Prüfer Domains of Integer-Valued Polynomials

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Abstract Let D be an integral domain with quotient field K. The ring $\mathrm{Int}(D) = \{f(x) \parallel f(D) \subseteq D\}$ has been studied as a ring for more than forty years. A major topic of interest during that time has been the question of when the construction yields a Prüfer domain. The principal question has been resolved, but interesting generalizations are still being worked on. This is a survey paper that traces the history of study of integer-valued polynomial rings with a focus on when they are Prüfer domains.

1 Introduction

Throughout this paper, D is an integral domain, K is its field of fractions, and E is a nonempty subset of K. A polynomial $f(X)$ with coefficients in K is *integer-valued* if every $d \in D$ satisfies $f(d) \in D$; i.e., $f(D) \subseteq D$. The collection of such polynomials is designated $\mathrm{Int}(D)$. One could also consider polynomials that are *integer-valued on a subset*; more precisely, the polynomial $f(X)$ is *integer-valued on the subset* $E \subseteq D$ if every $d \in E$ satisfies $f(d) \in D$; i.e., $f(E) \subseteq D$. The collection of these polynomials is designated $\mathrm{Int}(E, D)$.

The first studies of $\mathrm{Int}(D)$ were by Polya [18] and Ostrowski [16] both in 1919. Although it is easy to see that $\mathrm{Int}(D)$ is a ring, both of these papers dealt purely with the additive structure. In particular, they focused on the D-module structure of $\mathrm{Int}(D)$ where D is a ring of algebraic integers. For the next half century $\mathrm{Int}(D)$ was studied periodically, always still with focus on the additive/module structure. Study of the ring theoretic structure of $\mathrm{Int}(D)$ began almost simultaneously, and independently in three

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different places. Graduate students Paul-Jean Cahen and Jean-Luc Chabert at the University of Paris, professors Hiroshi Gunji and Donald McQuillan at the University of Wisconsin, and graduate student Demetrious Brizolis at UCLA all began study of $\text{Int}(D)$ as a ring in the early 1970s. Integer-valued polynomials have been well studied now; there are deep results in many different directions. One of the main topics from very close to the beginning has been the question of when the integer-valued construction yields a Prüfer domain. This article will trace, chronologically, the study of this specific question.

2 Int(*D*)

2.1 Noetherian Domains

The consideration of $\text{Int}(D)$ being a Prüfer domain began with the work of Brizolis [1]. This actually does not appear to have been his goal. The fact that $\text{Int}(D)$ can be a Prüfer domain proved to be useful to him in his study of problems involving generating ideals. He proved that $\text{Int}(D)$ is a Prüfer domain for a class of Dedekind domains which includes the rings of algebraic integers, and then used this result to generalize work of Skolem from the 1940s. He did find this "intermediate" result interesting though, and questioned what necessary and sufficient conditions on a domain D would be for $\text{Int}(D)$ to be a Prüfer domain.

Jean-Luc Chabert [4] and Donald McQuillan [13] pursued this aggressively in the succeeding years and each, independently, settled the characterization problem in the case where the ring D is Noetherian. In particular, they each essentially proved the following theorem.

Theorem 2.1 *If D is Noetherian, then* $\text{Int}(D)$ *is a Prüfer domain if and only if D is a Dedekind domain with all residue fields finite.*

In each case the method was to solve the problem locally and then globalize the solution. In particular, they each proved that $\text{Int}(V)$ is a Prüfer domain if V is a DVR with a finite residue field. The general case follows from this because when D is Noetherian $\text{Int}(D)$ behaves well with respect to localization. More precisely, let V be a DVR with maximal ideal M and residue field F. Let V^* be the M-adic completion of V. Then $\text{Int}(V)$ is a Prüfer domain. Moreover, the maximal ideals of $\text{Int}(V)$ lying over M all have the following form.

$$M_\alpha = \{f(x) \in \text{Int}(D) \parallel f(\alpha) \in MV^*\}$$

And these maximal ideals are all distinct. So the maximal ideals are indexed in a natural way by the M-adic completion of V. What McQuillan and Chabert were able to show is that this property can be globalized. Namely, if M is a maximal ideal of a Noetherian domain D, and $S = D - M$ then $\text{Int}(D_M) = S^{-1}\,\text{Int}(D)$. So, if D is a Dedekind domain with all residue fields finite, then the maximal ideals of $\text{Int}(D)$ lying over a maximal ideal M are naturally indexed by the elements of the M-adic completion of D exactly as in the DVR case, and this led to a proof of the above theorem.

