Multiplicative Ideal Theory in Noncommutative Rings

Evrim Akalan and Hidetoshi Marubayashi

Abstract The aim of this paper is to survey noncommutative rings from the viewpoint of multiplicative ideal theory. The main classes of rings considered are maximal orders, Krull orders (rings), unique factorization rings, generalized Dedekind prime rings, and hereditary Noetherian prime rings. We report on the description of reflexive ideals in Ore extensions and Rees rings. Further we give necessary and sufficient conditions (or sufficient conditions) for well-known classes of rings to be maximal orders, and we propose a polynomial-type generalization of hereditary Noetherian prime rings.

Keywords Maximal order · Reflexive ideal · Krull ring (order) · Unique factorization ring · Generalized Dedekind · Generalized Noetherian prime ring

1 Introduction

Multiplicative (arithmetic) ideal theory in algebraic number fields originated by Dedekind was developed by M. Sono, W. Krull, E. Noether, H. Prüfer, E. Artin during the period 1910–1930. In particular, E. Noether gave an axiomatic foundation on Dedekind's theory.

In the noncommutative setting, Dedekind–Noether's ideal theory was first extended to algebras by A. Speiser, H. Brandt, E. Artin, and H. Hasse (e.g., [9, 45, 69] and see also [31]), and then, in [10] K. Asano extended Noether's axiomatic foundation to noncommutative rings: Let R be a bounded order in its

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quotient ring Q. Then the set of all fractional R-ideals is a group if and only if the following three conditions hold:

(a1) R is a bounded maximal order in Q.
(a2) R satisfies the ascending chain conditions on integral ideals.
(a3) Any nonzero prime ideal is maximal,

which is the same axiomatic foundation as one of Noether in case of commutative domains. Furthermore, he extended many important ideal theories to orders satisfying (a1), (a2) and (a3) which is nowadays called bounded Asano rings (orders) [11, 12].

The aim of this article is to survey noncommutative rings from the viewpoint of multiplicative ideal theory.

In Sect. 2, we give the definitions of maximal orders, Asano, Dedekind, and hereditary which are the main topics in multiplicative ideal theory and give a classical ideal theory of maximal orders. Furthermore we discuss the maximal order properties of well-known noncommutative rings such as group rings, polynomial rings, universal enveloping algebras, and so on.

In Sect. 3, we define the concept of Krull orders in the sense of Chamarie and study the algebraic structure of Krull orders as well as the ideal theories of polynomial rings and Ore–Rees rings over Krull orders.

We give in Sect. 4 an overview of noncommutative unique factorization rings (UFRs for short) which is one of the important classes of maximal orders.

Section 5 contains a generalization of Dedekind and Asano orders which is called G-Dedekind (or G-Asano) and we give several characterizations of G-Dedekind. We also consider polynomial rings and Rees rings over G-Dedekind.

Hereditary prime rings are one of the most successful subjects in noncommutative rings during the years 1960–1970. In Sect. 6, we only discuss the ideal theory in HNP rings and propose a polynomial-type generalization of HNP rings.

We refer the readers to the books [63, 68] for terminologies not defined in this article.

Because of the page limit, we do not give the proofs of Propositions and Theorems and we quote the original papers or books for reader's convenience.

In the case of commutative rings and monoids we refer the reader to the books [34, 39] for commutative rings and [41] for monoids.

2 Maximal Orders

Throughout this paper, R is a prime Goldie ring unless otherwise stated with its quotient ring Q, which is a simple Artinian ring (in other words, R is an *order* in Q).

In this section, we define the concept of maximal orders in Q and give a classical ideal theory in maximal orders. Furthermore we give necessary and sufficient conditions (or sufficient conditions) for some well-known noncommutative rings to be maximal orders.

Definition 2.1 (1) Orders R and S in Q are *equivalent* if $aRb \subseteq S$ and $cSd \subseteq R$ for some units a, b, c, d in Q.
(2) An order is maximal if it is maximal in the set of all equivalent orders.

For a fractional right R-ideal I, $O_r(I) = \{q \in Q \mid Iq \subseteq I\}$, which is called a *right order* of I. It is easy to see that $O_r(I) \supseteq R$ and is equivalent to R. Similarly for a fractional left R-ideal J, $O_l(J) = \{q \in Q \mid qJ \subseteq J\}$, the *left order* of J, which contains R and is equivalent to R. Thus we have the following ideal theoretic characterizations of maximal orders:

Proposition 2.2 ([63, 68]) *Let R be an order in Q. The following conditions are equivalent:*

(1) R is a maximal order in Q.
(2) $O_l(J) = R$ for all fractional left R-ideals J and $O_r(I) = R$ for all fractional right R-ideals I.
(3) $O_l(A) = R = O_r(A)$ for all fractional R-ideals A.
(4) $O_l(A) = R = O_r(A)$ for all nonzero ideals A of R.

For a fractional right R-ideal I, let $I^* = \{q \in Q \mid qI \subseteq R\}$ and for a fractional left R-ideal J, let $J^+ = \{q \in Q \mid Jq \subseteq R\}$. If R is a maximal order, then for a fractional R-ideal A in Q, $A^* = A^{-1} = A^+$ by Proposition 2.2, here $A^{-1} = \{q \in Q \mid AqA \subseteq A\}$. Thus $A^{*+} = A^{+*}$, which contains A. If $A = A^{**}$, then A is called a *reflexive* fractional R-ideal in Q (some say a *divisorial* fractional R-ideal in Q).

Let $D(R) = \{A \mid A$ is a reflexive fractional R-ideal$\}$. For any A, B in $D(R)$, we define the multiplication "$\circ$" by $A \circ B = (AB)^{**}$. Then we have the following theorem which extends Asano's result:

Theorem 2.3 ([63, 68]) *Suppose R is a maximal order in Q.*

(1) R is a group with the multiplication "$\circ$".
(2) If R satisfies the ascending chain condition on reflexive ideals of R, then
(i) $D(R)$ is an Abelian group generated by maximal reflexive ideals.
(ii) Any maximal reflexive ideal is a minimal prime ideal (height-1 prime).
(3) The center of R is a completely integrally closed domain.

