Noetherian Semigroup Algebras and Beyond

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Abstract A selection of results on Noetherian semigroup algebras is presented. They are of structural, arithmetical, and combinatorial nature. Starting with the case of Noetherian group algebras, where several deep results are known, a lot of attention is later given to the case of algebras of submonoids of groups. The role of algebras of this type in the general theory of Noetherian semigroup algebras is explained and sample structural results on arbitrary Noetherian semigroup algebras, based on this approach, are presented. A special emphasis is on various classes of algebras with good arithmetical properties, such as maximal orders and principal ideal rings. In this context, several results indicating the nature and applications of the structure of prime ideals are presented. Recent results on the prime spectrum and arithmetics of a class of non-Noetherian orders are also given.

1 Introduction

The aim of this paper is to present selected representative results on Noetherian semigroup algebras $K[S]$, where S is a monoid and K is a field. The results are both of structural, arithmetical, and combinatorial nature. In particular, we present an approach exploiting in this context linear semigroups over division rings, and indicating the role of cancellative subsemigroups of S. An emphasis is therefore made on the case of Noetherian algebras $K[S]$ of submonoids S of polycyclic-by-finite groups. Certain concrete classes of such algebras that arise independently in other contexts and that motivate the general theory are presented. Hence, we first summarize some of the relevant results on Noetherian group algebras. Results on the structure of an arbitrary monoid S that yield certain necessary and sufficient conditions for $K[S]$ to be Noetherian are then presented. Some advantages of this

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approach are illustrated, in particular in the context of polynomial identities and the Gelfand–Kirillov dimension. A special attention is given to prime Noetherian orders in simple artinian rings, and especially maximal orders. We conclude with recent developments that indicate that certain non-Noetherian orders are of interest and importance, but at the same time they seem to be very difficult to study.

We start in Sect. 2 with important results on Noetherian group algebras that are relevant for the rest of the paper. In Sect. 3 an approach to the structure of general Noetherian semigroup algebras is presented. This is via semigroups of matrices over a field, or a division ring, and this is based on finite ideal chains of some specific type that arise in a natural way. In particular, this explains the role of cancellative subsemigroups of the given semigroup S in the general theory. And this also explains our focus on the case where S is a submonoid of a polycyclic-by-finite group. An intriguing class of examples arising from a different context is presented in Sect. 4. In Sect. 5 we discuss certain classical arithmetical properties, especially in the context of orders in simple artinian algebras. In particular, these include maximal orders and principal ideal rings. The special role that is played by the prime ideals is explained. In the final Sect. 6 a recent example of a non-Noetherian order, arising from considerations in noncommutative geometry, is presented. It motivates a new area of study, namely non-Noetherian orders coming from submonoids of nilpotent groups. We conclude with some recent results in this direction.

Throughout the paper, K will denote a field and S a monoid (a semigroup with a unity element) with operation written multiplicatively. The corresponding semigroup algebra is denoted by $K[S]$. If S has a zero element θ then $K\theta$ is a 1-dimensional ideal of $K[S]$ and $K_0[S] = K[S]/K\theta$ is called the contracted semigroup algebra of S over K. In other words, we identify the zero of S with the zero of the algebra.

Our basic references for the results and methods of the theory of group algebras and semigroup algebras are [38, 51, 54, 57, 58], while we refer to [27, 50] for an extensive background on noncommutative Noetherian rings.

2 Group Algebras—Introductory Results

The class of Noetherian group algebras is one of our starting points. Recall that a polycyclic-by-finite group is a group with a finite subnormal series whose every factor is either finite or cyclic, [61]. We have the following classical result.

Theorem 2.1 *Let G be a polycyclic-by-finite group. Then $K[G]$ is Noetherian.*

The idea of the proof is easy. It is based on an induction on the length of a subnormal chain of G with finite and cyclic factors. Let $H \subseteq F$ be two consecutive factors of such a chain. Assume that $K[H]$ is Noetherian. If $[F : H] < \infty$, then we have a finite module extension $K[H] \subseteq K[F]$. So $K[F]$ is Noetherian. If F/H is infinite cyclic, then an argument similar to that in the proof of Hilbert basis theorem is used to show that $K[F]$ is also Noetherian.

We note that it is not known whether there exist classes of Noetherian group algebras other than those described in Theorem 2.1.

The second point of departure is the class of commutative semigroup rings. The following result is attributed to Budach, see [23], Theorem 5.10.

Theorem 2.2 *Assume that S is a commutative monoid. Then $K[S]$ is Noetherian if and only if S is finitely generated.*

The proof of the nontrivial implication (the necessity) is based on a decomposition theory for congruences of a commutative monoid with acc on congruences, on properties of irreducible congruences and of cancellative congruences.

In many other important cases one can also show that 'noetherian' implies 'finitely generated'. Recall that the Gelfand–Kirillov dimension of a finitely generated algebra R over K is finite if the growth function $d_V(n)$ is bounded by a polynomial in n. Here V is a finite dimensional generating subspace of R and $d_V(n) = \dim_K(V + V^2 + \cdots + V^n)$. Then $\limsup(\log d_V(n)/\log(n))$ is called the Gelfand–Kirillov dimension of R and is denoted by $\operatorname{GKdim}(R)$, see [47]. In general, it is not an integer. On the other hand, in the class of commutative algebras, this dimension coincides with the classical Krull dimension.

Theorem 2.3 ([33, 38]) *Assume $K[S]$ is right Noetherian. Then S is finitely generated in each of the following cases:*

1. *S satisfies acc on left ideals (this holds in particular if $K[S]$ is also left Noetherian),*
2. *$K[S]$ satisfies a polynomial identity,*
3. *the Gelfand–Kirillov dimension of $K[S]$ is finite.*

It is not known whether the assertion of the above theorem is true for an arbitrary right Noetherian algebra $K[S]$. This is not known even in the case where S is cancellative (in this case, S has group of classical right quotients G, because of the acc on right ideals; whence $K[G]$ is a classical localization of $K[S]$ and it is also Noetherian).

There are several deep results on the prime spectrum of Noetherian group algebras. We mention some highlights, that will be also used in Sect. 6. The first is due to Zalesskii, see [57], Corollary 11.4.6. Recall that a prime ideal P of $K[G]$ is faithful if the normal subgroup $\{g \in G \mid g - 1 \in P\}$ of G is trivial. By $\mathrm{Z}(G)$ we denote the center of G.