2.2 Non-Noetherian Domains

While the Noetherian case was being settled, there remained the general case where D is not assumed to be Noetherian. The first step in this direction was a result of Chabert [4] in 1987.

Theorem 2.2 *Let D be an integral domain. If* $\text{Int}(D)$ *is a Prüfer domain, then D is an almost Dedekind domain with all residue fields finite.*

A domain D is said to be almost Dedekind provided the localization at any maximal ideal is a DVR. Noetherian almost Dedekind domains are then exactly the Dedekind domains. So the theorem seems to be a natural extension of the Noetherian necessary and sufficient condition. However, while Chabert's result gives a necessary condition for $\text{Int}(D)$ to be a Prüfer domain, there was no indication that the condition was sufficient. In fact, at the time it seemed that the condition might be vacuous; it seemed possible that the only almost Dedekind domains with all residue fields finite were actually Dedekind.

There were a few examples of non-Notherian almost Dedekind domains in the literature. The first example is due to Nakano [15]: the ring of integers A_K of the infinite algebraic extension $K = \mathbb{Q}(\zeta_2, \zeta_3, \zeta_5, \zeta_7, \dots)$ of $\mathbb{Q}$, where ζ_p is a primitive pth root of unity. Subsequently, there were several constructions of such domains, all due to Gilmer along with several co-authors. (A good summary of these constructions is contained in [6].) These constructions include, for example, those obtained as a Kronecker function ring or as a monoid ring. However, all of these non-Noetherian almost Dedekind domain examples contain at least one maximal ideal with infinite residue field, and hence fail Chabert's necessary condition.

In 1990 Gilmer [6] filled this gap by providing examples of non-Noetherian almost Dedekind domains which have all finite residue fields. The construction involves infinite degree algebraic extensions of algebraic number rings (or more general Dedekind domains). In the standard setting of algebraic number theory one takes a finite degree algebraic extension of a number field. In the corresponding rings of integers a prime in the smaller ring either extends to a prime in the upper ring (inertia), or extends to a power of a prime (ramification), or to a product of primes (splitting/decomposition), or to a combination of the three.

To see what is needed in such a construction consider three different cases. In each case, let V be a valuation domain with maximal ideal M generated by d, finite residue field of order q, and quotient field K. Let L be an algebraic extension of K of degree n.

1. (Ramification) Suppose that V extends to a valuation domain W in L, but that $MW = Q^n$ where Q is the maximal ideal of W. Then W will still have a principal maximal ideal, but it will not be generated by d. Rather, d generates the nth power of Q.
2. (Inertia) Suppose that V extends to a valuation domain W and that MW is the maximal ideal of W. Then d will generate the maximal ideal of W, but the residue field in W will have order q^n.
3. (Splitting/Decomposition) Suppose that V extends to a domain W which has n maximal ideals. Then each maximal ideal is locally generated by d and each residue field has order q.

If we start then with a Dedekind domain with all residue fields finite it is intuitively clear that the way to obtain an almost Dedekind domain with all residue fields finite from an infinite degree algebraic extension is to sharply curtail both inertia and ramification in the finite algebraic extensions. The "ideal" type of extension would be one where a prime in the extension field has the same residue field and is locally generated by the same element as the prime it lies over in the lower field. This is called an immediate extension. An infinite degree extension of a one-dimensional Prüfer domain is still a one-dimensional Prüfer domain. Begin with a DVR V with maximal ideal P and with a finite residue field and then consider an infinite degree extension. Each maximal ideal of the extension corresponds to a branch of a tree following the primes at successive stages, lying over P. But if one branch involves infinitely many stages with nontrivial ramification then localization at a maximal ideal will yield a non-discrete valuation domain rather than a DVR. And if there are an infinite number of stages in a single branch that involve inertia then the resulting domain will have infinite residue fields. It is not generally possible to control the behavior of an infinite number of primes in a finite extension. Gilmer's method however, employed a deep result of Krull [8], to start with a single valuation domain and then to build a tower of finite degree extensions such that at each stage the collection of all primes (necessarily finite) is completely controlled. In particular, if we start with the unique prime P in V then follow a single line of primes lying over it then we can arrange things so that on that single branch we have only immediate extensions from some finite stage onward. This will yield an almost Dedekind domain with finite residue fields.