Theorem 2.3 (3) shows maximal orders are nothing but completely integrally closed in case of commutative domains.

A fractional R-ideal A is said to be *invertible* if $A^*A = R = AA^+$. An order in Q is said to be *Asano* if each nonzero ideal is invertible, and is said to be *Dedekind* if it is Asano and hereditary (see [68] for more detailed results on Asano and Dedekind orders).

In case of commutative domains, invertible ideal is equivalent to projective. Hence Dedekind, Asano, and hereditary are all same. However, in the noncommutative setting, invertible ideal is projective and the converse does not necessarily hold. Thus Dedekind orders imply Asano and hereditary. The converse implications do not necessarily hold and there are no implications between Asano and hereditary (see [63, 68] for such examples). However, if we assume that R is *bounded*, that is any essential one-sided ideal contains a nonzero ideal (this concept is defined by Asano), then we have

Proposition 2.4 ([56]) *Bounded Noetherian Asano orders are Dedekind.*

In the rest of this section, we give necessary and sufficient conditions for some well-known noncommutative rings to be maximal orders (or a sufficient condition for well-known noncommutative rings to be maximal orders).

Proposition 2.5 (Algebra case) *Let Q be a simple Artinian ring with its center F and R as a subring of Q with its center D. R is called a D-order in Q if the following two conditions are satisfied:*

(i) F is the quotient field of D and $Q = FR$, that is, R is an order in Q.
(ii) Every element of R is integral over D.
(1) There always exists a maximal D-order by Zorn's lemma.
(2) If D is a Dedekind domain, then every maximal D-order is a bounded noncommutative Dedekind order [73].
(3) If D is a Krull domain, then every maximal D-order is a bounded noncommutative Krull order ([35, 63], see Sect. 3 for the definition of Krull orders).

Let σ be an automorphism of R and δ be a left σ-derivation on R. The noncommutative polynomial ring $R[x; \sigma, \delta] = \{ f(x) = a_n x^n + \cdots + a_0 \mid a_i \in R\}$ in an indeterminate x with multiplication : $xa = \sigma(a)x + \delta(a)$ for any $a \in R$ is called an *Ore extension*.

In [72], Ore defined noncommutative polynomial rings in case R is a skew field and studied the structure of them. It is easy to see that σ and δ are extended to an automorphism σ of Q and a left σ-derivation δ on Q. Since $Q[x; \sigma, \delta]$ is a principal ideal ring, that is, any one-sided ideal is principal [22], it has a quotient ring which is a simple Artinian ring and so $R[x; \sigma, \delta]$ has a quotient ring which is the same quotient ring of $Q[x; \sigma, \delta]$.

Let I be an invertible ideal of R with $\sigma(I) = I$. A subring $R[Ix; \sigma, \delta] = \sum_{n=0}^{\infty} \bigoplus I^n x^n$ of $R[x; \sigma, \delta]$ is called an *Ore–Rees* ring associated to I, where $I^0 x^0 = R$.

Proposition 2.6 (Ore extensions and Ore–Rees ring) *If R is a maximal order in Q, then so is $R[x; \sigma, \delta]$, and if R is a Noetherian maximal order then so is $R[Ix; \sigma, \delta]$ [23, 47].*

Proposition 2.7 (Strongly graded rings) *(1) Let $S = \sum_{n\in\mathbb{Z}} \bigoplus R_n$ be a strongly $\mathbb{Z}$-graded ring, where $\mathbb{Z}$ is the ring of integers. If R_0, the part of degree zero, is a maximal order, then so is S [65].*

(2) Let R be a semiprime $\mathbb{Z}$-graded ring. R is an Asano order if and only if
(i) Every gr-R-ideal is invertible, and
(ii) Every essential gr-maximal ideal is maximal [53].

A commutative Noetherian local ring D is *regular* if and only if $gl.dim(D) < \infty$. If D is regular, then it is a UFD and so it is a maximal order. In noncommutative setting, we have

Proposition 2.8 (Rings of finite global dimensions) *(1) Any local Noetherian ring of finite global dimension which is integral over its center is a maximal order [40].*
(2) Any Noetherian, prime, AR-ring of finite global dimension with enough invertible ideals is a maximal order [19].
(3) Let F be a field and R be a Noetherian F-algebra.
(i) If R is Auslander-regular, Cohen–Macaulay and stably free, then R is a maximal order in its quotient division ring [77].
(ii) If R is a graded ring of finite global dimension such that R is integral over its center, then R is a maximal order in its quotient division ring [77].

Let $T = \begin{pmatrix} R & V \\ W & S \end{pmatrix}$ be a ring of Morita contexts which is a prime Goldie ring, where R and S are prime Goldie rings with the quotient rings $Q(R)$ and $Q(S)$, respectively, V, W are an (R, S)-bimodule, an (S, R)-bimodule, respectively. It follows that $Q(R)V = VQ(S)$ and $Q(S)W = WQ(R)$, which are denoted by $Q(V)$ and $Q(W)$, respectively. Then the quotient ring of T is $\begin{pmatrix} Q(R) & Q(V) \\ Q(W) & Q(S) \end{pmatrix}$, denoted by $Q(T)$. Similar to maximal orders, we can define an (R, S)-*maximal module* in $Q(V)$ and an (S, R)-*maximal module* in $Q(W)$ (see [7] for the definition of maximal modules). Put $V^* = \{\tilde{w} \in Q(W) \mid \tilde{w}V \subseteq S\}$ and $V^+ = \{\tilde{w} \in Q(W) \mid V\tilde{w} \subseteq R\}$. Similarly we define W^* and W^+.