Theorem 2.4 *Assume that G is a finitely generated torsion free nilpotent group. There is a bijection between the set of faithful primes in $K[G]$ and faithful primes of $K[\mathrm{Z}(G)]$, given by:*

$$Q \longrightarrow Q \cap K[\mathrm{Z}(G)], \quad P \longrightarrow P \cdot K[G].$$

The above, together with a reduction to a torsion free subgroup of finite index, is one of the steps of the following result of Smith, [57], Theorem 11.4.9. Recall that the Hirsch length $\mathrm{h}(G)$ of a polycyclic-by-finite group G is defined as the number of infinite cyclic factors in a subnormal chain in G with cyclic or finite factors (which is independent of the chosen chain). By $\mathrm{clKdim}(R)$ we denote the classical Krull dimension of an algebra R.

Theorem 2.5 *Assume that G is a finitely generated nilpotent group. Then*

$$\mathrm{clKdim}(K[G]) = \mathrm{h}(G).$$

Let us note that in the more general polycyclic-by-finite case, a more complicated invariant, called the plinth length of G (in general, not exceeding $\mathrm{h}(G)$), see [58], page 192, plays the role of $\mathrm{h}(G)$, by a result of Roseblade [60].

The known Noetherian group algebras share a very important property of finitely generated commutative algebras, called catenarity. Recall that the latter means that every two saturated chains of primes between any two given prime ideals $P \subset P'$ have equal lengths.

Theorem 2.6 ([49]) *The group algebra $K[G]$ of a polycyclic-by-finite group G is catenary.*

The following is an immediate consequence of the fact that polycyclic-by-finite groups are finitely presented, see [61], Theorem 8.4.

Theorem 2.7 *If G is a polycyclic-by-finite group, then the algebra $K[G]$ is finitely presented.*

As a consequence of the structural characterization obtained in Theorem 2.11, one can prove the following corollary, which settles a general framework for the results presented in Sect. 4.

Corollary 2.8 ([34]) *Let S be a submonoid of a polycyclic-by-finite group. If S satisfies the ascending chain condition on right ideals, then S is a finitely presented monoid. In particular, the semigroup algebra $K[S]$ is finitely presented.*

From the point of view of the theory of orders in division rings, or more generally in simple artinian rings, the following classical results of Connell on prime rings, see [57], Theorem 4.2.10, and of Farkas and Snider, [57], Theorem 13.4.18, and Cliff [11] (domains of zero and positive characteristic, respectively) are of basic interest.

Theorem 2.9 *Let G be a group. Then*

1. *$K[G]$ is prime if and only if G has no nontrivial finite normal subgroups.*
2. *If G is polycyclic-by-finite, then $K[G]$ is a domain if and only if G is torsion free.*

Orders of the form $K[G]$ are interesting also from the point of view of the associated division rings, as they supply a rich class of examples.

Theorem 2.10 ([15]) *Let G, H be non-isomorphic finitely generated nilpotent torsion free groups. Then the classical division rings of quotients $Q_{cl}(K[G])$ and $Q_{cl}(K[H])$ are not isomorphic.*

As said above, if $K[S]$ is right Noetherian for a submonoid S of a group G then S has a group of right quotients isomorphic to $SS^{-1} \subseteq G$ and $K[G]$ is Noetherian. The case where S is a submonoid of a polycyclic-by-finite group is therefore of special interest; first because of Theorem 2.1, second, because of some important examples discussed in Sect. 4, third because of a general structural approach explained in Sect. 3.

The following complete result comes from [37], while some partial steps were earlier made in [33, 34].

Theorem 2.11 ([37]) *Let S be a submonoid of a polycyclic-by-finite group. Then the following conditions are equivalent:*

1. *$K[S]$ is right Noetherian,*
2. *S satisfies acc on right ideals,*
3. *S has a group of quotients G and there exists a normal subgroup H of G such that: $[G : H] < \infty$, $S \cap H$ is finitely generated and the derived subgroup $[H, H] \subseteq S$,*
4. *$K[S]$ is left Noetherian.*

In the above notation, let $F = [H, H]$. So, in some sense, such $K[S]$ can be approached in two steps: from the perspective of the Noetherian group algebra $K[F] \subseteq K[S]$ and of the Noetherian PI-algebra $K[S/F] \subseteq K[G/F]$. Recall that the general theory provides additional strong tools in the class of Noetherian PI-algebras, [50]. In particular, finitely generated PI-algebras are catenary, see [50], Corollary 13.10.13.

3 A General Structural Approach

In this section we present a structural approach to arbitrary Noetherian semigroup algebras $K[S]$. It is based on finite ideal chains of S of a very special type. Such chains arise naturally in the study of linear semigroups and for this reason they seem unavoidable in the context of Noetherian algebras $K[S]$. On the one hand, they allow to prove certain necessary and sufficient conditions for S in order that $K[S]$ is Noetherian. On the other hand, they allow to reduce several problems to submonoids of groups, and hence to group algebras. They also are very useful in the case of certain families of algebras arising from other contexts, which will be reflected in Sect. 4.

Let X, Y be arbitrary nonempty sets and let $P = (p_{yx})$ be a $Y \times X$-matrix with entries in $T^0 = T \cup \{0\}$, for a monoid T. So, strictly speaking, P is a mapping

$Y \times X \longrightarrow T \cup \{0\}$. Let $\mathscr{M}(T, X, Y, P)$ be the set of all $X \times Y$-matrices with entries in $T \cup \{0\}$ but with at most one nonzero entry. Such a nonzero matrix can be denoted by (g, x, y) (with $g \in T$ in position (x, y)). Multiplication, called sandwich multiplication, is defined as follows:

$$a \circ b = aPb$$

where in the right hand side one uses the standard matrix products.

Assume also that the 'sandwich matrix' P has no nonzero rows or columns and $T = G$ is a group. Then $M = \mathscr{M}(G, X, Y, P)$ is called a completely 0-simple semigroup over the group G with sandwich matrix P. It has no ideals other than M and $\{0\}$ and it can bee considered as a semigroup analogue of a simple artinian ring. Such semigroups play a prominent role in semigroup theory, see [12], §2.7 and §3.2. The nonzero maximal subgroups of $\mathscr{M}(G, X, Y, P)$ are all isomorphic to G, they are of the form $G_{xy} = \{(g, x, y) \mid g \in G\}$, where $p_{yx} \neq 0$.

A subsemigroup S of $\mathscr{M}(G, X, Y, P)$ such that S intersects nontrivially every set $M_{xy} = \{(g, x, y) \mid g \in G\}, x \in X, y \in Y$, is called a uniform (sub)semigroup.

The case when $X = Y$ and $P = \Delta$, the identity matrix, is of special interest. If $|X| = r < \infty$ then we write $\mathscr{M}(G, r, r, \Delta)$. In this case, the contracted semigroup algebra $K_0[\mathscr{M}(G, r, r, \Delta)]$ is isomorphic to the matrix algebra $M_r(K[G])$. A uniform subsemigroup S of $\mathscr{M}(G, r, r, \Delta)$ is called a semigroup of generalized matrix type. So $K_0[S] \subseteq M_r(K[G])$.

