# Kazimierz Kuratowski—Biography and Genesis of the Theorem on Planar Graphs

**J. Wojnarowski and S. Zawiślak**

**Abstract** The present paper discusses Professor Kazimierz Kuratowski's achievements, especially proving his theorem on planar graphs in 1930. Some facts of the biography are analysed, aiming for explanations of how it was possible for him to do this and to delineate the background of such success. The general situation of mathematics in Poland (especially in Lvov) in the 1920s and 1930s is roughly described. The way of proving the theorem is also analysed i.e., transformation of the problem from the graph theory field into the field of topology.

**Keywords** Kuratowski theorem—multidisciplinary approach · Planar graphs · Lvov school of mathematics

## 1 Introduction

The Kuratowski theorem [1, 2] for planar graphs [3–8] is cited in almost every book on graph theory. The proof of this problem was published in 1930 [2], but the first formal announcement had been published in 1929 [1]. One can ask the question of how it would be possible in Poland—with rebirthing taking place in 1918 after World War I (WW1) after the lack of statehood for over 100 years. Moreover, in fact, this peaceful period started after the war against Soviet Russia in 1920. Additionally, there are no other results of Kuratowski's that were dedicated to graph theory.

J. Wojnarowski
Faculty of Mechanics and Technology, Silesian TU, 18a Konarskiego St., 44-100 Gliwice, Poland
e-mail: jozef.wojnarowski@polsl.pl

S. Zawiślak (✉)
Faculty of Mechanical Engineering and Computer Science, University of Bielsko-Biala, 2 Willowa St., 43-309 Bielsko-Biala, Poland
e-mail: szawislak@ath.bielsko.pl

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There are several papers and books on Kuratowski's biography and scientific achievements. As for data connected with his life, the most important is his autobiography [9] followed by a number of historical books and papers [3, 4, 10–34].

The problem of planarity for graphs can be formulated in the following manner: to draw a graph on a plane in such a way that the edges have common points only in graph vertices so there are not any crossings of the graph edges. It is not clear when the problem was firstly formulated, whereas a historical approach to graph drawing (in general) was presented in a very interesting way in [26]. According to the thesis [17], the problem of planarity was mentioned e.g. by Moebius in 1840 and Dudeney in 1917 (also the bipartite $K_{3,3}$ graph problem was formulated). According to the thesis [17], the planarity problem is connected strictly with graph coloring problems (such as the 4-color theorem), and the graph duality problem as well as the study of polyhedra.

The goal of this paper is to outline a hypothetical explanation of how it was possible for Kuratowski to solve this problem. According to our best knowledge, the genesis of his proof of the problem was not point-blank analyzed despite the fact that several historical papers, theses and book had been published. The only existing paper emphasizing that the proof is not done in terms of graphs is [10], according to the best knowledge of the Authors of the present paper.

There are papers dedicated to the proof of the planarity problem [3, 4, 10, 12, 35–37] but sometimes they focus an attention on the question: 'was Kuratowski really the first to do this?' [12, 22]. The paper that traces versatile attempts of proving the planarity theorem is paper [22]. The authors described the fact that also American scientists Frink and Smith [15] did the proof but their paper was rejected because the Editors had knowledge about Kuratowski's publication. However, the announcement was even later published than the complete proof of Kuratowski, therefore their full work dedicated to this problem was finally not printed. There are also publications named the theorem on planarity such as the Kuratowski-Pontrjagin theorem [12] (mainly in Russian publications) but the paper [22] explains the history in details based upon relations of Kuratowski's co-workers i.e., only oral communications were exchanged between scientists from Poland and Russia. The early Polish version of the contribution on the transformed problem of planarity, translated recently into English, can be found in [38].

There is no separate book on Kuratowski's education path or his scientific curriculum vitae and achievements such as exist for other famous Polish mathematicians of this era e.g., Stefan Banach [19–21, 24, 31, 32]. However in book [11], there is a chapter written by R. Engelking entitled "Kazimierz Kuratowski (1896–1980) His life and work in topology"—pages: 431–452. There are several short papers [16, 22] dedicated to the biographical data. However, the books and the papers do not fully analyze the cases of this outstanding success according to planar graph problems.