Proposition 2.9 (Rings of Morita contexts) *The following conditions are equivalent:*

(1) *T is a maximal order in $Q(T)$.*
(2) *(i) R and S are maximal orders in $Q(R)$ and $Q(S)$, respectively, and*
(ii) *$V^* = W = V^+$ and $W^* = V = W^+$.*
(3) *(i) V is an (R, S)-maximal module in $Q(V)$ and W is an (S, R)-maximal module in $Q(W)$, and*
(ii) *$V^* = W = V^+$ and $W^* = V = W^+$ [7, 66].*

As a generalization of universal enveloping algebras, in [15], Bell and Goodearl defined a PBW extension as follows: An over-ring S of R is called a *Poincare–Birkhoff–Witt extension* of R (*PBW extension* for short) if there exist elements $x_1, x_2, \dots, x_n \in S$ such that

(i) the ordered monomials $x_1^{v_1} \dots x_n^{v_n}$, where v_i are non-negative integers, form a basis for S as a free left R-module,
(ii) $x_i r - r x_i \in R$ for each $i = 1, \dots, n$ and any $r \in R$, and
(iii) $x_i x_j - x_j x_i \in R + Rx_1 + \cdots + Rx_n$ for all $i, j = 1, \dots, n$.

Proposition 2.10 (Enveloping algebras) *Let D be a Noetherian integrally closed domain and $\mathfrak{g}$ be a Lie D-algebra which is a finite free D-module. Then the enveloping algebra $U(\mathfrak{g})$ is a maximal order [23].*
(2) If R is a maximal order in $Q(R)$, then the PBW extension $R < x_1, x_2, \dots, x_n >$ is a maximal order [64].

Proposition 2.11 (Semigroup algebras) *Let F be a field and S a submonoid of a torsion free finitely generated abelian-by-finite group. The semigroup algebra $F[S]$ is a Noetherian maximal order if and only if the following conditions are satisfied.*
(1) S satisfies the ascending chain condition on left and right ideals.
(2) For every minimal prime P in S the semigroup S_P is a maximal order with only one minimal prime ideal.
(3) $\cap S_P = S$, where P runs over all minimal prime ideals of S [49].

3 Krull Orders

Several noncommutative ring theorists defined Krull orders (Krull rings) and studied the ideal theory and polynomial extensions during the period 1970–1980 [23, 24, 51, 52, 54, 55, 58–61]. However, in case of orders having polynomial identities, these Krull orders coincide.

In this section, we only give a definition of Krull orders due to Chamarie and study ideal theory, polynomial extensions, and Ore–Rees rings over Krull orders.

Let $\mathcal{F}$ be a right Gabriel topology on R and $R_{\mathcal{F}} = \{q \in Q \mid qF \subseteq R$ for some $F \in \mathcal{F}\}$, which is called the *right quotients of R with respect to $\mathcal{F}$*. If I is a right ideal of R, then $I_{\mathcal{F}} = \{q \in Q \mid qF \subseteq I$ for some $F \in \mathcal{F}\}$ is a right ideal of $R_{\mathcal{F}}$, and I is said to be *$\mathcal{F}$-closed* if $I_{\mathcal{F}} \cap R = I$. Similarly for a left Gabriel topology $\mathcal{F}'$ on R we denote the left quotients of R with respect to $\mathcal{F}'$ by ${}_{\mathcal{F}'}R$ (see [79] for Gabriel topologies and quotients).

We now introduce a special Gabriel topology on R as follows.

Put $\mathcal{F}_{\mathcal{R}} = \{F \mid F$ is a right ideal such that $(r^{-1} \cdot F)^* = R$ for any $r \in R\}$ which is a right Gabriel topology on R, where $r^{-1} \cdot F = \{a \in R \mid ra \in F\}$. Similarly $\mathcal{F}'_{\mathcal{R}} = \{F' \mid F'$ is a left ideal such that $(F' \cdot r^{-1})^+ = R$ for any $r \in R\}$ is a left Gabriel topology on R.

A right (left) ideal $I(J)$ of R is called *τ-closed* if $I = I_{\mathcal{F}_{\mathcal{R}}} \cap R$ ($J = {}_{\mathcal{F}'_{\mathcal{R}}}J \cap R$). An order in Q is said to be *τ-Noetherian* if it satisfies the ascending chain conditions on τ-closed left ideals as well as τ-closed right ideals.

Definition 3.1 An order in a simple Artinian ring is called a Krull order (ring) in the sense of Chamarie [23, 24] if it is a maximal order and τ-Noetherian.

Note that Noetherian maximal orders are Krull orders. We start with ideal theory between a Krull order and its over-ring.

Proposition 3.2 ([23, 63]) *Let R be a Krull order in Q and R' be an over-ring of R such that $R_{\mathcal{F}} = R' = {}_{\mathcal{F}'}R$ for some right (left) Gabriel topology $\mathcal{F}(\mathcal{F}')$ on R. Then*
(1) R' is a Krull order in Q.
(2) For any fractional right R-ideal I, ${}_{\mathcal{F}'}(I^{-1}) = (IR')^{-1} = (I_{\mathcal{F}})^{-1}$, where $I^{-1} = \{q \in Q \mid IqI \subseteq I\}$.
(3) The map: $I \longrightarrow I_{\mathcal{F}}$ is a bijection between the set of reflexive $\mathcal{F}$-closed right ideals I of R and the set of reflexive right ideals of R' (I is called reflexive if $I = I^{*+}$).

Theorem 3.3 (Structure theorem for Krull orders, [63]) *Let R be a Krull order in Q. Then*

(1) The center of R is a Krull domain.
(2) Any reflexive prime ideal P is localizable and R_P, the localization of R at P is a local principal ideal ring.
(3) $R = \cap R_P \cap S(R)$, where P ranges over all maximal reflexive ideals and $S(R) = \bigcup\{A^{-1} \mid A$ is nonzero ideal of $R\}$ is a reflexively simple Krull order in Q.
(4) R has a finite character property, that is, any regular element $c \in R$ is a non-unit in only finitely many of R_P.
(5) For any essential right ideal I, $I^{*+} = \cap I R_P \cap (IS(R))^{*+}$.

In the remainder of this section, R is an order in Q with an automorphism σ and a left σ-derivation δ, and put $T = Q[x; \sigma, \delta]$.

We denote the prime spectrum of R by Spec(R) and $\mathrm{Spec}_0^*(R[x; \sigma, \delta])= \{P$: reflexive prime ideals $\mid P \cap R = (0)\}$. It is shown in [63] that R is τ-Noetherian if and only if so is $R[x; \sigma, \delta]$ (in [23], Chamarie proved that R is τ-Noetherian, then so is $R[x; \sigma, \delta]$).