However, once the desired domains had been constructed, it was apparent that their behavior was not necessarily like that of Dedekind domains. Note that a finite residue field must have order a power of some prime p. In a Dedekind domain there can only be finitely many maximal ideals with residue fields of characteristic p. But in an almost Dedekind domain there can be a prime number p such that there are infinitely many maximal ideals M_i with residue fields having order a power of p. And Gilmer was able to build such a domain in which the sizes of these residue fields of characteristic p are unbounded. The idea is that on each branch the extensions are immediate from some point on, but looking from one branch to another we can have inertial behavior happening at arbitrarily high levels. In such an almost

Dedekind domain D Gilmer was able to find a distinguished maximal ideal M such that $\text{Int}(D) \subseteq D_M[x]$. This demonstrates that $\text{Int}(D)$ is not a Prüfer domain since $D_M[x]$ is not a Prüfer domain and all overrings of a Prüfer domain are again Prüfer domains. On the other hand, Gilmer also constructed some non-Noetherian almost Dedekind domains for which the orders of the residue fields of characteristic p is a bounded set, and in such cases he proved that $\text{Int}(D)$ is a Prüfer domain. Accordingly, he posed the following question (slightly paraphrased here)?

Question 2.3 *If D is an almost Dedekind domain such that all residue fields of characteristic p are of bounded size, is* $\text{Int}(D)$ *a Prüfer domain?*

Note that the question only deals with the question of sufficiency. Within the setting of construction by means of infinite degree algebraic field extension, Gilmer had proven necessity of the boundedness condition.

Chabert [5] approached Gilmer's question and answered it negatively. Chabert made use of Hasse's existence theorem [7], which, along the same lines as Gilmer's use of Krull's theorem, allowed him to find an algebraic extension in which the behavior of a finite number of primes can be completely controlled. To understand Chabert's method, suppose first that we are working in characteristic zero. Now if D is an almost Dedekind domain with finite residue field then each maximal ideal must contain a rational prime number. Start with a DVR with finite residue field such that 2 is in the maximal ideal. Since D is a DVR then 2 generates some power M^n of the maximal ideal M. In an almost Dedekind extension the exponent n such that $(2)D_M = M^n D_M$ varies from one maximal ideal M to another. Chabert's method in this example however, was to shut inertia down completely in the algebraic extensions so that the residue fields stayed small, but to include enough ramification that the exponents n satisfying $(2)D_M = M^n D_M$ were unbounded as M ranged across the maximal ideals containing 2. As with Gilmer's negative examples, in Chabert's examples that had unbounded ramification he was able to prove that $\text{Int}(D)$ was not a Prüfer domain by finding a distinguished maximal ideal M such that $\text{Int}(D) \subseteq D_M[x]$. Following we explain Chabert's proposed modification of Gilmer's conjecture (somewhat paraphrased here).

First, consider the following two conditions on an almost Dedekind domain D with all residue fields finite.

1. Choose a prime integer p. We say that D satisfies the first boundedness condition if there is a bound on the cardinalities of the residue fields of order a power of p for each prime p.
2. The second condition is not as simply stated. We give it in two parts.

   - If D has characteristic 0 then each maximal ideal must contain exactly one prime number. If D has characteristic p then D must contain a finite field F. Choose F to have maximal order—note that D cannot contain an infinite field, because then the residue fields would not be finite. In the characteristic p case there must also be an element $t \in D$ such that t is transcendental over F. Hence, the polynomial ring $F[t] \subseteq D$. Then each maximal ideal of D must contain exactly one irreducible polynomial from $F[t]$. These irreducible polynomials play the same role as the prime numbers do in the characteristic 0 case.

