))\} \tag{16}$$

where λ_1 and λ_2 are the time dependent adjoint variables and λ_0 a non-negative constant.[5]

As the optimal control ϵ has to maximize the Hamiltonian, the first order condition leads to

$$\frac{\partial H}{\partial \epsilon} = -\lambda_0 + \lambda_2[1 + \theta'(C)]\frac{\partial C}{\partial \epsilon} = 0 \tag{17}$$

[4] $A \leq 4$ assures that $\mu \leq 1$ as the maximum value of $I(1-I)$ is $1/4$ and $\left(1 - \frac{1}{1+\epsilon}\right)$ is bounded by 1.

[5] Actually one only has to distinguish between the normal case $\lambda_0 = 1$ which follows for strictly positive values of λ_0 by rescaling, or the abnormal case $\lambda_0 = 0$.

Remarks

1. It is difficult to exclude the anormal case $\lambda_0 = 0$ in a formal way. Nevertheless it can be easily seen from the Hamiltonian given by (16) that in this case the optimal control ϵ would either be 0 or unbounded in case that $0 < I < 1$ (depending on the sign of the adjoint variable λ_2), and undeterminate otherwise. As these cases are unrealistic we restrict our further analysis to the normal case $\lambda_0 = 1$.
2. To determine the sign of the adjoint variable λ_2 notice that for positive values of λ_2 the Hamiltonian is strictly monotonically decreasing for increasing ϵ, such that the constraint $\epsilon \geq 0$ would become active leading to the boundary solution $\epsilon = 0$ as value of the optimal effort.

   The adjoint variable λ_2 can be interpreted as shadow price of the state variable $I(t)$ and by economic reasoning we assume that this has to be negative for the cases we are considering in the following.
3. As $\lim_{\epsilon \to \infty} H = -\infty$ there exists an interior maximizer ϵ of the Hamiltonian if

$$\left. \frac{\partial H}{\partial \epsilon} \right|_{\epsilon=0} > 0. \tag{18}$$

   This condition holds for sufficiently small values of λ_2, i.e. iff

$$\lambda_2 < \frac{-1}{\gamma GAI(1-I)^2[1+\theta'(C)]}. \tag{19}$$

4. Under the additional assumption that the effect of collateral damages on the increase of the insurgency is marginally increasing i.e. $\theta(C)$ being convex, the Hamiltonian is concave w.r.t. ϵ and the first order condition (17) leads to a unique interior solution iff (19) holds. A sigmoid (convex/concave) relationship may be more plausible to describe the effect of collateral damages, but then the Hamiltonian is not concave any more leading to problems when solving for the optimal control variable.

The adjoint variables satisfy the adjoint differential equations

$$\dot{\lambda}_1 = \rho\lambda_1 - \frac{\partial H}{\partial G} = \rho\lambda_1 + g\lambda_0 + \lambda_2 \left\{ \gamma - [1+\theta'(C)]\frac{\partial C}{\partial G} \right\} \tag{20}$$

$$\dot{\lambda}_2 = \rho\lambda_2 - \frac{\partial H}{\partial I} = \lambda_2 \left\{ \rho - [1+\theta'(C)]\frac{\partial C}{\partial I} \right\} + \alpha\lambda_1 + f\lambda_0 \tag{21}$$

and the transversality conditions

$$\lim_{t\to\infty} e^{-\rho t}\lambda_1 = \lim_{t\to\infty} e^{-\rho t}\lambda_2 = 0. \tag{22}$$

In analysing dynamical systems the existence of steady states and its stability is of major concern. In the following proposition a necessary condition for the existence of interior steady states is derived.

Proposition *Under the assumption that keeping an army of size G leads to costs gG (i.e. $g > 0$) and that the shadow price λ_1 of the army is positive, an interior steady state of the optimization model can exist only if the marginal effect of casualties on the increase of insurgents is smaller than the average effect, i.e. only if*

$$\theta'(C) < \frac{\theta(C)}{C} \tag{23}$$

at the steady state level.[6]

Proof The steady states of the canonical system are solutions of $\dot{G} = \dot{I} = \dot{\lambda}_1 = \dot{\lambda}_2 = 0$, where the optimal control is implicitly given by (17).

At an interior steady state $\dot{I} = 0$ implies

$$\dot{I} = G\left[-\gamma + \frac{C}{G} + \frac{\theta(C)}{G}\right] = 0 \Rightarrow \gamma - \frac{C}{G} = \frac{\theta(C)}{G} \tag{24}$$

As

$$\frac{\partial C}{\partial G} = \frac{C}{G} \tag{25}$$

$\dot{\lambda}_1 = 0$ leads to

$$\lambda_1\rho + g = -\lambda_2\left\{\gamma - [1 + \theta'(C)]\frac{C}{G}\right\} \tag{26}$$

Note that the LHS of (26) is positive, therefore

$$\left\{\gamma - \frac{C}{G} - \theta'(C)\frac{C}{G}\right\} \tag{27}$$

has to be positive. (27) together with (24) leads to

$$\frac{\theta(C)}{G} - \theta'(C)\frac{C}{G} > 0 \quad \text{or equivalently} \quad \frac{\theta(C)}{C} > \theta'(C). \tag{28}$$

Remarks

1. Assuming a power function $\theta(C) = C^\alpha$ condition (23) holds iff $\alpha < 1$, i.e. if the effect of collateral casualties on the inflow of insurgents is marginally decreasing.

[6] In economic theory this means that the elasticity of θ is smaller than 1.

For a linear function $\theta(C) = \theta_0 + \theta_1 C$ condition (23) holds iff the intercept θ_0 is strictly positive.

2. In Feichtinger et al. (2012) $\theta(C) = \theta C^2$ is assumed. In their model interior steady states only exist if the utility of keeping an army of size G is larger than the corresponding costs, i.e. if G enters the objective functional positively. Obviously an army of size G may also lead to some benefits as it can be seen as status symbol or it leads also to some level of deterrence. That these utilities, however, compared to keeping costs result in a net benefit seems to be rather unrealistic.
3. Note that condition (23) is also required for the existence of interior steady states in the more general model (5)–(9).

In the following we specify the function $\theta(C)$ as linear, i.e. as $\theta(C) = \theta_0 + (\theta_1 - 1)C$, with $\theta_0 > 0, \theta_1 > 1$. The first order optimality condition (17) then reduces to

$$\frac{\partial H}{\partial \epsilon} = -1 + \lambda_2 \theta_1 \frac{\partial C}{\partial \epsilon} = -1 - \frac{A\lambda_2 \theta_1 \gamma G I(1-I)^2}{(1+\epsilon)^2} = 0 \tag{29}$$

which leads to the optimal level of effort

$$\epsilon = \sqrt{-A\lambda_2\theta_1\gamma G I(1-I)^2} - 1 \tag{30}$$

As the partial derivative of casualities w.r.t. I is given by

$$\frac{\partial C}{\partial I} = \gamma G \left(-\frac{\partial \mu}{\partial I}(1-I) - 1 + \mu \right) = \gamma G \left(\underbrace{A \frac{\epsilon}{1+\epsilon}(1-I)}_{=\frac{\mu}{I}}(3I-1) - 1 \right) \tag{31}$$

we have to analyse the canonical system

$$\dot{G} = -\alpha I + \beta \tag{32}$$

$$\dot{I} = -\gamma G + \theta_0 + \theta_1 \gamma G(1-\mu)(1-I) \tag{33}$$

$$\dot{\lambda}_1 = \rho\lambda_1 + g + \lambda_2\gamma\left[1 - \theta_1(1-\mu)(1-I)\right] \tag{34}$$

$$\dot{\lambda}_2 = \alpha\lambda_1 + f + \lambda_2\left[\rho - \theta_1\gamma G\left(\mu\left(3 - \frac{1}{I}\right) - 1\right)\right] \tag{35}$$

$$\mu = AI(1-I) - \sqrt{-\frac{AI}{\lambda_2\theta_1\gamma G}} \tag{36}$$

Solutions of the differential equation system (32)–(35) with the level of intelligence μ given by (36) together with initial values $G(0)$ and $I(0)$ where also the transversality conditions hold are extremals and strictly speaking only candidates for an optimal solution as we could not verify any sufficiency conditions so far.

In the following we show that periodic paths may exist as solutions of the dynamical system (32)–(36) by applying the Hopf bifurcation theorem (see, e.g., Guckenheimer and Holmes 1983, for details). This theorem considers the stability properties of a family of smooth, nonlinear dynamic systems for variation of a bifurcation parameter. More precisely, this theorem states that periodic solutions exist if (i) two purely imaginary eigenvalues of the Jacobian matrix exist for a critical value of the bifurcation parameter, such that (ii) the imaginary axis is crossed at nonzero velocity. Note that stable or unstable periodic solutions may occur and to determine its stability further computations, either analytically or numerically are necessary.

Unfortunately, for the canonical system (32)–(35) it is not even possible to find steady state solutions explicitly and therefore we have to base our results on simulations given in the following numerical example.

3.2 Numerical Example

We choose the discount rate ρ as bifurcation parameter. Values for the other parameters are specified as follows:[7]

$$\alpha = 0.77, \beta = 0.58, \gamma = 0.26, f = 2.00, g = 0.68, \theta_0 = 2.83, \theta_1 = 1.60, A = 3.24 \tag{37}$$

The Jacobian possesses a pair of purely imaginary eigenvalues for the critical discount rate $\rho_{crit} = 0.14305948$ at the steady state

$$G^\infty = 13.5712, I^\infty = 0.7532, \lambda_1^\infty = 53.9293, \lambda_2^\infty = -40.2585. \tag{38}$$

The optimal effort to be invested into intelligence at the steady state level amounts to $\epsilon^\infty = 4.8115$ leading to the level of intelligence $\mu^\infty = 0.4986$.

According to the computer code BIFDD (see Hassard et al. 1981) stable limit cycles occur for discount rates less than ρ_{crit}.

To present a cycle and discuss its properties we used the boundary value problem solver COLSYS to find a periodic solution of the canonical system for $\rho = 0.127$ by applying a collocation method.

For $\rho = 0.127$ the steady state is shifted to $G^\infty = 13.655, I^\infty = 0.7532, \lambda_1 = 46.6175, \lambda_2 = -31.8485$, with an optimal effort of $\epsilon^\infty = 4.1850$ leading to a level of intelligence $\mu^\infty = 0.4861$. This steady state is marked as a cross in Figs. 2–5. Additionally solutions spiraling towards the persistent oscillation are depicted in these figures.

[7]Note that these values do not represent any real situation and are purely fictitious to illustrate the occurrence of periodic solutions in principle.

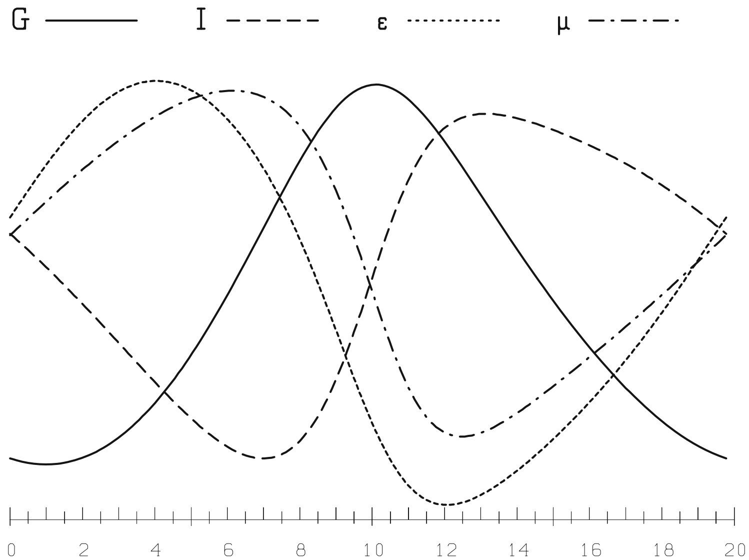

Fig. 1 Time path of control variable ϵ and the states together with the level of intelligence μ along one period of the cycle

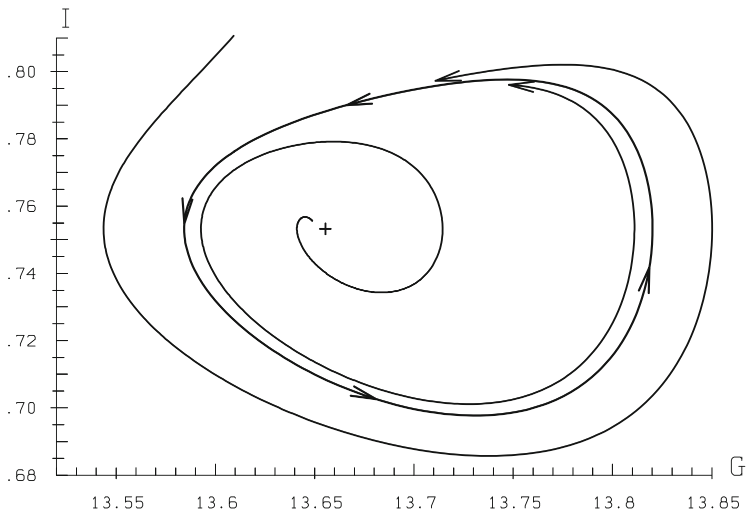

Fig. 2 Persistent oscillation in the G/I-diagram

In Fig. 1 the time path of the control ϵ as well as the resulting level of intelligence μ together with the two state variables G and I along one period of the cycle can be seen. The length of the period is 19.7762 time units.

As can be seen from Figs. 2, 3, 4, and 5 we can distinguish four different regimes along one cyclical solution. In the first phase which could be called *'increasing*

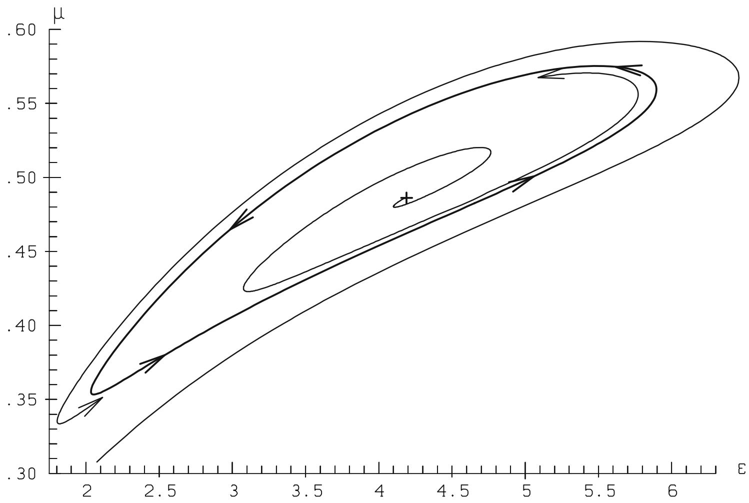

Fig. 3 Persistent oscillation in the ϵ/μ-diagram

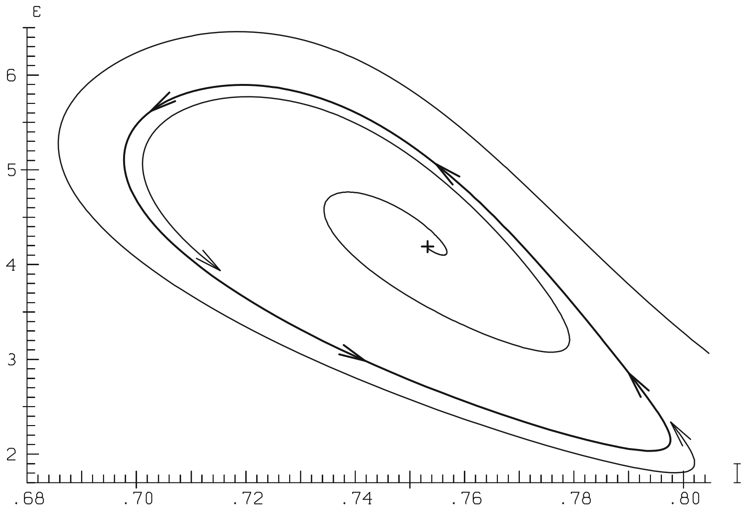

Fig. 4 Persistent oscillation in the I/ϵ-diagram

terror phase' both the strength of governmental troops $G(t)$ as well as the size of the insurgency $I(t)$ increase.

In the following phase, called *'dominant terror phase'* the insurgency increases, but the government reduces its counter-terror measures as this will also reduce the negative impact due to collateral damages.

As the negative effect of collateral damages is reduced also the number of insurgents decreases in the *'recovery phase'* where $G(t)$ and $I(t)$ both shrink.

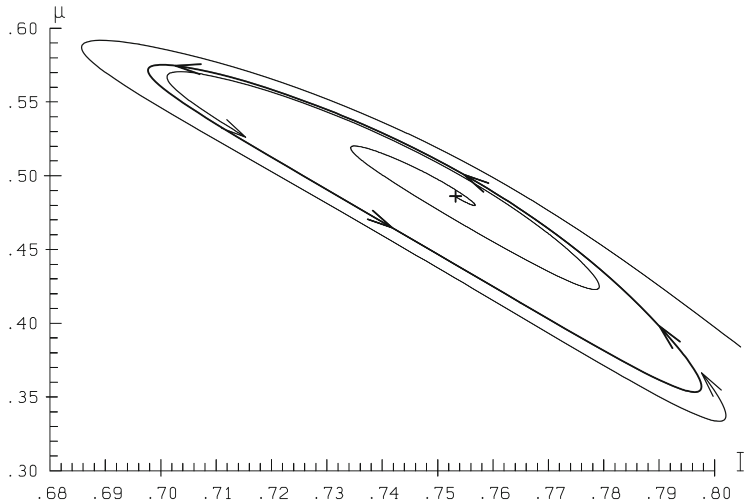

Fig. 5 Persistent oscillation in the I/μ-diagram

In the last phase the government can be seen as being dominant as $G(t)$ starts to increase again due to the low size of the insurgency, which is still decreasing. Due to an increasing effect of collateral damages the insurgency will start to become larger at the end and the cycle is closed.

Obviously the size of governmental troops $G(t)$ is the leading variable whereas $I(t)$ lags behind. This is caused by the effect of collateral damages and the induced inflow of insurgents.

Having a closer look at the effect of the effort on gathering information on the level of intelligence it turns out that along a fraction of 57.5 % of the cycle an increasing effort leads to a higher intelligence level, and in 30.0 % both the effort as well as intelligence are decreasing. Nevertheless there exist rather short phases along the cycle with opposing trends. The level of intelligence is increasing despite a falling effort along 10.5 % of the periodic solution due to the decreasing size of the insurgency, since then the effort becomes more effective. On the contrary along 2 % even an increasing effort in gathering information does not lead to a higher level of intelligence since, due to an increasing insurgency, it becomes harder to obtain reliable information.

4 Conclusions and Extensions

Surprisingly, Lanchester's pivotal attrition paradigm has never enriched the optimization scenario.[8] In the present paper we use optimal control theory to study how a government should apply intelligence efficiently to fight an insurgency. Due to the formal structure of the model, i.e. the multiplicative interaction of the two states with the control variable, complex solutions might be expected. As a first result in that direction a Hopf bifurcation analysis is carried out establishing the possibility of persistent stable oscillations.[9]

Since our analysis can be seen only as a first step, several substantial extensions are possible. The first is to include the 'fist', i.e. the attrition rate of the insurgents caused by the regime's forces, in addition to the 'eye', i.e. the intelligence level of the government. In particular, we intend to study the optimal mix of these two policy variables. A main question which arises in that context is whether these instruments might behave complementarily or substitutively.[10]

Among the further extensions we mention a differential game approach with the government and the insurgents as the two players. Moreover, it would be interesting to study the case where the regime has to fight against two (or more) opponents (insurgents). Additionally following the ideas by Udwadia et al. (2006) the combination of both direct military intervention to reduce the terrorist population and non-violent persuasive intervention to influence susceptibles to become pacifists could be analyzed under aspects of optimization.

Finally we should stress the fact that the model we discussed is not validated by empirical data. Even if this would be the case, the proposed model seems much to simple to deliver policy recommendations for concrete insurgencies.

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[8]There are, however, two remarkable exceptions. Kaplan et al. (2010) formulate an optimal force allocation problem for the government based on Lanchester's dynamics and develop a knapsack approximation and also model and analyse a sequential force allocation game. Feichtinger et al. (2012) study multiple long-run steady states and complex behaviour and additionally propose a differential game between terrorists and government.

[9]Persistent oscillations, more precisely, stable limit cycles, occur in quite few optimal control models with more that one state, see Grass et al. (2008), e.g. particularly in Sect. 6.2.

[10]E.g. the interaction between marketing price and advertising is a classical example of a synergism of two instruments influencing the stock of customers in the same direction. A further example is illicit drug control as analysed by Behrens et al. (2000), where prevention and treatment are applied but not at the same time.

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Drivers of the Change in Social Welfare in European Countries

Mikuláš Luptáčik, Eduard Nežinský, and Martin Lábaj

Abstract Recent discussions about the definition of economic growth in terms of welfare beyond GDP suggest that it is of urgent need to develop new approaches for measuring the economic performance of firms and national economies. For this purpose the DEA methodology is used by simultaneously taking into account economic as well as social and environmental indicators. SBM DEA models with restricted weights are used in order to analyse the impact of different strategies and goals of economic policy for measuring of social welfare. Using Malmquist productivity index the drivers of social welfare change over the period of 2003–2012 for 25 European countries are identified. This approach allows us to decompose the overall change to change in economic, environmental and social component. It can be shown that the main driver of the increasing social welfare in Europe was the environment-saving change of technology.

1 Introduction

The necessity of having economic performance measured in terms of welfare beyond GDP calls for new approaches capable of simultaneously taking into account economic as well as social and environmental indicators. Data envelopment analysis (DEA) proved to be a proper tool for measuring the economic performance and for assessing efficiency of firms and national economies in the situations of multiple

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inputs and multiple outputs where the indicators are expressed in different units and some of the outputs are undesirable (like pollutants). It was concept of eco-efficiency that first encompassed economic performance and its environmental impact. Introduced by Schaltegger and Sturm (1989), eco-efficiency was later identified by the OECD as one of the major strategic elements in its work on sustainability (OECD 1998).

For the purpose of measuring the eco-efficiency, the DEA approach can be extended by taking into account emissions as an undesirable output. First DEA models with undesirable outputs were developed by Färe et al. (1989, 1996) followed by Tyteca (1996, 1977) and Dyckhoff and Allen (2001) and others. Korhonen and Luptacik (2004) proposed different variants of DEA models for the evaluation of eco-efficiency in a single period. They show that the set of (strongly) efficient decision making units (DMUs) is the same for all the models. However, the different variants provide deeper insights into the underlying sources of eco-efficiency differential across DMUs and therefore show different ways of increasing eco-efficiency. Rao and Coelli (1999) analyze the social welfare encompassing growth in GDP as well as the changes in the distribution of income. They use a non-parametric method to measure productivity growth in different countries and generalize the approach to include both inequality and level of income as joint determinants of total welfare resulting from economic activity. Economic, environmental and social indicators were introduced into the DEA models by our contribution in Lábaj et al. (2014). Embracing three dimensions of the development—economic, ecological and social, we proposed two types of models—composite and complex—for bringing the three groups of indicators together.

The static approach to eco-efficiency was developed to an intertemporal setting by Caves et al. (1982) as s theoretical index called Malmquist productivity index. It was developed and popularized later by Fare et al. (1994a, b). Malmquist productivity index measures total factor productivity change and allows us to decompose it into the change in efficiency on the one hand side and to change in the frontier technology on the other. Mahlberg et al. (2011) used intertemporal comparison and Malmquist index decomposition to analyze eco-efficiency change over the period of 1995–2004 in EU15 countries indicating pollution-reducing type of technology.

In this paper, we assess the social welfare and drivers of social welfare over time in European countries. We contribute to current research by taking into account simultaneously economic, environmental and social dimensions of overall welfare. We refine our analysis given in Lábaj et al. (2014) by employing SBM rather than radial model to determine the composite score. Shadow prices coming out as optimal solutions from DEA model can be viewed as weights for particular economic social and environmental indicators reflecting the human judgments and priorities of economic policy that appear to be crucial for the social welfare. In other words, they provide the implicit weights for social welfare function (specified for every country) and are closely related to implicit weights of social welfare components discussed in van den Bergh (2015). To illustrate how *ex ante* determined weights affect the overall score, three variants of AR models are presented.

For intertemporal analysis, we use Malmquist productivity index to identify the drivers of social welfare change over the period of 2003–2012. Compared to Mahlberg et al. (2011), we extend the study by one dimension adding social indicators. We also analyze a broader set of 25 European countries. Use of SBM models prevents possible slack-related defects of radial models. This approach allows us to decompose the overall change to change in economic (technical) change, environmental change and social change.

The paper is organized as follows. First, we focus on unrestricted data envelopment models that measure economic, environmental and social efficiency. Then we combine the efficiency scores obtained from these models into overall welfare model. Second, we impose restrictions on weights in DEA models and analyse so called assurance region (AR) models. In this way we incorporate the policy preferences over different dimensions of social welfare. In the last section we analyse the drivers of social welfare change over time and conclude.

2 Assessing the Social Welfare and Drivers of Social Welfare Change Over Time

First, we analyse the social welfare taking into account simultaneously economic, environmental and social performance of European countries (Models I–IV). Second, we restrict the weights in DEA models (AR models). These restrictions represent the policy preferences. In the last section, we analyse the drivers of social welfare change over time by means of Malmquist productivity index based on Models I, II, III, and V. An overview of models employed is provided in Table 1. Due to data availability constraint, number of the European countries analysed amounts to 25. The static analysis is carried out for 2012, the intertemporal one covers the range of 2003–2012.

Much like in Lábaj et al. (2014), the economic (technical) performance is assessed by means of GDP (Y) measured in mil PPS—the output of production activities, and inputs—capital stock (K, mil PPS) and labour (L, thousands of persons employed). For ecological dimension we choose greenhouse gas emissions (GHG, thousand tonnes), energy consumption (EC, TJ), and material consumption (MC, thousand tonnes). Gini coefficient as the income inequality indicator, household disposable income (EUR), and active population rate act to evaluate the social dimension. We employ the Gini coefficient in the transformed form as subtracted from the unit so that the higher value corresponds to higher income equality. In some models, input equal unit for each DMU is used. For assessing particular dimensions as well as in Model V, variable returns to scale assumption was implemented due to considerable variance in size among the analysed economies. Calculations consisting in solving linear optimization problems were carried out by Matlab and Saitech software packages.

In Sects. 2.1–2.3 we describe DEA models and present empirical results.

Table 1 Overview of social welfare models

Model	Description	Type	Inputs			Outputs		
I	Technical	SBM-V	Capital	Labor		GDP		
II	Ecological	SBM-V	Material	Energy	Emissions	GDP		
III	Social	SBM-V	1			Income	Employment	1—Gini
IV	Welfare (composed)	SBM-C	1			Score I	Score II	Score III
ARx	Welfare (restricted weights)	SBM-C	1			Score I	Score II	Score III
V	Welfare	SBM-V	Capital	Labor		GDP		
			Material	Energy	Emissions	Income	Employment	1—Gini

Source: Authors' computations

2.1 Social Welfare with Unrestricted Weights

In DEA modelling, a considerable amount of data is processed. Subjects under evaluation called DMUs (Decision Making Units) are considered as transforming m inputs into s outputs. Denoting n number of DMUs, the data are arranged in the input matrix $\mathbf{X}$ and output matrix $\mathbf{Y}$ with elements x_{ij}, and y_{rj} ($j = 1, 2, \ldots, n$) respectively. In more technical detail, the approach to modelling social welfare is described in Lábaj et al. (2014). In contrast to radial measures of efficiency as proposed in Korhonen and Luptáčik (2004), the slack-based measure of efficiency (Tone 2001) is employed in this analysis to capture all sources of inefficiency that could be omitted in radial models.

SBM measure of efficiency is defined by the function $\rho = \frac{1-\frac{1}{m}\sum_{i=1}^{m} s_i^-/x_{i0}}{1+\frac{1}{s}\sum_{r=1}^{s} s_r^+/y_{r0}}$ meeting the requirements of unit invariance and monotonicity in slack variables $\mathbf{s}^-$ and $\mathbf{s}^+$. Moreover, it can be shown that $0 < \rho \leq 1$ (Cooper et al. 2007, p. 100). Evaluation of efficiency thus takes the form of a fractional program:

$\min\limits_{\boldsymbol{\lambda},\mathbf{s}^+,\mathbf{s}^-}$	$\rho = \frac{1-\frac{1}{m}\sum_{i=1}^{m} s_i^-/x_{i0}}{1+\frac{1}{s}\sum_{r=1}^{s} s_r^+/y_{r0}}$	(1)
s.t.	$\mathbf{x_0} = \mathbf{X}\boldsymbol{\lambda} + \mathbf{s}^-$	(2)
	$\mathbf{y_0} = \mathbf{Y}\boldsymbol{\lambda} - \mathbf{s}^+$	
	$\boldsymbol{\lambda} \geq 0, \mathbf{s}^- \geq 0, \mathbf{s}^+ \geq 0.$	

Using substitution $t = \frac{1}{1+\frac{1}{s}\sum_{r=1}^{s} s_r^+/y_{r0}}$ the fractional program can be linearized into conveniently computable form (Charnes and Cooper 1962). Variable returns to scale assumption are imposed by the additional constraint $\sum_{j=1}^{n} \lambda_j = 1$ on variables $\boldsymbol{\lambda}$ which are used to define production possibility set in the constraints (2). It is also possible to give the model output orientation by omitting output slacks in the objective function. SBM measure with variable returns to scale is denoted SBM-V type of model.

Primal and dual pair of linear programs where (3)–(4) represent *envelope* program is given below. Formulations (5)–(6) define *multiplier* program yielding shadow prices for individual outputs.

SBM: *primal and dual program*					
min	$\tau = t - \frac{1}{m}\sum_{i=1}^{m} S_i^-/x_{i0}$	(3)	max	ξ	(5)
s.t.	$t\mathbf{x_0} = \mathbf{X}\boldsymbol{\Lambda} + \mathbf{S}^-$	(4)	s.t.	$\xi + \mathbf{v}^{\mathrm{T}}\mathbf{x_0} - \mathbf{u}^{\mathrm{T}}\mathbf{y_0} = 1$	(6)
	$t\mathbf{y_0} = \mathbf{Y}\boldsymbol{\Lambda} - \mathbf{S}^+$			$-\mathbf{v}^{\mathrm{T}}\mathbf{X} + \mathbf{u}^{\mathrm{T}}\mathbf{Y} \leq \mathbf{0}^{\mathrm{T}}$	
	$\boldsymbol{\Lambda} \geq \mathbf{0}, \mathbf{S}^- \geq \mathbf{0}, \mathbf{S}^+ \geq \mathbf{0}, t > \mathbf{0}$			$\mathbf{v} \geq \frac{1}{m}[1/\mathbf{x_0}]\mathbf{u} \geq \frac{\xi}{s}[1/\mathbf{y_0}]$	

Unit value of the objective function (3) or (5) implies zero slack variables in the optimal solution. DMU under consideration (indexed by 0) is efficient. Otherwise, non-zero slacks would indicate benchmarks for potential improvement and inefficiency.

To assess each of the three dimensions of social welfare, we formulate three individual models I, II, and III. Model I measures the technical efficiency of European countries. Capital and labour are used as inputs to production process and gross domestic product measures the output. We measure the ecological efficiency by Model II describing transformation of material and energy into output (GDP) and pollutants. Following Korhonen and Luptacik (2004), we treat emissions (undesirable output) as an input variable in the model. Social efficiency is measured in Model III. It is determined by three selected indicators of social welfare—income, employment and Gini coefficient. We have transformed the Gini coefficient that measures the income inequality to 1—Gini coefficient that measures the income inequality. In this way, higher coefficient represents a better outcome (more equal income distribution). Model IV measures the overall welfare combining the efficiency scores from models I, II, and III. The results are presented in Table 2.

In columns labelled I, II and III, scores from models I, II and III respectively are displayed. Within the set of analysed countries, eight technically efficient countries with the unit score determine efficiency frontier—Belgium, Germany, Ireland, France, Cyprus, Latvia, Poland, and Slovakia. Lithuania is considerably lagging behind. The detailed results of optimization would be involved to determine specific sources of inefficiency.

Environmentally oriented assessment is given by model II yielding other set of efficient countries—German, Ireland, Spain, France, Italy, Cyprus, Latvia, Netherlands, Sweden, and United Kingdom. It is thus obvious that viewing GDP as a product of differently defined transformation processes results in different values of efficiency providing additional dimension of assessment.

The third model III only focuses on social indicators without regarding any inputs accounted for in previous two models. Germany, Netherlands, and Sweden appear efficient with this respect apparently in line with common view of having successfully implemented welfare state policies. Completing list of efficient countries from model III, Slovenia gains advantage from the lowest level of inequality contributing massively to its model III unit score. Greece and Lithuania appear to be performing countries with this regard achieving scores of 0.708 and 0.732 respectively.

The scores from the three models are used to obtain overall social welfare measure in model IV. In line with conceptual DEA model, DMUs are allowed to ascribe weights in the most favourable manner so that to achieve the maximal possible value of the resulting overall score IV. Obviously, this approach drives countries efficient in at least one dimension to give it the maximum weight neglecting their poor performance in the other dimensions. Thus all the efficient DMUs listed above form a set of efficient countries in model IV. For other countries, the best value from I, II or III scores is picked up to represent the social welfare efficiency. This approach does though not reflect the fact that in evaluating welfare

Table 2 Efficiency scores from models I–IV

DMU	Model			
	I	II	III	IV
Belgium	1	0.780	0.930	0.893
Bulgaria	0.806	0.392	0.559	0.537
Czech Republic	0.826	0.531	0.810	0.693
Denmark	0.856	1	0.931	0.925
Germany	1	1	1	1
Estonia	0.838	0.390	0.683	0.575
Ireland	1	1	0.830	0.936
Greece	0.702	0.677	0.708	0.696
Spain	0.918	1	0.786	0.893
France	1	1	0.921	0.972
Croatia	0.937	0.811	0.643	0.778
Italy	0.992	1	0.829	0.933
Cyprus	1	1	0.847	0.943
Latvia	1	1	0.541	0.780
Lithuania	0.263	0.221	0.732	0.309
Hungary	0.928	0.765	0.701	0.787
Netherlands	0.959	1	1	0.986
Austria	0.833	0.873	0.986	0.893
Poland	1	0.507	0.716	0.687
Portugal	0.764	0.804	0.765	0.777
Slovenia	0.932	0.758	1	0.884
Slovakia	1	0.688	0.776	0.801
Finland	0.812	0.551	0.967	0.735
Sweden	0.761	1	1	0.905
United Kingdom	0.976	1	0.910	0.960

Source: Authors' computations

no dimension should be allowed to be completely ignored which leads to imposing weight restrictions as a possible solution to the problem.

2.2 *Social Welfare with Restricted Weights*

An overall welfare measured in Model IV neglects the problems discussed in van den Bergh (2015). In unrestricted DEA models, countries can achieve high efficiency score simply by focusing on the strongest aspect of overall welfare and by neglecting all other dimensions. These solutions could not be optimal from the social welfare function perspective. We can get over these obstacles by imposing restrictions on weights in DEA models. In this way we can compute the overall welfare score with assurance region model (see Cooper et al. 2007 for detailed description of the AR model).

Taking a look back at the multiplier side of the SBM model and combining (5) and (6) yields max $\mathbf{u}^{\mathrm{T}}\mathbf{y}_0 - \mathbf{v}^{\mathrm{T}}\mathbf{x}_0$ for the objective function. This can be interpreted as the virtual profit expressed by means of shadow prices (multipliers) **u** and **v**. We Put scores from I to III together treating them as outputs y_1, y_2, y_3 (with input equal one) in the composite Model IV. Thus $\sum_{r=1}^{3} u_r y_{r0}$ stands for joint contribution of the particular scores to the total social welfare score. Multipliers u_1, u_2, u_3 representing weights assigned to respective dimensions of welfare are determined in the process of optimization. From the perspective of policy making, it could be desirable to impose *ex ante* weights on particular dimensions reflecting preferences of decision makers.

We present three variants of models with restricted multipliers. In the simplest version of assurance region model, Model AR1, weight of economic performance should not be lower than weight of ecological efficiency. There are no restrictions on social efficiency in this model. So, the policy preferences are indifferent with respect to social welfare and they value the economic performance more than environmental performance. Multipliers in Model AR2, are optimized so as to follow the requirements $u_1 \geq u_3 \geq u_3$ for implicit weights in the social welfare function with the focus on gross domestic product (based on van den Bergh 2015). Weights for economic performance should not be lower than weights for social performance and these should not be lower than weights for ecological performance. Model AR3 introduces stronger constraints to the hierarchy of multipliers as in AR2 demanding $u_1/u_3 \geq 2$ and $u_3/u_2 \geq 2$. Using SBM measure for bringing scores I–III together prevents problems with weak efficiency from which radial models could suffer.

Overview of AR models is in Table 3.

Complete results obtained from Models AR1 to AR3 are shown in Annex in Table 6. In Table 4 we highlight results for selected European countries and compare them with unrestricted Model IV.

There is a single efficient DMU—Germany achieving score of 1 under each of the imposed restrictions. It could prove the strength of the economy outperforming others even in case of shifting focus among dimensions as it performs at the score of 1 in models I–III. For each DMU it can be noticed that best-assessed dimension is assigned the highest importance (weight). Thus Slovakia or France weight technical

Table 3 Overview of assurance region models

Model	Description	Restrictions
AR1	Low preferences for environment performance and indifference with respect to social aspects	Weight I ≥ Weight II
AR2	Focus on economic performance	Weight I ≥ Weight III ≥ Weight II
AR3	Stronger AR2 preferences	2 ≤ Weight I/Weight III 2 ≤ Weight III/Weight II

Source: Authors' computations

Table 4 Efficiency score and weights for model IV and models AR1–AR3 for chosen countries

	Model IV				Model AR1				Model AR2				Model AR3			
	Score	U(1)	U(2)	U(3)	Score	U(1)	U(2)	U(3)	Score	U(1)	U(2)	U(3)	Score	U(1)	U(2)	U(3)
Germany	1	5.40	2.77	10.38	1	5.30	2.65	10.39	1	9.43	3.88	7.71	1	13.72	1.58	5.54
Estonia	0.57	0.40	0.85	0.49	0.55	0.85	0.85	0.49	0.52	0.85	0.85	0.85	0.38	3.42	0.85	1.71
France	0.97	16.75	6.29	0.36	0.97	10.83	7.00	0.36	0.97	20.68	0.33	0.36	0.95	25.07	0.33	0.67
Netherlands	0.99	0.35	4.60	12.79	0.99	0.35	0.34	20.95	0.99	0.35	0.34	0.34	0.95	1.33	0.33	0.67
Austria	0.89	0.40	0.38	0.34	0.89	0.40	0.38	0.34	0.89	0.40	0.38	0.38	0.76	1.53	0.38	0.76
Slovenia	0.88	0.36	0.44	16.41	0.88	0.44	0.44	17.92	0.88	0.44	0.44	0.44	0.82	1.76	0.44	0.88
Slovakia	0.80	15.37	0.48	0.43	0.80	18.73	0.48	0.43	0.79	21.42	0.48	0.48	0.73	24.26	0.48	0.97
Finland	0.74	0.41	0.60	0.34	0.72	0.60	0.60	0.34	0.71	0.60	0.60	0.60	0.57	2.42	0.60	1.21
United Kingdom	0.96	0.34	17.45	0.37	0.96	0.34	0.34	0.37	0.96	0.37	0.37	0.37	0.92	1.33	0.33	0.67

Source: Authors' computations

performance most ($u_1 = 15.37$ or 16.75 respectively) as contrasted to Netherlands or Slovenia stressing social dimension (u_3 equal 12.79 or 16.41). Restrictions in AR2 makes difference for countries with relatively weak technical performance, thus scores of Estonia, Finland, the UK, Bulgaria or Czech Republic deteriorate. Additional constraint imposed in AR2 makes it for some DMUs impossible to retain previous levels of efficiency. One can observe, for example, the efficiency of the UK deteriorate all the way moving from no restrictions through AR3 since the initial weights from Model IV went in just the opposite "importance" as required by AR3. Stronger requirements in AR4 worsen the UK's overall score even further to 0.75. Comparing Austria and Slovenia in no-restriction model, one can see a balanced distribution of weight for Austria outperforming (0.89–0.88 in total score) Slovenia which puts much more weight on social performance ($u_3 = 16.41$) that is not much affected by AR1 restriction. In AR3, the balanced distribution of Austria's optimal Model IV multipliers is distorted while Slovenia is still allowed to weight its III score twice more which results in even better AR3 performance for Slovenia. It is obvious that each additional or more stringent constraint forces the resulting overall score down revealing additional potential improvements corresponding to altered preferences.

2.3 Drivers of Social Welfare Change

To account for the social welfare efficiency change over time the static approach elaborated in previous section must be augmented. For intertemporal setting, we utilize productivity index which would present total factor productivity (TFP) change. We are also interested in decomposing the improvement in performance into individual country's effort and the general technology progress. DEA provides a method for measuring productivity and efficiency change over time. Intertemporal change of productivity is assessed by the Malmquist productivity index (MI) given by

$$MI = C \times F = \frac{d_o^2(\mathbf{x}_0, \mathbf{y}_0)^2}{d_o^1(\mathbf{x}_0, \mathbf{y}_0)^1} \times \left[\frac{d_o^1(\mathbf{x}_0, \mathbf{y}_0)^1}{d_o^2(\mathbf{x}_0, \mathbf{y}_0)^1} \cdot \frac{d_o^1(\mathbf{x}_0, \mathbf{y}_0)^2}{d_o^2(\mathbf{x}_0, \mathbf{y}_0)^2} \right]^{1/2} \tag{7}$$

where d_o denotes efficiency score related to activity ($\mathbf{x}_0$, $\mathbf{y}_0$) of the DMU under consideration in specific time period and within the production set of specific period of time. Malmquist index is expressed in (7) as a product of two subindices C and F. The former stands for efficiency change over time, the latter representing improvement in productivity ascribed to technological progress defined by DMUs performing at *best practice* level. Thus, in terms of DEA analysis, C and F represent two movements—one towards the efficiency frontier (catch-up effect) and the other movement of the frontier itself (frontier-shift). Values of Malmquist index as well as the component terms allow to infer on the type of technology change.

Efficiency change over time was analysed by Mahlberg et al. (2011) who conducted an analysis of eco-efficiency of EU-15 countries to identify drivers of technological change. Borrowing their idea, we used three social welfare dimensions described individually by models I, II and III to compute TFP change as well as components C and F for each country. Computation of (7) involves solving four auxiliary optimizations of the SBM-V type for each DMU. These were conducted under the "exclusive" scheme proposed by Cooper et al. (2007, p. 335). Finally, the MI and composite indices based on the compound Model V embracing all the variables (see Table 1) were determined. The results are displayed in Table 5.

A comprehensive picture of productivity change involving all inputs and outputs defining social welfare is provided by Malmquist index from model V (the rightmost column of the Table 5). On average, social welfare increased by 8.7 %. Slovakia, Finland, and Germany were the most successful in increasing the welfare. Taking a look at the MI outcomes of models I, II and III describing constituent parts of the welfare, it can be inferred that the most significant contribution to the total welfare was on the part of ecology—31.3 % on average while pure technical change or improvement in social sphere only went up by 1.6 % and 11.4 % respectively. Comparing countries individually, one can clearly see that the social welfare improvement was qualitatively different in Slovakia (driven mostly by ecology—56.5 % and about 30 % from both technical and social model) and Austria (3.7 % technical, 29.2 % ecology and 8.8 % social conditions).

Decomposition of Malmquist index into catch-up and frontier-shift terms provides information on whether the improvement should be ascribed to individual country's better performance or it was driven by the overall technology shift. Columns labelled C and F corresponding to models I, II, III and V contain values of MI subindices. Taking a look on average values, one can state that the social welfare improvement (model V) was nearly equally due to technological shift (4.8 %) and catching up (4.5 %) by individual countries. The results obtained from component models for technical, ecological and social dimensions though suggest that the increase in productivity is on average mainly technologically driven (F exceeds C).

Concentrating thus on the frontier-shift effect one can see that improvement in process involving transformation of the inputs into outputs relevant to social welfare presents average of 4.8 % resulting from model V. All the countries but Lithuania and Estonia experienced growth of social welfare productivity indicated by MI, Slovakia, Austria, Denmark, and Sweden were also the most pushed-forward by technology which contributed to their overall productivity increase.

A disaggregate look at the constituent dimensions reveals sources of improvement. A frontier-shift index from Model I reflects growth of productivity due to more efficient use of technical inputs while values from the column II describe productivity growth while saving environmentally-related inputs or reducing pollution. Thus the relative size of the effects can distinguish between input- or environment-saving technological progress types. It is obvious that the latter took place without exception since F-values of II are greater than those of I. The technological change was clearly environmentally biased. The same conclusion can be derived from the

Table 5 Malmquist index and decomposition terms

	I			II			III			V		
	C	F	MI	C	F	MI	C	F	MI	C	F	MI
Belgium	1.017	1.004	1.021	1.005	1.269	1.275	0.990	1.132	1.121	0.992	1.146	1.137
Bulgaria	0.774	1.049	0.812	1.113	1.200	1.336	1.111	1.076	1.196	0.602	1.097	0.661
Czech Republic	1.092	1.090	1.190	1.160	1.204	1.397	0.967	0.961	0.929	0.998	1.008	1.006
Denmark	1.036	0.972	1.007	1.223	1.158	1.416	0.923	1.062	0.980	1.016	1.050	1.066
Germany	1.013	1.077	1.091	1.013	1.210	1.226	1.060	1.103	1.169	1.067	1.174	1.253
Estonia	0.940	0.868	0.816	0.923	1.254	1.158	1.076	1.173	1.262	1.531	0.626	0.958
Ireland	0.971	1.016	0.987	1.014	1.322	1.340	0.910	1.126	1.024	1.001	1.037	1.038
Greece	0.930	0.971	0.903	0.913	1.196	1.092	0.853	1.131	0.965	0.893	1.107	0.989
Spain	0.986	0.955	0.941	1.250	1.182	1.477	0.895	1.123	1.005	0.999	1.178	1.176
France	1.121	0.970	1.088	1.033	1.226	1.266	0.947	1.134	1.074	0.988	1.060	1.047
Croatia	0.921	1.079	0.994	1.023	1.238	1.267	0.962	1.162	1.117	0.984	1.020	1.003
Italy	0.898	0.979	0.879	0.932	1.234	1.150	0.964	1.164	1.122	0.977	1.043	1.019
Cyprus	0.916	0.853	0.781	1.176	1.197	1.407	0.934	1.125	1.051	1.019	0.994	1.013
Latvia	1.114	0.918	1.022	0.990	1.309	1.296	1.138	1.201	1.367	1.021	0.968	0.988
Lithuania	1.020	1.172	1.196	1.136	1.256	1.427	1.120	1.161	1.300	1.151	0.906	1.042
Hungary	0.879	1.114	0.979	1.182	1.213	1.434	0.972	1.141	1.110	0.977	1.045	1.021
Netherlands	1.083	0.954	1.033	0.924	1.223	1.130	1.002	1.088	1.089	0.982	1.120	1.099
Austria	1.065	0.974	1.037	1.054	1.226	1.292	0.974	1.118	1.088	0.992	1.192	1.182
Poland	1.064	1.088	1.158	1.098	1.184	1.300	1.141	1.159	1.322	1.022	1.045	1.068
Portugal	0.938	1.086	1.019	0.968	1.264	1.223	0.925	1.140	1.055	0.980	1.024	1.004
Slovenia	1.024	1.020	1.044	1.058	1.300	1.375	1.001	0.959	0.960	1.006	1.027	1.033
Slovakia	1.226	1.069	1.310	1.269	1.234	1.565	1.171	1.144	1.340	1.417	1.160	1.643
Finland	1.059	0.971	1.028	1.132	1.211	1.371	1.046	1.105	1.156	1.498	1.075	1.610

Table 5 (continued)

	I			II			III			V		
	C	F	MI	C	F	MI	C	F	MI	C	F	MI
Sweden	1.094	0.979	1.071	1.058	1.217	1.288	0.995	1.039	1.034	1.013	1.067	1.081
United Kingdom	0.950	1.054	1.001	1.070	1.234	1.320	0.897	1.145	1.026	1.002	1.044	1.045
Average	**1.005**	**1.011**	**1.016**	**1.069**	**1.230**	**1.313**	**0.999**	**1.115**	**1.114**	**1.045**	**1.048**	**1.087**

Source: Authors' computations

comparison of frontier-shift indices from models II and III. On average, 1.1 % improvement due to technical input saving (Model I) and 11.5 % gain in the "social efficiency" (Model III) are outperformed by 23 % increase from the Model II. Some countries experienced overall deterioration due to worsening in particular dimension—Bulgaria and Estonia (technical), Greece (technical and social). It can be concluded that the social welfare in Europe in the span of 2003–2012 increased. This was mostly contributed by the change of technology that was for the most part environment-saving rather than technical-input-saving or social-conditions-improving.

3 Conclusions

In this study, a social welfare measures in several variants were presented. The basic model assessed three dimensions of social welfare—economic, environmental and social—individually to integrate them subsequently into a single social welfare indicator. Weight restrictions were introduced to prevent excluding of unfavourable performance in any dimensions from affecting the overall score. The approach is applicable in any case when preferences of policy makers should be taken into account in the process of evaluating performance outcomes. Such preferences can be estimated by multi-criteria analysis and thus provide an important support for policy makers.

The inter-temporal analysis using Malmquist productivity index revealed the prevailing role of technology in improving overall social welfare as well as its three constituent dimensions. Making use of the productivity indices derived from constituent models it was possible to expose environment-saving bias of social welfare performance change in technology, i.e. that the technology in Europe has for the most part been environment-saving rather than technical-input-saving or social-conditions-improving.

Annex

Table 6

Table 6 Efficiency score and weights from models IV and AR1–AR3

	SBM				SBM-AR1				SBM-AR2				SBM-AR3			
	Score	u1	u2	u3	Score	u1	u2	u3	Score	u1	u2	u3	Score	u1	u2	u3
Belgium	0.893	18.823	0.428	0.359	0.893	22.015	0.428	0.359	0.889	22.006	0.428	0.428	0.866	29.200	0.428	0.855
Bulgaria	0.537	0.414	0.850	0.597	0.514	0.850	0.850	0.597	0.486	0.850	0.850	0.850	0.342	3.400	0.850	1.700
Czech Republic	0.693	0.403	0.627	0.411	0.675	0.627	0.627	0.411	0.657	0.627	0.627	0.627	0.508	2.509	0.627	1.255
Denmark	0.925	0.390	12.768	0.358	0.925	0.390	0.363	0.358	0.925	0.390	0.353	0.358	0.807	1.333	0.333	0.667
Germany	1.000	5.397	2.770	10.380	1.000	5.300	2.648	10.386	1.000	9.431	3.882	7.710	1.000	13.720	1.577	5.542
Estonia	0.575	0.398	0.854	0.488	0.551	0.854	0.854	0.488	0.518	0.854	0.854	0.854	0.382	3.417	0.854	1.709
Ireland	0.936	14.713	6.026	0.402	0.936	12.937	8.129	0.402	0.936	21.091	0.398	0.402	0.898	21.918	0.333	0.667
Greece	0.696	0.475	0.492	0.471	0.693	0.492	0.492	0.471	0.690	0.492	0.492	0.492	0.492	1.970	0.492	0.985
Spain	0.893	0.363	10.310	0.424	0.893	0.363	0.361	0.424	0.889	0.424	0.405	0.424	0.799	1.333	0.333	0.667
France	0.972	16.750	6.293	0.362	0.972	10.828	7.000	0.362	0.972	20.678	0.335	0.362	0.950	25.072	0.333	0.667
Croatia	0.778	0.356	0.411	0.518	0.776	0.411	0.411	0.518	0.772	0.518	0.411	0.518	0.678	1.644	0.411	0.822
Italy	0.933	0.336	13.629	0.402	0.933	0.336	0.335	0.402	0.933	0.402	0.361	0.402	0.889	1.333	0.333	0.667
Cyprus	0.943	15.107	6.098	0.394	0.943	13.408	8.123	0.394	0.943	20.982	0.340	0.394	0.907	22.251	0.333	0.667
Latvia	0.780	23.620	0.333	0.616	0.780	16.854	0.333	0.616	0.780	19.911	0.333	0.616	0.766	21.413	0.333	0.667
Lithuania	0.309	1.268	1.509	0.456	0.293	1.509	1.509	0.456	0.271	1.509	1.509	1.509	0.134	6.036	1.509	3.018
Hungary	0.787	0.359	0.436	0.476	0.783	0.436	0.436	0.476	0.782	0.476	0.436	0.476	0.671	1.744	0.436	0.872
Netherlands	0.986	0.348	4.605	12.790	0.986	0.348	0.337	20.947	0.986	0.348	0.344	0.344	0.948	1.333	0.333	0.667
Austria	0.893	0.400	0.382	0.338	0.893	0.400	0.382	0.338	0.893	0.400	0.382	0.382	0.761	1.527	0.382	0.763
Poland	0.687	16.525	0.657	0.466	0.687	15.192	0.657	0.466	0.662	24.526	0.657	0.657	0.589	27.841	0.657	1.314
Portugal	0.777	0.436	0.415	0.436	0.777	0.436	0.415	0.436	0.777	0.436	0.415	0.436	0.600	1.659	0.415	0.829
Slovenia	0.884	0.358	0.440	16.411	0.880	0.440	0.440	17.921	0.880	0.440	0.440	0.440	0.816	1.760	0.440	0.880
Slovakia	0.801	15.371	0.485	0.430	0.801	18.734	0.485	0.430	0.794	21.421	0.485	0.485	0.731	24.259	0.485	0.969

Table 6 (continued)

	SBM				SBM-AR1				SBM-AR2				SBM-AR3			
	Score	u1	u2	u3	Score	u1	u2	u3	Score	u1	u2	u3	Score	u1	u2	u3
Finland	0.735	0.411	0.605	0.345	0.716	0.605	0.605	0.345	0.712	0.605	0.605	0.605	0.566	2.418	0.605	1.209
Sweden	0.905	0.438	4.246	12.530	0.905	0.438	0.395	16.605	0.905	0.438	0.402	0.425	0.758	1.333	0.333	0.667
United Kingdom	0.960	0.342	17.455	0.366	0.960	0.342	0.341	0.366	0.960	0.366	0.366	0.366	0.916	1.333	0.333	0.667

Source: Authors' computations

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Part II
Firm Management

Organizational Nimbleness and Operational Policies: The Case of Optimal Control of Maintenance Under Uncertainty

Ali Dogramaci and Suresh Sethi

Abstract The speed with which an organization takes action against unplanned failure and scrapping of its capital equipment is used as a measure of organizational nimbleness. Operational decisions at the plant level are studied in terms of the optimal control model of Kamien and Schwarz for maintenance policy. In this context it is shown how the form of optimal policies at the lower operational levels change, as the degree of nimbleness in decision making at higher echelons of the organization is increased.

1 Introduction

By organizational nimbleness in the context of this paper, we mean the speed with which an organization decides and takes action against a random event such as a failure that requires scrapping of capital equipment. Such a failure that occurs while the equipment is in service may require the mobilization of capital funds and the purchase, delivery and installation of new equipment to replace the failed one in order to resume the production function. If board or committee meetings are required for capital fund allocations, then the frequency of such meetings, or the speed with which "out of schedule" meetings can be arranged, their time to reach a decision, and the time for implementation of the decision, altogether, constitute the response time. This paper focuses on a specific optimal plant level operational policy, and investigates how optimal operations may be shaped by the response speed of the top management to random events that necessitate new capital investment decisions. The case under study will be the well-known Kamien–Schwartz (1971) model (KS model for short) for machine maintenance and replacement.

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In economic studies, often the empirical estimation of a relationship between variables of interest is preceded by a theory addressing the functional form of the relation. On the other hand in organizational science studies, choice of functional forms have not always enjoyed similar mathematical analyses prior to statistical estimation of the relation between the variables. Absence of a priori theory underlying the mathematical equation to be tested or estimated cannot necessarily be remedied by goodness of fit tests such as chi-square, or Kolmogorov–Smirnov, afterwards. This is because: (1) such goodness of fit tests need a large number of observations to rule out all other competing functional forms, and (2) there will be a large number of competing functional forms (including jumps, spikes, or changes of regimes) between the variables, unless one uses mathematical analysis to narrow down the number of alternatives to be studied. Such prior mathematical analyses may even serve as an alert for when not to use popular assumptions such as convexity or simple forms (such as linear, quadratic, logarithmic, or exponential relations). Thus for example each chapter of the econometrics textbook of Berndt (1991) begins with a functional form established by prior theory, the rest of the chapter then focusing on how such a functional form may be estimated. In this sense, the case studied in this paper serves as an illustration of how such functional forms may be established in organizational science. We focus on an attribute of an organization, namely organizational nimbleness, and the form of its relation (the timeline of intensity) for a specific operational activity, namely the maintenance of capital equipment.

The basic KS model applies in our terminology to a non-nimble organization, which junks the machine when it fails or sells it at its optimal salvage time, whichever comes first. KS proved the optimal machine maintenance policy $u(t)$ to be *non-increasing* over the lifetime of the machine. Figure 1a illustrates an example for such a profile. We will show that as organizational nimbleness increases, the intensity of maintenance may experience shocks of *increase* (Fig. 1b). Furthermore, as the organization becomes nimbler, so may the number of such shocks increase, albeit with smaller magnitudes. However, as we push further to its limit, i.e., for the nimblest organization capable of replacing the machine at any given instant, it will be shown that the shape of the maintenance intensity profile over the lifetime of the machine reverts to a non-increasing policy (possibly at a lower level than Fig. 1a).

2 Background

The basic building block to be used is the KS model, in which the intensity of maintenance can alter the probability of breakdown. KS showed for a single machine optimal control model that the maintenance intensity should be non-increasing over the lifetime of the machine. One of the assumptions of the KS model is that when the machine breaks down by chance, it is not replaced. Organizations that do not replace machines when they fail shall be termed non-nimble for the purposes of this study.

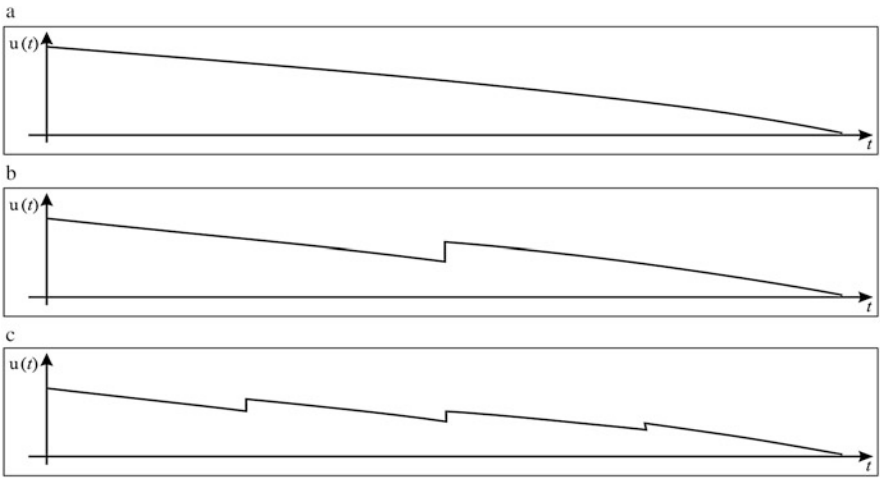

Fig. 1 Effect of organizational nimbleness on maintenance profile over the planned lifespan of an individual machine. (**a**) Maintenance profile for a non-nimble organization. (**b**) Maintenance profile for an organization with a little degree of nimbleness. (**c**) Maintenance profile for a more nimble organization

For a nimbler organization, we assume that replacement of failed machines can be considered in the repertoire of choices during the given planning horizon. What is the effect of incorporating into the KS model, replacement opportunities against in-service failures? How might the existence of such opportunities shape the profile of the maintenance effort? The ability of an organization to quickly reach capital equipment replacement decisions to cope with probabilistic breakdowns is considered as a measure of "organizational nimbleness", and its effect on the form of the optimal maintenance policy will be addressed for varying speeds of nimbleness.

Optimal control solutions for maintenance and replacement encompass a large body of literature going back to Thompson (1968) which led to many important extensions including those of Sethi and Morton (1972), Sethi (1973), Tapiero (1973), Tapiero and Venezia (1979) and Gaimon and Thompson (1984, 1989). The first optimal control model explicitly addressing hazard rates was that of Kamien and Schwartz (1971). Its extensions also multiplied over time and spread over diverse areas of applications, such as Carbone et al. (2005), Gjerde et al. (2005), and Kazianga and Masters (2006) as well as those listed in Dogramaci and Fraiman (2004) and Bensoussan and Sethi (2007). The focus of this paper on the other hand will be the maintenance and replacement of capital equipment in hierarchical organizations in the context of the KS model.

3 The Model

Notation:

$F_j(t)$ Probability that a machine of vintage j (bought when there are j periods to go until the end of the planning horizon) fails at or before t units of time from its purchase date.

$h_j(t) = [dF_j(t)/dt]/[1 - F_j(t)]$ The natural hazard rate of the machine at time t.

$u(t)$ Intensity of maintenance effort at time t. Denoting the minimum and maximum allowable levels for $u(t)$ as $\underline{U_j}$ and $\overline{U_j}$, respectively. $0 \le \underline{U_j} \le u(t) \le \overline{U_j} \le 1$.

$h_j(t)[1 - u(t)]$ The effective hazard rate of the machine at time t. The term $1-u(t)$ indicates the proportional reduction of the natural hazard rate and reduces the probability of failure at time t.

$M_j\,(u(t)) \cdot h_j(t)$ Cost of the maintenance effort $u(t)$ for a machine of vintage j at time t.

R_j Revenue rate, net of all costs except the cost of maintenance, from an operating machine of vintage j.

J_j Cash flow (other than loss of R_j) incurred by an in-service failure and scrapping of machine of vintage j. J_j denotes junk value of machine j minus the additional transition and switching costs arising from the in-service failure.

r Discount rate for present value calculations.

Z Length of the planning horizon.

n The number of time segments Z is divided into. In problems allowing only a finite number of machine replacements, n becomes the maximum number of potential replacements.

$S_j(t)$ Resale value of a working machine of vintage j at time t.

K_j The maximum number of periods a machine of vintage j can be allowed to operate: an integer upper bound on the operational life (based on technical, environmental or safety considerations).

T_j Planned retirement age yielding the maximum expected net present value for a machine of vintage j (purchased j periods before end of the planning horizon). $T_j \le K_j \le j$.

K_{T_j} Smallest integer larger or equal to T_j, $T_j \le K_{T_j} \le K_j \le j$.

$p_{\tau,i}$ Probability that the management will replace in period i, the machine that failed during the time interval $[\tau, \tau + 1)$, $\tau + 1 \le i$.

In problems involving two or more machines in tandem over time, to ensure that replacement takes place at some point in time, one may choose to set constraints such as $\sum_{i=\tau+1}^{K_{T_j}} p_{\tau,i} = 1$. We assume that these probabilities are given at the outset. Thus for a given value of n, the nimblest organization will have $p_{\tau,\tau+1} = 1$ and $p_{\tau,i} = 0$ for $i \neq \tau + 1$. On the other hand the least nimble organization will have $p_{\tau,K_{T_j}} = 1$, meaning it will not replace a broken machine before its originally intended retirement period. Note that these action times and probabilities apply to in-service failures, and not to "planned replacements".

Suppose that the plant floor operations are handled by an operations manager (OM for short), while capital investments are decided by a separate body of top managers. How might the OM choose his/her maintenance policy for a machine, if prospects for replacement of capital equipment is not only top management's prerogative, but also uncertain to the OM at the lower echelons? Such uncertainties will be reflected in the values of $p_{\tau,i}$. On one extreme, the OM may think that top management is totally non-nimble,[1] yet still wish to maximize expected cash flow. If so, the OM will have to favor the basic KS approach. The other extreme is that the OM may believe that top management will replace a broken machine at the earliest possible opportunity (i.e. at start of period $\tau + 1$). Accordingly, such nimbleness as reflected in the $p_{\tau,i}$ values of the model will influence the choice of maintenance policy at the lower levels of the organization.

3.1 The KS Formulation (A Single Machine Problem)

Cash flow at time t is $R_j - M_j h_j$ if the machine is up, and J_j if it fails and gets junked. The resulting net expected present value is $w = e^{-rt} \left\{ \left[R_j - M_j\left(u(t)\right) h_j(t)\right] \left(1 - F_j(t)\right) + J_j \frac{dF_j(t)}{dt} \right\}$.

The optimal control model of KS chooses $u(t)$ for each value of t so as to maximize

$$Y^* = \max_{u(t)} \int_{t=0}^{T_j} w dt + a\left[F_j(T_j), T_j\right], \qquad (1)$$

with $a\left[\right] = e^{-rT_j} S_j(T_j) \left(1 - F_j(T_j)\right)$ and subject to

$$\frac{dF_j(t)}{dt} = (1 - u(t)) h_j(t) \left(1 - F_j(t)\right), \quad F_j(0) = 0, \qquad (2)$$

with $0 \le \underline{U_j} \le u(t) \le \overline{U_j} \le 1$.

Maximization of (1) with respect to (2) involves choosing $u(t)$ for each point in time, so as to maximize the Hamiltonian $H = w + \lambda(t)[(1 - u(t))h_j(t)(1 - F_j(t)]$,

[1] That is, they will not replace a broken machine before its planned replacement time.

where $\lambda(t)$ denotes the shadow price related to constraint (2), and it satisfies $\lambda(T_j) = \frac{\partial a[\,]}{\partial F_j(T_j)} = -e^{-rT_j} S_j(T_j)$ and

$$\frac{d\lambda(t)}{dt} = -\frac{\partial H}{\partial F_j(t)} = e^{-rt}\left[R_j - M_j\left(u(t)\right)h_j(t) + J_j(1-u(t))h_j(t)\right] + \lambda(t)(1-u(t))h_j(t). \quad (3)$$

Considering the case of $0 \le J_j \le S_j(t) \le R_j/r$, $S_j'(t) < 0$, $0 \le h_j(t)$, $0 \le h_j{}'(t)$, $M_j(0) = 0$, $M_j'(u) > 0$, and $M_j''(u) > 0$, KS gave the necessary and sufficient conditions for optimal *u(t)* for any given T_j, and proved that this maintenance intensity is non-increasing over time. Maximization of the Hamiltonian involves only the terms that contain u. These terms are

$$g\left(u(t),t\right) \triangleq -M_j\left[u(t)\right] - (J_j + \lambda(t)e^{rt})u(t).$$

Optimal control $u(t)$ is chosen to maximize $g(u(t),t)$ at each $t \in \left[0, T_j\right]$. We define

$$g^*(t) \triangleq \max_{u(t)} \left\{-M_j\left[u(t)\right] - \left(J_j + \lambda(t)e^{rt}\right)u(t)\right\}. \quad (4)$$

KS showed that optimal $u(t)$ is continuous. Assuming $h_j'(t) < \infty$ and in view of (3), it holds that $\lambda(t)$ is smooth for $t \in \left[0, T_j\right]$.

In addition, for this single machine model with no replacement, KS provided the necessary condition for the optimal value of T_j. Recently, Bensoussan and Sethi (2007) proved that this condition along with an additional condition turn out to be sufficient for optimality of the maintenance and sale date of one machine. For a single machine problem, if the machine is still operational at T_j, then it is better to sell it at that time and receive $S_j(T_j)$ dollars and invest the money at the interest rate r.

3.2 Extended KS Model (Chain of Machines Over Time)

Incorporating machine replacement options into the KS model (1) and (2) relates to a nimbler organization, because it can also allow quicker responses to machine failures. For this purpose, the planning horizon [0, Z] may be divided into n segments such that the start of any time segment serves as a potential regeneration (i.e., replacement) point. At any such point, a new machine may be installed to replace the previous one. Nimbleness of an organization not only depends on its $p_{\tau,i}$ values, but also on the value of n. A finer mesh of regeneration points implies a nimbler organization.

3.2.1 Interrelated Replacement-Maintenance Optimization

Modeling the overall replacement problem via dynamic programming (DP) breaks the planning horizon Z into n stages. During the DP calculations, the unit of measurement for time is such that each time segment between two consecutive regeneration points is called one period. For example, if $Z = 4$ years, then for the model of a non-nimble organization, one period will denote a time span of four years (48 months), i.e. $n = 1$. A model spanning $52 \times 4 = 208$ weeks (using $n = 208$) would have each period one week long (from Monday to Sunday), and it will represent a nimbler organization. (In this nimbler organization, a machine that fails on Wednesday would be replaced the following Monday morning).

Incorporation of replacement choices into the KS model by Dogramaci and Fraiman (2004) and Dogramaci (2005) involved imbedding optimal control models into each other and studying the overall replacement of a chain of machines via DP. In the DP, the optimal value function j periods from the end of the planning horizon (i.e., when there are j more regeneration points to go) is denoted by $f_{(j)}$ with $f_{(0)} = 0$. For $j \geq 1$, let $f_{(j)}$ denote the expected net present value of an optimal regeneration and maintenance policy, from the point when a new machine is installed at stage *(j)* until the end of the planning horizon. Subscripts in parentheses indicate the stage number of DP calculations, rather than the equipment vintage. At stage (*j*), i.e., at time $Z - j$, the values of $f_{(j-1)}, f_{(j-2)}, \ldots, f_{(1)}$ are already at hand. Let $V(j, K)$ denote the optimal expected net present value for a vintage "*j*" machine purchased at time $Z - j$ at a cost of D_j dollars, with the intention of keeping it for K periods ($K \leq K_j$) and subsequent replacements (if any).[2] Unpredictability of the top management's decisions can be reflected through probabilities $p_{\tau,i}$. When a machine fails prematurely during period τ, the management may replace it in the immediate next opportunity with probability $p_{\tau,\tau+1}$. Or, the management may choose to stay idle for one period and replace in period $\tau + 2$ with probability $p_{\tau,\tau+2}$, or even in a later period.[3] In such a setting, the optimal control problem to be solved is:

$$
\begin{aligned}
V(j,K) = \max_{u(t),T_j} \sum_{\tau=0}^{K-1} \int_{\tau}^{\min\left[(\tau+1),T_j\right]} & \Big\{ e^{-rt} \Big\{ \left[R_j - M_j\left[u(t)\right] h_j(t)\right] \left[1 - F_j(t)\right] \\
& + J_j\left[1-u(t)\right] h_j(t) \left[1 - F_j(t)\right] \Big\} \\
& + \sum_{i=\tau+1}^{K} p_{\tau,i} \left[e^{-ri}\ f_{(j-i)}\right] \left[1-u(t)\right] h_j(t) \left[1 - F_j(t)\right] \Big\} dt \\
& + \ \left[1 - F_j(T_j)\right] \ \left[e^{-rT_j} S_j(T_j) + e^{-rK} f_{(j-K)}\right] - D_j
\end{aligned}
\tag{5}
$$

[2] K_j is an integer denoting an upper bound on intended life for a machine of vintage j, as determined by the management due to reasons such as safety, environmental issues, etc.. If there is no such constraint then one can set $K_j = j$, meaning the machine can (technically but not necessarily economically) be used until the end of the planning horizon.

[3] This may be due to unavailability of immediate funding, or supplier bottlenecks, or transportation delays, etc.

subject to

$$\frac{dF_j(t)}{dt} = [1-u(t)]\,h_j(t)\left[1-F_j(t)\right], \quad F_j(0)=0;\ t\in\left[0,\min\left(\tau+1,T_j\right)\right],$$
$$T_j \in [K-1,K], \tag{6}$$

with $0 \le \underline{U_j} \le u(t) \le \overline{U}_j \le 1, 0 \le K \le K_{T_j} \le K_j \le j$, and $\sum_{i=\tau+1}^{K} p_{\tau,i} = 1$.

In the objective function (5), for the last term of the summation ($\tau = K-1$), the value of T_j is to be obtained via numerical search as the one that yields the largest expected net present value of cash flow.

If a solution of (5)–(6) yields $V(j,K) < f_{(j-1)}$, then one can set $V(j,K) = f_{(j-1)}$ and stay idle, implying that we have an imaginary machine of zero cost and zero revenue from time $Z-j$ to $Z-j-1$, and consider buying a new machine at regeneration node $j-1$.

After the above values of $V(j,K)$ are obtained for each K, the value of $f_{(j)}$ can be obtained from

$$f_{(j)} = \max_K\left[V(j,K)\right], \qquad j=1,2,....,Z. \tag{7}$$

Problem (5)–(7) involves intertwined replacement and maintenance decisions. In order to obtain the values of $V(j,K)$, one has to deal with the objective function in (5), which has discontinuities from $t = 0$ to K due to different values of $f_{(j-i)}$. Following Dogramaci and Fraiman (2004), the targeted life span of the machine can be broken into segments, each of which is a single period long.

Consider a machine of vintage j which is planned to be replaced K periods after its acquisition. Just looking at this machine in isolation from the rest of the DP picture, the regeneration points may be denoted by $\tau = 0,1,2,\ldots,K$ as in Fig. 2. Suppose that for the given n, we have the most nimble organization, i.e., $p_{\tau,\tau+1} = 1$ for $\tau = 0,\ldots,K-1$ and $p_{\tau,i} = 0$ for $i \ne \tau+1$.

If the machine fails at time $t = 1.6$, then the production system will remain idle for 0.4 periods. Then at time $\tau = 2$, a new replacement would be installed and put into production. Our focus, however, is on the original machine, which at the time of its installation was planned to run for K periods. We will study

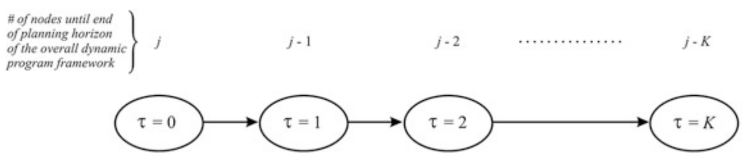

Fig. 2 Regeneration points in the life span of a machine begin with $\tau = 0$, and end with $\tau = K$ (K being the intended number of periods the machine is planned to serve.)

the maintenance on this machine, assuming it did not fail and got scrapped early. Suppose the machine is τ periods old, i.e., we are just on the regeneration point $j-\tau$ in the bigger overall DP picture, $\tau = 0, 1, \ldots, K-1$. The optimal expected net present value of the cash flow, from this "potential replacement node" τ to the end of the planning horizon is denoted by $Y_{j,\tau,K,F_j(\tau)}$. The subscript j indicates the vintage of the machine operational at node τ, and $F_j(\tau)$ the value of the state variable at this potential replacement node. For any given integer retirement age K, one can investigate whether the optimal retirement time lies in the interval $T_j \in [K-1, K]$. If this is done for $K = 1, \ldots, K_j$ the sweep will have covered all possible retirement points.

In the life of a machine, choose a time segment from age τ to $\tau+1$. For a maintenance model relating to this time segment, τ denotes the starting time of the local optimal control problem, K represents the number of periods the machine was intended to be used when it was bought. $F_j(\tau) \in [0, 1]$ is the initial value of the state variable. Let $b = \min\left(K, T_j\right) = K \wedge T_j$, with the value of T_j to be searched between *K-1* and *K*. The imbedded recursive optimal control problem is of the form:

$$\begin{aligned} Y_{j,\tau,K,F_j(\tau)} = \max_{u(t),T_j} \int\limits_{t=\tau}^{\min\left[(\tau+1),T_j\right]} & \Big\{ e^{-rt} \left\{ \left[R_j - M_j\left(u(t)\right) h_j(t)\right] \left[1 - F_j(t)\right] \right. \\ & \left. + J_j \left[1-u(t)\right] h_j(t) \left[1 - F_j(t)\right] \right\} \\ & + \sum_{i=\tau+1}^{K} p_{\tau,i} \left[e^{-ri}\ f_{(j-i)}\right] \left[1-u(t)\right] h_j(t) \left[1-F_j(t)\right] \Big\} dt \\ & + A\left(j, \tau+1, K, F_j(\tau+1)\right) \end{aligned} \tag{8}$$

with

$$\tau < K \le K_{T_j} \le K_j \le j, \quad \sum_{i=\tau+1}^{K} p_{\tau,i} = 1,$$

$$\begin{aligned} & A\left(j, \tau+1, K, F_j(\tau+1)\right) \\ & \quad = \begin{cases} \left[e^{-rT_j} S_j(T_j) + e^{-rK} f_{(j-K)}\right] \left[1 - F_j(b)\right] & \text{for } \tau = K-1,\ T_j \in [K-1, K] \\ Y_{j,\tau+1,K,F_j(\tau+1)} & \text{for } \tau = K-2, \ldots, 0 \end{cases} \end{aligned}$$

subject to

$$\frac{dF_j(t)}{dt} = \left[1-u(t)\right] h_j(t) \left[1 - F_j(t)\right] \tag{9}$$

with $F_j(\tau)$ given, $F_j(b)$ free, and $0 \le \underline{U_j} \le u(t) \le \overline{U_j} \le 1$, $0 \le u \le 1$.