Proposition 3.4 ([63]) *Suppose R is τ-Noetherian and put $S = R[x; \sigma, \delta]$.*
(1) There is a one-to-one correspondence between $\mathrm{Spec}_0^*(S)$ *and* Spec(T) *which is given by: $P' \longrightarrow P = P' \cap S$, where $P' \in$* Spec(T).
(2) If $P \in$ $\mathrm{Spec}_0^*(S)$*, then P is localizable and $S_P = T_{P'}$ which is a local principal ideal ring, where $P' = PT$.*

A fractional R-ideal $\mathfrak{a}$ is called *σ-stable* if $\sigma(\mathfrak{a}) \subseteq \mathfrak{a}$ and it is *σ-invariant* if $\sigma(\mathfrak{a}) = \mathfrak{a}$. An order R is called a *σ-maximal order* if $O_l(\mathfrak{a}) = R = O_l(\mathfrak{a})$ for any σ-invariant ideal $\mathfrak{a}$ of R, and R is a *σ-Krull order* if it is a σ-maximal order and τ-Noetherian. Similarly, a fractional R-ideal $\mathfrak{a}$ is called *δ-stable* if $\delta(\mathfrak{a}) \subseteq \mathfrak{a}$ and R is called a *δ-maximal order* in Q if $O_l(\mathfrak{a}) = R = O_r(\mathfrak{a})$ for any δ-stable ideal $\mathfrak{a}$ of R. An order is said to be a *δ-Krull order* if it is a δ-maximal order and τ-Noetherian.

In case $\delta = 0$ or $\sigma = 1$, we denote $R[x; \sigma, \delta]$ by $R[x; \sigma]$ or $R[x; \delta]$, respectively.

Theorem 3.5 ([23, 63]) *Let R be an order in Q.*

(1) If R is a Krull order, then so is $R[x; \sigma, \delta]$ (there are examples of orders R not Krull such that $R[x; \sigma, \delta]$ is a Krull order) [1, 62, 67].
(2) R is a σ-Krull order if and only if $R[x, \sigma]$ is a Krull order.
(3) R is δ-Krull order if and only if $R[x; \delta]$ is a Krull order.

Let $S = R[x; \sigma]$ or $S = R[x; \delta]$. We describe all the reflexive fractional S-ideals in case S is a Krull order.

Proposition 3.6 ([63]) *(1) Let $S = R[x; \sigma]$ and suppose R is a σ-Krull order in Q. Let P be an ideal of S with $P \cap R \neq 0$. Then P is a reflexive prime ideal if and only if $P = \mathfrak{p}[x; \sigma]$ for some σ-invariant reflexive ideal $\mathfrak{p}$ of R which is σ-prime* ($\mathfrak{p}$ *is σ-prime if, for σ-stable ideals $\mathfrak{a}, \mathfrak{b}$, $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{p}$ implies either $\mathfrak{a} \subseteq \mathfrak{p}$ or $\mathfrak{b} \subseteq \mathfrak{p}$*).

(2) Let $S = R[x; \delta]$ *and suppose* R *is a* δ*-Krull order in* Q*. Let* P *be an ideal of* S *with* $P \cap R \neq 0$*. Then* P *is a reflexive prime ideal if and only if* $P = \mathfrak{p}[x; \delta]$ *for some* δ*-stable reflexive ideal* $\mathfrak{p}$ *of* R *which is* δ*-prime* ($\mathfrak{p}$ *is* δ*-prime if, for* δ*-stable ideals* $\mathfrak{a}$, $\mathfrak{b}$*,* $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{p}$ *implies either* $\mathfrak{a} \subseteq \mathfrak{p}$ *or* $\mathfrak{b} \subseteq \mathfrak{p}$).

By Propositions 3.4 and 3.6, any maximal reflexive ideal P of S is either $P \in \mathrm{Spec}_0^*(S)$ or $P = \mathfrak{p}[x; \sigma]$ for a reflexive σ-prime ideal $\mathfrak{p}$ of R (in case $S = R[x; \delta]$, $P \in \mathrm{Spec}_0^*(S)$ or $P = \mathfrak{p}[x; \delta]$ for a reflexive δ-prime ideal $\mathfrak{p}$ of R).

We denote the set of σ-invariant reflexive fractional R-ideals by $D_\sigma(R)$ and by $D_\delta(R) = \{\mathfrak{a} \mid \mathfrak{a}$ is a δ-stable reflexive fractional R-ideal$\}$. Then $D_\sigma(R)$ is an abelian group generated by maximal σ-invariant reflexive ideals of R. Similarly, $D_\delta(R)$ is an abelian group generated by maximal δ-stable reflexive ideals of R.

Thus we have the following which describe all reflexive fractional S-idealsIdeal!sideal@s-ideal.

Theorem 3.7 ([63]) *(1) Suppose* R *is a* σ*-Krull order in* Q *and* $S = R[x; \sigma]$, $T = Q[x; \sigma]$*. Then*

$$D(S) \cong D_\sigma(R) \oplus D(T).$$

(2) Suppose R *is a* δ*-Krull order in* Q *and let* $S = R[x; \delta]$*,* $T = Q[x; \delta]$*. Then*

$$D(S) \cong D_\delta(R) \oplus D(T).$$

Let R be a Krull order in Q. The set of principal fractional R-ideals forms a subgroup $P(R)$ of $D(R)$, where a fractional R-ideal $\mathfrak{a}$ is principal if $\mathfrak{a} = aR = Ra$ for some $a \in \mathfrak{a}$. The factor group $D(R)/P(R)$ is called the *divisor class group* of R, which is denoted by $Cl(R)$. In case R is a σ-Krull order (δ-Krull order), we can similarly define $Cl_\sigma(R) = D_\sigma(R)/P_\sigma(R)$ ($Cl_\delta(R) = D_\delta(R)/P_\delta(R)$) which is called the σ*-divisor class group* of R (δ*-divisor class group* of R), respectively, where $P_\sigma(R)$ is the subgroup of σ-invariant principal fractional R-ideals ($P_\delta(R)$ is the subgroup of δ-stable principal fractional R-ideals).