Let $S \subseteq M = \mathscr{M}(G, X, Y, P)$ be a uniform subsemigroup. One can show that there exists a unique subgroup H of G and a sandwich matrix Q over H^0 so that $S \subseteq M \cong \mathscr{M}(H, X, Y, Q)$ and (if S is identified with a subsemigroup of $\mathscr{M}(H, X, Y, Q)$) every maximal subgroup of $\mathscr{M}(H, X, Y, Q)$ is generated as a group by its intersection with S. So, intuitively, one is tempted to think of $\mathscr{M}(G, X, Y, P)$ as a 'semigroup of quotients of S'. If one prefers, one can consider S as an order in $\mathscr{M}(G, X, Y, P)$. This can be given a very precise meaning if additionally H is a group of quotients of $S \cap H$, see [17]. The latter holds for example if $K[S]$ satisfies a polynomial identity or if S has acc on right ideals.

If I is an ideal of a semigroup S then the Rees factor S/I is defined as the set $(S \setminus I) \cup \{0\}$ with the operation $s \cdot t = st$ if $st \in S \setminus I$ and $s \cdot t = 0$ otherwise. A structure theorem, obtained in [52], see also [54], Theorem 3.5, reads as follows.

Theorem 3.1 *If S is a subsemigroup of the multiplicative monoid $M_n(F)$ of all $n \times n$-matrices over a field F, then S has a finite ideal chain $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_k = S$ with I_1 and every factor I_j/I_{j-1} nilpotent or a uniform semigroup. The same applies if F is a division ring and S satisfies the ascending chain condition on right ideals.*

In particular, the second part applies to the case where $K[S]$ is right Noetherian and embeds into $M_n(D)$ for a division ring D.

Clearly, the simplest example is $S = M_n(F)$, for a field F. Then the chain $M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n = M_n(F)$ defined by $M_j = \{a \in M_n(F) \mid rank(a) \leq j\}$ has all factors completely 0-simple. The maximal subgroups of S are of the form $H_e = \{a \in eM_n(F)e \mid \mathrm{rank}(a) = \mathrm{rank}(e)\}$, where $e = e^2 \in M_n(F)$ and they are isomorphic to the corresponding full linear groups $Gl_j(F)$, $j = rank(e)$.

The following important theorem is an extension of the classical result of Malcev saying that a finitely generated commutative algebra is embeddable in a matrix ring over a field.

Theorem 3.2 ([2]) *Let R be a finitely generated right Noetherian PI-algebra. Then R embeds into the matrix ring $M_n(F)$ over a field extension F of the base field K.*

So, in view of Theorem 2.3, Theorem 3.1 can be applied if $R = K[S]$ is right Noetherian and satisfies a polynomial identity. Moreover, since every semiprime Noetherian algebra has a semisimple artinian classical quotient ring, it follows that Theorem 3.1 applies also to $K[S]/B(K[S])$ ($B(K[S])$ denoting the prime radical of $K[S]$) as well as to every prime homomorphic image $K[S]/P$ of $K[S]$. So, such an S has a finite ideal chain with all factors nilpotent or uniform.

One can show that even more is true in certain other cases.

Theorem 3.3 ([38, 55]) *Let S be a monoid such that $K[S]$ is left and right Noetherian and* $\mathrm{GKdim}(K[S]) < \infty$. *If for every $a, b \in S$ one has*

$$a\langle a, b\rangle \cap b\langle a, b\rangle \neq \emptyset \neq \langle a, b\rangle a \cap \langle a, b\rangle b,$$

then S has an ideal chain $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_n = S$ such that S_1 and every factor S_i/S_{i-1} is either nilpotent or a semigroup of generalized matrix type.

More importantly, the following partial converse of this theorem holds.

Theorem 3.4 ([55]) *Let S be a finitely generated monoid with an ideal chain $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_n = S$ such that S_1 and every factor S_i/S_{i-1} is either nilpotent or a semigroup of generalized matrix type. If* $\mathrm{GKdim}(K[S]) < \infty$ *and S satisfies the ascending chain condition on right ideals, then $K[S]$ is right Noetherian.*

The assumptions in the theorem imply that cancellative subsemigroups of uniform factors S_i/S_{i-1} and S_1 have groups of quotients that are finitely generated and nilpotent-by-finite (so polycyclic-by-finite, in particular).

As an example of an application of this strategy to some problems of a combinatorial nature, we state the following result. Recall that the prime radical $B(K[S])$ of $K[S]$ is nilpotent if $K[S]$ is a right Noetherian algebra.

Theorem 3.5 ([53]) *Assume that $K[S]$ a right Noetherian algebra. Then*

1. *the Gelfand–Kirillov dimension of $K[S]$ is finite if and only if for every cancellative subsemigroup T of S we have* $\mathrm{GKdim}(K[T]) < \infty$.
2. *Moreover, in this case* $\mathrm{GKdim}(K[S]/B(K[S])) = \mathrm{GKdim}(K[T])$ *(and it is an integer) for a cancellative subsemigroup T of the image $\overline{S}$ of S in* $K[S]/B(K[S])$ *and*

GKdim$(K[S]) \leq r \cdot$ GKdim$(K[T])$, *where r is the nilpotency index of $B(K[S])$. Moreover, T has a finitely generated nilpotent-by-finite quotient group.*

Notice, that by a celebrated result of Gromov, the Gelfand–Kirillov dimension of a finitely generated group algebra $K[G]$ is finite if and only if G is nilpotent-by-finite, and in this case due to the formula of Bass it is an integer expressible in terms of the ranks of the (torsion free) factors of the upper central series of a nilpotent subgroup of finite index in G, see [47], Chap. 11. Moreover, GKdim$(K[T]) =$ GKdim$(K[G])$ if G is the group of quotients of its submonoid T, by a result of Grigorchuk, see [51], Chap. 8.

4 Important Motivating Examples—Algebras with Homogeneous Quadratic Relations

Important classes of examples of Noetherian semigroup algebras include algebras corresponding to the set theoretic solutions of the Yang-Baxter equation. Recall that by a set theoretic solution of the Yang-Baxter equation we mean a map $r : X \times X \to X \times X$, where X is a nonempty set, such that

$$r_{12}r_{13}r_{23} = r_{23}r_{13}r_{12},$$

where r_{ij} denotes the map $X \times X \times X \to X \times X \times X$ acting as r on the (i, j) factor and as the identity on the remaining factor. We will focus on the case where $X = \{x_1, \dots, x_n\}$ is finite.