Writing the Hamiltonian and selecting the terms that contain $u(t)$, we obtain an expression analogous to (4), namely,

$$G^*(t,\tau) \triangleq \max_{\underline{U_j} \le u(t) \le \overline{U_j}} G(u,t,\tau)$$
$$= \max_{\underline{U_j} \le u(t) \le \overline{U_j}} \left\{ -M_j[u(t)] - \left[J_j + \sum_{i=\tau+1}^{K} p_{\tau,i}\left[e^{-r(i-t)} f_{(j-i)}\right] + \lambda(t)e^{rt} \right] u(t) \right\}. \tag{10}$$

Similarly, analogous to (3) the time derivative of shadow price $\lambda(t)$ related to equation (9) is

$$\dot{\lambda}(t) = \frac{d\lambda(t)}{dt} = e^{-rt}\left\{R_j - M(u(t))\,h_j(t) + J_j[1-u(t)]\,h_j(t)\right\}$$
$$+ \sum_{i=\tau+1}^{K} p_{\tau,i}\left[e^{-ri} f_{(j-i)}\right]\,[1-u(t)]\,h_j(t) + \lambda(t)\,[1-u(t)]\,h_j(t). \tag{11}$$

As in the basic KS model of Sect. 3.1, we assume that between any two adjacent nodes $h(t)$ is finite. Since $0 \le \underline{U_j} \le u(t) \le \overline{U_j} \le 1$, from (11) it follows that $\lambda'(t)$ is finite and therefore $\lambda(t)$ is differentiable, finite, and continuous.

3.2.2 Solving for Optimal T_j

Solving (8)–(9) numerically begins at the far right end for the local optimal control problem between the node $\tau = K-1$ and T_j, with a search for optimal T_j. Optimal T_j can be found by calculating $Y_{j,K-1,K,F_j(K-1)}$ for different values of $T_j \in [K-1, K]$. For each choice of T_j, one first computes the terminal value of shadow price $\lambda(T_j) = \frac{\partial\left[\left(e^{-rT_j}S(T_j)+e^{-rK}f_{(j-K)}\right)\left(1-F_j(T_j)\right)\right]}{\partial F_j(T_j)} = -e^{-rT_j}S(T_j) - e^{-rK}f_{(j-K)}$ and uses it to find $u^*(T_j)$ via (10). One then substitutes these into (11) to obtain $\frac{d\lambda(T_j)}{dT_j}$. Using a numerical method such as Runge–Kutta, one can now compute $\lambda\left(T_j - \Delta\right)$, use it in (10) to get $u^*(t-\Delta)$, which feeds into (11), and repeat for $\lambda(t - 2\Delta)$ and so on, until the left end ($t = \tau = K-1$) of the local optimal control problem is reached. Thus all of the optimal values of $u^*(t)$ are now on hand. Next, we can calculate the value function $Y_{j,t_i,K,F_j(t_i)}$ using (8) for any starting time $t_i \in \left[K-1, T_j\right)$ by beginning with a given value of $F_j(t_i)$ and updating $F_j(t)$ via (9).

Dogramaci and Fraiman (2004, p. 790) noted that the solution of the differential equation (9) is of the form $F_j(t) = F_j(\tau)\,p(t) + q(t)$ and that substituting it into the objective function yields an expression that is a linear function of $\left[1 - F_j(\tau)\right]$ such that $Y_{j,\tau,K,F_j(\tau)} = \left[1 - F_j(\tau)\right] \cdot Y_{j,\tau,K,0}$. The model here is different because it allows for the possibility of an early retirement of the machine, and we need to explain that this property still holds in spite of the different upper limit of the integral in (8) and the need for a search of optimal T_j. Equations (10) and (11) do not involve $F_j(t)$ and the variables $\lambda, \dot{\lambda},$ and u^* do not depend on this state variable in the computations. In the objective function (8) for any choice of $t = K-1,\ K-1+\Delta, K-1+2\Delta, \ldots, T_j$, we can thus replace the state variable $F_j(t)$ by $F_j(t) = F_j\,(K-1)\,p(t)+q(t)$.

Replacing in (8) the state variable $F_j(t)$ by $F_j(t) = F_j\,(K-1)\,p(t)+q(t)$, we obtain

$$\begin{aligned} &Y_{j,K-1,K,F_j(K-1)} = \\ &= \int_{K-1}^{T_j} \left[1-\left\{F_j\,(K-1)\,p(t)+q(t)\right\}\right] \\ &\quad\times\left[\begin{array}{c} e^{-rt}\left\{\left[R_j - M_j\,(u^*(t))\,h_j(t)\,\right]+J_j\,[1-u^*(t)]\,h_j\big(t\big)\right\} \\ +\displaystyle\sum_{i=\tau+1}^{K} p_{\tau,i}\left[e^{-ri}\ f_{(j-i)}\right][1-u^*(t)]\,h_j(t) \end{array}\right] dt \\ &\quad+\ \left[1-\left\{F_j\,(K-1)\,p(T_j)+q(T_j)\right\}\right]\left[e^{-rT_j}S_j(T_j)+e^{-rK}f_{(j-K)}\right]. \end{aligned}$$

The "value function" $Y_{j,K-1,K,F_j(K-1)}$ linearly depends on the starting value of the state variable $F_j\,(K-1)$. This value function is the expected value of the cash flows from time K-1 until the end of the planning horizon. If by time K-1 the machine has failed, i.e., $F_j\,(K-1) = 1$, then the calculations prior to time K-1 pick up the junking and ensuing $f_{(\cdot)}$ values associated with the replacement machines (if any). In this case, $Y_{j,\,K-1,\,K,\,1} = 0$. As a function of $F_j\,(K-1)$, this is the smallest value $Y_{j,K-1,K,F_j(K-1)}$ can attain. At the other extreme, it reaches its maximum value when the starting probability of failure is zero (i.e., when $F_j\,(K-1) = 0$). Thus $Y_{j,K-1,K,F_j(K-1)} = \left[1-F_j\,(K-1)\right] Y_{j,K-1,K,0}$.[4] One only needs to compute for each T_j the value of $Y_{j,K-1,K,0}$ since the value of T_j that maximizes $Y_{j,K-1,K,0}$ will also yield the maximum $Y_{j,K-1,K,F_j(K-1)}$ for any $F_j\,(K-1)$. In conclusion, using a numerical search procedure, the T_j value that yields the largest $Y_{j,K-1,K,0}$ will be chosen.

If $K > 1$ and if $T_j = K-1$ is the best option found so far, then: (1) the value of K can be decreased by one, and (2) better choices for T_j will be explored for values smaller than the new K. If, however, $K = 1$ and the best choice for T_j is *0*, it means from node j to j-1 a machine should not be purchased and the system should stay idle.

3.2.3 Solving the Rest of the Problem

If $K = 1$, the whole problem will have been solved. Otherwise, if $K > 1$, then we will have solved the tail end local optimal control problem between the nodes

[4]This argument applies for any starting point $t_i \in \big(K-1, T_j\big)$ between the regeneration nodes. In (8), replacing the starting time τ (in the lower limit of the integral) with t_i and applying the same logic as in the previous two paragraphs yields the general relation $Y_{j,t_i,K,F_j(t_i)} = \left[1-F_j\,(t_i)\right] Y_{j,t_i,K,0}$.

$\tau = K - 1$ and K with an optimal value of $Y_{j,\tau,K,F_j(\tau)} = \left[1 - F_j(\tau)\right] \cdot Y_{j,\tau,K,0}$, which will feed as the salvage value $A\left[j, K-1, K, F_j(K-1)\right]$ for the adjacent upstream local optimal control problem. Now by decreasing the value of τ by 1 and applying it to (8)–(9), this immediate upstream problem can be solved[5] analogously along the lines of Dogramaci and Fraiman (2004, p. 790) and Sect. 3.2.2. Next, if $\tau > 0$, its value is decreased by 1 and the adjoining local optimal control problem on the left side is solved similarly. Thus, solving the problem stated in (8)–(9) recursively for $\tau = K-1, .., 0$, eventually yields $Y_{j,0,K,0}$, and $V(j,K)$ can be obtained from $V(j,K) = -D_j + Y_{n,0,K,0}$.

3.3 Features of the Extended KS Model

The issues addressed here differ from those in Dogramaci and Fraiman (2004) not only in terms of the additional features in (8)–(9), such as searching for T_j, but also in the following ways. Dogramaci and Fraiman (2004) provided a method of solution but did not address the form of the optimal maintenance policy, did not study the structure of the shadow price $\lambda(t)$ nor its variation over time, did not address the effects of the relative magnitudes of DP values at adjacent nodes in terms of how they relate to the maintenance profile, did not cover issues such as instantaneous replacements, and did not address any implications for organizational behavior. On the other hand, here in this paper, the whole model is evaluated in terms of organizational dynamics. A quantitative measure of organizational nimbleness is introduced and the response time of the decision makers is expressed in the $p_{\tau,i}$ terms. From this perspective: (1) the interplay between the behaviors at different administrative echelons will now be formally demonstrated, (2) a new way of proving the KS non-increasing maintenance policy will be developed using the properties of the shadow price and its derivative, (3) violations of the KS policy will be explained and linked to organizational nimbleness, and (4) an extreme case of nimbleness will be addressed using models of instantaneous replacements and their effect on the shape of the shop floor maintenance policy will be presented.

4 Properties of the Extended KS Model

In this section we study the optimal maintenance profiles of relatively nimble organizations (i.e., organizations with a finite number of replacement possibilities). Study of the variations of maintenance intensity relates to: (1) DP value functions

[5] By the same argument as in the previous footnote, the general relation $Y_{j,t_i,K,F_j(t_i)} = \left[1 - F_j(t_i)\right] Y_{j,t_i,K,0}$ will again hold.

$f_{(j)}$ for replacement decisions and (2) the imbedded optimal control value functions for maintenance during the life span of an individual machine.

4.1 Relative Magnitudes of $f_{(j)}$

DP values at the regeneration points are such that $f_{(j)} \geq f_{(j-1)}$ for $j = 1, 2, \ldots$ To see this, note that $f_{(0)} = 0$. $f_{(1)}$ cannot be less than or equal to $f_{(0)}$, since there is a better option of not buying a machine just one period prior to the end of the planning horizon: to simply stay idle at zero cost. By a similar argument, $f_{(2)} \geq f_{(1)}$, and so on.

4.2 Sign of the Shadow Price and Its Derivative: $\lambda(t) \leq 0$ and $\dot{\lambda}(t) \geq 0$

In Sect. 3.2.1 we saw that between two successive regeneration points, $\lambda(t)$ is continuous and differentiable. Here we will address the sign of $\lambda(t)$ and its derivative.

4.2.1 $\lambda(t) \leq 0$

From Sects. 3.2.2 and 3.2.3 we know that $Y_{j,t,K,F_j(t)} = \left[1 - F_j(t)\right] \cdot Y_{j,t,K,0}$. This in turn implies that for the shadow price of the problem (8)–(9), we have $\lambda(t) \leq 0$ as follows. $\lambda(t) = \frac{\partial Y_{j,t,K,F_j(t)}}{\partial F_j(t)} = \frac{\partial\left[\left[1-F_j(t)\right]\cdot Y_{j,t,K,0}\right]}{\partial F_j(t)} = -Y_{j,t,K,0} \leq 0$, since $Y_{j,t,K,0} \geq 0$ (otherwise we would rather retire the machine at t). Non-positive shadow price makes sense because the value of the objective function ought not to be larger for higher values of the cumulative probability of failure.

4.2.2 $\dot{\lambda}(t) \geq 0$

Now consider $\lambda(t) = -Y_{j,t,K,0}$. Here $Y_{j,t,K,0}$ is the net expected return from the system from time t until the end of the planning horizon, if we start at time t with $F_j(t) = 0$. Since we do not allow machine usage beyond the optimal value of T_j, $Y_{j,t,K,0}$ must be a smaller value if t is larger,[6] because it will cover a shorter

[6]That is, closer to the end of the planning horizon.

time period (see Appendix 2). Hence, $\frac{dY_{j,t,K,0}}{dt} \leq 0$ or $\frac{d(-Y_{j,t,K,0})}{dt} \geq 0$, and therefore $\frac{d\lambda(t)}{dt} = \dot{\lambda} \geq 0$.

4.3 Profile of u(t) Over Time

The behavior of $u^*(t)$ will be addressed first for the time segment between two adjacent potential replacement nodes, and then for the immediate left and right sides of a single node.

4.3.1 Variation of *u(t)* for $\tau \leq t < \tau + 1$

Using the properties of $\lambda(t)$ and $\dot{\lambda}(t)$, we will demonstrate the KS property of non-increasing $u^*(t)$ for $t \in [\tau, \tau + 1)$ between the replacement nodes $j - \tau$ and $j - (\tau + 1)$ for $\tau = 0, \ldots, K - 1$.

Lemma *For problem* (8)–(9), $u^*(t)$ *is non-increasing over time for* $\tau \leq t < \tau + 1$.

Proof In Sect. 4.2, it was shown that $\lambda(t) < 0$ and $\dot{\lambda} = \frac{d\lambda(t)}{dt} \geq 0$. From this it follows that $u^*(t)$ is non-increasing along the t-axis for $\tau < t < \tau + 1$. Reason: $u^*(t)$ can be found by solving for the value of u for which $\dot{G} = \frac{dG}{du} = 0$ (unless this value of u falls outside its permitted interval).[7] Note that $M_j, J_j, p_{\tau,i}, f_{(\cdot)} \geq 0$. The $u^*(t)$ that yields the minimum for

$$-G^* = \underset{\underline{U_j} \leq u(t) \leq \overline{U_j}}{Min} \left\{ M_j\,[u(t)] + \left(J_j + \sum_{i=\tau+1}^{K} p_{\tau,i} \left[e^{-r(i-t)}\ f_{(j-i)} \right] + \lambda(t) e^{rt} \right) u(t) \right\}$$

would have been the smallest allowable value of u, if $\lambda(t)$ were not negative. $\lambda(t)$ is the only reason why $u^*(t)$ may be larger than its lower bound $\underline{U}$. Since $\dot{\lambda} \geq 0$, the incentive for large $u^*(t)$ values will decrease or remain the same over time. Q.E.D.

4.3.2 Jumps in the Intensity of Maintenance: Deviation from the Policy of Non-Increasing *u(t)*

For convenience of exposition, from this point onward, we will focus on the case of $p_{\tau,\tau+1} = 1$ and study the dynamics of the optimal control in the neighborhood

[7] If the u obtained turns out to be greater than or equal to $\overline{U_j}$, then it will be set equal to $\overline{U_j}$. If the u obtained turns out to be less than or equal to $\underline{U_j}$, then it will be set equal to $\underline{U_j}$.

of the regeneration points and observe the possibilities of jumps in the values of the control variable.

An important conclusion of KS was that along an optimal path, the maintenance effort $u(t)$ is non-increasing over time over the lifetime of the machine. Introducing the *possibility* of installing another machine, in place of one that fails before the intended lifespan K, changes the structure of the optimal policy even when the machine does not fail at the potential replacement node.

Theorem *For the model in equations* (5) *and* (6), *the optimal maintenance effort* $u^*(t)$ *for a machine planned for K periods of use needs not be non-increasing, as* $u^*(t)$ *may increase at potential replacement nodes.*

Proof As stated in Sect. 3.2, for $\tau \le t < \tau + 1$, the optimal control for problem (5)–(6) is obtained by the maximization problem

$$\begin{aligned} G^*(t,\tau) &\triangleq \max_{\underline{U_j} \le u(t) \le \overline{U_j}} G(u,t,\tau) \\ &= \max_{\underline{U_j} \le u(t) \le \overline{U_j}} \left\{ -M_j[u(t)] - \left(J_j + e^{-r(\tau+1-t)} f_{(j-\tau-1)} + \lambda(t) e^{rt} \right) u(t) \right\}. \end{aligned} \tag{12}$$

On the other hand, from one interval $(\tau,\ \tau+1]$ to the next one $(\tau+1,\ \tau+2]$, the term $e^{-r(\tau+1-t)} f_{(j-\tau-1)}$ is replaced by $e^{-r(\tau+2-t)} f_{(j-\tau-2)}$ in the expression for G. It was shown in Sect. 4.1, as DP computations proceed from one stage to the next, the value function $f_{(\cdot)}$ will be non-decreasing $\left(f_{(j-\tau-1)} \ge f_{(j-\tau-2)} \right)$. Thus, for $r \ge 0$ and $\tau > 0$, we have $e^{-r(\tau+1-t)} f_{(j-\tau-1)} \ge e^{-r(\tau+2-t)} f_{(j-\tau-2)}$. Now focus on an intermediate node $\tau + 1$ (i.e. $\tau + 1 < K$). Consider any such intermediate regeneration point in time, and look at it from both left and right sides. The left side involves an optimal control problem from node τ to $\tau + 1$, and is of the form given in equations (5)–(6). For this problem on the left, denote the time parameter by t^-. As for the optimal control problem on the right (from node $\tau + 1$ to $\tau + 2$), denote the time parameter by t^+.

At time $t^- = \tau + 1$, we are finishing an optimal control problem (8)–(9) and at this finish time, the adjoint variable (or shadow price) must equal the partial derivative of the salvage term A in equation (8) with respect to the state variable $F_j(\tau + 1)$:

$$\lambda\left(t^- = \tau + 1\right) = \frac{\partial Y_{j,\tau+1,K,F_j(\tau+1)}}{\partial F_j(\tau + 1)}. \tag{13}$$

Now look at the shadow price as $t^+ \downarrow \tau + 1$ from left, for the problem on the right. Here we are beginning an optimal control problem whose optimal value at time $t^+ = \tau + 1$ is $Y^*\left[j, \tau + 1, K, F_j(\tau + 1)\right]$. Since the value of the adjoint variable (shadow price) at the start of an optimal control problem is equal to $\frac{\partial Y^*}{\partial F_j}$, we have

$$\lambda\left(t^+ = \tau + 1\right) = \frac{\partial Y^*_{j,\tau+1,K,F_j(\tau+1)}}{\partial F_j\left(\tau + 1\right)}. \tag{14}$$

Thus, as we pass over the node $\tau + 1$, the $\lambda()$ terms are the same for the problems on the left and right. Now we will address the question of "possible changes in the value of u". To determine the value of the optimal control u as we move from $G^*\left(t^-, \tau\right)$ to $G^*\left(t^+, \tau + 1\right)$, the $\lambda()$ terms in each are the same at time $\tau + 1$. As we advance from t^- region to t^+ region, the only significant change from $G^*\left(t^-, \tau\right)$ to $G^*\left(t^+, \tau + 1\right)$ (in short, G^- to G^+) is the possible decrease from $\lim_{t^- \uparrow \tau+1} e^{-r(\tau+1-t^-)} f_{(j-\tau-1)} = f_{(j-\tau-1)}$ to $\lim_{t^+ \downarrow \tau+1} e^{-r\left(\tau+2-t^+\right)} f_{(j-\tau-2)} = \frac{1}{e^r} f_{(j-\tau-2)}$. Denoting the time points and optimal controls at the immediate left and right neighborhoods of the transition at $\tau + 1$, respectively, by $t^-_{\tau+1}$ and $t^+_{\tau+1}$ and u^- and u^+, we shall show that $u^- \leq u^+$ to complete the proof. Note that $M_j''(u) > 0$, $M_j'(u) > 0$, and the rest of the terms in G, which we can collect as $Q \cdot u(t)$, are linear in u. At time $t^-_{\tau+1}$ we have

$$\begin{aligned}\frac{\partial G^-}{\partial u} &= M_j'\left[u\left(t^-_{\tau+1}\right)\right] + J_j + e^{-r\left(\tau+1-t^-_{\tau+1}\right)} f_{(j-\tau-1)} + e^{r\left(t^-_{\tau+1}\right)} \lambda\left(t^-_{\tau+1}\right) \\ &= M_j'\left[u\left(t^-_{\tau+1}\right)\right] + Q^- = 0.\end{aligned} \tag{15}$$

At time $t^+_{\tau+1}$ the corresponding equation is

$$\begin{aligned}\frac{\partial G^+}{\partial u} &= M_j'\left[u\left(t^+_{\tau+1}\right)\right] + J_j + e^{-r\left(\tau+2-t^+_{\tau+1}\right)} f_{(j-\tau-2)} + e^{r\left(t^+_{\tau+1}\right)} \lambda\left(t^+_{\tau+1}\right) \\ &= M_j'\left[u\left(t^+_{\tau+1}\right)\right] + Q^+ = 0\end{aligned} \tag{16}$$

Obtaining optimal u involves solving the equation $M_j'(u) = -Q$. If the value of u satisfying (15) or (16) turns out to exceed any of the bounds $\underline{U}$ or $\overline{U}$ in $0 \leq \underline{U} \leq u \leq \overline{U} \leq 1$, then u needs to be set to the value of that limiting boundary. This is shown in Fig. 3 where $\underline{U} = 0$, and $\overline{U} = 1$, and for (15) we set $u = 0$.

Since $M_j'(u) > 0$, so is $-Q$. Since $Q^- \geq Q^+$, it follows that $-Q^- \leq -Q^+$. This, in turn (for a function with a positive slope for all $u(t)$),[8] implies $u^- \leq u^+$. Q.E.D.

Naturally, it is possible for the left and the right problem to require the maximum intensity of maintenance, in which case there would be no possible jump in the u value. For example this would be the case where both Q^+ and Q^- correspond to u values greater than $\bar{U}$.

[8]The shape of the KS cost function with $\frac{d^2 M(u)}{du^2} > 0$ ensures this property.

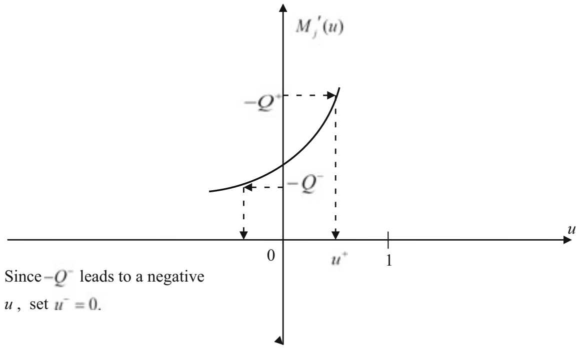

Fig. 3 An illustration of the curve $M_j'(u)$ and how larger value of $-Q^+$ leads to a corresponding u^+, which exceeds u^- (obtained in turn from its corresponding $-Q^-$)

5 The Case of the Nimblest Organization

The system addressed in this section replaces a failed machine immediately (i.e., with zero time delay). In this section we show that this extreme nimbleness removes any spike (of Sect. 4.3.2) and the optimal maintenance policy reverts to a non-increasing profile.

5.1 *Finite Horizon Problems*

Initial Problem: On the interval [0,*Z*], mark all points in time where new technologies will be introduced. We assume that each such date can be represented by a rational number. Choose *n* large enough so that every point in time, where a new technology gets introduced, coincides with a potential replacement node.

Step a) Solve the problem with given *n* using the method addressed in Sect. 3.2 above.

Step b) Note that

b.1) throughout the planning horizon [0,*Z*], if there are any idle segments of time in the solution, they will fall into the following two categories:

b.1.1) durations between T_j (the maximum date for planned retirement), and K_{T_j} (the next replacement opportunity), or,

b.1.2) between subsequent potential replacement nodes, say *j* and *j-1*, where it is better not to buy a new machine: $f_{(j)} = f_{(j-1)}$.

b.2) In the non-idle segments of the planning horizon, the integrand in equation (5) will be positive (see Appendix 1).

Step c) Increase *n*: Halfway between each pair of adjacent potential replacement nodes, insert a new node. Note that the technology of a machine bought at this node will be the same as that of the immediate upstream node. This is due to the way n is chosen initially.

Step d) Go to step a.

Suppose that the above steps are iterated for a very large number of times. Pick any machine in the obtained solution. Consider the in-between nodes from the installment of that machine to its replacement (or, time Z if that is the last machine). The large number of repetitions of steps a-d should display a pattern stated in the following lemma.

Lemma *In the DP model of Sect. 3.2, if using the procedure above increases the number of stages n toward infinity, then the durations of each period will approach zero, as well as the differences in the DP values of adjacent nodes. Namely,* $\lim_{n\to\infty} f_{(j)} - f_{(j-1)} = 0, \quad j = 1, 2, \ldots, n.$

Proof In any sweep of steps a-d, consider two adjacent nodes j, and *j-1*. During the period between the two nodes, the system may be in one of the following three cases:

1. Totally idle: If so, then due to the setup of the initial problem, in the subsequent sweeps of steps a-d, all new nodes inserted between the (original) nodes j and j-1 will keep having idle nodes between them. The machine introduced at a new node would have the same technology and profit-cost structure, as the one for the left (upstream) node. If the larger duration between nodes j and j-1 is not long enough to recoup the initial investment of the machine, nor will be the shorter duration between the newly introduced intermediate nodes. For all such additional nodes i and i-1 introduced in the repetition of steps a-d, we will have $f_{(i)} - f_{(i-1)} \triangleq \Delta f_i = 0$.
2. Totally Busy: In this case, the integrand of equation (5) is continuous since $u(t), F_j(t)$, $h_j(t)$, and $M(u(t))$ are continuous. In Appendix 1, it was shown that this integrand is non-negative. In a string of replacement or regeneration nodes, consider any two adjacent regeneration nodes i and i-1 such that node i is the original upstream (i.e. chronologically earlier) node and i-1 is the newly introduced node in step c, and node *i-2* is what used to be the downstream adjacent node to i before step c. Considering the non-negativity of the integrand of (5), $f_{(i)} \geq f_{(i-1)}$. Also $f_{(i)} - f_{(i-2)} \geq 0$, as well as $f_{(i)} - f_{(i-2)} \geq f_{(i)} - f_{(i-1)}$, while $f_{(i)} - f_{(i-2)} \geq f_{(i-1)} - f_{(i-2)} \geq 0$. In summary, increasing the number of nodes via steps a-d has diminished the differences between adjacent nodes if the difference was not already zero.
3. Busy until T_j and idle afterwards until the next node: Depending on whether the inserted node of step c falls in the idle segment or in the busy segment, a combination of the arguments in *i* and *ii* may be applied.

Thus, as we keep on introducing intermediate nodes between every adjacent pair, the differences (if any) become smaller and smaller: $\lim_{n\to\infty} f_{(j)} - f_{(j-1)} = 0 \; j = 1, 2, \ldots, n$. Q.E.D.

From Sect. 4.3.2 we know that the difference in the values of $u^{+}(\tau+1)$ versus $u^{-}(\tau+1)$ will be smaller as the associated differences $Q^{-}-Q^{+}=f_{(j-\tau-1)}-f_{(j-\tau-2)}\cdot e^{-r}$ diminish.

The discount factor e^{-r} is actually for the duration of one period. Recall that the length of a period is defined as the time interval between any two subsequent regeneration nodes. The length of the time between two such nodes will have to be explicitly taken into account and denoted by Δt, since we are now in the process of varying this value. Thus, the difference $Q^{-}-Q^{+}$ needs to be explicitly written as

$$Q^{-}-Q^{+}=f_{(j-\tau-1)}-f_{(j-\tau-2)}\cdot e^{-r\Delta t}.$$

As the number of DP stages increase to infinity, the length of the time segment Δt will approach zero, and $\lim_{\Delta t\to 0} e^{-r\Delta t}=1$.

If the difference between the $f_{()}$ values on each side of a time interval also approach zero, then the jumps in the u values would approach zero, washing out the violations of the non-increasing maintenance property.

5.2 Infinite Horizon Problems with Instantaneous Replacements

For the case of instantaneous replacement capability of a failed machine, in an infinite horizon problem, denote by $W(F,t)$ the value function starting at time t with the state variable value $F(t)=F$. Then,

$$W(0,0)=\max_{T,u}\left\{\begin{array}{c}\int_0^T e^{-rs}(R-M(u)h)(1-F)\,ds+\int_0^T e^{-rs}\dfrac{dF(s)}{ds}[J+W(0,0)]\,ds\\ +(1-F(T))\,e^{-rT}[S(T)+W(0,0)]-D\end{array}\right\}. \tag{17}$$

The above may be solved numerically by rewording it as a basic KS problem with the junk as well as the salvage values now including $W(0,0)$. One may choose a value of T. For this fixed T, starting with an initial guess for $W(0,0)$ used on the right side of equation (15), the optimal control problem can be solved numerically to yield a new $W(0,0)$, which then gets placed on the right side for a new iteration, until convergence. Repeating this for different values of T by a numerical search, the value of T yielding the largest *W(0,0)* may be obtained.

The KS model requires that $J\le S(t)\le R/r$ for each t, where R/r is the present value of revenues (excluding the cost of maintenance) from a machine running from time 0 to infinity. The first inequality is clear and the second inequality says that the salvage value of the machine at any time t had better not exceed R/r, or else it would be better for the firm to salvage the machine and go out of business. In our case, these inequalities need to be modified as $J+W(0,0)\le S(t)+W(0,0)\le R/r$ for each t. Note that if we had $S(t)+W(0,0)>R/r$ at some t, then once again it would be better to sell the machine (if it is still working) and get a profit of $S(t) + W(0,0)$. But $W(0,0)$ involves buying another machine which will be again be sold at age t if still functioning (or, scrapped and replaced in case of failure prior to age t). The second inequality $S(t) + W(0,0) \leq R/r$ is implicitly specified in terms of $W(0,0)$, which in turn can be expressed in terms of the problem parameters. In the context of equation (15), the cost of salvaging vs. producing involves $S(t) + W(0,0) \leq R/r$, which states that the net expected cash generated by the machine in T periods is not going to exceed $R/r - J$. Since $W(0,0)$ is the net expected cash flow *after* the original purchase price D is subtracted, and since $D > S(t) \geq J$, the structure and the spirit of the original KS model holds. This in turn means that the basic KS policy of *non-increasing u(t)* is optimal here as well.

In summary, in this section we showed that as the number of replacement opportunities increases, the number of potential jumps in u also increases, but with smaller and smaller sizes. With infinitely many regeneration points, the magnitude of the jumps are reduced to zero, and the KS result of non-increasing *u(t)* is restored for each machine, but this time as a part of a chain.

6 A Numerical Illustration

In this section we will illustrate the effect of nimbleness even when, throughout the planning horizon, technology does not change. Thus in this example, machine characteristics for all vintages are the same, and therefore we can drop the subscript j. Let the Weibull distribution underlie the hazard rate such that $h(t) = 1.5t^{0.5}$, interest rate $r = 0.05$, maintenance cost $M = 1.5\ \left(e^{5u(t)} - 1\right)$, $D = \$45$, $R = \$100$, $J = \$4$, $S = 0.88De^{-t/2}$, $n = 2$, and $K_j \equiv K = Z = 1$ year. Let $p_{\tau,\tau+1} = 1$ for $1 \leq \tau + 1 \leq n$.

As an initial case study, choose $n = 2$. This is the immediate next step up from being non-nimble: it is just nimble enough so that a machine failing in the first half of the year can be replaced on July 1st.

Using the procedure of Sect. 3.2 where imbedded optimal control models feed into larger dynamic programming, we obtain the following optimal control: $u(t)$ begins on January 1st at a value of 0.43 and gradually by end of June drops to 0.26. On July 1st, it jumps up to 0.38 and then gradually slides down reaching 0.20 at the end of the year. The overall pattern is same as in Fig. 1b and yields \$45.26.

If the lower management believed that the top management is non-nimble (will not replace any failed machine by July 1st) and if they chose to follow a maintenance policy commensurate with a non-nimble organization, their control pattern would be like the one in Fig. 1a, and yield \$40.90. On the other hand, increasing the nimbleness to $n = 4$ and implementing it would increase the returns to \$47.50 and result in the control pattern of Fig. 1c. Had it been possible to push nimbleness to the limit (replacing[9] a failed machine the minute it fails), then applying the policy of Sect. 5.2 for an infinite stream of models each with $T = 1$ year would yield a non-increasing maintenance profile, with lower $u(t)$ values than the non-nimble case, resulting in a higher expected net present value of \$53.77 per year. These values provide information to management wishing to choose its desired level of nimbleness.

7 Summary and Conclusion

We began with a question regarding the higher echelons of an organization, namely, the timings of top management decision making and their speed of implementation. We asked how such organizational attributes may influence optimal policies for the lower echelons (for plant/shop floor operations). In this context, we studied a maintenance function and how its profile may be influenced by the speed of strategic decisions for investments related to replacement of capital equipment.

The difference in the models between (1) a slow (non-nimble) organization, and (2) a nimbler one with shorter response time to the replacement needs generated by unplanned early scrapping of a machine, was reflected by the number of replacement opportunities within the firm's planning horizon. When we use more number of nodes in the dynamic programming model of Sect. 3, the duration between two subsequent nodes shortens, while the overall model still covers the same planning horizon (overall length of time).

The term $p_{\tau,i}$ was related to the number of periods $(i-\tau)$ needed for a replacement arising from an in-service failure. This term also implicitly incorporated into the model the "level of understanding" between different echelons of managerial hierarchy. If the lower level managers for the floor shop operations were not in a position to predict the capital investment behavior of the top management, the values of $p_{\tau,i}$ might be spread out evenly. Or, as stated in Sect. 4.3.2, if the lower level managers believe that top managers will never replace a machine before its original retirement time, then the maintenance intensity will be non-increasing over the life time of the machine. In Sect. 4.3.2 we also studied the case where the duration between potential replacement nodes is greater than zero: not only if the top management is nimble, but also if the operations managers believe that a failed machine will indeed be replaced at the earliest possible replacement node, then the chosen (optimal) maintenance intensity over time is likely to contain spikes (i.e. jumps).

For the nimbler organization (with a larger value of n), the difference between two subsequent values of f_j, and f_{j+1} is either smaller, or in unusual cases of

[9] By a new machine, which we will plan to serve for one full year (barring a failure which in turn would generate yet another replacement).

idleness, the same. The diminishing difference in turn can reduce the spike in the $u(t)$ values at the potential replacement/regeneration points.[10]

For the problem studied in this paper we reach the following conclusion. Operational policies not only depend on their immediate data such as costs, revenues, and characteristics of the physical machinery (their hazard rates), but also on the structure of the larger organizational environment. The demonstrated relationship between nimbleness of organizational decision making at the higher administrative echelons and the shape of the operational policies at the plant floor may be useful as a tool for future research on corporate decision processes across hierarchies.

Appendix 1: Optimal Policies with a Positive Integrand in the Objective Function

Can there be a problem for which optimal policy requires a negative or zero valued integrand in equation (5) for some time segment during the life of the machine, as an investment for larger profits later? If so, call this case A. We will now present another example, and call it case B, to argue why we can achieve just as good if not better by having a positive integrand.

To pull the value of the integrand below zero, one needs values of $u(t)$ so large that maintenance costs would exceed all other positive terms. Such expensive maintenance can only be justified if it would enable otherwise unattainable high profits later in the machine's life. Let b be the intended retirement time for the machine: $b = \min\left(K, T_j\right) = K \wedge T_j$. Denote the integrand of case A by $W_A(t)$. In the planned lifetime $[0, b]$, consider any integer $\tau_* \in (0, K]$ indicating a node (other than the starting node) in Fig. 2. We will examine whether, for some length of time before node τ_*, optimality ought to require $W_A(t) =$

$$\left\{\begin{matrix} e^{-rt}\left\{\left[R_j - M_j\left(u_A(t)\right)h_j(t)\right]\left[1 - {}_AF_j(t)\right] + J_j\left[1 - u_A(t)\right]h_j(t)\left[1 - {}_AF_j(t)\right]\right\} \\ + \left[e^{-r(\tau+1)}f_{(j-\tau-1)}\right]\left[1 - u_A(t)\right]h_j(t)\left[1 - {}_AF_j(t)\right] \end{matrix}\right\} \leq 0.$$

The control policy associated with case A is denoted by $u_A(t)$, $t \in \left[0, \min\left(K, T_j\right)\right]$, and the associated trajectory of the state variable by ${}_AF_j(t)$. If $W_A(b) \leq 0$, then cutting out the last non-positive segment and reducing b is better. The less obvious issue is when $t_* < b$, where t_* stands for the last point in time within $[0, b]$ and where $W_A(t)$ stops being non-positive. At that time, the state variable has the value ${}_AF_j(t_*)$. Note also: τ_* is chosen such that $t_* \leq \tau_*$.

[10] If the value of u is not at its upper bound, then the magnitude of spikes at the replacement nodes will diminish for larger values of n. On the other hand, if $u(t)$ is coasting at its upper limit $\bar{U}_j$, then there will be no spikes because there will be no room left for $u(t)$ to jump further up.

The integrand of equation (5) under this control policy can also be written as

$$W_A(t) = e^{-rt}\left\{\left[R_j - M_j\left[u_A(t)\right]h_j(t)\right]\left[1 - {}_AF_j(t)\right] + J_j\frac{d\left({}_AF_j(t)\right)}{dt}\right\} + e^{-r(\tau+1)}f_{(j-\tau-1)}\frac{d\left({}_AF_j(t)\right)}{dt}$$
$$= e^{-rt}\left\{\left[R_j - M_j\left[u_A(t)\right]h_j(t)\right]\left[1 - {}_AF_j(t)\right] + \frac{d\left({}_AF_j(t)\right)}{dt}\left[J_j + e^{-r(\tau+1-t)}f_{(j-\tau-1)}\right]\right\}.$$

The portion of equation (5) from time t_* to the intended retirement time b is:

$$v\left(t_*, b, {}_AF_j\left(t_*\right)\right) = \int_{t_*}^{\tau_*\wedge b} W_A dt + 1_{\tau_*<b}\sum_{\tau=\tau_*}^{K-1}\int_{\tau}^{(\tau+1)\wedge b} W_A dt$$
$$+ \left[1 - {}_AF_j(b)\right]\left[e^{-rK}S_j(b) + e^{-rK}f_{(j-K)}\right] - D_j, \quad (18)$$

where the indicator function $1_{\tau_*<b} = \begin{cases} 1 & \text{if } \tau_* < b \\ 0 & \text{otherwise} \end{cases}$ and $\frac{d\left({}_AF_j(t)\right)}{dt} = [1 - u_A(t)]\, h_j(t)\left[1 - {}_AF_j(t)\right]$ with $0 \le \underline{U_j} \le u_A(t) \le \overline{U_j} \le 1$.

Now, take a small time interval Δ to the left of t_* where $W_A(t)$ remains non-positive. Cut this time segment out and compare A (case before cut-out): $v(t_*, b, {}_AF_j(t_*))$ to B (case after cut-out): $v\left(t_* - \Delta, b - \Delta, {}_BF_j\left(t_* - \Delta\right)\right)$.

We assume that the control policy of A: $u_A(t), t \in [t_*, b]$ now applies to case B after the cut-out with $u_B\left(t - \Delta\right) = u_A(t),\ t \in [t_*, b]$. Note that ${}_BF_j\left(t_* - \Delta\right) \le {}_AF_j\left(t_*\right)$. The question is whether $v\left(t_*, b, {}_AF_j\left(t_*\right)\right) \le v\left(t_* - \Delta, b - \Delta, {}_BF_j\left(t_* - \Delta\right)\right)$? There are four factors involved and after the cut-out, for case B, the following can be observed in comparison to case A:

1. The integration of cash flows in (18) that used to start from t_*, now shifts Δ time units to the left and benefits from a smaller amount of discounting by the "net present value" factor e^{-rt}. Furthermore, the negative cash flow of Case A, $\int_{t_*-\Delta}^{t_*} W_A dt$, is avoided in Case B.
2. The planned retirement time of the machine shifts from b to $b - \Delta$, and since $S_j'(t) < 0$, a retired machine is a bit more valuable.
3. After the cut-out, the last $b - t_*$ duration of the journey begins with a value of the state variable (probability of failure) ${}_BF_j\left(t_* - \Delta\right) \le {}_AF_j\left(t_*\right)$, since $\dot{F}_j(t) \ge 0$. We know from Sect. 4.2.1 that $\lambda(t) = \frac{\partial Y}{\partial F} \le 0$, meaning smaller starting F values yield either the same or a higher value function.
4. Since $h_j{}'(t) > 0$, $h_j\left(t_* - \Delta\right) < h_j\left(t_*\right)$. So, the cut-out version B begins with a smaller hazard rate value $h\left(t_* - \Delta\right)$ as well. Furthermore, for the same passage of time from their respective starting points, version B's hazard rate value always "trails" that of version A. A larger hazard rate has two types of effects on the value function of case A: (1) it linearly increases the maintenance cost $M_j[u_A(t)]h_j(t)$, (2) through (6), for $u(t) < 1$, a higher hazard rate means larger values of $\dot{F}(t)$,

thereby increasing the probability of failure. Higher probability of failure causes increase in the expected junking revenues. This sort of cash increase, however, cannot claim optimality, since there is a better option than getting revenues by breaking and junking a machine: simply sell the machine without breaking it and enjoy $S_j(t)$, because by the KS assumptions we have $S_j(t) > J_j$. Furthermore, the procedure in Sect. 3.2.2 for choosing T_j assures that for $t \in \left[0, T_j\right)$, operating the machine yields a higher expected cash flow than retiring it early. In summary, the larger hazard rate of case A cannot be beneficial for the maximization of the expected net present value of cash flows.

Hence cutting out the last Δ bit of the negative cash flow should either maintain or improve total profitability. Similarly repeating the Δ cut-outs to the left, a totally positive cash flow picture emerges, with greater or equal cash flow, than the original counter-example we started with. In short, if the problem admits optimal solutions, then at least in one of them the integrand $W(t)$ is positive until the planned retirement date.

Appendix 2: $Y_{j,t_1,K,0} \geq Y_{j,t_2,K,0}$ if $t_1 \leq t_2$

t_1 is an earlier point in time than t_2. $Y_{j,t,K,0}$ is the optimal net expected discounted cash-flow from t until the end of the planning horizon as defined in (8) and (9) with zero for the starting value of F. Hence, $Y_{j,t_1,K,0}$ covers a longer time span than $Y_{j,t_2,K,0}$ does. For $t_1 \leq t_2$, we will investigate whether $Y_{j,t_1,K,0} \geq Y_{j,t_2,K,0}$.

Is it possible that during some time interval $t_2 \leq t \leq T$, the optimal policy of $Y_{j,t_1,K,0}$ requires cash flow $W_B(t)$ that is less than the cash flow of the policy under $Y_{j,t_2,K,0}$ (call it $W_A(t)$). In other words, is it possible that the condition $W_A(t) > W_B(t)$ leads to $Y_{j,t_1,K,0} \leq Y_{j,t_2,K,0}$?

Were the above true, we could improve on $Y_{j,t_1,K,0}$ by pulling the intended retirement date of the machine by $t_2 - t_1$ time units earlier and use the control policy of $Y_{j,t_2,K,0}$ from time t_1 onwards and make more money. Hence, such a smaller $Y_{j,t_1,K,0}$ cannot be optimal.

In conclusion, the value function cannot be less for a longer time span than a shorter alternative that starts later and ends at the same terminal time. In short: $Y_{j,t_1,K,0} \geq Y_{j,t_2,K,0}$ if $t_1 \leq t_2$.

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Safety Stocks in Centralized and Decentralized Supply Chains Under Different Types of Random Yields

Karl Inderfurth

Abstract Safety stock planning that focuses on risk protection to cope with demand uncertainties is a very well researched topic in the field of supply chain management, in central as well as in local decision making systems. In contrast, there is only few knowledge about safety stock management in situations where supply risks have to be covered that are caused by uncertainties with respect to production yields. In this study, a two-stage manufacturer-retailer supply chain is considered in a single-period context that allows for an analytical study of the impact of yield randomness on safety stock determination. In order to concentrate the analysis on the effects of yield uncertainty demand will be assumed to be deterministic.

We consider three basic types of yield randomness which represent different reasons for yield losses in production processes each, namely the stochastically proportional, binomial, and interrupted geometric yield type. It will be shown that these different yield risk specifications can bring about completely different properties concerning how safety stocks react on various input parameters in supply chain planning.

This holds especially for the impact of the demand size and for the influence of the level of product profitability in a supply chain. In an analytical model-based investigation it is demonstrated that the safety stock properties not only differ between the respective yield types, but also between systems of central and decentralized supply chain decision making. Thus, this study presents general insights into the importance of a correct yield type specification for an effective safety stock management and explains resulting differences in the stock distribution across supply chain stages in both centralized and decentralized settings.

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1 Safety Stocks in Supply Chains

Safety stocks are held in supply chains in order to provide an economically reasonable service of delivery for end customers if various types of risks may disrupt the flow of products. These risks can result from lacking predictability of external demand, from unreliability of production and transportation processes, or from service deficiencies of outside suppliers. Safety stocks are established at different stages of a supply chain by production or procurement decisions that result in planned inflow of products at certain stocking points which differ from the expected outflow. Thus, in a periodic-review planning environment a safety stock at some stock point can be generally defined as expected net stock at the end of a decision period (see Silver et al. 1998, p. 234) that results from respective operational decisions.

In literature, there is a vast amount of contributions that deal with supply chain safety stock planning aiming to cope with demand uncertainties (see Graves and Willems 2003; Axsäter 2003). The impact of supplier unreliability on safety stocks is also an issue that is widely addressed in scientific contributions, usually jointly with the supplier selection problem (see Minner 2003; Burke et al. 2009). Compared to that, a fairly limited number of contributions is devoted to the problem of safety stock determination in case of production yield risks in a supply chain setting (see Li et al. 2012; Inderfurth and Vogelgesang 2013).

Different from other risk environments, the specific problem concerning decision making under random yields is that in such a situation the procurement decision has an impact on the risk level. The basic context is illustrated in Fig. 1 where in a simple supply chain context the decision variable is the input Q at a production stage which is procured from an external supplier. This input quantity is randomly transformed into a production output $Y(Q)$ that is used as final product to fulfill some end customer demand D. In a centralized setting the decisions on manufacturing and sales are directly connected. In the case of a decentralized supply chain manufacturer and retailer are independent decision makers, and the retailer transforms the customer demand into an upstream order that constitutes the producer's demand. Outside supplier and end customer are supply chain external actors.

Since under regular process conditions the output quantity cannot exceed the input level, the so-called yield rate $Z(Q) = Y(Q)/Q$ is always a fraction between zero and one. According to different reasons for yield uncertainty there exist different types of yield randomness which are characterized by differences in the way the stochastic yield rate depends on the production level (see Yano and Lee 1995). In this paper it will be shown that it is critically important to identify the correct yield type in practical cases because different yield characteristics will cause different structural results for production and safety stock management. This holds for both

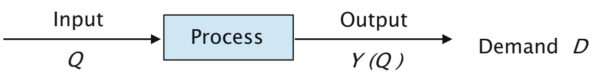

Fig. 1 The basic random yield setting

a centrally and a locally managed supply chain. A numerical study that reveals how yield type misspecification can harm the planning quality in a remanufacturing environment is found in Inderfurth et al. (2015).

In order to focus this study on the pure impact of yield randomness on safety stock decisions and to facilitate analytical results two major restrictions are imposed. First, only risks from the side of production yields are considered and data concerning external customers' demand and external suppliers' delivery processes are assumed to be deterministic. Second, only a single-period context is addressed so that it is possible to model and analyze decision making also for a decentralized supply chain with independent actors in a general way. The respective problem type has already been investigated for a limited range of yield models (see e.g. Keren 2009; Li et al. 2013). By concentrating on this type of model, we will consider a counterpart of the well-known newsvendor model with deterministic yield and random demand which is already very well researched (see Khouja 1999; Qin et al. 2011) including its extensions to decentralized supply chain settings (see Cachon 2003). From the newsvendor context under the objective of maximizing the expected profit it is well-known that the optimal procurement quantity is a critical fractile of the demand distribution. This critical ratio is high for high-margin products and low for low-margin ones. From the respective analysis it is also known that the optimal procurement level might be lower than expected demand in cases of low product profitability so that the safety stock becomes negative. Additionally, newsvendor research has revealed that in a decentralized supply chain under a simple wholesale price contract the procurement level and safety stock will always be below the respective value in the centralized case where the supply chain is managed by a single decision maker. This results from the so-called double marginalization effect (see Spengler 1950) that is usually observed in a simple wholesale price contract setting.

In the sequel, it will be investigated to which extent the newsvendor results carry over to a corresponding random yield model, and which role the specific yield type will play in this context. To this end the paper is organized as follows. In Sect. 2, before optimization procedures for centralized and decentralized decision making are explained, the three commonly used types of yield randomness are introduced and modeled, namely the stochastically proportional, the binomial, and interrupted geometric yield process. Next, for each yield type a specific section (Sects. 3–5) is dedicated in order to analyze safety stock determination under centralized and decentralized supply chain management and to reveal specific properties for different yield situations. Finally, Sect. 6 concludes this study by focusing on relevant insights and addressing open research questions.

2 Supply Chains with Random Yields

2.1 Types of Yield Randomness

There exist various reasons for randomness in the outcome of a production process. A comprehensive review of reasons and modelling approaches for randomness of production yields, also including literature referring to business examples, is found in Yano and Lee (1995) and in Inderfurth and Vogelgesang (2013). In these contributions three main types of yield randomness are identified.

First, in some cases a complete production batch Q is exposed specific uncertain processing conditions (like weather conditions in agricultural production) so that there is perfect correlation of defectiveness of units within a lot. This situation is described by a so-called *stochastically proportional* (*SP*) yield model, formulated as

$$Y(Q) = Z \cdot Q \tag{1}$$

with a random yield rate Z that is characterized by a *pdf* $\varphi(.)$ and *cdf* $\Phi(.)$ with given mean μ_Z and variance σ_Z^2.

The yield situation is completely different if single defective units are generated because of independent quality problems referring to single input materials or single manufacturing operations. In this case there is no correlation of defectiveness, and the total number of non-defective units in a lot Q follows a *binomial* (*BI*) distribution. With a success probability θ for each unit, in this *BI* yield case the probabilities of yield size realizations for $Y(Q)$ are given by

$$Pr\{Y = k\} = \binom{Q}{k} \cdot \theta^k \cdot (1-\theta)^{Q-k} \text{ for } k = 0, \ldots, Q. \tag{2}$$

Under *BI* yield the parameters of the yield rate Z are expressed by

$$\mu_Z = \theta \quad \text{and} \quad \sigma_Z^2 = \frac{\theta \cdot (1-\theta)}{Q} = \sigma_Z^2(Q). \tag{3}$$

A third yield model applies if production is affected by a risk which results in a move of the manufacturing process from an in-control to an out-of-control state, meaning that all produced units are good before this move while they are all defective afterwards. If the probability of staying in-control for any item is denoted by θ, the yield $Y(Q)$within a batch follows a so-called *interrupted geometric* (*IG*) distribution, characterized by

$$Pr\{Y = k\} = \begin{cases} \theta^k \cdot (1-\theta) & \text{for } k = 0, 1, \ldots, Q-1 \\ \theta^Q & \text{for } k = Q \end{cases} \tag{4}$$

In this *IG* yield case the following formulas hold for the yield rate parameters (see Inderfurth and Vogelgesang (2013))

$$\mu_Z = \frac{\theta \cdot \left(1 - \theta^Q\right)}{(1-\theta) \cdot Q} = \mu_Z(Q) \quad \text{and}$$

$$\sigma_Z^2 = \frac{\theta \cdot \left(1 - \theta^{1+2Q}\right) - (1-\theta) \cdot (1+2Q) \cdot \theta^{1+Q}}{(1-\theta)^2 \cdot Q^2} = \sigma_Z^2(Q). \tag{5}$$

Obviously, *IG* yield is characterized by some positive level of yield correlation within a production lot.

The three types of yield randomness described above are the basic ones that are widely used to model uncertainty in the output of production processes. For decision making it is critically important to consider which yield type is relevant in a specific case. This is because these basic yield models differ in the way the yield rate parameters are affected by the production input quantity Q. While μ_Z and σ_Z do not depend on the level of production under *SP* yield, things are very much different for the other yield types. Under *BI* yield the yield rate variance decreases with increasing production, and under *IG* yield additionally the mean of the yield rate becomes the smaller the larger the level of production will be. These effects that are visualized in Fig. 2 for a specific data set lead to qualitatively different conditions for optimal decision making concerning the size of production and safety stocks.

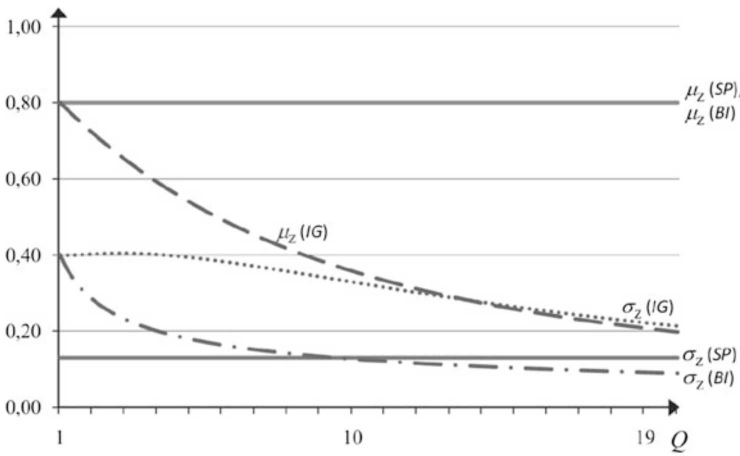

Fig. 2 Yield rate parameters and batch size for $\mu_Z(SP) = \theta = 0.8$ and $\sigma_Z(SP) = 0.13$

2.2 Optimization in Centralized and Decentralized Supply Chains

According to its definition as expected net stock at the end of a decision period, in a single-period context with firm demand the safety stock is directly determined by the choice of the production level. In the current analysis risk-neutral decision makers are assumed, and the optimal decisions are defined as those which maximize the expected profit. This optimization, however, proceeds quite different for centralized and decentralized supply chain settings.

2.2.1 Centralized Supply Chain

In a centralized supply chain production and retailing is in one hand, and the decision is how to determine the production input Q so that the total expected supply chain profit Π_{SC}under random production yield $Y(Q)$ and deterministic customer demand D is maximized . The respective profit function can be formulated as

$$\Pi_{SC}(Q) = p \cdot E\{min\,(Y(Q), D)\} - c \cdot Q. \tag{6}$$

Here, p stands for the retail price and c for the (input) cost per unit. Without loss of generality, it will be assumed in the following analysis that any excess stock after production has a zero salvage value. The optimal production decision Q^* results in a safety stock level SST^C for the centralized case amounting to

$$SST^C = E\left\{Y\left(Q^*\right)\right\} - D. \tag{7}$$

For the centralized supply chain the expected physical inventory (on-hand) IOH^C at the period's end is given by

$$IOH^C = E\left\{Y\left(Q^*\right) - D \middle| Y\left(Q^*\right) > D\right\} \tag{8}$$

and, thus, cannot exceed the safety stock.

2.2.2 Decentralized Supply Chain

Under decentralization of decision making, the producer and the retailer will maximize their local profits each. In this case, the retailer first decides on his order quantity D_R released to the manufacturer, and the producer determines her production input Q_P in response to this supply chain internal demand. Different from a standard newsvendor situation, in the case of random yield both supply chain actors bear a risk, namely a production risk at the manufacturer's stage and a supply risk at the retailer's stage. So the retailer needs to have some information about the manufacturer's yield process and resulting delivery performance in order to form some expectation on how the manufacturer will react on his order decision. To avoid some arbitrary estimation (like 'the manufacturer always inflates an order by the reciprocal of the mean yield rate') it is assumed that for the retailer the state of information is such that he can completely retrace the producer's optimization procedure. Thus, the interplay of decisions can be formulated as a Stackelberg game, meaning that the retailer will anticipate the producer's reaction when determining his order level. The interaction within the supply chain is assumed to be characterized by a simple wholesale price contract so that the retailer has to pay a respective (internal) price w for each unit he receives from the manufacturer. Because of the production yield uncertainty the manufacturer's output might exceed the retailer's order. In this case the producer only delivers the order quantity D_R and excess production is lost. Under these circumstances the producer's profit function equals

$$\Pi_P\left(Q_P\middle|D_R\right) = w \cdot E\left\{min\left(Y(Q_P), D_R\right)\right\} - c \cdot Q_P. \tag{9}$$

Maximizing this profit leads to an optimal conditional decision $Q_P\left(D_R\right)$. Under consideration of this decision the retailer maximizes his own profit $\Pi_R\left(D_R|Q_P\right)$ which is given by

$$\Pi_R\left(D_R\middle|Q_P\right) = p \cdot E\left\{\min\left(Y(Q_P), D_R, D\right)\right\} - w \cdot E\left\{\min\left(Y(Q_P), D_R\right)\right\}. \tag{10}$$

This maximization will result in an optimal order decision $D_R(Q_P)$. After inserting the respective conditional decisions, the effective optimal decisions $D_R^* = D_R\left(Q_P\left(D_R\right)\right)$ and $Q_P^* = Q_P\left(D_R^*\right)$ of both actors can be determined.

In a decentralized setting the global supply chain safety stock only depends on the producer's production output since under reasonable price conditions $w \leq p$ the retailer will never order below the (deterministic) demand level. So this safety stock SST^D is given by

$$SST^D = E\left\{Y\left(Q_P^*\right)\right\} - D. \tag{11}$$

A split of this overall safety stock to the producer and retailer side can only be carried out arbitrarily and, thus, will be left. This, however, is different for the expected stock on-hand which can be separated into a producer's share given by

$$IOH_P^D = E\left\{Y\left(Q_P^*\right) - D_R^*\middle|Y\left(Q_P^*\right) > D_R^*\right\} \tag{12}$$

and a retailer's share which is calculated as

$$IOH_R^D = E\left\{Y\left(Q_P^*\right) - D\middle|D_R^* \geq Y\left(Q_P^*\right) > D\right\}. \tag{13}$$

Price and cost parameters must meet some economic conditions in order to guarantee that the supply chain actors are able to make profits that are positive or at least zero. So it is assumed that $c/\mu_Z \leq w \leq p$. In this context, it has to be noted that due to the specific dependency $\mu_Z(Q)$ for *IG* yield the lower bound for *w* and *p* will depend on the choice of the production level. $1/\mu_Z$ can also be interpreted as minimum level of the product profitability that is defined by *w/c* for the manufacturer and *p/c* for the entire supply chain.

2.2.3 Yield Types and Safety Stock Properties

In order to find how different yield types affect the sign and level of safety stocks under centralized and decentralized decision making, the above optimization approaches have to be carried out for the different yield configurations (*SP*,*BI*,*IG*) and their respective modeling of the random variable $Y(Q)$. Before this will be investigated in detail in the forthcoming sections, some general results are presented which directly can be derived from the characteristics of the yield models. These results hold for both centralized and decentralized supply chains.

Property 1 If the production level Q does not exceed the demand size D (i.e., for $Q \leq D$), the supply chain safety stock is always negative ($SST < 0$) for each yield type.

This is simply because in all yield models $E\{Y(Q)\} < D$ holds due to $Y(Q) \leq Q$.

Property 2 In the case of *IG* yield the safety stock is always negative ($SST < 0$), irrespective of yield and price/cost parameters.

This property is a result of the specific probability distribution of yields in (4). The probabilities of *IG* yields with a value smaller than or equal to demand D do not change if the production level is increased above D, i.e. $Pr\{Y = k\}$ is independent of Q for $k \leq D$ and $Q > D$. This means that the expected revenues in the profit functions (6) and (9) cannot be increased by increasing the production level above demand. So always $Q \leq D$ will be chosen for economic reasons, and *Property* 1 directly applies in this case. The specific safety stock property under *IG* yield is a direct consequence of the underlying yield process that does not affect demand fulfilment in case of $Q > D$. It is easy to verify that this property also holds if in-control parameter θ is not constant but decreases with increasing number of produced units.

The detailed analysis of yield type effects on safety stock characteristics is facilitated if all variables can be treated as continuous. To this end, in the sequel we assume that for *SP* yield the yield rate Z is continuous and approximate the yield $Y(Q)$ in the *BI* yield case by a normal random variable with parameters from (3) (exploiting the De Moivre-Laplace theorem). Finally, under *IG* yield the respective yield expectation in (5) is treated as a continuous function in Q.

3 Safety Stocks under *SP* Yield

With *pdf* $\varphi(z)$ for the yield rate in the case of *SP* yield, the expected sales volume in (6) and (9) can be expressed as

$$E\{min\,(Y(Q), D)\} = \int_{0}^{D/Q} z \cdot Q \cdot \varphi(z) \cdot dz + \int_{D/Q}^{1} D \cdot \varphi(z) \cdot dz. \tag{14}$$

Exploiting this formulation, the profit maximization problem in the centralized and decentralized supply chain setting can be solved analytically as shown in Inderfurth and Clemens (2014). Thus, the respective production and ordering decisions can be analyzed with respect to their impact on safety stock holding, and general interrelationships can be detected.

3.1 Centralized Supply Chain

Maximizing the profit in (6) under the *SP* specific sales formula in (14) results in an optimal production quantity

$$Q^* = K^* \cdot D \quad \text{with} \quad K^* > 1 \tag{15}$$

where K^* is implicitly given by

$$\int_{0}^{1/K^*} z \cdot \varphi(z) \cdot dz = \frac{c}{p}. \tag{16}$$

From (15) it is evident that the optimal production quantity is always larger than demand with a demand inflation factor K^* that is constant. From (16) it follows that this inflation factor increases with increasing product profitability level *p*/*c*.

With Q^* from (15), according to (7) the safety stock in the centralized supply chain is given by

$$SST^C = \mu_Z \cdot Q^* - D = \left(\mu_Z \cdot K^* - 1\right) \cdot D. \tag{17}$$

Together with the K^* formulation in (16) this means that a positive safety stock will always be employed if the product profitability is sufficiently high and exceeds some critical level πc which is calculated from $\mu_Z \cdot K^* = 1$, resulting in

$$\pi c = \left[\int_0^{\mu_Z} z \cdot \varphi(z) \cdot dz \right]^{-1}. \tag{18}$$

With increasing profitability it is obvious that the supply chain safety stock will also increase. The expected on-hand inventory is given by

$$IOH^C = \int_{1/K^*}^{1} \left(K^* \cdot z - 1\right) \cdot D \cdot \varphi(z) \cdot dz \tag{19}$$

and obviously will increase with growing product margin.

3.2 Decentralized Supply Chain

The *manufacturer's optimization problem*, i.e. maximization of profit $\Pi_P\,(Q_P|D_R)$ in (9), equals that in the centralized problem except that external demand D is replaced by the retailer's order D_R and the sales price p by the wholesale price w. Accordingly, the buyer's optimal production is given by

$$Q_P\,(D_R) = K_P \cdot D_R \quad \text{with} \quad K_P > 1 \tag{20}$$

where K_P is defined by

$$\int_0^{1/K_P} z \cdot \varphi(z) \cdot dz = \frac{c}{w}. \tag{21}$$

From $w < p$ it is obvious that the producer's inflation factor is smaller than the respective factor under centralized optimization, i.e. $K_P < K^*$.

Anticipating the producer's reaction in (20), the retailer maximizes his profit $\Pi_R\,(D_R|Q_P)$ in (10) by ordering an amount that is equal to or above demand D according to

$$D_R^* = \begin{cases} D & \text{if } K_R \le K_P \\ D \cdot \frac{K_R}{K_P} & \text{if } K_R \ge K_P \end{cases}, \tag{22}$$

where the factor K_R is determined from

$$\int_0^{1/K_R} z \cdot \varphi(z) \cdot dz = \frac{c}{p} + \frac{w}{p} \cdot \frac{1 - \Phi(1/K_P)}{K_P} \tag{23}$$

so that also for K_R the relationship $K_R < K^*$ holds.

Given the retailer's ordering decision in (22), the supply chain interaction from (20) results in a manufacturer's production level that is characterized by

$$Q_P^* = \begin{cases} K_P \cdot D \textit{ if } K_R \leq K_P \\ K_R \cdot D \textit{ if } K_R \geq K_P \end{cases} . \tag{24}$$

Thus, also in a decentralized supply chain the effective production input is always proportional to the demand level, but with an inflation factor which is smaller than under centralized decision making. The system-wide safety stock SST^D from (11) is calculated analogously to (17) and, due to $Q_P^* < Q^*$, must be smaller than in the centralized supply chain setting.

The production and safety stock level in the decentralized supply chain very much depends on the level of the wholesale price w within its feasible range $c/\mu_Z \leq w \leq p$. From the definition of K_P in (21) it follows that K_P is increasing in w with $K_P = 1$ for $w = c/\mu_Z$ and $K_P = K^*$ for $w = p$. The K_R definition in (23) reveals that K_R is decreasing in w with $K_R = K^*$ for $w = c/\mu_Z$ so that there exists some wholesale price w (to be determined from equalizing the right-hand sides of equations (21) and (23)) for which both inflation factors K_P and K_R are equal. Given these interrelationships, the interplay of retailer and manufacturer decision in (22) and (24) leads to a producer's decision on production and safety stock level which is equal to the centralized solution (i.e. $Q_P^* = Q^*$ and $SST^D = SST^C$) for the border values of w, i.e. $w = c/\mu_Z$ and $w = p$. In between these limits, the retailer's order is decreasing with increasing w and stays constant at demand level D after K_R reaches the K_P value. Thereby, the manufacturer's production input will first decrease and then increase again. This relationship is visualized in the following Table 1 where the results for a numerical example are presented. This example is characterized by the following data: $c = 1,\ p = 14,\ D = 100$ and Z is uniformly distributed in [0,1] so that $\mu_Z = 0.5$ holds.

Table 1 Decisions and stocks in a decentralized supply chain for *SP* yield

	w	2	3	4	5	6	7	8	9	10	11	12	13	14
SP	Q_P	265	220	196	180	173	187	200	212	224	235	245	255	265
Yield	D_R	265	179	138	114	100	100	100	100	100	100	100	100	100
Z ∼ U[0,1]	***SST***	**32**	**10**	**−2**	**−10**	**−13**	**−6**	**0**	**6**	**12**	**17**	**22**	**27**	**32**
$\mu_Z = 0.5$	IOH_P	0	4	9	12	15	20	25	30	34	39	43	47	51
	IOH_R	51	29	14	6	0	0	0	0	0	0	0	0	0

In Table 1 also the values of the safety stock SST^D and the expected on-hand inventories are reported. The safety stock directly follows the manufacturer's production decision and thus reaches its highest value for the lower and the upper wholesale price limit where it just equals the stock level under centralization. Starting with the lowest w level, with increasing wholesale price the safety stock will always decrease first and will increase again after reaching a minimum value at some intermediate price level. Naturally, the total on-hand inventory in the supply chain follows the trend of the safety stock, but at some higher level. It is interesting, however, that the *IOH* distribution is very different for low and for high wholesale price values. This is specifically distinct in the extreme cases. For the lowest w level the on-hand inventory is completely held by the retailer because the manufacturer's production quantity coincides with the retailer's order size. At the highest w level the complete physical inventory is on the producer's side as the retailer's order does not exceed the external demand. With increasing wholesale price the inventory on hand is continuously increasing for the producer and continuously decreasing for the retailer reaching zero at the retail stage of the supply chain as soon as the retailer's order falls to demand level.

4 Safety Stocks under *BI* Yield

As mentioned in Sect. 2, for the *BI* yield analysis it will be assumed that the binomial distribution of yields can be properly approximated by a normal distribution. Based on this approximation, profit maximization in the centralized and decentralized supply chain can be conducted by means of mathematical calculus. The respective analysis is performed in Clemens and Inderfurth (2015) with results that will be exploited here for safety stock analysis in the following subsections.

Under the normality assumption in the case of *BI* yield, the expected sales volume in (6) and (9) can be expressed by

$$E\{min\,(Y(Q), D)\} = \int_0^D y \cdot f_Q(y) \cdot dy + \int_D^Q D \cdot f_Q(y) \cdot dy \tag{25}$$

where $f_Q(y)$ denotes the density function of a Normal distribution with parameters that—according to (3)—depend on the production level Q, i.e.

$$\mu_{Y(Q)} = \theta \cdot Q \text{ and } \sigma_{Y(Q)} = \sqrt{\theta \cdot (1-\theta) \cdot Q}. \tag{26}$$

A comparison with the corresponding expression for *SP* yield in (14) shows that the expected sales do not depend on the D/Q ratio in the same simple way, because production level Q has an impact on the yield variability.

4.1 Centralized Supply Chain

When the supply chain profit in (6) is maximized under the sales formula for *BI* yield in (25) the optimal production quantity $Q^*$is given as implicit solution from

$$M\left(D, Q^*\right) = \frac{c}{p}. \tag{27}$$

Here, $M(D, Q)$ is defined as

$$M(D, Q) := \frac{\theta}{2} \cdot \left[2 \cdot F_S\left(z_{D,Q}\right) - \frac{\sigma_{Y(Q)}}{\mu_{Y(Q)}} \cdot f_S\left(z_{D,Q}\right)\right] \tag{28}$$

where $F_S(.)$ and $f_S(.)$ stand for *cdf* and *pdf* of the standard normal distribution and $z_{D,Q}$ is the standardized variable

$$z_{D,Q} := \frac{D - \mu_{Y(Q)}}{\sigma_{Y(Q)}}. \tag{29}$$

From the properties of the $M(D, Q)$ function it can be shown that the optimal production quantity Q^*, and thus the respective safety stock level SST^C, will increase with increasing product profitability *p*/*c*. Different from the *SP* yield case, however, production level Q^* is no longer proportional to customer demand D and tends to the value $D/\mu_Z = D/\theta$ as the demand becomes larger and larger. This property is caused by the fact that under *BI* yield the yield rate variability is affected by the production quantity and that this variability—according to (3)—is approaching zero if the demand-triggered production volume becomes very large. This risk reduction is due to the missing correlation of defects within a production lot which, different from the *SP* situation, creates a risk-pooling effect.

With respect to the safety stock level $SST^C = \theta \cdot Q^* - D$ from (7) this means that this stock tends to zero when demand size D moves to a very high level. In general, the safety stock can be positive or negative, depending on the product profitability level *p*/*c*. The critical *p*/*c* ratio πc can be calculated from equation (27) by fixing $Q = D/\theta$ or $z_{D,Q} = 0$, respectively. This critical level then amounts to

$$\pi c = \left[\theta \cdot \left(0.5 - 0.2\sqrt{(1-\theta)/D}\right)\right]^{-1} \tag{30}$$

so that, different from the *SP* yield situation in (18), the sign of the safety stock also depends on the demand level. According to (8) the expected stock on hand IOH^C will always be somewhat larger than the safety stock level.

4.2 Decentralized Supply Chain

Like in the analysis for *SP* yield the *producer's optimal decision* corresponds to the optimal production decision in the centralized system given that demand and price are represented by the local data D_R and w. Thus the manufacturer's response function $Q_P\,(D_R)$is implicitly defined from

$$M\,(D_{R,}Q_P) = \frac{c}{w} \tag{31}$$

with the same properties as described for centralized decision making.

The retailer's reaction when maximizing his profit $\Pi_R\,(D_R|Q_P)$ from (10) is gained from the solution of the following equation

$$-w \cdot \left[1 - F_S\left(z_{D_R,Q_P}\right)\right] + \left[p \cdot M\,(D, Q_P) - w \cdot M\,(D_R, Q_P)\right] \cdot \frac{dQ_P\,(D_R)}{dD_R} = 0 \tag{32}$$

as long as the respective D_R value is larger than demand D. Otherwise $D_R^* = D$ is optimal. When exploiting the retailer-producer interaction in (31) the producer's optimal decision Q_P^*, unfortunately, cannot be expressed in a closed-form manner. From $M\,(D, Q^*) < M\left(D_R^*, Q_P^*\right)$ for $w < p$, however, it follows that $Q_P^* < Q^*$ so that also the supply chain safety stock SST^D in a decentralized supply chain will not exceed the respective stock level in a centralized system. It also can be shown that for the lower and upper wholesale price bound within the feasible range of w the manufacturer's optimal production level is equal to the optimal quantity in the centralized supply chain, i.e. $Q_P^* = Q^*$. For $w = p$ this results from (31) because the retailer's response equals external demand ($D_R^* = D$) in this case. For $w = c/\mu_Z = 1/\theta$ it can be derived from (32) that the retailer will order $D_R^* = Q^*$ and the manufacturer will choose her production level according to this order. The course of orders and production quantities for changing w values within the feasible range resembles that in the *SP* yield case. With increasing wholesale price w the retailer's order D_R^* is decreasing and reaches the size of external demand at some critical price, while the production quantity Q_P^* is first decreasing, but increasing again after it reaches some minimum level. This is also illustrated by a numerical example in Table 2 where the same data are used as in the *SP* yield example in Table 1 except for the yield description. For *BI* yield here a success probability of $\theta = 0.5$ is chosen which equals the μ_Z value in the *SP* yield case.

The values of safety stock SST^D and stock on hand *IOH* held by the producer and retailer that are reported in Table 2 follow directly from the production and order decisions. A comparison with Table 1 shows that the dependency of these stock values on the wholesale price w has the same structure as under *SP* yield. While the *BI* levels are considerably lower, their course is much smoother. This effect stems from the property of *BI* yield that, different from *SP* yield, the risk from the random yield rate (in terms of its variance) is decreasing with increasing order and production level.

Table 2 Decisions and stocks in a decentralized supply chain for *BI* yield

	w	2	3	4	5	6	7	8	9	10	11	12	13	14
BI	Q_P	215	211	207	205	205	207	209	210	211	212	213	214	215
Yield	D_R	215	109	104	101	100	100	100	100	100	100	100	100	100
$\Theta = 0.5$	***SST***	**8**	**6**	**4**	**3**	**3**	**4**	**5**	**5**	**6**	**6**	**7**	**7**	**8**
	IOH_P	0	2	3	4	4	5	6	6	6	7	7	8	8
	IOH_R	8	4	2	0	0	0	0	0	0	0	0	0	0

From the previous analysis it is apparent that both yield types, *SP* and *BI*, have an impact on safety stock management in centralized and decentralized supply chains which is different in terms of stock levels, but results in the same qualitative structure concerning the influence of prices on stocks.

5 Safety Stocks under *IG* Yield

From *Property* 2 we know that in case of *IG* yield the production level will never exceed the respective demand (i.e. $Q \leq D$) so that the expected sales quantity in (6) and (9) will reduce to

$$E\{min\,(Y(Q), D)\} = E\{Y(Q)\} = \frac{\theta \cdot \left(1 - \theta^Q\right)}{(1 - \theta)}. \tag{33}$$

From this sales function it follows that the respective profit functions which have to be optimized do not depend on the demand level, except for the condition that $Q \leq D$ holds. As mentioned in Sect. 2.2, under *IG* yield the lower bound $c/\mu_Z(Q)$ for w and p depends on the batch size Q. Since $\mu_Z(Q)$ in (5) is monotonously decreasing in Q, the minimum value of this bound is given for $Q = 1$ and has an amount of c/θ.

5.1 *Centralized Supply Chain*

Using the result in (33), the profit function from (6) can be expressed by

$$\Pi_{SC}(Q) = p \cdot \frac{\theta}{1 - \theta} \cdot \left(1 - \theta^Q\right) - c \cdot Q. \tag{34}$$

It is easy to show that $\Pi_{SC}(Q)$ is a concave function so that the optimal production level can by determined by exploiting the first-order optimality condition

$$\frac{d\Pi_{SC}(Q)}{dQ} = -p \cdot \frac{\theta}{1 - \theta} \cdot \ln\theta \cdot \theta^Q - c = 0.$$

This results in the following solution:

$$Q^+ = \frac{1}{\ln\theta} \cdot \ln\left[-\frac{1-\theta}{\theta \cdot \ln\theta} \cdot \frac{c}{p}\right]. \tag{35}$$

Thus, together with the restriction $Q \leq D$ the optimal production level in a centrally managed supply chain is given by

$$Q^* = \begin{cases} Q^+ & \text{if } Q^+ \leq D \\ D & \text{if } Q^+ \geq D \end{cases}. \tag{36}$$

From (35) it follows that Q^+ is steadily increasing with increasing product profitability *p*/*c* so that there exists some critical profitability level π_D for which Q^+ equals *D*. This level can be directly determined from (35) and is given by

$$\pi_D = -\frac{1-\theta}{\theta^{D+1} \cdot \ln\theta}. \tag{37}$$

Thus, if the profitability is sufficiently high, i.e. if $p/c \geq \pi_D$, the optimal production level will always be equal to external demand. In this context, it has to be mentioned that under *IG* yield the minimal profitability level which guarantees non-negative profits is equal to $1/\theta$.

The safety stock SST^C can be determined according to (7), resulting in the following closed-form expression

$$SST^C = \begin{cases} \frac{\theta}{1-\theta} \cdot \left(1-\theta^{Q^+}\right) - D & \text{if } Q^+ \leq D \\ \frac{\theta}{1-\theta} \cdot \left(1-\theta^{D}\right) - D & \text{if } Q^+ \geq D \end{cases}. \tag{38}$$

This confirms the finding in *Property* 2 that under *IG* yield the safety stock always must be negative, i.e. $SST^C < 0$. From (35) it is easy to see that the optimal production and, thus, the safety stock level is increasing with increasing product profitability *p*/*c* as long as $Q^* < D$. Like for the production quantity, the safety stock will be constant for each profitability level which exceeds the critical value π_D.