Proposition 3.8 ([63]) *(1) Suppose* R *is a* σ*-Krull order in* Q *and let* $S = R[x; \sigma]$*. Then the map* $\phi : D_\sigma(R) \longrightarrow D(S)$ *defined by* $\phi(\mathfrak{a}) = \mathfrak{a}[x; \sigma]$*, where* $\mathfrak{a} \in D_\sigma(R)$ *induces an isomorphism:* $Cl_\sigma(R) \cong Cl(S)$*.*
(2) Suppose R *is a* δ*-Krull order in* Q *and let* $S = R[x; \delta]$*. Then the map* $\psi : D_\delta(R) \longrightarrow D(S)$ *defined by* $\psi(\mathfrak{a}) = \mathfrak{a}[x; \delta]$*, where* $\mathfrak{a} \in D_\delta(R)$ *induces a surjective map:* $Cl_\delta(R) \longrightarrow Cl(S)$*. If* R *is a domain, then* $Cl_\delta(R) \cong Cl(S)$*.*

Let $S = R[Ix; \sigma, \delta]$ be an Ore–Rees ring, where R is a Noetherian prime ring as in Sect. 2. A fractional R-ideal $\mathfrak{a}$ is called $(\sigma; I)$*-invariant* if $I\sigma(\mathfrak{a}) = \mathfrak{a}I$.

An order R is a $(\sigma; I)$*-maximal order* if $O_l(\mathfrak{a}) = R = O_r(\mathfrak{a})$ for any $(\sigma; I)$-invariant ideal $\mathfrak{a}$ of R. If R is a $(\sigma; I)$-maximal order, then $D_{\sigma;I}(R)$, the set of all $(\sigma; I)$-invariant reflexive fractional R-ideals, is an Abelian group generated by maximal $(\sigma; I)$-invariant reflexive ideals of R (this is proved by standard way).

A fractional R-ideal $\mathfrak{a}$ is said to be $(\delta; I)$*-stable* if $I\delta(\mathfrak{a}) \subseteq \mathfrak{a}$ and $I\mathfrak{a} = \mathfrak{a}I$. We can define a $(\delta; I)$-maximal order in an obvious way and denote the set of all $(\delta; I)$-stable reflexive fractional R-ideals by $D_{\delta;I}(R)$. If R is a $(\delta; I)$-maximal order, then $D_{\delta;I}(R)$ is an Abelian group generated by maximal $(\delta; I)$-stable reflexive ideals of R.

In case $\delta = 0$ or $\sigma = 1$, we write $R[Ix; \sigma]$ for $R[Ix; \sigma, 0]$ and $R[Ix; \delta]$ for $R[Ix; 1, \delta]$, respectively. If S is a maximal order (in case $\delta = 0$ or $\sigma = 1$), then we completely describe the structure of reflexive fractional S-ideals as follows:

Theorem 3.9 ([47]) *Let R be a Noetherian prime ring and $S = R[Ix; \sigma]$ or $S = R[Ix; \delta]$. Then*
(1) In case $\delta = 0$.
(i) S is a maximal order if and only if R is a $(\sigma; I)$-maximal order.
(ii) If R is a $(\sigma; I)$-maximal order, then any reflexive fractional S-idealIdeal!sideal@s-ideal is of the form:

$$x^n w\mathfrak{a}[Ix; \sigma]$$

for some $\mathfrak{a} \in D_{\sigma;I}(R)$, $w \in \mathbb{Z}(Q(T))$, the center of $Q(T)$, and n is an integer.
(2) In case $\sigma = 1$.
(i) S is a maximal order if and only if R is a $(\delta; I)$-maximal order.
(ii) If R is a $(\delta; I)$-maximal order, then any reflexive fractional S-ideal is of the form:

$$w\mathfrak{a}[Ix; \delta]$$

for some $\mathfrak{a} \in D_{\delta;I}(R)$, $w \in \mathbb{Z}(Q(T))$.

Let G be a polycyclic-by-finite group and $R[G]$ be the group ring. A subset of G is called *orbital* if it has only finite many distinct G-conjugates. G is called *dihedral free* if it contains no orbital subgroup isomorphic to the infinite dihedral group $< a, b \in G \mid aba = b^{-1}, a^2 = 1 >$.

Proposition 3.10 (Group rings) *Let G be a polycyclic-by-finite group. The group ring $R[G]$ is a prime Krull order if and only if*

(i) R is a prime Krull order,
(ii) G has no nontrivial finite normal subgroup, and
(iii) G is dihedral free [17, 18, 20].

Remark (1) In the first paragraph of Sect. 3, we did not give the definitions of Krull rings different from Krull rings due to Chamarie. See [54, 55] for the definition of Ω-Krull rings, and see [61] for the definition in the sense of Marubayashi.

It is natural, in a sense, from the viewpoint of multiplicative ideal theory that an order is a Krull order (ring) if it is a maximal order and satisfies the ascending chain condition on integral reflexive ideals ([38, 49, 70] for monoids).

In case of rings having polynomial identities, those Krull rings all coincide, which is proved by using Posner's theorem [68, 13.6.5].

Krull orders in the sense of Chamarie are Krull orders in the sense of [70] ([63, Lemma 2.2.3]). However, it is still open whether each reflexive prime ideal of Krull orders in the sense of [70] is localizable or not, which is important to study the structure of orders. It is a remarkable result that an order R is Krull in the sense

of [70] if and only if the monoid of regular elements of R is a Krull monoid [38, Proposition 5.1].
(2) See [38, 75] for multiplicative ideal theory in noncommutative monoids.