The problem of finding all such solutions was posed in [13], and turned out to be very difficult. In particular, one considers solutions that are involutive ($r^2 = id$) and non-degenerate (this condition will be defined later). This area leads to a fascinating class of Noetherian algebras, referred to as Yang-Baxter algebras. Namely, one associates to r an algebra defined by the presentation $K\langle x_1, \dots, x_n\rangle/J$ where J consists of relations of the form $xy = x'y'$ if $r(x, y) = (x', y')$. This implies that J consists of $\binom{n}{2}$ relations and it follows also that every monomial xy, $x, y \in X$, appears in at most one relation.

These algebras arose independently in several other contexts, including homological methods developed for an important class of algebras, called Sklyanin algebras, [62]. The following theorem summarizes their main properties.

Theorem 4.1 ([21]) *These algebras are isomorphic to $K[S]$, where S is a submonoid of a finitely generated torsion free abelian-by-finite group. They are Noetherian PI domains of finite homological dimension and they are maximal orders.*

Actually, S has a group of quotients that is solvable [14], see also [31], and embeds into the semidirect product $F_n \rtimes S_n$, where S_n is the symmetric group acting on the free commutative group F_n of rank n by the natural permutation of the basis, [14, 38]. Simplest examples include commutative polynomial rings $K[x_1, \dots, x_n]$, arising from the free commutative monoids S, and the algebra of the monoid defined by the presentation $S = \langle x, y \mid x^2 = y^2 \rangle$.

These algebras have several other properties similar to the properties of commutative polynomial rings, including nice homological properties. Certain families of such algebras are known, but new examples are very difficult to construct.

Height one prime ideals P of these algebras have been described [31, 36]. In particular, if $P \cap S \neq \emptyset$ then $P = aK[S] = K[S]a$ for some $a \in S$, and there are finitely many such height one primes. While prime ideals of $K[S]$ not intersecting S come from primes of the group algebra $K[SS^{-1}]$ (see Sect. 5). In particular, this can be used to prove that $K[S]$ is a maximal order.

There exist important more general classes of semigroup algebras which fit in this context. We say that an algebra A is defined by homogeneous semigroup relations if it is defined by a presentation $A = K\langle x_1, \ldots, x_n \mid R \rangle$, with every relation of the set R of defining relations of the form $v = w$, where v, w are words in the free monoid on $x_1, \ldots, x_n$ and v, w have equal lengths. Exploiting the approach presented in the previous section, one can prove the following result.

Theorem 4.2 ([20]) *Assume that an algebra $A = K[S]$ is right Noetherian and* $\mathrm{GKdim}(K[S]) < \infty$. *If A is defined by homogeneous semigroup relations, then A satisfies a polynomial identity.*

In particular, consider the following class of quadratic algebras, that generalizes the Yang-Baxter algebras. These are semigroup algebras of monoids with generators $x_1, x_2, \ldots, x_n$ subject to $\binom{n}{2}$ quadratic relations of the form $x_i x_j = x_k x_l$ with $(i, j) \neq (k, l)$ and, moreover, every monomial $x_i x_j$ appears at most once in one of the defining relations. One of the origins of these algebras comes from [18]. Recently, further combinatorial aspects of such algebras have been studied in [19].

For every $x \in X = \{x_1, \ldots, x_n\}$, let

$$f_x : X \to X$$

and

$$g_x : X \to X$$

be the maps such that

$$r(x, y) = (f_x(y), g_y(x)).$$

One says that S is a non-degenerate quadratic monoid if each f_x and each g_x is bijective, with $x \in X$.

The following result was obtained in [20] in the special case of square-free defining relations, and in full generality in [41].

Theorem 4.3 *Let S be a non-degenerate quadratic monoid. Then $K[S]$ is right and left Noetherian, it satisfies a polynomial identity and embeds into a matrix algebra over a field extension of K.*

The proof uses the structural approach explained before and several other results. First, a finite ideal chain in S is constructed from the combinatorial data. Every factor of this chain is either of generalized matrix type or it is nilpotent. Independently, one shows that $K[S]$ has finite Gelfand–Kirillov dimension and that S satisfies acc on one-sided ideals. This allows us to prove that $K[S]$ is Noetherian, by applying Theorem 3.4. Then, using Theorem 4.2, one shows that the algebra satisfies a polynomial identity. Finally, using the embedding theorem of Anan'in, Theorem 3.2, we get the last assertion.

5 Prime Ideals and Arithmetical Properties of $K[S]$

There are several classical important arithmetical properties that have been extensively studied in the class of commutative semigroup rings. These include in particular Krull domains, integrally closed domains, principal ideal rings. For the main results and general techniques of this theory we refer to Gilmer's book [23]. And for general results on commutative orders, and integrally closed domains in particular, to [16]. Several methods and results of the multiplicative ideal theory are valid for both commutative rings and monoids, as they depend only on the multiplicative structure of the ring. The philosophy that such results should be derived as far as possible without making reference to the additive structure of the ring, is presented in particular in [28].

There has been also an extensive work done on noncommutative orders, that we will discuss in this section. Some of this is based on earlier general work on noncommutative orders (in particular, see [1, 8] and its bibliography), some has been developed for special classes of noncommutative semigroups in [64], and more recently in [22].

Recall that a monoid S which has a left and right group of quotients G is called an order. Then S is called a maximal order if there does not exist a submonoid S' of G properly containing S and such that $aS'b \subseteq S$ for some $a, b \in G$.

For subsets $A, B \subseteq G$ we define $(A :_l B) = \{g \in G \mid gB \subseteq A\}$, $(A :_r B) = \{g \in G \mid Bg \subseteq A\}$. Then S is a maximal order if and only $(I :_l I) = (I :_r I) = S$ for every fractional ideal I of S. A nonempty subset I of G is called a fractional ideal of S if $SIS \subseteq I$ and $cI, Id \subseteq S$ for some $c, d \in S$.

Assume now that S is a maximal order. Then $(S :_r I) = (S :_l I)$ for any fractional ideal I. One denotes this set as $(S : I)$. Define $I^* = (S : (S : I))$, the divisorial closure of I. If $I = I^*$ then I is said to be divisorial. Then S is said to be a Krull order (or a Krull monoid in the terminology of [22]) if S satisfies also the ascending chain condition on divisorial ideals contained in S. In this case the divisor group $D(S)$ (also defined as in ring theory) is a free abelian group with basis the set of prime divisorial ideals. The latter are minimal prime ideals of S.

The following result is our starting point. Notice in particular that the obtained description is expressed in terms of the underlying semigroup only. Here $U(S)$ denotes the unit group of the monoid S.