Since production always undershoots demand there is no stock on hand IOH^C that is held in the supply chain. This property also has consequences for the service level that can be guaranteed by the supply chain. Taking the fill rate *fr* as a possible service measure, its value is easy to analyze because under *IG* yield *fr* can simply be expressed by

$$fr\,(D,Q) = \frac{E\{Y(Q)\}}{D} = \frac{\theta \cdot \left(1-\theta^{Q}\right)}{(1-\theta) \cdot D}. \tag{39}$$

The highest possible service level is reached for all instances where Q^* equals D or, equivalently, when profitability p/c is larger than the critical level π_D. This means that a fill rate level $fr\,(D, D)$ cannot be exceeded even if the product profitability is arbitrarily high. This maximum fill rate is increasing with increasing success parameter θ and decreasing with increasing demand level D. As consequence, under *IG* yield the optimal service level can become extremely low if the customer demand reaches a very high level. The effect of decoupling between product profitability and service level does not exist under *SP* or *BI* yield where the production quantity exceeds the demand level and increases steadily with rising product profitability so that the service measure will do the same.

5.2 Decentralized Supply Chain

Like in the other yield situations, in a decentralized setting with local optimization the producer's profit function $\Pi_P\,(Q_P|D_R)$ has the same structure as the global one. The only difference to the profit in (34) is that the external price p is replaced by the wholesale price w and that the condition $Q_P \leq D_R$ has to be taken into account. Thus, as optimal production level for a given retailer's order we get

$$Q_P\,(D_R) = \begin{cases} Q_P^+ \text{ } if \text{ } Q_P^+ \leq D_R \\ D_R \text{ } if \text{ } Q_P^+ \geq D_R \end{cases} \tag{40}$$

with

$$Q_P^+ = \frac{1}{\ln \theta} \cdot \ln \left[-\frac{1-\theta}{\theta \cdot \ln \theta} \cdot \frac{c}{w} \right]. \tag{41}$$

The retailer knows that the manufacturer will never produce more than his own order D_R and that a production level above customer demand D will not affect his probabilities of receiving D or less units. Thus, he has no incentive to order more than D units (i.e., $D_R \leq D$) so that the profit function in (10) reduces to

$$\Pi_R \left(D_R \middle| Q_P \right) = (p - w) \cdot E\{Y(Q_P)\} \quad \text{with} \quad Q_P \leq D_R. \tag{42}$$

Due to the regular price relationship $w \leq p$ the retailer's optimal response is to order as much as possible under the restriction $D_R \leq D$. This results in an optimal order quantity of

$$D_R^* = D. \tag{43}$$

Thus, from the producer-retailer interaction in (40) we finally find as optimal production level of the manufacturer

$$Q_P^* = \begin{cases} Q_P^+ \ \textit{if}\ Q_P^+ \leq D \\ D \quad \textit{if}\ Q_P^+ \geq D \end{cases} \tag{44}$$

Since a comparison of (35) and (41) reveals that $Q_P^+ \leq Q^+$, it is obvious from the production levels in (36) and (44) that always $Q_P^* \leq Q^*$ holds. This means that also under *IG* yield the production level and hence the supply chain safety stock is smaller in a decentralized setting than under central decision making, except for $w = p$ where the results are identical. As consequence, also in a decentralized supply chain the safety stock is negative (i.e. $SST^D < 0$), and because of $Q_P^* \leq D_R^* = D$ neither the producer nor the retailer will hold any physical inventory ($IOH_P^D = IOH_R^D = 0$).

The properties concerning the impact of product profitability on production level and safety stock that were found for the centralized supply chain carry over to the decentralized setting if *w/c* is interpreted as profitability measure instead of *p/c*. Here, the lower bound on *w/c* is determined from the respective zero profit condition like in the centralized case at a production level of Q_P^+ instead of Q^+. Thus, different from the situation under *SP* and *BI* yield, with increasing wholesale price w in its feasible range $c/\theta \leq w \leq p$ the course of production and safety stock does not have a U-shape, but is characterized by a steady increase in the *IG* yield case until a maximum is reached when *w/c* exceeds the critical profitability $\pi_D$in (37). Furthermore, also under decentralization of decision making the fill rate *fr* can be calculated like in (39) with Q_P^* as production level Q. So the minimum service level is given for minimum product profitability $1/\theta$, and the maximum level holds for all cases with $w/c \geq \pi_D$.

These general results are illustrated by a numerical example where, except for the yield parameter θ, the same data are chosen like for the *SP* and *BI* yield examples. In order to report reasonable numerical results a high success probability of $\theta = 0.98$ is chosen for the *IG* example. The respective results (including fill rate data) for this example are presented in Table 3 and confirm that the safety stocks always remain below zero resulting in fill rates that range between 24 % for $w = 2$ and 43 % as highest level that is reached for $w/c > \pi_D = 7.7$. Even for the highest feasible wholesale price value this upper level will not be exceeded. The minimum wholesale

Table 3 Decisions and stocks in a decentralized supply chain for *IG* yield

	w	2	3	4	5	6	7	8	9	10	11	12	13	14
IG	Q_P	34	54	68	79	88	96	100	100	100	100	100	100	100
Yield	D_R	100	100	100	100	100	100	100	100	100	100	100	100	100
$\Theta = 0.98$	***SST***	**−76**	**−67**	**−63**	**−61**	**−59**	**−58**	**−57**	**−57**	**−57**	**−57**	**−57**	**−57**	**−57**
	fr (in%)	**24**	**33**	**37**	**39**	**41**	**42**	**43**	**43**	**43**	**43**	**43**	**43**	**43**

price level that guarantees non-negative profits is somewhat lower than $w = 2$ and can be calculated to be $w = 1/0.98 = 1.02$.

The safety stock analysis for situations with *IG* yield raises some critical questions with regard to the relevance of the modelled decision problem. The general negativity of safety stock values and especially the continuously decreasing service level for increasing demand, even for extremely high product profitability, make it doubtful that one would face such a decision context widely in practice. In the above example with a demand of $D = 100$ it needs a success probability of $\theta = 0.999$ to guarantee a fill rate of 95 %. So, under both centralized and decentralized supply chain conditions the solution of the modelled optimization problem will only lead to acceptable solutions if a very high process quality is existing and/or if demand has only a fairly low level. Under other planning conditions in case of *IG* yield the production process should be organized in such a way that more than a single production run per period can be carried out to satisfy the demand of a decision period. This facilitates the execution of smaller production batches with lower risk of large yield losses and helps to guarantee arbitrarily high service levels in cases of high product profitability. An overview of approaches that optimize the number and batch size of production runs for a given fixed cost per run is found in Grosfeld-Nir and Gerchak (2004).

6 Insights and Future Research

There is a bunch of findings that emerge from the above analysis of yield randomness and its impact on safety stock holding in supply chains under both central and local planning conditions. A first major insight is that it is not only the degree of yield risk but, even more importantly, the type of yield randomness that matters.

In general, in a centralized setting the safety stock size increases steadily with an increasing profit margin of the product. For *IG* yield, however, this increase is strictly limited because here production never exceeds demand so that the safety stock will never take on a positive value. Thus, under *IG* yield the service level might remain at a very low size even if the product profitability is extremely high. With *SP* and *BI* yield negative safety stock will only occur if the profit margin is relatively low. The safety inventory will always be positive if a critical profitability level is exceeded. Furthermore, the impact of demand is even more diverse for the different yield types. With increasing demand level the safety stock in continuously increasing or, in case of negative value, decreasing under *SP* yield while this stock tends to approach zero under *BI* yield. In the case of *IG* yield the safety stock will only change with increasing demand if the product profitability is sufficiently high.

In a decentralized setting the supply chain safety stock is directly determined by the manufacturer's production decision. Under all yield types this stock is lower than the respective safety level under central decision making as long as price and cost parameters are such that both supply chain members make positive profits.

This is a consequence of the double marginalization effect that is also well-known from other supply chain problem areas if a simple wholesale price contract is applied. Although the manufacturer's production problem has the same structure as the decision problem of the central supply chain planner, the general safety stock properties from the centralized setting do not simply carry over. This is because the manufacturer's response to the retailer's order is anticipated and causes a retailer's reaction that results in an order size that might exceed the external supply chain demand. One faces such a situation if from retailer's view the product profitability is relatively high. This effect is never found under *IG* yield, but it exists in case of *SP* and *BI* yield so that under these yield types an increase of the manufacturer's product margin (in terms of the ratio of wholesale price and unit production cost) will not necessarily lead to an increase of production and safety stock level, but can even result in a decrease. Interestingly, the distribution of the stock on hand between the supply chain members differs considerably with respect to the relative product profitability which depends on the wholesale price level. While the stock on hand concentrates on the retailer side for a low manufacturer's and high retailer's margin the opposite holds if the margin relationship is the other way around.

Further research is necessary in order to reveal to which extent the above insights carry over to more complex supply chain structures like those with several stages and multiple producers or retailers. The same holds for the investigation of situations where yield risks come along with demand uncertainty. Up to now, studies that address these cases like those by Gurnani and Gerchak (2007) and by Güler and Bilgic (2009), for example, only refer to problems with *SP* yield and need to be extended to the other yield types. An extension of the above research to multi-period problems can be based on already existing studies for centralized supply chains (see Inderfurth and Vogelgesang 2013), but needs the solution of highly complex game-theoretic problems in the case of local supply chain decision making. A very valuable extension would lie in the consideration of multiple production runs, particularly in situations with *IG* yield where the safety stock analysis in this paper has only limited practical relevance for cases with low process quality and high product margin. For this problem type solutions only exist for centrally coordinated supply chains (see Grosfeld-Nir and Gerchak 2004) while it is a major challenge to solve these problems in a decentralized setting with producer-retailer interaction. Finally, an additional field for future research is given if the current research is extended to manufacturer-retailer interactions that base upon more complex contract types than the simple wholesale price contract. For many problem areas a large variety of contracts has been analyzed which aim to coordinate local supply chain decisions to the optimal central solution (see Cachon 2003). From recent research contributions addressing the above random yield problem (see Inderfurth and Clemens 2014; Clemens and Inderfurth 2015) it is known that two contract types, namely the so-called overproduction risk sharing and the penalty contract, achieve coordination if their parameters are chosen such that the retailer always is incentivized to place orders according to the external demand. For these conditions the safety stock is equal to that of the centralized solution in case of *SP* and *BI* yield also under decentralized decision making. It is not clear, however, if this result also holds for other contract types that enable coordination like, for instance, the pay-back-revenue-sharing contract proposed in Tang and Kouvelis (2014). Furthermore, it is completely unknown which types of contracts will support coordination if *IG* yield is considered and how they affect safety stock management in the supply chain.

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Dynamic Pricing Under Economic Ordering and Strategic Customers with Forward Buying and Postponement

Stefan Minner

Abstract Dynamic pricing is an instrument to achieve operational efficiency in inventory management. We consider an economic ordering context with strategic customers who forward-buy or postpone their purchases in anticipation of price changes. By charging a lower price when inventories are high, significant profit improvements can be achieved. The paper analyzes a simple EOQ-type model with a single price change within each inventory cycle. Numerical examples illustrate its impact on profits, order cycle duration and optimal price discounts. In particular for slow moving and low margin products, improvements in retail-type environments are substantial.

1 Introduction and Motivation

Integrated pricing and ordering is a prominent topic at the marketing-operations interface. Nevertheless, both decisions are still frequently treated independently, each making simplistic assumptions with regard to the other domain. In pricing models, the operations domain is often represented by constant marginal costs, whereas demand in operations models is often assumed to be (probabilistically) known but independent of marketing activities such as advertising and pricing (Feichtinger et al. 1994). Research on integrated models shows the benefits of correctly anticipating the true operations cost function when optimizing prices (Thomas 1970; Kunreuther and Schrage 1973; Eliashberg and Steinberg 1993; Chan et al. 2004). With the rise of using revenue management ideas in operations contexts, dynamic pricing has received more attention (Elmaghraby and Keskinocak 2003). Feichtinger and Hartl (1985) analyze such an integrated production and pricing model. Rajan et al. (1992) and Transchel and Minner (2009) derive the optimal dynamic pricing and ordering decision in the context of the Economic Order Quantity (EOQ) model using an optimal control model. For an overview

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on the extensive EOQ literature, see Glock et al. (2014). The rationale of the analysis in Transchel and Minner (2009) (under price dependent demand) is that the optimal price charged increases over the inventory cycle as sold items remain longer in inventory, i.e. larger sales are triggered when inventory is high. One major limiting assumption of these investigations is the myopic behavior of the customers. Recently, the incorporation of strategically acting customers has received more attention (Aviv and Pazgal 2008; Cachon and Swinney 2009; Swinney 2011), see Gönsch et al. (2013) for a recent review. If the dynamic price trajectory is known to the customers, dynamically changing prices will trigger forward buying by some customers before prices increase and postponement by some customers who anticipate a lower price in the near future. Ramasesh (2010) provides an overview on lot-sizing models with temporary price incentives for the purchasing company. This manuscript analyzes the impact of strategic customers in a profit maximization EOQ context.

The manuscript is organized as follows. In Sect. 2, we introduce the model to be analyzed in Sect. 3. Numerical examples on optimal pricing and ordering are presented in Sect. 4, before Sect. 5 concludes the manuscript.

2 Model Assumptions

We assume the environment of a wholesaler or retailer in an Economic Order Quantity (EOQ) model with a known constant demand rate d being independent of sales prices. This assumption is motivated by a competitive market with customers having chosen a preferred supplier and a stable regular price. Demands need to be satisfied and backlogging of customer demands at the retailer is not permitted. The regular sales price p_2 is therefore an exogenous parameter. Products are replenished from an external supplier in lots of size Q at a unit procurement cost of c and a fixed cost A per order. All orders are delivered after a known constant lead time (in the following, without loss of generality, the lead time is zero). Units not sold directly are kept in inventory at a unit holding cost h_R per unit of time.

In order to reduce holding costs when inventories are large, in particular after receiving a large order, a reduced price $p_1 \leq p_2$ is offered for a time interval of length t_1. For the remaining inventory cycle between two consecutive replenishments t_2, the regular price p_2 is charged. The full inventory cycle length is therefore $T = t_1 + t_2$.

Although customers have constant needs over time, represented by the aggregate demand rate d, they are willing to buy in advance if the received price discount exceeds their holding cost. Let h denote the customer holding cost rate per unit per unit of time. Furthermore, customers are willing to postpone their purchase if the received price discount exceeds their internal backordering cost. Let v denote the backordering cost rate per unit per unit of time. For simplicity we assume that customers are homogenous with regard to their cost parameters. The backlogging (postponement) time interval t_B is determined by a customer being indifferent between (1) postponing the procurement in anticipation of a price decrease or (2) buying on demand, i.e. $t_B v = p_2 - p_1$. The forward buying time interval t_F is determined by a customer being indifferent between (1) buying units in advance of the requirement in anticipation of a price increase or (2) buying on demand, i.e. $t_F h = p_2 - p_1$. This changes the upfront assumption of a constant demand rate which now becomes dynamic, however, with the same long-run average.

Units are replenished in batches of size $Q = dT$ every $T = t_1 + t_2$ periods and the objective is to maximize the average profit per unit of time over an infinite planning horizon. The respective decision variables are the discounted price p_1 and the periods t_1 and t_2 where the different prices are charged (or implicitly the cycle length T).

3 Optimal Discount and Order Quantity

Figure 1 shows the price and inventory level development over time. When an order of size Q arrives, the backordered amount $q_B = dt_B$ is immediately delivered to the customers and not stored in inventory. During time interval t_1, products are sold at rate d and reduced price p_1. Just prior to the price increase from p_1 to p_2, forward buying customers take the amount $q_F = dt_F$ and therefore there will be no sales thereafter during a time interval of length $t_F = \frac{p_2 - p_1}{h}$. For those customers for whom it is not economical to buy at the discounted price, sales resume at rate d at time $t_1 + t_F$ at the regular price p_2 for a period of length $t_2 - t_B - t_F$ before customers will stop buying because they postpone their purchase until the next price discount for an interval of length $t_B = \frac{p_2 - p_1}{v}$.

Using these dynamics, we can express the sales at different prices and the resulting direct and overhead costs for ordering and holding inventory as follows. The sales at the two prices are

- at price p_1: $s_1 = d(t_1 + t_F + t_B)$,
- at price p_2: $s_2 = d(T - (t_1 + t_F + t_B))$.

The inventory over a single cycle (obtained from the two triangles and the two boxes in Fig. 1) are

$$Inv(p_1, t_1, T) = \frac{1}{2} d(t_1^2 + (T - t_1 - t_F - t_B)^2) + t_1(s_2 + q_F) + t_F s_2. \tag{1}$$

Then, the average profit per unit of time for the retailer, which consists of variable procurement costs, fixed ordering costs, sales revenues, and inventory holding costs, is

$$\begin{aligned}\Pi = -cd + \frac{1}{T}\Big(& -A + p_1 d \left(t_1 + \frac{p_2 - p_1}{h} + \frac{p_2 - p_1}{v}\right) \\ & + p_2 d \left(T - t_1 - \frac{p_2 - p_1}{h} - \frac{p_2 - p_1}{v}\right) - h_R \cdot Inv(p_1, t_1, T)\Big)\end{aligned} \tag{2}$$

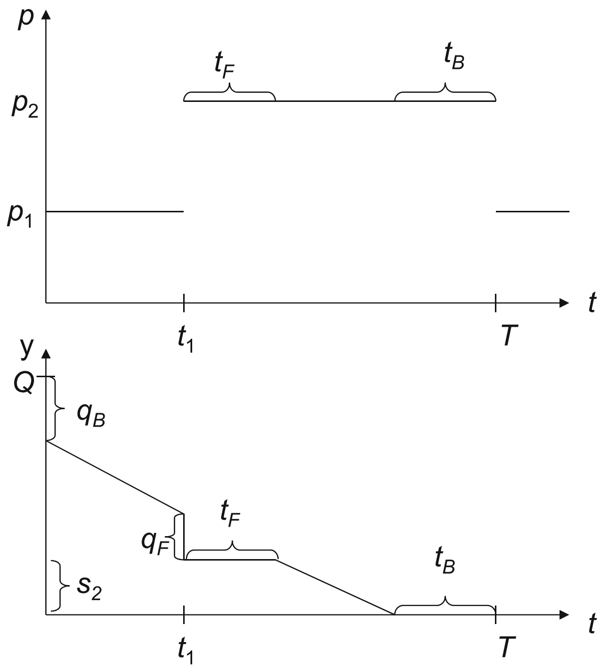

Fig. 1 Price and inventory dynamics

subject to the constraints

$$t_1 + \frac{p_2 - p_1}{h} + \frac{p_2 - p_1}{v} \leq T \Leftrightarrow t_2 \geq t_F + t_B \tag{3}$$

$$t_1 \geq 0,\ t_2 \geq 0,\ \Delta = p_2 - p_1 \leq p_2 - c. \tag{4}$$

The constraints ensure non-negativity of time intervals, an upper bound on the discount and that the regular sales time interval has to include the forward buying and postponement time intervals.

Inserting definitions and rearranging terms yields

$$\begin{aligned}\Pi = (p_1 - c)d + \frac{d(p_2 - p_1)}{t_1 + t_2}\left(t_2 - (p_2 - p_1)\left(\frac{1}{v} + \frac{1}{h}\right)\right) - \frac{A}{t_1 + t_2} \\ -\frac{h_R d}{t_1 + t_2}\left(\frac{1}{2}t_1^2 + \frac{1}{2}\left(t_2 - (p_2 - p_1)\left(\frac{1}{v} + \frac{1}{h}\right)\right)^2 \right. \\ \left. + \left(t_1 + \frac{p_2 - p_1}{h}\right)\left(t_2 - \frac{p_2 - p_1}{v}\right) - \left(\frac{p_2 - p_1}{h}\right)^2\right).\end{aligned} \tag{5}$$

The simplified profit function using the discount variable $\Delta = p_2 - p_1$ is

$$\Pi = (p_2 - c - \Delta)d + \frac{d\Delta}{T}\left(T - t_1 - \Delta\left(\frac{1}{v} + \frac{1}{h}\right)\right)$$
$$-\frac{A}{T} - \frac{h_R d}{2}\left(T - \frac{2\Delta}{v} + \frac{\Delta^2}{T}\left(\frac{1}{v^2} - \frac{1}{h^2}\right)\right). \tag{6}$$

Analyzing (6), we observe the following property. The profit function is decreasing in t_1:

$$\frac{\partial \Pi}{\partial t_1} < 0 \Rightarrow t_1^* = 0 \tag{7}$$

This implies that the discount price p_1 is only offered at the time of order arrival where the largest inventory reduction through forward buying and postponement can be achieved and that it is not beneficial to maintain the lower price for a longer time interval.

As a consequence, we can reduce the problem to only two remaining decision variables: the cycle length T and the price discount $\Delta = p_2 - p_1$. Next, we will analyze this problem by first relaxing the constraints and later showing that the free optimal solution either satisfies them or not. In the latter case, the bounded case is analyzed separately. The optimality conditions for the reduced problem are

$$\frac{\partial \Pi}{\partial \Delta} = -d + \frac{d}{T}\left(T - \Delta\left(\frac{1}{v} + \frac{1}{h}\right)\right) - \left(\frac{1}{v} + \frac{1}{h}\right)\frac{d\Delta}{T}$$
$$+\frac{h_R d}{v} - \frac{h_R d\Delta}{T}\left(\frac{1}{v^2} - \frac{1}{h^2}\right) = 0$$
$$\Leftrightarrow T = \Delta\left(\frac{1}{v} + \frac{1}{h}\right)\left(1 + v\left(\frac{2}{h_R} - \frac{1}{h}\right)\right), \tag{8}$$
$$\frac{\partial \Pi}{\partial T} = \frac{d\Delta^2}{T^2}\left(\frac{1}{v} + \frac{1}{h}\right) + \frac{A}{T^2} - \frac{1}{2}h_R d + \frac{1}{2}h_R d\Delta^2\left(\frac{1}{v^2} - \frac{1}{h^2}\right)\frac{1}{T^2} = 0$$
$$\Leftrightarrow T^2 = \frac{2A}{h_R d} + \Delta^2\left(\frac{1}{v} + \frac{1}{h}\right)\left(\frac{2}{h_R} + \frac{1}{v} - \frac{1}{h}\right). \tag{9}$$

From (8) we observe that the time interval constraint (3) is satisfied if $h_R \leq 2h$, i.e., the retailer's holding cost is less than twice the customer holding cost. Otherwise, it is optimal to set the discount such that the profit is maximized and no inventories are kept (zero inventory case). In this case, $T = \Delta\left(\frac{1}{v} + \frac{1}{h}\right)$ and we maximize the following (concave) function:

$$\Pi = (p_2 - c - \Delta)d - \frac{A}{\Delta\left(\frac{1}{v} + \frac{1}{h}\right)} \tag{10}$$

with optimal solution

$$\Delta^* = \sqrt{\frac{A}{d\left(\frac{1}{v} + \frac{1}{h}\right)}}. \tag{11}$$

The reduced objective function (6) is (jointly) concave in the relevant region because $\frac{\partial^2\Pi}{\partial T^2} < 0$, $\frac{\partial^2\Pi}{\partial \Delta^2} < 0$, and $\frac{\partial^2\Pi}{\partial T^2}\frac{\partial^2\Pi}{\partial \Delta^2} - \left(\frac{\partial^2\Pi}{\partial T \partial \Delta}\right)^2 > 0$ for $h_R < 2h$ and we have a unique profit maximum.

$$\frac{\partial^2\Pi}{\partial T^2} = -\frac{d\Delta^2}{T^3}\left(\frac{1}{v} + \frac{1}{h}\right)\left(2 + \frac{h_R}{v} - \frac{h_R}{h}\right) - \frac{2A}{T^3} \tag{12}$$

$$\frac{\partial^2\Pi}{\partial \Delta^2} = -\frac{d}{T}\left(\frac{1}{v} + \frac{1}{h}\right)\left(2 + \frac{h_R}{v} - \frac{h_R}{h}\right) \tag{13}$$

$$\frac{\partial^2\Pi}{\partial T \partial \Delta} = \frac{d\Delta}{T^2}\left(\frac{1}{v} + \frac{1}{h}\right)\left(2 + \frac{h_R}{v} - \frac{h_R}{h}\right) \tag{14}$$

The final solution for the unconstrained case is

$$T^* = \sqrt{\frac{2A}{h_R d}\left(1 - \frac{1}{\left(1 + \frac{2v}{h_R} - \frac{v}{h}\right)\left(1 + \frac{v}{h}\right)}\right)^{-1}}, \tag{15}$$

$$\Delta^* = \frac{vT^*}{\left(1 + \frac{2v}{h_R} - \frac{v}{h}\right)\left(1 + \frac{v}{h}\right)}. \tag{16}$$

As a consequence of reducing inventory costs through dynamic pricing, the optimal inventory cycle T^* is larger than the corresponding one of the EOQ model, $EOI = \sqrt{\frac{2A}{h_R d}}$.

4 Numerical Illustrations

In order to illustrate the impact of the price discount and the strategic customer behavior, we assume the following basic data set: $d = 100$, $p_2 = 10$, $A = 9$, $c = 8$, $h_R = 2$, $h = 2$, and $v = 2$. This data yields an $EOQ = 30$ and a quarterly Economic Order Interval $EOI = 0.3$. Annual revenues are 1000, direct costs 800 and ordering costs 60. This yields a profit of 140. We introduce the following three benchmarks for comparison:

- EOQ model without dynamic pricing

  In order to be profitable, the following condition has to hold:

$$(p_2 - c)d - \sqrt{2dAh_R} \geq 0 \Leftrightarrow \sqrt{\frac{2A}{h_R d}} \leq \frac{p_2 - c}{h_R}. \tag{17}$$

  The economic interpretation is that the contribution margin with the regular price needs to be larger than storing a (marginal) unit over the economic order interval.

- Zero inventory model with economic order interval (EOI)

  Here, we keep the optimal Economic Order Interval, but set the price discount such that no inventories are held, i.e. the inventory cycle only consists of the forward buying and the postponement time interval.

$$EOI = T = \Delta\left(\frac{1}{v} + \frac{1}{h}\right) \Leftrightarrow \Delta = \sqrt{\frac{2A}{h_R d}}\frac{hv}{h+v} \tag{18}$$

  Note that this strategy with a profit $\Pi(EOI)$ does not necessarily need to be beneficial, that is, it might yield a lower profit than the EOQ.

- Optimal zero inventory (ZI) model (see special case (11) above).

  Compared to the previous benchmark, we optimize the cycle length. Again, also this benchmark with a profit of $\Pi(ZI)$ does not need to be beneficial compared to the EOQ.

Furthermore, we also report the customer benefit W from exploiting the temporary price discounts, i.e. the price savings minus holding and postponement costs.

$$W = (p_2 - p_1)d(t_1 + t_F + t_B) - \frac{d}{2}\left(\frac{ht_F^2}{T} + \frac{vt_B^2}{T}\right) \tag{19}$$

Table 1 shows the results when varying the postponement parameter v from 0.25 to 4. This reflects a different willingness of postponing purchases to the time of the next incoming order, from high willingness to low. If there is a high willingness to postpone for small v, there is a considerable improvement in profits (comparing Π^* to $\Pi(EOQ)$) and the possibility to enlarge the order cycle accordingly. We observe that the order cycle, however, decreases in v and converges to EOI. The discount Δ does not show a monotonic behavior: it first increases and then decreases, whereas the discounts required to achieve a zero inventory policy always increase with v. This effect is shown in Fig. 2. With less willingness to postpone, larger incentives need to be provided. This effect, however, is overcompensated at a certain point by restricting the order cycle decrease. The customer benefit W shows a similar pattern, it is first increasing and then decreasing in v.

Table 1 Numerical results for a variation of the postponement parameter v

v	EOI	T^*	$T(ZI)$	Δ^*	$\Delta(EOI)$	$\Delta(ZI)$	$\Pi(EOQ)$	Π^*	$\Pi(EOI)$	$\Pi(ZI)$	W
0.25	0.30	0.65	0.64	0.13	0.07	0.14	140.00	172.51	163.33	171.72	7.19
0.50	0.30	0.50	0.47	0.16	0.12	0.19	140.00	164.00	158.00	162.05	9.60
0.75	0.30	0.44	0.41	0.17	0.16	0.22	140.00	158.82	153.64	155.69	11.03
1.00	0.30	0.40	0.37	0.18	0.20	0.24	140.00	155.28	150.00	151.01	11.93
1.25	0.30	0.38	0.34	0.18	0.23	0.26	140.00	152.71	146.92	147.38	12.47
1.50	0.30	0.37	0.32	0.18	0.26	0.28	140.00	150.76	144.29	144.45	12.79
1.75	0.30	0.35	0.31	0.18	0.28	0.29	140.00	149.25	142.00	142.03	12.95
2.00	0.30	0.35	0.30	0.17	0.30	0.30	140.00	148.04	140.00	140.00	12.99
2.25	0.30	0.34	0.29	0.17	0.32	0.31	140.00	147.06	138.24	138.26	12.96
2.50	0.30	0.33	0.28	0.17	0.33	0.32	140.00	146.25	136.67	136.75	12.86
2.75	0.30	0.33	0.28	0.16	0.35	0.32	140.00	145.58	135.26	135.44	12.73
3.00	0.30	0.33	0.27	0.16	0.36	0.33	140.00	145.01	134.00	134.27	12.57
3.25	0.30	0.32	0.27	0.15	0.37	0.33	140.00	144.52	132.86	133.24	12.39
3.50	0.30	0.32	0.27	0.15	0.38	0.34	140.00	144.11	131.82	132.31	12.19
3.75	0.30	0.32	0.26	0.15	0.39	0.34	140.00	143.75	130.87	131.48	11.99
4.00	0.30	0.32	0.26	0.14	0.40	0.35	140.00	143.43	130.00	130.72	11.79

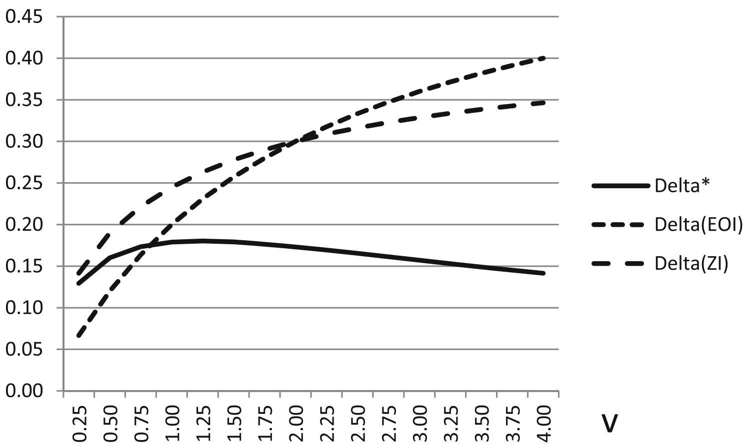

Fig. 2 Price and inventory dynamics

Table 2 shows the corresponding analysis when the forward buying parameter h i varied. For $h < 1$, the optimal solution is a zero inventory policy and therefore, these two solutions coincide. For $h \leq h_R$, i.e. the customer can store cheaper than the retailer, the order cycle T decreases, whereas in the opposite case, it (slightly) increases. The customer benefits are increasing in h. In the optimal zero inventory case, the optimal discount first increases and then shows a decreasing–increasing pattern as explained already for the postponement parameter variation.

Table 2 Numerical results for a variation of the forward buying parameter h

h	EOI	T^*	$T(ZI)$	Δ^*	$\Delta(EOI)$	$\Delta(ZI)$	$\Pi(EOQ)$	Π^*	$\Pi(EOI)$	$\Pi(ZI)$	W
0.25	0.30	0.64	0.64	0.14	0.07	0.14	140.00	171.72	163.33	171.72	7.07
0.50	0.30	0.47	0.47	0.19	0.12	0.19	140.00	162.05	158.00	162.05	9.49
0.75	0.30	0.41	0.41	0.22	0.16	0.22	140.00	155.69	153.64	155.69	11.08
1.00	0.30	0.37	0.37	0.24	0.20	0.24	140.00	151.01	150.00	151.01	12.25
1.25	0.30	0.35	0.34	0.19	0.23	0.26	140.00	148.90	146.92	147.38	12.44
1.50	0.30	0.35	0.32	0.18	0.26	0.28	140.00	148.29	144.29	144.45	12.53
1.75	0.30	0.35	0.31	0.17	0.28	0.29	140.00	148.08	142.00	142.03	12.73
2.00	0.30	0.35	0.30	0.17	0.30	0.30	140.00	148.04	140.00	140.00	12.99
2.25	0.30	0.35	0.29	0.17	0.32	0.31	140.00	148.07	138.24	138.26	13.27
2.50	0.30	0.35	0.28	0.18	0.33	0.32	140.00	148.13	136.67	136.75	13.54
2.75	0.30	0.35	0.28	0.18	0.35	0.32	140.00	148.20	135.26	135.44	13.81
3.00	0.30	0.35	0.27	0.18	0.36	0.33	140.00	148.29	134.00	134.27	14.06
3.25	0.30	0.35	0.27	0.18	0.37	0.33	140.00	148.37	132.86	133.24	14.31
3.50	0.30	0.35	0.27	0.18	0.38	0.34	140.00	148.46	131.82	132.31	14.53
3.75	0.30	0.35	0.26	0.18	0.39	0.34	140.00	148.54	130.87	131.48	14.75
4.00	0.30	0.35	0.26	0.19	0.40	0.35	140.00	148.62	130.00	130.72	14.95

Table 3 Numerical results for a variation of the Economic Order Interval through A

A	EOI	T^*	Δ^*	$\Pi(EOQ)$	Π^*	%	W	%
1	0.10	0.12	0.06	180.00	182.68	1.49	4.33	0.43
4	0.20	0.23	0.12	160.00	165.36	3.35	8.66	0.87
9	0.30	0.35	0.17	140.00	148.04	5.74	12.99	1.32
16	0.40	0.46	0.23	120.00	130.72	8.93	17.32	1.76
25	0.50	0.58	0.29	100.00	113.40	13.40	21.65	2.21
100	1.00	1.15	0.58	0.00	26.79	∞	43.30	4.53

Table 3 provides a sensitivity analysis for different Economic Order Intervals through varying the fixed cost A. Even though overall profits decrease as overhead costs for ordering and keeping inventory increase, the relative profit improvement through dynamic pricing increases. We even observe a case with $A = 100$ where a zero profit product can be turned into a positive one. The customer benefit W, due to increasing price incentives, increases in A.

The final sensitivity analysis when varying the profitability of a product through the procurement cost parameter c in Table 4 shows a similar behavior. Note that customer benefits are not affected by a change in c. Although overall profits decrease, the improvement percentage of dynamic pricing increases considerably and therefore shows significant benefits in particular for low-margin products, which is a common case in retailing.

Table 4 Numerical results for a variation of profitability through c

c	$\Pi(EOQ)$	Π^*	%
5	440	448.04	1.83
5.5	390	398.04	2.06
6	340	348.04	2.36
6.5	290	298.04	2.77
7	240	248.04	3.35
7.5	190	198.04	4.23
8	140	148.04	5.74
8.5	90	98.04	8.93
9	40	48.04	20.10

5 Conclusions

The simple integrated pricing and ordering model shows the benefits of a cost-oriented pricing under the presence of strategic customers, i.e. trigger in-advance purchasing when inventories are high and postponement to achieve larger economies of scale in ordering. The optimal solution of this very simple model, with a single (impulse) price discount at the order arrival time, provides coordination of procurement activities from a supply chain perspective and associated benefits for coordinating the procurement timing through forward buying and postponement in this context. The numerical examples highlight the considerable profit improvement potential through temporal (dynamic) price-differentiation in business-to-customer environments. The approach in particular yields improvements for slow moving items like spare parts, low margin products in retailing, or commodity manufacturing processes in the process industries with large lot-sizes and resulting inventories.

Because of the limiting assumptions of the model, several extensions are in line for future research. Instead of having a single price change, multiple or even continuous pricing are an option for further improvement. Other forms of price discounts (see e.g. Transchel and Minner 2008) also seem to be promising. So far, we assumed that customers are homogenous with respect to their demands and holding and backordering costs. Another natural extension would be to assume stochastic demand and a certain distribution of forward buying and postponement parameters among the group of customers and to further assume a general price dependency of demand through a price-response function, rather than having demand and the regular price being exogenously given. Finally, in the particular context of perishable products with a limited shelf-life (for a review see Karaesmen et al. 2011), dynamic pricing and ordering has the potential to considerably improve operations and profitability through age-category differentiated pricing (for a dynamic modelling idea, see Haurie et al. 1984) or to be integrated with delivery scheduling and second order opportunities (see Turan et al. 2015).

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The Effect of Remanufacturability of End-of-Use Returns on the Optimal Intertemporal Production Decisions of a Newsvendor

Marc Reimann

Abstract In this paper we study a joint manufacturing–remanufacturing problem of a manufacturer under demand uncertainty. Supply of remanufacturable units is constrained by the availability of used and returned cores, which depends on previous supply of new units. Potential cost savings due to remanufacturing in later periods may induce the manufacturer to reduce its short-term profits by artificially increasing its supply in earlier periods. For dealing with this trade-off we formulate an intertemporal optimization problem in a classical two-period newsvendor setting. Exploiting first period information when taking second period supply decisions we provide analytical insights into the optimal strategy and compare this optimal strategy with a previously proposed heuristic, where both first and second period decisions are committed to prior to period 1. Through a comprehensive numerical study we evaluate the associated profit implications.

1 Introduction and Related Work

Closed-loop supply chains have gained increased attention from both industry and academia over the last two decades. Companies from different industries, like automotive or consumer electronics, have identified the need for and potential of integrating their forward supply chain processes with the reverse flows of products. Take-back legislation for end-of-use and end-of-life items makes companies responsible for their products beyond the sales to their consumers. A typical example is the Waste Electrical and Electronic Equipment (WEEE) directive (Recast of the WEEE Directive 2014) in the EU dealing with electric and electronical devices post-use. On the other hand, companies like HP and IBM have understood the effects of return volumes exceeding 10 % of outgoing sales on cost and potential profits

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associated with efficient CLSC processes [see e.g. Fleischmann (2001), Guide and Van Wassenhove (2009) and Guide et al. (2006)].

The evolution of research in the domain of CLSCs has been laid out in Guide and Van Wassenhove (2009). More recently, in Souza (2013) a comprehensive review of the CLSC literature from a modelling perspective is given. The problem considered in this paper is most closely related to a stream of research dealing with product returns and the relationship between new and remanufactured products. A group of papers focuses on the demand side interaction of new and remanufactured products while ignoring their supply side link. Using dynamic programming the optimal manufacturing and remanufacturing decisions under perfect substitutability are studied in Li et al. (2010). With a focus on supply and demand uncertainty, robust optimization is applied in Wei et al. (2011). In Shi et al. (2011) a stochastic model for deciding optimal production and remanufacturing quantities for a product portfolio is presented. The optimal acquisition and remanufacturing policies of a model with uncertain quality of returns are determined in Teunter and Flapper (2011). The case of downward substitution between new and remanufactured products is studied in Inderfurth (2004). As mentioned above, none of these papers addresses explicitly the link between previous sales of new products and associated returns of used products.

An early paper focusing on this link is Kelle and Silver (1989) for the case of planning reusable containers. Returns of containers are stochastic but depend on past sales, and due to loss sometimes new containers must be acquired. The different quality of returns is considered in a single-product, finite multi-period inventory model with stochastic demands in Zhou et al. (2011). Dynamic pricing for influencing the uncertain supply of used products in a production-remanufacturing model with random customer demands is introduced in Zhou and Yu (2011). In Ferrer and Swaminathan (2006, 2010) the optimal supply quantities of (imperfectly) substitutable new and remanufactured products are derived in a deterministic monopoly setting as well as under simple demand competition.

A model extending this literature by jointly considering uncertain product demand, manufacturing and remanufacturing decision making, as well as the explicit link between sales in earlier periods and subsequent availability of returns for remanufacturing was presented in Reimann and Lechner (2012). In a setting, where returned and remanufactured end-of-use products can be used as a less costly but perfect substitute for newly manufactured units, a two-period newsvendor problem is investigated. In the first period, the manufacturer can supply new products anticipating the uncertain demand. In the second period the manufacturer can choose a supply mix of new production and remanufacturing returned cores for satisfying the uncertain demand. Under the assumption that all first and second period production decisions have to be taken at the beginning of period 1, it is shown that the option to remanufacture may induce the manufacturer to give up first period profits for improving overall profitability. Specifically, first period supply will in general exceed the optimal myopic newsvendor quantity in order to stimulate (expected) sales and consequently enlarge the supply of returned remanufacturable cores. Thereby second period profitability is improved, since less or no production of new units will be necessary, and second period supply and expected sales are increased.

Note that the assumption about the timing of all the supply decisions prior to the first period can in principle be justified by lead times or by the necessity to reserve capacity in advance. However, particularly when the two model periods correspond to two generations of a product such a long term commitment may be impractical. Rather, the manufacturer may be able to decide its second generation supply mix after observing the first period demand.

Thus, in this paper we revisit the model presented in Reimann and Lechner (2012) but allow the manufacturer to take its second period supply decisions upon observing the first period outcome. We derive the manufacturers optimal policy and compare it with the results from Reimann and Lechner (2012). Besides analytical structural results we provide numerical insights into the profit gains associated with the delay of the second period decisions.

The remainder of the paper is organized as follows. Section 2 deals with the formal model definition and the theoretical insights. Results from a numerical study are presented in Sect. 3. Section 4 concludes the paper with a short summary and an outlook on extensions of the presented work.

2 The Model

We study the supply decisions of a manufacturer over two generations of a product with uncertain demands D_1 and D_2. Demand D_1 for the first generation can only be satisfied through production of new units at cost c_1. All production has to take place before uncertainty resolves. The per-unit market price for the first product generation is given by $p_1 > c_1$. Storage costs are assumed to be prohibitive, such that unsold first generation units are not available in period two. At the end of the first generation's lifecycle an exogeneously given fraction $\gamma > 0$ of sales will be returned as end-of-use items. These items can be remanufactured and upgraded to as-new second generation units.[1] Besides, the manufacturer can produce new second generation units. The cost of new production is c_2, while remanufacturing and updating can be done at a discounted cost of $c_2 - \delta > 0$, where $\delta > 0$ models the efficiency of remanufacturing over new production. Again these production decisions have to be taken before the demand uncertainty of the second generation resolves. The per-unit market price for the second product generation is given by $p_2 > c_2$. Demand uncertainty for first (second) generation units is captured by the PDF $f_1(D_1)$ ($f_2(D_2)$) and CDF $F_1(D_1)$ ($F_2(D_2)$), with $F_1(0) = F_2(0) = 0$. The manufacturer is risk-neutral and wants to maximize its expected total profit π over

[1] Note that in order to ensure comparability with the original model in Reimann and Lechner (2012) we assume that units can be remanufactured several times.

the two product generations, where the second generation profits are discounted by the factor β, i.e. $\pi = \pi_1 + \beta\pi_2$.

2.1 Timeline of the Model

1. Manufacturer decides first generation new production quantity q_1
2. Realization d_1 of demand D_1
3. Manufacturer incurs sales $s_1 = \min\{q_1, d_1\}$ and at the end of the lifecycle associated returns γs_1
4. Manufacturer decides its remanufacturing quantity $q_2^r \leq \gamma s_1$ and second generation new production quantity q_2
5. Realization d_2 of demand D_2
6. Manufacturer incurs sales $s_2 = \min\{q_2 + q_2^r, d_2\}$

2.2 Model Formulation

Let

$$S_1 = \int_0^{q_1} u f_1(u) du + q_1[1 - F_1(q_1)] \tag{1}$$

and

$$S_2 = \int_0^{q_2+q_2^r} u f_2(u) du + (q_2 + q_2^r)[1 - F_2(q_2 + q_2^r)] \tag{2}$$

denote the expected sales of generation 1 and generation 2 units, respectively. Given the timeline of decisions, the optimization problem can be defined as follows:

$$\max_{q_1,q_2,q_2^r} \pi = \pi_1 + \beta\pi_2 = -c_1 q_1 + p_1 S_1 + \beta\left[-c_2\,(q_2 + q_2^r) + \delta\, q_2^r + p_2 S_2\right] \tag{3}$$

s.t.

$$q_2^r \leq \gamma s_1 \tag{4}$$

$$q_2^r \geq 0 \tag{5}$$

$$q_2 \geq 0 \tag{6}$$

Clearly the supply decision of generation 1 units not only influences sales and profits in the first generation, but also the volume of returns available for remanufacturing to as-new second generation units. Since remanufacturing reduces the supply cost of generation 2 items, the manufacturer faces an intertemporal problem in trading off generation 1 and generation 2 profits.

To solve this problem, we exploit the fact that the manufacturer's second generation decisions are taken under knowledge of both q_1 and d_1 and consequently the sales s_1 and associated returns γs_1. Thus, we start with the discussion of the optimal second generation strategy.

2.3 Optimal Supply Strategy for Product Generation 2

It is easy to verify that π_2 is concave in both q_2 and q_2^r. Moreover, since supply q_2^r is less expensive than q_2, absent any constraints it would be optimal to only use remanufactured items and set $q_2 = 0$. Then, the optimal supply of remanufactured items is given by $F_2(q_2^r) = \frac{p_2-(c_2-\delta)}{p_2}$. Let us refer to this maximum total second generation supply quantity as q_{max}.

However, since only returned end-of-use items can be remanufactured and upgraded, this optimal supply may not be feasible. Rather, the supply of remanufactured and upgraded units q_2^r is constrained by the volume of first generation returns as shown in (4).

Finally, observe that when $s_1 = 0$ it follows that $q_2^r = 0$ and the problem reduces to a classical newsvendor problem. In that case the optimal supply of new second generation units is given by $F_2(q_2) = \frac{p_2-c_2}{p_2}$. Let us refer to this minimum total second generation supply quantity as q_{min}.

Using these insights, Proposition 1 describes the optimal supply strategy for product generation 2.

Proposition 1 *For given s_1, the optimal second generation supply strategy induced by maximizing π_2 under constraint (4) can be obtained by applying the KKT conditions and is given by*

Case 1: $\gamma\ s_1 > q_{max}$
Only remanufacturing, ***not*** *all the returned cores are used.*
In this case $q_2^r = q_{max}$ and $q_2 = 0$.
Case 2: $q_{min} \leq \gamma\ s_1 \leq q_{max}$
Only remanufacturing, all the returned cores are used.
In this case $q_2^r = \gamma \cdot s_1$ and $q_2 = 0$.
Case 3: $q_{min} > \gamma\ s_1$
Remanufacturing and new production, all the returned cores are used.
In this case $q_2^r = \gamma \cdot s_1$ and $q_2 = q_{min} - q_2^r$.

All proofs are provided in the appendix. Using these results for q_2 and q_2^r, the expected second generation profit $\pi_{2,i}$ for case i= 1, 2, 3 is given by

$$\pi_{2,1} = p_2 \int_0^{q_{max}} u f_2(u) du, \tag{7}$$

$$\pi_{2,2} = -(c_2 - \delta)\gamma s_1 + p_2 \int_0^{\gamma s_1} uf_2(u)du + p_2\gamma s_1[1 - F_2(\gamma s_1)] \tag{8}$$

$$\pi_{2,3} = \delta\gamma s_1 + p_2 \int_0^{q_{min}} uf_2(u)du. \tag{9}$$

Clearly, the expected second generation profits induced by the optimal second generation decision satisfy $\pi_{2,1} \geq \pi_{2,2} \geq \pi_{2,3}$. Moreover, we know that the total second generation supply is bounded by $q_{min} \leq q_2 + q_2^r \leq q_{max}$.

2.4 *Optimal Supply Strategy for Product Generation 1*

Using the results for the optimal supply for product generation 2 we can now address the decision problem for the first generation product supply q_1.

As shown above, the optimal second generation strategy depends on the interaction between q_1 and d_1 through $s_1 = \min\{q_1, d_1\}$. Depending on the actual level of q_1, the optimal second generation strategy for the same level of d_1 may change.

Thus, while for a given q_1 we can specify the overall expected profit π as a function of d_1, the actual shape of this function will depend on q_1. Moreover, d_1 and consequently s_1 are unknown at the time first generation supply q_1 is decided.

To handle this, we can use the following insight. Observe that Case 1 described in Proposition 1 above can only occur when $\gamma s_1 > q_{max}$. Since $s_1 = \min\{q_1, d_1\}$, this is only possible if $\gamma q_1 > q_{max}$. Similarly, Case 2 can only happen if $\gamma q_1 \geq q_{min}$. These observations give rise to three regions over q_1, for which specific profit functions π can be defined. Proposition 2 formalizes this insight using the following additional definitions to simplify the exposition:

$$d_1^0 := \frac{q_{min}}{\gamma} \tag{10}$$

$$d_1^1 := \frac{q_{max}}{\gamma} \tag{11}$$

$$g(w) := -(c_2 - \delta)w + p_2 \int_0^{w} uf_2(u)du + p_2 w[1 - F_2(w)] \tag{12}$$

$$k(w) := \delta w + p_2 \int_0^{q_{min}} uf_2(u)du \tag{13}$$

Proposition 2 *The expected profit function π for different levels of q_1 is given by*

Case 1: $\gamma q_1 > q_{max}$

$$\pi = \pi_1 + \beta\left[[1 - F_1(d_1^1)]\, \pi_{2,1} + \int_{d_1^0}^{d_1^1} g(\gamma\, x) f_1(x)\, dx + \int_0^{d_1^0} k(\gamma\, x) f_1(x)\, dx \right] \tag{14}$$

Case 2: $q_{min} \leq \gamma\, q_1 \leq q_{max}$

$$\pi = \pi_1 + \beta \left[[1 - F_1(q_1)]\, g(\gamma\, q_1) + \int_{d_1^0}^{q_1} g(\gamma\, x) f_1(x)\, dx + \int_0^{d_1^0} k(\gamma\, x) f_1(x)\, dx \right] \tag{15}$$

Case 3: $q_{min} > \gamma\, q_1$

$$\pi = \pi_1 + \beta [1 - F_1(q_1)]\, k(\gamma\, q_1) + \beta \int_0^{q_1} k(\gamma\, x) f_1(x)\, dx \tag{16}$$

It is again easy to verify that the expected profit function π is concave in all three cases. Moreover, for $\gamma q_1 = q_{max}$ expected profits in cases 1 and 2 are identical, while for $\gamma q_1 = q_{min}$ expected profits in cases 2 and 3 are identical.

Proposition 3 *The optimal, unconstrained supply quantities $q_{1,i}$ in the three cases $i = 1, 2, 3$ are given by*

$$F_1(q_{1,1}) = 1 - \frac{c_1}{p_1} \tag{17}$$

$$F_1(q_{1,2}) = 1 - \frac{c_1}{p_1 + \beta\, \gamma\, [p_2\, [1 - F_2(\gamma\, q_{1,2})] - (c_2 - \delta)]} \tag{18}$$

$$F_1(q_{1,3}) = 1 - \frac{c_1}{p_1 + \beta\, \gamma\, \delta} \tag{19}$$

Note that Case 1 corresponds to the optimal myopic, first generation newsvendor quantity, while the optimal supply quantities given by Eqs. (18) and (19) satisfy $q_{1,1} \leq q_{1,2} \leq q_{1,3}$. Thus, in these latter two cases, the manufacturer increases its first generation supply beyond the short-term optimum to exploit gains from remanufacturing in generation 2.

Observe further that case 2 does not give us a closed form solution. However, a simple iterative procedure based on bisection will produce the solution for $q_{1,2}$. It is worth noting, that replacing $F_2(\gamma q_{1,2})$ in this equation by $F_2(q_{max})$ and $F_2(q_{min})$ gives $F_1(q_{1,2}) = \frac{p_1 - c_1}{p_1}$ and $F_1(q_{1,2}) = \frac{p_1 - c_1 + \beta\, \gamma\, \delta}{p_1 + \beta\, \gamma\, \delta}$, respectively. Thus, there are no discontinuities in the optimal first period production quantity between the cases. Intuitively, this also applies to the expected profits at the boundaries between the cases.

Using this insight, we can obtain the overall, constrained optimal solution q_1^* as described in Proposition 4.

Proposition 4 *The iterative procedure given by the flowchart shown in Fig. 1 yields the overall optimal first generation supply q_1.*

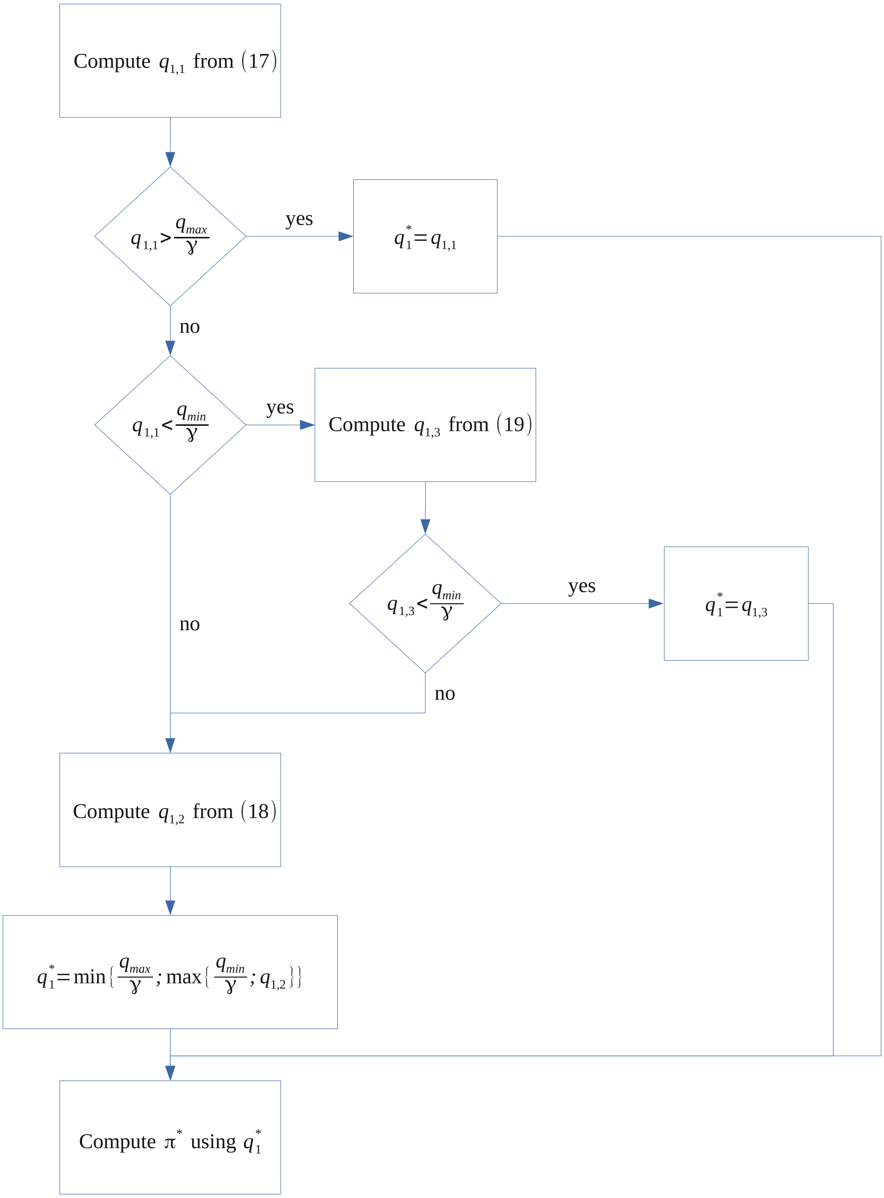

Fig. 1 Flowchart of the iterative procedure for obtaining the optimal strategy

2.5 Comparison of this Optimal Strategy with the Heuristics from Reimann and Lechner (2012)

The problem tackled in this paper has been treated in a slightly modified form before in Reimann and Lechner (2012). In that paper, it is assumed that the manufacturer has to take all decisions—q_1 as well as q_2 and q_2^r—prior to the resolving of the first generation demand uncertainty. Obviously, in that setting the manufacturer uses less information for deciding its second generation supply. Rather than exploiting the knowledge about first generation demand realization, the manufacturer has to work with his expectation about demand. This leads to the modified constraint

$$q_2^r \leq \gamma S_1, \tag{20}$$

while the objective function is unchanged [compare objective function (9.1) and constraint (9.2) on page 225 in Reimann and Lechner (2012)]. The resulting strategy can thus be seen as a heuristic with respect to the setting discussed in this paper.

In this section we want to analyze, how the optimal strategy differs from the heuristic strategy presented in Reimann and Lechner (2012). Let us first restate the strategy from Reimann and Lechner (2012) in terms of the notation used in this paper.

1. Case 1: $\gamma S_1 > q_{max}$
   Optimal myopic newsvendor supply in generation 1. Exclusive, but limited remanufacturing to satisfy generation 2 demand:
   $q_1 = F_1^{-1}(\frac{p_1-c_1}{p_1})$ and $q_2^r = F_2^{-1}(\frac{p_2-c_2+\delta}{p_2})$ and $q_2 = 0$
2. Case 2: $q_{min} \leq \gamma S_1 \leq q_{max}$
   Excess supply in generation 1. Exclusive, full remanufacturing to satisfy generation 2 demand:
   $q_1 = F_1^{-1}(\frac{p_1-c_1+\gamma\,\lambda_R}{p_1+\gamma\,\lambda_R})$ and $q_2^r = \gamma\, S_1(q_1) = F_2^{-1}(\frac{\beta(p_2-c_2+\delta)-\lambda_R}{\beta\, p_2})$ and $q_2 = 0$
3. Case 3: $q_{min} > \gamma S_1$
   Excess supply in generation 1. Dual sourcing with full remanufacturing and new production to satisfy generation 2 demand:
   $q_1 = F_1^{-1}(\frac{p_1-c_1+\gamma\beta\delta}{p_1+\gamma\beta\delta})$ and $q_2^r = \gamma\, S_1(q_1)$ and $q_2 = F_2^{-1}(\frac{p_2-c_2}{p_2}) - q_2^r$

For determining q_1 in Case 2, a bisection over the shadow-price λ_R of the remanufacturing constraint (20) is necessary. Clearly $\lambda_R = 0$ is the lower bound inducing Case 1. Case 3 arises when $\lambda_R = \beta\delta$.

We first observe that the first generation supply strategies in Cases 1 and 3 are identical in both the exact and the heuristic model. However, in the heuristic model the strategy switches earlier from Case 1 to Case 2 and from Case 2 to Case 3. Specifically, the heuristic switches as soon as $\gamma S_1 \leq q_i, i = \{\min, \max\}$, while the exact approach prescribed to switch when $\gamma q_1 \leq q_i$. Since we know that $q_1 > S_1$ by definition, it follows that in the region $\gamma S_1 \leq q_{max} < \gamma q_1$, the heuristic yields excessive supply q_1, while the exact approach still sticks with the classical newsvendor quantity for q_1. Similarly, in the region $\gamma S_1 \leq q_{\min} < \gamma q_1$, the heuristic already utilizes supply of new second generation units q_2, while the exact approach still prescribes single sourcing with remanufactured items.

Thus, the heuristic strategy systematically exaggerates first and second generation new supply q_1 and q_2, respectively. In the next section we will analyze the expected profit implications of using the exact approach shown in this paper, when compared with the heuristic approach.

3 Numerical Analysis

As shown above the exact approach utilizes its information on first generation outcomes for its second generation decision making. Thereby, it generally reduces its first generation supply q_1. In this section we will evaluate what expected profit gains are possible when compared with the heuristic approach. To get a comprehensive picture of this effect we ran a total of 12,100 problem instances. In all of these instances we kept the following parameters fixed: $p_1 = p_2 = 10$, $\beta = 0.9$ and $D_1 \sim U(0, 100)$.

Over the remaining parameters we performed a full factorial analysis with the parameter ranges shown in Table 1.

These parameter ranges were chosen to reflect very different settings. We varied c_1 to study the influence of varying optimal newsvendor quantities for the first product generation. The values chosen for c_2 are meant to reflect different types of on-the-job learning, thereby making second generation supply of new units more efficient. Similar to that, δ models different efficiencies of remanufacturing. The variation in second generation demand should model different life cycle stages of the product category. Finally, γ reflects the ease or efficiency of the used core collection.

Let us first discuss some aggregate results. Over all the 12,100 instances we found that in around 18.5 % of the cases the exact solution differs from the heuristic solution, with an average profit gain of 0.26 % and a maximum profit increase of 2.56 %. In the majority of the remaining cases (more than 81 %) both approaches prescribe to use dual sourcing of new and remanufactured items. This also implies that first generation supply should exceed the optimal myopic newsvendor quantity in almost all cases to exploit the efficiency gains associated

Table 1 Parameters and their values used in the numerical experiments

Parameter	Range of values
c_1	5, 6, . . ., 9
c_2	2, 3, . . ., c_1
δ	1, 2, . . ., c_2-1
D_2	$U(a_2, b_2)$, where $a_2 = 0$ and $b_2 = 50, 60, \dots, 150$
γ	0.1, 0.2, . . ., 1

with remanufacturing. These results show the importance of jointly optimizing manufacturing and remanufacturing decisions.

By disaggregating these results we can now gain insights into the conditions under which the exact and the heuristic approach differ. We start by looking at each of the parameters individually. In doing so we compute two measures. On the one hand we use the relative frequency with which the exact strategy outperforms the heuristic strategy. Let n_i correspond to the number of instances for which the parameter of interest i takes a certain value and let n_i^+ denote the number of those instances for which $\pi_{exact} > \pi_{heuristic}$. The relative frequency is then given by $h(\pi_{exact} > \pi_{heuristic}) = \frac{n_i^+}{n_i}$.On the other hand, we are interested in the associated average improvement in expected profit—termed $\Delta\pi$ below—induced by applying the exact strategy. For a given instance j the percentage improvement in expected profit induced by the exact strategy is computed by $\frac{\pi^j_{exact}-\pi^j_{heuristic}}{\pi^j_{heuristic}}100$. Let N_i denote the set of instances for which the parameter of interest i takes a certain value. Then $\Delta\pi = \frac{\sum_{j\in N_i}\frac{\pi^j_{exact}-\pi^j_{heuristic}}{\pi^j_{heuristic}}100}{n_i^+}$.

Figures 2, 3, 4, 5, and 6 show these two measures for different levels of γ, b_2, c_1, c_2 and δ, respectively. In terms of γ and b_2 Figs. 2 and 3 show very similar results. With an increase (decrease) in γ (in b_2) the preferability of the exact approach becomes more pronounced in terms of both measures. In both cases, the average improvement in expected profit goes up to around 0.4 %. The underlying mechanism at work is that with more efficient collection of used items or reduced second generation demand, the manufacturer can switch from dual sourcing to exclusive remanufacturing for satisfying second generation demand.

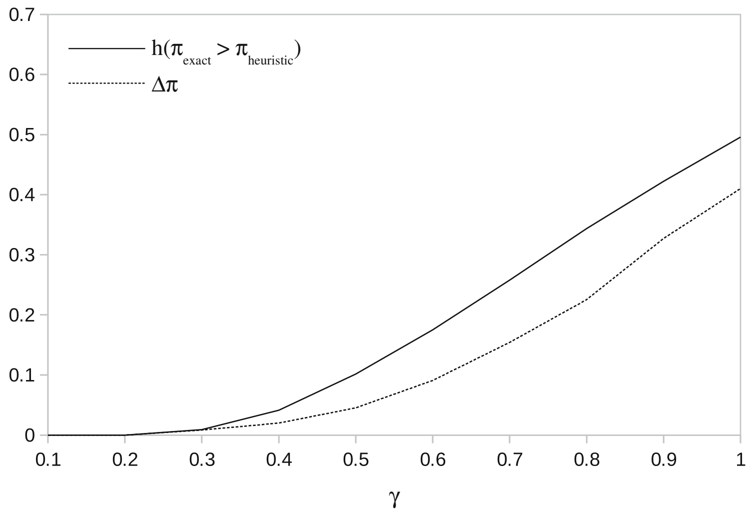

Fig. 2 Comparison of the exact and the heuristic approach for different levels of return rates γ

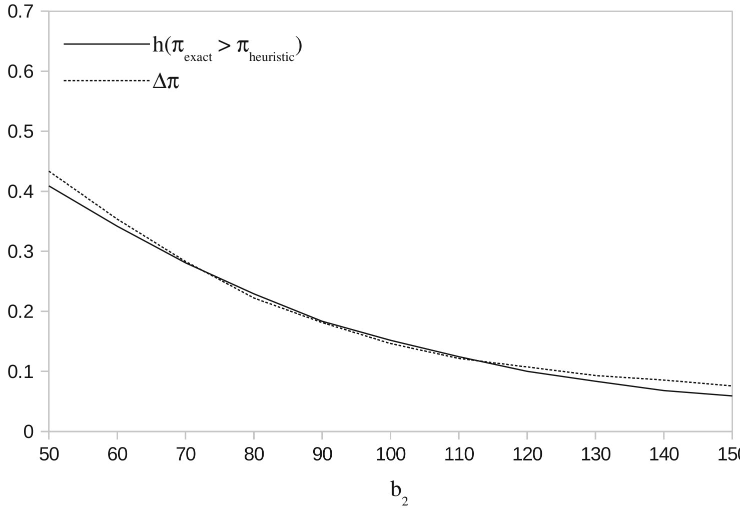

Fig. 3 Comparison of the exact and the heuristic approach for different levels of maximum second generation demand b_2

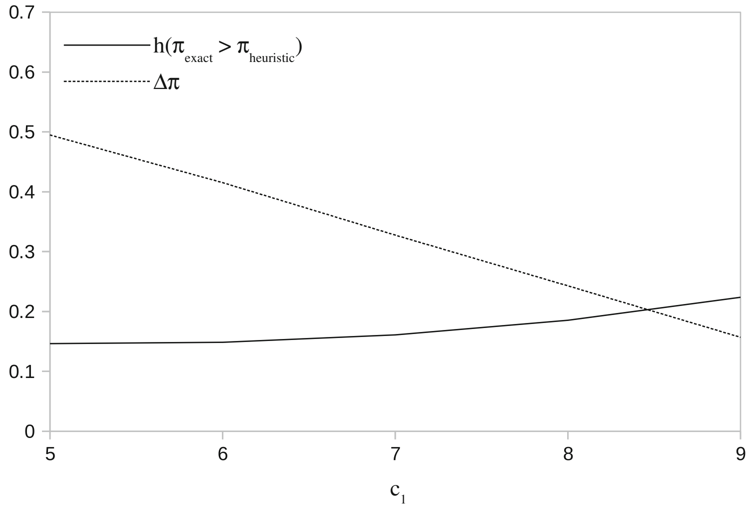

Fig. 4 Comparison of the exact and the heuristic approach for different levels of first generation production cost c_1

The results for c_1, c_2 and δ show a somewhat similar pattern in that the relative frequency of increased expected profits induced by the exact approach goes up with increases in these parameters. Contrary to that the associated average improvement in expected profit goes down. To understand this, let us first observe how these three parameters are linked to each other in our experimental setup. As mentioned

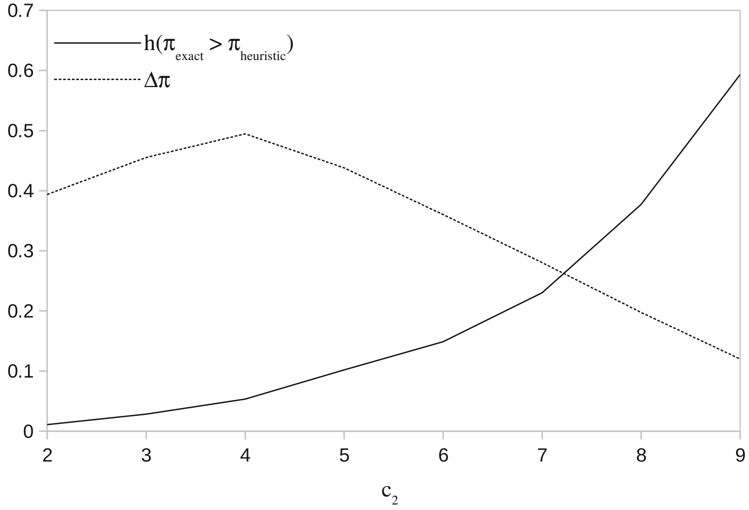

Fig. 5 Comparison of the exact and the heuristic approach for different levels of second generation production cost c_2

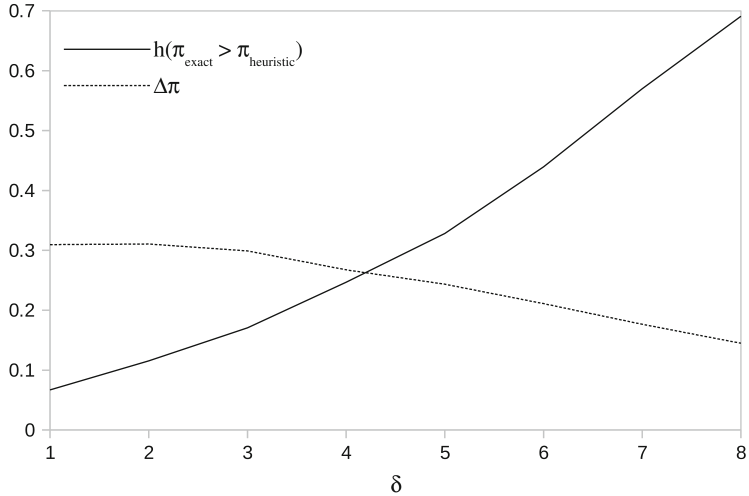

Fig. 6 Comparison of the exact and the heuristic approach for different levels of remanufacturing cost savings δ

$\delta < c_2 \leq c_1$. Thus, when c_1 is small, this implies that c_2 and δ also will be small, while when c_1 is large, we have larger variations in c_2 and δ over the associated instances.

Now let us start with considering the effects of reducing c_1. When c_1 goes down the optimal newsvendor supply of first generation units and consequently the expected first generation sales go up. Yet it also increases the probability that first generation demand is below the supply q_1 and thus drives the actual sales. As a result, the heuristic, which bases its second period decisions on expected returns from generation one, more often overestimates the value of increasing q_1. Not only does increased q_1 (beyond the optimal first generation single-period newsvendor quantity) reduce first generation profits, the profits of the second generation are also smaller since actual returns are lower than expected thus requiring the manufacturer to dual source and supply new second generation units as well. It is this effect that leads to larger relative gains $\Delta\pi$ of using the exact approach over the heuristic one when c_1 is smaller as shown in Fig. 4.

The same effect just described also drives the results when δ goes up since this implicitly reduces the range of c_1. In terms of c_2, Fig. 5 shows that an increase in c_2 first increases $\Delta\pi$ and only then leads to the above discussed reduction. Thus, a second effect is at work here, which is again driven by the parameter settings of the numerical study. When c_2 is very small, increasing it does not change the possible values of c_1 and the above described effect is not at work. Rather, larger c_2 reduces second generation supply thereby increasing the possibility to single source for generation two with remanufactured items only, i.e. Cases 1 and 2 become more prevalent in both approaches, but more so in the exact approach. As profits also go down, the percentage improvement of the exact approach over the heuristic approach increases. Only when c_2 increases further, the other effect described before takes over and reduces the associated average improvement in expected profit.

Finally, as mentioned above, the relative frequency of increased expected profits induced by the exact approach goes up with increases in all three parameters c_1, c_2 and δ. This is mainly driven by the fact that when δ is smaller, remanufacturing gains are rather small. Consequently there are smaller incentives to increase q_1 in order to reduce new second generation supply q_2. On the other hand, when δ increases both approaches will more often prescribe Case 2, where remanufacturing is the only source of second generation supply. In that case the heuristic approach always (slightly) exaggerates first generation supply q_1 and thus produces smaller profits. This can be seen directly from Fig. 6. Since increases in c_1 and c_2 implicitly allow for a larger range of δ as well, the effect is also observable in Figs. 4 and 5 albeit to a smaller extent. Particularly in terms of c_1, the effect is not very strong since a large c_1 alone is not sufficient for allowing large δ.

To conclude this section let us visualize the specific results for some instances with varying second generation demand. For given values $c_1 = c_2 = 5$, $\gamma = 1$ and $\delta = 1$ Table 2 nicely captures most of the possible effects. Let us first consider the case of the growth phase of the product categories lifecycle. When demand increases, i.e. $b_2 > 100$, both approaches are identical in suggesting excess (over the unconstrained newsvendor quantity) supply of first generation units and dual sourcing with both new and remanufactured units. Contrary to that, in case of the maturity phase, i.e. when demand is stationary ($b_2 = 100$), the heuristic still prescribes dual sourcing while the exact approach suggests sourcing only remanufactured units to satisfy second generation demand. Moreover, the exact approach produces a much smaller first generation supply q_1.

Table 2 Comparison of the exact and the heuristic approach for varying levels of second generation demand $D_2 \sim U(0, b_2)$ and $p_1 = p_2 = 10$, $\beta = 0.9$, $D_1 \sim U(0, 100)$, $c_1 = c_2 = 5$, $\gamma = 1$ and $\delta = 1$

	Exact			Heuristic			
b_2	q_1	Case	π	q_1	Case	π	$\Delta\pi$
50	50	1	202.59	50	1	202.59	0
60	50	1	217.29	50	1	217.29	0
70	50	1	231.71	52.13	2	228.11	1.58
80	50	1	245.86	54.13	3	243.99	0.76
90	51.31	2	259.48	54.13	3	258.89	0.23
100	52.97	2	271.92	54.13	3	271.82	0.04
110	54.13	3	283.43	54.13	3	283.43	0
120	54.13	3	294.68	54.13	3	294.68	0
130	54.13	3	305.93	54.13	3	305.93	0
140	54.13	3	317.18	54.13	3	317.18	0
150	54.13	3	328.43	54.13	3	328.43	0

Finally, let us consider the case of a declining market for the product category, i.e. falling demand ($b_2 < 100$). As second generation demand decreases the differences between the two strategies at first becomes more striking as the exact strategy starts to prescribe the optimal unconstrained first generation supply (Case 1) while the heuristic exaggerates first generation supply by more than 8 %. Only when second generation demand is very small, the heuristic also yields the optimal unconstrained first generation supply (Case 1) and both approaches are again identical.

4 Conclusion

In this paper we have analyzed a joint manufacturing–remanufacturing problem over two product generations. We have confirmed the importance of integrating the forward and the reverse SC for maximizing expected profits. Our results have also shown the value of information through adequately modeling the intertemporal aspect of the problem. By exploiting knowledge of first generation demand when taking second generation supply decisions, a reduction in first generation supply was generally found. This effect was particularly pronounced when used core collection is very efficient or second generation demand is tight.

A key extension of this line of research is to consider the impact of the operational CLSC processes on product or process improvement decisions. For example, in the present work we consider potentially reduced production cost for new items of the second generation. These could stem from on-the-job learning or process innovation. Yet, if the former mechanism were at work, our model oversimplifies the real situation by assuming that learning effects are independent of first generation output. If the latter mechanism were at work, our model assumes that process innovation is costless. Clearly, both interpretations can be challenged and inclusion of one or both mechanisms could lead to interesting questions concerning the allocation of R&D budget. For the case of the return rate, the effect of an active acquisition process has been studied in e.g. Kleber et al. (2012), Lechner and Reimann (2014) and Minner and Kiesmüller (2012). Jointly analyzing these strategic product and process design decisions will further extend our understanding of the value of Closed-loop Supply Chains for achieving sustainability.

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Appendix

Proof Proof of Proposition 1:

Let λ_s, λ_r and λ_2 correspond to the shadow prices of constraints (4), (5) and (6), respectively. Then, for given $s_1 > 0$, the KKT conditions associated with the problem

$$\max_{q_2, q_2^r} \pi_2 \tag{21}$$

subject to constraints (4)–(6) are given by

$$F_2(q_2 + q_2^r) = \frac{p_2 - (c_2 - \delta) + \lambda_r - \lambda_s}{p_2} \tag{22}$$

$$F_2(q_2 + q_2^r) = \frac{p_2 - c_2 + \lambda_2}{p_2} \tag{23}$$

$$q_2 \geq 0 \tag{24}$$

$$q_2^r \geq 0 \tag{25}$$

$$\gamma s_1 - q_2^r \geq 0 \tag{26}$$

$$\lambda_s \geq 0 \tag{27}$$

$$\lambda_r \geq 0 \tag{28}$$

$$\lambda_2 \geq 0 \tag{29}$$

$$\lambda_s(\gamma s_1 - q_2^r) = 0 \tag{30}$$

$$\lambda_r q_2^r = 0 \tag{31}$$

$$\lambda_2 q_2 = 0 \tag{32}$$

Let us first prove that $q_2^r > 0$ is always true. We will show this by contradiction. Let $q_2^r = 0$. Given $\gamma > 0$ and $s_1 > 0$ this implies that $\gamma s_1 - q_2^r > 0$ and consequently $\lambda_s = 0$. From (22) and (23) we get $F_2(q_2) = \frac{p_2-(c_2-\delta)+\lambda_r}{p_2} = \frac{p_2-c_2+\lambda_2}{p_2}$. When $q_2 = 0$ the resulting $F_2(0) = \frac{p_2-c_2+\lambda_2}{p_2} > 0$ contradicts our assumption $F_2(0) = 0$ since $p_2 > c_2$. Alternatively, $q_2 > 0$ implies $\lambda_2 = 0$, which yields $F_2(q_2) = \frac{p_2-(c_2-\delta)+\lambda_r}{p_2} = \frac{p_2-c_2}{p_2}$, which holds when $\delta + \lambda_r = 0$. Since $\delta > 0$ and $\lambda_r \geq 0$ this can never be true. Thus, $q_2^r = 0$ always leads to a contradiction and it follows that $q_2^r > 0$ and consequently $\lambda_r = 0$.