4 Unique Factorization Rings

Noncommutative unique factorization rings were defined by various ring theorists with two different approaches. In 1963, P.M. Cohn generalized the notion of commutative unique factorization domains (UFD) to noncommutative rings with an element-wise approach, [29]. In 1984, A.W. Chatters introduced unique factorization for elements in the context of Noetherian rings which are not necessarily commutative with both element-wise approach and ideal theoretic approach, [25], and published a series of papers on the subject with his co-authors (D.A. Jordan, M.P. Gilchrist, and D. Wilson). In [1], authors gave a more general definition to noncommutative unique factorization rings and introduced connections to Krull orders. In this section, we give a summary of all approaches mentioned above.

A commutative unique factorization domain (UFD) is an integral domain satisfying the following three conditions (e.g. [81]):

1. Every element of R which is neither zero nor unit is a product of primes.
2. Any two prime factorizations of a given element have same number of factors.
3. The primes occurring in any factorization of a are completely determined by a, except for their order and for multiplication by units.

In [29], Cohn generalizes the notion of UFD to noncommutative rings by taking 1–3 as starting point. By an integral domain we understand a ring (not necessarily commutative) in which $1 \neq 0$, and without zero-divisors. Thus in an integral domain R, the nonzero elements form a semigroup under multiplication which will be denoted by R^*. Two elements a, b of a ring R are said to be associated, if $b = uav$, where u, v are units in R. An irreducible element in R is a non-unit which is not a product of two non-units. Clearly, if a is irreducible, or unit, or zero, then so is any element associated to a. Two elements a, b of R are said to be right similar, if $R/aR \cong R/bR$, as right R-modules [48].

Lemma 4.1 ([33]) *Two elements in an integral domain are right similar if and only if they are left similar.*

Let $a, b \in R$ and consider any factorizations of a and b:

$$a = u_1 u_2 \dots u_r,$$
$$b = v_1 v_2 \dots v_s.$$

These factorizations are said to be isomorphic, if $r = s$ and there is a permutation π of (1,…,r) such that u_i is similar to $v_{i\pi}$.

Proposition 4.2 ([29, Proposition 2.2]) *Let a, b be nonzero elements of an integral domain R which are similar. Then any factorization of a gives rise to an isomorphic factorization of b.*

A factorization of a may be regarded as a chain of cyclic submodules from R to aR, and by the isomorphism $R/aR \cong R/bR$ this gives a chain from R to bR, in which corresponding factors are isomorphic.

Definition 4.3 ([29]) A unique factorization domain (UFD for short) is an integral domain R such that every non-unit of R^* has a factorization into irreducibles and any two factorizations of a given element are isomorphic.

Since in a commutative integral domain R, a and b are associated if and only if $R/aR \cong R/bR$ holds, we have the following theorem:

Theorem 4.4 ([29, Theorem 2.3]) *A commutative integral domain is a UFD if and only if it satisfies 1–3 above.*

Noncommutative principal ideal domains [48] are given as an example of a noncommutative UFD. This includes in particular the skew polynomial rings studied by Ore [72] and the ring of integral quaternions. Moreover, any free associative algebra is a UFD [29, Theorem 6.3].

In 1984, A.W. Chatters defined unique factorization domains in the context of (not necessarily commutative) Noetherian rings which also has an equivalent elementwise definition.

Let R be a prime ring. A height-1 prime ideal of R is a prime ideal P of R such that P is minimal among nonzero prime ideals of R. An element p of R is *completely prime* if $pR = Rp$ is a height-1 prime of R and R/pR is a domain. If I is an ideal of R then $\mathcal{C}(I)$ is the set of elements of R which are regular (i.e. not zero-divisors) modulo I. Set $\mathcal{C} = \bigcap \mathcal{C}(P)$, where P ranges over the height-1 primes of R.

Proposition 4.5 ([25, Proposition 2.1]) *Let R be a prime Noetherian ring with at least one height-1 prime ideal, then the following conditions on R are equivalent:*
1. Every height-1 prime of R is of the form pR for some completely prime element p of R.
2. R is a domain and every nonzero element of R is of the form $cp_1p_2 \dots p_n$ for some $c \in \mathcal{C}$ (as defined above) and for some finite sequence $p_1, \dots, p_n$ of completely prime elements of R.

Definition 4.6 ([25]) A Noetherian unique factorization domain (Noetherian UFD for short) is a Noetherian integral domain which has at least one height-1 prime ideal and which satisfies the equivalent conditions of Proposition 4.5.

Examples of Noetherian UFDs include Noetherian UFDs of commutative algebra and also the universal enveloping algebras of solvable Lie algebras.

We can deduce from Sect. 2 that a commutative Noetherian domain is a maximal order if and only if it is integrally closed. In the case of Noetherian UFDs we have the following theorem:

Theorem 4.7 ([25, Theorem 2.10]) *Let R be a Noetherian UFD such that every nonzero prime ideal of R contains a height-1 prime; then R is a maximal order.*

The Noetherian UFDs defined as in [25] has one respect which is not analogous to the commutative case, and that is the existence of Noetherian UFDs R such that the polynomial ring $R[x]$ is not a UFD. Because of this reason, Chatters and Jordan gave a more general definition of Noetherian unique factorization rings.

Definition 4.8 ([27]) A ring R will be called a Noetherian unique factorisation ring (Noetherian UFR, for short) if R is a prime Noetherian ring such that every nonzero prime ideal of R contains a nonzero principal prime ideal.

The class of Noetherian UFRs includes all Noetherian UFDs as defined in [25]. If D is the division algebra of rational quaternions and $R = D[x]$ then R is a Noetherian UFR and $(x^2 + 1)R$ is a height-1 prime of R, but R is not a Noetherian UFD because $R/(x^2 + 1)R$ is not a domain.

Following are some of the important results obtained by Chatters and Jordan.

Theorem 4.9 ([27]) *If R is a Noetherian UFR then R is a maximal order.*

Theorem 4.10 ([27]) *If R is a Noetherian UFR then $R[x]$ is a Noetherian UFR.*

Let $R[x; \sigma]$ and $R[x; \delta]$ be defined as in Sect. 2. Then;

Theorem 4.11 ([27]) *Let R be a Noetherian UFR with an automorphism of finite order. Then $R[x; \sigma]$ is a Noetherian UFR.*

Theorem 4.12 ([27]) *Let R be a Noetherian UFR and let δ be a derivation of R such that every nonzero δ-prime ideal contains a nonzero principal δ-ideal. Then $R[x; \delta]$ is a Noetherian UFR.*

However, if R is a Noetherian UFR in the sense of [27], then $R[x; \sigma]$ and $R[x; \delta]$ are not necessarily Noetherian UFRs in the sense of [27].