Theorem 5.1 ([10]) *A commutative monoid algebra $K[S]$ is a Krull domain if and only if S is a submonoid of a torsion free abelian group which satisfies the ascending chain condition on cyclic subgroups and S is a Krull order in its group of quotients.*

Furthermore, S is a Krull order if and only if $S = U(S) \times S_1$, where S_1 is a submonoid of a free abelian group F such that S_1 is the intersection of the quotient group of S_1 with the positive cone of F. Moreover, in this situation the class group of $K[S]$ coincides with the class group of S.

This result extended an earlier work of Anderson, [3, 4], on commutative Noetherian maximal orders. Notice that the class of commutative Noetherian maximal orders $K[S]$ coincides with the class of finitely generated integrally closed domains.

The last property mentioned in the theorem allows one to simplify the calculation of the class group in several concrete classes of algebras, and it also shows that the height one primes of $K[S]$ determined by the minimal primes of S are crucial. In particular, for certain concrete finitely presented commutative algebras this invariant was calculated in [26].

Theorem 5.2 *Let A be a finitely generated commutative algebra over a field K with a presentation $A = K[X_1, \ldots, X_n | R]$, where R is a set of monomial relations in the generators $X_1, \ldots, X_n$. So $A = K[S]$, the semigroup algebra of the monoid $S = \langle X_1, \ldots, X_n | R \rangle$. A characterization, purely in terms of the defining relations, is given of when A is an integrally closed domain, provided R contains at most two relations. Also the class group of such algebras A is calculated.*

Also within the noncommutative ring theory, Noetherian orders in simple algebras form an important class of rings. Maximal orders have been studied in this context, in particular for the class of group algebras of polycyclic-by-finite groups. Recall that the infinite dihedral group $\langle a, b \mid ba = a^{-1}b,\ b^2 = 1 \rangle$ is denoted by D_∞. A group G is said to be dihedral-free if the normalizer of any subgroup H isomorphic with D_∞ is of infinite index in G, (equivalently, H has infinitely many conjugates in G).

Theorem 5.3 ([5]) *Let G be a polycyclic-by-finite group. The group algebra $K[G]$ is a prime maximal order if and only if*

1. *G has no nontrivial finite normal subgroups,*
2. *G is dihedral-free.*

The first condition in the theorem is equivalent with the group algebra being prime, see Theorem 2.9. Brown also determined when the height one prime ideals are principal. By $\Delta(G)$ we denote the finite conjugacy subgroup of G.

Theorem 5.4 ([5]) *Let G be a polycyclic-by-finite group. If $K[G]$ is a prime maximal order, then the following conditions are equivalent for a height one prime ideal P of $K[G]$:*

1. *P is right principal,*
2. *P is invertible, that is, $Q_{cl}(K[G])$ contains a $K[G]$- bimodule J with $IJ = JI = K[G]$,*
3. *P is right projective;*
4. *$P = K[G]n = nK[G]$ for some $n \in K[\Delta(G)]$,*
5. *P contains a nonzero central element,*
6. *P contains a nonzero normal element,*
7. *P contains an invertible ideal.*

If these conditions hold for all height one primes of $K[G]$, then $K[G]$ is a UFR (unique factorization ring) in the sense of Chatters and Jordan, [9]. Some earlier partial results on this topic can be found in [42, 43, 63]. The following consequence for the case of PI-algebras that are domains is of special interest.

Theorem 5.5 ([5]) *Let G be a finitely generated torsion free abelian-by-finite group. Then the group algebra $K[G]$ is a Noetherian maximal order. Moreover, all height one primes of $K[G]$ are principally generated by a normal element.*

Only for very few classes of noncommutative semigroups S it has been determined when the semigroup algebra is a Noetherian maximal order. Apart from the Yang-Baxter algebras, see Sect. 4, Wauters in [64] dealt with cancellative semigroups S consisting of normal elements (so $aS = Sa$ for every $a \in S$) and with the cancellative semigroups of the regular elements of a prime Goldie ring. Various aspects of arithmetical properties of noncommutative monoids were recently studied in [22]. We will summarize results obtained on algebras of submonoids of a polycyclic-by-finite group G, obtained in [24, 25, 32, 36].

Recall that G has a normal subgroup of finite index H that is torsion free. Then $K[G]$ can be considered as a ring graded by the finite group G/H in a natural way. Therefore, known deep results on the correspondence of prime ideals for rings graded by finite groups [58], Theorem 17.9, allow to establish a strong link between the primes in $K[G]$ and in $K[H]$ (incomparability, going up, going down). Hence, the information on prime ideals in the torsion free case is essential, [24]. Crucial results on prime ideals in case $K[S]$ is Noetherian and $G = SS^{-1}$, based also on Theorem 2.11, were proved in [24, 34].

Proposition 5.6 ([24]) *Let S be a submonoid of a torsion free polycyclic-by-finite group. Assume that $K[S]$ is right Noetherian. Then*

1. *$K[S \cap P]$ is a prime ideal in $K[S]$ for any prime ideal P in $K[S]$ with $P \cap S \neq \emptyset$.*
2. *$K[Q]$ is a prime ideal in $K[S]$ for any prime ideal Q in S.*
3. *the set of height one prime ideals of $K[S]$ intersecting S nontrivially coincides with the set of the ideals of the form $K[Q]$, where Q is a minimal prime ideal of S.*

In view of the above theorem, the study of prime ideals of $K[S]$ splits into two cases, one leading to primes of the group algebra $K[SS^{-1}]$ and one leading to the primes of the monoid S. Recall that if C is a right Ore subset consisting of regular elements in a ring R, we denote by R_C the classical localization of R with respect to C. If either R is right Noetherian or R satisfies a polynomial identity and R_C is right Noetherian, then the maps $P \mapsto PR_C$, $J \mapsto J \cap R$ are inverse bijections between the sets of prime ideals in R not intersecting C and the set of primes in R_C, see [27], Theorems 10.18 and 10.20 and its proof. This also holds in the following case.

Lemma 5.7 ([29]) *Let S be a submonoid of a nilpotent group and let G be the group of quotients of S. Assume that P is a prime ideal of $K[S]$. If $P \cap S = \emptyset$, then*

1. *$PK[G]$ is a two-sided ideal of $K[G]$,*
2. *$Q = PK[G]$ is a prime ideal of $K[G]$, $Q \cap K[S] = P$ and $K[G]/Q$ is a localization (with respect to an Ore set) of $K[S]/P$.*

We say that a prime Goldie ring R is a Krull order if R is a maximal order that satisfies the ascending chain condition on divisorial integral ideals. In the next theorem we collect some of the essential properties of these orders in the case of algebras satisfying a polynomial identity. In this case, our definition coincides with that of Chamarie. For details we refer the reader to his work [7, 8]. The prime spectrum of R is denoted by $Spec(R)$, and the set of height one prime ideals of R by $X^1(R)$.