- For ease of exposition assume that D has characteristic 0. Choose a prime number p. For each maximal ideal M containing p consider the integer n such that $pD_M = (M^n)D_M$. Call n a ramification index. We say that D satisfies the second boundedness condition if the collection of ramification indices is bounded for each prime p.

An almost Dedekind domain which satisfies the above conditions is said to be doubly-bounded. This then led Chabert to the following question.

Question 2.4 *Suppose D is an almost Dedekind domain with all residue fields that is doubly-bounded. Is* $\mathrm{Int}(D)$ *Prüfer?*

Chabert's question turned out eventually to precisely give the necessary and sufficient conditions for $\mathrm{Int}(D)$ to be a Prüfer domain. As with Gilmer's question, Chabert's questions dealt only with sufficiency. The reason for this is that both were able to prove the necessity of the boundedness conditions in the special setting of the constructions they employed. In particular, they began with a Dedekind domain, took a countably generated algebraic extension of the quotient field, and produced the desired almost Dedekind domain in the field extension. So the sufficiency question was still outstanding, and the necessity question would be still outstanding if it could be shown that non-Noetherian almost Dedekind domains with finite residue fields could be constructed that were built without utilizing a countably generated algebraic field extension.

At the same time as he analyzed a two-part condition which he knew to be necessary under certain conditions, Chabert also considered a condition which he could prove was sufficient

- For M a maximal ideal of D let $S = D - M$. Then $\mathrm{Int}(D)$ is said to *behave well under localization* if $S^{-1}\,\mathrm{Int}(D) = \mathrm{Int}(D_M)$ for each maximal ideal M of D. Chabert proved:

Theorem 2.5 *Let D be an almost Dedekind domain with finite residue fields. If* $\mathrm{Int}(D)$ *behaves well under localization, it is a Prüfer domain.*

This clearly leads to a question about necessity:

Question 2.6 *If* $\mathrm{Int}(D)$ *is a Prüfer domain, does it necessarily behave well under localization?*

In some sense, the property of good behavior under localization would not be a satisfactory resolution of the characterization question because it attempts to equate two properties of $\mathrm{Int}(D)$ rather than equating the Prüfer property of $\mathrm{Int}(D)$ with a property of D. However, in the particular case of almost Dedekind domains defined by countably infinite degree algebraic field extensions, Chabert was able to show that good behavior under localization was equivalent to a property of D which he called the *immediate subextension property*. This property imposed a strong finiteness condition on the manner in which properties of valuation domains could be modified as one went up and down the ladder of an infinite degree field extension. We explain more precisely below.

- Let K_0 be a field and let K be a countably generated algebraic extension of K_0. Let D_0 be a Dedekind domain with quotient field K_0 and let D be an almost Dedekind domain with quotient field K such that every maximal ideal of D lies over a maximal ideal of D_0.
- Choose a maximal ideal M of D. Then we can associate other maximal ideals M_i of D with M by
  - Choose an intermediate field K^* between K_0 and K.
  - Contract the valuation domain D_M to a valuation domain V^* contained in K^*.
  - Consider all the valuation overrings of D which are extensions of V^*. Consider these valuation domains to be associated with D_M.
- Then we say that D has the immediate subextension property if for every D_M we can find a field K^* which is finitely generated over K_0 such that when we restrict D_M to a valuation domain V^* of K^* and then pull back up to all the valuation overrings of D which are extensions of V^*, then for all D_M and all the valuation domains thus associated with it the extensions are immediate.

A modified form of Theorem 2.5 is then

Theorem 2.7 *Let D be an almost Dedekind domain with finite residue fields. If D is constructed using a countably infinite algebraic field extension and satisfies the immediate subextension property then* $\mathrm{Int}(D)$ *is a Prüfer domain.*

So we pose a modification of Question 2.6.