Moreover, in the present paper, the historical background will be studied. This background is embedded in the history of creation of the so-called 'Lvov school of mathematics' [13], Warsaw school of mathematics or finally Polish school of mathematics within the period 1918-1939. In that time, the internet, on-line

electronic versions of scientific journals and even television were not available. Some facts from earlier history of Poland are also shortly mentioned.

The considerations are partly based on memoirs of Kuratowki himself [9], but also Hugo Steinhaus [33], Kazimierz Szałajko [34] and Stanisław Ulam [39].

### *1.1 General Historical Background*

Poland was erased from the map of Europe in 1795. It was conquered by neighboring countries Russia, Prussia and Austria. In those times, the territory of Poland enclosed some lands of the current Lithuania, Belarus, Ukraine and partly Latvia. It was due to peaceful treaties with Lithuania—so called Unions (years—1385 (Krewo), 1413 (Horodło), 1501 (Mielnik) and 1569 (Lublin)). Poland was created anew in 1918. The borders of the country were established during the Versal (Paris) conference, the uprisings (Silesia and Poznan regions), local voting (plebiscites) as well as—finally—the war with Soviet Russia (1920) and the post-war peace treaty. The borders, which were finally established, encircled Vilnius and Lvov as Polish towns.

During the time of division, Poles were citizens of the states of invaders. They had to learn the appropriate state languages. Simultaneously they could study in the best Universities of these countries.

## 2 Mathematics in New Established Poland

The countries that conquered Poland performed a policy of assimilation of citizens —consisting in: introducing state languages in the school system from grammar schools up to universities, in courts as well as in the state offices and institutions. At the beginning, the job carriers of Poles in Russia, Prussia and Austria were rather rare in such domains as politics, army, kingdom or empire officers (courts, authorities) or science. However, some Poles were active in France, Switzerland or Great Britain and other European countries. However, within early years of the XX-th century, the Austro-Hungarian Empire changed the internal national policy, smaller changes took place also in Russia and Prussia. For example, the Universities in Austrian province ‘Galicja’ e.g. Lvov and Cracow could give lectures in Polish.

In 1918, in the new-established Poland, 5 universities were created: Warsaw, Cracow, Poznan, Lvov and Vilnius, moreover some technical universities were also established in Warsaw, Cracow, Lvov and a lower rank technical school in Poznan. The university in Warsaw started its works in 1915, just during the war, not waiting for formal establishing of an independent country. It caused that the curricula or/and contents of lectures were not supervised by government, ministry or even based on inter-university exchange.

The basics for such immediate and efficient opening of the universities are versatile activities performed at the end of XIX century within the former territory of Poland:

- foundation of societies: Cracow Academy of Science in Cracow (Austro-Hungary Empire), Society of Science in Warsaw (Russia),
- foundation of journals (Mathematical and Physical Works, Bulletin of Academy Skills in Cracow, Mathematical Newsletter),
- opening of Scientific Courses (Warsaw, Russia), introducing Polish language in universities in Austro-Hungarian Empire.

The new Polish Universities and Technical Universities hired as professors or lecturers many outstanding mathematicians:

(a) returning from other Universities after studies or doctorate. Here only a few exemplary names are listed:

- Kazimierz Żorawski (pupil of Sophus Lie from Norway)—differential geometry firstly in Cracow, then in Warsaw (from 1919),
- Stanisław Zareba—study in Petersburg (engineering), University of Paris (mathematics), doctor thesis at the Sorbonne University in Paris,
- Zygmunt Janiszewski—doctor thesis in France, doctor exam commission consisted of: H. Poincare, H. Lebesgue and M. Frechet,
- studying abroad: Stefan Mazurkiewicz (Munich, Göttingen), Hugo Steinhaus, Waclaw Sierpiński (Göttingen);

(b) returning to Poland being professors in other countries:

- Antoni Przeborski—Charkov to Warsaw,
- Jan Sleszyński—Odessa to Cracow,
- Wiktor Emeryk Staniewicz—Petersburg to Vilnius,
- Leon Jan Staniewicz—Petersburg to Warsaw (after WW2 in Gdansk).