In view of the structural result on Noetherian algebras $K[S]$, Theorem 2.11, it is natural to consider first the case where G is a finitely generated abelian-by-finite group. Recall that the group algebra of a finitely generated group G satisfies a polynomial identity if and only if G is abelian-by-finite, see [57], Theorems 5.3.7 and 5.3.9.

Theorem 5.8 ([24]) *Let R be a prime Krull order satisfying a polynomial identity. Then the following properties hold:*

1. *The divisorial ideals form a free abelian group with basis $X^1(R)$, the height one primes of R.*
2. *If $P \in X^1(R)$ then $P \cap \mathrm{Z}(R) \in X^1(\mathrm{Z}(R))$, and furthermore, for any ideal I of R, $I \subseteq P$ if and only if $I \cap \mathrm{Z}(R) \subseteq P \cap \mathrm{Z}(R)$.*
3. *$R = \bigcap R_{\mathrm{Z}(R)\setminus P}$, where the intersection is taken over all height one primes of R, and every regular element $r \in R$ is invertible in almost all (that is, except possibly finitely many) localizations $R_{\mathrm{Z}(R)\setminus P}$. Furthermore, each $R_{\mathrm{Z}(R)\setminus P}$ is a left and right principal ideal ring with a unique nonzero prime ideal.*
4. *For a multiplicatively closed set of ideals M of R, the (localized) ring $R_M = \{q \in Q_{cl}(R) | Iq \subseteq R$, for some $I \in M\}$ is a Krull order, and $R_M = \bigcap R_{\mathrm{Z}(R)\setminus P}$, where the intersection is taken over those height one primes P for which $R_M \subseteq R_{\mathrm{Z}(R)\setminus P}$.*

If S is a monoid with a torsion free abelian-by-finite group of quotients G (so $K[S]$ is a PI-domain), the maximal order property of $K[S]$ is determined by the structure of S and can be reduced to some 'local' monoids S_P, with P a minimal prime ideal of S. Here

$$S_P = \{g \in G \mid Cg \subseteq S \text{ for some } G\text{-conjugacy class } C \text{ of } G \text{ contained in } S \text{ with } C \not\subseteq P\}.$$

The next theorem comes from [35], see also [38], Theorems 7.2.5 and 7.2.7.

Theorem 5.9 *Let S be a submonoid of a finitely generated torsion free abelian-by-finite group. Then the monoid algebra $K[S]$ is a Noetherian maximal order if and only if the following conditions are satisfied:*

1. *S satisfies the ascending chain condition on one-sided ideals,*
2. *S is a maximal order in its group of quotients,*
3. *for every minimal prime ideal P of S the monoid S_P has only one minimal prime ideal.*

Furthermore, in this case, each S_P is a maximal order satisfying the ascending chain condition on one-sided ideals.

This result was extended in [24] to the case of a submonoid of an arbitrary finitely generated abelian-by-finite group. The final step was made in [25], where a further extension was obtained.

Theorem 5.10 ([25]) *Let S be a submonoid of a polycyclic-by-finite group such that the semigroup algebra $K[S]$ is Noetherian, i.e., there exist normal subgroups F and N of $G = SS^{-1}$ such that $F \subseteq S \cap N$, N/F is abelian, G/N is finite and $S \cap N$ is finitely generated. Suppose that for every minimal prime P of S the intersection $P \cap N$ is G-invariant. Then, the semigroup algebra $K[S]$ is a prime maximal order if and only if the monoid S is a maximal order in its group of quotients G, the group G is dihedral-free and has no nontrivial finite normal subgroups.*

Suppose that in the previous theorem one also assumes that the group G is abelian-by-finite. Then, in [24], it is shown that the condition 'for every minimal prime P of S the intersection $P \cap N$ is G-invariant' is necessary for $K[S]$ to be a maximal order. However, no example of a maximal order S in a polycyclic-by-finite group $G = SS^{-1}$ (with G dihedral-free and $K[G]$ prime) is known so that $K[S]$ is Noetherian but not a maximal order. We note that for a submonoid S of a torsion free polycyclic-by-finite group certain necessary and certain sufficient conditions for a Noetherian $K[S]$ to be a unique factorization ring in the sense of Chatters and Jordan were studied in [44, 45].

The following result allows to construct several concrete examples of maximal orders in the PI-case. As we shall see, this is in contrast to the situation described in Sect. 6, where no such a general construction is known.

Proposition 5.11 ([24, 38]) *Let A be an abelian normal subgroup of finite index in a group G. Suppose that B is a submonoid of A so that $A = BB^{-1}$ and B is a finitely generated maximal order. Let S be a submonoid of G such that $G = SS^{-1}$ and $S \cap A = B$. Then S is a maximal order that satisfies the ascending chain condition on right ideals if and only if S is maximal among all submonoids T of G with $T \cap A = B$.*

Substantial results have been also obtained on semigroup algebras that are principal ideal rings. This story begins with the case of group algebras, settled by Passman in [56], and concludes with the results obtained in [30]. References to several partial intermediate results can be found in [23, 38]. The rest of this section is devoted to a presentation of these results. We will always assume that a principal ideal ring contains an identity element, though in this section S is not necessarily a monoid. First we state Passman's result on the group algebra case. We follow [46], where this result is stated in the slightly more general context of matrices over group algebras, that will be needed later.

Proposition 5.12 ([56]) *Let G be a group and $R = M_n(K)$, a matrix ring over K. The following conditions are equivalent:*

1. *$R[G] = M_n(K[G])$ is a principal right ideal ring,*
2. *$R[G]$ is right Noetherian and the augmentation ideal $\omega(R[G])$ is a principal right ideal,*
3. *if char $K = 0$, then G is finite or finite-by-infinite cyclic,*
   if char $K = p > 0$, then G is finite p'-by-cyclic p or G is finite p'-by-infinite cyclic.

This result was then extended to semigroup algebras of cancellative monoids as follows.

Proposition 5.13 ([46]) *Let T be a cancellative monoid and K a field of characteristic p (possibly zero). The following conditions are equivalent:*

1. *$K[T]$ is a principal right ideal ring,*
2. *T is a semigroup satisfying one of the following conditions:*
   a. *T is a group satisfying the conditions of Proposition 5.12,*
   b. *T contains a finite p'-subgroup H and a nonperiodic element x such that $xH = Hx$, $T = \bigcup_{i \in \mathbf{N}} Hx^i$ and the central idempotents of $K[H]$ are central in $K[T]$.*

As explained in Sect. 3, the structure theorem for linear semigroups provides a link between a linear semigroup and some of its cancellative subsemigroups. In order to apply this approach to semigroup algebras of arbitrary semigroups that are principal ideal rings one first has to reduce the problem to linear semigroups. This is guaranteed by Theorem 3.2 together with the following result.