Question 2.8 *If* $\mathrm{Int}(D)$ *is a Prüfer domain, does D have the immediate subextension property?*

This question focuses on a property of D, but it is restricted to just those domains built using countably infinite algebraic field extensions. In any case, the properties of behaving well under localization and immediate subextension turned out not to be the properties that characterize when $\mathrm{Int}(D)$ is a Prüfer domain. Nonetheless, they are important because they illustrate the topological nature of resolving the classification problem in the general case. In particular, it seems reasonable that for a Dedekind domain, since any nonzero element is contained in only finitely many prime ideals then perhaps the only convergence that could happen with prime ideals is convergence to the zero ideal. But if an almost Dedekind domain were not

Noetherian then a nonzero element could be contained in infinitely many prime ideals and nontrivial convergence of some sort could happen. A model for this idea is the behavior of $\text{Int}(Z)$. The maximal ideals containing a given prime p are naturally indexed by the p-adic numbers. Hence, one might expect these maximal ideals to have topological properties relative to each other matching the topology of the p-adic integers. In the negative examples of Gilmer and Chabert the proof that $\text{Int}(D)$ was not a Prüfer domain was accomplished by finding a maximal ideal M of D such that $\text{Int}(D) \subseteq D_M[x]$. Also, in both cases there were infinite collections of maximal ideals for which a particular index was unbounded on the collection. So it seems plausible to try to locate the distinguished maximal ideal as a limit of a sequence of maximal ideals which has the relevant index going to infinity. With this intuitive idea in mind, Loper defined what seemed to be perhaps the simplest possible class of non-Noetherian almost Dedekind domains.

A *sequence domain* is a non-Noetherian almost Dedekind domain D with finite residue fields and field of fractions K such that the following conditions hold:

1. There exists a collection of maximal ideals $S = \{P_i\}_{i=1}^{\infty}$ of D such that
   a. $D = \cap_{i=1}^{\infty} D_{P_i}$,
   b. each residue field D/P_i has the same characteristic p,
   c. the collection $\{P_i\}_{i=1}^{\infty}$ does not constitute all of the maximal ideals of D.
2. There exists a collection $\{v_i\}_{i=1}^{\infty}$ of valuations on K such that
   a. $v_i^{(N)}$ is the normed valuation on K corresponding to P_i for each i,
   b. for all $d \in D\backslash\{0\}$, the sequence $\{v_i(d)\}_{i=1}^{\infty}$ is eventually constant,
   c. for all $d \in D\backslash\{0\}$, $v^*(d) = \lim_{i\to\infty} v_i(d) \in \mathbb{Z}^+ \cup \{0\}$,
   d. there is $\pi \in D$ such that for all $i \in \mathbb{Z}^+$, $v_i(\pi) = 1$.

Set $P^* = \{d \in D \parallel v^*(d) > 0\} \cup \{0\}$. It turns out that if the residue field of each P_i is finite, then the set $\{P^*, P_1, P_2, \cdots\}$ comprises all of the maximal ideals of the sequence domain D. Moreover, the primes P_i are all principal while P^* is not finitely generated. The idea here is to view P^* as the limit of the sequence $\{P_i\}$. Then the maximal ideals of $\text{Int}(D)$ lying over P^* inherit their properties from sequences of maximal ideals lying over the P_i's rather than from the structure of $\text{Int}(D_{P^*})$. In particular

Theorem 2.9 *If D is a sequence domain, then* $\text{Int}(D)$ *is Prüfer if and only if D is doubly-bounded.*

For sequence domains, double-boundedness translates to the set $\{|D/P_i|\}_{i\in\mathbb{Z}^+\cup\infty}$ being bounded and, for each $d \in D\backslash\{0\}$, the set $\{v_i^{(N)}(d)\}_{i\in\mathbb{Z}^+\cup\infty}$ being bounded. Hence in the setting of sequence domains, the classification question has a complete answer.

This setting also allows insight into whether $\text{Int}(D)$ behaving well under localization is necessary for it to be Prüfer. When D is a sequence domain, the following result characterizes when $\text{Int}(D)$ behaves well under localization:

Theorem 2.10 *If D is a sequence domain, then* $\mathrm{Int}(D)$ *behaves well under localization if and only if both of the following conditions hold:*

1. $q_i = |D/P_i| = |D/P^*|$ *for all but finitely many* $i \in \mathbb{Z}^+$.
2. $v_i = v_i^{(N)}$ *for all but finitely many* $i \in \mathbb{Z}^+$.