Now assume that $q_2 > 0$. This implies that $\lambda_2 = 0$ and from (23) we get $F_2(q_2 + q_2^r) = \frac{p_2-c_2}{p_2}$. Together with (22) this yields $\delta - \lambda_s = 0$. It follows that $\lambda_s = \delta > 0$ and consequently $\gamma s_1 - q_2^r = 0$. Since $\gamma > 0$ and $s_1 > 0$ we get $q_2^r = \gamma s_1 > 0$. Using $F_2(q_{min}) = \frac{p_2-c_2}{p_2}$ it follows that $q_2 + q_2^r = q_{min}$. Thus, $q_2^r = \gamma s_1 = q_{min} - q_2 < q_{min}$. This concludes the proof of Case 3.

Now consider $q_2 = 0$, i.e. $\lambda_2 \geq 0$. From (22) and (23) we get $F_2(q_2^r) = \frac{p_2-(c_2-\delta)-\lambda_s}{p_2} = \frac{p_2-c_2+\lambda_2}{p_2}$. We continue by distinguishing two scenarios. First, let $q_2^r < \gamma s_1$. It follows that $\lambda_s = 0$. This implies that $F_2(q_2^r) = \frac{p_2-(c_2-\delta)}{p_2}$, i.e. $q_2^r = q_{max}$. As a result, $q_{max} < \gamma s_1$. This concludes the proof of Case 1. Second, let $q_2^r = \gamma s_1$. Now $\lambda_s \geq 0$ gives $F_2(q_2^r) = \frac{p_2-(c_2-\delta)-\lambda_s}{p_2} = \frac{p_2-c_2+\lambda_2}{p_2}$. Thus, $F_2(q_{max}) \geq F_2(q_2^r) \geq F_2(q_{min})$ and consequently $q_{max} \geq q_2^r = \gamma s_1 \geq q_{min}$, which concludes the proof of Case 2.

Proof Proof of Proposition 2

Let us first consider the case where $\gamma q_1 > q_{max}$. In that case, there are the following four possible first generation outcomes leading to different optimal second generation strategies.

1. $d_1 > q_1$ $\pi = -c_1\, q_1 + p_1\, q_1 + \beta \pi_{2,1}$ Here first generation sales are given by supply q_1 and consequently returns are sufficient to cover q_{max} in generation 2 yielding expected profit $\pi_{2,1}$.
2. $d_1 \leq q_1$ & $\gamma \cdot d_1 > q_{max}$: $\pi = -c_1\, q_1 + p_1\, d_1 + \beta \pi_{2,1}$ In this setting, demand is limiting first generation sales. Yet as above, the associated returns are sufficient to cover q_{max} in generation 2 yielding expected profit $\pi_{2,1}$.
3. $q_{max} \geq \gamma \cdot d_1 \geq q_{min}$: $\pi = -c_1\, q_1 + p_1\, d_1 + \beta g(\gamma\, d_1)$ Again demand is limiting first generation sales but associated returns are now insufficient to cover q_{max} in generation 2. Yet, the associated returns are sufficient to cover q_{min} in generation 2. Thus, we are in Case 2 described in Proposition 1. However, now second generation profits depend on the actual level of d_1 as shown by the term $g(\gamma\, d_1)$.
4. $q_{min} > \gamma \cdot d_1$: $\pi = -c_1\, q_1 + p_1\, d_1 + \beta k(\gamma\, d_1)$ In this final setting, demand is limiting first generation sales and the associated returns are insufficient to even cover q_{min} in generation 2. Thus, dual sourcing is optimal and we are in Case 3 described in Proposition 1. Similarly as above, second generation profits depend on the actual level of d_1 as shown by the term $k(\gamma\, d_1)$.

We can now turn to the second case where $q_{max} \geq \gamma\, q_1 \geq q_{min}$. Now, first generation sales are no longer sufficient to cover q_{max} in generation 2 and we are left with three possible first generation outcomes leading to different optimal second generation strategies.

1. $d_1 > q_1 \;\; \pi = -c_1\, q_1 + p_1\, q_1 + \beta g(\gamma\, q_1)$ Here first generation sales are given by supply q_1. The associated returns are sufficient to cover q_{min} in generation 2. Thus, we are in Case 2 described in Proposition 1. However, now second generation profits depend on the actual level of q_1 as shown by the term $g(\gamma\, q_1)$.
2. $d_1 \leq q_1 \quad \& \quad \gamma \cdot d_1 \geq q_{min} : \pi = -c_1\, q_1 + p_1\, d_1 + \beta g(\gamma\, d_1)$ Here first generation sales are given by demand d_1. The associated returns are sufficient to cover q_{min} in generation 2. Thus, we are still in Case 2 described in Proposition 1. However, now second generation profits depend on the actual level of d_1 as shown by the term $g(\gamma\, d_1)$.
3. $q_{min} > \gamma \cdot d_1 : \pi = -c_1\, q_1 + p_1\, d_1 + \beta k(\gamma\, d_1)$ In this final setting, demand is limiting first generation sales and the associated returns are insufficient to even cover q_{min} in generation 2. Thus, dual sourcing is optimal and we are in Case 3 described in Proposition 1. Similarly as above, second generation profits depend on the actual level of d_1 as shown by the term $k(\gamma\, d_1)$.

The third case arises when $q_{min} > \gamma\, q_1$, i.e. first generation supply and consequently sales are insufficient to even cover the minimum supply of generation 2 units. In this case, dual sourcing is always necessary and optimal and we are left with two possible first generation outcomes leading to different optimal second generation strategies.

1. $d_1 > q_1 \;\; \pi = -c_1\, q_1 + p_1\, q_1 + \beta k(\gamma\, q_1)$ Here first generation sales are given by supply q_1. Since the associated returns are insufficient to even cover q_{min} in generation 2 we are in Case 3 described in Proposition 1. Second generation profits depend on the actual level of q_1 as shown by the term $k(\gamma\, q_1)$.
2. $d_1 \leq q_1 \;\; \pi = -c_1\, q_1 + p_1\, d_1 + \beta k(\gamma\, d_1)$ Now demand is limiting first generation sales and second generation profits depend on the actual level of d_1 as shown by the term $k(\gamma\, d_1)$.

Integrating over the possible values of d_1 again yields the proposed expected profit function in all three cases.

Proof Proof of Proposition 3 The optimality conditions for $q_{1,1}$, $q_{1,2}$ and $q_{1,3}$ are readily obtained from setting the first derivative of the associated expected profit functions given by Proposition 2 equal to zero.

Proof Proof of Proposition 4 Let us first observe that at most one of the candidate solutions $q_{1,1}$, $q_{1,2}$ and $q_{1,3}$ can fall within its admissible range, since $q_{1,3} \geq q_{1,2} \geq q_{1,1}$ while at the same time their boundaries prescribe the opposite order. Furthermore, we know that at the boundaries between two cases their respective profits are identical. Thus, from concavity it follows that whenever one of the candidate solutions falls within its admissible range, it is the optimal solution, since its associated profit exceeds the respective boundary profits of the other two cases.

Using this logic, we can begin with computing an arbitrary candidate solution $q_{1,1}$, $q_{1,2}$ or $q_{1,3}$. The flowchart in Fig. 1 shows this approach starting with $q_{1,1}$. When this solution falls within its admissible range, the optimum has been found and the algorithm stops. Otherwise, the next case needs to be considered. To compute as little solutions as possible the algorithm exploits the information on $q_{1,1}$ as well as the fact that $q_{1,3} \geq q_{1,2} \geq q_{1,1}$ when deciding which case to consider next. Specifically, whenever $\gamma q_{1,1} > q_{min}$ Case 3 can never be optimal and needs not be considered, since $\gamma q_{1,3} > q_{min}$ violates its admissible range.

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Comparing Modeling Approaches for the Multi-Level Capacitated Lot-Sizing and Scheduling Problem

Christian Almeder

Abstract Determining lot sizes is an essential step during the material requirements planning phase influencing total production cost and total throughput time of a production system. It is well-known that lot-sizing and scheduling decisions are intertwined. Neglecting this relation, as it is done in the classical hierarchical planning approach, leads to inefficient and sometimes infeasible plans. In this work we compare different approaches for integrating the lot-sizing and the scheduling decisions in multi-stage systems. We show their abilities and limitations in describing relevant aspects of a production environment. By applying the models to benchmark instances we analyze their computational behavior. The structural and numerical comparisons show that there are considerable differences between the approaches although all models aim to describe the same planning problem. The results provide a guideline for selecting the right modeling approach for different planning situations.

1 Introduction

In the recent years a trend towards integrating multiple levels of the classical hierarchical and sequential planning concept of the production planning and control systems into a single planning step can be observed. This trend results in a demand for new modeling concepts that include lot-sizing as well as capacity planning and scheduling decisions within the material requirements planning. That means planning models are needed which respect the relations between raw materials, intermediate items, subassemblies, and final products (given by the bill-of-materials), capacity restrictions, and timing of production lots simultaneously. Of course, the price to pay for the increased complexity and more detailed models is that (1) the models are usually tailored to a specific situation and they are not

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general-purpose anymore, and (2) the computational effort to solve these models is increasing despite the technological progress in hardware and software.

In this paper we have a more detailed look on the different model classes discussed in the research literature regarding the aforementioned aspects of tailored models and computational effort. In particular, we investigate models for the capacitated multi-level lot-sizing and scheduling problem and analyze the underlying assumptions and the limitations of those models and their computational properties using a set of benchmark instances.

The main classification of lot-sizing models for dynamic deterministic demand is related to the underlying time structure. The small-bucket models are characterized by a partitioning of the planning horizon into short time buckets (micro-periods) during which only a single setup operation on a machine is allowed. This class of models is solving simultaneously the lot-sizing and the scheduling problem. Big-bucket models use quite long time periods (macro-periods) during which several production lots can be processed on a machine, but each product only once. Here the scheduling aspect is not considered within the basic model concept. Surprisingly, the core of the research literature on multi-level lot-sizing deals with big-bucket formulations (cf. a recent review on big-bucket models by Buschkühl et al. 2010). Nevertheless, there are recent multi-level formulations based on small-bucket models, which we will consider in our study. The goal of this paper is to analyze those different modeling approaches. Hence, the contribution of this paper is twofold:

- We show and compare the limitations of different modeling concepts due to the underlying assumptions. These results will help to select an appropriate modeling approach for a specific situation.
- We provide a numerical comparison of all models in order to underline the limitations and differences discussed in the previous point and to determine more promising model approaches regarding the computational tractability using a standard optimization software.

The paper is organized as follows: in Sect. 2 the relevant literature is discussed. In Sect. 3 the different models are presented and compared regarding their internal structure, whereas in Sect. 4 the computational comparison is provided. Section 5 concludes the paper summarizing the major findings of this study.

2 Literature Review

All basic lot-sizing models for dynamic demand — based on the formulation introduced by Wagner and Whitin (1958) — are driven by the trade-off between the setup and holding costs. Introducing capacity limitations and considering several products simultaneously led to the capacitated lot-sizing problem (CLSP), which Bitran and Yanasse (1982) have shown to be NP-hard. Billington et al. (1983) were among the first researches extending the big-bucket formulation of the CLSP towards the multi-level situation of the material requirements planning which led to the multi-level capacitated lot-sizing problem (MLCLSP). The disregard of the scheduling problem in connection with a multi-level structure in the MLCLSP potentially leads to capacity and inventory problems as shown by Almeder et al. (2015). Therefore, the models have to be extended either considering a one period long delay between the production of a predecessor and a successor item, or allowing a detailed sequencing of lots within a period. The first approach has been used by many researchers (cf. Buschkühl et al. 2010) but it produces unnecessary high work-in-process levels (cf. Almeder et al. 2015). The latter approach is rather complex. A very detailed non-linear model was formulated by Fandel and Stammen-Hegene (2006), but this model is beyond any possibility to be solved. A formulation as a mixed integer linear program by Almeder et al. (2015) is used in this paper.

The continuous setup and lot-sizing problem (CSLP) and the proportional lot-sizing and scheduling problem (PLSP) are the most prominent model approaches in the class of small-bucket problems (cf. Drexl and Kimms 1997). An extension to multi-level systems is straight forward. Nevertheless, not many researchers have been working on this multi-level extension. Stadtler (2011) formulated a multi-level PLSP model which allows the production of a predecessor and a successor product within the same time period. But he restricts his model to the rather simple single-machine case.

Seeanner and Meyr (2013) extended the general lot-sizing and scheduling problem (GLSP) to the multi-level multi-machine case. The GLSP is in some sense a hybrid model concept combining small-bucket and big-bucket ideas. The planning horizon is divided into longer time periods (big buckets), but each such period is subdivided into several smaller periods (small buckets) of variable length. Another approach to the simultaneous lot-sizing and scheduling problem in a multi-level system described in the research literature is a scheduling based one. Kim et al. (2008) proposed an integrated model, where each demand is transformed into production jobs for the requested product and all its predecessors. The resulting scheduling problem is solved using a genetic algorithm. The main criticism might be that this approach does not consider lot-sizing possibilities at their full extent. In fact, lots are formed by combining different (subsequent) demands of the same item, but no demand is split into several lots. Dauzère-Pérès and Lasserre (2002) enhanced a multi-level lot-sizing problem by special scheduling constraints as a substitute for the capacity constraints. They do not consider any setup processes and do not solve the model as an integrated one, but split it into a lot-sizing and a capacity refining step.

3 Model Formulations

3.1 Overview

In this paper, a comparison of three model classes is conducted. The focus lies on the comparison of deterministic capacitated multi-level lot-sizing problems with dynamic demand, a finite planning horizon and a general product structure. The objective is to minimize the sum of setup and inventory costs.

First, the class of small-bucket models is considered. Here the continuous setup lot-sizing problem (MLCSLP) and the proportional lot-sizing problem (MLPLSP) for the multi-level case are analyzed. One of the main disadvantages of these formulations is that they are not adequate for problems where long setup operations occur, as the duration of these operations must not exceed one micro-period. However, enabling multi-period setup operations may cause a substantial increase in model complexity. A recent formulation (MLPLSP$_{SS}$) by Stadtler (2011) is considered in addition, that is built on the MLPLSP and accounts for period overlapping setup times to enable more flexibility in planning. Moreover, Stadtler (2011) assumes a zero lead time offset, whereas for the MLCSLP and the MLPLSP a delay of one micro-period between production of predecessor and successor items is necessary.

The second class reflects the big-bucket models. The basic multi-level capacitated lot-sizing problem (MLCLSP) does not allow to carry over the setup state of a machine from one period to the next one, i.e., in each period a setup is necessary to start production. In order to be consistent and comparable with the small-bucket models, a version with linked lots between periods is used (MLCLSP$_L$). Furthermore, two variants presented in Almeder et al. (2015) are considered. One covers the lot-streaming case (referred to as MLCLSP$_{LS}$) and the other allows production in batches (MLCLSP$_{BS}$), where the production of a batch has to be finished before the items can be used as input for the production of successor items.

The third class is the general lot-sizing and scheduling problem for multiple stages (GLSPMS) proposed by Seeanner and Meyr (2013) with their underlying two-stage time structure. The model accounts for both, setup operation and production quantity splitting over consecutive variable micro-periods. Quantity splitting means that the whole production amount of one micro-period is virtually split into two parts, one is available to subsequent processing in the on-going micro-period and the other in the following micro-period at the earliest.

In order to keep this paper comprehensive we refer the reader for detailed model descriptions for MLPLSP$_{SS}$, MLCLSP$_{BS}$, MLCLSP$_{LS}$, and MLGLSP to the respective literature (Stadtler 2011; Almeder et al. 2015; Seeanner and Meyr 2013).

3.2 Basic Notation

Indices and index sets

N	number of products, $i, j \in \{1, \ldots, N\}$
M	number of resources, $m \in \{1, \ldots, M\}$
T	number of periods, $t \in \{1, \ldots, T\}$
S	number of micro periods, $s \in \{1, \ldots, S\}$
Φ	set of micro periods
$\Phi^t \subset \Phi$	set of micro periods forming the last micro period within a macro period
$\Gamma(i)$	set of immediate successors of product i based on BOM
$\phi(m)$	set of items assigned to resource m

Decision variables

X_{mis}/X_{mit}	production quantity of item i in period s or t on machine m
I_{is}/I_{it}	inventory of item i at the end of period s or t
$\alpha_{mis}/\alpha_{mit}$	$\begin{cases} 1 & \text{if resource } m \text{ is set up for item } i \text{ at the beginning of period } s \text{ or } t \\ 0 & \text{otherwise} \end{cases}$
T_{mijs}/T_{mijt}	$\begin{cases} 1 & \text{if there is a set up from item } i \text{ to } j \text{ on machine } m \text{ in period } s \text{ or } t \\ 0 & \text{otherwise} \end{cases}$

Parameters

a_{ij}	amount of item i, required for production of one unit of j
p_{mi}	time required for production of one unit of i on resource m
st_{mij}	time required to setup resource m from item i to item j
c_{mij}	cost incurring on resource m when setup from item i to item j
E_{is}/E_{it}	external demand of item i in period s or t
I_{i0}	initial inventory of item i
h_i	inventory costs of product i
L_{ms}/L_{mt}	available production time on machine m in period s or t
l_i	lead time of item i in number of periods
G	a sufficient large number.

3.3 Small-Bucket Models

We use the MLCSLP and MLPLSP formulations proposed by Kimms and Drexl (1998) and incorporate setup times. While in the MLCSLP only one setup is allowed to be performed at the beginning of each micro-period, in the MLPLSP this very restrictive assumption is relaxed by allowing the setup to be performed at any time within a period and therefore inducing the possibility to produce two distinct items within one micro-period. Hence, these models implicitly determine also a production schedule. In order to enable a numerical comparison between the micro-period and macro-period structure, a constant number of micro-periods of the same length are embedded into macro-periods. Holding cost and demand will only occur in the last micro-period of a macro-period (if $s \in \Phi^t$). The model formulation for the MLCSLP reads:

$$min \sum_{i,s} c_i \cdot T_{is} + \sum_{i,s \in \Phi^t} h_i \cdot I_{is} \tag{1}$$

subject to

$$I_{is} = I_{i(s-1)} + X_{is} - E_{is} - \sum_{j \in \Gamma(i)} a_{ij} \cdot X_{js} \qquad \forall\ i, s \tag{2}$$

$$I_{is} \geq \sum_{j \in \Gamma(i)} \sum_{\tau = s+1}^{min[s+l_i;S]} a_{ij} \cdot X_{j\tau} \qquad \forall\ i, s = 0, \ldots, S-1 \tag{3}$$

$$\sum_{i \in \phi(m)} \alpha_{is} \leq 1 \qquad \forall\ m, s \tag{4}$$

$$T_{is} \geq \alpha_{is} - \alpha_{i(s-1)} \qquad \forall\ i, s \tag{5}$$

$$p_i \cdot X_{is} + s_i \cdot T_{is} \leq L_{ms} \cdot \alpha_{is} \qquad \forall\ i, s \tag{6}$$

$$I_{is}, X_{is}, T_{is} \geq 0, \alpha_{is} \in \{0, 1\} \qquad \forall\ i, s \tag{7}$$

The objective to be minimized consists of setup and holding cost (1). External and internal demands have to be fulfilled without delay which is shown by the inventory balance equations (2). Lead times between predecessors and successors are incorporated by constraints (3). Constraints (4) enforce a unique setup state and constraints (5) describe the setup decisions. Either a setup for i takes place in s ($T_{is} = 1$) or a setup state for i is carried over from $s - 1$ to s. Constraints (6) constitute the capacity restrictions and finally, (7) are the non-negativity constraints and binary variables restrictions. Substituting (6) by constraints (8), describing a setup to be made at any time within a micro-period and new capacity constraints (9) leads to the MLPLSP:

$$p_i \cdot X_{is} \leq L_{ms} \cdot (\alpha_{i(s-1)} + \alpha_{is}) \qquad \forall\ i, s \tag{8}$$

$$\sum_{i \in \phi(m)} (p_i \cdot X_{is} + s_i \cdot T_{is}) \leq L_{ms} \qquad \forall\ m, s \tag{9}$$

A major design decision when employing small-bucket models is the appropriate number of micro-periods for each macro-period. The appropriate number depends on several circumstances: How many distinct items are to be produced, is there a lead time to be considered, of how many stages consists the production process, does the length of the shortest period at least correspond to the length of the longest setup operation? If too few micro-periods are embedded into a macro-period, the detailed planning offered by the model is not exploited and finding a feasible solution may become quite difficult. On the other hand, a very large number of micro-periods leads to very long computational times for real-world instances (cf. Seeanner and Meyr 2013).

The two models above consider a lead time of at least one micro-period. However, situations may occur where even one micro-period lead time is not realistic. Stadtler (2011) extended the single machine PLSP to a multi-level single machine PLSP with zero lead time ($MLPLSP_{SS}$) that accounts for the problem arising in a specific pharmaceutical company. In order to make this model comparable with the other models analyzed, we adapted the original model to the multi-machine case by defining a set of products $i \in \varphi(m)$ assigned to a machine m.

3.4 Big-Bucket Models (MLCLSP)

The second class of lot-sizing models we are discussing are the big-bucket models, where the planning horizon is divided into a few large macro-periods. The MLCLSP by Billington et al. (1983) has some drawbacks compared with the small-bucket models. The scheduling is completely neglected in the MLCLSP. Instead, it is assumed that the production scheduling is carried out in an additional subsequent planning level. This causes serious feasibility problems in multi-level planning when coordination between the production stages is neglected (cf. Almeder et al. 2015). Considering a full macro-period lead time[1] allows to generate feasible production plans, but it leads to overestimated cycle times and huge work-in-process (WIP).

Furthermore, the classical MLCLSP does not preserve the setup state of a machine from one period to the next one, i.e., if the same product is produced at the end of a period and at the beginning of the next period an additional setup is necessary. Hence, a logical extension of the MLCLSP is to introduce so-called linked lots, denoted by $MLCLSP_L$ (cf. Suerie and Stadtler 2003; Haase 1994). We will focus on a formulation by Sahling et al. (2009) that accounts for setup carryovers over multiple periods and lead times.

Since we deal with macro-periods in the MLCLSP, we replace the index s by t for macro-periods. In addition to the variables listed in the basic notation table in Sect. 3.2, a variable for the initial inventory level $\hat{y}_i$ and an auxiliary variable v_{mt} is used in the model formulation.

[1]The actual processing time of a lot usually takes only a small fraction of a macro-period.

The objective function of the MLCLSP$_L$ reads:

$$min \sum_{i,t} (c_i \cdot (\alpha_{it} - T_{it}) + h_i \cdot I_{it}) \tag{10}$$

subject to:

$$I_{i(t-1)} + X_{it} - \sum_{j \in \Gamma(i)} a_{ij} \cdot X_{j(t+1)} - I_{it} = E_{it} \qquad \forall\ i, t = 2, \dots, T-1 \tag{11}$$

$$I_{i(T-1)} + X_{iT} - I_{iT} = E_{iT} \qquad \forall\ i \tag{12}$$

$$\hat{y}_i - \sum_{j \in \Gamma(i)} a_{ij} \cdot X_{j1} - I_{i0} = 0 \qquad \forall\ i \tag{13}$$

$$I_{i0} + X_{i1} - \sum_{j \in \Gamma(i)} a_{ij} \cdot X_{j2} - I_{i1} = E_{i1} \qquad \forall\ i \tag{14}$$

$$\sum_i (p_i \cdot X_{it} + s_i \cdot (\alpha_{it} - T_{it})) \leq L_{mt} \qquad \forall\ m, t \tag{15}$$

$$X_{it} \leq G \cdot \alpha_{it} \qquad \forall\ i, t \tag{16}$$

$$\sum_{i \in \phi(m)} T_{it} \leq 1 \qquad \forall\ m, t \tag{17}$$

$$T_{it} \leq \alpha_{i(t-1)} \qquad \forall\ i, t \tag{18}$$

$$T_{it} \leq \alpha_{it} \qquad \forall\ i, t \tag{19}$$

$$T_{it} + T_{i(t+1)} \leq 1 + v_{mt} \qquad \forall\ m, i \in \phi(m) \tag{20}$$

$$(\alpha_{it} - T_{it}) + v_{mt} \leq 1 \qquad \forall\ i \tag{21}$$

$$T_{i1} = 0 \qquad \forall\ m, i \in \phi(m) \tag{22}$$

$$I_{it}, X_{it}, I_i^0, v_{mt} \geq 0;\ \alpha_{it}, T_{it} \in \{0, 1\} \qquad \forall\ m, i, t \tag{23}$$

Constraints (11)–(14) reflect the inventory balance equations where a one macro-period lead time is incorporated to achieve coordination between the production levels. The initial inventory of item i is denoted by $\hat{y}_i$. Constraints (15) are again the capacity restrictions enforcing the capacity limits to be kept. Note, that T_{it} now has a different meaning from the small bucket models discussed above, as it constitutes the setup carry-over variable, so that $T_{it} = 0$ if there is no setup carryover and thus a setup operation for i takes place. A product can only be produced if a machine is in the appropriate setup state (16). For each period and machine at most one setup carryover is allowed (17) . The setup carryover itself is described by (18) and (19) stating that a setup carryover is only possible ($T_{it} = 1$) if the resource is set up for item i in both period t and $t-1$ ($\alpha_{it} = 1$ and $\alpha_{i,t-1} = 1$). Constraints (20) and (21) model the multi-period setup carryover characteristic of the formulation.

As mentioned earlier, incorporating a full macro-period lead time may result in long cycle times and huge amounts of WIP. Two extensions of the MLCLSP are proposed by Almeder et al. (2015) dealing with the problem of incorporating lead times in the MLCLSP, one based on batch production and the other on lot-streaming. In contrast to the MLCLSP$_L$, where scheduling is neglected and the small-bucket models, where sequencing is done implicitly, scheduling is done simultaneously to lot-sizing in these MLCLSP extensions. This can be achieved by using a formulation based on models for sequence-dependent setups. Furthermore, the beginning of production of an item i in period t is captured by a continuous decision variable μ_{it}.

The first model MLCLSP$_{BS}$ is based on batch production, where a lot of products can only be further processed when the production of the whole lot is finished. At the start of production of item i the whole amount of predecessor j needed for production of i must be available. The second formulation MLCLSP$_{LS}$ assumes lot-streaming, meaning that each single item produced may be further processed immediately in the next stage without delay times. Strictly speaking, it allows even simultaneous production of predecessors and successors on different machines. However, one might account for a minimum production lead time τ_i that could be added to the starting time μ_{it} of product i. For both versions the in-period inventory levels are tracked using the new decision variables μ_{it}. Additional constraints ensure that this in-period inventory does not become negative.

3.5 Mixed Models (GLSPMS)

The last model considered here is the GLSPMS by Seeanner and Meyr (2013), which is an extension to the single-level GLSP introduced by Fleischmann and Meyr (1997).

The main characteristic of this model is the two-level time structure. The external dynamics within a planning horizon are described by demand data and holding cost and are defined on a fixed discrete time frame of a few macro-periods. Each macro-period t is split into a predetermined number of non-overlapping micro-periods $s \in \Phi^t$. These micro-periods are in contrast to the pure small-bucket models of variable length depending on the lot sizes and the setup operations. Thus, the scheduling problem is solved implicitly by assigning lots to the different micro-periods. In order to enable synchronization over all levels and machines, the GLSPMS enforces the same micro-period structure (i.e. equal starting times and equal length) on all machines. As only one product is allowed to be produced within a micro-period, scheduling is performed by determining the starting times of the variable micro-periods. As in the MLPLSP$_{SS}$, in the full variant of GLSPMS, Seeanner and Meyr (2013) split the lot of one micro-period into two parts, one allowed to be further processed in the actual period, the other in the next period at the earliest. In such a way it allows immediate usage of an item for the production of a successor without any delay.

3.6 Comparison of Model Characteristics

In Table 1 some of the major characteristics of the discussed models are summarized. One important distinguishing characteristic of the models regards the way to organize synchronization between production stages. Because of their detailed structure, considering a minimum lead time of one micro-period in the MLCSLP and MLPLSP is a good compromise when lot-streaming is not realizable. The MLPLSP$_{SS}$ allows going beyond this lead time of one micro-period, because it allows the production of predecessor and successor items in the same micro-period, but only if production is in the first and second campaign, respectively. Furthermore, with none of the small-bucket models it is possible to realize a batching constraint, i.e. starting a successor only after the finishing of the complete batch of the predecessor, because usually a production batch lasts for several micro-periods. For the big-bucket models MLCLSP$_{L,BS,LS}$ the picture is clear. Here we have either synchronization via a lead time of one macro-period, a lot-streaming possibility or a batching constraint. The GLSPMS enables partial lot-streaming (only possible where the production rate of the successor is not higher than of the predecessor) as a result of the variable micro-periods and because of the identical structure of these periods on all machines at all different stages.

Regarding the timing of the setup operations only the MLCSLP is quite restrictive, because it allows only setups at the beginning of the fixed micro-periods. All other models have considerably more flexibility. For the MLPLSP it may be at any time within a micro-period, for the MLPLSP$_{SS}$ and the GLSPMS it is also possible to split setup operations between micro-periods. All big-bucket models as well as the GLSPMS allow setup operations at any time during a macro-period. (Due to the variable micro-period length also the GLSPMS allows this flexibility.)

Considering the possible number of different products and the possible number of setups within a macro-period, there is a notable difference. All small-bucket models allow a setup each micro-period and in the GLSPMS a micro-period can contain two fractions of different setup operations. Hence, it is possible to produce S/T ($S/T+1$ for the MLPLSP and MLPLSP$_{SS}$) different lots within a macro-period. These lots might be of different products or of the same products. The big-bucket models allow only a single setup and a single lot for each product per macro-period.

Table 1 Comparison of model characteristics (x/− indicates that a feature is present/missing in the respective model; ≈ indicates that a feature is partially considered.)

	Small-bucket			Big-bucket			
	MLCSLP	MLPLSP	MLPLSP_{SS}	MLCLSP_{L}	MLCLSP_{BS}	MLCLSP_{LS}	GLSPMS
Time structure							
Macro-periods	−	−	−	x	x	x	x
Micro-periods	x	x	x	−	−	−	x
Lot transfer							
Batch production	−	−	−	x	x	−	−
Lot-streaming	−	−	≈	−	−	x	≈
Fixed lead times	L_{ms}	L_{ms}	−	L_{mt}	−	−	−
Setup times							
Time of setup op.	begin	any	any	any	any	any	begin, end
Duration of setup	L_{ms}	L_{ms}	$2L_{ms}$	L_{mt}	L_{mt}	L_{mt}	$2L_{ms}$
Splitting of setup	−	−	x	−	−	−	x
Number of products and setups per macro-period							
Max # of products	S/T	$S/T+1$	$S/T+1$	N	N	N	S/T
Max # setups per product	S/T	S/T	S/T	1	1	1	S/T

4 Computational Results

4.1 Test Instances

In order to enable a comparison between the models a subset of the class 1 instances from Tempelmeier and Buschkühl (2009) are used. In total we use 480 instances, each representing a problem with 10 products produced on 3 machines in 3 production stages over a planning horizon of 4 macro-periods. The test instances have the following properties:

Product structure: general and assembly
Machine assignment: cyclic and acyclic
Coefficient of variation (CV) of the demand: 0.2, 0.5, 0.8
Capacity utilization (CU): 50, 70, 90 %, increasing and decreasing from the final to the first production stage.
Time-between-orders (TBO): 1, 2, 4, and varying.

All instances are capacity feasible considering the $MLCLSP_L$ formulation. Due to the complexity of some of the model formulations, it makes no sense to consider bigger instances. Some pretests on a smaller subset of instances have shown that for small-bucket models 5 micro-periods per macro-period provide a good balance between solution quality and computational tractability. All test instances were solved on a INTEL X5570 processor with 4 GB memory and with a Linux operating system using IBM ILOG CPLEX 12.4.

4.2 Results

All test instances were solved using all seven model variants (MLCSLP, MLPLSP, $MLPLSP_{SS}$, $MLCLSP_L$, $MLCLSP_{BS}$, $MLCLSP_{LS}$ and GLSPMS). A maximum runtime of one hour was applied. In Table 2 we report the obtained results. MAPD denotes the mean average percentage deviation of all instances solved with a specific model to the best solution found. In particular, the model which provides the lowest

Table 2 Computational results of all test instances

	Small-bucket			Big-bucket			GLSPMS
	MLCSLP	MLPLSP	$MLPLSP_{SS}$	$MLCLSP_L$	$MLCLSP_{BS}$	$MLCLSP_{LS}$	GLSPMS
MAPD (%)	6.5	2.1	1.3	34.1	9.2	0.2	4.4
$MAPD_{hc}$ (%)	6.9	2.3	1.6	41.4	10.5	0.4	0.4
$MAPD_{sc}$ (%)	9.3	4.0	1.5	20.3	12.1	0.2	10.2
GAP (%)	5.6	10.6	4.1	0.0	8.9	9.1	29.6
Optimal (%)	13.8	0.0	22.7	100	26.7	35.0	0.0
RT (s)	3329	3600	3039	<0.01	2809	2548	3600

cost for a certain instance is used as the benchmark. MAPD_{sc} and MAPD_{hc} are showing the deviation of holding cost and setup cost, respectively. GAP reports the average gap returned by CPLEX after one hour runtime and the row optimal shows the percentage of instances that could be solved to optimality. The final row RT indicates the average run time.

MLCLSP_{LS} provides in most cases the solution with the lowest cost. This is not surprising, because this model allows simultaneous production of predecessor and successor items and full scheduling flexibility within a macro-period. The small-bucket models are providing results with slightly higher cost, but the cost are decreasing with the additional flexibility when switching from MLCSLP to MLPLSP and to MLPLSP_{SS}. The results of the MLCLSP_{BS} indicate that the batching assumption leads to an average cost increase of 9.2 %. Since the MLCLSP_{L} includes a lead time of one macro-period a lot of additional inventory is necessary such that the total cost increase by 34.1 %. Surprisingly the quality provided by the GLSPMS is between the MLCSLP and the MLPLSP although it provides more flexibility. But this lower quality might be caused by the extraordinary high average gap of 29.6 %. In particular, the computational behavior regarding gap and average runtime is quite similar with two exceptions. MLCLSP_{L} can be solved for each instance without any problem and GLSPMS cannot be solved for any instance within the time limit.

In Table 3 we disaggregate the gap and the MAPD results for different instances classes. The reported gaps show that for all model variants the instances with a general product structure and an acyclic machine assignment are the easiest to solve. This is surprising because an assembly structure where each item has at most one successor should simplify the scheduling problem. The CV has a clear impact on the solvability—the higher the variation, the easier to solve. An explanation might be that for a uniform demand pattern there exist multiple solutions with similar cost which avoid the early pruning of the branch-and-bound tree. Only the GLSPMS is an exception, where the CV seems to have no effect. The TBO aspect has a different effect on the small- and big-bucket models. For the small-bucket models it is better to have a small TBO, i.e. setup costs are small compared to holding cost. For the GLSPMS this effect is quite dramatic. For the MLCLSP_{BS} and MLCLSP_{LS} the picture is contrary. Here it seems that a higher TBO leads to easier problems, which is in contrast to the usual findings for lot-sizing problems. When looking at the CU results, the small-bucket models are not affected by a change of the CU, whereas the big-bucket versions behave like expected, namely, they are easier to solve if the CU is low.

The results for the MAPD show that an increased CV reduces the differences between the model variants, but regarding the TBO and CU we cannot identify a clear picture. The results of the MLCLSP_{L} indicate that neglecting the scheduling aspect leads to worse results, if the problem instance allows a lot of flexibility in the solution, i.e., a low CV, TBO or CU.

Table 3 Resulting average GAP and MAPD for different groups of test instances (Bold numbers indicate the lowest gap in the respective column.)

	MAPD							GAP						
	Small-bucket			Big-bucket				Small-bucket			Big-bucket			
	MLCSLP	MLPLSP	$MLPLSP_{SS}$	$MLCLSP_L$	$MLCLSP_BS$	$MLCLSP_LS$	GLSPMS	MLCSLP	MLPLSP	$MLPLSP_{SS}$	$MLCLSP_L$	$MLCLSP_BS$	$MLCLSP_LS$	GLSPMS
Total	6.5	2.1	1.3	34.1	9.2	0.2	4.4	5.6	10.6	4.1	0.0	8.9	9.1	29.6
Product and process structure														
Assembly-acyclic	7.1	2.7	1.6	**30.7**	12.4	0.0	**3.8**	5.9	11.0	3.7	0.0	9.2	6.6	29.6
Assembly-cyclic	7.5	2.3	1.6	36.4	10.3	0.0	4.8	6.6	12.7	4.0	0.0	8.6	8.4	32.8
General-acyclic	6.0	2.0	1.0	34.5	8.7	0.0	**3.8**	**2.6**	**6.8**	**1.5**	0.0	**6.3**	**2.7**	**24.2**
General-cyclic	**5.5**	**1.5**	**0.9**	35.0	**5.3**	0.9	5.1	7.3	12.3	7.3	0.0	11.4	18.9	31.7
CV (Coefficient of Variation)														
0.2	8.1	2.8	1.8	42.7	11.6	**0.2**	4.8	7.9	12.1	6.2	0.0	14.8	15.0	**28.4**
0.5	7.0	2.4	1.5	33.6	9.7	0.3	5.0	5.8	11.0	4.3	0.0	7.8	7.7	30.8
0.8	**4.5**	**1.2**	**0.6**	**26.0**	**6.3**	**0.2**	**3.2**	**3.0**	**8.9**	**1.8**	0.0	**4.1**	**4.7**	29.6
TBO (time between orders)														
1	**5.5**	**1.5**	**1.2**	38.8	**7.9**	0.3	1.7	**2.7**	**4.8**	**2.0**	0.0	9.3	13.7	**11.7**
2	6.7	2.3	1.4	34.7	9.6	0.2	**3.8**	6.1	10.6	4.8	0.0	10.5	11.9	26.9
4	6.6	2.3	**1.2**	**27.3**	8.7	**0.1**	7.3	8.7	16.8	5.7	0.0	**7.3**	5.7	46.1
var	7.2	2.4	1.3	35.9	10.5	0.3	4.6	4.8	10.5	4.0	0.0	8.4	**5.3**	33.7
CU (capacity utilization)														
50	6.3	2.7	1.7	48.3	7.7	**0.1**	4.5	5.4	11.7	3.3	0.0	**4.3**	**4.0**	31.7
70	7.6	2.1	1.2	34.9	10.0	0.2	5.3	5.8	10.1	3.8	0.0	8.7	7.8	29.5
90	5.6	**1.7**	1.0	**20.6**	8.9	0.3	4.1	5.1	10.0	4.7	0.0	12.7	14.1	27.8
Increasing	**5.5**	**1.7**	**0.9**	27.2	**7.1**	**0.1**	**3.5**	**4.6**	**9.2**	**2.9**	0.0	9.5	9.3	**26.4**
Decreasing	7.6	2.4	1.6	39.7	12.1	0.4	4.3	7.1	12.4	5.9	0.0	9.3	10.4	32.5

5 Conclusions

We have shown various ways, derived from literature, to model a multi-level capacitated lot-sizing and scheduling problem. Although all the models are aiming to solve the same problem, there are substantial differences. The first aspect is the model structure itself. Here we have seen different ways to represent the scheduling and synchronization aspects in the multi-level product structure. This ranges from allowing simultaneous production of predecessor and successor items (MLCLSP_{LS}) to batch production, where a lot has to be completed before it can be used for further processing (MLCLSP_{L}, MLCLSP_{BS}). In between there are models with a relation between successor and predecessor which depends on the internal time structure of the model (MLCSLP, MLPLSP, MLPLSP_{SS}, GLSPMS). Also the timing of setup operations is an issue in most model variants due to the enforced decomposition of the planning horizon into time buckets.

The numerical results have shown, that all models, which solve the full scheduling problem, demand a high computational effort. Nevertheless, it seems that rather complex big-bucket formulations are worth to consider, because of their additional flexibility regarding the timing and their computational tractability. A surprising result is, that the GLSPMS as a flexible small-bucket model drops behind when solving it. Of course, all models provide different solutions and thus also different objective values. This can be seen as measure of flexibility allowed by the model formulation. Summarizing the results it seems that the MLPLSP_{SS}, the MLCLSP_{LS}, and the MLCLSP_{BS} are the most favorable formulations, regarding flexibility, detailed modeling and computational tractability. But still a significant demand of new and fast solution algorithms is given for these complex models in order to be applied to bigger problem instances and probably sometimes to real-world cases.

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Hybrid Metaheuristics for Project Scheduling and Staffing, Considering Project Interruptions and Labor Contracts

Thomas Felberbauer, Karl F. Doerner, and Walter J. Gutjahr

Abstract This article extends a recently developed model for project scheduling and staffing by addressing two practically important features, namely the possibility of interruptions between the execution periods of a project on the one hand, and decisions between different types of labor contracts on the other hand. A hybrid metaheuristic employs a decomposition of the problem into a project scheduling problem and a personnel planning problem. For the scheduling decision, a guided variable neighborhood descent search is applied, whereas for the personnel planning decision, a greedy staffing heuristic is used to obtain initial solutions. In a post-processing phase, information about the best greedily evaluated schedule triggers a re-evaluation of the staffing decision by means of an exact solver. To test the approach, we compare the outcome of the developed hybrid metaheuristic with the results obtained by applying only the exact solver to the considered optimization problem. The numerical tests show that the metaheuristic performs well for small to medium-sized test instances and offers good solutions for larger instances where the exact solver fails to return a feasible solution.

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1 Introduction

Deploying human resources efficiently, especially in high-wage countries and labor-intensive industries, is critical to global competitiveness. Economic considerations show that for many companies, labor is the primary direct cost component. Cutting this cost, even if just by a few percentage points, by implementing a more effective scheduling or staffing decision therefore can prove very beneficial. Yet the related, inevitable resource planning tasks pose a huge challenge to companies that manage vast human resources with heterogeneous skills. Aggregate planning models, which typically use optimization techniques such as linear programming (LP) or mixed integer programming (MIP), can support managers during their decision-making process. Without doubt, optimization models, solved by means of standard solvers as, e.g., CPLEX, are advantageous compared with manual planning (e.g., spreadsheet-based planning applying simple greedy heuristics). Empirical evidence on the advantages of optimization models compared to manual planning approaches for project scheduling and staffing can be found in Heimerl and Kolisch (2010a). However, as the size of the company increases, the problem tends to grow more complex (e.g., Bruecker et al. 2015), and standard solvers may fail to find a feasible solution within a reasonable time. Therefore, companies need alternatives, and one of them is to use metaheuristics.

In this study, we consider a combined personnel planning and project scheduling problem, in an effort to minimize internal and external labor costs. We extend the MIP formulation introduced by Heimerl and Kolisch (2010a) by (1) allowing for an interruption between project periods, and (2) introducing decisions about different labor contracts, namely, full-time, part-time, or no employment. The key questions include when to start the project periods, which type of labor contract per internal human resource should be applied, and how work packages should be assigned to human resources. For our numerical study, we use synthetically generated test instances, obtained by an instance generator (inspired by a real-world situation) that has been presented by Heimerl and Kolisch (2010a). Using these instances, we can test the performance of a commercial MIP solver for the basic model and for the two developed model extensions. For large instances, the high problem complexity leads the state-of-the-art MIP solver to fail to find a feasible integer solution within a reasonable time. To cope with this situation, we propose an innovative hybrid metaheuristic solution procedure that seeks good solutions for complex instances.

Figure 1 presents the framework of the developed hybrid metaheuristic. The main components are the iterated local search metaheuristic (ILS) (e.g., Lourenço et al. 2010) and the exact post-processing. After generating an initial solution, we use variable neighborhood descent (VND) (e.g., Mladenovic and Hansen 1997) as our guided local search (GLS) algorithm. The neighborhood operators then use information about the capacity profiles that guide the VND search to prioritize time periods with high external or overtime costs. To evaluate the predefined project period starting times, according to the neighborhood function, we use a greedy staffing heuristic. The perturbation operator is applied if all neighborhoods fail to

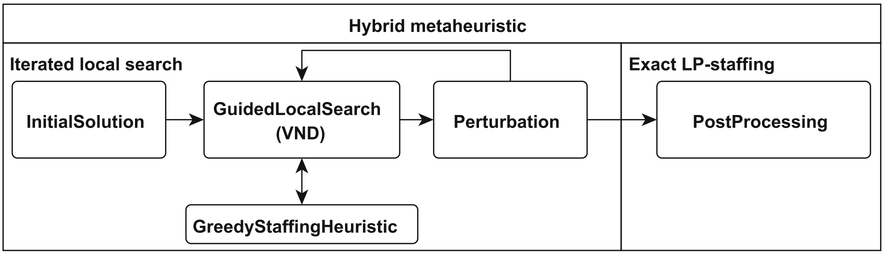

Fig. 1 Hybrid metaheuristic framework

improve the incumbent solution. It generates a new solution randomly and helps escape from any local optimum. The ILS approach with the implemented VND is repeated until the stopping criterion is reached. Finally, in a post-processing step, the staffing decisions of the best schedules obtained by the ILS are re-evaluated by means of the exact LP solver. The top project schedules become the input for the linear staffing problem, which is optimized using CPLEX.

Substantial literature focuses on project scheduling, with and without the possibility of preemption or interruption, personnel planning, different kinds of labor contracts, staffing, and heterogeneous skills. However, to the best of our knowledge, no previous studies have combined these aspects and provided an appropriate solution method. The contribution of our article thus is threefold:

1. We advance a recently published project scheduling and staffing model by two model extensions.
2. We investigate when an exact solver is an appropriate method to solve the considered models and at which point a metaheuristic solution method becomes necessary.
3. We provide a hybrid metaheuristic, which performs well for small and medium-sized test instances and offers good solutions also in cases where the MIP solver fails to return a feasible solution.

The paper is organized as follows: Section 2 gives a short review of research into the optimization of personnel planning and project scheduling decisions. In Sect. 3, we describe the basic model and the two proposed model extensions. Section 4 presents the heuristic solution methods for the model extensions. The test instance structure and the parametrization of the new hybrid metaheuristic is described in Sect. 5. Finally, in Sect. 6, we present the results of our computational experiments, designed to compare the heuristic technique to the MIP solver.

2 Literature Review

Surveys on the resource-constrained project scheduling problem (RCPSP) and its extensions as well as on exact and heuristic solution methods for this class of problems are available in Kolisch and Padman (2001) and Hartmann and Briskorn (2010). Bergh et al. (2013) also offer a state-of-the-art literature review on personnel scheduling, analyzing 293 articles from 2004 to 2012. They classify personnel according to their labor contracts, distinguishing, in particular, between full-time and part-time workers. The vast majority of research focuses exclusively on full-time personnel scheduling problems; one of the contributions of our study consists in overcoming this restriction. The survey (Weglarz 2011) focuses on solutions for deterministic, single-project, single-objective project scheduling problems using a finite or infinite number of modes. These authors conduct in-depth analyses of the exact and heuristic solution methods for the RCPSP and its multi-mode generalization (MRCPSP), is recently attracting much attention (see, e.g., Schnell and Hartl 2016). A solution approach using variable neighborhood search, as presented by Fleszar and Hindi (2004), instead focuses on the non-preemptive and non-renewable RCPSP problem, with the goal of minimizing the total makespan.

Messelis and De Causmaecker (2014) propose an automatic algorithm selection approach for the MRCPSP, also with a minimal makespan objective. Their result is a super-algorithm that outperforms all of its individual components. Given a new instance, the super-algorithm selects an algorithm from a portfolio of state-of-the-art algorithms (tabu search and hybrid genetic algorithm), based on simple characteristics of the instance. Barz and Kolisch (2014) describe resource assignments in the telecommunication industry, where a discrete Markov decision process offers a model for incoming jobs over an infinite time horizon, and investigate a hierarchical, multi-skill resource assignment.

To deal with project selection, project scheduling and staff assignment in an integrated framework, Gutjahr et al. (2008) propose a hierarchical optimization model. On the upper hierarchy level, a portfolio of projects is selected, taking the effects of competence development over time (learning and knowledge depreciation with respect to different skills) into account. A MIP formulation is achieved by a linear approximation of the originally nonlinear multi-period optimization model. On the lower level, for a candidate set of projects to be selected, project scheduling decisions as well as staff assignment decisions are optimized. Exact solutions are compared to solutions obtained from a combination of two metaheuristics for the project portfolio decision with greedy search for the scheduling/staffing decision. In Gutjahr (2011), a related model is analyzed from a more theoretical perspective. In particular, the mathematical results address the question of whether an optimal project selection policy considering competence development would rather specialize in a narrow class of projects related to core competencies, or rather retain a mixed project portfolio.

The project scheduling and staffing model proposed by Heimerl and Kolisch (2010a) deals with the problem of assigning multi-skilled human resources to work, while accounting for their specific skills which result in heterogeneous efficiencies, with the aim of minimizing the costs for internal and external personnel. After presenting a basic problem formulation, the authors reformulate the problem and compare the performance of a state-of-the-art solver applied to the two model formulations. They also investigate the influence of different parameters—such as time windows, utilization, and the distribution of skills over resources—on the solution value. In another contribution, Heimerl and Kolisch (2010b) investigate staffing when the human resources possess multiple skills and experience learning or knowledge depreciation, depending on the tasks to which they are assigned. In their effort to find strategic target values for company skill levels, the authors present a constrained, nonlinear optimization model, solved using COIN-OR's Ipopt. In Kolisch and Heimerl (2012), these same authors develop a solution method, using a genetic algorithm and tabu search, for the intermediate planning level of project scheduling and staffing. They separate the problem into a scheduling and a staffing problem, where the latter is solved by a generalized network simplex algorithm. The developed hybrid metaheuristic outperforms the MIP solver in that it is able to solve also larger instances.

The contributions from Heimerl and Kolisch (2010a) and Kolisch and Heimerl (2012) constitute the starting point of our study. However, unlike these previous studies, our work addresses an application context where the execution of a project does not have to follow a rigid time scheme, but, instead, shorter or longer interruptions of the execution are allowed, as long as an overall time window for the project is respected. For some applications in fields as, e.g., construction or software engineering, this seems to be a more realistic assumption. Especially in cases with independent projects where a specific due date per project is defined and the project management is entirely the responsibility of the service provider this is an important assumption which lead to more flexibility and significant cost savings. Secondly, our model extends the approach in the above-mentioned papers by the inclusion of a possible decision about the different types of labor contracts. For the latter model extension, we calculate internal labor costs not based on scheduled work time, but rather based on available works hours. We generate test instances based on the same test instance generator as Kolisch and Heimerl, but include instances up to a maximum of 300 projects instead of only 50 as in the previous work. We test the limits of the exact solver and develop a metaheuristic solution approach for the two model extensions. Whereas Kolisch and Heimerl (2012) address a short-term planning level, we rather focus on a context of a tactical planning level.

3 Problem Formulations

Table 1 lists the various problem formulations investigated in this paper. The basic model from which we start is the model MIP-II presented in Heimerl and Kolisch (2010a). It addresses the simultaneous scheduling and staffing of multiple projects with multi-skilled human resources that exhibit heterogeneous skill efficiencies. We extend this model in a stepwise manner.

First, in model extension *MIP-P*, we adapt the *MIP-II* model by allowing that the execution of a project is interrupted for one or several time periods and resumed at a later time. Example project scheduling decisions that highlight the differences between *MIP-II* and *MIP-P* are given in Sects. 3.2 and 3.3 and in Fig. 2. Second, with *MIP-SP*, we further extend the model by allowing decisions on the applied labor contracts for each internal resource and period.

3.1 Model Formulation MIP-II

Let us start with a description of the constants and variables of MIP-II.

Projects: P is the set of projects that must be completed during the given planning horizon (e.g., 1 year). All projects are independent but compete for the same

Table 1 Problem formulations

Abbreviation	Description	Features
MIP-II	Basic model	Project scheduling and staffing
MIP-P	Model extension one	Interruptive project scheduling and staffing
MIP-SP	Model extension two	Labor contracts, interruptive project scheduling and staffing

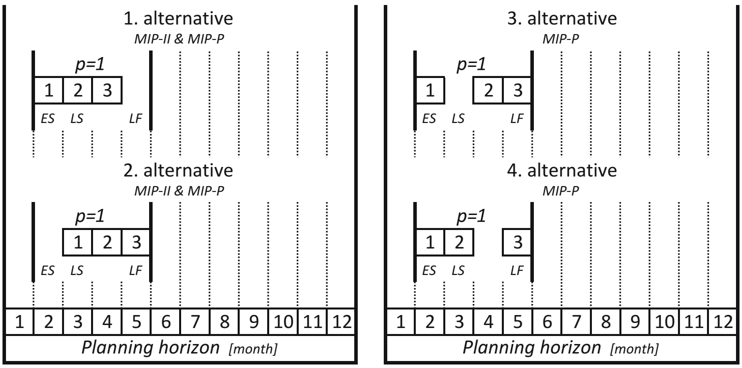

Fig. 2 Visualization of project schedule alternatives for *MIP-II* and *MIP-P*, assuming $|P| = |S| = |K| = 1$, $ES_p = 2$, $LS_p = 3$, and $d_p = 3$

resources. The planning horizon comprises T discrete time periods $t = 1, \ldots, T$ (e.g., months). To each project $p \in P$, a work schedule is assigned, consisting of d_p periods. For its description, the periods are indexed by $q = 1, \ldots, d_p$. During period q of project p ($1 \leq q \leq d_p$), project p requires r_{psq} work units of skill s. This amount of work represents a *work package*. The set of all skills s is denoted by S. Furthermore, MIP-II assumes that to each project p, a time window $[ES_p, LS_p]$ is given, wherein ES_p and LS_p are the indices of the earliest period and the latest period in which project p can be started, respectively. The latest finish period for project p is defined as $LF_p = LS_p + d_p - 1$. The time window size of project p is defined as $\gamma = LS_p - ES_p$.

Resources: The model MIP-II distinguishes between internal and external resources. Each internal resource k, an element of the set K of all internal human resources, holds a subset of skills $S_k \subseteq S$. Vice versa, from the perspective of skills, the subset $K_s \subseteq K$ is defined as the set of all internal resources k that possess skill s. The efficiency of resource k in performing skill s is denoted by the variable η_{sk}, which is known at the beginning of the planning process and does not change during the planning horizon T. The higher the efficiency value η_{sk}, the quicker a work package r_{psq} assigned to resource k can be executed. We call the amount r_{psq} of work the *effective work time* and assume that the *real work time* is given by r_{psq}/η_{sk}. For example, suppose that the work package r_{psq} needs 10 time units (effective work time) to be processed, then employee k, with efficiency $\eta_{sk} = 1.25$ in skill s, needs $r_{psq}/\eta_{sk} = 10/1.25 = 8$ time units of real work time to complete this work package. The availability of an internal, full-time equivalent human resource k in time period t is R^r_{kt} in regular working time and R^o_{kt} in overtime. The cost per time unit (cost rate) of the internal human resource k during regular time is c^r, whereas it is c^o during overtime. An important assumption is that external resources are unlimited for each skill s, at the expense of a comparably high cost rate c^e_s. The cost rates for each skill s are ordered as follows: $c^e_s > c^o > c^r$.

Decision Variables: In the model *MIP-II*, decision variable $x^r_{ptsk} \geq 0$ defines the amount of (effective) work performed by internal resource k during regular working time using skill s in period t for project p. Analogously to the first decision variable, $x^o_{ptsk} \geq 0$ describes the amount of (effective) work performed by internal resource k during *overtime*, using skill s in period t for project p. Finally, the decision variable $y_{pts} \geq 0$ describes the amount of (effective) work performed by *external* resources using skill s in period t for project p. To define the start times of the projects, we introduce the binary decision variables $z_{pt} = 1$ if project p starts in period t and $z_{pt} = 0$ otherwise.

3.2 MIP-II Model

$$\min_{z,x,y} \sum_{p \in P} \sum_{t=ES_p}^{LF_p} \sum_{s \in S} \Big(c_s^e y_{pts} + \sum_{k \in K_s} \frac{1}{\eta_{sk}} (x_{ptsk}^r c^r + x_{ptsk}^o c^o) \Big) \tag{1}$$

subject to

$$\sum_{t=ES_p}^{LS_p} z_{pt} = 1 \qquad p \in P \tag{2}$$

$$\sum_{(\tau,q) \in \tau_{pt}} r_{psq} z_{p\tau} \leq y_{pts} + \sum_{k \in K_s} (x_{ptsk}^r + x_{ptsk}^o) \qquad p \in P,\ t = ES_p, \ldots, LF_p,\ s \in S \tag{3}$$

$$\sum_{p \in P} \sum_{s \in S} \frac{1}{\eta_{sk}} x_{ptsk}^r \leq R_{kt}^r \qquad k \in K,\ t = 1, \ldots, T \tag{4}$$

$$\sum_{p \in P} \sum_{s \in S} \frac{1}{\eta_{sk}} x_{ptsk}^o \leq R_{kt}^o \qquad k \in K,\ t = 1, \ldots, T \tag{5}$$

$$\sum_{t=ES_p}^{LF_p} \sum_{s \in S} \sum_{k \in K_s} (x_{ptsk}^r + x_{ptsk}^o) \geq e_p \sum_{t=ES_p}^{LF_p} \sum_{s \in S} y_{pts} \qquad p \in P \tag{6}$$

$$x_{ptsk}^r, x_{ptsk}^o \geq 0 \qquad p \in P,\ t = ES_p, \ldots, LF_p,\ s \in S,\ k \in K_s \tag{7}$$

$$y_{pts} \geq 0 \qquad p \in P,\ t = ES_p, \ldots, LF_p,\ s \in S \tag{8}$$

$$z_{pt} \in \{0, 1\} \qquad p \in P,\ t = ES_p, \ldots, LS_p \tag{9}$$

The objective function in Eq. (1) minimizes the scheduled costs of internal and external employees, where the internally scheduled costs depend on the efficiency values η_{sk} since for cost calculation, effective work times have to be transformed to real work times. External employees do not have different efficiency values. The constraints in Eq. (2) force each project p to start exactly once in the time window $[ES_p, LS_p]$ between the earliest and the latest starting time of project p. In Eq. (3),

$$\tau_{pt} = \{(\tau, q) \in \{ES_p, \ldots, LS_p\} \times \{1, \ldots, d_p\} \mid \tau + q - 1 = t\} \tag{10}$$

denotes the set of all possible combinations (τ, q) of project start periods τ and project period indices q of project p that lead to resource demand in time period t. Thus, the equation guarantees that for all such combinations of project start periods and project period indices, the *required* effective work time of project p in time period t is equal to or lower than the sum of *scheduled* internal and external effective work times of project p in time period t. (An example follows in the next paragraph.) The constraints in Eqs. (4) (for regular time) and (5) (for overtime) ensure that for each resource k in each time period t, the scheduled *real* work time is equal to or lower than the available time. For each project p, a minimum ratio e_p of work processed by internal human resources is defined by Eq. (6). Finally, the domains of the decision variables are defined in Eqs. (7)–(9).

Heimerl and Kolisch (2010a) show that the *MIP-II* formulation is a generalization of the multiprocessor scheduling problem. This leads them to the finding that the basic problem *MIP-II* can be classified as NP-hard too. Due to the fact that the model extensions *MIP-P* and *MIP-SP* can be reduced to the basic problem they are also NP-hard.

3.2.1 Project Scheduling Example, *MIP-II* Formulation

Figure 2 depicts an example of the project schedule alternatives, assuming one project ($|P| = 1$), one employee ($|K| = 1$), one skill ($|S| = 1$), an earliest start time of $ES_p = 2$, a latest start time of $LS_p = 3$, a project duration of $d_p = 3$, and consequently a latest finish time of $LF_p = 5$. Alternatives 1 and 2 indicate the possible project plans for the *MIP-II* model (the other alternatives in the picture will be explained later). For the *MIP-II* model formulation, the sets τ_{pt} defined in Eq. (10) are given as follows:

$$\begin{aligned}
\tau_{12} &= \{(2, 1)\} \\
\tau_{13} &= \{(2, 2), (3, 1)\} \\
\tau_{14} &= \{(2, 3), (3, 2)\} \\
\tau_{15} &= \{(3, 3)\}.
\end{aligned}$$

Using τ_{pt} in Eq. (3) and omitting the indices p, s and k (which are all equal to 1), we obtain the following constraints for each time period t within the time window $[ES_p, LF_p]$:

$$r_1 \cdot z_2 \leq y_2 + x_2^r + x_2^o \; (t = 2)$$
$$r_2 \cdot z_2 + r_1 \cdot z_3 \leq y_3 + x_3^r + x_3^o \; (t = 3)$$
$$r_3 \cdot z_2 + r_2 \cdot z_3 \leq y_4 + x_4^r + x_4^o \; (t = 4)$$
$$r_3 \cdot z_3 \leq y_5 + x_5^r + x_5^o \; (t = 5)$$

3.3 Model Extension MIP-P

According to the model formulation of *MIP-II*, the start period of a project p can be scheduled in the time window $[ES_p, LS_p]$ between the project's earliest and its latest start time. In this model, the time for the start period of the project already defines when the subsequent periods $q = 2, \ldots, d_p$ of the project are scheduled, namely consecutively following the start period $q = 1$ without the possibility of any break. To model now the project scheduling and staffing problem with *allowed* interruption, we replace the binary decision variable z_{pt} by binary decision variables $z_{pqt} \in \{0, 1\}$. These decision variables identify the start times of all *periods* q of all projects p. The new variable z_{pqt} is equal to 1 if project period q of project p is scheduled in time period t, and equal to 0 otherwise. For the new formulation, the definitions of the quantities ES_p, LS_p, and LF_p assigned to project p remain unchanged, whereas the earliest start time and the latest start time for period q of project p are now defined as $ES_{pq} = ES_p + q - 1$ and $LS_{pq} = LS_p + q - 1$, respectively. Equation (11) defines the new decision variable:

$$z_{pqt} \in \{0, 1\} \qquad p \in P, \; q = 1, \ldots, d_p, \; t = ES_{pq}, \ldots, LS_{pq} \tag{11}$$

Thus, whereas the start times are defined once per project p in the *MIP-II* model formulation, they are defined per project p and per project period q in the *MIP-P* formulation. Therefore, Eq. (9) is replaced by Eq. (11), and Eq. (2) is replaced by Eq. (12) below, which forces each period q of project p to be scheduled exactly once within its start time window $[ES_{pq}, LS_{pq}]$:

$$\sum_{t=ES_{pq}}^{LS_{pq}} z_{pqt} = 1 \qquad p \in P, \; q = 1, \ldots, d_p \tag{12}$$

The constraint that project periods q must be scheduled in ascending order is expressed by Eq. (13):

$$1 + \sum_{t=ES_{pq}}^{LS_{pq}} tz_{pqt} \leq \sum_{t=ES_{p,q+1}}^{LS_{p,q+1}} tz_{p,q+1,t} \qquad p \in P, \quad q = 1, \ldots, d_p - 1 \tag{13}$$

Because the new decision variables z_{pqt} define the start times per project p and project period q, the set τ_{pt} changes. The possibility of interruptions implies that it is no longer feasible to use the formula (10) for those combinations (τ, q) of project start times τ and project period indices q that have an impact on time period t. Instead, we introduce the set τ'_{pt} which includes all possible project periods q of project p that can (but need not necessarily) lead to demand in time period t:

$$\tau'_{pt} = \{q \in \{1, \ldots, d_p\} \mid ES_{pq} \leq t \leq LS_{pq}\}. \tag{14}$$

We adjust the constraints in Eq. (3) according to the new decision variables z_{pqt} and the modified sets τ'_{pt}. The new formulation is

$$\sum_{q \in \tau'_{pt}} r_{psq} z_{pqt} \leq y_{pts} + \sum_{k \in K_s} (x^r_{ptsk} + x^o_{ptsk}). \qquad p \in P, \quad t = ES_p, \ldots, LF_p, \quad s \in S \tag{15}$$

Note that the model extension MIP-P is a special case of the problem discussed by Kolisch and Heimerl (2012). Using their model formulation MIP-P can be obtained by setting all minimum time lags to 1 and the maximum of the time lags to the time window size γ.

3.3.1 Project Scheduling Example, *MIP-P* Formulation

For the model extension *MIP-P*, the possible project schedules in the previously presented example extend from two to four alternatives, as depicted in Fig. 2. The binary decision variables z_{pt} change to z_{pqt}, which effects that for all periods q of project p, the start times are defined separately, subject to the dependence constraints in Eq. (13). These constraints ensure that the start time of (p, q) must be lower than the start time of its successor $(p, q + 1)$. The new sets τ'_{pt} are now the collections of those project periods q of project p that could possibly impact time period t. For our example, we get:

$$\begin{aligned} \tau'_{12} &= \quad \{1\} \\ \tau'_{13} &= \quad \{2, 1\} \\ \tau'_{14} &= \quad \{3, 2\} \\ \tau'_{15} &= \quad \{3\} \end{aligned}$$

Using the new τ'_{pt} in Eq. (15) and omitting again the indices p, s, and k, we obtain the following constraints for the time periods $t = ES_p, \ldots, LF_p$:

$$r_1 \cdot z_{12} \leq y_2 + x_2^r + x_2^o \; (t = 2)$$
$$r_2 \cdot z_{23} + r_1 \cdot z_{13} \leq y_3 + x_3^r + x_3^o \; (t = 3)$$
$$r_3 \cdot z_{34} + r_2 \cdot z_{24} \leq y_4 + x_4^r + x_4^o \; (t = 4)$$
$$r_3 \cdot z_{35} \leq y_5 + x_5^r + x_5^o \; (t = 5)$$

3.4 Model Extension MIP-SP

As a second modification, we add decisions on labor contracts which determine the term of employment and therefore the available hours of internal resources. In order to represent these decisions, we introduce new binary decision variables v_{ktw} to specify which labor contract w out of a set of possible alternatives W (including the option "no employment") is chosen for the internal resource k in time period t. The variable v_{ktw} equals 1 if, for resource k in time period t, labor contract w is applied. Equation (16) below defines the binary decision variables, and Eq. (17) guarantees that exactly one type of labor contracts is activated per resource k and time period t:

$$v_{ktw} \in \{0, 1\} \qquad k \in K, \; t \in T, \; w \in W \tag{16}$$

$$\sum_{w \in W} v_{ktw} = 1 \qquad k \in K, \; t \in T \tag{17}$$

In particular, we consider three possible types of labor contracts: $w = 1$ (no employment), $w = 2$ (part-time work), and $w = 3$ (full-time work). The constants f_w ($w \in W$) define to each labor contract w the corresponding factor of available time, compared to a regular full-time job equivalent. The factors were chosen as $f_1 = 0$ in the case where the employee is not occupied, $f_2 = 0.5$ in the case of a part-time contract, and $f_3 = 1$ in the case of a full-time labor contract. In the model *MIP-SP*, we adapt the capacity constraints of the basic model *MIP-II* concerning the availability of internal resources during regular working time (Eq. (4)) and during overtime (Eq. (5)). Equations (18) and (19) below give the new constraints for regular working time and overtime, respectively. The left-hand sides of Eqs. (18) and (19), which represent the scheduled time, are the same as in the model *MIP-II*. The available hours on the right-hand sides, however, are adapted according to the chosen labor contract by multiplication with $\sum_{w \in W} f_w v_{ktw}$. The constant R^r_{kt} denotes now the working time of human resource k if this resource works under a full-time contract in time period t during regular working time, and R^o_{kt} denotes the analogous value during overtime. By multiplication by the binary decision variable v_{ktw} and the labor contract factor f_w and by summation over the three possible contract types, the actual availability of resource k in time period t is obtained.

$$\sum_{p\in P}\sum_{s\in S}\frac{1}{\eta_{sk}}x^r_{ptsk} \le \sum_{w\in W} R^r_{kt} f_w v_{ktw} \qquad k \in K,\ t \in T \tag{18}$$

$$\sum_{p\in P}\sum_{s\in S}\frac{1}{\eta_{sk}}x^o_{ptsk} \le \sum_{w\in W} R^o_{kt} f_w v_{ktw} \qquad k \in K,\ t \in T \tag{19}$$

As an example, assume a resource k with a maximum available time of $R^r_{kt} = 40$ h during regular working time in time period t and $R^o_{kt} = 12$ h during overtime. In this situation, giving resource k a full-time contract would result in 40 h available during regular time and 12 h available during overtime; a part-time contract would provide 20 h during regular time and six time hours during overtime, and deciding for "no employment" would of course set both availability values to zero.

The second modification in the model *MIP-SP* is that in the objective function, the calculation of the costs for internal resources during regular working time is not done based on scheduled time units (actual work), but based on regular time units according to the selected contract (contractual work). Therefore, the objective function in Eq. (1) is replaced by the following new objective function:

$$\min_{z,x,y,v} \sum_{p\in P}\sum_{t=ES_p}^{LF_p}\sum_{s\in S}\Big(c^e_s y_{pts} + \sum_{k\in K_s}\frac{1}{\eta_{sk}}(c^o x^o_{ptsk})\Big) + \sum_{k\in K}\sum_{t\in T}\sum_{w\in W} c^r f_w R^r_{kt} v_{ktw} \tag{20}$$

This objective function minimizes the labor costs of external resources, internal resources during overtime, and internal resources during regular working time. The decision variables are the project scheduling decisions represented by z, the staffing decisions represented by x and y, and the labor contract decisions represented by v. This modification is especially important for moderate levels of utilization; for many cases, it describes the internal labor costs of a company in a more realistic way. In the model *MIP-SP*, companies pay for the available hours of their personnel, as agreed upon by the employment contracts, not just for those hours that are assigned to work packages. Overtime and external resources can then be hired as required for a fee per time period, so we model the internal overtime as well as the external personnel costs according to the *MIP-II* model.

4 Heuristic Solution Methods

To provide a hybrid metaheuristic solution method for the problems *MIP-P* and *MIP-SP*, we use an adapted version of ILS (e.g., Lourenço et al. 2010), where VND is the local search component (e.g., Mladenovic and Hansen 1997). The general framework of the developed solution approach can be seen in Fig. 1, and the algorithmic outline is given in Algorithm 1. For instances in real-world scale, we seek a solution method that can find good solutions within a reasonable time and is able to outperform the MIP solver regarding solvability and solution quality. In this section, we present detailed information about the proposed solution approaches. The design decisions reflect our in-depth analysis of multiple (pre-)tests. We present the most successful framework for further discussion.

4.1 MIP-P Solution Method

To solve the project scheduling and personnel planning problem with the possibility of interruption, we separate the challenge into two distinct problems. The project schedules are determined by means of a metaheuristic, whereas the underlying staffing sub-problem is solved by a greedy heuristic. After generating an initial solution, the VND neighborhood operators determine the start times per project p and project period q. We use a two-dimensional solution representation by a matrix X. The rows of X correspond to the $|P|$ projects p, and the columns correspond to the project periods $q = 1, 2, \ldots$. The entry X_{pq} of the matrix X contains the start time for project period q of project p. The time windows $[ES_{pq}, LS_{pq}]$ and the start time relationships $X_{p,q-1} < X_{pq}$ (cf. Eqs. (13) and (15)) have to be respected. Note that X is an equivalent alternative encoding of the binary project scheduling decision variables z_{pqt}: For given X, each variable z_{pqt} can be determined in an obvious way, and conversely, for given $z = (z_{pqt})$, the start time X_{pq} of project period q of project p can be calculated by $X_{pq} = \sum_{t=ES_{pq}}^{LS_{pq}} t z_{pqt}$. In our metaheuristic approach, we work with the representation X, letting the developed metaheuristic determine the start times X_{pq} of the projects periods (p, q). Given a current scheduling solution X (or, equivalently, z), the demand of work package r_{psq} in time period t can be computed as

$$r'_{pst} = \sum_{q \in \tau'_{pt}} r_{psq} z_{pqt}. \tag{21}$$

Using this calculation, the underlying staffing sub-problem for a fixed schedule X of starting times reduces to a linear program:

$$\min_{x,y} \sum_{p\in P} \sum_{t=ES_p}^{LF_p} \sum_{s\in S} \Big(c_s^e y_{pts} + \sum_{k\in K_s} \frac{1}{\eta_{sk}} (x_{ptsk}^r c^r + x_{ptsk}^o c^o) \Big) \tag{22}$$

subject to

$$r'_{pst} \leq y_{pts} + \sum_{s\in K_s} (x_{ptsk}^r + x_{ptsk}^o) \qquad \begin{array}{l} p \in P, \\ t = ES_p, \ldots, LF_p, \\ s \in S \end{array} \tag{23}$$

and the constraints in Eqs. (3)–(8).

This staffing sub-problem can be solved by a standard LP solver. In a similar vein, Kolisch and Heimerl (2012) solve a similar staffing sub-problem by means of a generalized network simplex (GNS) algorithm which outperforms the CPLEX solver in terms of average computation times. Nevertheless, they address the short-term planning level where projects are disaggregated to activities and activity periods rather we focus on the tactical planning level. Concerning solution performance they show that doubling the number of projects from five to ten requires three times as much computational effort. Solving real-size problems, with hundreds of projects and resources, would create substantial problems because the metaheuristic solving the upper-level scheduling problem would then not be able to explore a sufficient number of project schedules within the given time budget. Preliminary test have shown that the computation time of the solvers strongly depends on the problem size, so the exact solver, i.e., CPLEX or GNS, appears too time-consuming for being used for each evaluation of a scheduling solution generated by the upper-level metaheuristic. For this reason, we implemented a greedy heuristic for the solution of the staffing sub-problem resulting from a predefined project schedule X. Our greedy heuristic provides a good approximation and is essentially faster than the solution of the linear program by CPLEX. For example, solving the underlying staffing problem, with $|P| = 20$ and $\gamma = 2$, for an arbitrary project period schedule the test shows that the LP solver CPLEX requires $\approx$ 85 ms and the greedy heuristic needs 2.5 ms to solve the staffing problem. The Solving a big problem with $|P| = 300$ and $\gamma = 2$ the solution time of CPLEX increases to $\approx$ 4100 ms whereas the greedy heuristic needs 42 ms to evaluate the predefined project schedule. The solution value of CPLEX is 1.06 %/0.55 % lower than the solution value of the greedy heuristic with respect to the two test instances.

We depict the pseudocode of the overall heuristic solution method in Algorithm 1 and explain its building blocks afterward.

Initial Solution The first step of the solution approach generates an initial solution X. The goal of the initial solution heuristic is to generate a constant capacity demand, where the number of project periods, defined by $u = \sum_{p\in P} d_p$ are scheduled to achieve

Algorithm 1 ILS search scheme for the MIP-P problem

$X \leftarrow \textit{InitialSolution}()$
$X^*, X' \leftarrow X$
while $t < t_{max}$ **do**
  $h \leftarrow 1$
  repeat{//GLS-VND}
    $X'' \leftarrow \textit{FirstImprovement}(X', h)$ {//find the best neighbor}
    $X, X^*, X' \leftarrow \textit{NeighborhoodChangeS}(X, X'', h)$ {//neighborhood change and accept solution}
  until $h \leq h_{max}$
  $X' \leftarrow \textit{Perturbation}(X, \alpha|P|)$ {//Perturbation}
  $t \leftarrow \textit{CpuTime}()$
end while
$X^* \leftarrow \textit{PostProcessing}(X_1^*, \ldots, X_n^*)$ {//reevaluate staffing for the top n heuristically evaluated schedules}
return X^*

balance across the time horizon T. The time window of each project period (p, q) between $[ES_{pq}, LS_{pq}]$ represents possible start times. To construct the initial solution, we observe the requirement of Eq. (13), namely, that the start period of project period q must be prior to the start period of $q + 1$. For the construction of the initial solution, we use an aggregated view of the demand situation. Therefore, the number of project periods q per time period t is crucial to project scheduling, but the detailed required capacities per work package r_{psq} can be neglected. The solution value of the initial solution $f(X)$ is calculated with the greedy staffing heuristic and set as incumbent solution X^* and solution X'. After the construction of the initial solution, the ILS procedure starts. The stopping condition in our case is the maximum allowed CPU time.