Theorem 5.14 ([30]) *Let $K[S]$ be a principal right ideal ring. Then $K[S]$ satisfies a polynomial identity.*

Using the structure theorem of linear semigroups, explained in Sect. 3, one now can prove the following results.

Proposition 5.15 ([30]) *If $K[S]$ is a principal right ideal ring, then the Gelfand–Kirillov dimension of $K[S]$ is equal to its classical Krull dimension and* *it is* 0 *or* 1. *In the former case S is finite. Moreover, every prime artinian homomorphic image of $K[S]$ is finite dimensional over K.*

Theorem 5.16 ([30]) *Let S be a semigroup and K a field of characteristic p (possibly zero). The following conditions are equivalent:*

1. *$K_0[S]$ is a principal (left and right) ideal ring;*
2. *there exists an ideal chain*

$$I_1 \subseteq \cdots \subseteq I_t = S$$

such that I_1 and every factor I_j/I_{j-1} is of the form $\mathcal{M}(T,n,n,P)$ for an invertible over $K_0[T]$ sandwich matrix P, and one of the following conditions holds:

   a. *T is a group of the type described in Proposition 5.12;*
   b. *T is a monoid with a finite group of units H such that $T = \bigcup_{i\geq 0} Hx^i$ for some $x \in T$, and either this union is disjoint or $x^n = \theta$ for some $n \geq 1$. Also $Hx = xH$, the central idempotents of $K[H]$ commute with x, and $p = 0$ or $p \nmid |H|$.*

In case the equivalent conditions are satisfied it follows that

$$K_0[S] \cong K_0[I_1] \oplus K_0[I_2/I_1] \oplus \cdots \oplus K_0[I_t/I_{t-1}].$$

Moreover, $K_0[S]$ is a finite module over its center, which is finitely generated.

It is not known whether the left-right symmetric hypothesis in Theorem 5.16 is essential.

The above theorem applies to finite dimensional algebras $K[S]$, since a finite dimensional algebra is a principal right ideal ring if and only if it is a principal left ideal ring. One can also show that semiprime principal right ideal semigroup algebras are necessarily principal left ideal rings as well.

Theorem 5.17 ([30]) *Let S be a semigroup and K a field of characteristic p (possibly zero). Then $K_0[S]$ is a semiprime principal right ideal ring if and only if there exists an ideal chain*

$$I_1 \subseteq \cdots \subseteq I_t = S$$

such that I_1 and every factor I_j/I_{j-1} is of the form $\mathcal{M}(T,n,n,P)$ for an invertible over $K_0[T]$ sandwich matrix P and a monoid T such that

1. *either T is a group as in Proposition 5.12 so that $K[T]$ is semiprime,*
2. *or T is a monoid with finite group of units H such that $T = \bigcup_i Hx^i$ is a disjoint union, for some $x \in T$. Also $Hx = xH$, the central idempotents of $K[H]$ commute with x, and $p = 0$ or $p \nmid |H|$. Furthermore, for every primitive central idempotent $e \in K[H]$, either $K[H]ex = 0$ or $K[H]ex^i \neq 0$ for all $i \geq 1$.*

Moreover, if the equivalent conditions are satisfied, then $K_0[S]$ is a principal left ideal ring.

Corollary 5.18 *$K_0[S]$ is a prime principal right ideal ring if and only if*

$$S \cong \mathscr{M}(\{1\}, n, n, Q), \quad S \cong \mathscr{M}(\langle x \rangle, n, n, Q) \quad or \quad S \cong \mathscr{M}(\langle x, x^{-1} \rangle, n, n, Q)$$

where Q is invertible in $M_n(K)$, $M_n(K[x])$ or $M_n(K[x, x^{-1}])$ respectively. Hence, $K_0[S] \cong M_n(K), M_n(K[x])$, or $M_n(K[x, x^{-1}])$.

6 Why Should We Look at the Non-Noetherian Case? Motivation and First Results

We start with an interesting example of a finitely presented algebra, denoted by $R(\mathbb{1})$, that has recently played an important role in certain aspects of noncommutative geometry. This algebra is not Noetherian, but it leads to a family of deformations that consists of Noetherian algebras [59]. It turns out that it is based on a relatively simple construction of a semigroup algebra of a submonoid of the Heisenberg group (a nilpotent group of class 2):

$$G = gr(a, b, c \mid ac = ca, ab = ba, bc = acb).$$

On one hand, this example shows that computations in such algebras may be quite difficult. On the other hand, it seems to be a good motivation for studying non-Noetherian orders coming from finitely generated nilpotent groups. After explaining the nature of this example, we present some recent general results on this class of algebras.

Let

$$M = \langle x, y, z, t \mid xy = yx, zt = tz, yz = xt = zx, zy = tx = yt \rangle,$$

a finitely presented monoid, defined by homogeneous relations. So $K[M]$ carries some similarity to Yang-Baxter algebras, considered in Sect. 4. Namely, it has the 'correct' number of quadratic relations ($\binom{n}{2}$ relations), however some monomials appear in two different relations.

It can be shown that: $\phi : M \longrightarrow G$ defined by

$$x \mapsto c, y \mapsto ac, z \mapsto bc, t \mapsto abc$$

is a homomorphism which also is an embedding. Hence

$$M \cong \phi(M) \subseteq G.$$

Note that $K[M]$ is an Ore domain, but it is not Noetherian (use Theorem 2.11: G is not abelian-by-finite while M has trivial units; but this is also easy to check directly).

$K[M]$ is the algebra used by Yekutieli and Zhang [65] (as a counterexample in the context of Artin-Schelter regular rings), and recently by Rogalski and Sierra, where it plays a key role in the classification of 4-dimensional non-commutative projective surfaces, [59]. Namely, a family of deformations of $K[M]$ is considered. They are of the form:

$$R(\rho,\theta) = K\langle x_1, x_2, x_3, x_4 \mid f_i = 0, i = 1, 2, 3, 4, 5, 6\rangle$$

where

$$\begin{aligned}
f_1 &= x_1(cx_1 - x_3) + x_3(x_1 - cx_3)\\
f_2 &= x_1(cx_2 - x_4) + x_3(x_2 - cx_4)\\
f_3 &= x_2(cx_1 - x_3) + x_4(x_1 - cx_3)\\
f_4 &= x_2(cx_2 - x_4) + x_4(x_2 - cx_4)\\
f_5 &= x_1(dx_1 - x_2) + x_4(x_1 - dx_2)\\
f_6 &= x_1(dx_3 - x_4) + x_4(x_3 - dx_4)
\end{aligned}$$

for $c = (\theta - 1)/(\theta + 1)$ and $d = (\rho - 1)(\rho + 1)$.