The key to both the Prüfer characterization and the good behavior under localization for sequence domains is the same. The behavior of maximal ideals of $\mathrm{Int}(D)$ that lie over P^* is determined by sequences of maximal ideals lying over the P_i's. Consider just the residue field part of this. If the sizes of the residue fields of the P_i's are unbounded then, even though the residue field of P^* is finite, we have $\mathrm{Int}(D) \subseteq D_{P^*}[x]$. which proves that $\mathrm{Int}(D)$ is not Prüfer. So since P^* is a limit of primes with residue fields of cardinalities going to infinity then $\mathrm{Int}(D)$ behaves as if the residue field of P^* was infinite even though it is actually finite. Similarly, examples can be built such that the residue field of each P_i has order p^2 but P^* has residue field of order p and then $\mathrm{Int}(D)$ will have maximal ideals lying over P^* with residue field of order p^2. The integer-valued polynomial ring for such a domain is a Prüfer domain but does not behave well under localization. Thus Question 2.6 has a negative answer. The key again is that $\mathrm{Int}(D)$ respects the limiting process of the maximal ideals of D even when D does not.

The complete classification of all domains D such that $\mathrm{Int}(D)$ is a Prüfer domain came not long after the paper on sequence domains. Loper's proof that double-boundedness is sufficient in [10] was expanded by Cahen and Chabert in [2]. While not presented as such, their proof actually demonstrates sufficiency for the general case. Chabert proved necessity in the case where D is built using a countably infinite algebraic field extension. So what was left was to prove necessity in a general setting. This was done in [9] using the topological ideas in [10]. In particular, ultrafilters were used to find limit primes of unbounded sequences, yielding a maximal ideal M of D such that $\mathrm{Int}(D) \subseteq D_M[x]$.

If D has characteristic zero the final theorem is as follows.

Theorem 2.11 *Let D be an almost Dedekind domain with finite residue fields. Then the following conditions are equivalent.*

1. $\mathrm{Int}(D)$ *is a Prüfer domain.*
2. *For each prime number p which is a nonunit in D, the two sets*

$$F_p = \{|D/P| \parallel p \in P\}$$

and

$$E_p = \{v_P^{(N)}(p) \parallel p \in P\}$$

are bounded sets.

The theorem remains true for fields with nonzero characteristic, provided a suitable irreducible polynomial replaces the prime number p.

3 Int(E, D)

Recall that Int(E, D) is the set of polynomials with coefficients in K that map a subset E of D into D; that is, $\text{Int}(E, D) = \{f \in K[X] \mid f(E) \subseteq D\}$. As with Int$(D)$, the question of when Int(E, D) is a Prüfer domain has been studied; however, it is far from being resolved. Recall that $E \subseteq K$ is a *fractional subset* of D if there is some nonzero element d of D such that dE is a subset of D. In almost all cases, $\text{Int}(E, D) = D$ if $E \subseteq K$ is not a fractional subset of D. In the few cases where Int(E, D) is different from D when E is not a fractional subset, D is not integrally closed. It is easy to see that Int(E, D) is not a Prüfer domain in this case. Moreover, if E is a fractional subset of D and $dE \subseteq D$ with $d \neq 0$ then Int(E, D) is naturally isomorphic to Int(dE, D). We will then assume henceforth that $E \subseteq D$.

There is then a very easy necessary condition. Choose an element $d \in E$. Then the set $\{f(x) \in \text{Int}(E, D) \mid f(d) = 0\}$ is easily seen to be a prime ideal of Int(D). It is also easy to see that the quotient of Int(D) by this prime ideal is D. Hence our necessary condition is

- If E is a fractional subset of D and Int(E, D) is a Prüfer domain, then D is a Prüfer domain.