First Improvement For the GLS-VND, we used the *FirstImprovement()* function. The algorithm uses the solution X' and the current neighborhood seize h. It stops if the current neighborhood solution has a lower objective function value than the current incumbent solution $f(X'') < f(X^*)$ or if all neighborhoods are evaluated.

Neighborhood Function In the first step of the neighborhood function used within the *FirstImprovement()* operator, we can calculate the number of changes applied to the project period starting times X. The number of changes depends on the number of projects $|P|$ and the current neighborhood size h. The neighborhood function *FirstImprovement()* uses the capacity profiles of solution X', which indicate how much capacity is scheduled per time period t, separated into regular, overtime, and external capacity. In the local search operator $N_h()$, the starting times of the project periods q are first changed in time periods t where the external capacity consumption is very high. The solution modification principle shifts the starting time of the project period q forward or backward. Then, the *RepairSolution()* function checks all predecessor and successor project periods (p, q) for the required time window $[ES_{pq}, LS_{pq}]$ and its starting time relationship $X'^{i}_{p,q-1} < X'^{i}_{pq}$; if necessary, they are repaired. According to the calculated changes, this procedure continues, or else the schedule X'^i is returned for evaluation in the *FirstImprovement()* function.

Greedy Staffing Heuristic The greedy heuristic evaluates the project schedule X'^i_{pq} according to the neighborhood function. For the greedy staffing heuristic, an efficiency threshold value per skill s can be calculated for the time mode, that is, regular t^r_s or overtime t^o_s, according to $t^r_s = 1/(c^e_s/c^r)$ and $t^o_s = 1/(c^e_s/c^o)$. The threshold values in the staffing procedure identify an efficiency level at which the external resource is cheaper than the internal one. Therefore, we do not assign an employee k with skill s to a work package r'_{pst} if the resource efficiency is $\eta_{sk} < t^r_s$ during regular working time or $\eta_{sk} < t^o_s$ during overtime.

Before the resource assignment starts, the internal resources k are ordered according to their skill efficiencies in descending order. Employees with the highest η_{sk} values—that is, the most profitable ones—are scheduled first. Because costs during regular time c^r and overtime c^o differ, we introduce a normalized skill efficiency factor η'_{sk} for overtime, which is $\eta'_{sk} = \eta_{sk}/(c^o/c^r)$. The skill efficiencies η_{sk} and η'_{sk} are sorted in descending order in the set N_{sk}, with information about the time mode (regular or overtime). According to the predefined solution X^i, the work packages r'_{pst} get added to the set WP_{ts} for time period t and skill s.

In the greedy staffing heuristic, we iterate through all resource efficiency combinations in the set N_{sk} and all time periods t. For the respective time period t, we start the staffing with the most efficient employee $n_{sk} \in N_{sk}$ and check its efficiency threshold value t^r_s and t^o_s, as well as its availability in the respective time period. Next, a work package $r'_{pst} \in WP_{ts}$ that starts in the respective time period t and requires skill s is identified. According to the availability of resource k in time period t, all or a part of the work package r'_{pst} is assigned to resource k using skill s. After that, the availability of resource k in time period t is reduced, according to the previously made staffing decision, and we continue the algorithm with the next work package $r'_{pst} \in WP_{ts}$ or the next employee skill combination $n_{sk} \in N_{sk}$. Finally, we check if there are work packages $r'_{pst} \in WP_{ts}$, or parts thereof, that are still not assigned to resources and allocate them to the unlimited external resources y_{pts}.

Neighborhood ChangeS In the *neighborhoodChangeS()* function, a candidate solution X'', which is the best neighborhood solution, becomes the new current incumbent solution X^* if it is better than the current incumbent solution X^*. If $f(X'') < f(X^*)$, the parameter h defining the size of the neighborhood returns to its initial value; otherwise, if $f(X'') > f(X^*)$, k is increased by 1. When $f(X'') > f(X^*)$, we distinguish two different cases. First, with a probability of β, the best neighborhood solution X'' becomes the new incumbent solution for the next iteration within the search procedure. Otherwise, the current incumbent solution X^* is used for the further course of our search procedure. If $k+1 \leq k_{max}$, the subsequent search process continues with the *FirstImprovement()* operator using neighborhood $h + 1$. If $h + 1 > h_{max}$, we use the *Perturbation()* operator to escape from the local optimum and continue the search with the *FirstImprovement()* operator where $h = 1$.

Perturbation The function *Perturbation*() in line 9 of Algorithm 1 generates a new solution X' according to the input parameters, namely, the solution X and the number of swaps. The *Perturbation*() is applied if none of the h neighborhoods of the GLS-VND can improve the incumbent solution, because it helps the local search algorithm escape from any local optimum. The number of swaps is the product of the perturbation factor α times the number of projects $|P|$. Therefore, the number of swaps depends on the problem size. The perturbation function swaps the schedules (starting times of project periods $q = 1, \ldots, d_p$) of two randomly chosen projects and repairs the solution afterward, if the new schedule is beyond the allowed time windows, as determined by $[ES_{pq}, LS_{pq}]$.

Post-Processing Finally, when the stopping criterion is reached, a post-processing step reevaluates the best n project period schedules $X_1^*, \ldots, X_n^*$, obtained by the ILS with the included greedy staffing heuristic, using the LP solver of CPLEX. Therefore, the project period start times of the best solutions provide the input for the linear staffing problem (cf. the objective function in Eq. (22), and the constraints in Eqs. (23) and (3)–(8)). The best exactly evaluated schedule represents the final solution for the metaheuristic.

4.2 MIP-SP Solution Method

For the MIP-SP model, the main framework of the metaheuristic is the same as in the MIP-P case, except that there are additional binary decision variables v_{ktw} to be considered, which define the labor contract type per resource k and time period t. However, there are some notable modifications.

The evaluation of the greedy staffing heuristic is now threefold: First, the greedy staffing heuristic assumes that all employees have full-time labor contracts. Second, we analyze this solution, and apply a *RoundingHeuristic()* to set the suitable labor contract per resource k and time period t, according to their current utilization during regular working time. Therefore, resources with a utilization $\leq 25\,\%$ are set to represent no employment, resources with a utilization $\geq 25\,\%$ and $\leq 75\,\%$ are set to the half-time, and resources with a utilization $\geq 75\,\%$ are set to a full-time labor contract. Third, after adapting the available time per resource k and time period t during regular and overtime, according to the *RoundingHeuristic()* results, we conduct the staffing again, and the objective value of the feasible solution is returned as $f(X'^i, v)$.

In the *Post-Processing()* step, in addition to the staffing decision, we optimize the type of labor contract per time period t and resource k. Therefore, the problem changes from a purely linear model to a mixed integer problem. However, the small number of binary decision variables $\big(|K| * T\big)$ means that the solution of the MIP model that includes the labor contract decision is only slightly slower than that of the pure linear staffing problem.

5 Computational Experiments

For the computational experiment, we generated test instances, using the same test instance generator as applied by Heimerl and Kolisch (2010a). The test instances are inspired by data from the IT department of a large semiconductor manufacturer. We investigate several experimental levels for the number of projects $|P|$ and the time window size $\gamma = LS_p - ES_p$.

Table 2 lists the parameters for the experimental design ranging from 10 to 300 projects and a time window size γ from 1 to 3 periods. For each problem setting, which represents one combination of the two factors and the master data in Table 3, five random instances are generated. Therefore, we consider 195 test instances with 39 different problem structures.

The earliest start periods of projects ES_p are drawn from a uniform distribution between 1 and 7. The deterministic project length d_p of six periods and the latest start time of time period 7 led to a planning horizon of $T = 12$, which could represent strategic annual personnel planning and project scheduling. The skills required per project period are three, and the total number of skills per project is limited to four. The 100 internal resources each owning 4 of 25 available skills have different efficiencies per skill, drawn from a truncated normal distribution with an expected value of $\mu = 1$, a standard deviation of $\sigma = 0.25$, and minimum and maximum threshold values of 0.5 and 1.5, respectively. The internal cost rate c^r is 500 during regular working time and $c^o = 600$ during overtime. The external cost rates c_s^e, differ per skill, and are drawn from a truncated normal distribution $TN_{a,b}(\mu, \sigma)$ with $\mu = 800$, $\sigma = 100$, $a = 600$, and $b = 1000$. The available time per internal resource increases in a stepwise function, until the full capacity R_{kt}^r is

Table 2 Factors and experimental levels for the test instance generation

Factor	Experimental levels
$\lvert P\rvert$	10, 20, 30, 40, 50, 70, 90, 110, 130, 150, 200, 250, 300
γ=$LS_p - ES_p$	1, 2, 3

Table 3 Master data for the test instance generation

Master data	
$ES_p \sim U(1, 7)$	$\lvert S_k\rvert = 4$
$d_p = 6$	$\lvert K\rvert = 100$
$T = 12$	$R_{kt}^r = 20(t = 6, \ldots, 12)$
$\lvert S\rvert$ *per period and project*$= 3$	$R_{kt}^o = 0.3R_{kt}^r = 6(t = 6, \ldots, 12)$
$\lvert S\rvert$ *per project* ≤ 4	$\eta_{sk} \sim TN_{0.5,1.5}(1, 0.25)$
$e_p = 0.2$	$c^r = 500$
$\rho = 1.0$	$c^o = 1.2c^r = 600$
$\lvert S\rvert = 25$	$c_s^e \sim TN_{600,1000}(800, 100)$
$f_w = \{0.0, 0.5, 1.0\}$	

reached in the sixth time period, to mimic a dynamic situation in which projects that started in the past have already been assigned to internal resources. Utilization ρ is defined as the ratio of expected resource demand to available internal resource capacity in time period $t = d_p = 6$. On the basis of the utilization ρ and a resource supply R^r and R^o, we can calculate the expected resource demand $E(r_{psq})$ of the work packages. The actual resource demand is drawn from a normal distribution with $\mu = E(r_{psq})$ and a coefficient of variation $CV = 0.1$. For the *MIP-SP* problem, three labor contract possibilities are available, as described in Sect. 3.3.

The tests where performed using an Intel Xeon E78837 processor (Frequency: 2.66 GHz; 24MB L3 Cache) with 8 kernels and 64 GB of working memory. For the exact results, we used ILOG CPLEX version 12.4.

The parameter tuning for the implemented metaheuristic was done by means of the irace package (López-Ibáñez et al. 2011), a software for automatic algorithm configuration. By defining a set of instances out of all test instances, irace performs the parameter tuning for the numerical parameter perturbation factor α, and the solution acceptance probability β. The optimal values, according to the racing software, are $\alpha = 15\,\%$, and $\beta = 2\,\%$. In preliminary tests, we determined the number of neighborhoods, $h_{max} = 3$, in line with the suggestion of Mladenovic and Hansen (1997). The number of staffing reevaluations n in the post-processing step, is 20.

6 Results

For performing the tests we solve the test instances according to the description in Sect. 5 by the state-of-the-art solver CPLEX and the developed metaheuristics to generate insights into the performance of the solution methods for the proposed models. Specifically, we investigate the following questions:

1. What influence do the possibility of interruptions (*MIP-P*) and the introduction of labor contracts (*MIP-SP*) have on the solution value, compared with *MIP-II*?
2. Is the MIP solver a reliable solution method for the various models? From which point on is a heuristic solution method necessary?
3. How do the developed metaheuristics for the *MIP-P* and *MIP-SP* model formulation perform compared with the MIP solver?
4. How do the performance values change if the time budget of the MIP solver is increased by the factor 10, relative to that used within the metaheuristic?

Table 4 Relative solution value difference $\Delta sv'$ and solution gap sg with respect to the different problem formulations and the time window size of γ

Problem	CPLEX(360 s) $\Delta sv'$ and sg in %					
	$\gamma = 1$		$\gamma = 2$		$\gamma = 3$	
	$\Delta sv'$	sg	$\Delta sv'$	sg	$\Delta sv'$	sg
MIP-II	0.00	0.01	0.00	0.06	0.00	0.16
MIP-P	−4.07	0.06	−5.65	0.30	−8.65	0.75
MIP-SP	13.79	1.19	38.07	4.16	32.54	4.99

6.1 Comparison of Model Extensions: Solution Value, and Solution Gap

In our first numerical study, we consider the influence of the model extensions *(MIP-P,* and *MIP-SP)* on the average solution value of test instances, solved by CPLEX. Table 4 shows the key performance indicators, solution value difference ($\Delta sv'$) and solution gap (sg), both measured in percentage terms for the three different problem formulations. For the solver, a time limit of 360 s is predefined, and we investigate a time window of size $\gamma \in \{1, 2, 3\}$. The measures are calculated for all feasible, solved test instances, generated using the data from Tables 2 and 3. The instances are considered as solved if CPLEX finds a feasible integer solution within the given time budget. For the comparison of the *MIP-II, MIP-P*, and *MIP-SP* problems, we adjusted the solution values according to the minimum costs C_{min}. In the presented model formulations, all work packages must be processed within the time horizon T, so there exist C_{min} values of considerable size. These minimum costs are inevitable, regardless of simultaneous project scheduling and personnel planning, and their detailed calculation is as described by Heimerl and Kolisch (2010a). Equation (24) describes the calculation of adjusted solution values:

$$sv' = (sv - C_{min}). \tag{24}$$

For the calculation of the relative solution value difference $\Delta sv'$ in Table 4, all model formulations are compared to the *MIP-II* model. Therefore, $\Delta sv'$ is calculated for each instance of the respective problem formulation *MIP-xx* $\in$ {*MIP-II, MIP-P, MIP-SP*} according to Eq. (25), and the results are reported as average values of all feasibly solved instances:

$$\Delta sv' = (sv'_{MIP-xx} - sv'_{MIP-II})/sv'_{MIP-II} \tag{25}$$

The second performance measure listed in Table 4 is the solution gap sg. It is an output of the *MIP* solver, returned at the end of the optimization process. The solution gap also acts as a stopping criterion. By its default setting, the optimizer stops the optimization process when we reach a solution gap ≤ 0.01 %.

In our first numerical study, we obtain the result that the model extension *MIP-P* leads to cost improvements of −4.07 %, −5.65 %, and −8.65 % for time window sizes $\gamma = 1, 2, 3$, respectively. Another finding from the *MIP-P* model reveals a positive correlation between the time window size γ and the cost improvements and a positive correlation of the time window size γ and the solution gaps. The implementation of labor contracts (*MIP-SP*) leads to an increase in costs. The reason for the costs increase is, that the *MIP-SP* model mimics the more realistic situation where the available hours during regular working time, are crucial to the objective function, whereas in *MIP-II* and *MIP-P* only the scheduled time is used. Furthermore, there is a positive correlation between the time window size γ and the solution gap sg, which means that the complexity of the problem increases with γ.

6.2 Comparison of Model Extensions: Solution Time, Solution Gap, and Solvability

In the next numerical study, we investigate the solution time, solution gap, and solvability of the*(MIP-II, MIP-P,* and *MIP-SP)* models, to determine how the solver performs for large test instances and different time budgets. We also investigate when the exact solver is an appropriate method to solve the various models and from which point on a heuristic solution method is necessary.

Table 5 shows the solution time, solution gap, and solvability for the three different model formulations and three different time limits of the MIP solver (360 s, 1 h, and 10 h). For the study with a time limit of 360 s per instance, all 195 test instances is solved once; for the study with a time budget of 1 h per instance, one combination of the master data in Table 2 for the number of projects $|P|$ and the time window size γ, or 39 instances, are solved. Finally, for the numerical study with a time limit of 10 h per instance, only the test instances with a time window of size $\gamma = 3$ from the subset of 39 instances are solved, due to the limited experimental time budgets. All test instances are solved using CPLEX, parameterized with default values. In preliminary tests we tried to improve performance of CPLEX by setting non default parameters. Examining the node log, we find that CPLEX spends a lot of time solving the root relaxation. This leads us to test different settings for the startalgorithm parameter. Also different settings for the probing and MIP emphasis parameter does not yield to significant performance improvements compared to the default settings. The solution time (st) is the average solution time in seconds for all feasibly solved test instances. The average solution gap (sg), measured as a percentage, is listed in Table 5. The key performance indicator, solvability (sa), reveals the percentage of feasible solved test instances.

For *MIP-II*, the average solution time and solution gap for all test instances solved by CPLEX parametrized with default values and a time limit of 360 s are 292 s and 0.08 %, respectively. In addition, CPLEX can find a feasible solution for all test instances of the *MIP-II* formulation. A increase of the time budget from 360 s to 1 h reduces the average solution gap from 0.08 to 0.03 %. Another increase to 10 h per instance does not further improve the average solution gap of 0.03 %.

Table 5 Solution time (st), solution gap (sg), and solvability (sa) for the different problem extensions, CPLEX time limits, and time window size of γ

	CPLEX(360 s)			CPLEX(1 h)			CPLEX(10 h)		
	$\gamma \in \{1, 2, 3\}$			$\gamma \in \{1, 2, 3\}$			$\gamma = 3$		
Problem	*st* (s)	*sg* (%)	*sa* (%)	*st* (s)	*sg* (%)	*sa* (%)	*st* (s)	*sg* (%)	*sa* (%)
MIP-II	292	0.08	100	2064	0.03	100	28,337	0.03	100
MIP-P	351	0.26	72.30	3472	0.68	95	36,000	0.12	100
MIP-SP	353	0.83	66.15	3482	1.5	89.7	36,000	3.45	100

Investigating only those instances for which the time window $\gamma = 3$ (not reported in Table 5), we find increasing the solution time hundred times leads to an average improvement of the solution gap from 0.16 to 0.03 %.

For the feasible solved *MIP-P* test instances, the average solution time is 351 s, and the solution gap is 0.26 %. For 27.7 % of these instances, CPLEX cannot find a feasible integer solution. The solvability problem starts with test instances in which the number of projects exceeds 150, combined with a time window size of $\gamma = 2$, or project amounts of $|P| \geq 50$ combined with a time window size of $\gamma = 3$. A change of the time budget from 360 s to 1 h reduces the percentage of unsolved instances to 5 % and raises the average solution time and solution gap to 2472 s and 0.68 %, respectively. The increase of this solution gap looks counterintuitive, but might be explained as follows: Due to the increase of the time limit, a larger number of test instances can be solved, but they are harder to solve, so the average solution gap is larger than that for the shorter time limit. Another increase of the time limit to 10 h (instances with time window sizes $\gamma = 3$) improves the solvability of the *MIP-P* model to 100 %, increases the average solution time to the maximum of 36,000 s, and reduces the average solution gap to 0.12 %, compared with 0.75 % for the time limit of 360 s, as long as we include only instances with the same time window size $\gamma = 3$.

For the *MIP-SP* model, CPLEX, with a default parameterization and a time limit of 360 s, finds solutions for 66.15 % of all test instances within an average solution time of 353 s and an average solution gap of 0.83 %. Yet for instances with $|P| \geq 200$ and $\gamma = 1$, $|P| \geq 130$ and $\gamma = 2$, or $|P| \geq 50$ and $\gamma = 3$, CPLEX fails to find feasible solutions. In such cases, the exact solver is not a reliable solution method. Increasing the CPLEX time limit to 1 h raises the solvability percentage to 89.7 %, the solution time to 3482 s, and the solution gap to 1.5 %. A time budget of 10 h and test instances with $\gamma = 3$ results in an average solution time of 36,000 s, and an average solution gap of 3.45 %.

This numerical study therefore confirms that the MIP solver is an appropriate solution method for the *MIP-II* problem, solving instances of sizes as they appear in Table 2 and 3. For the model extensions *MIP-P* and *MIP-SP*, the exact solver usually becomes unable to solve medium-sized and large test instances, even with time limits of 10 h. This result confirms the need for an appropriate heuristic solution method for the *MIP-P* and *MIP-SP* model formulations.

6.3 Heuristic Solution of MIP-P

We summarize the average solution gap sg' for the *MIP-P* model with respect to the number of projects $|P|$ and the time window size γ in Table 6. Each of the five generated instances with the same combination of $|P|$ and γ are solved by the heuristic, using 10 random replications. For this heuristic solution method 360 s represents the stopping criterion, and we executed *Post-Processing()* afterwards. For a fair comparison of the developed heuristic to the MIP solver, we made the average solution time of the heuristic solution method (including the required post-processing time) the new time limit for a new experiment, that solved all test instances with the exact solver. Therefore, the results regarding solvability differ now slightly from the results reported in Tables 4 and 5. To tackle the problem that the MIP solver possibly cannot find any feasible integer solution within the time budget, together with the need for a key performance indicator that measures the performance of the heuristic for large and complex instances, we calculate the solution gap (sg') with respect to the lower bound per instance. This bound is determined by relaxing all integrality constraints of the decision variables and solving the continuous linear model with CPLEX. To reveal the variance in the results, we report the 95 % confidence interval $\pm CI$ of the solution gap sg' in Table 6. For the heuristic solution gap, the variance has two reasons. First, it arises from the five different test instances with one combination of $|P|$ and γ (cf. Sect. 5). Second, variance emerges from the ten replications used to solve one test instance.

Table 6 *MIP-P*: ILS vs. CPLEX(st-ILS) solution gaps (sg') compared with the lower bound and its 95 % confidence interval ($\pm CI$) with respect to the number of projects $|P|$ and time window of size γ

	$\gamma = 1$				$\gamma = 2$				$\gamma = 3$			
	ILS		CPLEX		ILS		CPLEX		ILS		CPLEX	
$\|P\|$	sg'	$\pm CI$	sg'	$\pm CI$	sg'	$\pm CI$	sg'	$\pm CI$	sg'	$\pm CI$	sg'	$\pm CI$
10	2.07	0.49	2.01	0.51	3.28	0.33	2.87	0.32	4.15	0.25	3.41	0.40
20	0.89	0.08	0.68	0.08	1.57	0.10	0.99	0.12	2.35	0.22	1.55	0.22
30	0.69	0.05	0.39	0.04	1.13	0.11	0.60	0.06	1.24	0.13	0.80	0.13
40	0.59	0.13	0.27	0.06	0.76	0.05	0.36	0.08	0.94	0.06	0.69	0.12
50	0.48	0.03	0.22	0.03	0.64	0.09	0.28	0.03	**0.78**	0.12	1.10*	
70	0.34	0.04	0.10	0.02	0.48	0.07	0.24	0.05	**0.50**	0.12		
90	0.33	0.06	0.08	0.02	0.39	0.05	0.30	0.05	**0.36**	0.08		
110	0.28	0.08	0.05	0.01	0.33	0.06	0.22	0.05	**0.46**	0.13		
130	0.24	0.03	0.04	0.01	0.37	0.05	0.20	0.03	**0.39**	0.06		
150	0.25	0.04	0.05	0.01	**0.30**	0.04	0.31	0.08	**0.32**	0.03		
200	0.18	0.04	0.04	0.01	**0.25**	0.04	0.18*	0.05	**0.27**	0.05		
250	0.18	0.04	0.03	0.01	**0.26**	0.03	0.31*		**0.35**	0.14		
300	**0.16**	0.03	0.03*	0.01	**0.25**	0.07	0.20*		**0.22**	0.03		
Avg.	0.51	0.09	0.31	0.06	0.77	0.08	0.54	0.08	0.95	0.11	1.61	0.22

For the calculation of a confidence interval $\pm CI$ using the exact solver, we do not use replications, so the variance is caused only by the five test instances with the same problem characteristics of $|P|$ and γ. The solution gaps of the MIP solver results marked by an asterisk (*) are those combinations of $|P|$ and γ where CPLEX cannot find a feasible integer solution for all five instance combinations within the predefined time limit. If only one problem characteristic could be solved, the confidence interval is empty. In the bottom row, the average values over all number of projects $|P|$ are reported per time window size γ. The gap values in bold face indicate instance combinations $|P|$ and γ in which the heuristic outperforms the exact solver, in the sense of either solvability or the solution gap.

The results in Table 6 show that the average solution gap over all our test instances for the heuristic and the MIP solver decreases with the number of projects $|P|$. Applying the same utilization ρ for all test instances, an increase of the number of projects leads to a linear increase of work packages and a linear decrease of the demand per work packages r_{psq}. The increase in the number of work packages and lower demand per work package have positive smoothing effects on capacity demand, which leads to a solution gap decrease. The time window size γ exhibits a positive correlation with the solution gap for both solution methods. Using the MIP solver as a solution method, these results show that the percentage of unsolved instances $(1-sa)$ increases with the number of projects $|P|$ and the time window size γ. For small and medium-sized instances, with $|P| \leq 250$ and $\gamma = 1$ or $|P| \leq 150$ and $\gamma = 2$, the MIP solver returns marginally better results than our developed metaheuristic. Specifically, the heuristic solution method solves all test instances (averaging all $|P|$ and γ values) with an average solution gap of 0.74 %, whereas the MIP solver reaches, for the approximately 73 % feasible solved instance, a solution gap of 0.61 %. Generally, the results in Table 6 show that our developed solution heuristic performs well of small and medium-sized instances and delivers a feasible solution for large and complex instances, with a small solution gap when the exact solver fails to return a solution.

6.4 *Heuristic Solution of MIP-SP*

The results of the developed heuristic solution method for the *MIP-SP* problem, are shown in Table 7, whose structure mimics that of Table 6. The metaheuristic performs significantly better than the MIP solver. Specifically, the heuristic solution method solves all generated test instances with an average solution gap of 1.03 %, whereas the exact solver returns an average solution gap of 1.39 % for the approximately 70 % of feasibly solved instances. The detailed values reflect the number of projects $|P|$ and time window period γ. For a time window size of $\gamma = 1$, the MIP solver is slightly better (0.57 %) than the developed heuristic (0.75 %). Nevertheless, the solver fails to find a feasible solution for approximately 55 % of instances. For time window sizes of $\gamma = 2$ and $\gamma = 3$, the developed heuristic outperforms the solver in terms of both the solution gap sg' and solvability sa.

Table 7 *MIP-SP*: ILS vs. CPLEX(st-ILS) solution gaps (sg') compared with the lower bound and its 95 % confidence interval ($\pm CI$) with respect to the number of projects $|P|$ and time window of size γ

	$\gamma = 1$				$\gamma = 2$				$\gamma = 3$			
	ILS		CPLEX		ILS		CPLEX		ILS		CPLEX	
$\|P\|$	sg'	$\pm CI$	sg'	$\pm CI$	sg'	$\pm CI$	sg'	$\pm CI$	sg'	$\pm CI$	sg'	$\pm CI$
10	2.31	0.54	2.04	0.51	3.90	0.41	3.14	0.30	4.87	0.47	3.87	0.54
20	**1.08**	0.10	0.78*	0.08	1.90	0.04	1.41	0.11	**2.74**	0.33	2.40*	0.17
30	**0.98**	0.14	0.61*	0.06	1.46	0.09	1.08	0.15	**1.58**	0.18	1.91*	1.06
40	**0.76**	0.15	0.65*	0.05	**1.02**	0.10	0.67*	0.22	**1.21**	0.13	2.71*	0.39
50	**0.67**	0.07	0.45*	0.03	**0.84**	0.11	0.87*	0.40	**1.04**	0.11	3.57*	0.47
70	**0.54**	0.07	0.33*	0.02	**0.64**	0.06	1.04*	0.16	**0.72**	0.11		
90	**0.58**	0.08	0.36*	0.14	**0.65**	0.06	1.00*	0.21	**0.61**	0.12		
110	**0.52**	0.06	0.35*	0.09	**0.58**	0.05	1.25	0.24	**0.76**	0.13		
130	**0.48**	0.05	0.29*	0.10	**0.59**	0.05	1.22*	0.50	**0.62**	0.12		
150	**0.56**	0.07	0.34*	0.05	**0.49**	0.05	0.92*	0.22	**0.55**	0.04		
200	**0.45**	0.09	0.29*	0.10	**0.50**	0.07	0.95*	0.17	**0.49**	0.06		
250	**0.45**	0.13	0.43*	0.20	**0.55**	0.03			**0.92**	0.74		
300	**0.43**	0.05	0.44*		**0.47**	0.10			**0.51**	0.17		
Avg.	0.75	0.12	0.57	0.12	1.05	0.09	1.23	0.24	1.28	0.21	2.89	0.53

In summary, within a predefined computation time, the complexity of the *MIP-SP* model is too high for an application of the MIP solver as an appropriate solution method. Even for small instances, the solver sometimes fails to return a feasible solution within a reasonable time. However, for small instances, the heuristic solution method performs as well as the exact solver does and also solves real-size instances, providing a good and stable solution gap.

Table 8 presents another interesting comparison of the developed heuristic and the MIP solver. In contrast with the results presented in Table 7, we increase solution time ten times for the exact solver. For example, if the average solution time of one test instance for the heuristic has been 381 s, the same test instance is now given to the MIP solver using a time budget of 3810 s. The tenfold solution time for the MIP solver decreases the average solution gap from 0.57 to 0.43 % for instances with a time window of $\gamma = 1$ and from 1.23 to 0.61 % for instances with a time window of $\gamma = 2$. In the latter subset ($\gamma = 1$ and $\gamma = 2$), the exact solver is able to solve all generated test instances, whereas for test instances with a time window size of $\gamma = 3$, the MIP solver has problems solving instances in which the number of projects exceeds 150. In addition, test instances in which $|P| \geq 90$ causes the solver to return solutions with a considerable solution gap, whereas the heuristic (using only 10 % of the computation time) returns solutions with an acceptable solution gap. Therefore, considering the tenfold increase in runtime, the MIP solver decreases solution gaps for small and medium-sized instances but still cannot find good solutions for large and complex ones.

Table 8 *MIP-SP*: ILS vs. CPLEX(10x st-ILS) solution gaps (sg') compared with the lower bound and its confidence interval ($\pm CI$) with respect to the number of projects $|P|$ and time window of size γ

	$\gamma = 1$				$\gamma = 2$				$\gamma = 3$			
	ILS		CPLEX		ILS		CPLEX		ILS		CPLEX	
$\|P\|$	sg'	$\pm CI$	sg'	$\pm CI$	sg'	$\pm CI$	sg'	$\pm CI$	sg'	$\pm CI$	sg'	$\pm CI$
10	2.31	0.54	2.04	0.51	3.90	0.41	2.87	0.32	4.87	0.47	3.45	0.41
20	1.08	0.10	0.74	0.08	1.90	0.04	1.06	0.12	2.74	0.33	1.67	0.24
30	0.98	0.14	0.48	0.05	1.46	0.09	0.68	0.05	1.58	0.18	0.85	0.14
40	0.76	0.15	0.42	0.06	1.02	0.10	0.52	0.10	1.21	0.13	0.72	0.05
50	0.67	0.07	0.33	0.04	0.84	0.11	0.46	0.09	1.04	0.11	0.67	0.20
70	0.54	0.07	0.22	0.02	0.64	0.06	0.35	0.02	0.72	0.11	0.66	0.12
90	0.58	0.08	0.24	0.04	0.65	0.06	0.38	0.05	**0.61**	0.12	14.28	26.44
110	0.52	0.06	0.22	0.03	0.58	0.05	0.34	0.01	**0.76**	0.13	29.69	35.01
130	0.48	0.05	0.21	0.03	0.59	0.05	0.29	0.07	**0.62**	0.12	43.21	34.22
150	0.56	0.07	0.21	0.03	0.49	0.05	0.24	0.01	**0.55**	0.04	0.31*	0.06
200	0.45	0.09	0.19	0.03	0.50	0.07	0.24	0.09	**0.49**	0.06	17.58*	30.25
250	0.45	0.13	0.16	0.04	0.55	0.03	0.22	0.09	**0.92**	0.74	38.06*	46.91
300	0.43	0.05	0.17	0.03	0.47	0.10	0.28	0.03	**0.51**	0.17	0.35*	
Avg.	0.75	0.12	0.43	0.08	1.05	0.09	0.61	0.08	1.28	0.21	11.65	14.50

7 Conclusion and Outlook

We extended a model for simultaneous project scheduling and staffing with heterogeneous skills by proposing two practically relevant model extensions, which led to new models *MIP-P* and *MIP-SP*. After giving mathematical programming formulations of the new optimization models, we proposed metaheuristic solution techniques. For synthetically generated test instances, obtained by an instance generator inspired by a real-world situation, we compared the solution approaches based on key performance indicators: solvability, solution gaps, and solution times, relative to a state-of-the-art MIP solver.

Investigating real-size instances, we found that the MIP solver is an appropriate solution method for the basic *MIP-II* problem. However, for our model extensions *MIP-P* and *MIP-SP*, our study reveals the need for a heuristic solution procedure.

The developed metaheuristic based on iterated local search decomposes the problem into a project scheduling master problem and a staffing sub-problem. Observing computation time advantages, we used a greedy staffing heuristic to solve the sub-problem. A variable neighborhood descent algorithm uses capacity profiles to identify time periods with high external capacity costs, which improves the efficiency of our metaheuristic. In a post-processing step, we also reevaluated the top heuristic solutions by means of the exact solver to further improve the staffing decision. For the problems *MIP-P* and *MIP-SP*, the proposed heuristic technique yields high-quality solutions for small to medium-sized instances, and provides solutions with a good and stable solution gap for large and complex instances.

A further research direction could be a multi-criteria extension, obtained by adding an objective function to address job satisfaction. Another interesting research topic would be to introduce a hierarchical skill profile structure and to investigate its benefits in different job market situations.

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From Incomplete Information to Strict Rankings: Methods to Exploit Probabilistic Preference Information

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Abstract Decision makers are often not able to provide precise preference information, which is required to solve a multicriteria decision problem. Thus, many methods of decision making under incomplete information have been developed to ease the cognitive burden on decision makers in providing preference information. One popular class of such methods, exemplified by the SMAA family of methods, uses a volume-based approach in parameter space and generates probabilistic statements about relations between alternatives. In the present paper, we study methods to transform this probabilistic information into a strict preference relation among alternatives, as such strict preferences are needed to actually make a decision. We compare these methods in a computational study, which indicates a trade-off between accuracy and robustness.

1 Introduction

The solution of decision problems with multiple criteria involves trade-offs between the different criteria under consideration. Thus, these problems do not have one objectively optimal solution, but require information about subjective preferences of the decision maker. However, providing this information is often a difficult task. During the last decades, a considerable stream of literature has emerged that aims at simplifying this task for decision makers. These methods are often denoted as approaches for decision making under incomplete information, or sometimes as partial or imprecise information, since they do not require an exact specification of the decision maker's preferences. For a seminal review of early developments in this field, see e.g., Weber (1987), more recent developments are summarized in Ehrgott et al. (2010).

Since the preference information is only incomplete, many methods in this field also provide only incomplete outputs, either in the form of incomplete

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preference relations, e.g. in Robust Ordinal Regression (Greco et al. 2008), or in the form of probabilistic preference statements, e.g. in the Stochastic Multiattribute Acceptability Analysis (Lahdelma et al. 1998). However, decision makers often need a clear choice, or a complete and unambiguous ranking of alternatives. In this paper, we develop methods to bridge the gap from incomplete preference information to a strict and complete ranking of alternatives.

Our approach extends the Stochastic Multiobjective Acceptability Analysis (SMAA) family of methods (Lahdelma et al. 1998; Lahdelma and Salminen 2001). The main output of these methods is a set of indices, which describe probabilities that an alternative occupies a certain rank, or that one alternative is considered to be better than another one, and we present methods to obtain a complete ranking of alternatives from these indices. We compare the results of these methods to other methods of decision making with incomplete information, in particular another SMAA-based approach using average parameters, and an optimization based approach, which directly estimates unique preference parameters.

The remainder of the chapter is structured as follows: Section 2 gives a brief overview of decision methods under incomplete information. Our methods are developed in Sect. 3. A computational study to compare them with other approaches is described in Sect. 4. Section 5 presents the results of the computational study. These results are discussed in Sect. 6, which also concludes the paper and gives an outlook to future research.

2 Decision Making Under Incomplete Information

Methods for multicriteria decision making typically employ parameters such as weights for attributes, partial utility values for outcomes, or threshold levels, to represent the decision maker's preferences. Methods for decision making under incomplete information acknowledge that sometimes decision makers are not able to specify these parameters exactly. Many types of incomplete information were considered in literature. Some methods (e.g., Weber 1987; Park and Kim 1997) assume that decision makers specify preference parameters in the form of intervals, or ordinally by stating e.g. a ranking of attribute weights. Other approaches, which are often labeled as preference disaggregation techniques (Jacquet-Lagreze and Siskos 2001) use comparisons between alternatives to establish constraints on preference parameters. In general terms, incomplete information on preferences thus defines a set of constraints on *admissible* preference parameters.

Literature on decision making under incomplete information can roughly be classified into three streams. The first stream uses incomplete information to derive robust conclusions, which are compatible with all parameter information available. This approach originated in the study of unknown probabilities in decision problems under risk (Kmietowicz and Pearman 1984), and was then also adapted to decision problems with multiple attributes (Kirkwood and Sarin 1985; Hazen 1986). More recently, these concepts were taken up under the name of Robust

Ordinal Regression (ROR) methods (Greco et al. 2008; Tervonen et al. 2013). These methods use incomplete information on preferences to calculate *necessary* and *possible* preference relations. Necessary preference is established between two alternatives A_i and A_j, if alternative A_i is considered to better than A_j for all admissible parameter vectors. The necessary preference relation is usually not a complete order relation on the set of alternatives. Possible preference between two alternatives A_i and A_j is established, if there exists at least one admissible parameter vector for which A_i is preferred to A_j. Possible preference usually is not an asymmetric relation, any asymmetric element of the possible preference relation is also an element of the necessary preference relation.

Another stream of literature selects one particular admissible parameter vector as an approximation to the true parameter vector of the decision maker. An early reference in this stream is Srinivasan and Shocker (1973), who developed a model to estimate attribute weights from pairwise comparisons of alternatives. The best known approach is the UTA method (Jacquet-Lagreze and Siskos 1982), which estimates piecewise linear utility functions from preference statements. Greco et al. (2011) integrated these concepts with the ROR approach and introduced the concept of a *representative value function* to ROR models. These methods generate a complete order of alternatives.

The third stream of literature considers regions in parameter space, in which certain conditions (e.g. a certain ranking between two alternatives, or optimality of one alternative) hold. The volumes of these subsets are interpreted as probabilities and are used to make probabilistic statements about the ranking of alternatives. This concept goes back to the domain criterion of Starr (1962), which was applied to the context of multi-criteria decisions by Charnetski and Soland (1978) and Eiselt and Laporte (1992). A similar approach was later on provided in the VIP software of Dias and Climaco (2000). More recently, SMAA has evolved into a popular approach for decision making under incomplete information (Lahdelma et al. 1998; Lahdelma and Salminen 2001). These methods use Monte-Carlo simulation to sample the set of admissible parameter vectors and obtain probabilistic preference statements.

Two types of such statements can be obtained. *Rank acceptability indices* r_{ik} indicate the probability that alternative A_i obtains rank k among all alternatives. *Pairwise winning indices* p_{ij} indicate the probability that alternative A_i is preferred to A_j. Since the precision of simulation is limited by sample size, recent literature (Kadzinski and Tervonen 2013) proposes to combine SMAA with optimization based approaches to obtain exact results.

Since the SMAA method generates probabilistic information, it does not provide a strict order relation among alternatives. However, as SMAA generates a representative sample of the set of admissible parameter vectors, it is possible to determine a central parameter vector within this set, similar to the concept of a representative utility function (Beuthe and Scannella 2001; Kadzinski et al. 2012). This vector can than be used to rank alternatives.

3 Model Formulation

3.1 Models Based on Rank Acceptability Indices

We consider a decision problem in which N_{alt} alternatives $A_i(i = 1, \ldots, N_{alt})$ are evaluated according to N_{crit} attributes. Each alternative is represented by the vector of attribute values $A_i = (a_{i1}, \ldots, a_{iN_{crit}})$.

Assigning each alternative to a unique rank in the set $\{1, \ldots, N_{alt}\}$ is a standard assignment problem (Wagner 1975). We define binary variables x_{ik} to indicate that alternative A_i is assigned to rank k. Each alternative must be assigned to one rank, and to each rank, one alternative must be assigned:

$$\sum_{k=1}^{N_{alt}} x_{ik} = 1 \quad \forall i = 1, \ldots, N_{alt} \tag{1}$$

$$\sum_{i=1}^{N_{alt}} x_{ik} = 1 \quad \forall k = 1, \ldots, N_{alt} \tag{2}$$

$$x_{ik} \in \{0, 1\} \quad \forall i, k \tag{3}$$

Different objective functions could be used in this problem. The average probability of assignments is maximized via the objective function

$$\max \sum_i \sum_k r_{ik} x_{ik} \tag{4}$$

which also most clearly points out the relationship to the standard assignment problem. Another possibility is to consider the lowest probability of any assignment

$$\min_{i,k:x_{ik}=1} r_{ik} \tag{5}$$

which can be written as a linear optimization problem:

$$\max z \tag{6}$$

s.t.

$$z \leq r_{ik} + (1 - x_{ik}) \ \forall i, k \tag{7}$$

$$\sum_{k=1}^{N_{alt}} x_{ik} = 1 \ \forall i \tag{8}$$

$$\sum_{i=1}^{N_{alt}} x_{ik} = 1 \ \forall k \tag{9}$$

$$x_{ik} \in \{0, 1\} \tag{10}$$

Although this model is indifferent between assignments of alternatives which do not have the lowest probability, and thus might have many different optima, it provides a focus on the weakest part of the available information and thus on robustness of the resulting ranking.

3.2 Models Based on Pairwise Winning Indices

A complete order relation of alternatives must be complete, asymmetric, irreflexive, and transitive. To represent the relation, we introduce binary variables y_{ij} indicating that alternative A_i is preferred to alternative A_j. The following constraints represent the required properties of a complete order relation:

Completeness and asymmetry:

$$y_{ij} + y_{ji} = 1 \ \forall i \neq j \tag{11}$$

Irreflexiveness:

$$y_{ii} = 0 \ \forall i \tag{12}$$

Transitivity:

$$y_{ij} \geq y_{ik} + y_{kj} - 1.5 \ \forall k \neq i, j \tag{13}$$

For this model, the same objective functions as for the model using rank acceptability indices can be used. It would also be possible to combine both approaches by considering the ranks of alternatives. Denote the rank assigned to alternative A_i by R_i. From the model based on rank acceptability indices, the rank can be obtained as

$$R_i = \sum_k k \cdot x_{ik} \tag{14}$$

and from the model based on pairwise winning indices by counting the number of alternatives which are preferred to A_i:

$$R_i = 1 + \sum_j y_{ji} \tag{15}$$

The order relations represented by x_{ik} and y_{ij} can be linked by equating the right hand sides of (14) and (15). However, our computational study indicated that the rankings obtained via the two methods are very similar, thus, the extra effort involved in using both sets of binary variables in one model seems to be unnecessary.

4 Computational Study

4.1 Overview

In this computational study, we compare the resulting rankings from the methods introduced in the previous section to each other and to two benchmarks. The comparison is based on random problem instances of a given size of N_{alt} alternatives and N_{crit} attributes, which were created by drawing $N_{alt} \times N_{crit}$ partial utility values of all alternatives and attributes from a uniform distribution. These partial utility values were subsequently rescaled to the zero-one interval. Alternatives were not filtered for dominance.

We assume that the decision maker's true preferences can be represented by an additive value function of the form

$$U(A_i) = \sum_k w_k v_k(a_{ik}) \tag{16}$$

where w_k is the weight of attribute k, and $v_k(a_{ik})$ is the partial utility value generated for alternative i in attribute k. The simulation then generated a "true" weight vector, uniformly distributed across the $N_{crit}-1$-dimensional unit simplex, using the method of Butler et al. (1997).

Using the "true" ranking of alternatives, the simulation then generated incomplete preference information in the form of preference statements between pairs of alternatives. We used two strategies to provide such information. The first strategy used the numbering of alternatives, and created preference information for pairs of alternatives with neighboring numbers. This approach provides $N_{alt} - 1$ information sets containing one to $N_{alt} - 1$ constraints. In the second strategy, alternatives were sorted according to the true ranking, and then pairwise comparisons of neighboring alternatives in the ranking were successively added to the constraint set. This process started with the pair with the largest utility difference (assuming that this would be the "easiest" comparison for the decision maker), up to the pair of neighboring alternatives with the smallest utility difference. When all such comparisons are made, this approach generates a setting with complete preference information.

For each of the information sets, the benchmarks described in the next section were solved to obtain an estimated weight vector. Using this estimated weight vector and the true partial utilities, utility values of all alternatives and their ranking were calculated. Then an SMAA analysis was performed using the respective constraints set, and the rankings of alternatives using the average of all sampled admissible weight vectors as well as the solutions of the various methods were calculated. Finally, we calculated the Kendall rank correlation coefficient between the true ranking of alternatives and each of the rankings previously generated. This correlation coefficient serves as the measure of quality for all methods.

4.2 Benchmark Models

The first benchmark fits a multi-attribute value function of the form (16) to generated preference statements. Using the additive value function (16), preference for alternative A_i over alternative A_j generates the linear constraint

$$\sum_k w_k(a_{ik} - a_{jk}) > 0 \tag{17}$$

or, by explicitly introducing the slack (or violation) variable z_{ij}:

$$\sum_k w_k(a_{ik} - a_{jk}) - z_{ij} = 0 \tag{18}$$

z_{ij} can take on positive or negative values. If z_{ij} is positive, the constraint is fulfilled and z_{ij} represents the slack. If z_{ij} is negative, the constraint is violated and z_{ij} represents the amount (in terms of utility) by which it is violated. To find a parameter vector at the center of the admissible set, we follow the approach of Beuthe and Scannella (2001) and Graf et al. (2013) and maximize the minimum of z_{ij}. For unknown attribute weights, we thus obtain the model

$$\max z \tag{19}$$

s.t.

$$z \le z_{ij} \forall i, j \tag{20}$$

$$\sum_k w_k(a_{ik} - a_{jk}) - z_{ij} = 0 \quad \forall i, j : A_i \succ A_j \tag{21}$$

$$\sum_k w_k = 1 \tag{22}$$

which can be simplified to

$$\max z \tag{23}$$

s.t.

$$\sum_k w_k(a_{ik} - a_{jk}) - z \ge 0 \quad \forall i, j : A_i \succ A_j \tag{24}$$

$$\sum_k w_k = 1 \tag{25}$$

In the following, we will refer to this approach as a distance-based benchmark to distinguish it from the second benchmark. The second approach employs the average of all admissible weight vectors generated in the SMAA simulation. This benchmark model will be referred to as the average weights benchmark.

4.3 Measurement and Parameter Settings

In the simulation, different levels of preference information are generated by adding constraints to the model. However, depending on the location of the additional constraint, adding a constraint might have very different effects on the admissible set in parameter space. We therefore measure the amount of information available by the relative volume of the admissible parameter set under given constraints. This measure can conveniently be obtained during the SMAA simulation. We denote this variable by "Vol". A higher volume indicate less information (since a larger fraction of the original parameter space is still admissible). Our analysis pools results from both methods to generate constraints, and uses this volume as consistent indicator of the amount of information.

To study the effects of different problem sizes, we performed simulation experiments for problems with $N_{alt} = 6, 9, 12, 15$ alternatives and $N_{crit} = 3, 5, 7$ criteria. For each problem size, 500 problems were generated. To test whether this sample size is sufficient for results to be independent of the random number stream used, we repeated all runs for 5 attributes with a different random stream and compared resulting correlations using a nonparametric Wilcoxon test. Out of the 304 compared samples, only one exhibited a significant difference at a significance level of 1 %. Applying a Bonferroni-Holmes correction for alpha error accumulation, no significant differences remained.

In the SMAA analysis, 10,000 admissible weight vectors were usually generated. To avoid excessive computation times in highly constrained problems, we limited the total number of weight vectors generated to 30 million for each problem, which in some cases led to a lower number of admissible vectors. Furthermore, solution time for each mixed integer problem was limited to 180 s, and some problems which could not be solved in that time were discarded. The entire simulation and all statistical analyses were performed in R (R Core Team 2013). Optimization problems were solved using LPSolve, which was called from R via the LPSolveAPI package (Konis 2013).

5 Results

Figure 1 provides a first glimpse at the simulation results. It shows the average value of the Kendall rank correlation coefficients for the different methods and different levels of the information variable *Vol* for one specific problem dimension. In this figure, values of *Vol* were rounded to the nearest tenth to enable aggregation across experiments involving different levels of *Vol*.

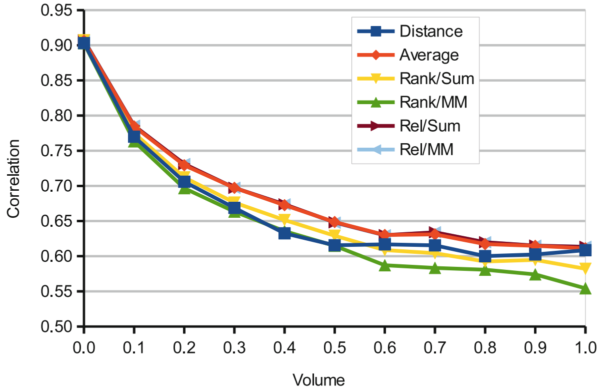

Fig. 1 Rank correlations between true rankings and rankings obtained for different methods for 12 alternatives and 5 criteria

There is a clear effect of information. The lowest volume, corresponding to the highest level of information, naturally provides the best fit. The figure also shows clearly that there are differences between methods. Methods are labeled according to the type of indices used (indicated by “Rel” for pairwise winning indices and “Rank” for rank acceptability indices) and “Sum” or “MM” to indicate objective functions (4) and (5), respectively. Methods based on the pairwise winning indices perform as well as the benchmark based on average weights from the SMAA simulation, methods based on rank acceptability indices perform somewhat worse than the benchmark and also exhibit a clear difference for the two objective functions.

To statistically test the significance of these differences, and also to analyze the impact of other parameters such as problem size, we performed a regression analysis. The dependent variable in the regression models is the Kendall rank correlation coefficient, which is explained by problem size, information and the different methods. To take into account possible nonlinear effects of problem dimensions, we model them as factors and provide a dummy variable for each possible number of alternatives and criteria in the regression model. To study how the different methods are able to process additional information, interaction terms between information and methods are also included in the analysis. To clearly illustrate the effect of these variables, we estimated a set of nested regression models, starting with problem dimensions in M1, then including information (M2), methods (M3) and interactions (M4). In models M3 and M4, methods are coded as dummy variables, using the distance-based benchmark as baseline.

Table 1 provides the results of the four nested regression models. The first and second models clearly show the importance of the information variable. Model M1 provides only a poor fit ($R^2 = 0.0527$) to the data, and contains implausible positive coefficients for the number of alternatives. These positive coefficients are mainly due to the fact that experiments for problems with a larger number of alternatives also contained a larger number of preference statements, which provided more precise information and allowed for a better approximation of the true ranking.

Table 1 Regression models on correlation to true rankings

Model	M1	M2	M3	M4
Intercept	0.8267***	0.9870***	0.9813***	0.9808***
$N_{Alt} = 9$	0.0031***	−0.0352***	−0.0352***	−0.0352***
$N_{Alt} = 12$	0.0187***	−0.0423***	−0.0423***	−0.0423***
$N_{Alt} = 15$	0.0301***	−0.0459***	−0.0460***	−0.0460***
$N_{Crit} = 5$	−0.0627***	−0.0860***	−0.0860***	−0.0860***
$N_{Crit} = 7$	−0.1011***	−0.1307***	−0.1307***	−0.1307***
Vol		−0.3531***	−0.3531***	−0.3512***
Method				
Avg. weights			0.0118***	0.0089***
Rank/Sum			0.0030***	0.0069***
Rank/MM			−0.0067***	0.0016°
Rel/Sum			0.0132***	0.0101***
Rel/MM			0.0132***	0.0100***
Interaction Vol with …				
Avg. weights				0.0114***
Rank/Sum				−0.0154***
Rank/MM				−0.0331***
Rel/Sum				0.0125***
Rel/MM				0.0126***
adj. R^2	0.0527	0.3174	0.3190	0.3196

∘: $p < 10\,\%$, *: $p < 5\,\%$, **: $p < 1\,\%$, ***: $p < 0.1\,\%$

Model M2 in contrast provides a considerably better fit, and also contains the expected negative coefficients for problem size.

Model M3 shows that there is a significant, albeit small, effect of methods. All methods except Rank/MM perform on average better than the distance-based approach. Methods using pairwise winning indices yield a correlation coefficient which is on average about one percentage point larger than the one obtained with the distance-based approach.

Coefficients of the methods in model M3 refer to the average value of *Vol* (0.253), while coefficients in model M4 refer to the hypothetical situation that the volume has a value of zero. Taking this into account, the coefficients in the two models are almost identical. The interaction terms show that increasing information has a different effect on the methods. Additional information improves the performance of methods based on rank acceptability indices, but worsens performance of methods based on pairwise winning indices.

These interactions are illustrated for different problem sizes in Fig. 2. If much information is provided (i.e., *Vol* has a value close to zero) the differences between methods diminish. This is quite plausible: If more information on preferences is available, results can no longer vary strongly between methods and are most likely close to the true preferences of the decision maker. With less information, the

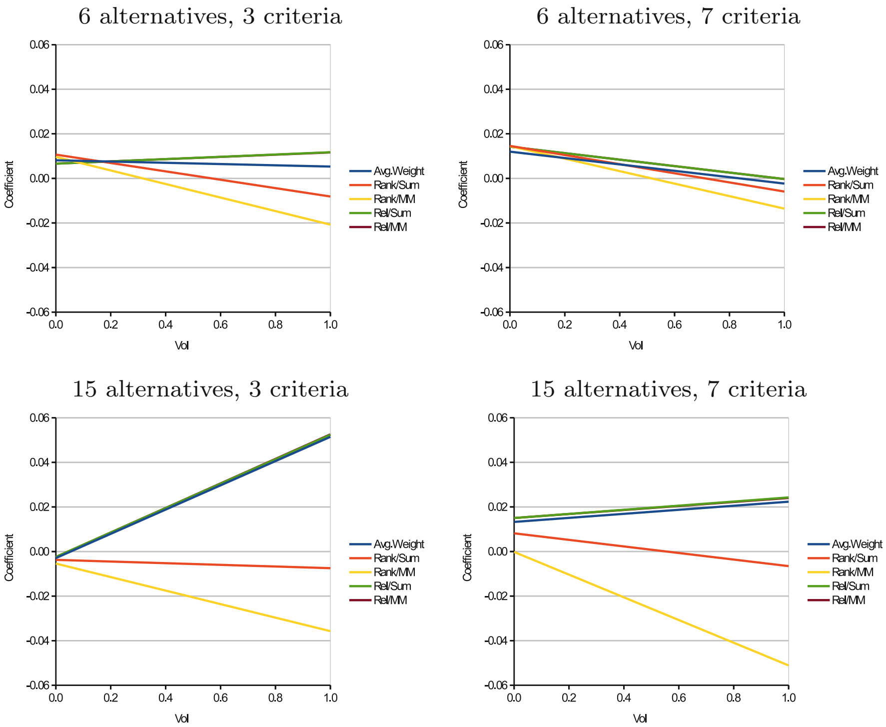

Fig. 2 Different information effects on methods for selected problem dimensions

differences between methods become more pronounced if more alternatives are present. Increasing the number of criteria, however, seems to make the performance of methods more similar.

In most cases, methods using pairwise winning indices lead to similar results as average weights from the SMAA method. For these methods, the choice of objective function seems to make no difference. In the case of 15 alternatives and 3 criteria, the difference in correlation coefficients between these methods and the distance-based approach reaches about five percentage points, while most other differences stay in the range of less then two percentage points. Methods based on rank acceptance indices perform worse, and this difference is also more pronounced when little information is available.

All these regression models imply a monotonic relationship between information and the quality of fit to the true ranking. This relationship holds on average, but in individual experiments, the opposite effect can sometimes be observed: Adding a constraint leads to a ranking which has a lower correlation to the true ranking than the previous one. Even if the admissible parameter set is quite large, it might by chance give a good approximation of the true ranking. This is likely to happen if errors on average cancel out, i.e., when the true parameter vector is close to the center of the admissible parameter set. If the additional constraint destroys this symmetry, it can lead to a worse approximation of the true ranking.

As Table 2 shows, such anomalies occur in 5 % to almost 30 % of all cases. To study the effects of problem parameters on the occurrence of information anomalies, we used logistic regression analysis on a binary variable indicating whether an information level in an experiment indeed led to an information anomaly (i.e., a decrease rather than an increase in correlation with the true ranking when adding a constraint).

The regression results shown in Table 3 indicate a strong impact of problem characteristics. As was already observable in Table 2, information anomalies are more likely in larger problems (involving more alternatives or criteria). Additional information reduces the number of information anomalies. There are also significant differences between methods. Compared to the baseline of the distance-based method, model M3 indicates that most methods lead to a slightly larger number of information anomalies. However, the interaction effects in model M4 indicate that this disadvantage will be influenced by information. Figure 3 shows the interaction effects in model M4 graphically and indicates that for low levels of information (high values of *Vol*), the model based on rank acceptability indices and average probabilities leads to less information anomalies than the baseline.

6 Discussion, Conclusions and Future Research

Volume-based decision methods like SMAA provide several useful indices, that offer insight into the uncertainty involved in making decisions based on incomplete preference information. The methods which we have introduced and analyzed in this paper take this information one step further and provide complete rankings of alternatives.

Our simulation experiments have revealed small, but significant differences in the performance of these methods. In particular, they have identified a trade-off between efficiency and robustness. While methods based on pairwise winning indices typically provide a better approximation of the decision maker's "true" ranking of alternatives, they are also more likely to exhibit information anomalies.

The fact that such information anomalies occur at a quite substantial rate, in particular for more complex problems, is another major result of our study. Providing more information about a decision maker's preferences does not automatically lead to better decisions, which more closely reflect the actual preferences of the decision maker. Of course, when solving a real problem, detecting that such an information anomaly has occurred is not possible. However, the possibility that this happens should be kept in mind both by decision makers and by analysts.

The trade-off between accuracy and robustness that is present in the different methods we studied here can be overcome by applying several methods in parallel. In the computational study performed in this paper, most of the computational effort was used for the SMAA simulations. In comparison, solution times for the

Table 2 Fraction of information anomalies

N_{Alt}	N_{Crit}	Distance (%)	Avg. weights (%)	Rank/Sum (%)	Rank/MM (%)	Rel/Sum (%)	Rel/MM (%)
6	3	9.78	8.33	6.00	5.45	8.43	8.43
	5	10.15	10.85	7.33	7.68	10.70	10.70
	7	7.68	10.30	7.58	8.58	10.03	10.03
9	3	13.60	12.10	9.01	9.01	12.53	12.53
	5	13.94	16.47	13.59	15.11	15.99	15.96
	7	13.77	17.61	15.10	17.97	17.14	17.16
12	3	15.02	14.11	11.06	13.30	14.39	14.42
	5	15.52	18.79	16.09	20.54	18.58	18.57
	7	15.59	21.28	19.10	23.96	20.32	20.32
15	3	15.67	16.10	12.97	15.67	16.56	16.55
	5	16.71	20.73	19.25	24.10	20.99	21.00
	7	17.65	24.89	23.94	29.11	24.52	24.58

Table 3 Logistic regressions on occurrence of information anomalies

Model	M1	M2	M3	M4
Intercept	−2.6194***	−3.1600***	−3.3102***	−3.3157***
$N_{Alt} = 9$	0.5581***	0.6844***	0.6852***	0.6857***
$N_{Alt} = 12$	0.7781***	0.9788***	0.9800***	0.9806***
$N_{Alt} = 15$	0.9606***	1.2130***	1.2151***	1.2157***
$N_{Crit} = 5$	0.3209***	0.4208***	0.4215***	0.4217***
$N_{Crit} = 7$	0.4674***	0.5944***	0.5957***	0.5959***
Vol		1.3074***	1.3100***	1.3317***
Method				
Avg. weights			0.2029***	0.1986***
Rank/Sum			0.0204	0.0675***
Rank/MM			0.2510***	0.2264***
Rel/Sum			0.1965***	0.2046***
Rel/MM			0.1971***	0.2041***
Interaction Vol with ...				
Avg. weights				0.0198
RankSum				−0.2135***
RankMM				0.1122*
RelSum				−0.0363
RelMM				−0.0314
null.deviance	548449	548449	548449	548449
deviance	539350	531560	530807	530758
aic	539362	531574	530831	530792

∘: $p < 10\,\%$, *: $p < 5\,\%$, **: $p < 1\,\%$, ***: $p < 0.1\,\%$

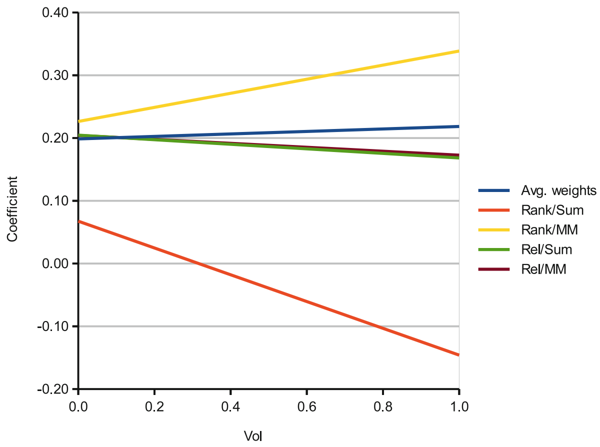

Fig. 3 Interaction effect of method and information level on the occurrence of anomalies

optimization models used to derive rankings from the SMAA results were quite moderate. Solution time for these models can further be reduced considerably, since the average weight vector from SMAA already provides a ranking of alternatives and thus a feasible starting solution to all the optimization models. Thus, a parallel application of several different methods is easily possible.

Applying different methods in parallel would be beneficial both if the methods provide identical results, and if their results are different. If results are identical, this would increase confidence in the ranking obtained. If the results are different, one can specifically seek for preference information about the alternatives which are ranked differently. This could lead to an interactive process which converges towards a unique ranking of alternatives.

The results we have presented here thus offer some insights into the possibilities to obtain strict rankings of alternatives from the probabilistic information obtained from SMAA and similar methods. However, these results still have several limitations. In this study, we have assumed that preferences of the decision maker can be represented by an additive value function, and that only the attribute weights in this function are unknown.

In fact, the optimization models we have formulated in Sect. 3 are independent of the underlying preference model. They only require stochastic indices as inputs, and their output is a general order relation. Although SMAA and similar methods are based on some explicit model of preferences, they can be applied to a more general setting than the one studied here. It is possible to consider not only weights but also value in additive value functions as uncertain (Sarabando and Dias 2010), and similar methods were also developed for other preference models like outranking relations (Tervonen et al. 2009).

Of course, any computational study is restricted to a specific problem setting. This concerns the dimensions of problems studied, as well as the specific methods e.g. to obtain preference statements from the ‘true” ranking. Furthermore, the methods we have studied also used specific objective functions, and were limited to strict preference. All these limitations provide opportunities to extend these methods in future research.

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The After-Effects of Fear-Inducing Public Service Announcements

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Abstract Messages using fear appeals often appear in social marketing, to promote causes such as smoking cessation, healthcare and driving accident prevention. Fear appeals can enhance the effectiveness of such communications, but they also may have unintended side effects. This study investigates the effect of fear-inducing public service announcements on evaluations of subsequent commercials in a commercial break. In two laboratory experiments, the authors measured participants' evaluations of advertisements using a program analyzer. In line with affective priming theory, the results showed that fear-inducing public service announcements can negatively affect evaluations of subsequent commercials.

1 Introduction

Scholars have long been interested in the use of fearful messages designed to persuade or ensure compliance. Even ancient studies of rhetoric mention fear appeals—albeit as a logical fallacy, or *argumentum ad metum* (Walton 2002). A more recent iteration of fear appeals was effectively exemplified by a television commercial by the Japanese tire manufacturer Autoway (2013), which rapidly went viral and received more than 3 million hits on YouTube: It depicted a car driver's encounter with a grisly figure during a snowstorm, preceded by a warning: 'Not for the faint of heart. Please refrain from watching the content if any of the following applies to you: Have any mental or physical health concern and may have to see a doctor regularly. We shall not be liable for any injuries, illness, and damages claimed to be caused by watching the contents'. The popularity of the commercial, which explicitly highlighted its fearful message, attests the power of fear appeals to create

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attention. Scary advertisements date as far back as in the 1950s (Hunt and Shehryar 2002).

Such appeals are particularly prevalent in social marketing, which seeks to use marketing strategies and techniques to change the attitudes and behavior of the general public (Kotler and Levy 1969). According to Kotler et al. (2002), social marketing might aim to improve health, prevent injuries, protect the environment or promote community involvement. Social marketing campaigns can appeal to consumers' self-interest through rational appeals or to their feelings through positive (e.g. erotic) or negative (e.g. fear) emotional appeals (Lennon et al. 2010). Because so many social marketing issues (e.g. cancer screening, seat belt use, environmental pollution, organ donations) are closely linked to negative consequences though, they lend themselves well to fear appeals, as demonstrated by campaigns for road safety (Transport Accident Commission 2002), health care (Benet et al. 1993) and smoking (DeJong and Hoffman 2000). However, such appeals also could have unintended side effects (Hastings et al. 2004). One unresolved controversy pertains to the relationship between the intensity of fear induced and advertising effectiveness.[1] According to the parallel response model (Janis 1967), the relationship between fear and advertising effectiveness is curvilinear (inverted U shape): As fear intensity increases, so does the persuasiveness of the advertisement up to a certain point, after which the advertisement's effectiveness decreases. Some studies accordingly indicate that medium levels of fear are most effective (e.g., Krisher et al. 1973), though others argue that higher levels of fear tend to be more effective (e.g., Baron et al. 1994), and at least one meta-analysis affirms this linear model (Witte and Allen 2000). Yet for ethical and practical reasons, extremely high levels of fear have not been investigated (Boster and Mongeau 1984; Ruiter et al. 2001).

In studies of the effectiveness of fear appeals, an often overlooked question is the potential effect of scary advertisements on how consumers perceive the other commercials that surround them in the same pod. Social marketers might achieve their goals with fear appeals in public service announcements (PSAs), but the PSAs then could adversely influence the effectiveness of commercials that follow them. We investigate the carryover effects of PSAs that use fear appeals, to determine how fear appeals influence evaluations of other commercials. An econometric model is developed to map the decay of these carryover effects over time.

In the next sections, we detail the influence that advertisers have over commercial scheduling, as well as prior literature on advertising context effects. We then develop a hypothesis and two research questions[2] regarding the effects of fear-inducing PSAs on evaluations of subsequent commercials. After outlining our study

[1]We use the expression advertising effectiveness in a broad, common sense here but concede that there are many different measures of advertising effectiveness used in the literature: attitude towards to ad, awareness, liking, recall, persuasiveness etc.

[2]Differences in depth of theoretical reasoning and level of detail cause the terminological difference between hypothesis and research question.

methodology, we present the results of two laboratory experiments and discuss our findings in detail.

2 Effects of Public Service Announcements on Evaluations of Other Advertisements

2.1 Advertisers' Influence Over Commercial Scheduling

Media planning, or finding the best means to deliver an advertiser's message, entails a variety of decisions, in addition to budget considerations and decisions about reach and frequency. For example, advertisers that use television commercials must make two basic choices about where to place their advertisement. First, they choose the program context, that is, the television show that will be interrupted by their commercial. This decision entails finding what they believe will be the most appropriate vehicle for their promotional message. The effect of the program context on the effectiveness of television advertising has been studied widely; Kennedy (1971) provided an early examination of how the program environment influences viewers' processing of commercials. Many similar studies have followed, revealing that during a commercial break, mental reactions to the television programs persist. Thus, viewers' reactions to the program carry over to commercials (Goldberg and Gorn 1987; Schumann and Thorson 1990; Lloyd and Clancy 1991; Norris and Colman 1993; Aylesworth and MacKenzie 1998).

Second, marketers must decide where to place the commercial within a commercial break (Bruhn 1997). Unlike the substantial research attention devoted to program context effects (Sissors and Bumba 1989), relatively fewer studies address the placement of commercials within a commercial break. In some countries, advertisers may select where their commercial will be positioned in a block of commercials (Pieters and Bijmolt 1997), and the limited research into the resulting effects generally indicates that placing a commercial at the beginning or end of a commercial break can significantly increase viewers' retention (Crano 1977), in line with primacy and recency effects, respectively (e.g. Brosius and Fahr 1996; Pieters and Bijmolt 1997). However, compared to the research stream on the effect of the program on the evaluation of advertisements, relatively little is known regarding the carryover effects of a focal commercial on other commercials within the same pod. With the exception of some proprietary research carried out in the 1950s, a study by Poncin and Derbaix (2009) and an investigation of the context effects of erotic ads (Ebster et al. 2009), this question has received little attention in the extant literature.

The reason for this gap might be the seemingly limited influence advertisers have over the placement of their commercials in a block, which represents something of a misconception. Television networks offer advertisers two purchase options: participation and sponsorships. Most advertising relies on the participation basis, such that all available advertising time within a program gets divided among several advertisers (Boveé and Arens 1982). In this case, the advertiser indeed has relatively little control over the placement of its commercial. With a sponsorship, which is more costly and less frequently used, advertisers have substantial control over the number, placement and content of all advertisements that appear in the program (Belch and Belch 2004). Networks, for example, when offering special presentations (e.g. Schindler's List for the first time), include limited commercial breaks that are more like intermissions, and the sponsoring company is the only one featured. Also, some presentations of sports events are using this tactic (e.g. Multichannel 2014). Such advertisers could benefit from a better understanding of the carry-over effects induced by other commercials or PSAs.

2.2 Hypotheses Development

This study seeks to initiate research into the effects of PSAs that use fear appeals on viewers' processing of subsequent commercials. We define fear appeals as emotional advertising executions that present a threat to create feelings of anxiety and tension and provide recommendations of behaviors that can relieve those feelings (LaTour and Zahra 1988; Dijker et al. 1997; Ruiter et al. 2001). The threat might relate to social acceptance (social fear) or bodily harm (physical fear) (Schoenbachler and Whittler 1996; Laroche et al. 2001). In addition, we predict that fear-inducing PSAs influence consumers' attitudes toward subsequent commercials, likely in a negative direction. This prediction reflects the theory of affective priming (Fazio et al. 1986) and cognitive psychological research, which reveals that consumers' interpretations of information often depend on their currently activated knowledge structures or concepts (Higgins and King 1981; Wyer and Srull 1981). More recently activated concepts tend to be more accessible (Higgins and King 1981), and priming can activate a particular mental knowledge structure, through exposure to a stimulus, and thereby influence interpretations of subsequent information. For example, being presented with the word 'eye' activates a sensory modality in people and primes them to spell the spoken word 'sight/site/cite' as s-i-g-h-t (Sternberg 2006). Such priming can occur without awareness, such that the person does not actively recognize the activated concepts (Higgins et al. 1985) but still interprets a stimulus in accordance with whatever primed concept is most easily accessible (Srull and Wyer 1980). However, when priming occurs without awareness, the effects tend to be weaker than the effects of conscious priming (Bargh and Chartrand 2000).

In addition, when exposed to ambiguous stimuli, consumers often interpret them according to the surrounding context (Wyer and Srull 1981). In experiments in which participants saw a series of print advertisements, two of which referred to personal computers, Yi (1991) demonstrated that the content of the first advertisement influenced the evaluation of the brand advertised second. Specifically, if participants saw an initial advertisement that emphasized versatility, a subsequent target advertisement that promoted a computer with many features led them to evaluate the target computer more positively than did participants who previously had been primed with an advertisement that emphasized ease of use. Beyond such cognitive effects, affective priming relies on emotional stimuli to prompt congruent emotional evaluations (Fazio et al. 1986). For example, listening to a sad story may trigger negative emotions that in turn influence subsequent appraisals (Wauters and Brengman 2013), though Fazio (2001) cautions that the duration of affective priming effects tends to be relatively short. Even changes in temperature can invoke emotional priming: Participants holding a hot drink perceived that other people had "warmer" personalities than those holding a cold drink (Williams and Bargh 2008).

In television advertising, both visual stimuli (Avero and Calvo 2006) and words (Fazio et al. 1986) can help prime positive or negative emotions. As Fazio (2001, p. 117) asserts, affective priming is 'a robust and replicable phenomenon apparent in experiments using a variety of priming stimuli, target stimuli and specific task requirements'. Furthermore, affective priming effects can be enhanced by similarity between the prime and probe, but they function even if the two are not similar (Avero and Calvo 2006). Thus affective priming can reinforce feelings of anxiety and threat (Wauters and Brengman 2013), and these negative forms of affect can influence consumers' attitudes toward an advertisement (Edell and Burke 1987). In empirical support of the effectiveness of negative affective priming in an advertising context, Brooker (1981) shows that a print advertisement containing a fear appeal negatively influence evaluations of the product promoted in an advertisement that immediately follows the fear-inducing message. The emotions generated by advertising transfer to the consumer's attitude toward the ad (Yi 1990), which in turn can alter brand evaluations (MacKenzie et al. 1986). The effect of fear appeals in public service announcements on surrounding commercials has not been investigated previously, though Ebster et al. (2009) find, in accordance with the valence of the primed emotion, that positive emotional appeals (erotic appeals) in television commercials improve evaluations of subsequent commercials. Furthermore, the results of a study by Poncin and Derbaix (2009) suggest that commercials can invoke negative affective reactions to subsequent commercials. In accordance with this combined theory and prior empirical results, we hypothesize:

H1: If a neutral commercial is preceded by a PSA containing a fear appeal, viewers evaluate it more negatively than if it were preceded by a neutral commercial. Neutral commercials distinguish from emotionally arousing commercials (containing for instance fear appeals) by their informative character and cognitive appeals.

Many studies, particularly in social marketing literature, focus on the effectiveness of repetitive advertisements (e.g. MacKinnon et al. 2000; Hitchman et al. 2014) and offer mixed evidence. However, an advertising wear-out effect appears to exist, such that campaign effectiveness decreases with increased exposures (e.g. Fry 1996; Fox et al. 1997; Janiszewski et al. 2003). For our research, this general effect implies that a fear-inducing message that is already known to viewers might create less attention and induce smaller context effects. Although a comprehensive investigation of multiple exposures of fear-inducing PSAs on subsequent advertising is beyond the scope of our study,[3] we pose the following research question:

RQ1: Do the context effects of repeatedly shown PSA containing a fear appeal wear out, compared with single exposures?

Witte and Allen (2000) offer an extensive meta-analysis of the effects of fear appeals for public health campaigns, in which they show that individual differences in the target audience do not appear to influence the processing of fear appeals, with rare exceptions. Concentrating on fear appeals related to distracting driving, Lennon et al. (2010) suggest that some distractions (e.g. talking on a phone, texting, eating while driving) appear more relevant for different audiences, depending on their gender, age and parenting status. Geller (2003) also identifies age as a moderator of the effect of fear-inducing advertising. None of these studies analyses the influence of demographic variables on the context effects of fear appeals though. Therefore, we pose a second research question:

RQ2: Do the demographic characteristics of the audience moderate the after-effects of public service announcements containing fear appeals?

3 Method

We conducted two experimental studies, using the same design and a pre-study. In them, participants watched a videotaped news program, interrupted by a commercial break that contained 16 commercials (Fig. 1), each of which lasted about 30 s. This stimulus material was identical in all conditions, except that spots 3 and 8 varied for a control group (CG) and two experimental groups (EG1, EG2). The third spot was neutral in the CG and EG1, whereas in EG2, it was a fear-inducing PSA. Then in both experimental groups, the eighth spot contained the same fear-inducing PSA, whereas the CG saw another neutral commercial.

To measure participants' evaluations of the spots, we used a program analyzer (Baumgartner et al. 1997), specifically, a 'hand held unit to indicate the degree to which they "feel positive [negative]" about what is on the television screen at that moment' (Fenwick and Rice 1991, p. 25). Lazarsfeld and Stanton originally developed this instrument in the late 30s of the twentieth century in order to record people's reaction to radio programs (Levy 2006). In the course of time, this device underwent several refinements and improvements and was mainly used for testing whether people did or did not like television shows or commercials. As main advantage, the program analyzer allows measurement of spontaneous reactions and thus replaces the use of self-report scales. As a disadvantage, however, a test

[3]The empirical research analyzes the effect of two exposures of the same fear-inducing PSA on the subsequent neutral commercial. Because of this rather limited scope we do not claim, however, this project to be a comprehensive investigation of effects of multiple (i.e. more than double) exposures.

CG	EG1	EG2
News program	News program	News program
Commercials 1 – 2	Commercials 1 – 2	Commercials 1 – 2
Commercial 3	Commercial 3	**Fear-inducing PSA**
Commercial 4	Commercial 4	Commercial 4
Commercials 5 – 7	Commercials 5 – 7	Commercials 5 – 7
Commercial 8	**Fear-inducing PSA**	**Fear-inducing PSA**
Commercial 9	Commercial 9 ← *RQ*1 →	Commercial 9
Commercials 10 – 16	Commercials 10 –- 16	Commercials 10 – 16

Fig. 1 Experimental design. H1 is tested by comparing lightly, RQ1 by comparing *darkly shaded cells*

effect might come into play because subjects are aware that they are observed (e.g. Schweiger and Schrattenecker 2013, p. 377). As Baumgartner et al. (1997) detail the program analyzer measures affective reactions which are antecedents of overall judgments and other consequences. There is a huge body of literature dealing with the importance of emotions for subsequent psychological constructs or behavioral responses which might result in more concrete effects of the advertising effort (e.g. Kroeber-Riel and Gröppel-Klein 2013, p. 168). In our case we restrain on spontaneous evaluations mainly for two reasons: first, responses are expected to be small and should not be confounded with other drivers of advertising effectiveness which might come into play with more elaborate evaluations at a later stage of cognitive processing; and, second, it would not be clear whether to analyze behavioral responses to the fear-inducing PSA or the subsequent neutral commercial.