There were also other professors (who returned) from other fields of knowledge related to mathematics e.g.:

- physics: Franciszek Zienkowski (Göttingen to TU Warsaw) who at the beginning was a Kuratowski's teacher in the secondary school in Warsaw [9]; Wojciech Rubinowicz (Muenchen to Lvov) co-worker of Arnold Sommerfeld [13, 18–20, 23], Józef Wierusz-Kowalski (Switherland to Warsaw University);
- astronomy: Marcin Ernst [23] (diplomas: Berlin and Vienna—work at the Lvov University).

There were also a reverse directions of migration of scientists in 1930s i.e. newly educated Polish young graduates or doctors emigrated to USA or Europe (especially of Jewish origins) e.g.: Stanisław Ulam (Lvov—Los Alamos, Princeton, USA), Otto Nikodym (Lvov—Kenyon College, USA), Marek Kac (USA), Leopold Infeld (Cambridge, U.K., Princeton, USA). However, they periodically returned to Lvov for seminars or during summer holidays. In every case, meetings with all

former co-workers were a must. It allowed for immediate exchange of ideas and achievements.

Alfred Tarski, mathematician from Warsaw University, emigrated to USA in 1939, working e.g. in Harvard University, Princeton and Berkeley.

It can be stated that mathematics in Poland in this time was on a very high level which can be confirmed by the possibility of working of Polish mathematicians in the best world-wide universities as well as working on problems of extreme novelty like e.g. Stanisław Ulam worked on the atom bomb in Los Alamos (being educated and doctor promoted in Lvov).

Moreover, many outstanding mathematicians from all-over the world visited Poland especially the Universities in Lvov and Warsaw (see Chaps. 2 and 3).

## *2.1 Methods of Working and the Scottish Book*

The methods of co-operations between the scientists of this period in Poland were amazing and rather outstanding. They established an informal [39] collective, groups of fellows. They called themselves "colleague", they discussed problems in different cross-domain and multi-university groups—for example the informal meeting of the Polish Mathematical Society took place regularly almost every Saturday evening, besides the seminars organized during working days at the Universities [39, p. 67]. Cooperation of the Lvov branch with branches in Cracow, Poznan or Vilnius was live and frequent. The report on formal meetings in the Lvov branch covering the period 1928–1938 [13] had shown that there were 180 meetings and 360 talks which means that there were on average 16–17 official meetings per year and usually two talks were given in every case. In total, there were 65 authors including 16 authors from abroad.

Many famous mathematicians from other countries were registered as foreign members of the Society. The presidents of the Society were consecutively chosen from different Polish Universities (Stanisław Zaremba-Cracov, Wiktor Staniewicz-Vilnius, Zdzislaw Krygowski-Poznan, Waclaw Sierpiński-Warsaw, Kazimierz Bartel-Lvov) [21, 27, 40].

The state congresses of the Polish Mathematical Society started to be organized in 1927. The first took place in Lvov (J. von Neumann (USA), M. Luzin and N. Bari (Russia), V. Hlavaty (Prague) were the guests), the following in 1931 (Vilnius) and in 1937.

The informal meetings of mathematicians took place in different places but one of Lvov's restaurants was very characteristic and attended regularly. It was the Scottish Café placed in Fredro square. The meetings among others consisted in posing and discussing mathematical problems. If the problem was solved, those who discussed designated one who wrote down the results, all real contributors who really developed the considerations were automatically the authors of the immediately prepared paper. Since 1933 (or 1934 [39] or 1935 [13]) the problems had been registered in a special copybook which was called the "Scottish Book". The register (from a copied original version) was translated into English and edited by S.

Ulam [41] in USA. One of the authors of the present paper had the privilege of reading the original. Kazimierz Szałajko, a co-worker of Stefan Banach, was a director of the Institute of Mathematics at the Silesian University of Technology in Gliwice in 1970s. Upon his invitation, son of Stefan Banach came for a meeting at the Faculty of Applied Mathematics, showing the original copybook which was preserved so many years also during the World War II (WW2). The xerox copy of the book is stored in the Library of the Polish Academy of Science Institute of Mathematics in Warsaw. The problems were solved by different mathematicians, sometimes after several years later.

The papers of Polish mathematicians were published sometimes in Polish but mainly in foreign languages. Moreover some papers were published not only in the created Polish journals but also in the best world-wide journals, e.g., Bulletin of the American Mathematical Society [39, p. 71].