Let G be a polycyclic-by-finite group. A *plinth* in G is a torsion-free abelian orbital subgroup H of G such that $H \otimes_{\mathbb{Z}} \mathbb{Q}$ is an irreducible $\mathbb{Q}T$-module for every subgroup T of a finite index in $N_G(H)$, where $\mathbb{Q}$ is the field of rationals. The plinth H is *centric* if its centralizer $C_G(H)$ has a finite index in G. We denote by $\Delta(G)$ the FC-subgroup, that is $\Delta(G) = \{g \in G : |G : C_G(g)| < \infty\}$, where $C_G(g)$ is the centralizer of g in G.

Proposition 4.13 ([17, 26]) *Let R be a commutative ring and G be a polycyclic-by-finite group. Then $R[G]$ is a Noetherian UFR in the sense of [27] if and only if*

(1) R is a UFD,
(2) G has no nontrivial finite normal subgroup,
(3) G is dihedral free, and
(4) Every plinth of G is centric.

Proposition 4.14 ([17, 26]) *Let R be a commutative ring and G be a polycyclic-by-finite group. Then $R[G]$ is a UFD in the sense of [25] if and only if*

(1) R is a UFD,
(2) G is torsion free,
(3) All plinths of G are central, and
(4) $G/\Delta(G)$ is torsion free.

Let S be a monoid with a polycyclic-by-finite group of quotients G. S is called *normalizing* if every element in S is normal, that is, $cS = Sc$ for all $c \in S$ and S is called *UF-monoid* if every prime ideal of S contains a principal prime ideal P, that is, $P = Sr = rS$ for some $r \in S$ as one in [27].

Proposition 4.15 ([50]) *Let S be a normalizing monoid with a torsion-free polycyclic-by-finite group of quotients G and let K be a field. Assume that S satisfies the ascending chain condition on left and right ideals. Then the semigroup algebra $K[S]$ is a Noetherian UFR in the sense of [27] if and only if $K[G]$ is a Noetherian UFR in the sense of [27] and S is a UF-monoid.*

In [28] Chatters, Gilchrist and Wilson developed a theory of noncommutative UFRs without the Noetherian condition.

Let R be an associative ring with identity element. An element x of R is *normal* if $xR = Rx$. A principal ideal of R is an ideal of the form xR for some normal element x of R. Let R be a prime ring, a prime element of R is a nonzero normal element p such that pR is a prime ideal.

Definition 4.16 ([28]) A ring R is called a unique factorization ring (UFR for short) if every nonzero prime ideal of R contains a prime element.

If R is a UFR as in [28] then the set of principal ideals of R is closed under finite intersections and satisfies the ascending chain condition, and the polynomial ring over R in an arbitrary number of central indeterminates is also a UFR. Restricting to the case of UFRs which satisfy a polynomial identity (PI) gives several genuinely noncommutative examples such as trace rings of generic matrix rings [21], the ring of n by n matrices over a commutative Dedekind domain of finite class number n; and the group ring $R[G]$ where R is any UFR which satisfies a PI and G is a torsion-free abelian group which satisfies the ascending chain condition for cyclic subgroups [28].

Another definition of unique factorization rings and its connections to Krull orders are given in Abbasi et al. [1]. Noetherian UFRs in the sense of [27] are Krull orders in the sense of Marubayashi [61] by [1, Proposition 1.9], and Krull orders in the sense of [61] are Krull orders in the sense of Chamarie [24]. Existence of examples of Krull orders which are not Krull orders in the sense of [61] and being natural that UFRs are closed under the polynomial extensions were the motivation of the authors of [1] to give a new definition of UFRs.

Definition 4.17 ([1]) Let R be a τ-Noetherian order with an automorphism σ in a simple Artinian ring Q. Then R is called a σ-unique factorization ring (a σ-UFR for short) if any σ-prime ideal P of R such that $P = P^{*+}$ or $P={}^{+*}P$ is principal.

In case σ is the identity mapping on R, R is said to be a UFR.

It turns out [63] that R is a UFR in the sense of [1] if and only if
(i) R is a maximal order and
(ii) Any reflexive ideal is principal.

Noetherian UFRs in the sense of [27] are UFRs in the sense of [1], however the converse is not true in general (see [1] for examples). Furthermore, we have the following:

Proposition 4.18 ([1]) *Let R be a UFR in the sense of [1]. Then R is a Noetherian UFR in the sense of [27] if and only if R_N is a simple ring, where N is Ore set consisting of all normal elements in R.*

Let δ be a derivation on R. Replacing σ-prime ideals by δ-prime ideals, we can naturally define δ-UFRs. Of course, if R is a UFR in the sense of [1], then R is a σ-UFR and δ-UFR, and we have the following characterizations:

Proposition 4.19 ([1]) *(1) R is a σ-UFR if and only if $R[x; \sigma]$ is a UFR in the sense of [1].*
(2) If R is a δ-UFR, then $R[x; \delta]$ is a UFR in the sense of [1]. In case R is a domain, the converse is also true.

In [1], they obtained the following characterizations of group ring $R[G]$ (independent on [26]).

Proposition 4.20 ([1]) *Let R be a UFR in the sense of [1] and G be a polycyclic-by-finite group. Then $R[G]$ is a UFR in the sense of [1] if and only if*
(1) G has no nontrivial finite normal subgroup, and
(2) G is dihedral free.

Proposition 4.21 ([1]) *Let R be a Noetherian UFR in the sense of [27] and G be a polycyclic-by-finite group. Then $R[G]$ is a Noetherian UFR in the sense of [27] if and only if*
(1) G has no nontrivial finite normal subgroup,
(2) G is dihedral free, and
(3) every plinth of G is centric.

We refer the readers to [76] for more examples and detailed survey of unique factorization rings.