The program analyzer used in our study measured evaluations ranged from 0 (very negative) to 10 (very positive) and were recorded every 0.5 s. At the middle position (5), the analyzer created a slight hitch that users could feel, whereas the lever moved smoothly over all the other positions. By comparing the participants' evaluations of the fourth and ninth spots across groups, we tested H1.

4 Results

4.1 Pre-Study

In a pre-study, we collected stimulus materials. To avoid any potential effects of familiarity, we needed commercials that were unknown to the participants. Therefore, we selected 20 advertisements that had an informative (rather than an emotion arousing) character, then asked a convenience sample of 78 respondents to evaluate these ads using the program analyzer, as well as state whether they had seen each advertisement previously. We chose 16 neutral spots on the basis of the liking (mean evaluation of about 5) and familiarity (more than 95 % of respondents did not know the spots) ratings.

We used a similar method to identify 5 fear-inducing PSAs related to driving behaviors. The same sample evaluated these announcements with the program analyzer, then responded to a questionnaire that collected their assessments of the fear appeals, their evaluations and their level of familiarity. The order of the presentation of the spots varied systematically. We selected a spot entitled '30 for a reason', which depicts a dead girl, lying beside the road. While the camera zooms in on her face, her voice-over says, 'If you hit me at 40 miles an hour, there's about 80 % chance I'll die'. Then her bodily injuries heal quickly, and a drop of her blood recedes into her ear. The girl slides back to the road, suddenly start breathing again and says: 'Hit me at 30 and there's around an 80 % chance I'll live'. This dramatic spot was liked least (3.48), according to the program analyzer measurements, and evoked the highest percentage of negative emotions (62 %). It also prompted high scores on self-report measures of whether it was shocking (5.00 on seven-point scale) and fear-inducing (4.69) but low on the liking measure (3.00). Finally, it was unknown to all subjects.

4.2 Study 1

The 300 students who participated in Study 1 (56 % women; mean age, 23 years) were informed, after arriving in the laboratory setting, that their task would be to watch and evaluate a television program. One of the authors instructed them on how to use the program analyzer. In a training session with three commercials, they became familiar with the measurement instrument. Next, they were randomly assigned to one of the three treatment conditions, viewed the stimulus material and completed a questionnaire (which mainly gathered demographic data). Finally, they were debriefed about the goals of the study.

The data collected through the program analyzer feature three dimensions (X_{sit}): $s = 1, \ldots, 16$ identifies the different spots; $i = 1, \ldots, 300$ indicates the respondents (100 per group); and $t = 1, \ldots, T_s$ denotes their evaluations over time (T_s is approximately 60 but depends on the length of each spot s). For convenience,

Table 1 Manipulation check, evaluations of spots 3 and 8

	Study 1		Study 2	
	Spot 3	Spot 8	Spot 3	Spot 8
CG	5.98	5.33	6.27	5.13
EG1	5.92	4.14	6.00	4.47
EG2	4.53	4.27	4.61	3.42
F	14.73	11.54	17.99	15.39
p	<0.01	<0.01	<0.01	<0.01

Table 2 Evaluations of spot 9, study 1

	Average Evaluations after . . .				Total
	5 s	10 s	15 s	20 s	
CG	5.58	5.51	5.45	5.44	5.23
EG1	4.51	4.78	4.90	4.99	4.98
EG2	4.60	4.72	4.75	4.81	4.78
F	7.00	5.43	4.29	3.51	1.79
p	<0.01	<0.01	0.02	0.03	0.17

we present the results aggregated over time ($X_{si.}$), with respondents as the unit of analysis. (In Study 2, we report the results aggregated over respondents ($X_{s.t}$), such that half seconds are the unit of analysis.)

As the mean evaluations in Table 1 (columns 2 and 3) indicate, the evaluations of spot 3 were much higher among participants in the CG (5.98) and EG1 (5.92) conditions than by those participants in EG2 who watched the fear-inducing PSA (4.53). These differences were statistically significant, according to an analysis of variance. The manipulation was also successful for spot 8; the second exposure to the fear-inducing PSA invoked even worse evaluations (4.27) in EG2.

For a more concise presentation of the results, we focus on spot 9. Considering the potential for a short duration of affective priming effects, in Table 2 we present the participants' evaluations of spot 9 averaged over the first 5, 10, 15 and 20 s, as well as for the total duration of the spot, with corresponding tests for differences in means (rows 6 and 7). After 5 s, members of the two experimental groups evaluated spot 9 significantly worse (4.51 and 4.60) than members of the control group (5.58). This after-effect of the preceding fear-inducing PSA decreased over time; evaluations averaged over the whole duration of the spot did not differ significantly anymore. These findings support H1, but only for a short period of time. With respect to RQ1, we found no significant differences between single (EG1) and double (EG2) exposures to the fear-inducing spot. Moreover, our detailed analysis of the potential moderator effects of age and gender (RQ2) offered no consistent, significant results.

4.3 Study 2

The student population used in Study 1 could constitute a limitation, in that students might be less diligent or respond more spontaneously. According to Lennon et al.'s (2010) findings, the '30 for a reason' spot also should be more effective among parents with children. Therefore, we replicated Study 1 with a gender-based quota sample, using identical stimulus material and measurement instruments and similar data collection procedures. Research assistants approached potential participants at different locations (e.g. office buildings, shopping malls), where they could show the stimulus material on a screen and record evaluations with the program analyzer. These respondents again were assigned randomly to the three treatments, but some technical difficulties interfered with some of the data recording. We thus achieved a sample of 296 usable data records: 107 for the CG, 83 for EG1 and 106 for EG2. Among the respondents, 52 % were women, and 43 % were younger than 35 years, 42 % between 35 and 55 years and 15 % older than 55 years.

In Table 1 (columns 4 and 5), we present the mean evaluations of spots 3 and 8, which reinforces our finding that the manipulation was successful. In addition, the evaluations of the two samples were similar. The fear-inducing PSA also was evaluated as significantly worse when the Study 2 participants viewed it again (EG2: 4.61 vs. 3.42).

We present the evaluations of spots 4 and 9 in Tables 3 and 4, respectively. These results again affirmed our Study 1 findings, indicating a pronounced after-effect of a preceding fear-inducing PSA for both single (EG1 spot 9; EG2 spot 4) and double (EG2 spot 9) exposures. This after-effect decreased with the passage of time, such that evaluations averaged over the entire duration of the spots were no longer significantly different. We found no consistent, significant moderator effects of age or gender. These results reinforce the outcomes of Study 1 with respect to H1, RQ1 and RQ2.

To explicate this diminishing magnitude of the after-effects, we aggregated the data over respondents ($X_{s,t}$) and show, in Figs. 2 and 3, the evolutions of the three groups' ratings of the spots over time. These visual presentations clearly substantiate the identified pattern. The fear-inducing PSA (spot 3 for EG2, spot 8 for both experimental groups) was less liked than a neutral advertisement. An after-effect of spots 3 and 8 on spots 4 and 9 arose among participants who watched the preceding fear-inducing PSA, but after about 20 s, the effect levelled off.

Table 3 Evaluations of spot 4, study 2

	Average Evaluations after…				
	5 s	10 s	15 s	20 s	Total
CG	5.71	5.04	4.57	4.23	3.77
EG1	5.46	4.71	4.20	3.86	3.46
EG2	4.14	4.05	3.84	3.69	3.49
F	12.58	5.57	3.30	1.93	0.71
p	<0.01	<0.01	0.04	0.16	0.49

Table 4 Evaluations of spot 9, study 2

	Average Evaluations after . . .				Total
	5 s	10 s	15 s	20 s	
CG	5.28	5.23	5.23	5.27	5.15
EG1	4.62	4.62	4.71	4.86	5.02
EG2	3.60	3.60	3.95	4.24	4.85
F	12.19	12.19	9.01	7.05	0.76
p	<0.01	<0.01	<0.01	<0.01	0.49

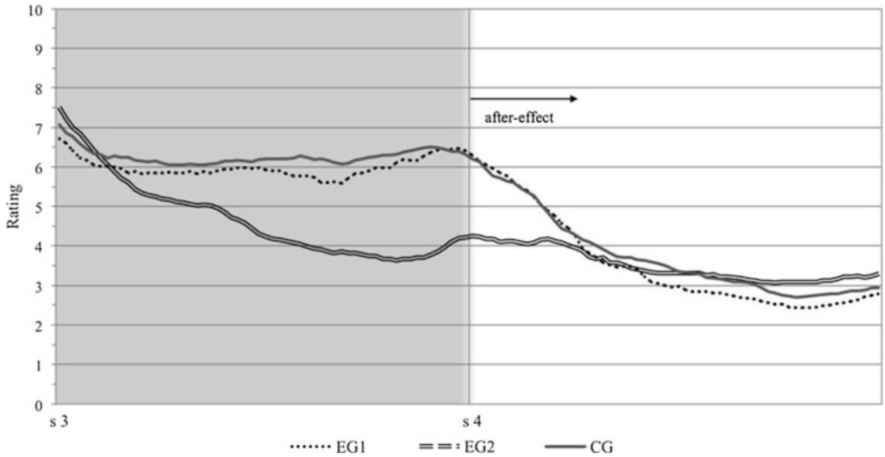

Fig. 2 Ratings of spots 3 and 4, study 2

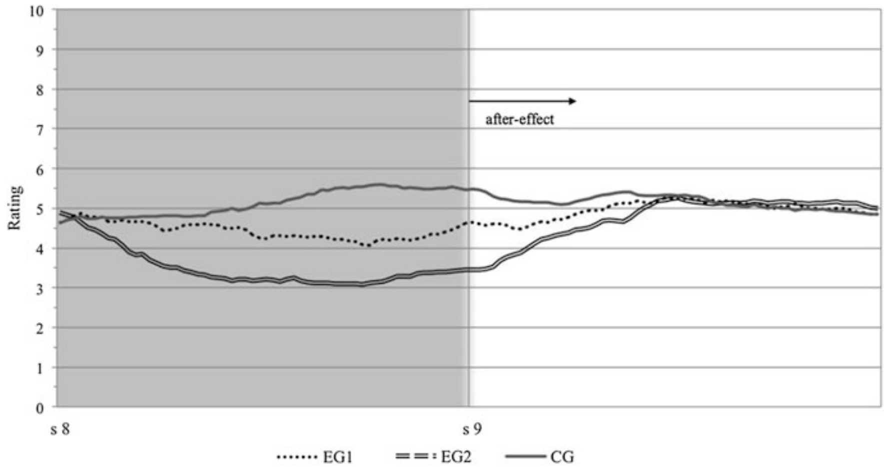

Fig. 3 Ratings of spots 8 and 9, study 2

Finally, we investigated the pattern by postulating the following model:

$$X_{s.t}^{g} = \alpha + \beta \cdot X_{s.t}^{CG} - \exp(-\gamma \cdot t) + \varepsilon_t$$

ε_t : random error term
g = EG1 or EG2
$s = 4$ or 9
α, β, γ : model parameter

That is, we assumed a linear relation between evaluations $X_{s.t}^{CG}$ of a certain spot (i.e. spot 4 or 9) by members of CG and by members of EG1 or EG2 ($X_{s.t}^{EG1}, X_{s.t}^{EG2}$). If ($\alpha = 0, \ \beta = 1$), it would indicate perfect accordance. The linear relation is further assumed to be moderated by a random error term ε_t and an exponentially decreasing shock $\exp(-\gamma \cdot t)$ that represents the after-effect of the preceding spot. Therefore, we expected $\gamma > 0$.

Table 5 presents the results when we estimated $\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}$ using nonlinear regression analysis. The R-square values (row 6, columns 2, 5, and 8) indicate satisfactory fit for all three cases. In a nonlinear regression, the F-statistic is under certain assumption only asymptotically Fisher distributed (Greene 2003, p. 176), though some researchers dispute its application. Nevertheless, for this study, the statistic was well above the appropriate critical value of an F-distribution for a Type-I error of 0.01 in all three cases (row 6, columns 3, 6, and 9). All but one parameter estimate were statistically highly significant and face valid; the postulated exponential decay of the after-effect seemed to match particularly well. The exception was the $\hat{\beta}$ of spot 9 associated with the EG2 evaluations, as a dependent variable. Figure 3 clarifies the situation: The almost constant regressor variable (CG evaluations) did not contribute explaining variations in the dependent variable, so $\hat{\beta}$ was estimated to be zero. Finally, the magnitude of the after-effect after 5, 10 and 20 s showed a decreasing trend; its magnitude was estimated to be 0.11 for spot 4 after 20 s (cf. row 8 of Table 5). Thus, the evaluations of participants in EG2 after 20 s were only slightly lower than the evaluations of members of the CG. This deviation was within the confidence interval of the error component of the model and statistically negligible (cf. row 6, column 4: the estimate of the error standard deviation $\hat{\sigma} = .11$ resulted in a 95 % error confidence interval of [−0.22, +0.22]). Thus, using half seconds as the unit of analysis, we again found support for H1 and no substantial differences between single and double exposures (RQ1).

5 Discussion

The results of Studies 1 and 2 were very similar, despite their distinct sampling populations (i.e. students vs. general population). This comparison offers a further preliminary response to RQ2; the demographic characteristics of the audience

Table 5 Results of nonlinear regression analysis, study 2

Dependent variable	Spot 4 ($s = 4$), EG2 evaluations ($g = \text{EG2}$)			Spot 9 ($s = 9$), EG1 evaluations ($g = \text{EG1}$)			Spot 9 ($s = 9$) EG2 evaluations ($g = \text{EG2}$)		
	Parameter estimate	Standard error	p-level	Parameter estimate	Standard error	p-level	Parameter estimate	Standard error	p-level
$\hat{\alpha}$	1.53	0.06	<0.01	0.87	0.23	<0.01	5.08	0.07	<0.01
$\hat{\beta}$	0.59	0.01	<0.01	0.85	0.05	<0.01	0		
$\hat{\gamma}$	0.06	0.01	<0.01	0.08	0.01	<0.01	0.07	0.02	<0.01
	$R^2 = .92$	$F = 217.62$	$\hat{\sigma} = .11$	$R^2 = .95$	$F = 340.31$	$\hat{\sigma} = .05$	$R^2 = .77$	$F = 94.34$	$\hat{\sigma} = .21$
	after 5 s	after 10 s	after 20 s	after 5 s	after 10 s	after 20 s	after 5 s	after 10 s	after 20 s
$\exp(-\hat{\gamma} \cdot t)$	0.57	0.33	0.11	0.46	0.21	0.05	0.51	0.26	0.07

probably do not moderate the after-effects of PSAs. Across both studies, fear-inducing PSAs influenced viewers' assessments of subsequent commercials in the block, but only those that appeared immediately following the fearful message—or even more specifically, the first few seconds of these immediately subsequent commercials. The effect rapidly decreased with the passage of time, which is consistent with the decay of the arousal-spillover effect observed by Mattes and Cantor (1982). These findings are also in line with affective priming studies that indicate priming effects are fleeting. The results thus enable us to specify a time limit for the duration of emotional priming effects in the context of television commercials (i.e. less than 20 s).

The findings with regard to RQ1 indicate that previous exposures to a fear-inducing message do not reduce its effectiveness. This result might reflect the great strength of the stimulus we used in these studies. We might expect an eventual wear-out effect after many repeated exposures, but a PSA featuring a dead, young girl lying on the side of the road did not lose its shock appeal after just one exposure.

The results of this study should be relevant to advertising practitioners, because even though the after-effects of the fear-inducing messages persisted for only the first few seconds of subsequent commercials, it could be enough to be a concern. Particularly in markets such as Europe, where commercials often are only 15 s long, every second counts. Therefore, we advise against scheduling advertisements directly following shocking PSAs or commercials that employ fear appeals.

A notable limitation of this study is methodical, in that respondents' latency when moving the lever on the program analyzer might have affected the results. However, the evidence available from previous research that also has used program analyzers to measure carry-over effects (Ebster et al. 2009) leads us to believe that the effects we uncovered are real, not some kind of methodical artefact. It might be insightful for additional research to investigate this issue though. Another limitation might stem from the forced exposure design, including a somewhat artificial environment and stimulus material that was unknown to participants, which could have affected their involvement.

In addition to research to address these limitations, we suggest that further research investigate the carry-over effects of different types of fear appeals (e.g. social fear). For example, does it make a difference if the fearful situation gets resolved at the end of the commercial or PSA? We hope the results of this study are of use to advertisers and media planners who strive to achieve the most effective positions for their commercial communication.

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Decision Support for Strategic Disaster Management: First Release of a Wiki

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Abstract For successful emergency management (EM) it is crucial that all stakeholders, especially health care emergency responders, use the same terminology. Throughout the emergency management lifecycle it is necessary for individual agencies to work together, sharing information and resources. Emergency management is already a complex process, but a multi-agency response comes with added difficulties. Each agency has its own organisational cultures, structures, and technologies in place, managed by internal processes and systems. To address some of the challenges associated with a multi-agency response (e.g., lack of coordination, information, and interoperability), standardisation is promoted. By ensuring the use of shared terms, operational inefficiencies and delays can be reduced and a shared vocabulary can be promoted across multiple agencies. For this reason, the S-HELP Strategic Disaster Management wiki has been developed by University of Vienna, Austria (UNIVIE). The wiki provides main glossary terms, definitions, and standards to improve decision making. It is implemented as a part of the FP7-EU S-HELP (Securing Health.Emergency.Learning.Planning) project, which develops a Decision Support (DS) tool for EM and is coordinated by University College Cork (UCC), Ireland.

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1 Introduction

Operations Research and Management Science models have successfully supported policy making in disaster management as illustrated by the reviews of Altay and Green (2006), Caunhye et al. (2012), Galindo and Batta (2013), Manopiniwes and Irohara (2014), and Hoyos et al. (2015). For example, Doerner and Hartl (2008) investigated new challenges for routing problems for health care logistics, emergency preparedness, and disaster relief with a focus on the Austrian situation. In addition, distribution of critical goods plays a crucial role in disaster relief such as water (Nolz et al. 2010) or food, shelter, and medicine (Nolz et al. 2009).

Decision support systems are essential in supporting activities undertaken during an emergency response. For example, "spatial decision support systems were used routinely in the rescue and relief operations in the World Trade Center disaster. These ranged from micro-level risk assessments (shifts in the debris pile and temperature hot spots at the site) to the spatial status of lifelines (electric, water, telephone, transportation networks), all of which changed almost daily" (Cutter 2003, p. 441). For emergency management, these systems need to integrate advanced computer and communication technologies with components such as Geographic Information Systems (GIS), storm tracking tools, damage projection and flooding models, and models for evacuating an affected population (Tufekci 1995), to name a few. These systems help emergency responders to make better decisions based on real-time data and the use of modelling tools. Using the evacuation scenario as an example, various decisions must be made based on how many people are at risk, where people can be evacuated to, how casualties can be transported, and deciding on how the process can be managed (Carver and Turoff 2007). As a result, integrated real-time decision support needs to be effective under complex and changing conditions (Tufekci 1995), as well as provide easy to understand displays that do not 'overload' the end-users with too much information and ultimately reduce an agency's ability to find information when it is needed (French and Turoff 2007; Manoj and Baker 2007).

In order to address these challenges, and the challenges associated with emergency management, the FP7-EU S-HELP (Securing Health.Emergency.Learning. Planning) is developing a decision-support tool-set with function-specific modules that integrate with existing tools and systems. S-HELP is coordinated by University College Cork (UCC), Ireland (http://www.fp7-shelp.eu/). As agencies that respond to emergencies already have legacy systems in place, it is vital that new solutions can be integrated with old, as well as add value through new capabilities and functionalities. Hence, the S-HELP project seeks to create a tool-set to support rapid and effective decision making across all stages of an emergency lifecycle (i.e. mitigation, preparation, response, and recovery), which can be effectively implemented for a cross-border multi-agency response. To achieve this, a holistic framed approach has been developed that integrates a number of components for consideration in the development of an EM decision support system. Such components include interoperability standards (cf. Waugh and Streib 2006), modular end-user focused tool development (cf. Carver and Turoff 2007), real-life emergency scenarios (cf. Reznek et al. 2003), end-user training in decision making (cf. Alexander 2003; Kowalski-Trakofler et al. 2003), as well as risk communication through information communication technologies and social media (cf. Sutton et al. 2008; Veil et al. 2011). Furthermore, for the design of the decision support tool-set, concepts such as spatial modelling and mapping (cf. Cutter 2003), psychological frameworks and information processing for user interface design have been investigated and incorporated (cf. Chen and Lee 2003).

Consequently, S-HELP enhances the protection of public health and common grounds for interoperability by significantly advancing the existing knowledge base required for the development of a user-centred decision support (DS) tool-set for the management of emergency situations. By gathering the collective knowledge of industry and research practice to identify current constraints in the development of emergency management solutions, S-HELP addresses the capability gaps that exist in currently available commercial tool-sets. Literature has highlighted gaps in the activities undertaken by emergency managers and the capabilities of commercially available tools, as well as a lack of focus and integration between the complimentary phases of the emergency management (EM) lifecycle (Altay and Green 2006). Hence, S-HELP seeks an advantage over other projects and tools, by developing a solution that accounts for the interplay among the entire EM lifecycle and that addresses the capability gaps identified in the literature and by EM end-users. S-HELP will define an interoperability standard to enable communication and coordination of multi-agencies across different geographical areas and cultural settings and facilitate a collaborative end-user driven solution to meet the needs of these varied users and countries. In addition, S-HELP advances the design and application of currently available solutions, to improve preparedness, response and recovery in emergency situations and provide a decision support tool-set that has been tested, evaluated, and enhanced through quality, end-user designed emergency scenarios.

For EM stakeholders, it is of highest importance to obtain a list of a controlled vocabulary and general structure for the main areas of strategic disaster management. Hereby, a glossary of terms and definitions with common grounds and standards for interoperability are essential. For this reason, we have provided a taxonomy for strategic disaster management in the form of a wiki. Main glossary terms for emergency management are incorporated based on international standard terms, scientific literature, and practice. Furthermore, main stakeholder groups and associated common grounds are considered. For the specific disaster examples, we focus on the three disaster policy scenarios of S-HELP (flooding, chemical explosion, and biological hazard).

Taxonomies or systems of classification are often used in the natural sciences (e.g., the Periodic Table, International Classification of Diseases) to apply order to “an apparently perplexing variety of elements of a scientific field or specialty” (Heidenberger and Roth 1998). In addition, taxonomies may help stakeholders “in a systematic search for hitherto unknown, and hopefully better” components, structures, and systems of a certain area (Heidenberger and Roth 1998, p. 337).

The taxonomies can be "extracted or identified along multiple dimensions and ordered to result in a coherent frame of reference. This frame of reference facilitates the recombination of the essentials of previous approaches and may result in new approaches of enhanced analytical power" (Heidenberger and Roth 1998, p. 337).

Our taxonomy hierarchically structures both controlled terms and the main areas of strategic disaster management by creating a visual representation (cf. Zarate 2013). For certain areas, thesaurus terms are provided if needed. Furthermore, it is very useful for education and training purposes for emergency responders. Please note, that the main content of this taxonomy is implemented in the S-HELP DSS.

The development of wiki for the strategic disaster management glossary and taxonomy was deemed most appropriate. As a wiki has a "significant potential to improve knowledge work and information sharing within organizations" (Jackson and Klobas 2013). A wikis' advantage of usability and simplicity are useful and appropriate to support the decision making of EM stakeholders. The wiki platform provided by University of Vienna, Austria was used for the implementation by UCC. The strategic disaster management wiki can be accessed via https://wiki.univie.ac.at/display/SHELP/ Please note that this is the first release of the strategic disaster management wiki that is planned to be further expanded.

In Sect. 2, we explain the taxonomy for strategic disaster management in detail and refer to related work. Examples for glossary terms are discussed in Sect. 3. Section 4 concludes the paper and outlines further research.

2 Taxonomy for the Strategic Context of Disaster Management[1]

Figure 1 contains the top level of the *taxonomy for strategic context of disaster management* that includes both external and internal components/criteria. This taxonomy and the related glossary terms serve as an essential part of the S-HELP DSS for disaster management policy makers. The first release's structure of the taxonomy contains 29 Figures and 806 glossary terms as explained before and can be expanded if needed.

For EM decision makers who use the S-HELP DSS, it is essential to understand the external *emergency management environment* influence on policy making (cf. Sect. 2.1). Next, EM decision makers need to know what can happen and how disasters can be defined (cf. Sect. 2.2), which is a key component of the strategic management. *Disasters* also impact the *emergency management environment*. This knowledge is incorporated in the S-HELP DSS and is needed for training of

[1]This section is based on University of Vienna, Austria (UNIVIE): Rauner, M., Niessner, H., Sasse, L., Tomic, K. (2014) Securing Health Emergency Learning Planning, S-H.E.L.P., Collaborative Project FP7-SEC-2013-1, Project no. 607865, Deliverable No. 2.1, Glossary of terms and definitions & common grounds and standards for interoperability.

Fig. 1 Taxonomy for the Strategic Context of Disaster Management

emergency responders with a special focus on health care responders (*stakeholders*) who are responsible for coping with different disaster policy scenarios (*disaster definition*).

Once this external information is gathered, the internal *decision making* can take place by EM *stakeholders.* First, we analyse the decision making process (*stakeholders, decision making levels, decision support systems, command and control systems*) in Sect. 2.3. EM policy makers have to decide what has to be done (*emergency management cycle*, *related interventions*) [cf. Sect. 2.4], which staff and material are needed (*resources*) [cf. Sect. 2.4]. Note that *decision making* will also impact the *emergency management environment*. These logically-clustered taxonomy components can be easily expanded by further taxonomies. Pease note that a main focus is given on disasters in the field of flooding, chemical explosion, and biological hazards because of the S-HELP project. This information is the basis for emergency responder (*stakeholder*) training depending on the disaster scenarios (*disaster definition*). These disaster scenarios also impact on the *related interventions* initialized by EM decision makers and the required *resources*.

On the left-hand side of our strategic disaster management wiki, EM policy makers can:

1. search for specific glossary terms,
2. look-up figures and glossary terms related to certain chapters of the *taxonomy for strategic context of disaster management* (cf. Sect. 2),
3. select definitions of glossary terms listed in alphabetical order (cf. Sect. 3), and
4. display the literature of the strategic disaster management wiki.

Main figures for classification are developed and linked to the definitions (glossary terms). Each section of the taxonomy contains at least one figure and a further set of the topic related-terms, either below each figure or below the chapter heading on the left-hand side in the wiki. We included general terminology used by international EM Organizations such as the World Health Organization (WHO) or specialized health care agencies such as the Centers for Disease Control and Prevention (CDC).

During the performance of the literature research and interviews of policy makers as well as consideration of end-user requirements, a decision was taken to restrict the strategic disaster management wiki to key topics on a general non-country specific level based on general international standards and guidelines for disaster management to enhance interoperability and cross-border communication. Restricting the wiki to essential topics was necessary due to a limited development timeframe (9 months). Furthermore, by restricting the wiki to essential topics the first release could easily be incorporated into the S-HELP DSS which is highly essential for end-user training. It is envisaged that this strategic disaster management wiki will become a living repository of core EM terms. The use of wiki technology ensures that this resource can extended over time.

The research was performed as a desktop research of academic literature (books, journal articles, and conference papers), the EU, other EU projects, and existing domain-specific skill descriptions in practice. Furthermore, domain-specific skills of paramedics were investigated in detail at the Enquete “Paramedics: International Education Concepts” at the University of Applied Sciences, St. Pölten (October 2014). UNIVIE presented the S-HELP project and conducted three interviews with Austrian end users regarding the functionality of the S-HELP Decision Support System: (1) deputy head of Division Operation, Innovation, and Subsidiaries of the Austrian Red Cross, (2) head of special course of studies on “Ambulance Service Management”, Danube University of Krems, and (3) policy makers at the Coordination Center for Emergency Calls, Lower Austria. The academic and practitioner literature was supplemented with other sources including:

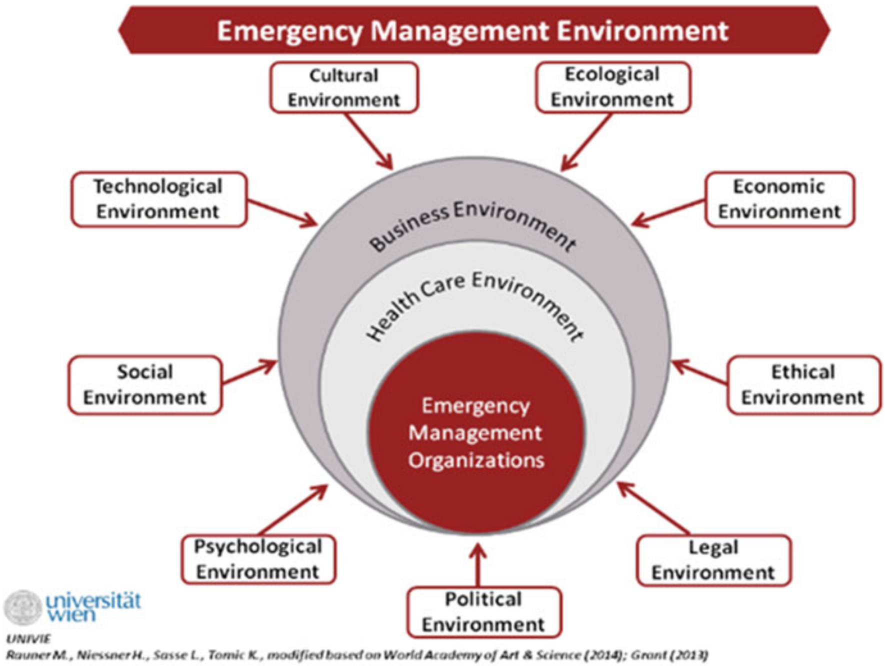
- Centers for Disease Control and Prevention, CDC,
- Centre for Research on the Epidemiology of Disaster, CRED,
- Digital Humanitarian Network, DHN,
- Emergency Response Coordination Centre, ERCC,
- European Commission, Flood Risk Management,
- United Nations Office for the Coordination of Humanitarian Affairs, OCHA, and
- World Health Organization, WHO.

The project builds on widely accepted international standards for interoperability based on ISO norms and EU guidelines such as:

- European Union Civil Protection Modules, EU CP Modules,
- International Organization for Standardization, ISO, and
- Interoperable Delivery of European eGovernment Services to Public Administrations, Businesses, and Citizens, IDABC.

2.1 Emergency Management Environment

The general *emergency management environment* is defined in Fig. 2 which was modified based on World Academy of Art and Science (2014) and Grant (2013). Please note that *disasters, decision making*, and the *emergency management environment* influence each other. For example, improved decision making such as the S-HELP DS tool can lower the *extent of an event* and the *vulnerability* of a society to disasters.



- Business Environment
- Cultural Environment
- Ecological Environment
- Economic Environment
- Emergency Management Organizations
- Ethical Environment
- Health Care Environment
- Legal Environment
- Political Environment
- Psychological Environment
- Social Environment
- Technological Environment

Fig. 2 Emergency Management Environment

Cultural, *ecological, economic*, *ethical*, *legal*, *political*, *psychological*, *social*, and *technological environments* are sub-environments that impact on the *business environment* and further on the *health care environment* up to the *emergency management organizations*. For example, the *technological environment* is defined as the surrounding field “resulting from improvements in technical processes that increase productivity of machines and eliminates manual operations or operations done by older machines” (Merriam-Webster Dictionary 2014). Of highest importance are the *emergency management organizations* that are responsible for “the organization and management of resources and responsibilities for addressing all aspects of emergencies, in particular preparedness, response and rehabilitation” (United Nations Office for the Coordination of Humanitarian Affairs 2008). These emergency management organizations are also part of the *stakeholders* who make decisions in EM situations (cf. Sect. 2.3.1).

A list of the components (i.e., sub-environments) for the *emergency management environment* is explained below the related figure in the wiki, as displayed in Fig. 2. All related definitions are linked to the glossary.

Health care organizations/individuals (cf. Ginter et al. 2013) comprise (cf. Fig. 3): (1) *organizations that regulate primary and secondary providers*, (2) *secondary providers (resources)*, (3) *primary providers (health services)*, (4) *organizations that represent primary and secondary providers*, and (5) *individuals and patients (consumers)*. For example, patients and consumer groups are consumers of health care services. Thus, they are an essential part of the external health care

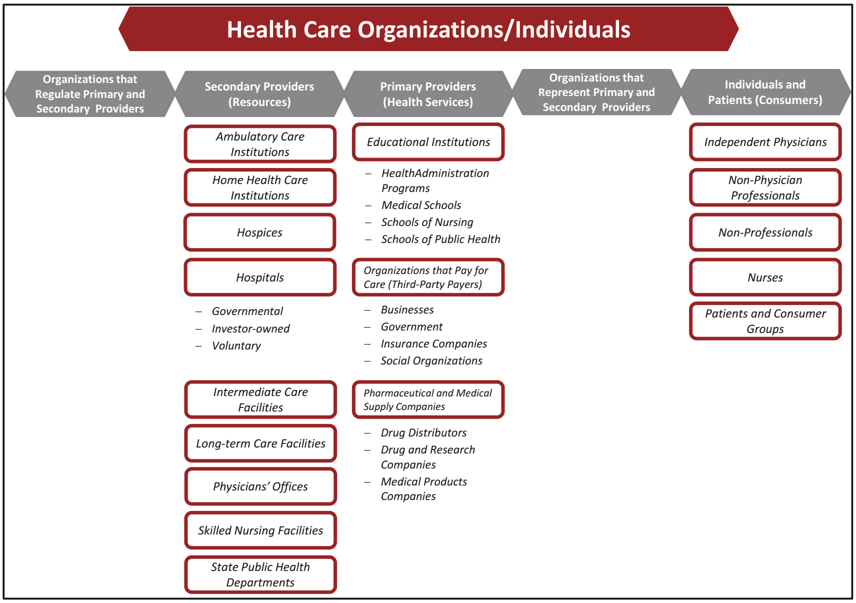


Fig. 3 Health Care Organizations/Individuals

environment. All of the groups mentioned above influence technological, social, regulatory, political, economic, and/or competitive issues within the health care sector. Note that *health care organizations/individuals* might also be part of the *stakeholders* in the EM environment (Sect. 2.3.1). Especially the primary and secondary providers that are involved in care and treatment of injured people by using their staff and material (*resources*).

2.2 Disaster Definition

For EM *stakeholders* it is of highest importance to first categorize a disaster according to Fig. 4 (cf. Birkmann 2007): (1) *disaster types* (Sect. 2.2.1), (2) *extent of event* (Sect. 2.2.2), (3) *vulnerability* (Sect. 2.2.3), and (4) *disaster risk* (Sect. 2.2.4). The *disaster risk* is dependable on *disaster types*, *extend of event*, and *vulnerability*.

In addition, the wiki provides the main general *disaster definitions* by international EM agencies:

1. "A serious disruption of the functioning of a community or a society causing widespread human, material, economic or environmental losses which exceed the ability of the affected community or society to cope using its own resources" (Office for the Coordination of Humanitarian Affairs 2008).

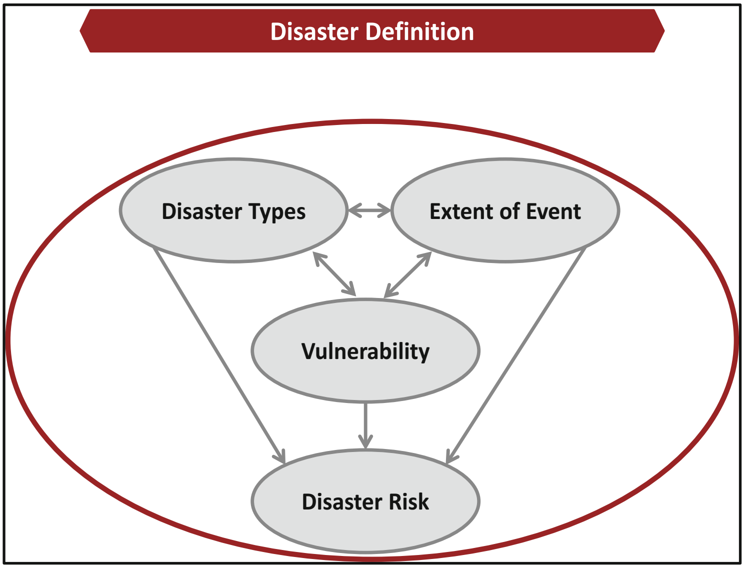


Fig. 4 Disaster Definition

Comment: Disasters are often described as a result of the combination of a natural hazard, the conditions of vulnerability, and insufficient capacity or measures to reduce or cope with the potential negative consequences. A disaster also may be seen as an outcome of the 'risk process', the interactions of the above three factors over time that lead to the development of disaster risks and the expression of that risk through disaster events (Office for the Coordination of Humanitarian Affairs 2008).

2. "Situation or event, which overwhelms local capacity, necessitating a request to national or international level for external assistance (definition considered in EM-DAT); An unforeseen and often sudden event that causes great damage, destruction and human suffering. Though often caused by nature, disasters can have human origins. Wars and civil disturbances that destroy homelands and displace people are included among the causes of disasters. Other causes can be: building collapse, blizzard, drought, epidemic, earthquake, explosion, fire, flood, hazardous material or transportation incident (such as a chemical spill), hurricane, nuclear incident, tornado, or volcano (Disaster Relief)" (The International Disaster Database, EM-DAT 2014).

The wiki also describes the criteria for a *disaster*—and at least one of the following criteria must be fulfilled (The International Disaster Database, EM-DAT 2014):

- "Ten (10) or more people reported killed.
- Hundred (100) or more people reported affected.
- Declaration of a state of emergency.
- Call for international assistance."

2.2.1 Disaster Types

The *disaster types* can be categorized into *natural*, *man-made*, and *hybrid disasters* that can also cause *subsequent disasters* (cf. Fig. 5) which are modified based on the international disaster database (EM-DAT 2014; Shaluf 2007).

Natural disasters contain *biological*, *cosmological*, *geophysical*, and *hydro-meteorological disasters*, while *man-made disasters* include *socio-technical disasters* and *human conflicts*.

For example, *biological disasters* might be an *epidemic*. *Epidemics* can be categorized by: (1) *types of epidemics*, (2) *chain of infection*, (3) *infection disease stages of individuals*, and (4) *evolution of epidemics*.

Types of epidemics include *bacterial*, *fungal*, *parasitic*, *prion*, or *viral infections* (*infectious diseases*). The wiki defines infectious diseases as "diseases that can be spread, directly or indirectly, from one person to another" (WHO 2014). "Zoonotic diseases are infectious diseases of animals that can cause disease when transmitted to humans" (WHO 2014). Examples of viral infectious diseases are listed in the wiki such as hepatitis and Marburg virus based on WHO (2014). Several diseases are further exemplarily explained.

To fight against certain diseases by selecting related interventions, policy makers must understand the *chain of infection* by identifying the mode of transmission, portal of entry/exit, as well as the reservoir, infectious agent, and susceptible host as illustrated by Fig. 6 (cf. Krämer et al. 2010).

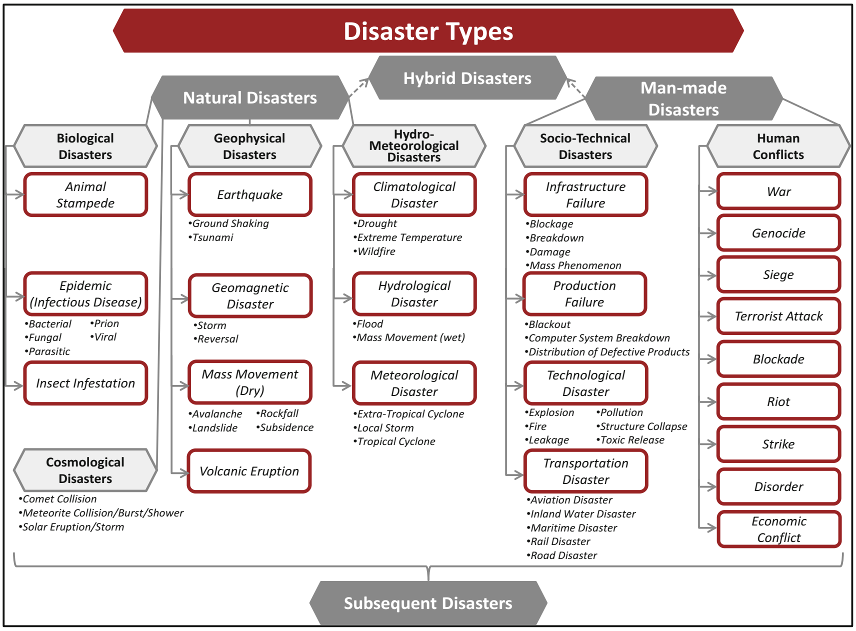


Fig. 5 Disaster Types

The main *infectious disease stages of an individual* are represented in Fig. 7 (cf. Anderson and May 1991). Please note, that there are several infectious diseases such as HIV/AIDS from which an individual cannot recover from yet. In addition, a recovery from a certain disease does not automatically result in immunity.

Figure 8 (cf. Institute of Medicine 2009) displays the *evolution of epidemics* for infection diseases from the initial *exposure* to a population (level 1) up to an *epidemic spread* (level 4). Policy makers can fight against the spread of *epidemics* by initiating *interventions* using *resources* (staff and material).

2.2.2 Extent of Event

The *extent of event* is most crucial for EM *stakeholders* as it highly impacts *disaster risk* (cf. Fig. 9). The main components of the extent of event cover (modified based on De Smet et al. 2012): (1) *forewarning*, (2) *time*, (3) *location & size*, (4) *level*, (5) *intensity*, and (6) *duration*.

For example, the *intensity* of a disaster can be measured by established standards, depending on the disaster type. The *discharge* or *stage* can be used to measure the severity for *floods* (The Science Education Resource Center at Carleton College, SERC 2014).

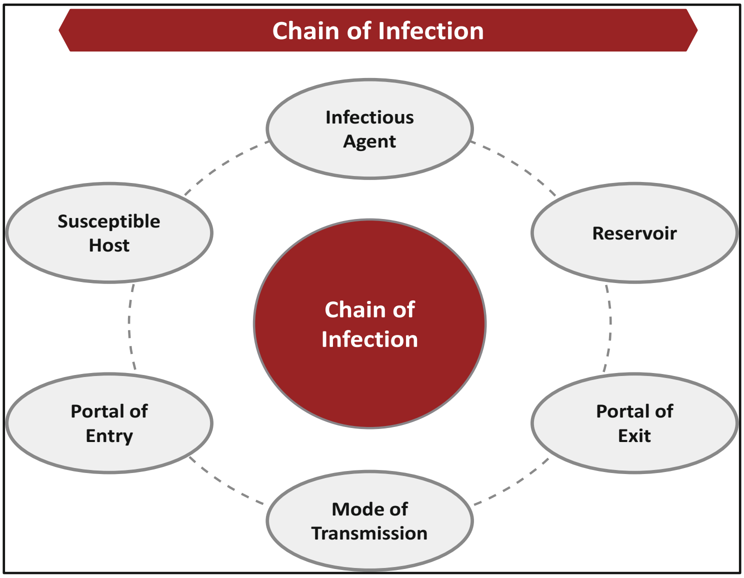


Fig. 6 Chain of Infection

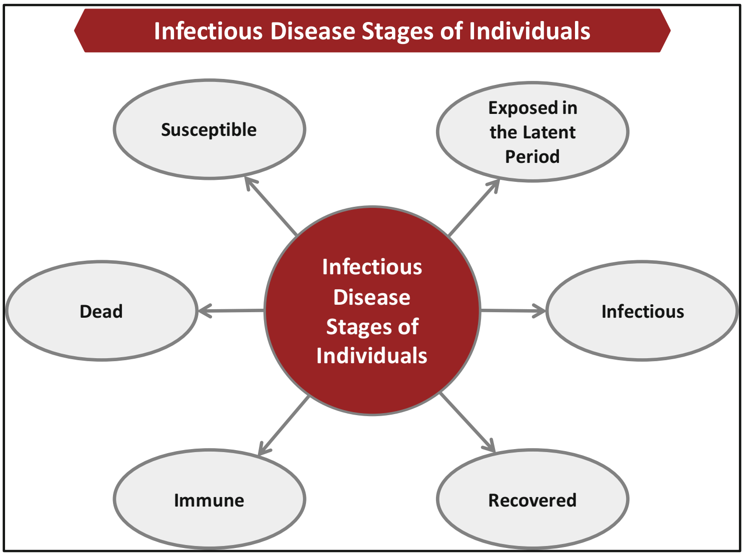


Fig. 7 Infectious Disease Stages of Individuals

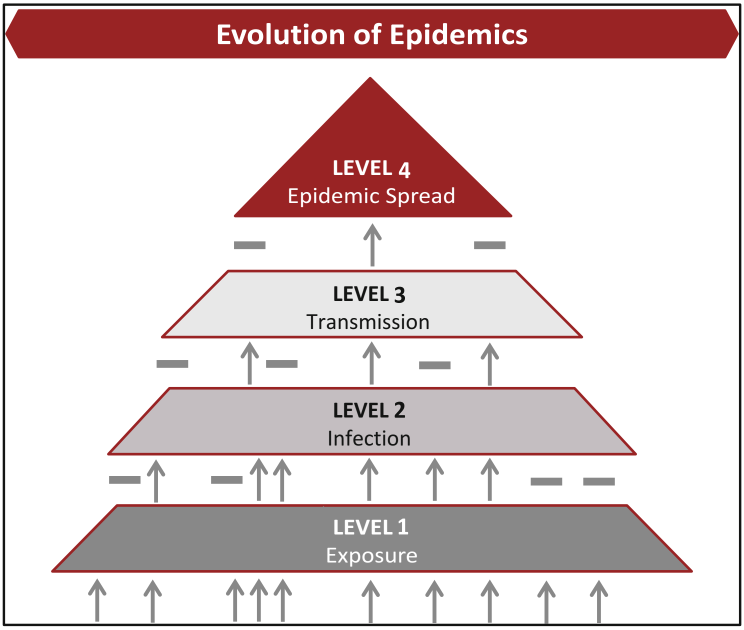


Fig. 8 Evolution of Epidemics

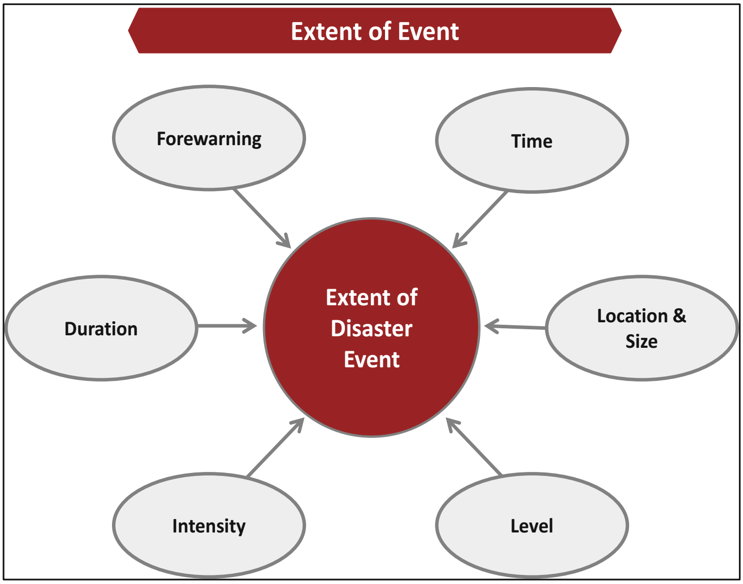


Fig. 9 Extent of Event

An emergency incident can happen on five levels based on the regional extent: (1) *local*, (2) *regional*, (3) *national*, (4) *cross-border*, and (5) *international level* (Irish National Steering Group 2006). Especially, a biological-hazard scenario or a chemical explosion policy scenario might be at a cross-border or even international level.

2.2.3 Vulnerability

The *disaster types* (cf. Sect. 2.2.1), the *extent of event* (cf. Sect. 2.2.2), and the *vulnerability* (cf. Fig. 10) influence each other, and together define the *disaster risk* (cf. Sect. 2.2.4).

The *vulnerability* is "determined by *physical*, *social*, *economic* and *environmental factors* or processes, which increase the *susceptibility* of a community to the impact of hazards (Office for the Coordination of Humanitarian Affairs 2008). It is dependent on the *exposure*, *susceptibility & fragility*, and *lack of resilience* (Birkmann et al. 2013). EM stakeholders need to increase the *resilience* capabilities and thus lower the *susceptibility* and *fragility* of societies by using *DSS.*

Exposure is defined as "the extent to which a unit of assessment falls within the geographical range of a hazard event. Exposure extends to fixed physical attributes of social systems (infrastructure) but also human systems (livelihoods, economies,

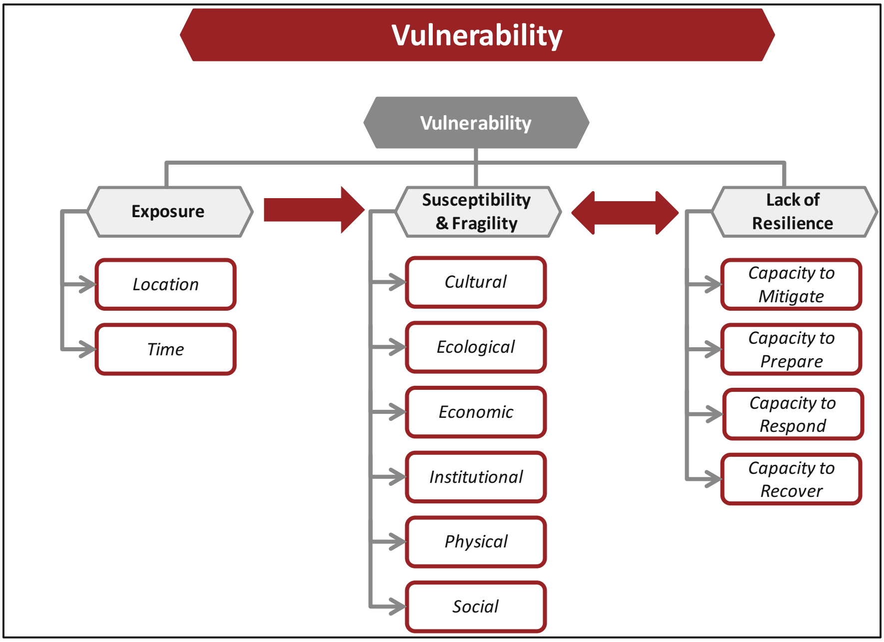


Fig. 10 Vulnerability

cultures) that are spatially bound to specific resources and practices that may also be exposed. Exposure is then qualified in terms of spatial and temporal patterns" (Birkmann et al. 2013, p. 200).

Exposure then affects the *susceptibility/fragility*, the "predisposition of elements at risk (social and ecological) to suffer harm—often independent of exposure" (Birkmann et al. 2013, p. 200). It has a *cultural*, *economic*, *environmental*, *institutional*, *physical,* and *social* component. For example, with respect to the *cultural dimension*, policy makers want to protect the "potential for damage to intangible values including meanings placed on artefacts, customs, habitual practices and natural or urban landscapes" (Birkmann et al. 2013, p. 200).

The *lack of resilience* and the *susceptibility/fragility* interact. *Resilience* is "the capacity of a system, community, or society potentially exposed to hazards to resist, adapt, and recover from hazard events, and to restore an acceptable level of functioning and structure" (Office for the Coordination of Humanitarian Affairs 2008). Comment: "Resilience means to 'resile from' or 'spring back' after a shock. The resilience of a social system is determined by the degree to which the system has the necessary resources and is capable of organizing itself to develop its capacities, to implement disaster risk reduction and to institute means to transfer or manage residual risks" (Office for the Coordination of Humanitarian Affairs 2008). The *lack of resilience* can be divided into the four elements of the emergency management cycle (cf. Sect. 2.4): (1) *lack of capacity to mitigate*, (2) *to prepare*, (3) *to respond*, and (4) *to recover* (Birkmann et al. 2013).

2.2.4 Disaster Risk

Disaster risk is affected by the *disaster types* (cf. Sect. 2.2.1), the *extent of event* (cf. Sect. 2.2.2), and the *vulnerability* (cf. Sect. 2.2.3).

Disaster risk is "the magnitude of potential disaster losses, in lives, livelihoods and assets, which could occur to a particular community or group, arising from their exposure to possible future hazard events and their vulnerability to these hazards" (Office for the Coordination of Humanitarian Affairs 2008). Comment: "The concept of disaster risk shifts the viewpoint from disasters as events randomly affecting places, to that of negative potential conditions continuously affecting all areas. Disaster risk encompasses several different types of potential losses—in lives, livelihoods and financial and other assets—and is often difficult to quantify. Nevertheless, with knowledge of the prevailing hazards and the patterns of population and socio-economic development, it can be assessed and mapped, in broad terms at least, and the factors contributing to the risks can be made subject to public and private risk-reducing actions" (Office for the Coordination of Humanitarian Affairs 2008). Thus, *humans*, *fauna*, *flora*, and the *infrastructure* might be impacted/harmed by a potential disaster.

For policy makers it is of highest importance to protect the *critical infrastructure* against disasters and to restore destroyed *critical infrastructure* as soon as possible

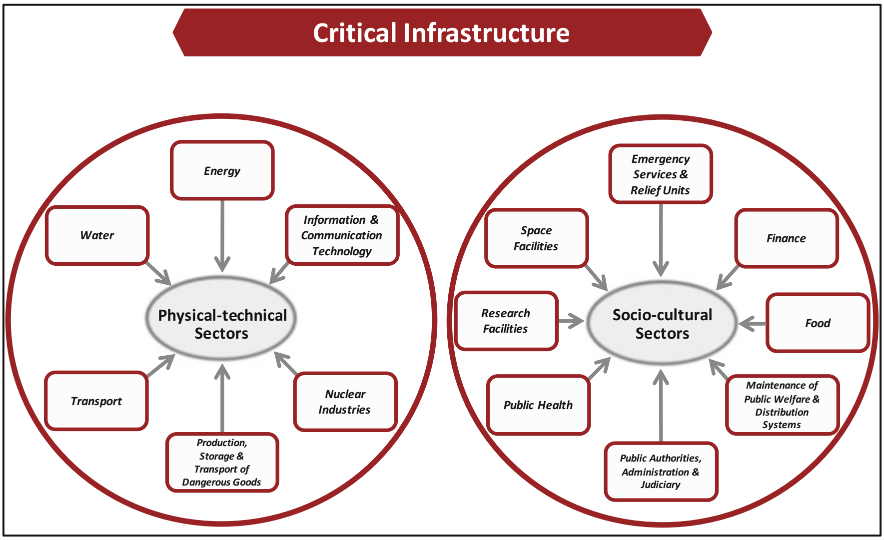


Fig. 11 Critical Infrastructure

after a disaster (cf. Fig. 11). The magnitude of the material destruction and/or contamination after an impact of an event can be divided into four categories:

1. "**Small**: one destructed or contaminated site (e.g., plant, building)
2. **Medium**: more than one destructed or contaminated site, partial destruction of a village, small town or small region (<50 km^2)
3. **Big**: partly destruction or contamination of more than one town
4. **Very big**: destruction or contamination of a large region (several towns are heavily disrupted or affected)" (De Smet et al. 2012, p. 143).

The *critical infrastructure* (cf. Fig. 11) can be categorized into *physical-technical sectors* (*energy, information & communication technology, nuclear industries, production, storage & transport of dangerous goods, transport, water*) and *socio-cultural sectors (emergency services & relief units; finance; food; maintenance of public welfare & distribution systems; public authorities, administration & judiciary; public health; research facilities; space facilities*) (Stangl et al. 2012).

From the *physical-technical sectors*, the *water sector* plays a key role. According to WHO (2014), "safe and readily available water is important for public health, whether it is used for drinking, domestic use, food production or recreational purposes". Kleiner (1999) suggests that "the average male should consume a minimum 2.9 litres per day and the average female 2.2 litres Approximately one-third of this fluid was considered likely to be derived from food" (Howard and Bartram 2003, p. 5). WHO (2014) summarizes that "a minimum of 15 litres per person per day should be provided as soon as possible after a disaster. During emergencies, people may use untreated water for laundry or bathing". In the S-

HELP flooding policy scenario, the flooding might reduce both the quality and quantity of potable water resulting in dehydration of humans and animals, damage to agriculture, impairment of hygienic measures, and occurrence of infectious diseases etc. (Stangl et al. 2012).

From the *socio-cultural sector*, the *public health sector* (e.g., hospitals, health care facilities, laboratories, drugs, search and rescue services, emergency services) is most important for the S-HELP project, especially for the biological-hazard policy scenario. Not only infected people are treated by the health sector but also a strong effort must be made to contain emerging epidemics by keeping in mind that the workforce for all critical infrastructures has to be provided and protected against infection.

2.3 Decision Making

Stakeholders (Sect. 2.3.1) perform *decision making* on certain *levels* (Sect. 2.3.2) applying *decision support systems* (Sect. 2.3.4). EM organizations use *command and control systems* (Sect. 2.3.3) to accomplish their command and control duties for disaster management. Generally, a *decision* "is a choice among options" (Power 2014). For interoperability in disaster management it is of highest importance to explore in detail the decision making for disaster management of EM *stakeholders*.

2.3.1 Stakeholders

Digital Humanitarian Network (2014) defines EM *stakeholders* to include "all those – from agencies to individuals – who have a direct or indirect interest in the humanitarian intervention, or who affect or are affected by the implementation and outcome of it. Within the context of the Quality Pro Forma, primary stakeholders refers to both beneficiaries and non-beneficiaries within the affected population."

Digital Humanitarian Network (2014) provides a comprehensive overview on different Humanitarian Stakeholders which is displayed in Fig. 12. We have included this categorization into our taxonomy. First, we displayed the main *stakeholder* groups (cf. Fig. 12): *donors, individuals, international organizations, media, military, non-governmental organizations, private sector*, and *public sector*. Next, we developed one sub-Figure for each of the eight EM *stakeholder* categories of the Humanitarian Decision Makers Taxonomy (Digital Humanitarian Network 2014). These eight sub-Figures are displayed in detail in the wiki. As the Digital Humanitarian Network (2014) does not provide any explanations for the stakeholders illustrated in at http://digitalhumanitarians.com/content/decision-makers-needs, we explain all main EM *stakeholders* listed for EM policy makers in the strategic disaster management wiki. In addition, this classification could also be improved. Please note that we are currently expanding this taxonomy for stakeholders to include all essential emergency responder categories for the S-HELP

Fig. 12 Stakeholders

project regarding the three emergency scenarios (flooding, chemical explosion, and biological hazard). This expanded classification will be included in the second release of the strategic disaster management wiki in Spring 2016 as discussed in the conclusion section.

2.3.2 Decision Making Levels

Decision Making can be performed on a *strategic*, *tactic*, and *operational level* (Mintzberg 1994) by EM *stakeholders* (cf. Sect. 2.3.1).

The *strategic level* "is concerned with the broader and long-term implications of the emergency and which establishes the policies and framework within which decisions at the tactical level are taken" (Irish National Steering Group 2006).

At the *tactical level* "the emergency is managed, including issues such as, allocation of resources, the procurement of additional resources, if required, and the planning and co-ordination of ongoing operations" (Irish National Steering Group 2006).

The *operational level* is "the level at which the management of hands-on work is undertaken at the incident site(s) or associated areas" (Irish National Steering Group 2006).

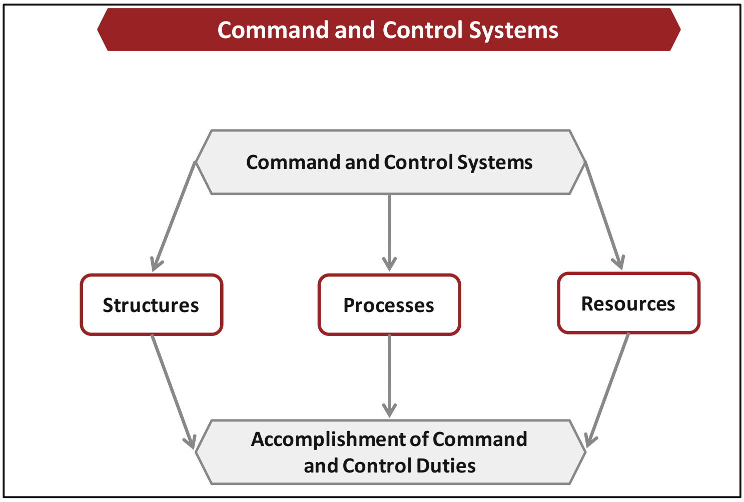


Fig. 13 Command and Control Systems

2.3.3 Command and Control Systems

Command and control systems (cf. Fig. 13) are highly essential for the decision making of EM *stakeholders* and are defined as "systems that support effective emergency management of all available assets in a preparation, incident response, continuity and/or recovery process" (International Organization for Standardization 2011).

For interoperability in disaster management, it is of highest importance to compare different command and control systems (structure, process, and resources) among selected countries. This feature of the wiki could be expanded in the future.

2.3.4 Decision Support Systems

In the wiki, general glossary terms/definitions of *decision support systems* are explained based on Power (2014) to support EM *stakeholders* to generally understand DSS. Power (2014) classifies *decision support systems* as follows: "A DSS is an interactive computer-based system or subsystem intended to help decision makers use communications technologies, data, documents, knowledge and/or models to identify and solve problems, complete decision process tasks, and make decisions. DSS is a general term for any computer application that enhances a person or group's ability to make decisions and also refers to an academic field of research that involves designing and studying DSS and their context of use."

In general, DSS are a class of computerized information system that support decision-making activities. The five more specific DSS types (cf. Power 2014) are included and explained in the wiki:

1. *Communications-driven DSS*
2. *Data-driven DSS*
3. *Document-driven DSS*
4. *Knowledge-driven DSS*
5. *Model-driven DSS*

2.4 Emergency Management Cycle

The *emergency management cycle* "includes sum total of all activities, programmes and measures which can be taken up before, during and after a disaster with the purpose to avoid a disaster, reduce its impact or recover from its losses" (Vasilescu et al. 2008, p. 46). The *emergency management* cycle consists of two phases, namely a *pre-disaster phase* and a *post-disaster phase* (cf. Alexander 2012), which are further divided into *mitigation* and *preparedness (pre-disaster) as well as response* and *recovery* (post-disaster).

Mitigation "involves reducing or eliminating the likelihood or the consequences of a hazard. It is used to treat the hazard and resulting incident in order to reduce its impact on society" (cf. Coppola 2011). Vaccination of individuals against diseases might be a *mitigation* measure in the biological-hazard policy scenario of S-HELP.

Preparedness can be explained as "the readiness of an organization and/or community to respond to an emergency/disaster/crisis in a coordinated, timely, effective, and efficient manner. It involves equipping responders, decision-makers, and the public with the tools and mechanisms necessary to increase their chance of survival and to minimize losses" (Coppola 2011). The DSS developed by S-HELP and the training of EM stakeholders might be such an example for preparedness.

Response covers the "sum of decisions and actions taken during and after the event of an emergency/disaster/crisis to reduce or eliminate the impact of the disaster in order to prevent further health suffering, financial loss, or a combination of both" (Coppola 2011). Isolation of infected individuals might be a *response* measure in the biological-hazard policy scenario of S-HELP.

Recovery "involves returning victim's lives back to the normal state they were before the disaster. This usually begins immediately after the incident but it can last for months or even years" (Coppola 2011). The reconstruction of critical infrastructure after a major disaster might be such an example (e.g., chemical explosion policy scenario).

2.5 Related Interventions and Resources

The related *interventions* that EM *stakeholders* select using their given *resources* (*staff* and *material*) are not yet incorporated in the first release of the wiki.

In the S-HELP project, we developed a skills taxonomy template to interlink emergency interventions/tasks and emergency responders/skills. Furthermore, we provided an overview which emergency interventions/tasks can be covered by EU Civil Protection Modules by incorporating availability, start of operation, self-sufficiency, and operation time (cf. University of Vienna, Austria, UNIVIE 2015a). Next, the resource taxonomy template contained the linkage of emergency interventions/tasks and emergency responders/skills to emergency equipment/materials needed (cf. University of Vienna, Austria, UNIVIE 2015b). The skills and resource taxonomy templates considered the complex and multi-disciplinary nature of health services in emergency preparedness, response, and recovery. They are included in the DSS of S-HELP. In addition, they might be also included in the strategic disaster management wiki in a future release.

3 Glossary Terms for the Strategic Context of Disaster Management[2]

In the following, we illustrate how EM *stakeholders* can use our wiki for looking up definitions (glossary terms). Our taxonomy currently consists of 29 figures and 806 glossary terms that further explain the entire structure of our taxonomy and the essential terminology used based on 136 references to the literature. A glossary term can be entered or selected from the list of 806 definitions at the left hand side of the wiki screen which is displayed in Fig. 14. For each glossary term, a title,

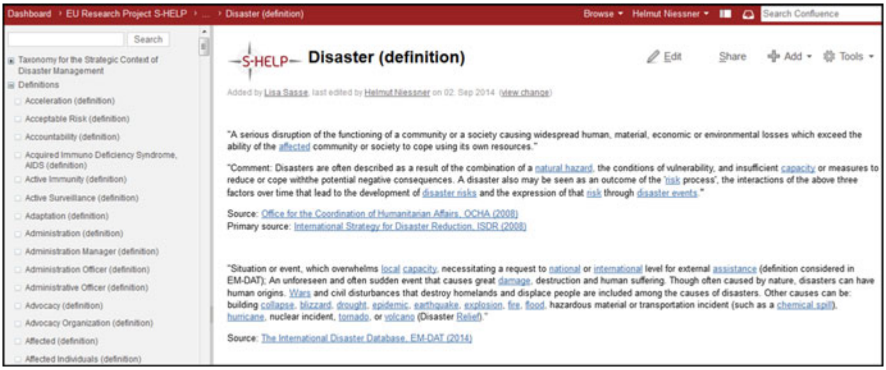


Fig. 14 Screenshot of the Wiki for Glossary Term "Disaster". *Source*: https://wiki.univie.ac.at/display/SHELP/

[2]This section is based on University of Vienna, Austria (UNIVIE): Rauner, M., Niessner, H., Sasse, L., Tomic, K. (2014) Securing Health Emergency Learning Planning, S-H.E.L.P., Collaborative Project FP7-SEC-2013-1, Project no. 607865, Deliverable No. 2.1, Glossary of terms and definitions & common grounds and standards for interoperability.

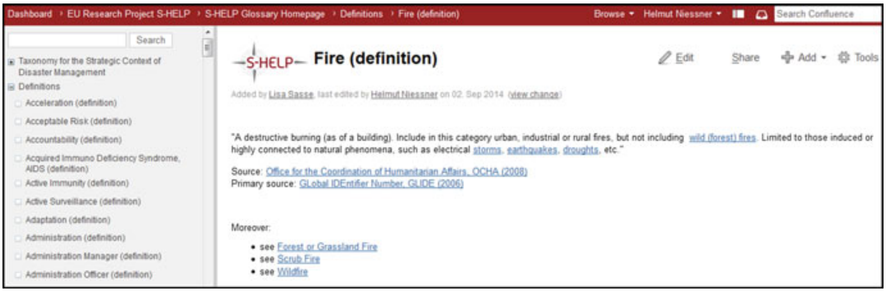


Fig. 15 Screenshot of the Wiki for Glossary Term "Fire". *Source*: https://wiki.univie.ac.at/display/SHELP/

Fig. 16 Screenshot of the Wiki for a Selected Literature Source. *Source*: https://wiki.univie.ac.at/display/SHELP/

description, and references to the literature is provided (i.e., source, primary source). In the description field, several terms are highlighted which are hyperlinks to other glossary terms in our wiki.

Figure 14 shows how the glossary term "Disaster" is explained in the wiki. We used standard definitions provided by international organizations/institutions such as the Office for the Coordination of Humanitarian Affairs (2008) and The International Disaster Database, EM-DAT (2014). If other glossary terms of our wiki are used for the explanation of a certain glossary term, links will be provided to these terms (e.g., collapse, fire, flood).

If an EM policy maker wants to get further information on certain glossary terms such as "fire" by clicking on the glossary term, the following explanation for the glossary term "fire" will appear (cf. Fig. 15).

Next, if a policy maker is interested in the source of the explanation for the glossary term "fire" (Office for the Coordination of Humanitarian Affairs 2008) they can click on the related source link, which opens the following wiki window (cf. Fig. 16).

For all sources, the full reference is given. For several sources, the entire source as a pdf-File is provided and the link to the related web-site is displayed.

4 Conclusion and Further Research

EM stakeholders are supported by a glossary/taxonomy of strategic management disaster terms and definitions & common grounds and standards for interoperability. For this reason, we developed a strategic disaster management wiki using the platform provided by University of Vienna, Austria: https://wiki.univie.ac.at/display/SHELP/

The main parts of this strategic disaster management wiki are incorporated into the DSS of S-HELP (http://www.fp7-shelp.eu/). The wiki contains 29 figures and 806 glossary terms for strategic disaster management limited to the following topics, which can be further expanded in the future:

- Disaster Definition (What can happen?)
- Decision Making
  - Stakeholders (Who makes the decisions?)
  - Decision Making Levels & Command and Control Structures (How are the decisions made?)
  - Decision Support Systems (How are decisions supported?)
- Emergency Management Cycle
- Emergency Management Environment

In the S-HELP project, we developed a skills taxonomy template to interlink emergency interventions/tasks and emergency responders/skills (University of Vienna, UNIVIE 2015a). Furthermore, we provided an overview which emergency interventions/tasks can be covered by EU Civil Protection Modules (European Commission, EC 2014) by incorporating availability, start of operation, self-sufficiency, and operation time. Next, the resource taxonomy template contained the linkage of emergency interventions/tasks and emergency responders/skills to the required emergency equipment/materials (University of Vienna, UNIVIE, 2015b). The skills and resource taxonomy templates take into account the complex and multi-disciplinary nature of health services in emergency preparedness, response, and recovery. The main parts are implemented in the DSS of S-HELP by University College of Cork which also might be included in the strategic disaster management wiki in the future.

Moreover, the content on command and control systems for disaster management might be expanded to help decision makers compare different approaches among countries (University of Vienna, UNIVIE 2015c). For example, the international EM guidelines (e.g., ISO 22320—International Organization for Standardization 2011; ISO 22300—International Organization for Standardization 2012) play an important role. In addition, further interoperability standards are investigated by the European Interoperability Framework for Pan-European eGovernment Services that are published by the Interoperable Delivery of European eGovernment Services to Public Administrations, Businesses and Citizens (2004). The current strategic disaster management wiki focus on organisational and semantic interoperability, while the technical interoperability (data-sets and data exchange) might be essential for implementing DSS based on standards such as the international EM guideline ISO 22351 (International Organization for Standardization (under development)).

For the second release of the strategic disaster management wiki for S-HELP in spring 2016, the team will include the following additional content (glossary terms, figures, and tables) based on essential parts of the skills taxonomy template (University of Vienna, UNIVIE 2015a), resources taxonomy template (University of Vienna, UNIVIE 2015b), and the interoperability model (University of Vienna, UNIVIE 2015c):(1) main emergency interventions, (2) main emergency resources, (3) main human emergency resources (emergency responders): (a) national emergency responders, (b) incident command-related emergency responders, and (c) international emergency responders, (4) main non-human emergency resources, and (5) EU Civil Protection Modules (including the general EU Emergency and Crisis Coordination Arrangements). This additional content may prove essential for policy makers to better plan skills and resources needed for emergencies under consideration of interoperability.

To conclude, this work contributes to both research and industry practitioners. By putting forth a strategic disaster management wiki that includes a taxonomy of terms and definitions, it addresses some of the issues identified in the literature that are critical in cross-border multi-agency response. As our susceptibility to hazards and related disaster scenarios increases due to urbanisation and increasing populations, it is more crucial than ever to address interoperability issues like standardisation in advance of emergency situations. By providing this work, S-HELP is helping to reduce some of the operational inefficiencies and delays associated with not having a shared vocabulary across agencies. In addition, the findings can be used to extend the current knowledge base on emergency management interoperability. Emergency responders have been provided with an extensive glossary of emergency management terms, which are being applied for use in the S-HELP DSS solution. This solution addresses the challenges of interoperability while providing a set of tools that can integrate with legacy systems and afford essential capabilities and functionalities in the management of cross-border emergency situations. It is envisaged that this strategic disaster management wiki will become a living repository of core EM terms. The use of wiki technology ensures that this resource can extended over time.

Acknowledgement This project has received funding from the European Union's Seventh Framework Programme for research, technological development and demonstration under grant agreement N607865 (http://www.fp7-shelp.eu/).

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Overview of Optimization Problems in Electric Car-Sharing System Design and Management

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Abstract Car-sharing systems are increasingly employing environmentally-friendly electric vehicles. The design and management of Ecar-sharing systems poses several additional challenges with respect to those based on traditional combustion vehicles, mainly related with the limited autonomy allowed by current battery technology. We review the main optimization problems arising in Ecar-sharing systems at strategic, tactical and operational levels, and discuss the existing approaches often developed for similar problems, for example in car-sharing systems with traditional vehicles. We also outline open problems and fruitful research directions.

1 Introduction

The purpose of this paper is to summarize the main contributions to the definition and solution of optimization problems arising in the design and management of car-sharing systems which use electric vehicles (EV).

Car-sharing is a general public mobility mode that is based on the shared use of vehicles by a set of users, who are generally subscribers of the service and pay flat and per-use fees. These systems were introduced around 1970–80 in some limited pilot implementations (see Shaheen et al. 1998), but have only recently seen

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a considerable development in urban areas. In huge cities congestion and parking costs make the ownership of private cars much less attractive for citizens who rely on public transportation for their regular commuting, and need cars only for special purposes. For a general overview of car-sharing systems we refer to Shaheen et al. (1998) and Millard-Ball et al. (2005), whereas a recent survey on optimization problems arising in such context is given by Jorge and Correia (2013). Finally, the important aspect of demand estimation for car-sharing systems is discussed in Stillwater et al. (2009) and Schmöller and Bogenberger (2014).

Car-sharing systems are increasingly employing environmentally friendly vehicles that may reduce the overall negative impact of the mobility on the environment and may have easier access to congested urban areas. For car-sharing systems the most commonly used environmentally friendly vehicles are indeed electric ones. In this paper, for short we indicate car-sharing systems employing electric vehicles as Ecar-sharing systems.

As described in Pelletier et al. (2014, 2016), several types of electric vehicles actually exist and their characteristics may heavily influence their use possibilities in general and in relation to shared transportation systems. In particular, we consider plug-in electric vehicles (PEVs) that may be charged by plugging-in them into the electric grid. In turn, these vehicles can be classified into plug-in battery electric vehicles (PBEVs), which use the power provided by the battery only, and plug-in hybrid electric vehicles (PHEVs) which also have an internal combustion engine. Both vehicle types are able to recover energy generated during travel (from braking and driving downhill) to recharge the battery. Whenever no specific distinction is required, we call all these vehicles electric vehicles.

For what concerns the organizational issues, an important distinction has to be made between *two-way* (or *roundtrip*) systems, in which the vehicle must be returned to the station where it has been picked up, and *one-way* systems in which vehicles may be also returned to a different station. The second model is clearly more flexible for the users but, as we will extensively discuss in the following, it requires a rebalancing of the vehicles at different stations during the service. We finally mention that recently some car-sharing systems in which vehicles are no longer based at specific stations were introduced. Such systems are generally called *free-floating* (see e.g., car2go and BMW DriveNow).

Designing and operating car-sharing systems which use electric vehicles poses additional technological and practical challenges with respect to the systems employing traditional combustion vehicles. For example, the relatively limited autonomy of currently available electric cars requires recharging the vehicles during the day, which has to be performed at specific charging stations. In addition, due to the high costs involved, not too many charging stations have been built, and charging times can be quite long unless expensive fast-charging stations are present. Finally, the electricity consumption is considerably affected by the driving and environmental conditions (e.g., the speed profile or the outside temperature) that need to be accurately modeled to better estimate the actual charge status of the vehicles during the day.

In the following sections we examine the main problems that are relevant for the optimal design and management of electric car-sharing systems. We note that the existing literature on Ecar-sharing is very limited. Therefore, on the one side we highlight the optimization problems that arise in this context. On the other side, we examine the relevant literature on related problems, such as works focusing on electric vehicles (privately owned, taxis, etc.) or on car-sharing systems with conventional vehicles. For each such problem we both describe the characteristics that have been faced so far in the literature and discuss the components of real-world systems that have not been examined so far, so as to provide interesting and practically motivated research directions.