## 2.2 Journals

Immediately after creation of independent Poland, Polish scientific journals of international meaning were established [9, 13, 27, 32] i.e.: "Fundamenta Mathematicae" in 1920 in Warsaw and a little later "Studia Mathematica" in Lvov in 1929 as well as "Acta Arithmetica" in Warsaw in 1935. The advantages of these journals were as follows: the published works were of very high scientific level, they were published only in foreign languages (French, German and English), 18 % of papers were published by authors from abroad [13]. The additional benefit was that almost 200 items of one issue (500 items of the first mentioned title) were subjected to exchange with other world-wide journals and university libraries [13, p. 56–58] due to lack of financial sources just for buying. A fresult was that not only Polish achievements were distributed around the Europe but also Polish scientists could have access to the newest achievements across Europe. The editors of the first issue of "Studia Mathematica" were: Stefan Banach and Hugo Steinhaus.

## 2.3 Congresses, Conferences and Visits

In the discussed times i.e. 1920s and 1930s, there were no such social media as the internet (e.g.: 'Scholar Google', SpringerLink data bases) or mobile phones, therefore live meetings and establishing of personal connections between scientists had been a base of knowledge about the state-of-art in a particular field of science. Circulation of journals was not so fast as today's on-line journals which causes that nowadays availability is immediate. Participation in congresses, conferences and

lectures was an effective way to be informed on the current trends and achievements. Moreover, it is well worth emphasizing that Polish mathematicians were not only participants in those events but also invited speakers. It was due to knowledge and appreciation of their contribution to particular fields of science. There were several lectures given by Poles in some famous and prestigious universities e.g.:

(i) Kazimierz Kuratowski: 1927—invited lectures Heidelberg and Goettingen (meetings with P. Aleksandrov, H. Hopf, E. Noether, D. Hilbert), 1935—Princeton, Harvard, USA,
(ii) Stanisław Ulam: 1934–visit in Vienna—Professor K. Menger (topology, geometry) formerly invited to Poland by Kazimierz Kuratowski, 1934–visit in Paris—Professor E. Cartan, 1934—visit in Cambridge, U.K.—Professor G.H. Hardy.

Visits of foreign mathematicians in Lvov were relatively frequent. Some famous visitors were as follows: Austria (M. Jacob,), France (E. Borel, J. Leray, P. Montel, H. Lebegue), Czech Republic (E. Čech, W. Hlavaty), U.K. (G.T. Whyburne, A. C. Offord, A.J. Ward), USA (J. von Neuman), Germany (E. Zermelo, L. Lichtenstein), Swiss (R. Wawre), Romania (P. Segrescu, S. Stoimov), Russia (N. Bari, N.N. Bogolubov, L.A. Lusternik, N. Luzin, S.L. Sobolev) [3, 17, 23, 26]. The title of Doctor Honoris Causa of Jan Kazimierz, University of Lvov was granted to the famous French mathematician Henri Lebesgue in May 1938.

Polish scientists took part in three Mathematical Congresses which were held in: (a) Bologna (1928), (b) Zurich (1932) and (c) Oslo (1936). Professor Kuratowski took part in all of these congresses. In the first of them Hugo Steinhaus and Stefan Kaczmarz had contributions in the conference sections. In Zurich, Professor Waclaw Sierpiński gave a plenary lecture. In total, there were 20 Polish participants [13]. In Oslo, the plenary lecture was given by Stefan Banach. Other congresses in which Poles took part were as follows: 1930—Charkov Mathematical Meeting (Antoni Przeborski, Jerzy Spława-Neyman) [27], 1934—Conference on Differential Geometry in Moscov (Antoni Hoborski, Stanislaw Gołąb, Aleksander Wundheiler), 1935—Conference on Topology in Moscov (Karol Borsuk, Witold Hurewicz, Stefan Mazurkiewicz, etc.) [27].

## 3 Short Kuratowski's Biography (1896–1980)

The facts about Kazimierz Kuratowski are collected mainly from his autobiography. He starts the book with a statement on the first page (!) that He began to learn foreign languages from early childhood [9, 42, 43]. In fact, French was taught simultaneously with Polish as a second language at home. At school English and German were taught. Living in the Russian Empire, Kuratowski had to speak Russian which was used at schools and in administration. To attend and finish

school, Russian should be known fluently. Then, throughout his life, Kuratowski continued to study and consequently spoke all these 5 languages perfectly.