McQuillan [14] completely settled the case when E is finite. He has shown:

Theorem 3.1 *If E is finite, then* Int(E, D) *is a Prüfer domain if and only if D is a Prüfer domain.*

Since a necessary condition is that D be a Prüfer domain, and it is reasonable to approach the problem locally, the next results consider Int(E, V) where V is a valuation domain. If V is a DVR with finite residue field and $E \subseteq V$ then Int(E, V) is an overring of the Prüfer domain Int(V) and hence a Prüfer domain.

Along this line, Cahen, Chabert, and Loper [3] considered the case of Int(E, V), where E is an infinite subset of a valuation domain V with quotient field K with particular focus on the cases where Int(V) is not a Prüfer domain. Let $I \subset V$ be an ideal of V such that $\cap(I^n) = (0)$, and consider the I-adic completions $\widehat{E}, \widehat{K}, \widehat{V}$ of E, K, V, respectively. We say that E is precompact if $\widehat{E}$ is compact in $\widehat{K}$. The main result of the paper connected to the Prüfer property is a sufficient condition.

Theorem 3.2 *If E is a precompact subset of V, then* Int(E, V) *is Prüfer.*

The key to this theorem is that if E is precompact then E hits only finitely many cosets modulo any nonzero ideal. In this regard, E has many properties in common with the collection of all elements of a DVR with finite residue field. There was no proof of the necessity of this condition.

There is also a curious example in the paper. Let T be the ring of entire functions. It is well known that T is a Bezout domain and that it has many maximal ideals of infinite height. In fact, the height of such a maximal ideal is large enough that the intersection of a chain of prime ideals contained in it cannot be the zero ideal. One consequence of this is that if we localize T at a maximal ideal of infinite height, we obtain a valuation domain V such that $\mathrm{Int}(E, V)$ is a Prüfer domain if and only if E is finite.

Recently, Loper and Werner proved that precompactness is not a necessary condition.

To understand this result let V be a one-dimensional valuation domain that is not discrete. Let $\{d_i\}$ be a sequence of elements of V such that $v(d_i - d_{i+1})$ is an increasing sequence, but does not increase to infinity. We say then that the sequence is pseudo- convergent. If $\{d_i\}$ is a pseudo-convergent sequence and $\alpha \in V$ is such that $v(\alpha - d_i)$ is an increasing sequence then we say that α is a pseudo-limit of the sequence.

It can happen in such a valuation domain V that pseudo-convergent sequences that have pseudo-limits or that do not have pseudo-limits can both exist, with the sequences in both cases not converging in the classical sense. Consider the following examples.

1. Let k be a field. Consider the ring $k[\{x^\alpha\}]$ where α runs over the positive real numbers. We can either think of this as a polynomial ring in powers of x or as a semigroup ring over k. In any case, localize the ring at the maximal ideal generated by the powers of x. The result is a one-dimensional valuation ring V with value group the field of real numbers under addition. For a given power of x, the value is simply the exponent.
2. Consider the sequence $\{x^{\beta_i}\}$ where $\{\beta_i\}$ is an increasing sequence of real numbers converging to 2. Then the sequence $\{x^{\beta_i}\}$ is a pseudo-convergent sequence with x^2 as a pseudo-limit.
3. Let $\{\beta_i\}$ be as above. Then define $y_1 = x^{\beta_1}$ and for $n > 1$ define $y_n = x^{\beta_1} + x^{\beta_2} + \cdots + x^{\beta_n}$. The sequence $\{y_i\}$ is then pseudo-convergent, but does not have a pseudo-limit in V.

Using this type of setup Loper and Werner [12] proved:

Theorem 3.3 *There exists a nondiscrete one-dimensional valuation domain V with a subset E consisting of a pseudo-convergent sequence which does not have a pseudo-limit in V such that* $\mathrm{Int}(E, V)$ *is a Prüfer domain, even though E is not precompact.*

Hence the question of when $\mathrm{Int}(E, D)$ is a Prüfer domain is very far from settled. There is no complete classification for when $\mathrm{Int}(E, D)$ is a Prüfer domain even in the special case where D is a valuation domain. And in the case where D is a valuation domain it is clear that the solution will not mirror the characterization for $\mathrm{Int}(D)$.