More precisely, we organized the exposition into two separate sections. The first part (Sect. 2) is devoted to strategic and tactical problems, which are appropriate in the design of the systems. Within such category falls mainly the problem of locating the charging stations for the electric vehicles and for privately owned cars (Sect. 2.1). Section 2.2 discusses the tactical problem of defining the allocation strategies for the assignment of vehicles to the stations.

In the second part (Sect. 3) we present operational problems that arise in the short-term management of Ecar-sharing systems. Section 3.1 introduces the relocation of vehicles between the available stations, which is required to balance the supply and demand patterns. Section 3.2 examines the possibilities offered by battery-swap technologies and Sect. 3.3 considers the computation of shortest paths specifically designed to incorporate the main characteristics of electric vehicles. Section 3.4 deals with the definition of multi-stop travels for electric vehicles that typically occur in freight distribution. Finally, Sect. 4 draws some conclusions.

2 Strategic and Tactical Problems

As their name suggests, the problems of this class deal with making good high-level, big-picture decisions. These determine the overall structure of the underlying car-sharing system and can therefore have a great impact on how well the system performs. Decisions made at this level are usually long-term, i.e., once they are made, they cannot easily be reversed. As they often imply high cost, they also have a significant impact on the car-sharing operator. Thus, a high solution quality is of great importance for these problems. Combined with the fact that strategic decisions need not be made very frequently, this suggests the use of exact or combined methods for solving them.

Although some pilot systems are already in use, not much scientific literature dedicated to the study of the design and operational challenges of Ecar-sharing systems (from a general perspective) exists. Notably, Barth and Todd (1999) were among the first to consider the use of electric cars in the context of car-sharing systems. Based on a case study from a resort in Southern California, they concluded that (already) 3–6 vehicles are sufficient per 100 trips of each day to satisfy customer waiting times, but approximately 18–24 vehicles would be necessary to also minimize the necessary number of relocations. Besides the number of vehicles per trip, they conclude that the relocation algorithm and the used charging scheme are the main factors for successfully using such a system. Note that particular characteristics of the considered use case include the fact that trips are shorter than 5 miles on average, thus, the charging state of cars never drops below approximately 70 %.

Considering a real-world use case from Genoa, Cepolina and Farina (2012) are concerned with the design of a flexible, multi-station Ecar-sharing system for pedestrian areas. Their aim is to optimize the size and distribution of the fleet among a set of stations at the beginning of operation, so that the sum of total transportation and waiting costs is minimized. Particular characteristics of the system include the possibility for instant access, open ended reservation and one-way trips. A simulated annealing approach that uses a microscopic simulation of user behavior and waiting times is developed, in which a subset of users is assumed to be flexible in the sense that they have an associated set of acceptable stations. Recharging is not explicitly treated but simply assumed to occur in idle times and no explicit relocation actions are considered (i.e., relocation by users). The authors analyze the cost changes with respect to the total number of vehicles and, as in Barth and Todd (1999), the influence of the vehicle-to-trip ratio on the total average waiting time.

Other pilot implementations are that of the Kyoto public car system project described in Kitamura (2002), and the system with different types of electric vehicles discussed in Luè et al. (2012).

Strategic decisions arising in Ecar-sharing systems mainly involve planning locations and sizes (i.e., numbers of charging slots) of charging stations throughout the operational region. The operator's main goal is to minimize their cost arising from building the stations while at the same time ensuring that the profit obtained from satisfied user requests during operation is maximized. Since users will only consider using a car-sharing system if their requests are accepted with a relatively high probability, an operator is facing a difficult trade-off between the initial costs to set up the car-sharing system (long term investment) and the profits obtained later on (operational phase), especially since the latter are highly uncertain.

Tactical decisions are instead related to mid-term planning horizons. Within this time horizon the main optimization problem that is relevant in Ecar-sharing systems is that of allocating the vehicles to the charging stations. Such a problem is mainly relevant for two-way models in which the initial position of the vehicles is critical and may need to be adjusted whenever substantial changes in the demand distribution patterns occur.

2.1 Location of Stations

As mentioned above, a key factor determining the performance of a car-sharing system is the location of each currently unused car within the system, as it determines which customers can actually use it. Since many car-sharing systems are station-based (i.e., cars are always picked up from and returned to a fixed set of parking spots owned by the car-sharing company), the location of these stations

Table 1 Classification of the literature related with location of charging stations

Category		Methodology	
Type	Vehicle type	Exact	Heuristic/simulation
Car-sharing	Electric	Boyacı et al. (2015)	
Private fleet	Electric	Baouche et al. (2014), Cavadas et al. (2015), Chen et al. (2013), Frade et al. (2011), González et al. (2014), Wang and Lin (2013), Worley et al. (2012), and Xu et al. (2013)	Chen et al. (2013), Ge et al. (2011), Hess et al. (2012), and Wang et al. (2010)
Taxi cabs	Electric	Asamer et al. (2016)	Sellmair and Hamacher (2014)
Car-sharing	Traditional	Correia and Antunes (2012) and Correia et al. (2014)	Fassi et al. (2012)

becomes equally important. This is especially true for those systems which use electric cars, since they must usually be recharged at the aforementioned stations during the day in addition to (fully) recharging them overnight.

In the following, existing studies on strategic decisions are classified into four categories: (1) location of charging stations in Ecar-sharing systems; (2) location of charging stations to serve privately owned cars; (3) location of charging stations for electric taxi cabs; and (4) location of stations for car-sharing systems with non-electric cars. Note that we include literature related to the latter three categories, as the literature on Ecar-sharing systems is still sparse and as the arising optimization problems share many characteristics. A first brief overview which acts as a guideline to this section's content is given in Table 1.

2.1.1 Location of Charging Stations for Ecar-Sharing Systems

Boyacı et al. (2015) describe a bi-objective mixed integer programming (MIP) model for a station-based one-way system. Potential sites for the charging stations are first found by solving a set covering problem. The authors then seek to optimize the location and size of the stations, together with the number of vehicles, their initial allocation and relocation during the system's operation with respect to both the operator's revenue and the users' benefit. To reduce the size of their model, they use an aggregated model where all relocations happen from or to imaginary hubs, each representing a set of stations, instead of between individual stations. The charge state of each vehicle's battery is not explicitly considered in the model—instead, the necessary pauses for recharging must be provided as an input. The authors evaluate their model for the Nice region by using data from an existing two-way car-sharing system and analyze the effects of various parameters like increased demand on the optimal solution. A preliminary study on the design of a comprehensive vehicle-sharing system involving various types of electric vehicles and different types of ownership is described in Luè et al. (2012).

2.1.2 Location of Charging Stations for Privately Owned Cars

The most studied case is that of the location of charging stations for privately owned cars. Frade et al. (2011) provide an MIP formulation to decide on the location and capacity of electric vehicle charging stations with the objective of maximizing the demand covered under a certain service level and budget constraints. They conduct a case study based on real-world data from Lisbon (Portugal). A similar model is later developed by Cavadas et al. (2015) and improved in order to provide a better coverage when some portion of the demand can be transferred between the successive stops of a trip. In addition to transfer of demand, the model is further adapted to a more realistic case where the variation of demand during the day is modeled by splitting the day into time intervals. The comparison of the models using data from Coimbra (Portugal) under different parameter settings reveals two important findings: (1) if there is a possibility of transferring demand, its inclusion in the model might provide significant improvements of the solution; and (2) independently from its transferability, the consideration of the demand based on different time intervals prevents solutions with overcapacity, which might be the case if demand is aggregated.

Wang and Lin (2013) consider a similar objective under budget constraints to decide on the location of multiple types of charging stations which differ in charging speed, and provide a MIP formulation for this problem. They also consider a variant in which the total cost to satisfy all demands is minimized. Both formulations are tested on a network from Penghu Island (Taiwan) and the test results show that the consideration of mixed stations yields benefits in terms of objective values compared to using a single station type only.

Minimization of the total cost is adopted also by Baouche et al. (2014) when deciding on the optimal locations of the charging stations. Based on a survey on the metropolitan area of Lyon (France), they split the surveyed region into several demand clusters and calculate the energy demand at each of them. The MIP formulation they propose then finds the minimum-cost set of potential charging stations that covers all energy demands. The cost takes into account both the construction of the stations and the energy demand for traveling to them. In addition, each station has a fixed type that determines how much charging they can provide. The individual state of vehicles, namely their location or charge state, and the temporal component of demand is only considered in an aggregated way.

A similar approach is used by Chen et al. (2013) for the Seattle (WA, USA) area. Their MIP model determines which charging stations should be opened to minimize the total walking distance required for satisfying all demand. The authors note that a simple greedy heuristic finds solutions of similar quality, but with a significantly higher maximum walking distance.

González et al. (2014) seek to find an optimal charging schedule for private electric vehicles in the Flanders region of Belgium with respect to the cost of electricity used. To estimate the recharging demand, traffic data for conventional vehicles is used. While the locations of charging stations that are opened are not considered in their problem variant (they assume that charging can happen at any time and place), the authors note that in their optimal solution, some zones show a charging demand significantly above the average, which suggests that they are prime candidates for the construction of public charging infrastructure. They also show that over 80 % of all current trips could be performed with electric vehicles without requiring any charging outside of the owner's home and note that much of the charging required for the remaining vehicles could be done while the owners are at their workplace.

In contrast to the exact methods used above, Ge et al. (2011) employ a genetic algorithm to partition a planning area into zones and assign each of them a charging station of appropriate size, using the required energy expenditure as a quality criterion. Their algorithm is then evaluated on a test instance. Similarly, Hess et al. (2012) describe a genetic algorithm for placing charging stations to minimize the total trip distances. They use a traffic simulator, modified to account for electric vehicles, to generate data for the inner city of Vienna, on which they evaluate their algorithm.

Wang et al. (2010) describe a heuristic algorithm for finding good locations for charging stations serving private electric vehicles, considering both existing gas stations and entirely new spots as potential sites. Their approach considers a number of objectives including demand coverage, factors relating to the power grid and municipal planning factors (which seek to keep the stations away from places where they might impact other traffic). The algorithm is evaluated on data gathered from the city of Chengdu.

An integrated MIP model that optimizes both the location of charging stations and the routing of electric vehicles is given by Worley et al. (2012), with the objective being the minimization of the total cost, which consists of the costs for building stations, charging vehicles and driving. Another MIP-based algorithm for finding the optimal charging station locations is presented by Xu et al. (2013), who consider customer accessibility (both spatial and temporal), number of charging slots and crime safety as relevant factors.

2.1.3 Location of Stations for Electric Taxi Cabs

Electric taxi cab stations represent a good combination of the two previous categories. Sellmair and Hamacher (2014) consider the problem of selecting existing taxi stands as possible locations for charging stations and determining the number of charging points per station. By using simulation techniques, customer trips between taxis stands are generated. The simulation is based on the GPS data collected from five conventional taxis in the city of Munich in Germany. The simulation takes the state of charge into account for deciding whether trips can be accepted or not. An iterative heuristic approach is used to determine the number and location of the charging stations.

Asamer et al. (2016) present a study based on operational data of a radio taxi provider in the city of Vienna in Austria. Positioning data of approximately 800 taxis over 12 weeks, one for each calendar month, is used. The authors aim to find locations for a limited number of charging stations dedicated to taxis. Instead of assuming taxi stands as possible locations, regions are considered and the exact locations within the selected areas are identified in a post-optimization phase, where various soft constraints need to be considered. The spatially-distributed charging demand is aggregated, meaning that start and end locations of taxi trips within each region are summed up. Based on this data, a set-covering approach is used to model the location problem with the goal of maximizing the coverage of the aggregated demands. The problem is modeled as a MIP and solved using the IBM CPLEX solver.

2.1.4 Location of Stations for Non-Electric Car-Sharing Systems

As noted in this section introduction, the problem of finding the optimal locations of vehicle depots in conventional (i.e., non-electric) car-sharing systems is closely related to that of finding the locations of charging stations for electric vehicles, since the factors determining a station's quality are similar (e.g., proximity to areas of high demand). One key difference between these two problems is that models for conventional car-sharing usually do not consider the vehicles' fuel state, since gasoline-powered vehicles can be refilled comparatively quickly.

Correia and Antunes (2012) describe MIP formulations that optimize the operator's profit by finding the optimal set of vehicle depots that should be opened, as well as their size and the allocation of vehicles among them. Three different models that maximize the operators' profit are studied, in which (1) the operator has full freedom to decide whether or not to accept a potential trip; (2) all trips need to be accepted; or (3) trips may only be rejected by the operator if no vehicle is available at the pick-up station. The authors evaluate their model on input data for the Lisbon area in Portugal, and conclude that the operator's profits decrease significantly when all trip requests must be fulfilled. In another publication, Correia et al. (2014) analyze the effects of increased user flexibility on the operator's profit. They develop an MIP formulation that allows users to select one of several potential starting and ending vehicle depots for each trip, with the additional option of providing them with information about the availability of cars or parking spaces at the relevant depots. By applying the model to the Lisbon data set from their previous paper, the authors find that the flexible models improve vehicle usage, but increase walking and total travel times.

In contrast to the aforementioned publications, which deal with finding an optimal solution with respect to some measures of quality, others deal exclusively with the simulation and evaluation of solutions. Fassi et al. (2012) evaluate the effects of several growth strategies (like increasing the size of stations and opening new ones) on the activity of stations and members, as well as the members' satisfaction with the service.

2.1.5 Summary, Open Problems and Possible Research Directions

The main objectives in station location problems for (electric and non-electric) car-sharing systems are to minimize the total cost or maximize the total profit of the car-sharing companies. The characteristics of the location of charging stations for privately owned electric cars can be mainly considered in two categories: problems that aim to minimize total cost while satisfying all demand, and problems that aim to maximize demand coverage under budget constraints. Additionally, objectives pertaining to user satisfaction are sometimes considered. This includes, in addition to the aforementioned demand coverage, objectives like minimizing the walking distance of customers.

The objective of maximizing demand coverage in Ecar-sharing systems seems to be an open problem in the literature and has yet only been addressed in the context of electric taxi cabs (Asamer et al. 2016). As suggested by Wang and Lin (2013), multiple types of charging stations can be included in location decisions. Such models could also be extended to consider certain characteristics of the electric grid, like varying charging capacity throughout the day. Improved solutions are obtained when possible transfer of charging demand is considered by Cavadas et al. (2015) for the stations dedicated to privately owned electric cars. Adaptation of this idea to the Ecar-sharing systems might be worthwhile to investigate. To better capture aspects related to the particular characteristics of electric cars (i.e., very limited range, long recharging times) integrated models combining strategic and operational aspects seem worth investigating. In that respect, we particularly refer to variants that include detailed tracking of battery-state and recharging times. The high degree of uncertainty in terms of energy usage for individual trips also suggests further investigations of robust or stochastic problem variants. Furthermore, explicitly capturing the trade-off between naturally arising conflicting objectives (such as long term investment costs, short term profits, relative number of accepted user requests) in terms of bi- or multi-objective problem variants seem worth further studies.

More generally, an aspect that is worth investigating is the study of inter-modal people transportation problems that include (electric) car-sharing systems, i.e., to study the integration of (electric) car-sharing with public transportation and other means of transportation. Besides, considering the likely relatively short distances of many car-sharing trips within cities, a study of the trade-off between vehicle cost and vehicle range seems relevant for the case of electric cars.

Another possible avenue of research would be the development of a flexible pricing scheme that considers the variation of demand throughout the network at different times. This might eventually lead to a system where relocation of vehicles is mostly user-based. It is, however, unclear whether such a system would find acceptance among its potential users.

2.2 Allocation of Vehicles to Existing Stations

Besides relocating vehicles between stations (as described in the next sections), most papers do not seem to explicitly optimize the assignment of vehicles to stations. On the contrary, it is typical that vehicles are considered as origin of a given demand and stations are built and dimensioned to satisfy that demand, see, e.g., Chen et al. (2013), Ge et al. (2011), and González et al. (2014). Whenever the actual positions of vehicles throughout a certain planning period (typically a day) are considered in an approach (that, e.g., considers a location-routing problem combining the planning of stations or relocations), an (initial) allocation of vehicles is implicitly optimized by not fixing the (initial) status, see, e.g., aforementioned articles by Correia et al. (2014) and Boyacı et al. (2015). On the contrary, other articles (such as Baouche et al. (2014)) do not consider these temporal components, but simply design a set of stations (with their capacity) in order to be able to fulfill the demand corresponding to the set of vehicles. Clearly, the latter, which in turn is not so different from other classical assignment problems (p-center, set-covering), is more appropriate for car-sharing systems in which only round trips are allowed and issues such as relocation are not important.

One example of a model that considers the initial allocation of vehicles as a decision variable to be optimized is given by Nakayama et al. (2002). The authors describe a genetic algorithm to optimize, among other factors, the number of vehicles within the car-sharing system and their location at the beginning of each day, given a fixed set of charging stations with a similarly fixed number of parking spots. The algorithm is then evaluated on data from an electric car-sharing operator from Kyoto.

2.2.1 Summary, Open Problems and Possible Research Directions

Since the initial placement and allocation of vehicles to existing stations is rarely considered as an explicit optimization problem but rather assumed to be given, no particular objectives and general constraints have been identified.

An interesting aspect that needs further investigation concerns the integration of vehicle allocation with general location and relocation aspects.

3 Operational Problems

We consider here the optimization problem arising in the operational management of Ecar-sharing systems. Such problems may be grouped into two main classes. The first one is related to the within-day optimal relocation of vehicles while the second considers the possibility of exchanging the battery at charging stations so as to restore vehicle autonomy. We also consider some relevant operational problems that have potential connections with the management of Ecar-sharing systems, namely, the electric vehicle shortest path and vehicle routing problems.

3.1 Relocation of Vehicles for Multiple-Stations Car-Sharing

During the last years, the offer of one-way trip mode has experienced an increased popularity in car-sharing services with fleets of conventional or electric vehicles. One-way car-sharing systems can be free-floating, in the absence of fixed parking spots, or *station-based*: in the latter case, reservations may be asked from the users. Since literature on free-floating services is very scarce, this section is focused on station-based systems. However, many issues described in this section apply to the free-floating case as well. The one-way option allows for a considerable increase in the number of potential customers interested in shared-use cars. This enhanced flexibility has a strong impact on the vehicle distribution in the service-provider network. Without the imposition of round-trips, an imbalance situation can occur and make the problem of ensuring vehicle availability in under-supplied stations a key issue for the system provider. In order to limit the unserviced trips and restrict economic losses of the car-sharing company, two types of relocation strategies may be implemented. In the first one, called *user-based* (UB) strategy, the relocation is decided by the customer itself, whereas in the second one, called *operator-based* (OB) strategy, relocation decisions are made by staff operators at a centralized or distributed level. The main characteristics of the papers examined here are presented in Table 2.

3.1.1 User-Based Strategies

From the system provider point of view, the organization of staff-relocation operations can carry an important economic load and cause operational difficulties. In order to alleviate such burden, Barth et al. (2004) introduce two user-based relocation mechanisms called trip joining (or ride-sharing) and trip splitting. Reduced prices are offered to customers willing to accept these modifications of their trip mode. The trip demand data they consider is generated from the University of California-Riverside Campus fleet (UCR IntelliShare) historical database. The system offers trip joining when multiple users want to travel from one low-vehicle-quantity station to a high-vehicle-quantity station, and trip splitting in the opposite situation. Given the demand, a discrete-event time-step simulation model is presented. The simulation allows to calculate the reduction in operator-based relocations thanks to trip joining, trip splitting and the two techniques concurrently. Simulation results show that, in most cases, trip splitting proved to be more effective than trip joining in reducing the staff operators workload. Using these user-based techniques, a 42 % reduction in the number of relocations is reported.

Table 2 Classification of the literature related with vehicles relocation (UB: user-based relocation strategy, OB: operator-based relocation strategy)

Reference	Strategy	Objective	Methodology
Barth et al. (2004)	UB	min. relocation costs	Simulation
Clemente et al. (2013)	UB	max. revenue and max. user's benefit	Simulation
Kek et al. (2006) and Kek et al. (2009)	OB	min. relocation cost and rejected demand	Exact/Heuristic/Simulation
Nair and Miller-Hooks (2011)	OB	min. relocation costs	Exact
Lee and Park (2013)	OB	min. relocation distance	Simulation
Jorge et al. (2014)	OB	max. profit	Exact/Simulation
Bruglieri et al. (2014)	OB	max. number of relocations served	Exact
Boyacı et al. (2015)	OB	max. revenue and max. user's benefit	Exact

Clemente et al. (2013) apply information and communication technology to the management of a one-way Ecar-sharing system. Real-time monitoring tools are used in order to propose economic incentives to the users, and help the rebalancing of vehicles in the network stations throughout the day. The authors used a timed Petri Net Framework to model the Ecar-sharing system. The customers response to the proposed trip alternatives modifies the random switches in the Petri Net. The proposed simulation model compares the "as-is" situation (no incentives), with two potential "to-be" strategies. In the "to-be" scenarios, users are encouraged to return cars as soon as possible (offline scenario) or to head to empty stations (online scenario); the latter situation requires the online monitoring of the system. Results on the Ecar-sharing system of Pordenone (Italy) are presented where the online scenario proves to be more profitable for the service provider. The authors conclude that relocation decisions rely on appropriate high-level strategic decisions; when such decisions are not accurately taken (e.g., the station fleet size), the relocation policy is not likely to be effective in solving the congestion problems.

To the best of our knowledge, user-based relocation strategies are not currently implemented by car sharing providers. Although the aforementioned papers simulate the impact of such strategies on profit, their actual potential is yet to be evaluated. However, nowadays some incentives to users are proposed in order to reduce the workload of providers (e.g. car2go gives free riding time if the users re-fuels the car).

3.1.2 Operator-Based Strategies

Existing car sharing providers usually perform overnight relocation. In the literature different practical relocation methods are described. Examples of such techniques can be found in Barth et al. (2004):

- Moving EVs with a truck (troublesome in cities)
- Towing a single EV to a "service" car
- Transporting operators to relocation positions by using a "service" car

Notice that, unless otherwise stated, the following papers evaluate the benefits of introducing relocation during the daily service, regardless of the specific technique that will be implemented.

Contributions by Kek et al. (2006) and Kek et al. (2009) are motivated by the development of four shared-use vehicle companies in Singapore. The focus is on a multiple-station company that allows one-way trips; the customer also has the flexibility to modify the previously specified return station en-route. In the first paper, a relocation time-stepping simulation model is proposed and applied on a real set of shared-use vehicle data from commercial operations. Two operator-based relocation techniques are proposed. When service level is the main concern, the vehicle relocation from a neighboring station to an under-supplied station should be performed in shortest time (i.e., travel time to the over-supplied station and relocation duration). The inventory balancing strategy aims instead to relocate vehicles in order to gain an equilibrium in the vehicle distribution in the stations. Cost efficiency is the objective of such technique. The simulation model is validated with real commercial data trips over a typical one-month period. The performance is measured in terms of number of relocations; besides, Kek et al. (2006) measures time in which parking slots in a station are either full (full port time) or empty (zero vehicle time). The simulated indicators show fidelity in replicating the trends occurring in the real situation; besides, they provide information on the potential cost savings which could be achieved without impacting the level of service. The authors observe that the individual change of the car-sharing systems parameters has no significant performance impact: this is due to the strong interrelation of operating parameter in such systems.

In Kek et al. (2009), the authors present a three-phase optimization-trend-simulation (OTS) decision support system for car-sharing operators to determine a set of near-optimal manpower and operating parameters. A MIP in a time-space network determines the lowest-cost resource allocation and vehicle scheduling, given inputs on station characteristics, vehicle relocation costs and historical customer usage patterns. In the second phase of Trend Filtering, the suggested staff and vehicle activities output from phase one are filtered through several heuristics in order to produce a recommended set of operating parameters. Such output parameters are finally used in the relocation simulator previously described in Kek et al. (2006). The solution approach has been tested on real operational data from Singapore. Results show remarkable improvements in the system performance according to the proposed measure of effectiveness.

Considering the same case study of Kek et al. (2006) and Kek et al. (2009) in Singapore, in Nair and Miller-Hooks (2011) the aim is finding a least-cost fleet redistribution plan such that most demand scenarios are satisfied. The probability distribution of users demand is defined by data collected with an Intelligent Transportation System infrastructure which enables monitoring of the trips. A stochastic MIP with joint chance constraints is formulated. The feasible region of the problem is nonconvex. Two solution methods are presented: when demand at stations is correlated, an enumeration procedure based on the concept of p-efficient points is applicable; when the demand at each station is assumed to be independent, a cone-generation solution method is used. Solutions of the proposed case study proved to be robust in simulation studies.

Jorge et al. (2014) present two methods for implementing operator-based relocation strategies. The strategic decision of location of stations is taken by adapting the model proposed in Correia and Antunes (2012) to the case in which the demand between existing stations is not always satisfied. The first relocation method is based on a novel MIP formulation in a time-space network which aims to maximize the daily profit of the car-sharing system. The second method is a discrete event time-driven simulation for testing two real-time relocation policies. Such strategies consider different frequencies for checking whether a station is a supplier (vehicles in excess) or a demander (vehicles shortage). The two solution approaches were applied, both independently and in a combined way, to several realistic scenarios in a case study in Lisbon. The optimized relocation decisions for these networks indicated significant potential profit gain with respect to the case of no relocation actions. The optimal solutions of the mathematical model provide upper bounds on the economic gains that are achievable with relocations since its input data are based on full knowledge of future daily trip demands. Even though trip reservation is necessary in the considered system, the simulation results based on real-time policies are remarkable.

Lee and Park (2013) propose an operation planner for relocation staff operations in Ecar-sharing systems. The relocation scheme consists of three steps covering the relocation strategy, the action planning and the staff operation planning, respectively. The demand is estimated by using the extensive Jeju City dataset on actual trips consisting of pick-up and drop-off points collected from a taxi telematics system. Relocation is assumed to be carried out during non-operation hours. The third phase is the main focus of the paper. It implements the relocation staff operations (i.e., moving from an initial to a final station). Single relocation team scheduling is considered for simplicity. The scheduling phase is tackled by using a genetic algorithm in which the relocation distance is the main performance metric considered.

In Bruglieri et al. (2014), the authors claim that relocation activities which rely on a truck for auto transport may not be practically implementable in urban environment, since stations may be hardly reachable by the trucks. To overcome this problem, they propose the use of folding bicycles for staff operators relocation movements from an under-supplied station (drop-off) to an over-supplied station (pick-up). Such relocation approach generates a specific pick-up and delivery problem called the Electric Vehicle Relocation Problem (EVRP). Given a set of pick-up and drop-off requests defining the network graph, the relocation is formulated as a Vehicle Routing Problem aiming to maximize the total number of requests served. Their MIP model explicitly considers the battery degradation profile using linear assumption. The estimation of the demand has been performed by studying historical data on private car movements in the city of Milan, and restricting these data to the estimated percentage of users interested in using the car-sharing service. A car-sharing simulator has estimated the unbalances due to the projected travel demand. Computational results on realistic instances show that using two workers with a duty time of 5 h is sufficient to satisfy a high percentage (about 86 %) of the relocation requests.

Boyacı et al. (2015) present an integrated (strategic, tactical and operational) framework to decide on the location of stations (see Sect. 2.1.1), on the number of parking slots to satisfy the uncertain user demand, on the assignment of users to slots and on the operator-based relocation actions. The considered Ecar-sharing system is one-way, non-free-floating and reservation-based: both the beginning and the ending station of the trip have to be specified. Demand centers represent sites that can be served by the same set of candidate stations; demands are obtained by an aggregation of orders of rentals, sharing the same set of origin and destination points and common departure and arrival time intervals. The considered graph is a time-space network. A set of scenarios is considered for coping with the stochasticity and seasonality of the demand. The authors develop a bi-objective MIP model.

An aggregated model which uses the concept of virtual hubs is presented for the practical solution of instances based on the large-scale car-sharing system in Nice. Extensive sensitivity analysis for relevant parameters is performed. The model evaluates the trade-off between operator benefit and users' level of service, showing that the investment in relocation personnel is worthy both from the company and customers point of view.

3.1.3 Summary, Open Problems and Possible Research Directions

We now summarize the main constraints and optimization objectives considered in the literature for relocation in Ecar-sharing systems.

At each network node, each activity is restricted to begin after the previous one is completed (see Kek et al. 2009). Taking into account relocation action and maintenance activities, the number of available vehicles is updated during the operating day. A limit on the number of rejected demands and vehicle returns is imposed.

There are a number of capacity constraints present in these models. In Kek et al. (2009) and Boyacı et al. (2015), station capacity constraints are imposed: in each time discretization step, the sum of available and unavailable vehicles in a station can not exceed the station capacity.

In Kek et al. (2009), Nair and Miller-Hooks (2011), and Boyacı et al. (2015), the authors limit the number of vehicles relocated out of a station with the number of vehicles available at the start of the planning period; also, the number of vehicles relocated to a station cannot exceed the number of available slots. These conditions are called capacity constraints.

When time-space network representation is used (see Jorge et al. 2014), the vehicle flow at each node in the time-space network must be preserved. The stations must have enough parking spaces for vehicles present at each minute. Flow conservation constraints are also considered in Bruglieri et al. (2014) and Boyacı et al. (2015). In Boyacı et al. (2015), atom-coverage constraints are introduced. An atom is a small geographic area that is eligible to receive the car-sharing service. The number of operating parking spaces in all open stations constitutes an upper bound to the number of relocation actions.

In Nair and Miller-Hooks (2011), the probabilistic level-of-service constraints state that the redistribution plan must result in inventories that satisfy p-proportion of all demand scenarios in the planning horizon. The resulting system is called a p-reliable system.

In some cases (see Bruglieri et al. 2014) time windows for customers requests are present. Therefore, specific service limitations, such as imposing precedence constraints in the visit time of nodes and bounding the duration of a route are considered.

Finally, specific restrictions characterizing Ecar-sharing systems are imposed in Bruglieri et al. (2014) and Boyacı et al. (2015). In the first paper, the distance traveled by an electric vehicle is assumed to be linearly proportional to the residual charge: it is imposed that an electric vehicle needs to have minimum residual charge (level) in order to perform a trip. In the second paper, the electric vehicles are required to be recharged in the arriving station after each rental operation. In addition, the number of vehicles in the station should be greater than or equal to the number of vehicles requiring charging.

In this specific area there are several open research directions. Regarding the simulation approaches for the impact of user-based relocation strategies, Barth et al. (2004) and Clemente et al. (2013) underline the interest of estimating user participation rate in the proposed relocation activities. The first paper suggests to collect extensive statistical data for making this forecast. The second one proposes a detailed behavioral analysis of the users willingness to accept real time trip suggestions which would permit a more precise trip pricing policy.

Other research directions are represented by integrating the relocation action in the strategic planning phase of car-sharing management and to investigate the adoption of real-time relocation policies. In addition, using multiple relocation teams and combining operator-based relocation approach with pricing policies on the parking stations offered to the users, all seem promising options.

Several papers have underlined the strong interrelation between the different levels of decision-making in car-sharing systems problems. As already mentioned, the strategic decision of the location of stations has a huge impact on the tactical and operational issues, such as the routing of the shared-use vehicle fleet, in order to satisfy users requests. An integrated modeling approach seems a promising line of future research.

Car-sharing problems might be considered as real-world applications in which a location-routing scheme is directly present or at least identifiable. The location-routing problem is a research category which considers the integrated solution approaches for tackling location problems in which the tour planning aspects are strongly interrelated with the strategic decisions. To the best of our knowledge, in literature, car-sharing problems have not been explicitly stated in location-routing framework yet and we refer the reader to the survey by Nagy and Salhi (2007), which provides a good introduction to the problem. More recently, Prodhon and Prins (2014) update the first survey presenting the multi-echelon problems and several other variants. Finally, the survey by Drexl and Schneider (2015) proposes future research directions from the methodological and modeling point of view, such as the integration of revenue management in location-routing formulations.

3.2 Battery Swap

One main challenge for the large-scale spreading of battery-electric vehicles is their limited range and the fact that in contrast to traditional vehicles, re-charging operations take a significant amount of time (with the exception of expensive and not yet very widespread fast-charging stations such as Tesla Superchargers and CHAdeMO). Especially for long distance travel, overnight recharging is not sufficient. Thus, battery swapping (rather then recharging) has been considered as a viable alternative, in which the batteries are owned by a company and users simply exchange their currently used (nearly empty) battery with a fully charged one at predefined battery swapping stations (BSSs). A main advantage from a users perspective is that this process can be done in a few minutes (i.e., approximately in the same time frame needed for refueling a traditional car). Even if such technological approach is made difficult by the lack of standardization on batteries and by the huge investments required to set up the system, some interesting studies were presented in the literature.

Yang and Sun (2014) study a location-routing problem arising in the delivery of goods to customers using a fleet of electric vehicles (EVs). Given a set of customer demands and of potential BSSs, the goal is to simultaneously determine the location of the battery swapping stations, the allocation of customers to EVs as well as that of EVs to BSSs. In addition, tours from the single depot to serve all customers are designed that consider the selected BSSs and the driving range of the vehicles. The objective is to minimize the total costs arising from the construction of BSSs and the service of the demands with the EVs. Energy consumption and maximum vehicle range are considered to be proportional to the traveled distance. Two flow-based integer programming models are proposed; only the second one allows to revisit BSSs (i.e., to pass at a station / customer multiple times). In addition, two heuristic approaches are studied. The first one is a tabu search which mainly focuses on the location of BSSs and uses a modified Clarke and Wright Clarke and Wright (1964) savings algorithm to heuristically compute a set of routes based on the currently selected swapping stations. A radius-covering method is applied to find an initial set of BSSs. In addition, a hybrid heuristic combining various approaches (namely, modified sweep heuristic, iterated greedy and adaptive large neighborhood search), is described. The main idea is to initially ignore most of the constraints (i.e., battery driving range, BSS location) and subsequently refine a candidate solution to satisfy all conditions. Finally, a last phase aims at improving solutions that are already feasible for the considered problem. Computational experiments are performed using data sets from the CVRP in which all nodes are considered as potential BSSs. Results show that revisits often pay off. The influence of different maximum driving ranges is also analyzed.

Mak et al. (2013) aim to optimize location and sizing of BSSs at strategic locations along a network of freeways. They argue that the strategic network decisions need to be taken before observing the actual demand. Therefore, they propose distribution-robust optimization problems where in a first phase the location of BSSs needs to be decided while the number of batteries stored at each BSS can be determined after the uncertain factors are realized. Two variants in which either the expected building and operating costs are minimized ("cost-concerned" model) or a robust estimate of the probability to meet a certain return-on-investment target is maximized ("goal-driven" model) are considered. Models based on mixed-integer second-order cone programming are derived and potential impacts of battery standardization and advancements on the deployment strategy are studied. Computational experiments are performed using instances based on the San Francisco

Bay Area freeway network. It is also pointed out that there exist real world cases (Israel) in which the set of candidate BSSs corresponds to the set of existing gas stations and that upper bounds on the number of batteries per location need to be considered. This restriction arises from the capacity of the electrical grid. Furthermore, the number of arising swap-demanding EVs are treated by a Poisson process, the swapping is assumed to be instantaneous, and a heuristic first-in-first-out strategy for battery selection is considered.

Li (2014) studies the scheduling of electric transit buses when either battery swapping or fast charging is employed. An exact branch-and-price algorithm (including stabilization and an initial construction heuristic) as well as heuristic variants based on truncated column generation, variable fixing, and local search are developed. A computational study is performed on instances that are based on publicly available real-world transit data. Besides comparing variants of the proposed algorithms, the results achieved are benchmarked against approaches for other types of buses (gas, diesel, hybrid). Despite the main disadvantage of electric buses, such as the need of deadhead travels to battery stations, the author concludes that the total operational costs of electric buses are smaller than those of the other options. The use of electric buses, therefore, represents a viable alternative also because they produce zero emissions during operation.

Other authors (see, e.g., Chen and Hua 2014) focus on the placement of battery swapping stations without discussing too many aspects that differ from the planning of other re-charging stations; we therefore refer to Sect. 2.1 for more details.

Another stream of research concerned with battery-swapping deals with the replacement of degraded batteries within a fleet of vehicles by new ones. Almuhtady et al. (2014) study different swapping and replacement policies within maintenance of a fleet by a mathematical model as well as two metaheuristic approaches: genetic algorithm and simulated annealing. Experimental results using data inspired from real world are shown.

3.2.1 Summary, Open Problems and Possible Research Directions

Existing approaches in the literature are mainly concerned with either minimizing the total costs in installing (and possibly maintaining) battery-swapping stations. In addition, total routing costs are partially considered in case of classic vehicle routing applications. One exception to this trend is given by Mak et al. (2013) who also consider a variant in which the probability to meet a certain return-on-investment goal is maximized. Most of the related works consider constraints limiting maximum travel ranges (whenever a location-routing problem is considered) and restrictions to relatively small sets of potential swapping stations (often only existing “traditional” gas stations). Besides, upper bounds on the numbers of batteries per location arising from limitations of the electric grid are considered (in particular if fast-charging is employed).

Open problems in this area include the appropriate integration of charging times within the overall models and the potential consideration of charging at different speeds instead of assuming a given number of available, charged batteries. Furthermore, integration of aging and replacing aspects of batteries (with respect distance traveled, charging cycles) into battery-swapping problems can be a relevant topic.

3.3 Electric Vehicle Shortest Path Problems

This section discusses optimal path problems involving electric vehicles—with focus on PBEVs—and their specifics. In the car-sharing context these problems might be relevant when the provider wants to estimate the energy consumption of customer trips or when navigation services are offered to customers.

In general one can think of many different practical problem variants of finding an efficient path from A to B while respecting the battery limits (lower and upper bound) of PBEVs. Among them, the following objectives might be relevant:

- minimize energy consumption,
- minimize travel time, and/or
- minimize total costs including costs for traveling, charging, drivers, etc.

Several additional aspects may be considered, e.g.:

- visits to charging stations,
- charging times,
- energy recuperation, i.e., negative energy values on arcs, and/or
- charging station capacities.

An extensive survey on EV shortest path problems and algorithms can be found in Pelletier et al. (2016). In the following, we review important works and extend this survey.

Artmeier et al. (2010) minimize energy consumption while allowing recuperation. Since lower and upper bounds of the battery charge have to be respected, the resulting problem is a variant of the constrained shortest path problem which is NP-hard in general. However, here the optimized and constrained resource are the same, finally leading to a polynomial-time algorithm, i.e., a modified Bellman-Ford algorithm. Since the energy consumption on links also depends on the speed on the previous link on the selected path, applying the label-setting algorithm on the original graph is not possible. Thus, the authors describe the construction of an energy graph in which nodes are replicated for each velocity value on incoming arcs. Since the node degree in street network is three on average, the corresponding energy graph is not much larger than the original one.

Eisner et al. (2011) extend the work by Artmeier et al. (2010) by applying an adaptation of Johnson's potential shifting technique to obtain non-negative edge costs and finally run Dijkstra's algorithm to execute queries in polynomial time. Additionally, the idea of contraction hierarchies is used to further dramatically speed-up shortest path queries.

Sachenbacher et al. (2011) also improve the work by Artmeier et al. (2010) by considering an A*-related shortest path algorithm. They show that an energy consumption function depending on distance, elevation, and speed provides a consistent heuristic for the A* algorithm, i.e., an energy-optimal route can be found. Their approach significantly outperforms the standard Bellman-Ford and Johnson variants and additionally allows to use dynamic energy information at query-time.

Cassandras et al. (2014) consider the problem of finding a path from A to B of a single PBEV with minimal total time while respecting the battery constraints and determining which and how long charging stations are visited. The total time includes both travel and charging times. A non-linear MIP is presented and under several assumptions the authors transform it to an LP: (1) at each node there is a charging station with a fixed charging rate, and (2) all energy consumption values on arcs are non-negative. The authors also study the path routing problem with multiple vehicles involving traffic congestion issues and assuming that all vehicles are controlled by a central system. Several non-linear MIPs are proposed to solve this problem.

Arslan et al. (2014) deal with an NP-hard minimum-cost path problem for plug-in hybrid electric vehicles (PHEVs) (with both combustion and electric engine) with intermediate fueling/charging stations. They transform the original graph in a way that only origin, destination, and fueling/charging nodes are left. Edges represent the shortest paths between the corresponding nodes in the original graph. When considering only PBEVs, it is possible to find a minimum-cost path from A to B in this graph in polynomial time (e.g., by Dijkstra's algorithm), visiting fueling/charging stations if necessary. For PHEVs, the additional decision of choosing the driving mode makes the problem NP-hard. In an extended problem variant the authors additionally consider vehicle depreciation, stopping, and battery degradation costs. An exact MIP model with quadratic constraints, a dynamic programming and a shortest path based heuristic are presented to solve this problem.

3.3.1 Summary, Open Problems and Possible Research Directions

In earlier works, the main objective is to minimize the energy consumption on the total path. More recently, researchers often consider the minimization of the total travel time while respecting the energy limits, which might be more relevant in practical applications. Additionally, complex cost functions are used combining the (time-dependent) costs for traveling, charging, battery degradation, etc.

The most important common constraints are based on the physical limits of the battery of PBEVs. Because of the currently still quite small battery capacities, PBEVs quickly run out of energy. Recuperation, i.e., the recovery of energy when breaking, may compensate partly for this deficiency. This, however, leads to negative energy values on links and thus to more complicated optimization problems.

The systemic battery limits of PBEVs may also lead to further related constraints. If visits to a given set of charging stations are allowed, then corresponding charging times and station capacities have to be considered, which may also be time-dependent based on the overall state of the underlying electrical grid.

Many authors use simplified formulas to calculate the energy consumption on links. Here, more realistic (possibly non-linear) functions involving a large number of influencing factors may be considered. For some applications, such detailed energy consumption models may not be needed, but nevertheless it should be clear which components mostly contribute to the energy consumption. A sensitivity analysis for a complex energy model might be performed to identify the crucial aspects.

Most works consider only a single vehicle and search for the best path in an egocentric point of view. For governmental stakeholders and local authorities, however, it might be more relevant to consider a global system optimum rather than a local egocentric optimum. Thus, more sophisticated models involving multiple vehicles and complex evaluation functions may be considered in the future.

Realistic energy consumption models and cost functions often involve non-linear terms. Finding accurate linear approximations for these functions might be a way to finally obtain efficient solution approaches for these problems. Discretization might be a promising candidate to reach this goal.

3.4 Electric Vehicle Routing Problem

This section discusses works on vehicle routing problems in which traditional vehicles are either replaced by or mixed with PBEVs. Such problems might be relevant for car-sharing providers if navigation services are offered which involve finding routes visiting a set of locations given by the customer.

Since the battery capacity of electric vehicles is strongly limited, it may be necessary to re-charge the battery along a single route, possibly multiple times. In the literature, this limitation is handled quite differently, as discussed in the next paragraphs. An early survey on sustainable VRP variants can be found in Lin et al. (2014). The survey by Pelletier et al. (2016) summarizes several aspects of electric vehicles, i.e., different types of electric vehicles, market penetration, incentives, OR related works, and research perspectives. More details on the specifics of electric vehicles can be found in Pelletier et al. (2014). Since the survey by Pelletier et al. (2016) is quite extensive, here we only discuss papers which are particularly relevant or not mentioned in the survey.

In the green VRP introduced by Erdoğan and Miller-Hooks (2012), routes for alternative-fuel powered vehicles are determined. A compact MIP based on Miller-Tucker-Zemlin Miller et al. (1960) subtour elimination constraints (Big-M) is presented, minimizing the traveled distance while considering the limited distance, possible visits to alternative fuel stations, and upper bounds on the number of tours and their duration. In contrast to classical VRP variants, vehicles are assumed to be uncapacitated here. Refueling time is assumed to be constant, which is usually not the case for electric vehicles. The authors also propose two construction heuristics to create feasible solutions. The results indicate that as the number of fuel stations increases, costs decrease for the same number of served customers, more customers can be served, and the total distance traveled decreases.

Van Duin et al. (2013) examine the fleet size and mix Vehicle Routing Problem with Time Windows with special focus on different types of electric vehicles for goods distribution. The battery limitations are considered by setting a maximal tour length which can be completed with a single battery charge, i.e., recharging at specific stations is not allowed. A compact MIP based on Big-M constraints is presented without solving the model. To find solutions for a case study in Amsterdam, the authors developed a simple construction heuristic which provides satisfying results in their application.

Schneider et al. (2014) extend the green VRP by integrating time windows (VRPTW), customer demands, and capacity constraints to the problem, while focusing exclusively on PBEVs. As a result, recharging times depend on the vehicles battery charge when arriving at a recharging station, and assuming a full recharge. The authors consider a hierarchical objective function first minimizing the fleet size and second minimizing the total travel distance. A hybrid metaheuristic combining variable neighborhood search with tabu search yields small gaps compared to a compact MIP model with Big-M constraints solved by CPLEX.

Frank et al. (2014) consider the same problem as Schneider et al. (2014), but involve load-dependent energy consumption: each arc is associated with an energy consumption value both for an empty vehicle and a single load unit. Then, the total energy consumption on an arc is linearly dependent on the amount of cargo loaded. The authors provide several MIP models for this problem variant: (1) a compact model with Big-M constraints, (2) a compact two/three-index-formulation with Big-M constraints allowing at most one charging station visit between two clients, and (3) a set-partitioning model. The same authors present in Preis et al. (2013) a more detailed energy consumption model based on distance, altitude, load, and several vehicle properties. In a compact MIP model with Big-M constraints for the electric VRPTW, they minimize the total energy consumption. Additionally, the authors use tabu search heuristics to solve this problem.

Felipe et al. (2014) also consider the same problem as Schneider et al. (2014) except that (1) partial recharges at charging stations are allowed, (2) different charging station technologies can be used at a station (faster charging is more expensive), and (3) the objective is to minimize the charging and battery cycle costs. A compact MIP model with Big-M constraints and a simulated annealing approach incorporating local search in several neighborhood structures are proposed.

Goeke and Schneider (2015) extend the work by Schneider et al. (2014) by considering a mixed fleet with both traditional vehicles and PBEVs in the electric VRPTW. The main contribution of this article is that the energy consumption does not only depend on the distance but involves more parameters, i.e., travel speed, gradient of link, and current load. Here, the energy consumption may also be negative, allowing recuperation and recovery of energy on downward slopes and in breaking events. However, the battery is still fully recharged at a charging station visit. The authors provide a compact MIP model similar to the one in Schneider et al.

(2014) based on Big-M constraints but including non-linear parts related to load-dependent energy consumption. Additionally, an Adaptive Large Neighborhood Search algorithm is presented. Tests are performed on newly generated instances and on the Solomon-based instances by Schneider et al. (2014). The authors also consider different objective functions not only involving the traveled distance, but also fuel and battery depreciation costs.

Hiermann et al. (2014) tackle the same problem as Schneider et al. (2014) but additionally consider a mixed fleet of different PBEVs varying in the load and battery capacity. A compact MIP model and an adaptive large neighborhood search are presented to solve this variant.

Desaulniers et al. (2014) consider a generalization of the classical VRPTW using only electric vehicles: additional nodes represent charging stations which may be visited an arbitrary number of times. The authors also consider several special variants of this problem: (1) at most one charging station can be visited on each route, and (2) at each charging station visit the battery is fully loaded. In the more general variant, there is no limit on the number of visited charging stations and the battery may also be partially loaded at a charging station. The results of these variants are compared, leading to the conclusion that in the unrestricted variant routing costs and the number of needed vehicles can be reduced. The authors present exact branch-price-and-cut approaches based on a classical set-partitioning formulation for the considered problem variants. Much effort is put into the development of efficient solution methods for the pricing subproblem, which often represents a performance bottleneck in these approaches. Mono- and bi-directional labeling algorithms are presented for the different variants, enhanced with acceleration strategies based on ng-route relaxations and reduced graphs. To decrease the integrality gap, two sets of valid inequalities defined on the route variables are added: (1) the 2-path cuts, and (2) the subset row inequalities. The presented approaches are tested on a benchmark set introduced in Schneider et al. (2014) and generated from the classical Solomon VRPTW instances. All instances can be solved in reasonable time. To the best of our knowledge, these approaches represent the computational state-of-the-art for many variants of the electric VRPTW.

Worley et al. (2012) consider a combination of location of charging stations and routing of electric vehicles. They present an MIP model with variables for all route segments (no intermediate depot or charging stations) but do not mention how this model with an exponential number of variables is solved. The objective is to minimize the total costs consisting of the costs for building stations, charging vehicles, and driving.

Table 3 gives an overview of the different problem variants discussed in the last two sections.

Table 3 Classification of the literature related with EV routing problems (SP: shortest path problem, VRP: vehicle routing problem)

Reference	Type	Objective	Energy calculation	Charging	Methodology
Artmeier et al. (2010), Eisner et al. (2011), and Sachenbacher et al. (2011)	SP	min. energy consumption	Predefined	No	Exact
Cassandras et al. (2014)	SP	min. travel + charging time	Predefined	Partial	Exact
Arslan et al. (2014)	SP	min. travel + charging costs	Distance	Full	Exact/heuristic
Erdoğan and Miller-Hooks (2012)	VRP	min. distance	Distance	Constant	Exact/heuristic
Van Duin et al. (2013)	VRP	min. travel + vehicle + driver costs	Distance	No	Heuristic
Schneider et al. (2014)	VRP	min. distance	Distance	Full	Exact/heuristic
Frank et al. (2014)	VRP	min. distance	Predefined + load	Full	Exact
Preis et al. (2013)	VRP	min. energy	Predefined + load	Full	Exact
Felipe et al. (2014)	VRP	min. charging + battery costs	Distance	Partial	Exact/heuristic
Goeke and Schneider (2015)	VRP	min. distance/battery costs/energy + driver costs	Predefined + load	Full	Heuristic
Hiermann et al. (2014)	VRP	min. travel + vehicle costs	Distance	Full	Exact/heuristic
Desaulniers et al. (2014)	VRP	min. distance	Predefined	Partial/full	Exact
Worley et al. (2012)	VRP	min. building + charging + travel costs	Distance	Full	Exact

3.4.1 Summary, Open Problems and Possible Research Directions

Most works consider the minimization of the total traveled distance, or more generally the total costs including costs for traveling, fleet investments, battery degradation, etc. Often, the number of vehicles used is minimized in a hierarchical way (in contrast to a weighted objective or a multi-objective formulation). Some authors, however, focus on the minimization of the total energy consumption which seems to be less relevant for practical needs.

Common for many problem variants is the consideration of customer demands, maximal vehicle load capacities, customer time windows, and clearly the highly restricted battery limits. In more strategic problems, the vehicle fleet is heterogeneous in terms of propulsion type (combustion/electric), battery size (if applicable), and/or load capacity.

Similar to Sect. 3.3, different (more or less detailed) energy consumption models are used. Additionally, for VRP variants it is relevant to also consider the current load for the energy consumption since it may change throughout the tour. The battery limits for PBEVs are considered differently: either simply the tour length is limited or the vehicles are allowed to visit charging stations within the tour. In the second case, different models for charging are implemented: (1) constant charging times, (2) full charging based on the current state of charge, or (3) partial charging. Different technologies and therefore charging speeds and capacities may be available at the stations to choose from.

In recent works, the researchers consider more integrated problem variants, e.g., by combining the location of charging stations with the routing part. Here, also the technology, the number of charging points, and the electric capacity may need to be decided for a new charging station.

There are existing models and exact approaches for load-dependent energy consumption. However, there seems to be some room for improvement in terms of model strength and efficiency of solution methods. Also more detailed energy consumption models may be considered in the VRPs, cf. Sect. 3.3.1.

When considering capacities and technologies of charging stations the corresponding electrical grid and its time-dependent load may be considered. In the area of smart energy grids, researchers brought up the idea of using PBEVs as a temporary energy storage to compensate high demands in peak hours (Kempton et al. 2001). The integration of such features in existing VRP variants may lead to even more complicated problems but probably would also improve their relevance in real-world applications. The combination of the location of charging stations and vehicle routing goes into a similar direction.

4 Conclusions

In this paper, we reviewed the main optimization problems arising in the design and management of car-sharing systems based on electric vehicles. For each problem class, the relevant literature and the main practical issues arising from real-world applications are discussed.

The most relevant research directions for each problem are:

- Location problems (see Sect. 2.1.5)
  - Simultaneous consideration of different station types (e.g., slow and fast charging stations)
  - Incorporate detailed battery-state modeling in electric location-routing problems
- Relocation of vehicles for multiple-station car-sharing (see Sect. 3.1.3)
  - Assess users willingness to modify the trip when incentives are offered
  - Investigate the integration of user-based techniques in staff relocation
  - Use real-time information for online relocation
- Electric vehicle shortest path problems (see Sect. 3.3.1)
  - Use more realistic functions to calculate the vehicle's energy consumption
  - Find system-optimal paths in complex traffic networks rather than optimal paths in an egocentric point of view
- Electric vehicle routing problems (see Sect. 3.4.1)
  - Use more practically relevant objective functions
  - Use more realistic energy consumption models, e.g., involving the vehicle's load
  - Consider the (time-dependent) capacity and load of charging stations and the underlying electrical grid

Besides tackling each of these problems individually, the study of combined approaches (e.g., simultaneously optimizing the location of charging stations and relocation decisions) is a worthwhile goal for future research.

Many open problems are discussed, indicating Ecar-sharing systems as a rich and promising research area for optimization methods.

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The Golf Tourist Problem

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Abstract Tourism and travel with the purpose to do sports is gaining in popularity and the golf tourism market is considered to be one of the largest. Motivated by this phenomenon we model and solve the golf tourist problem which generalizes the orienteering problem with time windows. It aims at providing decision support for the traveling golfer by concurrently optimizing two objective functions: travel cost on the one hand and attractiveness of the generated travel plans on the other hand. Travel costs consist of flight cost, hotel cost, car rental cost, green fees as well as petrol cost for traveling between the selected golf courses. Attractiveness is measured by the total par scores of the visited golf courses. We assume that the traveling golfer provides a selection of regions in Europe that he or she is equally inclined to visit on his or her next trip. A feasible travel plan selects one region, contains only golf courses of this region and starts and ends at the respective airport. We solve the golf tourist problem to optimality by means of a recent bi-objective branch-and-bound algorithm and by means of the ϵ-constraint method. Furthermore, we devise a decomposition approach that solves each regional problem separately and then combines the obtained Pareto sets. The proposed methods are applied to several real world instances with up to nine regions and between 57 and 227 golf courses per region. Our results show that the decomposition approach is significantly more efficient than the holistic approach. They also show that the bi-objective branch-and-bound algorithm performs better than the ϵ-constraint scheme.

1 Introduction

Tourism and travel with the main purpose to do sports is an expanding global phenomenon (Higham and Hinch 2002; Barros et al. 2010).The golf tourism market is regarded as one of the largest sports-related tourism industries. There are several reports on the economic impact of the golf industry (cf. Kim et al. 2008). Although

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there are no official figures, the number of golfers is estimated to be around 80 million. They are playing on approximately 40,000 courses across the planet. The golf tourism market is claimed to be worth more than €10 billion (Golf's 2020 Vision: The HSBC Report 2012; Kim et al. 2008). An increasing number of amateur golfers are choosing to travel abroad to play golf in regions different from their domestic clubs (Barros et al. 2010). These destinations are selected based on various reasons, often personal in nature, and driven by a number of internal and external forces (cf. Ong et al. 2010).

The majority of studies on golf tourism are analyzing customer satisfaction levels from a marketing point of view. The focus is to help regions in attracting golfers to travel to their domestic clubs, and stay there as long as possible (e.g Moital et al. 2013; Kim and Ritchie 2012; Tassiopoulos and Haydam 2008; Kim et al. 2005). However, this paper comes up with decision support for the traveling golfers themselves. We mainly address the group of *intensive golfers* as it is defined by Kim and Ritchie (2012). This type of golf tourist tends to travel for relatively long periods in order to be able to play at a great variety of attractive golf courses. A common possibility to quantify the attractiveness of a golf course is its *par* score. This is the average number of strokes a very good player (called *0 handicap player*) needs to complete this course. In particular for the group of intensive golfers, it can be assumed that the attractiveness of a course depends on its difficulty, i.e. its par value.

Finding a route, limited in length, that visits a number of golf courses such that the sum of the *par* scores is maximized corresponds, from a mathematical point of view, to an orienteering problem (Chao et al. 1996; Vansteenwegen et al. 2011). However, we assume that the traveling golfer is not only interested in the best travel plan based on par scores for a given budget but also in the trade-off relationship between attractiveness (total par scores) and travel costs. Therefore, as a second objective, we assume that the golfer wants to minimize his or her travel expenses. In addition, we consider that the traveling golfer has not yet made up his mind as to where to spend his or her next golf vacations but is able to specify a set of equally appealing regions to choose from. We formulate this as a bi-objective optimization problem. In order to solve real-world instances we use the branch-and-bound framework proposed by Parragh and Tricoire (2015) which generates the complete set of efficient (or Pareto-optimal) solutions. In what follows, we assume that the reader is familiar with basic concepts of multi-objective optimization. For a detailed introduction to the topic, we refer to Ehrgott (2005).

The remainder of this paper is structured as follows. Section 2 gives a literature review. The mathematical model and the solution methods are described in Sects. 3 and 4, respectively. The computational study is presented in Sect. 5. Conclusions and further research are given in Sect. 6.

2 Literature Review

The generation of personalized routes for tourists has received considerable attention in the past years. A survey on tourist trip planning tools and their key functionalities is presented by Souffriau and Vansteenwegen (2010). Li et al. (2009) develop a user model based on information retrieval techniques to rank tourist preferences of points-of-interest. They generate tours using a guided local search metaheuristic. Solutions are produced dynamically in order to be able to meet tourist's real-time requirements. A personal navigation system for tourists is proposed by Maruyama et al. (2004). The user can specify multiple destinations with desired arrival/stay time and preference degree. For the resulting route search problem, the authors develop an efficient genetic algorithm. They are able to calculate a semi-optimal route almost in real-time. Rodríguez et al. (2012) study the existing tourist support systems and analyze their advantages and weaknesses. Based on this analysis they offer the tourist a tool which makes up for the shortcomings of the other systems. It refers to a mathematical model which considers a large part of the tourist's objectives. The authors face a multi-objective problem of activity selection and sequencing which they solve using a metaheuristic method based on tabu search.

The tourist trip planning problem is commonly formulated as an extension of the orienteering problem. This allows to deal with a vast number of practical applications (Souffriau and Vansteenwegen 2010). Automated selection of the most interesting tourist attractions by means of a hand-held personalized electronic tourist guide is investigated by Vansteenwegen (2009). The planning problems that have to be solved are called tourist trip design problems. They are considered as extensions of the team orienteering problem proposed by Chao et al. (1996b), and they are solved with different metaheuristics. Li and Fu (2012) formulate the tourist trip design problem by a mathematical programming model introducing a time-aggregated graph. A novel label correcting algorithm is presented to solve this problem based on the idea of network planning and dynamic programming.

Souffriau et al. (2011) work on the planning of trips for recreational cyclists. The problem is defined as a variant of the arc orienteering problem in which the score of a route in a directed graph has to be maximized by visiting arcs. The problem is solved using a greedy randomized adaptive search procedure.

A mobile tourist decision support system that suggests personal trips, while taking the opening hours of the points of interest and the available time budget into account, is proposed by Souffriau et al. (2009). The planning problem is modeled as an orienteering problem with time windows and solved by an iterated local search algorithm.

Along these lines, we model the golf tourist problem as an orienteering problem with two concurrent objectives. Other works that deal with bi-objective orienteering problems are those of Schilde et al. (2009) and Parragh and Tricoire (2015). Schilde et al. (2009) consider tourist trip planning applications where the tourist may specify profits in two different categories, such as leisure and culture. The proposed algorithms (an ant colony algorithm and a variable neighborhood search algorithm both combined with path relinking) generate an approximation of the efficient frontier. Parragh and Tricoire (2015) consider the bi-objective team orienteering problem with time windows, i.e. instead of a single route a given number of routes is planned. They concurrently optimize cost and profit. Optimal Pareto sets are obtained by means of column generation embedded into a branch-and-bound algorithm. They compare the performance of the proposed branch-and-bound algorithm to using the same ingredients in an ϵ-constraint scheme and they are able to show that the branch-and-bound algorithm is much faster in many cases. In this paper, we use the bi-objective branch-and-bound scheme of Parragh and Tricoire (2015) to solve the golf tourist problem and we also compare it to the ϵ-constraint scheme. However, in contrast to Parragh and Tricoire (2015) and since in the golf tourist problem only a single route is to be planned, we do not rely on column generation.

3 Problem Formulation

In the golf tourist problem (GTP), the aim is to generate efficient travel plans that concurrently optimize attractiveness and travel cost. The attractiveness of a travel plan is the sum of the par scores of all the golf courses visited in the plan; it is maximized. Travel costs are composed of flight, hotel, car rental, and petrol cost for visiting the selected golf courses as well as green fees; they are minimized. A set of regions that the traveling golfer is equally inclined to visit (it may also only be a single region), each associated with an airport is given. Depending on distances to nearby airports, a course can belong to more than one region. All selected golf courses of a given travel plan have to belong to the same region and their respective opening and closing times have to be respected, i.e. some golf courses are accessible on all days of the week while others are only open to visitors from Sunday to Thursday, Monday to Friday or any other consecutive sequence of days. We assume that the traveling golfer starts his or her trip at the airport of the selected region to which he or she has to return to within a given number of days. In addition, we assume that the traveling golfer plays at most one course per day. The travel time between each pair of golf courses is also given in days, where a value of 1 indicates that two golf courses can be played on consecutive days while a value of 2 means that an additional day of traveling is required. In addition to opening times (denoted as time windows in the following) at the golf courses, the traveling golfer may also specify a minimum as well as a maximum number of golf courses to be included into the plan. Golf courses are considered incompatible if they are in the same location and only differ in, e.g., the number of holes. In order to formulate the GTP, we use the following sets and parameters:

Sets		Parameters	
N	Set of golf courses	f_k	Fixed cost for region k (hotel, flight, car rental)
R	Set of regions	g_i	Green fee of golf course i
R_i	Set of regions of course i	p_i	Par score (attractiveness) of golf course i
D^+	Set of starting depots	c_{ij}	Cost for traveling from i to j (petrol costs)
D^-	Set of ending depots	t_{ij}	Time needed to travel from i to j (in days)
A	Set of existing arcs	e_i	Beginning of the time window at i
B	Set of the sets of incompatible golf courses	l_i	End of the time window at i
B_l	Set of incompatible courses	d_k^+	Starting depot of region k
		d_k^-	Ending depot of region k
		m	Minimum number of courses that should be visited
		q	Maximum number of courses that should be visited

as well as the following decision variables:

$$z_k = \begin{cases} 1, \text{ if region } k \text{ is selected} \\ 0, \text{ otherwise,} \end{cases}$$

$$x_{ij} = \begin{cases} 1, \text{ if arc } (i,j) \text{ is traversed,} \\ 0, \text{ otherwise,} \end{cases}$$

$$y_i = \begin{cases} 1, \text{ if golf course } i \text{ is visited} \\ 0, \text{ otherwise,} \end{cases}$$

$$u_i = \text{day on which course } i \text{ is visited.}$$

The GTP can now be formally stated as follows:

$$f_1 = \max \sum_{i \in N} p_i y_i \tag{1}$$

$$f_2 = \min \sum_{k \in R} f_k z_k + \sum_{(i,j) \in A} c_{ij} x_{ij} + \sum_{i \in N} g_i y_i \tag{2}$$

subject to

$$\sum_{k\in R} z_k = 1 \tag{3}$$

$$y_i \le \sum_{k\in R_i} z_k \qquad \forall i \in N \tag{4}$$

$$y_i = \sum_{(i,j)\in A} x_{ij} = \sum_{(j,i)\in A} x_{ji} \qquad \forall i \in N \tag{5}$$

$$z_k \le \sum_{(d_k^+,j)\in A} x_{d_k^+ j} \qquad \forall k \in R \tag{6}$$

$$z_k \le \sum_{(j,d_k^-)\in A} x_{jd_k^-} \qquad \forall k \in R \tag{7}$$

$$u_j \ge u_i + t_{ij}x_{ij} - (1 - x_{ij})M \qquad \forall (i,j) \in A \tag{8}$$

$$e_i \le u_i \le l_i \qquad \forall i \in N \cup D^+ \cup D^- \tag{9}$$

$$m \le \sum_{i\in N} y_i \le q \tag{10}$$

$$\sum_{i\in B_l} y_i \le 1 \qquad \forall B_l \in B \tag{11}$$

$$z_k \in \{0, 1\} \qquad \forall k \in R \tag{12}$$

$$x_{ij} \in \{0, 1\} \qquad \forall (i,j) \in A \tag{13}$$

$$y_i \in \{0, 1\} \qquad \forall i \in N \tag{14}$$

Objective function (1) maximizes the total attractiveness of the travel plan and objective function (2) minimizes the total cost. Constraint (3) makes sure that exactly one region is chosen from the available set of regions. Constraints (4) guarantee that only those golf courses can be visited that are located in the chosen region. Equalities (5) make sure that if a golf course is selected it is entered and left. In the selected region the starting depot (which corresponds to the city where the airport is located) has to be visited at the beginning of the tour and returned to at the end. This is ensured by constraints (6) and (7). Constraints (8) set the visiting times. Note that M can conveniently be set to the maximum number of days. Inequalities (9) take care of time windows and (10) set bounds on the minimum and maximum number of courses that should be visited. Constraints (11) take care of incompatible golf courses. This constraint is mainly used to make sure that from a set of golf courses at the same location, e.g. with a different number of holes, only one is chosen. It can, however, also be used to model any kind of user defined clique condition. For instance, a user may want to visit no more than one golf course within a certain city or region, or no more than one golf course with 18 holes, spending time on smaller courses the rest of the time.

4 Solution Methods

We use two different exact algorithms to solve the GTP. The first relies on a bi-objective branch-and-bound algorithm and the second uses the ϵ-constraint framework. For the sake of simplicity, they are both described as operating on a bi-objective mixed-integer program (MIP) where both objectives functions f_1 and f_2 have to be minimized. The GTP can easily be transformed to fit that context by multiplying every term in objective (1) by -1 and considering it as a minimization objective.

4.1 Bi-objective Branch-and-Bound

The first method is based on the bi-objective branch-and-bound (BIOBAB) algorithm for integer programs introduced by Parragh and Tricoire (2015). It is a generalization of the concept of branch-and-bound to the bi-objective context. Instead of single numerical values, upper and lower bound sets are considered, relying on the concept of *bound set* described in Ehrgott and Gandibleux (2006). At each node of the branch-and-bound tree, a lower bound (LB) set is calculated using an algorithm similar to that from Aneja and Nair (1979). This algorithm consists in solving a succession of single-objective weighted sum problems, using linear combinations of objective functions f_1 and f_2. Each weighted-sum problem can be solved separately. The algorithm of Aneja and Nair enumerates a set of weight combinations that is sufficient to produce the corner points of the convex hull of the LB set in objective space. These points allow us to describe a continuous LB set for the bi-objective problem. This LB set is then filtered using an upper bound (UB) set, which is a non-dominated subset of all the feasible (integer) solutions obtained during the search. In order to prune a node from the branch-and-bound tree, the whole LB set at this node must be dominated by the UB set, i.e. it has to be a subset of the space dominated by the UB set. The main loop of BIOBAB is summarized in Algorithm 1. It relies on function $push(C, x)$, adding node x to an existing collection of nodes C and on function $pop(C)$, retrieving a node from a collection of nodes. These functions are used to store and retrieve the nodes of the branch-and-bound tree. Different tree exploration strategies can be obtained by using different data structures for C. A node is simply a set of branching decisions. Additionally, the algorithm can take a starting UB set in order to speed up the search, just like single-objective branch-and-bound can consider a starting UB value to prune more nodes. The UB set is updated with newly found integer solutions during the bounding procedure.

Algorithm 1 *BIOBAB*(*UB*)

```
rootNode ← ∅
C ← ∅
push(C, rootNode)
while C ≠ ∅ do
    node ← pop(C)
    LB ← bound(node, UB)
    LB ← filterLB(LB, UB)
    if LB ≠ ∅ then
        newBranches ← branch(LB)
        for all decision ∈ newBranches do
            push(C, node ∪ {decision})
        end for
    end if
end while
return UB
```

A key feature of the BIOBAB algorithm is that nodes can be partially filtered in order to speed up the search, through what is called *objective space branching*. Simply put, information from the UB set is used in order to reduce the size of the LB set. If the resulting LB set is discontinuous then branching occurs, each branch corresponding to a different continuous subset of the discontinuous LB set. Another key feature is the addition of filtering rules relying on the fact that the objective values of integer feasible solutions only take integer values.

In the current implementation, no starting UB set is generated and in terms of tree exploration strategy, breadth-first search is used. This is achieved by using a FIFO queue to implement C in Algorithm 1. Furthermore, we note here that contrary to the previous uses of BIOBAB, the MIP is not relaxed in the bounding procedure. This means that the corner points produced by function $bound(node, UB)$ correspond to supported feasible integer solutions. First, we tried to solve the linear relaxation at this step of the algorithm but it led to poor results. We believe it is due to the fact that there are various groups of binary variables, making it difficult to decide on which variable binary branching should occur. Another explanation is that the LB set obtained by considering the linear relaxation is of poor quality when considering both objectives. However, BIOBAB also works if the MIP is not relaxed, as it still provides a convenient way to split the objective space and enumerate all Pareto-optimal solutions.

Since we do not solve a linear relaxation in the bounding procedure, only objective space branching is performed. After filtering, a LB set can be discontinuous. This is due to the fact that, in the general case, the UB set dominates a subset of the LB set. The aforementioned filtering rules, based on integrality of objective functions for feasible solutions, mean that some space around each already found integer solution is filtered out of LB sets. This generates more discontinuity in LB sets. For instance, at the root node, where all supported solutions are found, there is a discontinuity in the LB set around each of these supported solutions. In the general case, the LB set after filtering is discontinuous. At this point, each non-dominated continuous section of the LB set gives rise to a new branch.

4.2 ϵ-Constraint Framework

The second method is the well-known ϵ-constraint method, first introduced by Haimes et al. (1971). It consists in iteratively solving single-objective versions of the original bi-objective problem, enumerating the Pareto front sequentially by increasing values of one objective and decreasing values of the other objective. It relies on a certain value ϵ which represents the smallest interval in objective value between two Pareto-optimal solutions. In case of a problem where objective values are always integer, 1 is a valid ϵ. A possible implementation of the ϵ-constraint framework is summarized in Algorithm 2.

The ϵ-constraint framework is very simple and easy to implement, given access to a solver for the single-objective versions of the MIP. In our case, this can be done with any commercial MIP solver. An interesting property of that method is that its complexity is linear in the number of solutions in the Pareto set. However, there is a lot of overlapping in the objective space considered in the successive single-objective MIPs.

We note here that the objectives can be interchanged, i.e. it is possible to optimize first f_2 then f_1 at every step. This can actually have an impact on the performance of the algorithm: one way is faster than the other. In the following we always optimize profit (f_1) first then costs, as it proved overall faster in our preliminary experiments. This means that the Pareto front is enumerated from the most profitable solution to the cheapest one.

Algorithm 2 ϵ-constraint

$\Lambda \leftarrow \emptyset$
ϵ-constraint $\leftarrow f_2 \leqslant \infty$
add ϵ-constraint to MIP
repeat
    $x \leftarrow Min(f_1)$
    $localObjectiveBound \leftarrow f_1 = f_1(x)$
    add $localObjectiveBound$ to MIP
    $x \leftarrow Min(f_2)$
    $\Lambda \leftarrow \Lambda \cup \{x\}$
    remove $localObjectiveBound$ from MIP
    update ϵ-constraint: ϵ-constraint $\leftarrow f_2 \leqslant f_2(x) - \epsilon$
until MIP cannot be solved
return Λ

4.3 Aggregate Solving

We now exploit the structure of the GTP to devise a third approach. Because only one region can be visited in a feasible solution, it is actually possible to decompose the GTP by solving each region separately. Then all solutions obtained this way have to be considered. The procedure is very simple. In a first step, every region is solved separately (using for instance BIOBAB or ϵ-constraint). In a second step, we consider the non-dominated union of all the sets of solutions obtained for individual regions. The resulting set is the Pareto set of the original instance.

5 Computational Study

The previously described algorithms are implemented using Python and Gurobi 6.0. The CPU time is mostly spent on solving MIPs, which is done by Gurobi, so the Python part of the code has little impact on the overall CPU time. The algorithms are run on a cluster where each node consists of two Intel Xeon CPUs at 2.50 GHz with ten cores each. Multi-threading is disabled in Gurobi.

5.1 Data

We consider several regions in Europe, each associated with a given airport. E.g., region VIE consists of all the golf courses that are reachable from the airport of Vienna. Golf courses are considered as reachable if their latitude and longitude values are within +/- 3 of latitude and longitude of the respective airport. For region VIE, the distance to the farthest point from the airport is about 215 km.

The list of golf courses has been retrieved from Albrecht Golf Guide (2014). This database contains around 28,000 courses in 98 countries all over the world. For our study we focus on European destinations, leading to a set of around 4400 courses. Also information on par scores, fees, and opening hours has been taken from this website. Airport coordinates were taken from the international airport database (OpenFlights.org 2014). Golf courses for which the par score could not be retrieved were not considered. Those for which the green fee could not be determined, the average green fee across all golf courses in the same region is used.

Travel times and distances were generated using MapPoint (©Microsoft), optimizing with respect to travel time. If the travel time between two locations A and B is lower or equal to 5 h, we assume that B can be reached from A within the same day, i.e., we consider a travel time of one day. If the travel time is more than 5 h, we consider a travel time of 2 days.

Average hotel prizes per region are based on data from The Hotel Price Index (2012). In terms of travel costs, we consider that we need 6 L of petrol per 100 km.

The resulting value is then multiplied by the average petrol cost in the country where a journey is starting. We use data from Fuel prices, Europe List (2014). From this database we retrieve prizes per liter for Super 95 (unloaded 95 RON). Flight costs are determined using the online platform checkfelix (©KAYAK). Prizes refer to return flights in economy class departing from Vienna. We assume booking 3 months in advance and each journey includes a weekend. The same platform is used to retrieve car rental costs. Here we assume booking 2 months in advance for a minimum term of lease of 14 days. All prizes refer to economy class vehicles.

Finally, in order to obtain integer cost values, we multiply all cost values by 100 and consider them integers.

5.2 *Instances*

In order to produce test instances we consider nine different regions, based on the set of airports with international codes BCN, IBZ, KRK, LYS, NCE, OSL, TLL, VIE and ZAG. Considering all possible subsets of airports minus the empty set, we generate 511 different instances. The regions associated to these airports vary in size from 57 to 224 golf courses. For each region, Table 1 provides the following information: the number of golf courses, the return flight costs from VIE, average per night hotel cost, car rental cost per day, and the number of cliques, i.e. sets of incompatible golf courses. Although the model can handle golf courses that are reachable from different airports, we duplicate them in practice for ease of implementation. Furthermore, we assume that the traveling golfer plans to travel for 1 week and arrives on a Saturday.

Figure 1 shows some of the golf courses located around the city of Vienna, Austria, which represents the largest region.

Table 1 Characteristics of considered regions

Region	Nbr of golf courses	Flight cost	Avg. hotel cost	Car rental cost	$\|B\|$
BCN	109	114.0	106.27	8.0	13
IBZ	61	155.0	106.27	7.0	9
KRK	57	141.0	78.76	22.0	9
LYS	190	203.0	128.77	15.0	21
NCE	159	132.0	128.77	16.0	20
OSL	211	139.0	148.77	43.0	18
TLL	110	159.0	85.01	19.0	21
VIE	224	0.0	116.27	22.0	32
ZAG	183	158.0	146.27	10.0	34

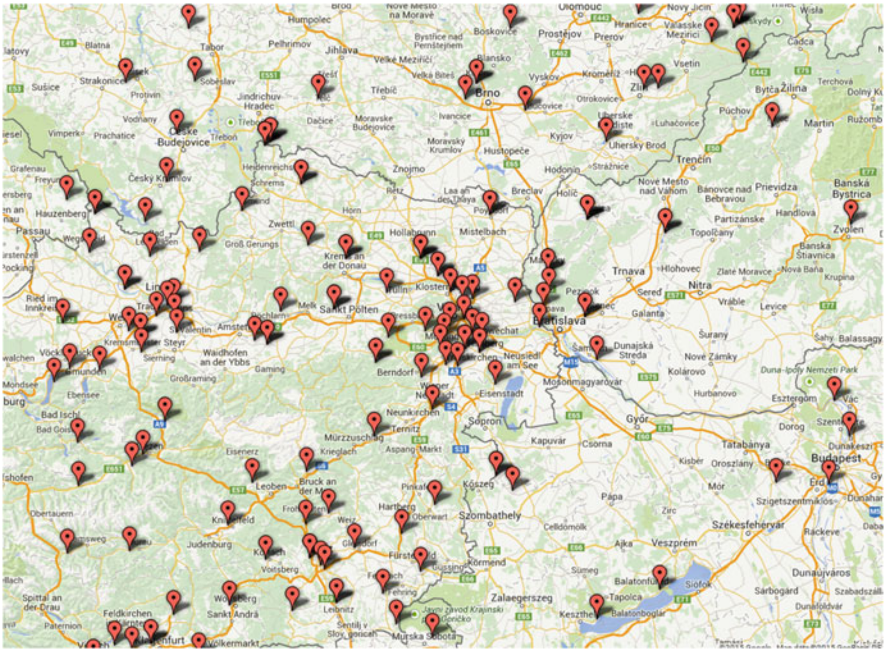

Fig. 1 Golf courses located around the city of Vienna, Austria. Picture created with the Fusion Tables web app (©Google): http://tables.googlelabs.com

5.3 A Note on the ϵ-Constraint

As mentioned in Sect. 4.2, the ϵ-constraint framework is faster when it enumerates Pareto-optimal solutions from the most profitable one to the cheapest one (ϵ_1) than when going the other direction, from cheapest to most profitable (ϵ_2). However, this also appears to require more memory. In our initial set of experiments, we had set a limit of 3 GB of RAM. This was sufficient for BIOBAB and for ϵ_2, but in the case of ϵ_1 one instance could not be solved (when considering regions BCN, KRK, LYS, OSL, TLL and ZAG). This instance could then be solved with 8 GB, which is how we produced the result for that instance.