Parents as well as their relatives were lawyers, medical doctors, historians, engineers etc. They were well educated and rich. His father was also a journalist in the local newspaper for lawyers, moreover he was involved in versatile social organizations. Brother Roman was a lawyer. Kuratowski was born in Warsaw and spent his youth in this town. He attended schools of an extremely high level of education. In 1913 he started to study at the Glasgow University at the Engineering Faculty. It was popular that rich citizens of Polish nationality sent daughters and sons to study abroad but Glasgow was not too popular. Kuratowski's choice was supported by a will to go with one of his best colleagues. He finished first year with the second rank and returned for holiday to his parents. He was satisfied by the lectures of Andrew Gray formerly assistant to Lord Kelvin. Additionally, in Glasgow mathematical lectures were common in the first year for engineers and mathematicians—so the level was really high. In 1914 World War I began and after the war he did not return to Glasgow but started to attend University in Warsaw which was opened in 1915. It was during the war therefore that the study curricula were not obeyed and lectures were arranged according to availability of professors. A result was that he was unbalanced on the side of geometry having all of its kinds: analytical, projective, differential, descriptive etc. He also gave lectures on topology which was a new branch of mathematics (Zygmunt Janiszewski, Wacław Sierpiński). Kuratowski's first scientific results were achieved during his studies therefore publishing his work at the University was an obvious option. He had published several papers as a student and additionally other results were randomly collected, so the problem was only in formal paper writing. He obtained a doctoral diploma (PhD) on the 5th January 1921. In the Fall, Kuratowski obtained the title of D.Sc. (habilitation) on set theory and started to work as an associate professor. Within the period 1917–1926, he published approximately 40 scientific papers of great importance and therefore obtained a proposal to move to Lvov. Kuratowski accepted the opportunity. In 1927, He traveled to Heidelberg and Göttingen in Germany for invited lectures. Work at the Lvov University lasted within the period 1927–1933. Among Kuratowski's co-workers were e.g. Edward Otto (who was involved in descriptive geometry) and Stanisław Ulam. The Chair of Descriptive Geometry was directed by Stanisław Bartel. Kuratowski published many new papers there e.g. common papers with Stanisław Ulam on the Hausdorff measure problem and Edward Otto on topology. But the most famous is the graph planarity theorem published in 1930.

Kuratowski returned to the University of Warsaw where he worked till the end of his work duties. During World War II the Polish universities were closed but he took part in the so-called underground system of higher education which could be punished by death. After World War II, Kuratowski flourished in both areas: science and organization work. He was an active fellow of the Polish Mathematical Society as well as a founder and a member of the Polish Academy of Science. He had also world-wide recognition, giving lectures in the USA (1948/1949 and 1968/1969) in Harvard, Princeton, Yale, Columbia, Berkeley and Baton Rouge. Other destinations of his scientific visits were: China, India (1956), Germany (1957) and Russia.

Kuratowski received honorary degrees from many universities including Glasgow (U.K), the Sorbonne (Paris, France), Prague (Czech Republic) and Wrocław (Poland).

## 4 Proof of Kuratowski's Theorem—Genesis

It seems that the circumstances allowing for the successful solving of planarity problem were as follows:

(i) personal:

- gifted, from the family of lawyers, doctors, teachers, well educated,
- perfect knowledge of foreign languages: Russian—certificate of secondary school in Moscow, English—one academic year of study at Glasgow University (1913–1914) and lectures in USA, French—many publications, known from childhood, German—several visits in Germany, which allowed for reading, publishing scientific papers from versatile journals, conferences' proceedings as well as giving and hearing conference talks,
- probably the most important—continuation of studies in Warsaw since 1915—during the World War I based on a study curricula arranged to the available professors. It was a non-classic and a non-traditional study course. Therefore he studied at first several subjects connected with geometry in extended hours—especially topology and even descriptive geometry,
- mobility, many scientific trips abroad, attendance at mathematical international conferences and meetings, being informed on the current state-of-art,
- a period of work at Lvov University, descriptive geometry specialization of his doctoral student Edward Otto,
- lectures abroad e.g. in Germany (1927) and in the USA;

(ii) scientific and social:

1. rebirth of Poland—the majority of professors came to the newly-opened Universities from Austria, Russia and Prussia (outstanding and very high-level best Universities, doctors came also from France, Germany and Switzerland—obtaining PhD titles from the best European (or world-wide) professors,
2. high, international level of lectures and knowledge of current trends,
3. creation of three Polish mathematical journals of international meaning and via purchasing and also exchange—availability of foreign journals,
4. international co-operation on the highest level, the best world-wide mathematicians in Lvov, Polish mathematicians visited leading Universities in USA and Europe,

5. taught by outstanding professor and teacher—so-called personalities—which is in sound opposition to modern tendencies of distance learning. It seems that meetings with such supervisors who inspire was a trigger of his own will of self-improvement,
6. active work in Mathematical Society, organizing of visits of foreign professors, an active part in European Mathematical Meetings, which gave additional opportunity for exchange ideas,
7. contacts with Hugo Steinhaus [39, p. 72] who frequently proposed unusual approaches consisting in representation of geometrical problems via combinatorial approaches—i.e. reverse attitude from Kuratowski's way. However, nowadays, it is worth underlining the need of posing reverse problems. It could allow for seeing both of them in a new light or from different points of view. Other multi-disciplinary approaches were considered by S. Ulam and J. Schreier—merging problems from topology and algebra.

The proof of the graph planarity theorem was published in 1930 [2] in French. However, some introductory results were published even earlier in Polish which

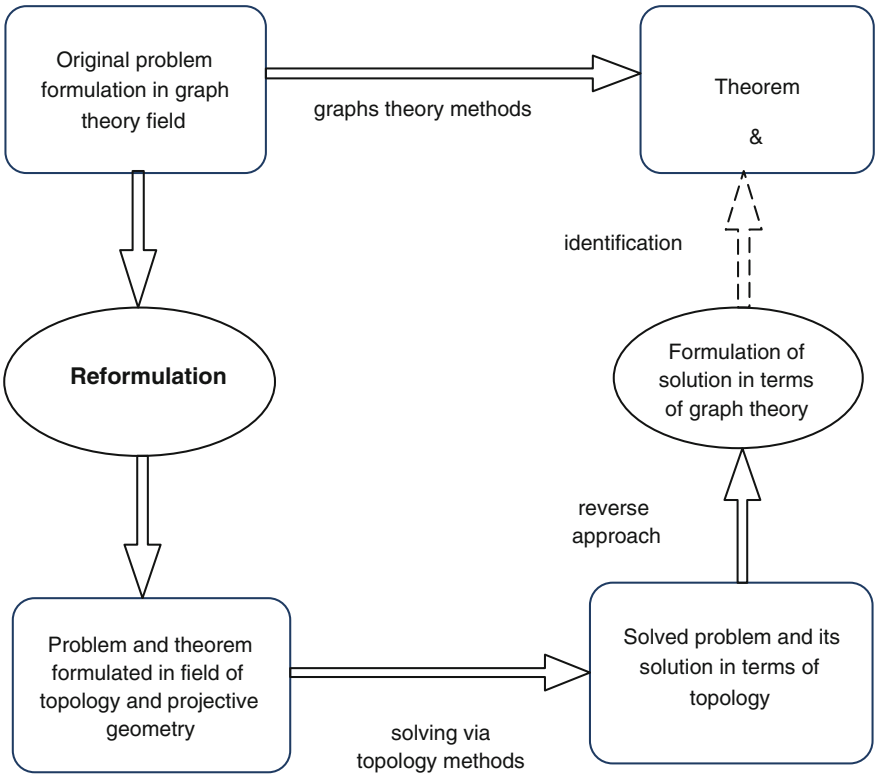


**Fig. 1** Multi-disciplinary approach to solving the problems via their cross-field transformation upon planar graph problem

was described in historical publications [44, 45]. The theorem says that a graph is planar if it does not contain as subgraphs any graphs equivalent to $K_5$ and $K_{3,3}$. It had been proved in fact as a problem of topology or even projective geometry [10] (!) as it is mentioned in the French title of the paper which is available e.g. from webpage [42]. The combinatorial proofs of the theorem can be found in [29, 30].