4 Generalizations

Let D be domain with quotient field K and let $\overline{K}$ be an algebraic closure of K. If we let $f(x)$ be a polynomial in $K[x]$ and let $\alpha \in \overline{K}$ be integral over D then it is reasonable to ask whether $f(\alpha)$ is still integral over D. Along this line of thought we can define a generalized form of integer-valued polynomial ring.

1. Let A_α be the ring of algebraic integers in the finite degree extension $Q[\alpha]$ of the rational numbers.
2. Let A_∞ be the ring of all algebraic integers.
3. Let A_n be the set of all algebraic integers in A_∞ of degree $\leq n$ over Q.
4. Let $\mathrm{Int}_Q[A_\alpha] = \{f(x) \in Q[x] \parallel f(A_\alpha) \subseteq A_\alpha\} = \mathrm{Int}(A_\alpha) \cap Q[x]$
5. Let $\mathrm{Int}_Q(A_n) = \{f(x) \in Q[x] \parallel f(A_n) \subseteq A_n\} = \bigcap_{[Q[\alpha]:Q]\leq n} \mathrm{Int}(A_\alpha) \cap Q[x]$

Using the above constructions Loper and Werner [11] proved the following theorem.

Theorem 4.1 *Let A_α and A_n be as above. Then* $\mathrm{Int}_Q[A_\alpha]$ *and* $\mathrm{Int}_Q(A_n)$ *are Prüfer domains.*

Moreover, they give a strong answer to a question posed by Brizolis in the paper where Prüfer rings of integer-valued polynomials were introduced. Brizolis wondered whether a proper subring of $\mathrm{Int}(Z)$ existed which had $Q(x)$ as quotient field and was a Prüfer domain. Chabert answered this question in [4] by demonstrating that if we let $E = \frac{1}{2}Z$ be the fractional ideal of Z generated by $1/2$ then $\mathrm{Int}(E, Z)$ is a proper subring of $\mathrm{Int}(Z)$ and is isomorphic to $\mathrm{Int}(Z)$. Note however, that $2x$ lies in the ring $\mathrm{Int}(E, Z)$ but x does not. A stronger question is whether there exists a Prüfer domain which lies properly between $Z[x]$ and $Q[x]$. The theorem above demonstrates that such domains do exist.

The paper [11] also generalizes a little farther. Let I be the $n \times n$ identity matrix, let α be a rational number and identify α with the diagonal matrix αI. With this identification we can choose a polynomial $f(x)$ over the rational numbers and evaluate at an $n \times n$ matrix M with integer entries. It is then reasonable to ask which polynomials with rational coefficients map integral $n \times n$ matrices to integral $n \times n$ matrices. Let $M_n(Z)$ be the ring of $n \times n$ matrices over the integers. We then define $\mathrm{Int}_Q(M_n(Z))$ to be the ring of all polynomials over the rational numbers which map $M_n(Z)$ to $M_n(Z)$. Since each such matrix satisfies a monic polynomial over the integers it seems natural to identify this ring with $\mathrm{Int}_Q(A_n)$. However, let M be a nonzero matrix such that $M^2 = 0$. Then $f(x) = x^2/n^2$ will map M to 0 for any positive integer n, but for all but finitely many integers $g(x) = x/n$ will map M to a matrix with entries not lying in the integers. This suggests that $\mathrm{Int}_Q(M_n(Z))$ is not integrally closed. Accordingly, Loper and Werner proved the following theorem.

Theorem 4.2 $\mathrm{Int}_Q(M_n(Z))$ *is not integrally closed but has integral closure equal to* $\mathrm{Int}_Q(A_n)$*, which is a Prüfer domain.*

Along the same lines as the above results, Peruginelli [17] has very recently extended McQuillan's results concerning integer-valued polynomials over finite sets.

Theorem 4.3 *Let D be an integrally closed domain with quotient field K, and let A be a torsion-free, finitely generated D-algebra. Let $E \subseteq A$ be a finite set of elements and consider the ring* $\mathrm{Int}_K(E, A)$ *of polynomials with coefficients in K which map E into A. Then the integral closure of* $\mathrm{Int}_K(E, A)$ *is a Prüfer domain if and only if D is a Prüfer domain.*

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