5.4 Results

Given that all the methods provide the same result for any given instance, which is the Pareto front for that instance, we compare CPU times. Since the number of instances is high, we provide synthetic views of these results in box plots. First, we compare the BIOBAB and the ϵ-constraint framework in Fig. 2. Each box corresponds to a certain method for a certain instance size, and represents the

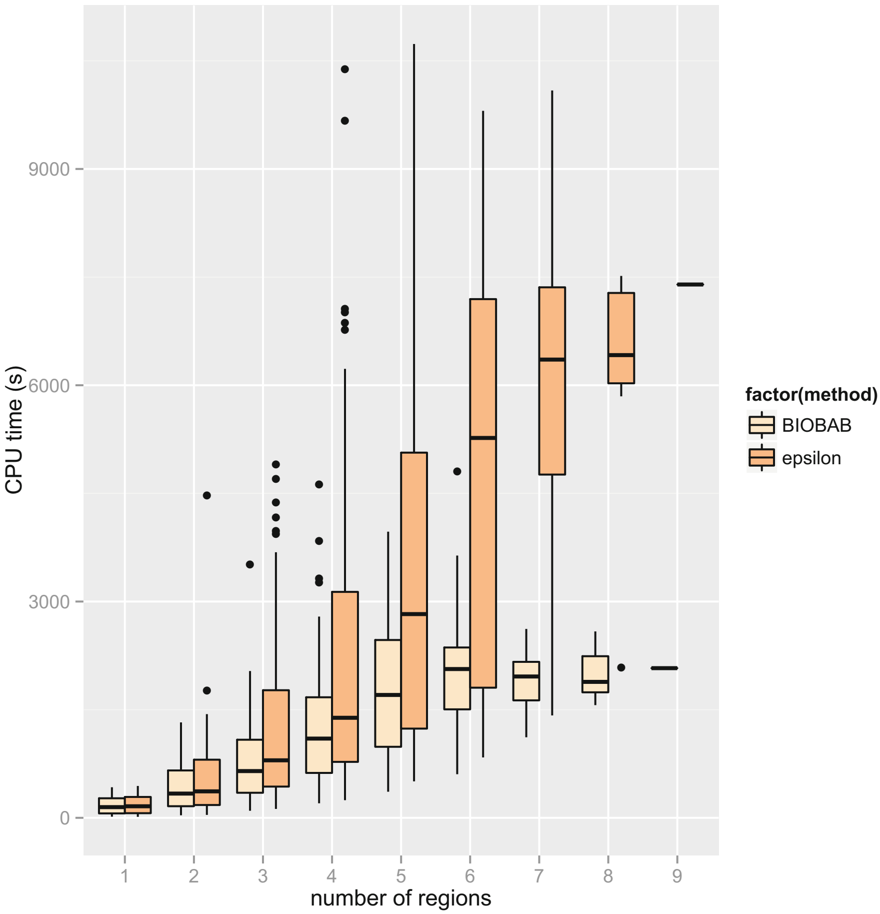

Fig. 2 Compared CPU time of BIOBAB and ϵ-constraint framework, for each instance size

data on all instances of that size for that method. Boxes of the two methods are grouped together for each instance size for easy comparison: the first two boxes from the left are for instances with one region, the next two for instances with two regions, and so on. The BIOBAB is consistently better than the ϵ-constraint. Although the differences do not appear to be very large for small instances, they become significant starting with three regions.

We now compare the respective aggregated versions of the BIOBAB and ϵ-constraint framework, based on the decomposition explained in Sect. 4.3. This decomposition provides results that are overall much faster than the non-aggregated versions. We present these results in a similar box plot, in Fig. 3. Again, the BIOBAB is consistently better than the ϵ-constraint, although not by much this time. This is due to the fact that for one-region instances both methods have relatively close performance.

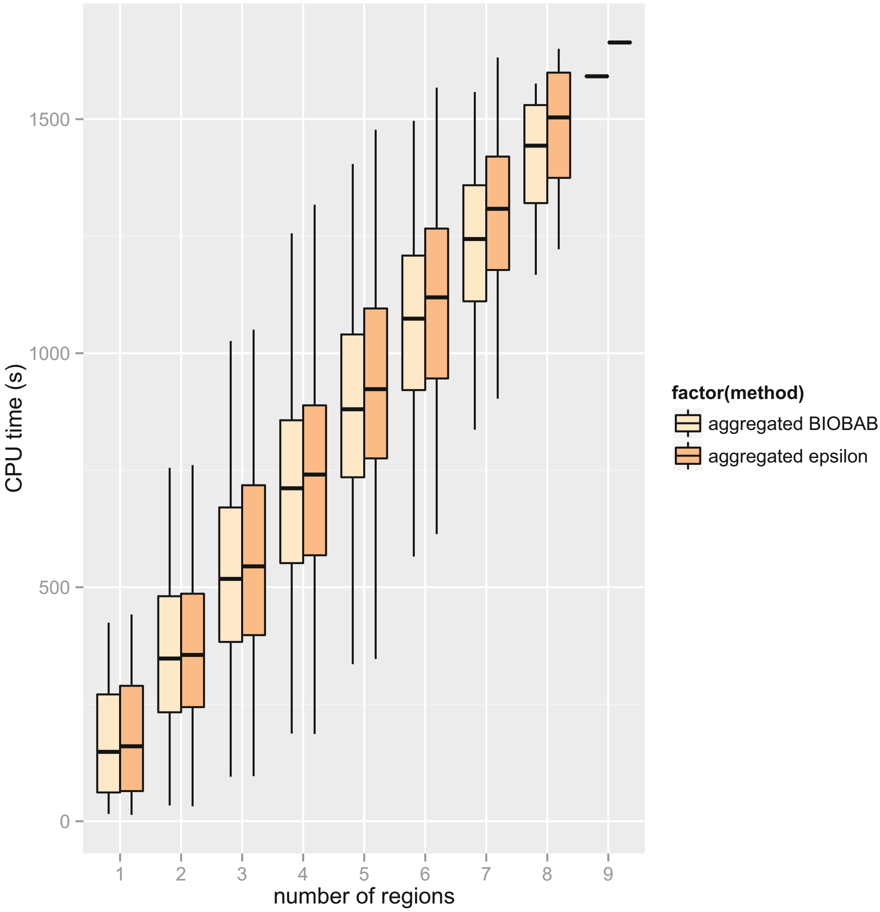

Fig. 3 Compared CPU time of aggregated BIOBAB and aggregated ϵ-constraint framework, for each instance size

Finally, we provide a summary of the comparative performance of ϵ-constraint versus BIOBAB as well as the performance of each base method versus its aggregated version. For that purpose, we compute relative gaps between methods. Considering two methods t_1 and t_2 associating a CPU time to any instance x, the gap is defined as $gap(t_1(x), t_2(x)) = (t_1(x) - t_2(x))/t_2(x)$. A strictly negative gap means that $t_1(x) < t_2(x)$, i.e. that t_1 is faster than t_2 on instance x. We now look at how often a method is better than another method, as well as how often a method is at least 30 and 50 % better than another. Table 2 presents these results over all 511 test instances. From this table it is again clear that the BIOBAB is better than the ϵ-constraint framework: it is faster in 462 cases out of 511, at least 30 % faster in 300 out of 511 cases. It is also clear that the aggregated versions are better than their

Table 2 Summary of pairwise CPU time comparisons between different methods, over 511 instances

	gap(ϵ, B)	gap(ϵ, agg. ϵ)	gap(B, agg. B)
gap < 0	49	41	75
gap $\geqslant 0$	462	470	436
gap $\geqslant 30\,\%$	300	390	346
gap $\geqslant 50\,\%$	248	352	274

B denotes the BIOBAB, ϵ denotes the ϵ-constraint framework, agg. ϵ denotes the aggregated ϵ-constraint framework and agg. B denotes the aggregated BIOBAB

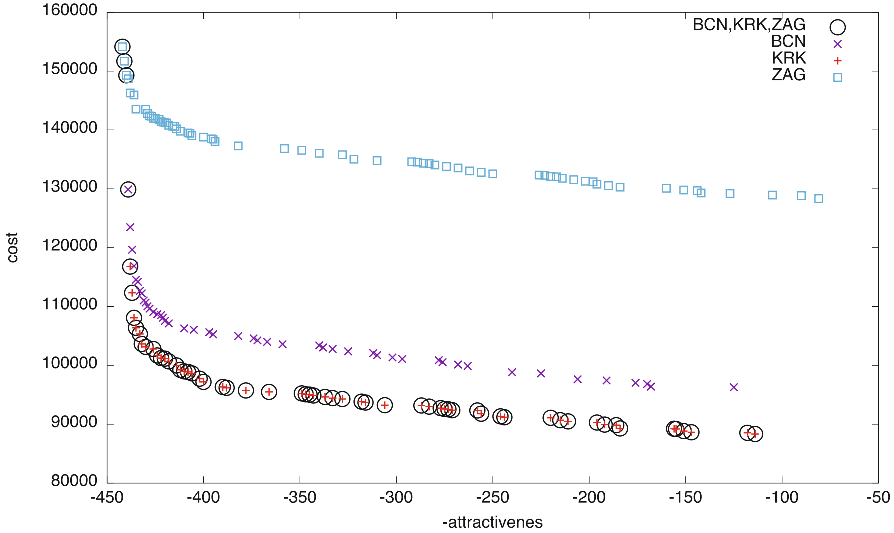

Fig. 4 Contribution of the Pareto front of separate regions to the Pareto front of the instance consisting of the union of these regions

basic counterpart, as they are faster most of the time and at least 50 % faster more than half of the time.

Finally, in order to illustrate how the aggregation works, we provide the Pareto front for an example considering three regions. Figure 4 shows on the same diagram the individual Pareto fronts of three different regions, as well as the Pareto front of the instance consisting of the union of these three regions. In this example, most solutions on the union instance come from the same region (KRK). However, we can also see that there are Pareto-optimal solutions from each other region.

6 Conclusion and Future Work

In this paper, we propose the golf tourist problem, in which the objectives of travel cost and attractiveness are concurrently optimized. We adapt two existing exact frameworks of the literature to this new problem and devise a straightforward decomposition approach that we name aggregated approach. When applied to several real-world instances with up to nine regions and between 57 and 224 golf courses per region, the aggregated approach, using the bi-objective branch-and-bound algorithm to solve the individual subproblems, proves to be the most efficient method.

The aggregated approach, given that the Pareto frontiers of each individual region can be precomputed offline, would lend itself well to the implementation of a mobile or online app for golf tourists flying from Vienna, producing trade-off solutions in real time: for each new selection of regions, the only task is to compute the non-dominated union of all respective Pareto frontiers, which can be done very quickly. This is assuming that available approximated cost values can be used. In a more sophisticated version, where the user specifies concrete dates for their travel plans, the correct data should be retrieved each time anew and precomputing region-wise Pareto frontiers is no longer possible. If such an implementation is envisaged, efficient heuristic algorithms have to be devised that are able to produce good approximations of the optimal Pareto set quickly.

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Twenty Years of Vehicle Routing in Vienna

The Many Challenges of the VRP Solved by Richard F. Hartl

Karl F. Doerner, Alexander Kiefer, and David Wolfinger

Abstract The vehicle routing problem was formulated more than 50 years ago and has attracted great attention since then, not least due to its high practical relevance and its computational complexity. Throughout the years, various generalizations and solution techniques were proposed. The purpose of this survey is to describe the developments in this particular field. Starting with a basic model, several generalizations to the classical vehicle routing problem are explained by gradually extending the initial model. A special focus lies on the contributions to this field of study by Richard F. Hartl and his colleagues at the University of Vienna, particularly with regard to developed solution methods.

1 Introduction

The Capacitated Vehicle Routing Problem (CVRP), initially called *Truck Dispatching Problem*, was introduced by Dantzig and Ramser (1959) as a generalization of the Travelling Salesman Problem (TSP). In the TSP, a salesperson has to visit a set of nodes, where each node has to be visited exactly once, such that the total distance travelled is minimized. Dantzig and Ramser (1959) proposed a generalization in a way, that a fleet of vehicles is considered and the trucks face capacity restrictions. Furthermore, the total demand of all delivery nodes exceeds the capacity of a single truck, since otherwise the problem would be equivalent to a regular TSP. The task consists of assigning delivery nodes to vehicles, such that the demand of each node

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is satisfied and capacity restrictions hold. Still, the objective is to minimize the total distance covered.

The CVRP and its generalizations are particularly interesting because of their practical relevance. Moreover, finding high quality solutions is a challenging task, due to the inherent computational complexity of Vehicle Routing Problems (VRP). The TSP is NP-hard as demonstrated by Karp (1972). More precisely, Karp (1972) showed the NP-completeness for the problem of finding a Hamiltonian Cycle which can be reduced to the TSP. In turn, the CVRP is NP-hard as a generalization of the TSP, and consequently all generalizations of the CVRP are NP-hard too.

Over the years, various generalizations of the CVRP were proposed and discussed in the literature. Furthermore, several surveys and books about VRP have been published, including the books by Toth and Vigo (2001b, 2014) and Golden et al. (2008). Recently, Kritzinger et al. (2015a,b) surveyed solution methods for VRP. On the occasion of the 50th anniversary of the CVRP, Laporte (2009) wrote a survey particularly about the development of exact and heuristic solution methods for the CVRP. Other surveys, that deal with particular aspects or generalizations of the CVRP, will be mentioned in the respective sections.

In our review, we want to highlight advances in the field of VRP with a focus on publications which originated in the past 20 years at the University of Vienna. Figure 1 gives an overview of the number of papers with Richard F. Hartl as a co-author that originated within the field of VRP. The horizontal axis refers to the

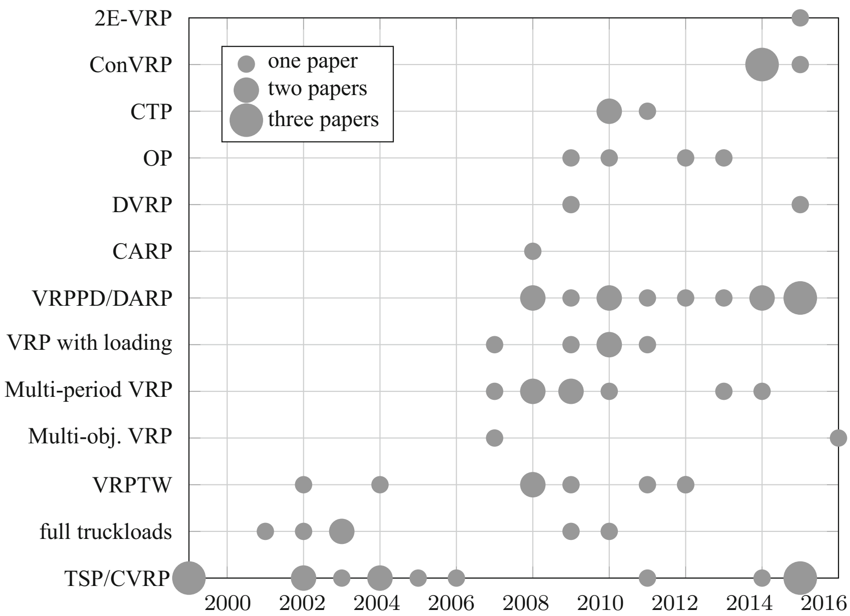

Fig. 1 Richard F. Hartl's Contribution to VRP

years and the vertical axis indicates the problem. Hence, the figure illustrates the prevailing routing topics over the years and gives an impression of the considerable work of Richard F. Hartl in VRP. In case the attribution of a paper to a particular problem was ambiguous, the apparent main feature was decisive. The acronyms along the vertical axis will be introduced and explained later in the respective subsections. A list of acronyms can be found at the end of this review.

The survey is structured in a way, that starting from the CVRP, various extensions are presented in subsequent sections. Again, as many analyzed problems comprise multiple features, we decided to categorize the corresponding papers regarding their most significant feature. In Sect. 2 a model for the CVRP is presented and some minor extensions are discussed. The basic model is then extended in Sect. 3, where several generalizations are considered in its subsections, including the Vehicle Routing Problem with Time Windows (VRPTW) and the Dial-A-Ride Problem (DARP). In Sect. 4 other routing problems that are no extensions of the CVRP are briefly introduced. Section 5 concludes and gives an outlook of possible future research directions. Due to the variety of solution concepts employed by Richard F. Hartl and co-authors references to papers where the respective concept is described in further detail are given in brackets whenever a concept is mentioned for the first time. Furthermore, since many tackled routing problems at the University of Vienna originated from projects with the industry, the studied issues are sometimes only loosely connected and therefore it could not be avoided that some paragraphs have a more enumerative character.

2 Basic Variants

In this section a Mixed Integer Programming (MIP) model for the CVRP is presented. Developments regarding solution techniques and minor extensions to the CVRP are then discussed within subsections. The problems dealt with in this section include the CVRP, VRP with loading constraints, and VRP with multiple objectives. Moreover, multiple depots, split deliveries, and different distribution levels are briefly mentioned.

2.1 CVRP

2.1.1 Model

In the CVRP, truck routes have to be defined, such that each node (except for the depot) is visited exactly once, demands of the customers are satisfied, capacity constraints hold and the total travelled distance is minimized. The model for the CVRP is based on the DARP formulation by Cordeau (2006) because this basic model will be modified in the upcoming sections, with the DARP as one of its most comprehensive extensions. The problem is modelled on a complete directed graph. The notation described in Table 1 will be used throughout this paper. The

Table 1 Notation

n	Number of customer nodes
$\hat{n}$	Number of pickup nodes
$\tilde{n}$	Number of delivery nodes, $\hat{n} + \tilde{n} = n$
	in case of paired pickup and deliveries: $\hat{n} = \tilde{n}$
N	Set of all nodes, including start depot 0 and end depot $n + 1$
P	Set of pickup nodes
D	Set of delivery nodes, $N = P \cup D \cup \{0, n + 1\}$
K	Set of vehicles
c_{ij}^k	Route cost on arc $(i, j) \in N \times N$ of vehicle $k \in K$
t_{ij}^k	Travel time on arc $(i, j) \in N \times N$ of vehicle $k \in K$
q_i	Demand of node i, $q_i < 0, i \in D,\ q_0 = q_{n+1} = 0$
	Supply of node i, $q_i > 0, i \in P$
C_k	Capacity of vehicle $k \in K$
d_i	Service duration at node i, $d_i \geq 0, i \in N$, $d_0 = d_{n+1} = 0$
T_k	Maximum route duration of vehicle $k \in K$
e_i	Earliest start time of service at node $i \in N$
l_i	Latest start time of service at node $i \in N$
L	Maximum ride time of passengers

Table 2 Decision variables

x_{ij}^k	$= \begin{cases} 1 & \text{if arc } (i,j) \in N \times N \text{ is traversed by vehicle } k \in K \\ 0 & \text{otherwise} \end{cases}$
Q_i^k	Load of vehicle $k \in K$ leaving node $i \in N$
B_i^k	Time of vehicle $k \in K$ beginning service at node $i \in N$

light-shaded cells refer to components of the model that will be used in Sect. 3.1 (VRPTW) and onwards, while the darker shaded cells refer to notation used only in Sect. 3.2 (DARP).

Typically, the solution is represented by binary decision variables x_{ij}^k, indicating whether arc (i, j) is traversed in the tour of vehicle k or not. We use the vehicle index due to its simplicity and expandability, however, there exist formulations for the CVRP without a vehicle index, e.g., the model described by Toth and Vigo (2001a).

As capacity considerations have to be taken into account, another decision variable Q_i^k is needed to represent the load of vehicle k leaving node i. In case time windows have to be modelled, the decision variable B_i^k is used to define the starting time of the service of vehicle k at customer i. The decision variables are summarized in Table 2.

Using the above notation, the objective function and the constraints are formulated as follows.

$$\min \sum_{k \in K} \sum_{i \in N} \sum_{j \in N} c_{ij}^k x_{ij}^k \tag{1}$$

s.t.

$$\sum_{k \in K} \sum_{j \in N} x_{ij}^k = 1 \quad \forall\, i \in N \setminus \{0, n+1\} \tag{2}$$

$$\sum_{j \in N} x_{0j}^k = 1 \quad \forall\, k \in K \tag{3}$$

$$\sum_{i \in N} x_{i,n+1}^k = 1 \quad \forall\, k \in K \tag{4}$$

$$\sum_{j \in N} x_{ij}^k = \sum_{j \in N} x_{ji}^k \quad \forall\, i \in N \setminus \{0, n+1\}\ \forall\, k \in K \tag{5}$$

$$Q_j^k \leq Q_i^k + q_j + M \cdot (1 - x_{ij}^k) \quad \forall\, i, j \in N,\ k \in K \tag{6}$$

$$\max\{0, q_i\} \leq Q_i^k \leq \min\{C_k, C_k + q_i\} \quad \forall\, i \in N,\ k \in K \tag{7}$$

$$x_{ij}^k \in \{0, 1\} \quad \forall\, i, j \in N,\ k \in K \tag{8}$$

The objective function (1) seeks to minimize the total routing costs referring to the travelled distance. Constraints (2) ensure that each customer is visited exactly once. Constraints (3) guarantee that each vehicle leaves the depot and finally, each vehicle has to return to the depot, due to Constraints (4). For each vehicle and each customer node, the inflow and outflow is balanced by Constraints (5).

Constraints (6) ensure that whenever a vehicle traverses a link (i, j), corresponding to customer j being served immediately after customer i, the load of vehicle k is reduced by the demand of customer j, as q_j is negative for all delivery nodes, where M represents a sufficiently large number. Finally, Constraints (7), ensure that the capacity restrictions are satisfied. Moreover, in case of solely delivery nodes in the network, Constraints (6) and (7) prohibit subtours. If a subtour occurred, there would be no feasible value for the decision variables representing the load of the vehicle, since the load variables are continually reduced along a subtour.

Several other possibilities for prohibiting subtours exist in the literature. For a model without a vehicle index for the binary decision variable x_{ij}^k some variants are mentioned by Toth and Vigo (2001a). In particular, whenever one has to incorporate time-related restrictions such as time windows or a maximum tour duration, one has to keep track of the time a vehicle visits a node, as explained in Sect. 3.1 in more detail. Thus, each node-vehicle pair has a decision variable associated representing the visiting time. Along a tour, these variables are gradually increased. As a side effect, subtours are avoided. Furthermore, for each vehicle, the constraints that prohibit overloading would simplify to the following inequalities.

$$-\sum_{i \in N} \sum_{j \in N} \sum_{k \in K} \left(x_{ij}^k \cdot q_i\right) \leq C_k \quad \forall\, k \in K \tag{9}$$

2.1.2 Contribution of Richard F. Hartl and His Team

VRP and particularly their solution approaches have been extensively studied in the literature. Due to their complexity, metaheuristics are viable approaches for solving these problems. In the context of the TSP, Bullnheimer et al. (1999c) apply the standard Ant System (AS) (Dorigo et al. 1996) approach and extend it by an elitist strategy and a ranking strategy. In Bullnheimer et al. (1999a,b) the authors apply a modified version of their AS approach on the CVRP. Doerner et al. (2002) and Reimann et al. (2002b) further develop the AS approach of Bullnheimer et al. (1999b) by substituting the nearest neighbor algorithm as a tour construction mechanism with the savings algorithm of Clarke and Wright (1964). They call their new approach the Savings-based Ant System (SbAS) and apply it to the CVRP.

The unified AS, proposed by Reimann et al. (2003), is capable of solving several VRP, including the CVRP, the VRPTW, the Vehicle Routing Problem with Backhauls (VRPB) and the Vehicle Routing Problem with Backhauls and Time Windows (VRPBTW). In the VRPB, in addition to delivery customers (linehaul), also pickup customers (backhaul), who send goods to the depot, have to be serviced. While the SbAS are further developed in Reimann et al. (2004) by incorporating a decomposition approach, called Decomposition Ants (D-Ants), Doerner et al. (2004) analyze the parallelization of three different Ant Colony Optimization (ACO) (Dorigo and Stützle 2010) paradigms (rank based AS, max-min AS and Ant Colony System (ACS) Dorigo and Gambardella 1997), which leads to parallelized versions of the SbAS and D-Ants approaches in Doerner et al. (2005, 2006). In all the above mentioned papers the respective solution method is tested on standard benchmark instances from the literature (e.g., Christofides et al. 1979, compared with state-of-the-art heuristics at that time, and able to provide competitive results. Gussmagg-Pfliegl et al. (2011) formulate the mail delivery problem as a CVRP and solve it with a two-stage solution approach, where first a number of promising routes is generated heuristically (in a cluster-first route-second manner), and then a set covering problem is solved to optimality.

Kritzinger et al. (2014) propose a generic Variable Neighborhood Search (VNS) (Mladenović and Hansen 1997; Hansen et al. 2010) algorithm to tackle various fixed fleet VRP, including the CVRP, the Open Vehicle Routing Problem (OVRP), the VRPTW, the Open Vehicle Routing Problem with Time Windows (OVRPTW), and the Time-Dependent Vehicle Routing Problem with Time Windows (TDVRPTW). The OVRP refers to a routing problem, where vehicles do not necessarily have to return to the depot, while in the case of a TDVRPTW, the travel time between two locations does not only depend on the distance but also on the time of the day. The authors apply their approach to several benchmark instance sets from the literature.

Table 3 provides a short summary. In column *Benchmark* the reference for the employed benchmark instances is given, whereas Christofides, Golden, Solomon, Jacobs-Blecha, Gélinas, Fisher, Li, Ichoua, and Balseiro refer to the instances proposed by Christofides et al. (1979), Golden et al. (1998), Solomon (1987), Jacobs-Blecha and Goetschalckx (1992), Gélinas et al. (1995), Fisher (1994), Li et al. (2007), Ichoua et al. (2003), and Balseiro et al. (2011), respectively. The

Table 3 Richard F. Hartl's contribution to the CVRP

Reference	Problem	Approach	Benchmark
Bullnheimer et al. (1999a)	CVRP	AS	Christofides
Bullnheimer et al. (1999b)	CVRP	AS	Christofides
Doerner et al. (2002)	CVRP	SbAS	Christofides
Reimann et al. (2002b)	CVRP	SbAS	Christofides
Reimann et al. (2003)	CVRP, VRPTW, VRPB, VRPBTW	unified AS	Christofides, Solomon, Jacobs-Blecha, Gélinas
Reimann et al. (2004)	CVRP	D-Ants	Christofides, Golden
Doerner et al. (2004)	CVRP	Ranking AS, ACS, max-min AS	Christofides
Doerner et al. (2005)	CVRP	D-Ants	Christofides
Doerner et al. (2006)	CVRP	SbAS, D-Ants	Christofides, Golden
Gussmagg-Pfliegl et al. (2011)	mail delivery as CVRP	2-phase: heuristic, set covering	real-world
Kritzinger et al. (2014)	CVRP, OVRP, VRPTW, OVRPTW, TDVRPTW	VNS	Golden, Fisher, Christofides, Li, Solomon, Ichoua, Balseiro

instances by Ichoua et al. (2003) and Balseiro et al. (2011) for the TDVRPTW are themselves based on those by Solomon (1987).

Basic extensions to the CVRP include the Multi-Depot Vehicle Routing Problem (MDVRP) and the Split Delivery Vehicle Routing Problem (SDVRP). Breunig et al. (2015) analyze the Two-Echelon Vehicle Routing Problem (2E-VRP), where two levels in the distribution process are considered. Initially, trucks deliver the goods to intermediate facilities, where the goods are consolidated and transshipped to smaller city freighters which deliver them to the customers. Thus, the first level corresponds to a SDVRP, while the latter refers to a MDVRP. The authors solve the problem by means of Large Neighborhood Search (LNS) (Shaw 1998; Pisinger and Ropke 2010), based on iterative destructions and subsequent reparations of the incumbent solution.

2.2 *VRP with Loading Constraints*

CVRP plus packing combines two fundamental problems in distribution logistics, namely the loading of freight into vehicles (loading problem) and the subsequent routing of vehicles (CVRP) to satisfy the demand of each customer. In the Two-Dimensional Loading Capacitated Vehicle Routing Problem (2L-CVRP), proposed by Iori (2004), vehicles have a single rectangular loading surface and a maximum

Table 4 Richard F. Hartl's contribution to VRP with loading constraints

Reference	Problem	Approach	Benchmark
Doerner et al. (2007)	MP-VRP	ACO, TS	Doerner et al. (2007)
Fuellerer et al. (2009)	2L-CVRP	ACO	Iori (2004)
Fuellerer et al. (2010)	3L-CVRP	ACO	Gendreau et al. (2006)
Strodl et al. (2010)	2L-CVRP	VNS	Iori (2004)
Tricoire et al. (2011)	MP-VRP	B&C, VNS	Doerner et al. (2007)

weight capacity. Customers demand sets of rectangular items with a given size and weight. The aim is to find routes of minimum total cost, such that the maximum weight capacity is not exceeded and a feasible two-dimensional allocation of the items into the rectangular loading surface exists for each vehicle. The 2L-CVRP generalizes the CVRP, and is itself generalized by the Three-Dimensional Loading Capacitated Vehicle Routing Problem (3L-CVRP), introduced by Gendreau et al. (2006). In the 3L-CVRP both the items and the loading space of the vehicles have a three-dimensional rectangular shape.

Fuellerer et al. (2009, 2010) develop an ACO based algorithm and successfully apply it to the 2L-CVRP and 3L-CVRP, respectively, while Strodl et al. (2010) apply a VNS with various indexing mechanisms to the 2L-CVRP. All three works test their solution method on benchmark instances from the literature described in Iori (2004) and Gendreau et al. (2006) for the 2L-CVRP and 3L-CVRP, respectively.

Another CVRP with loading constraints, the so called Multi-Pile Vehicle Routing Problem (MP-VRP), was introduced by Doerner et al. (2007). Like in the 3L-CVRP, each vehicle has a three-dimensional loading space. However, in this problem, the length of the loading space is divided into a fixed number of piles (or stacks) of equal size, which are parallel and right next to each other. Customers demand sets of items where the width always equals the width of a pile but the length and height differ. The objective is to feasibly load the items into the vehicles and deliver them to the customers with minimum total cost. While Doerner et al. (2007) solve the MP-VRP with both an ACO and a Tabu Search (TS) algorithm (Glover 1986; Gendreau and Potvin 2010), Tricoire et al. (2011) use a Branch and Cut (B&C) algorithm, as well as a VNS algorithm. Both test their solution methods on randomly-generated and real-world test instances. Richard F. Hartl's contribution to VRP with loading constraints is summarized in Table 4.

2.3 Multi-Objective VRP

In multi-objective optimization more than one objective is optimized simultaneously. A typical bi-objective variant of the CVRP is the Capacitated Vehicle Routing Problem with Route Balancing (CVRPRB), whereby route balancing refers to balancing the length of tours in a solution for the CVRP. This means that in addition to the classic objective of minimizing the sum of the total distance travelled by each vehicle, the CVRPRB also seeks to minimize the difference between the longest and the shortest vehicle tours. The CVRPRB was introduced by Jozefowiez et al. (2002). Pasia et al. (2007) propose a population-based local search for solving the CVRPRB and perform a computational study on several of the CVRP instances of Christofides et al. (1979).

In the context of home care services, Braekers et al. (2016) analyze the trade-off between operating costs and client satisfaction in the so-called bi-objective home care routing and scheduling problem. The problem consists of generating routes and schedules for nurses, such that the nurses can perform the demanded tasks at the patients' homes. In particular, the problem takes the qualifications of the nurses, work regulations, time windows, client preferences, and different cost factors into account. Braekers et al. (2016) propose a LNS embedded in a multi-directional local search framework (Tricoire 2012) as a solution method.

Other authors also deal with multi-objective routing problems, including Schilde et al. (2009), Parragh et al. (2009), Nolz et al. (2010a,b, 2011), and Kovacs et al. (2015). However, their analyzed problems incorporate various features and are thus explained in more detail in the respective section with the most evident connection.

3 Generalizations

This section deals with generalizations to the CVRP discussed in Sect. 2. As many problems analyzed in the literature are often motivated by real-world situations, they typically incorporate various features, including a heterogeneous fleet (e.g., the capacity or travel speed of vehicles differs), multiple depots, and backhauls. The more substantial generalizations are discussed in the following subsections, including VRP with time windows, pickup and delivery problems, and multi-period VRP.

3.1 Time Windows

The VRPTW considers that customers may only be served within a certain time interval. In general, one can distinguish between soft, semi-hard, and hard time window constraints. The first penalizes deviations from the allowed service time window in the objective function, while in the last case the vehicle has to wait at a customer if it arrives early and arriving too late is prohibited. In case of semi-hard time windows, only one end point of the interval is hard, while the other one is soft. In this section we focus on a model with hard time window constraints. Additionally, the total tour length of a vehicle might be bounded, possibly due to work regulations of the drivers.

Early articles that consider time windows within vehicle scheduling are case studies by Pullen and Webb (1967) and Knight and Hofer (1968). For a review of the VRPTW with a focus on heuristic solution methods we refer to the surveys by Bräysy and Gendreau (2005a,b), where also references to earlier surveys in this field are given. A recent discussion about the VRPTW can be found in Desaulniers et al. (2014).

3.1.1 Model Extensions

$$B_j^k \geq B_i^k + d_i + t_{ij} - M \cdot \left(1 - x_{ij}^k\right) \quad \forall\, i, j \in N, k \in K \tag{10}$$

$$B_{n+1}^k - B_0^k \leq T_k \quad \forall\, k \in K \tag{11}$$

$$e_i \leq B_i^k \leq l_i \quad \forall\, i \in N, k \in K \tag{12}$$

Time windows are incorporated in the model in a similar way as capacity restrictions. Constraints (10) state that if customer j succeeds customer i on the tour of vehicle k, its service has to start after the start of the service of customer i plus the service duration and the time required for travelling from customer i to customer j. The maximum tour length of vehicle k is restricted by Constraints (11). Finally, the start of the service at a customer has to be within its specified time interval, guaranteed by Constraints (12). As the service time is increased from customer to customer along a tour and each customers service time has to be unique, subtours are prohibited by Constraints (10).

3.1.2 Contribution of Richard F. Hartl and His Team

Reimann et al. (2002a) and Reimann and Ulrich (2006) analyze the VRPBTW where both linehaul and backhaul customers are constrained by time windows. Both solve this problem with an ACO approach and test it on the benchmark instances developed by Gélinas et al. (1995). In Polacek et al. (2004) the authors solve the Multi-Depot Vehicle Routing Problem with Time Windows (MDVRPTW) with a VNS approach, whereby they are the first to ever apply a VNS to this problem. Polacek et al. (2008a) extend and parallelize this VNS approach. In both cases the solution method is tested on the benchmark instance described in Cordeau et al. (2001). Ostertag et al. (2008, 2009) solve the MDVRPTW as well. They propose a heuristic approach based on the POPMUSIC framework, where a VNS and a Memetic Algorithm (MA) (Moscato and Cotta 2010) serve as optimizers, respectively, and test it on real-world test instances.

In Kritzinger et al. (2011, 2012) a TDVRPTW is discussed. In contrast to the previously mentioned authors, Kritzinger et al. (2011, 2012) consider soft time windows. They use a VNS approach to solve the problem and test their solution method on the benchmark instances of Solomon (1987). The VRPTW might be

Table 5 Richard F. Hartl's contribution to the VRPTW

Reference	Problem	Approach	Benchmark
Reimann et al. (2002a)	VRPBTW	ACO	Gélinas et al. (1995)
Polacek et al. (2004)	MDVRPTW	VNS	Cordeau et al. (2001)
Polacek et al. (2008a)	MDVRPTW	VNS	Cordeau et al. (2001)
Ostertag et al. (2008)	MDVRPTW	POPMUSIC with VNS	Real-world
Ostertag et al. (2009)	MDVRPTW	POPMUSIC with MA	Real-world
Kritzinger et al. (2011)	TDVRPTW	VNS	Solomon (1987)
Kritzinger et al. (2012)	TDVRPTW	VNS	Solomon (1987)
Hiermann et al. (2014)	E-FSMVRPTW	ALNS	New, Schneider et al. (2014), Liu and Shen (1999)

extended by considering multiple time windows, as done by Tricoire et al. (2010) in the context of the Orienteering Problem (OP), described in Sect. 4.2.

The Electric Fleet Size and Mix Vehicle Routing Problem with Time Windows and Recharging Stations (E-FSMVRPTW) was introduced by Hiermann et al. (2014). It generalizes the VRPTW by the consideration of electric vehicles of different types that may be recharged on the tour using recharging stations and decisions about the fleet composition. As a solution approach, the authors propose an Adaptive Large Neighborhood Search (ALNS) (Ropke and Pisinger 2006) extended by a local search and a labelling procedure. They apply their approach to the benchmark instances by Liu and Shen (1999) and Schneider et al. (2014), which are based on the instances by Solomon (1987), and the vehicle type descriptions by Liu and Shen (1999). Furthermore, the authors test their algorithm on new instances based on those by Schneider et al. (2014). The papers by Richard F. Hartl, where the consideration of time windows is one of the most prominent features, are given in Table 5.

3.2 Pickup and Delivery

Time windows are regularly used in particular within Vehicle Routing Problems with Pickup and Deliveries (VRPPD), where goods or people have to be transferred from a pickup node to a delivery node. In the context of the TSP, Lokin (1979) analyzed precedence constraints for route planning for the first time, which are generally also necessary for VRPPD. For an extensive review about different types of VRPPD, we refer to the surveys by Parragh et al. (2008a,b). Within the class of VRPPD, Parragh et al. (2008b) distinguishes between the classical Pickup and Delivery Problem (PDP) with paired pickup and delivery points, the Pickup and

Delivery Vehicle Routing Problem (PDVRP) with unpaired pickup and delivery nodes, and the DARP. The latter considers the transportation of people, whereby their inconvenience is taken into account, i.e., the riding time of passengers is restricted. A compact survey by Parragh et al. (2010a) about the DARP discusses real-world extensions and solution approaches.

3.2.1 Model Extensions

In this subsection, a model for the DARP based on the one by Cordeau (2006) is presented. Other pickup and delivery problems can be modelled in a similar way, with typically less constraints, though. The extended model of Sect. 3.1 is modified by adding the Constraints (13) to (15) and adjusting the loading constraints (6). As noted by Parragh et al. (2010b), a standardized problem definition for the DARP can hardly be found in the literature, since these problems are often motivated by practical applications and hence incorporate some situation-specific constraints or objectives. The model by Cordeau (2006), contains very elementary features, though.

$$\sum_{j \in N} x_{ij}^k = \sum_{j \in N} x_{\hat{n}+i,j}^k \quad \forall\, i \in P, k \in K \tag{13}$$

$$t_{i,\hat{n}+i} \leq \underbrace{B_{\hat{n}+i}^k - \left(B_i^k + d_i\right)}_{\text{ride time of user } i} \quad \forall\, i \in P, k \in K \tag{14}$$

$$B_{\hat{n}+i}^k - \left(B_i^k + d_i\right) \leq L \quad \forall\, i \in P, k \in K \tag{15}$$

$$Q_j^k \geq Q_i^k + q_j - M \cdot (1 - x_{ij}^k) \quad \forall\, i,j \in N,\ k \in K \tag{16}$$

Constraints (13) refer to the feature of paired pickup and delivery nodes, i.e., if vehicle k picks up customer i, the same vehicle has to visit the corresponding destination node $\hat{n}+i$. The ride time of the customers is bounded by Constraints (14) and (15), where the former guarantee that a pickup node is visited before the respective delivery node. Whenever pickup nodes are considered, i.e., nodes with a strictly positive quantity, the loading constraints of Sect. 2.1 have to be reformulated, as shown by Constraints (16). Each time a pickup node is visited, the vehicle load has to be increased by the picked up quantity. Conversely, whenever a demand node is visited, the vehicle load is decreased. The load of each vehicle is still restricted by Constraints (7) of Sect. 2.1. Note, that in this model the capacity constraints are not prohibiting subtours any more, as the load of the vehicle is not strictly increasing or decreasing along a tour, but the time window constraints still do.

3.2.2 Contribution of Richard F. Hartl and His Team

At the University of Vienna, a substantial part of research related to the DARP was induced by the Austrian Red Cross (ARC). For an analysis of problems in health care logistics, we refer to Doerner and Hartl (2008). These problems include patient transport and (periodic) blood collection and deliveries, in particular.

Motivated by a dispatching problem of the ARC, Parragh et al. (2009) deal with a DARP with two objectives, i.e., minimization of costs and user inconvenience, called Multi-Objective Dial-A-Ride Problem (MO-DARP). Therefore, they propose a two phase solution procedure, consisting of an iterated VNS with an acceptance scheme based on Simulated Annealing (SA) (Kirkpatrick et al. 1983; Nikolaev and Jacobson 2010) and a Path Relinking (PR) method (Glover and Laguna 1997; Resende et al. 2010), in order to generate Pareto efficient solutions. Moreover, the authors identify exact efficient sets by a B&C algorithm embedded in an ϵ-constraint framework for small instances. Their heuristic procedure is tested on the benchmark instances by Cordeau (2006).

Parragh et al. (2010b) apply a VNS with a SA acceptance scheme to a DARP, where solely the covered distance has to be minimized, and test their algorithm on the instance set by Cordeau and Laporte (2003). In addition, the authors solve a modified problem, where most of the hard constraints are relaxed to soft ones, whose penalties are then incorporated into a weighted objective function.

Parragh (2011) considers the Dial-A-Ride Problem with Heterogeneous Users and Vehicles (HDARP) and analyzes waiting times of passengers aboard. For solving this problem, valid inequalities for the DARP are adapted and incorporated in B&C algorithms and the VNS employed by Parragh (2009) is adjusted to the HDARP. The HDARP is extended by Parragh et al. (2012) by driver-related constraints such as lunch breaks and shift lengths, leading to the Heterogeneous Dial-A-Ride Problem with Driver-Related Constraints (HDARPD). As solution methods, they propose a column generation algorithm, a VNS with a SA acceptance scheme, and a collaborative framework comprising both approaches. They apply their solution approaches to real-world instances from the ARC and the benchmark instance set by Cordeau (2006).

Parragh and Schmid (2013) propose a LNS approach, a column generation algorithm with an integrated VNS column generator, and hybridization strategies for the DARP and test their approaches on the instance sets by Cordeau and Laporte (2003) and Cordeau (2006). The same benchmark instances are tackled by Ritzinger et al. (2014) with an exact Dynamic Programming (DP) algorithm, a restricted DP algorithm, and a LNS integrating the DP approaches.

Dynamic extensions of the DARP are discussed by Schilde et al. (2011, 2014). Furthermore, Schilde et al. (2011) incorporate a stochastic aspect, by considering that patients requesting a transport to the hospital, eventually induce a return transport on the same day. The authors solve the Dynamic and Stochastic Dial-A-Ride Problem (DSDARP) with a dynamic VNS, a dynamic Stochastic Variable Neighborhood Search (S-VNS) (Gutjahr et al. 2007), a Multiple Plan Approach (MPA) (Bent and Van Hentenryck 2004), and a Multiple Scenario Approach (MSA)

Table 6 Richard F. Hartl's contribution to the DARP

Reference	Problem	Approach	Benchmark
Parragh et al. (2009)	MO-DARP	2 phase: VNS, PR	Cordeau (2006)
Parragh et al. (2010b)	DARP	VNS	Cordeau and Laporte (2003)
Schilde et al. (2011)	DSDARP	Dynamic VNS, dynamic S-VNS, MPA, MSA	Real-world
Parragh et al. (2012)	HDARPD	VNS, column generation	Adapted Cordeau (2006), Real-world
Ritzinger et al. (2014)	DARP	DP + LNS	Cordeau and Laporte (2003), Cordeau (2006)
Schilde et al. (2014)	Dynamic DARP, stochastic time-dependent speed	Dynamic VNS, dynamic S-VNS, MPA, MSA	Real-world

(Bent and Van Hentenryck 2004), and apply their approaches to instances based on real-world data. Schilde et al. (2014) extend the dynamic DARP by stochastic time-dependent travel speeds, again employ dynamic VNS, dynamic S-VNS, MPA, and MSA, and therewith solve real-world instances. Richard F. Hartl's contribution to the DARP is summarized in Table 6.

The ambulance routing problem discussed by Kiechle et al. (2009) can be interpreted as a DARP with vehicle breakdowns. In particular, they incorporate dynamically occurring emergencies. In this case, the nearest empty vehicle, eventually being on its way to a scheduled regular transport, has to serve the emergency request. Thus, the original problem has to be re-optimized with one vehicle less.

In the context of a pickup and delivery problem with full truckloads and time windows, Gronalt et al. (2003) propose four different savings-based heuristics. Doerner et al. (2001, 2003) and Dawid et al. (2002) consider a full truckload pickup and delivery problem with the objectives of minimizing the sum of the total distance travelled and the fleet size. The authors solve this problem with different variations of ACO.

Schmid et al. (2009, 2010) deal with problems arising in concrete logistics. Their routing problems are characterized by a heterogeneous fleet, full truckloads, and time windows. Schmid et al. (2009) propose a matheuristic solution approach consisting of an integer multi-commodity flow optimization method and a VNS, while Schmid et al. (2010) present two hybrid procedures combining VNS and MIP embedded in Very Large Neighborhood Search (VLNS).

Hartl and Romauch (2013) analyze the single commodity lateral transshipment problem with one vehicle that extends the Pickup and Delivery Travelling Salesman Problem (PDTSP). They propose a LNS heuristic as solution method.

3.3 Periodic VRP

The Periodic Vehicle Routing Problem (PVRP) was first proposed by Beltrami and Bodin (1974) in the context of waste collection. It extends the CVRP by considering several days within a planning period, whereby routes have to be designed for each day. Customers have to be visited on one or several days. Feasible combinations of visits are specified for each customer, e.g., visits are allowed either for the combination Monday–Wednesday or Wednesday–Friday. Moreover, some authors consider the extension, where the availability of some vehicles might depend on the day (e.g., Christofides and Beasley 1984).

3.3.1 Model Extensions

The model of the PVRP presented in this subsection is based on Christofides and Beasley (1984). The additional notation and decision variables are listed in Table 7. Basically, the previous model is extended in a way that the binary decision variable x_{ij}^k gets an additional day index t in order to design the routes for each day independently. Analogously, the decision variable corresponding to the vehicle load Q_i^k and, if needed, the time decision variable B_i^k are adjusted by adding a day index t, as well. New decision variables y_{it} and z_{ci} are required to identify on which days customers are visited. Most of the routing constraints stay essentially the same, however. To avoid confusion in the previous models, we decided to omit the extra notation for the PVRP in Sect. 2.1. Hence, the relevant additional notation and (adapted) decision variables are introduced at this point.

$$\sum_{c \in \mathcal{C}_i} z_{ci} = 1 \quad \forall\, i \in N \setminus \{0, n+1\} \tag{17}$$

Table 7 Additional notation and decision variables

$\mathcal{T}$	Set of days of the considered planning period
K_t	Fleet available on day $t \in \mathcal{T}$
$\mathcal{C}_i$	feasible combinations of service days of customer $i \in N \setminus \{0, n+1\}$
δ_{cit}	$= \begin{cases} 1 & \text{if day } t \in \mathcal{T} \text{ is part of combination } c \in \mathcal{C}_i \text{ of customer } i \\ 0 & \text{otherwise} \end{cases}$
x_{ijt}^k	$= \begin{cases} 1 & \text{if arc } (i,j) \in N \times N \text{ is traversed by vehicle } k \in K \text{ on day } t \in \mathcal{T} \\ 0 & \text{otherwise} \end{cases}$
y_{it}	$= \begin{cases} 1 & \text{if customer } i \in N \setminus \{0, n+1\} \text{ is visited on day } t \in \mathcal{T} \\ 0 & \text{otherwise} \end{cases}$
z_{ci}	$= \begin{cases} 1 & \text{if customer } i \text{ is visited on the days specified by combination } c \in \mathcal{C}_i \\ 0 & \text{otherwise} \end{cases}$

$$y_{it} = \sum_{c \in \mathcal{C}_i} z_{ci} \cdot \delta_{cit} \quad \forall\, i \in N \setminus \{0, n+1\}\ t \in \mathcal{T} \tag{18}$$

$$\sum_{k \in K_t} \sum_{i \in N} x_{ijt}^k = y_{jt} \quad \forall\, j \in N \setminus \{0, n+1\}, t \in \mathcal{T} \tag{19}$$

$$z_{ci} \in \{0, 1\} \quad \forall\, c \in \mathcal{C}_i, i \in N \tag{20}$$

Constraints (17) ensure that for each customer exactly one combination of service days is selected. Customer i is visited on day t if and only if that day is part of the chosen service combination, as specified by Constraints (18). Due to Constraints (19), an arc directing towards customer i has to be traversed if and only if that customer has to be served on that day. Note, that the binary property of variable y_{it} is granted by a combination of the other constraints and is thus not explicitly stated in the model.

The remaining parts of the model basically correspond to the routing problem of each day and coincide with those of Sect. 2.1 and, in case of time windows, Sect. 3.1, except for the following minor adjustments. First, the decision variables and the fleet set K have to be extended by the day index t. Second, in the objective function (1) one has to sum over all days. Third, the constraints have to be modified in a way that they have to hold for each day t. Finally, Constraints (2) may be omitted due to the PVRP-specific constraints.

3.3.2 Contribution of Richard F. Hartl and His Team

In Polacek et al. (2007) the authors consider the one-vehicle variant of the PVRP (i.e., the Periodic Travelling Salesman Problem (PTSP)) and solve it with a VNS approach, which they apply to real-world instances. Hemmelmayr et al. (2009a) solve the classical PVRP with a VNS approach, and are the first ones to do so. They test their solution method on both the *old* and the *new* data set from the literature. With regard to the old instance set, ten instances were proposed by Christofides and Beasley (1984), which are adapted CVRP instances originally proposed by Eilon et al. (1971). Furthermore, the *old* set includes 1 instance by Russell and Igo (1979), 2 by Russell and Gribbin (1991), and 19 by Chao et al. (1995). The ten *new* instances were proposed by Cordeau et al. (1997).

In Doerner et al. (2008b) the problem faced by the ARC of delivering blood products from a central blood bank to several hospitals is discussed. They formulate the problem as a stochastic 2-day CVRP with known probabilities for urgent and regular delivery requests and develop three ACO based solution methods to solve the problem.

Hemmelmayr et al. (2009b) also deal with the problem of supplying blood products to hospitals, however, formulated as Inventory Routing Problem (IRP). Essentially, the IRP combines routing decisions with inventory control. The IRP and the PVRP have in common, that costumers may be supplied on several days within the planning horizon. However, they differ from each other, as in the case of the IRP, customers consume the product at a particular rate and have a local inventory to store delivered products. For a detailed description of the IRP we refer to Bertazzi et al. (2008). Hemmelmayr et al. (2009b) propose three different solution procedures for their considered IRP, including a VNS algorithm, where the problem is treated as PVRP. This approach is further developed and extended in Hemmelmayr et al. (2010) to cope with the stochastic element of the problem. In both studies, the solution methods are tested on real-world data.

In the context of solid waste collection Hemmelmayr et al. (2013) study a generalized version of the PVRP, i.e., the Periodic Vehicle Routing Problem with Intermediate Facilities (PVRP-IF), for which they present a MIP formulation. In this problem intermediate facilities serve as unloading stations where vehicles can go to when they are full and continue their trip afterwards. The authors develop a hybrid solution method based on VNS and DP to solve the problem and test it on adapted instances described in Crevier et al. (2007). Hemmelmayr et al. (2014) further extend the problem by incorporating bin allocation, i.e., how many bins of which size to place at each service site. Their solution method is a hybrid approach, which uses VNS for the routing part and a MIP approach for the bin allocation part.

A CVRP with multiple periods can be extended by incorporating constraints to guarantee consistency, in the form of arrival-time consistency (i.e., servicing customers always at the same time of the day), driver consistency (i.e., servicing a customer with the same driver), region consistency (assigning each driver to the same service region), or delivery consistency (replenishing each customer's stock at regular intervals with similar delivery quantities). For a recent survey on the so-called Consistent Vehicle Routing Problem (ConVRP) we refer the reader to Kovacs et al. (2014b). Kovacs et al. (2014a) present a Template-Based Adaptive Large Neighborhood Search (TALNS) for the ConVRP. The ConVRP and the PVRP are related in a way that both problems consider several periods. In contrast to the PVRP, however, the ConVRP takes the days with service requests as given.

Kovacs et al. (2014c) study the Generalized Consistent Vehicle Routing Problem (GenConVRP), where the variation in the arrival times is penalized in the objective function and each customer is serviced by a limited number of drivers, as opposed to one in the ConVRP. Here, the authors forego the template based approach and implement a flexible LNS approach instead. The solution approaches by Kovacs et al. (2014a,c) are tested on benchmark instances from the literature and new self-generated ones, which are available on their website.[1]

The GenConVRP is extended by Kovacs et al. (2015) by the Multi-Objective Generalized Consistent Vehicle Routing Problem (MOGenConVRP). The conflicting objectives include time consistency, driver consistency, and routing costs. For large instances, the authors propose a Multi-Directional Large Neighborhood Search (MDLNS), combining LNS with multi directional local search. Small instances are solved by two exact approaches, i.e., the Two Dimensional Adaptive ϵ-Constraint

[1] http://prolog.univie.ac.at/research/ConVRP/

Table 8 Richard F. Hartl's contribution to the PVRP and multi-period VRP

Reference	Problem	Approach	Benchmark
Polacek et al. (2007)	PTSP	VNS	Real-world
Doerner et al. (2008a)	VRPmiTW	heuristics, B&B	Real-world
Doerner et al. (2008b)	2-day CVRP	ACO	Real-world, random
Hemmelmayr et al. (2009a)	PVRP	VNS	*Old* & *new* data sets
Hemmelmayr et al. (2009b)	PVRP/IRP	VNS	Real-world
Hemmelmayr et al. (2010)	stochastic PVRP/IRP	VNS	Real-world
Hemmelmayr et al. (2013)	PVRP-IF	VNS + DP	Crevier et al. (2007)
Hemmelmayr et al. (2014)	PVRP-IF, bin allocation	VNS + MIP	Crevier et al. (2007)
Kovacs et al. (2014a)	ConVRP	TALNS	Groër et al. (2009), new
Kovacs et al. (2014c)	GenConVRP	LNS	Kovacs et al. (2014a), Groër et al. (2009), new
Kovacs et al. (2015)	MOGenConVRP	MDLNS, 2D, 3D	Kovacs et al. (2014c), new

Method (2D) and the Three Dimensional Adaptive ϵ-Constraint Method (3D). The GenConVRP benchmark instances by Kovacs et al. (2014c) and new small instances are used.

Doerner et al. (2008a) discuss the Vehicle Routing Problem with Multiple Interdependent Time Windows (VRPmiTW), which shows similarities to the PVRP and the IRP. In order to solve the problem, they propose heuristic approaches and a Branch-and-Bound (B&B) algorithm. Another problem being related to the PVRP is analyzed by Archetti et al. (2013), i.e., the free newspaper delivery problem. Table 8 lists Richard F. Hartl's contribution to the PVRP and other multi-period VRP.

4 Further Routing Problems

In this section, further routing problems which do not extend the CVRP are briefly introduced. A focus lies on those problems, that were tackled by Richard F. Hartl and his colleagues, including the Capacitated Arc Routing Problem (CARP), the OP, and the Covering Tour Problem (CTP).

4.1 Arc Routing

The CARP, proposed by Golden and Wong (1981), is formulated on an undirected network, where each arc has an associated travelling cost and a non-negative demand. The objective is to find tours, such that the travelling costs are minimized, the demand on the edges is satisfied, and the capacity restrictions hold. As in the CVRP, each tour has to start and end at the depot, each customer has to be serviced by exactly one vehicle, and a homogeneous fleet is considered.

Polacek et al. (2008b) present a VNS for the CARP and the CARP with intermediate facilities. Even though their work is the only contribution to this topic by the team at the University of Vienna, their research has been very fruitful as they found several new best known solutions.

4.2 Orienteering

The OP was proposed by Tsiligirides (1984). It differs from the TSP in a way that not every node has to be visited. Instead, each node has a certain score associated and the sum of the scores of the visited nodes is sought to be maximized. Typically, the total covered distance is bounded by a hard constraint. For a survey about TSP with profits, which includes the OP in particular, we refer to Feillet et al. (2005). The OP is generalized by Chao et al. (1996b) by considering a team of people, for whom tours have to be generated. A score for a visited node can be obtained at most once by a team, though. The objective is to maximize the total score of the team. This problem is denoted as Team Orienteering Problem (TOP).

Schilde et al. (2009) propose two metaheuristics for the bi-objective OP, including Pareto ACO and Pareto VNS, whereas both incorporate a PR procedure. Their algorithms are tested on the benchmark instance set by Chao et al. (1996a) and also on real-world cases.

Tricoire et al. (2010) present a generalization to the TOP, namely the Multi-Period Orienteering Problem with Multiple Time Windows (MuPOPTW). As a solution method for this problem, the authors propose a VNS incorporating an exact algorithm for the route feasibility check and the time window selection. Tricoire et al. (2013) released an addendum to the original paper with improved results.

If the objective additionally incorporates minimization of travel cost it is commonly referred to as Profitable Tour Problem (PTP). Gansterer et al. (2015) propose and formally define the Multi Vehicle Profitable Pickup and Delivery Problem (MVPPDP), where it is assumed that carriers serve less than truckload paired pickup and delivery requests. To solve this problem the authors develop two variants of General Variable Neighborhood Search (GVNS) and test them on newly created instances with up to 1000 customer requests.

In Hartl and Romauch (2015) the authors deal with an extension of the PTP, namely the Single Route Lateral Transshipment Problem (SRLTP). In this problem the objective is to find the most profitable route for redistributing (or balancing) quantities of goods amongst different retailers, with respect to the corresponding surplus or shortage and dependent on the related profits and costs. The authors propose a LNS framework which includes a Variable Neighborhood Descent (a variant of VNS) component and test their solution approach on benchmark instances adapted from instances (for the OP) from Chao et al. (1996b) and Tsiligirides (1984).

Service technician routing and scheduling problems are analyzed by Kovacs et al. (2012), where a number of technicians have to carry out tasks. Each service technician possesses certain skills that might be requested by some tasks. Moreover, the tasks have to be performed within given time windows. One problem variant further considers the grouping of technicians into teams before the start of their tours. The problem is related to the TOP, as the service technicians do not necessarily have to cover all customers. Instead, some requests may be outsourced. The objective is to minimize the routing and the outsourcing costs. Kovacs et al. (2012) present an ALNS approach, which is then tested on artificial and real-world instances.

4.3 Covering Tour

The CTP, presumably first proposed by Current (1981) (see Gendreau et al. 1997), is formulated on an undirected graph with a vertex set comprising three different types of vertices: vertices that can be visited, vertices that must be visited, and vertices that must be covered. The CTP consists then in finding a minimum length tour over a subset of vertices, such that the tour contains all vertices that must be visited and every vertex that must be covered lies within a maximum distance from a vertex of the tour. Note that such a tour may not always exist.

In the context of disaster relief operation planning Nolz et al. (2010a) study the problem of water distribution to the affected population in a post disaster environment and formulate it as a multi-objective CTP. The authors develop two alternative solution approaches: an ϵ-constraint method and a memetic approach based on the Nondominated Sorting Genetic Algorithm II (NSGA-II) (Deb et al. 2002), including a VNS and PR, which they test on real-world instances.

Nolz et al. (2010b, 2011) study a different version of the same problem. In Nolz et al. (2010b) a heterogeneous fleet instead of a homogeneous one is considered, whereas Nolz et al. (2011) additionally incorporate risk, defined as the possibility that delivery tours become impassable after a disaster. For solving the problem the former refine the two solution approaches by Nolz et al. (2010a) and again apply them to real-world instances, while the latter develop a two-phase solution approach based on a MA for the first phase and a shortest path labelling algorithm (Martins' algorithm) for the second phase and also apply it to real-world instances.

5 Conclusion

Since the formulation of the CVRP 50 years ago, several generalizations to the original problem have been proposed and many researchers have focused their attention on this field. The CVRP and its extensions are particularly interesting, as these problems are characterized by a substantial practical relevance on the one hand and are remarkably hard to solve on the other hand.

Among the many possible directions for future research in vehicle routing we want to highlight two in particular, i.e., disruption management and multi-modal transportation, due to both their increasing importance in contemporary transportation logistics and the lack of research in these areas. When dealing with unforeseen events in vehicle routing, one may refer to stochastic and/or dynamic VRP. In the recent survey by Ritzinger et al. (2015), the authors consider both, dynamic and stochastic VRP, and highlight their benefits.

In the context of disruptions, Bektas et al. (2014) lists several applications of dynamic vehicle routing, including service cancellations, vehicle breakdowns, unexpected congestions, cargo damages and changes in customer locations or demands. The literature about disruption management for VRP is relatively scarce, particularly with vehicle breakdowns, as noted by Mu et al. (2011). Vehicle breakdowns are disruptions that require real-time adjustments to the operational plan. Mu et al. (2011) highlight the differences of a disrupted CVRP to the classical CVRP. Most notably, these include that the vehicles typically are located at any point in the network at the time the disruption occurs, i.e., the vehicles do not start at the depot.

Multi-modal transportation is particularly relevant in long-distance transportation. Contrary to the above discussed VRP, vehicle routing in a multi-modal context has to consider several modes of transportation and their interaction simultaneously when searching for minimum cost routes to satisfy customer requests. To the best of our knowledge, so far, mostly problems on a strategic or tactical level (e.g., terminal network design, service network design or timetable planning) have been studied in the literature, while research on operational level problems—multi-modal routing in particular—is still scarce. For a recent review on inter- and multi-modal transportation planning we refer the reader to the surveys of Steadie Seifi et al. (2014) and Caris et al. (2013).

Within this survey, we described the basic CVRP and several generalizations and presented a model for different variants that was gradually extended throughout the paper. Furthermore, the contribution of Richard F. Hartl and his team to the field of VRP was highlighted. Clearly, Richard F. Hartl tackled a multitude of variants of VRP but there will for sure be many more interesting and exciting extensions waiting to be solved.

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List of Acronyms

2D	Two Dimensional Adaptive ϵ-Constraint Method
2E-VRP	Two-Echelon Vehicle Routing Problem
2L-CVRP	Two-Dimensional Loading Capacitated Vehicle Routing Problem
3D	Three Dimensional Adaptive ϵ-Constraint Method
3L-CVRP	Three-Dimensional Loading Capacitated Vehicle Routing Problem
ACO	Ant Colony Optimization
ACS	Ant Colony System
ALNS	Adaptive Large Neighborhood Search
ARC	Austrian Red Cross
AS	Ant System
B&B	Branch-and-Bound
B&C	Branch and Cut
CARP	Capacitated Arc Routing Problem
ConVRP	Consistent Vehicle Routing Problem
CTP	Covering Tour Problem
CVRP	Capacitated Vehicle Routing Problem
CVRPRB	Capacitated Vehicle Routing Problem with Route Balancing
D-Ants	Decomposition Ants
DARP	Dial-A-Ride Problem
DP	Dynamic Programming
DSDARP	Dynamic and Stochastic Dial-A-Ride Problem
DVRP	Dynamic Vehicle Routing Problems
E-FSMVRPTW	Electric Fleet Size and Mix Vehicle Routing Problem with Time Windows and Recharging Stations
GenConVRP	Generalized Consistent Vehicle Routing Problem
GVNS	General Variable Neighborhood Search
HDARP	Dial-A-Ride Problem with Heterogeneous Users and Vehicles
HDARPD	Heterogeneous Dial-A-Ride Problem with Driver-Related Constraints
IRP	Inventory Routing Problem
LNS	Large Neighborhood Search
MA	Memetic Algorithm
MDLNS	Multi-Directional Large Neighborhood Search
MDVRP	Multi-Depot Vehicle Routing Problem
MDVRPTW	Multi-Depot Vehicle Routing Problem with Time Windows
MIP	Mixed Integer Programming
MO-DARP	Multi-Objective Dial-A-Ride Problem
MOGenConVRP	Multi-Objective Generalized Consistent Vehicle Routing Problem
MPA	Multiple Plan Approach
MP-VRP	Multi-Pile Vehicle Routing Problem
MSA	Multiple Scenario Approach
MuPOPTW	Multi-Period Orienteering Problem with Multiple Time Windows
MVPPDP	Multi Vehicle Profitable Pickup and Delivery Problem
NSGA-II	Nondominated Sorting Genetic Algorithm II
OP	Orienteering Problem
OVRP	Open Vehicle Routing Problem
OVRPTW	Open Vehicle Routing Problem with Time Windows
PDP	Pickup and Delivery Problem
PDTSP	Pickup and Delivery Travelling Salesman Problem
PDVRP	Pickup and Delivery Vehicle Routing Problem
PR	Path Relinking
PTP	Profitable Tour Problem

PTSP	Periodic Travelling Salesman Problem
PVRP	Periodic Vehicle Routing Problem
PVRP-IF	Periodic Vehicle Routing Problem with Intermediate Facilities
SA	Simulated Annealing
SbAS	Savings-based Ant System
SDVRP	Split Delivery Vehicle Routing Problem
SRLTP	Single Route Lateral Transshipment Problem
S-VNS	Stochastic Variable Neighborhood Search
TALNS	Template-Based Adaptive Large Neighborhood Search
TDVRPTW	Time-Dependent Vehicle Routing Problem with Time Windows
TOP	Team Orienteering Problem
TS	Tabu Search
TSP	Travelling Salesman Problem
VLNS	Very Large Neighborhood Search
VNS	Variable Neighborhood Search
VRP	Vehicle Routing Problems
VRPB	Vehicle Routing Problem with Backhauls
VRPBTW	Vehicle Routing Problem with Backhauls and Time Windows
VRPmiTW	Vehicle Routing Problem with Multiple Interdependent Time Windows
VRPPD	Vehicle Routing Problems with Pickup and Deliveries
VRPTW	Vehicle Routing Problem with Time Windows

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Part III
Appendix

Achilles and the Theory of Games

Alexander Mehlmann

Abstract This small note offers a "tongue-in-cheek" route to some game theoretical facets of Greek mythology. By using an apocryphal bilingual poem as well as mathematical arguments, we finally explain how Ulysses succeeded in detecting Achilles among the daughters of Lykomedes.

Explanatory Notes *From the vast treasury of heroic sagas commonly written by life itself, only a single one could be truly narrated about Richard Hartl. Did he or did he not vegetate, disguised as a mathematician, in the academic ruins of an ancient Viennese university, before being finally unmasked as a management scientist by the cunning tenure committee at the Otto von Guericke University Magdeburg?*

As his former roommate in Vienna I could easily enumerate all his countless weak points. However, due to the lack of blackmailing tendencies, I will rather remain silent by flatteringly describing him as a profane variant of Achilles, the son of Peleus.

Up to now the story of Achilles and his exposure did definitely not belong to the canon of education in mathematics or even business administration and one could argue that its unexpected popping out at the centralized Austrian school leaving examination would certainly generate more causalities than the Trojan war as a whole.

This follows from the deplorable fact, that the reception of heroic sagas has been left, in a quite uncritical manner, mainly to linguists, philosophers, and mythologists, instead of entrusting it to the only discipline fit to reveal the truth behind myths: mathematics.

Each reader, inclined to fully understand the narrative of Achilles and his exposure without using a Latin dictionary, will either have to rely on Wikipedia (searching for "Achilles on Skyros") or abandon all prosaic versions by following the

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interpretation subsequently offered by a lighthearted and polyglot (German and English) piece of poetry, the validity of which is finally and convincingly proved by means of a game theoretic model.

Hyginus Mythographus FABULA XCVI. ACHILLES

Thetis Nereis cum sciret Achillem filium suum, quem ex Peleo habebat, si ad Troiam expugnandam isset, periturum, commendavit eum in insulam Scyron ad Lycomedem regem, quem ille inter virgines filias habitu femineo servabat nomine mutato; nam virgines Pyrrham nominarunt, quoniam capillis flavis fuit et Graece rufum pyrrhon dicitur. Achivi autem cum rescissent ibi eum occultari, ad regem Lycomeden oratores miserunt qui rogarent ut eum adjutorium Danais mitteret. Rex cum negaret apud se esse, potestatem eis fecit ut in regia quaererent. Qui cum intellegere non possent quis esset eorum, Ulixes in regio vestibulo munera feminea posuit, in quibus clipeum et hastam, et subito tubicinem jussit canere armorumque crepitum et clamorem fieri iussit. Achilles hostem arbitrans adesse vestem muliebrem dilaniavit atque clipeum et hastam arripuit. Ex hoc est cognitus suasque operas Argivis promisit et milites Myrmidones.

Die Entlarvung des Achilles[1]

Im ganzen Universe
Kennt man die rechte Ferse
Als meinen wunden Punkt.

Es hat stets Mama Thetis,
Was halt a patzn Gfrett is,
Vom Tod im Krieg geunkt.

Am Hof des Lykomedes
Vernaschte ich schon jedes
Von seinen Töchterlein.

Im Frauenoberkleide
Aus Wolle, nicht aus Seide,
Da steckt' ich nur zum Schein.

Als die Achäer kamen,
Da trafen sie nur Damen
Und keinen Sagenheld.

Dem Sohne des Laertes
Fiel ein was Unerhörtes
Danach als Hinterlist

Er brachte von Mykene
Die beispiellos mondäne
Gewandung aus Batist

Mit Fibeln disponibel
Aus Elfenbein (sensibel
Erjagt in Côte d'Ivoire).

Von Troja bis nach Delphi
Fehlt wohl in keinem Selfie
Ein solches Accessoire.

Wenn Mädchen es erblicken
Ob Fürstinnen, ob Zicken
Schrei'n sie im Modewahn.

[1]This lyric gem belongs to an apocryphal collection of mathematical ballads: Alexander Mehlmann, *Mathematische Moritaten*, CreateSpace Independent Publishing Platform, 2016.

Bei Schlachtlärm kreischen Misses
(Das wusste auch Ulysses)
Und geben Fersengeld.

Die Griechen kehrten wieder
Mit Weibertand und Flieder
Samt Waffengarnitur.

Bestrebt mich einzukreisen,
Taucht auf am kalten Eisen
Ein Fingerabdruck nur.

Dann bliesen sie Fanfaren,
Um endlich zu erfahren
Wer nun Achilles sei.

Doch keine von den Schnecken
Griff nach den Eisenstecken
Und tanzte aus der Reih.

Gewillt dies anzubringen,
Ließ ich den Kehlklopf klingen
Im schallenden Päan.

Jedoch die blöden Miezen
Im Glauben mich zu schützen,
Vollführten keinen Ton.

Als Spottgestalt der Mode
Besingt mich kein Rhapsode.

Drum auf nach Ilion!

The exposure of Achilles[2]

Aside from sex appeal
I also have a heel
Well known as my weak spot.

As mother Thetis said,
"Live as a girl instead;
In Troy it's bad to rot."

At Lykomedes' court
I did myself comport
By sleeping with his girls.

I loved it just to pose
Disguised in womens' clothes
And wearing costly pearls.

Fluent in ancient Greek,
Ulysses came to seek
A hero for the war.

Then Penelope's man
Designed a cunning plan
To catch the guy disguised.

He brought as dastard lure
A piece of fine couture
In Argos highly prized.

Emblazoned with a brooch
Of ivory (to poach
In Côte d'Ivoire for free).

Each maiden would shout out
(In madness without doubt)
For this accessory.

According to this thought
I started all for naught
An ancient battle roar.

[2] A roughly adequate translation of the German language poem printed on the previous page. (see: Alexander Mehlmann, *Mathematische Moritaten*, CreateSpace Independent Publishing Platform, 2016).

The only thing he found
Confined to this compound:
Nine girls and nothing more.

He briskly came then back
With womens' bric a brac
Concealing sword and spear.

Since his detection test
Needs choices to be guessed
And anything to fear.

A blare of clarion
(A bit Wagnerian)
Did scare the pants of all.

But no one from the nine
Did as a hoplite shine
By answering the call.

But echo came there none.
(Ulysses having fun
Rolled laughing on the floor.)

A hero to protect,
That's why chicks did defect
Oppressing any sound.

To quit as laughing stock,
Let's take the Trojan walk
And get my mortal wound!

In the original variant of the story (see Hyginus Mythographus for a short Latin summary) the reason why Achilles is exposed seems to contradict any game theoretic rule. Disguised as one of the daughters of king Lykomedes, Achilles is revealing itself to Odysseus, by picking a weapon from the gifts offered, as soon as the alarming sound of a clarion, indicating danger, is heard.

This behavior is, however, counterintuitive. Achilles' strategy should not in fact be distinguishable from the presumed behaviour of the king's daughters. Thus the cunning plan of Odysseus (Ulysses for the Latin aficionado) is from the first doomed to failure.

The second variant, which is both more erudite and poetical valuable than the original story, can be consistently modelled as an extensive game (see Fig. 1 above). Ulysses will only be able to expose Achilles, if his behaviour is fully distinguishable from the reaction shown by the daughters of Lykomedes.

Since any collusion between our hero and the Lykomädis (i.e. Lykomedes-girls) is strictly excluded, the only reasonable equilibrium will let Achilles oscillate his vocal fold (P) with probability 0.5, whereas the chicks will equiprobable remain silent (S). Since both coordination and repetition are excluded, the pure equilibria (P, P) and (S, S), which would be preferred by standard textbooks of game theory, have to remain out for consideration.

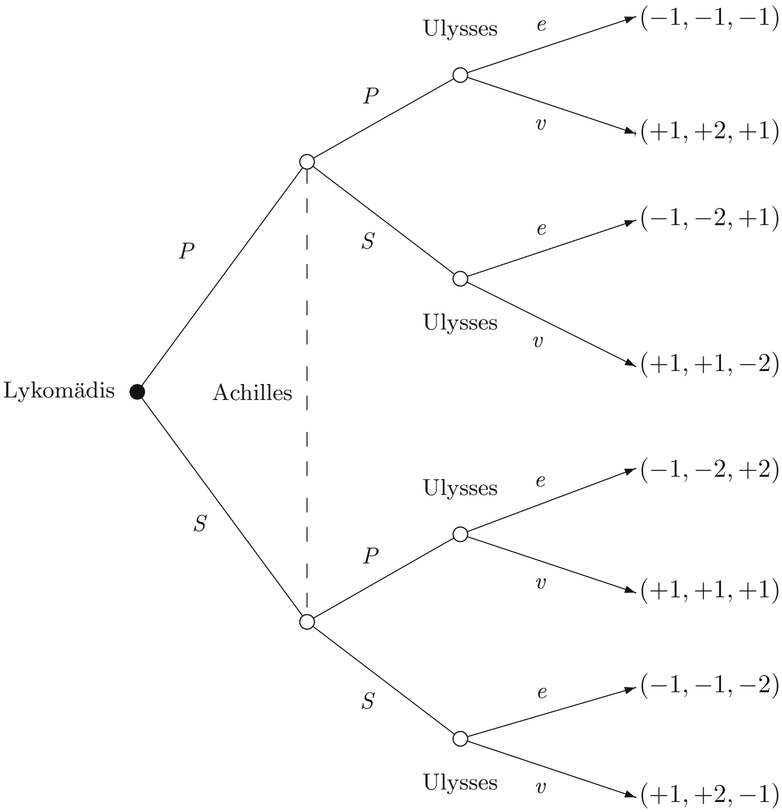

Fig. 1 Achilles and his game tree

Final Comment *Due to the lack of space and the banishment of this contribution to the appendix, all necessary deductions will remain undone. We will leave them over as a last homework assignment to Richard Hartl, who is enjoyned to work out the details in ancient Greek and to deliver the results at his retirement festivity.*