The utilized approach can be symbolically presented in the scheme shown in Fig. 1. Based upon the original text of [2] one can see that the scheme presented in Fig. 1 was utilized by Kuratowski. For example: nowadays called Kuratowski's graphs: $K_5$ and $K_{3,3}$ (Fig. 2c, d) had been converted into two adequate polyhedra—pictures 1 and 2 [2, p. 272] which are redrawn in the present paper in the first row in Fig. 2. The numbers of polyhedral nodes are assigned to adequate graphs' vertices presented in the second row in Fig. 2c, d. In case of the first polyhedral, one edge was added to the first skeleton (Fig. 2a), i.e. (4, 6) and diagonals (1, 3) and (2, 4) were added in the second case (Fig. 2b).

As it was previously mentioned, there are no papers of Kuratowski dedicated to graph theory, written in the nomenclature of graph theory, which we know checking his complete bibliography [43].

In fact, the multi-disciplinary approach can be utilized only by scientists who are open-minded, have a wide scope of interest, and are eager to acquire novelty and originality. This idea has been propagated recently but to our surprise there are still opponents fighting for purity of some fields of knowledge. We do hope that the above considerations confirm the necessity of a wider and more interdisciplinary attitude to scientific problems, especially graph theory and its applications.

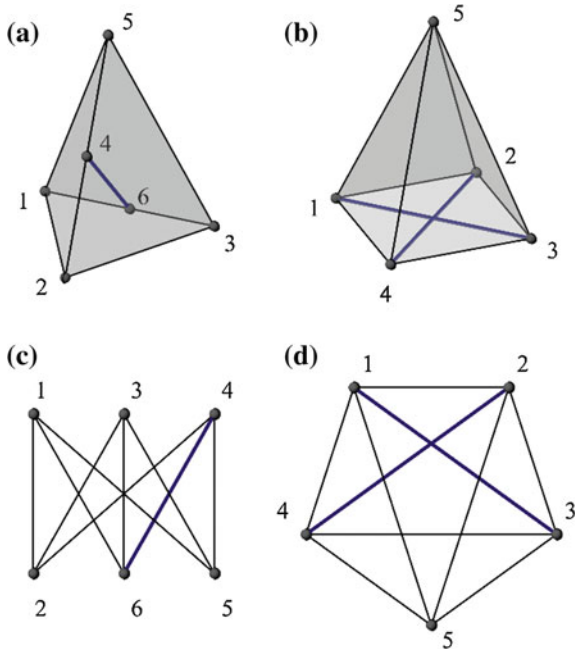


**Fig. 2** Two polyhedrals originally considered by Kuratowski in [2] (*first row*, (**a**)(**b**)) and two adequate graphs—known as Kuratowski's graphs (*second row* (**c**)(**d**)) where the numbers of nodes are equal to the equivalent graphs' vertices names

Further investigation into this problem consisted in: a planar graph drawing by means of straight lines only [14, 46], a planar graph drawing in grids [47], algorithms for checking planarity [4, 48], some generalization on infinite graphs [49] and dual form of the discussed theorem [35]. Recent papers, where comparisons of planarity checking algorithms can be found, are [4, 50], even of polynomial complexity.

## 5 Conclusions

The paper discusses Kuratowski's theorem on planar graphs. The origins are analyzed taking into account the fact that Kazimierz Kuratowski was born in the period when Poland was not present on the map of Europe. The state was re-established in 1918 and almost immediately many outstanding Polish mathematicians were actively working in Poland and publishing works on a world-wide level. In the case of Kuratowski, he had been a professor of mathematics in Warsaw and Lvov since 1920s. The present considerations enclose the hypothetical genesis of the successes of the Polish school of mathematics and Kuratowski himself.

It seems that paradoxically Kuratowski's study during World War I could have been the base for his interest in planar graphs because he was taught in an imbalanced and untypical way on topology and geometry due to availability of adequate professors just at this time, additionally spending his first year at the Glasgow University.

Moreover, during partitioning of Poland citizens had to know foreign languages and Poles had to view their scientific carriers within temporary home countries (Russia, Prussia or Austria) or wherever in Europe and even in the USA. Many professors, due to their patriotic backgrounds, started to work for an independent Poland since 1918 after rebirthing of the state. It is worth emphasizing that Polish mathematicians organized their professional lives and organizations in outstanding efficiency, scope and manner. It gave the background for achieving a high international level and in some fields allowed even for world-wide leadership.

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