Notice that $R(1, 1) \cong K[M]$ and it is embeddable in the skew polynomial ring $K(u, v)[t, \sigma]$ over the rational function field $K(u, v)$, where $\sigma(v) = v$, $\sigma(u) = uv$.

Theorem 6.1 ([59]) *Assume that K is algebraically closed and uncountable. If ρ, θ are algebraically independent over the prime subfield of K, then $R(\rho, \theta)$ is a Noetherian domain of global dimension 4 and Gelfand–Kirillov dimension 4. And it is birational to $\mathbb{P}^2$.*

Here, for a Noetherian domain R such that $R = \bigoplus_{i\geq 0} R_i$ is connected $\mathbb{N}$-graded (meaning that the zero component $R_0 = K$ and $\dim(R_i) < \infty$ for every i), it is known that the graded ring of quotients $Q_{gr}(R) \cong D[t, t^{-1}, \sigma]$, for a division ring D. So, $Q_{gr}(R)$ is obtained by localizing with respect to the set of nonzero homogeneous elements in R. If the division ring D is a field (then $D = K(X)$ for a projective variety X), then R is said to be birational to X.

Hence, this provides a new motivation to study algebras of submonoids of nilpotent groups that are not necessarily Noetherian. The starting case is where the quotient group is nilpotent of class 2. Then we have the following surprising and very useful result.

Lemma 6.2 ([29]) *A prime ideal P of a submonoid S of a nilpotent group of class two is completely prime; that is, $st \in P$ implies $s \in P$ or $t \in P$, for $s, t \in S$. In particular, if S is finitely generated, then S has only finitely many prime ideals.*

Using also Lemma 5.7 and Proposition 5.6, one can then get a partial extension of the classical result on the classical Krull dimension of a group algebra of a nilpotent group, stated in Theorem 2.5.

Theorem 6.3 ([40]) *Let S be a submonoid of a nilpotent group of class two. If the group of quotients $G = SS^{-1}$ of S is finitely generated then* $\mathrm{clKdim}(K[S]) = \mathrm{h}(G)$. *Moreover, if P is a prime ideal of $K[S]$, then $K[S]/P$ is a Goldie ring.*

In order to indicate a striking contrast with the case of higher nilpotency classes, we will construct some prime ideals in the algebra $K[S]$ of the submonoid $S = \langle b, c\rangle$ of the free nilpotent group $\mathrm{F}_3(b, c)$ of class 3. In other words, $\mathrm{F}_3(b, c)$ is defined by the following relations:

$$bc = acb,\ ab = dba,\ ac = eca,$$
$$db = bd,\ dc = cd,\ eb = be,\ ec = ce.$$

Lemma 6.4 *For positive integers k and n, the word $(bc^k)^n$ cannot be rewritten in $S = \langle b, c\rangle \subseteq \mathrm{F}_3(b, c)$.*

Recall that a doubly infinite word in b and c is a sequence $x = (x_i)_{i\in\mathbb{Z}}$ with $x_i \in \{b, c\}$. One says that x is recurrent if every (finite) subword of x appears in x at least twice (thus, it appears infinitely many times). For example, the cyclic word $(bc^k)^\infty$ is of this type. Then,

$$J = \{s \in S\ :\ s \neq t \text{ in } S \text{ for every subword } t \text{ of } x\}$$

is an ideal of S and it is easy to check that J is a prime ideal of S. Since $\mathrm{F}_3(b, c)$ is torsion free, this, together with Proposition 5.6, is used to derive the following consequence.

Theorem 6.5 ([40]) *The submonoid $S = \langle b, c\rangle$ of the group $\mathrm{F}_3(b, c)$ has infinitely many prime ideals P that are not completely prime. Furthermore, each $K[P]$ is a prime ideal of $K[S]$ such that $K[S]/K[P]$ is an algebra satisfying a polynomial identity and* $\mathrm{clKdim}(K[S]/K[P]) = \mathrm{GKdim}(K[S]/K[P]) = 1$.

This result shows that the situation is quite different than the one in the case of nilpotency class 2, where all primes are completely prime.

A natural open question that arises is whether there exist other, more exotic, primes in $K[S]$ for a submonoid S of a finitely generated nilpotent group G of nilpotency class exceeding 2. In particular, do there exist prime homomorphic images of $K[S]$ that are not Goldie? Can $K[S]$ have infinite classical Krull dimension?

As mentioned in Sect. 5, prime ideals provide one of the main tools in dealing with maximal orders, and with related classes of algebras with nice arithmetical properties. We state some results in this direction.

Theorem 6.6 ([32]) *Let S be a submonoid of a finitely generated torsion free nilpotent group. Then the following properties hold.*

1. *S is a maximal order if and only if $K[S]$ is a maximal order.*

2. *If S satisfies the ascending chain condition on right ideals and is a maximal order, then all elements of S are normal (meaning that $aS = Sa$ for every $a \in S$).*

So, in the latter case, the theorem below applies.

Theorem 6.7 ([32]) *Let S be a submonoid of a torsion free polycyclic-by-finite group. Assume that all elements of S are normal. Then the following conditions are equivalent:*

1. *$K[S]$ is a Krull domain,*
2. *S is a Krull order,*
3. *$S/U(S)$ is an abelian Krull order.*

Using the special features of groups of nilpotency class 2, and applying Theorem 6.7, one can prove the following result. Here we define $N(S) = \{a \in S \mid aS = Sa\}$, the submonoid of normal elements of S.

Theorem 6.8 ([39]) *Assume that S is a submonoid of a torsion free nilpotent group of class two. Assume that S is a Krull order. Then*

(i) the derived subgroup G' of the quotient group G of S is contained in S,
(ii) $S = N(S)$,
(iii) S/G' is a commutative Krull order,
(iv) if G is finitely generated, then $K[S]$ is a Krull domain for every field K; moreover S is finitely generated and $K[S]$ is right and left Noetherian.

On the other hand, if $G' \subseteq S$ and S/G' is a Krull order then S is a Krull order.

So, in some sense, the class of such orders is quite restricted and carries a lot of commutative flavor. It is an open problem whether there exist maximal orders that do not satisfy the property $N = N(S)$ and, in higher nilpotency classes whether there exist Krull orders of this type.

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