5 G-Dedekind Prime Rings

The class of rings in which every reflexive (fractional) R-ideal (right or left) is invertible was first defined by Cozzens and Sandomierski in [30] with the name *RI-orders*. In [2], following the commutative version of the theory, Akalan characterized the class of rings in which $(AB)^* = B^*A^*$ is satisfied for all R-ideals A, B and gave the name *Generalized Dedekind prime rings* (G-Dedekind prime, for short) to this class of rings. It turns out that in a G-Dedekind prime ring every reflexive R-ideal is invertible and therefore is an RI-order.

The class of G-Dedekind prime rings is a broad class including both the class of Dedekind prime rings and the class of Noetherian UFRs [27]. Moreover, Noetherian maximal orders with gld ≤ 2 are examples of G-Dedekind prime rings. This assertion follows from Bass' characterization of Noetherian rings with gld ≤ 2 as rings over which duals of finitely generated modules are projective (see [30] and [14, Proposition 5.2]).

Definition 5.1 ([2]) A prime Noetherian maximal order satisfying $(AB)^* = B^*A^*$ for all R-ideals A and B, is called a generalized Dedekind prime ring (G-Dedekind prime ring).

As we have mentioned in Sect. 2 (Theorem 2.3), the set $D(R)$ of all reflexive R-ideals becomes an Abelian group with multiplication "$\circ$". We denote the divisor class group of R by $Cl(R) = D(R)/P(R)$ where $P(R)$ is the subgroup of $D(R)$ which consists of principal R-ideals, and the Picard group of R by $Pic(R) = Inv(R)/P(R)$ where $Inv(R)$ is the group of invertible R-ideals.

Theorem 5.2 ([2, Theorem 3.1]) *For an order R, the following conditions are equivalent:*
(1) $A^{**}A^* = R$ and $A^+A^{++} = R$ for each R-ideal A.
(2) R is a maximal order and $(AB)^* = B^*A^*$ for all R-ideals A and B of R.
(3) R is a maximal order and the product of reflexive R-ideals is reflexive.
(4) R is a maximal order and $D(R)$ is a group with the usual product.
(5) R is a maximal order and every reflexive R-ideal is invertible.
(6) R is a maximal order and $Cl(R) = Pic(R)$.

In [8, 80], many examples of commutative maximal orders with reflexive ideals which are not invertible are given. Following is a noncommutative example of a prime Noetherian maximal order with a reflexive ideal which is not invertible.

Example 5.3 By [16, Example 35] there exists a prime Noetherian smooth PI ring R which is also a maximal order with a unique height one prime ideal P which is not a projective R-module on either side. This height one prime ideal P is maximal reflexive by [3, Theorem 3.1]. However since P is not projective, it is not invertible.

The class of G-Dedekind prime rings is closed under the formation of $n \times n$ full matrix rings and moreover if R is a G-Dedekind prime ring then so is the ring eRe where e is an idempotent such that $ReR = R$. Thus, being a G-Dedekind prime ring is a Morita invariant.

Theorem 5.4 ([2, Theorem 5.4]) *If R is a PI G-Dedekind prime ring then so is the polynomial ring $R[x]$.*

In [4], Akalan showed that the PI condition can be waived from Theorem 5.4.

Theorem 5.5 ([2, Theorem 6.2]) *If R is a PI G-Dedekind prime ring then so is the Rees ring $R[Ix]$ where I is an invertible ideal of R.*

In Marubayashi et al. [67], authors use the terminology "generalized Asano prime rings" for "generalized Dedekind prime rings". Let σ be an automorphism of R, they call R a *σ-generalized Asano* prime ring (a σ-G-Asano prime ring for short) if it is a σ-Krull prime ring whose σ-invariant reflexive R-ideals are invertible. In case σ is identity, R is said to be a *G-Asano* prime ring.

Theorem 5.6 ([67, Theorem 2.8]) *Let R be an order in Q. R is a σ-G-Asano prime ring if and only if $R[x; \sigma]$ is a G-Asano prime ring.*

Definition 5.7 ([46]) A ring R is called δ-generalized Asano prime ring if R is a δ-Krull prime ring whose δ-stable reflexive R-ideals are invertible.

Theorem 5.8 ([46, Theorem 2.6]) *Let R be an order in Q. Then R is a δ-generalized Asano prime ring if and only if $S = R[x; \delta]$ is a generalized Asano prime ring.*

A generalized Asano prime ring is a Krull prime ring, but the converse of this does not necessarily hold [67] and [36, Example 1.10].

6 Hereditary Noetherian Prime Rings (HNP Rings) and a Generalization of HNP Rings

Hereditary Noetherian prime rings (HNP for short) are a very interesting class of rings and a lot of research has been done on them, especially for 1960–1990. In 1960, Auslander and Goldman found an example of HNP rings which is not Dedekind in crossed product algebras [13]. Since then, in case of algebras, Harada had studied the structure of HNP rings including ideal theory [42–44]. In 1970, Eisenbud and Robson studied the ideal theory of HNP rings which are not necessarily algebras. In this section, we mainly discuss the ideal theory in HNP rings and propose a polynomial-type generalization of HNP rings.

One of the important results on HNP rings is that the invertible ideals in an HNP ring generate an Abelian group as in Dedekind orders, which is obtained under the condition: every ideal is projective (left and right projective). The followings are the key propositions to prove this result.

Proposition 6.1 ([32, Proposition 2.1]) *Let R be an order in a simple Artinian ring such that each ideal of R is projective. Then every invertible ideal of R is a product of maximal invertible ideals (ideals maximal amongst the invertible ideals).*

Proposition 6.2 ([32, Proposition 2.2]) *Let R be an order in a simple Artinian ring such that each ideal of R is projective. Then each maximal ideal of R is either idempotent or invertible.*

A finite set of distinct idempotent maximal ideals $M_1, \ldots, M_n$ such that $O_r(M_1) = O_l(M_2), \ldots, O_r(M_n) = O_l(M_1)$ is called a *cycle*. An invertible maximal ideal is considered to be a trivial case of a cycle.

Theorem 6.3 ([32, Theorem 2.6]) *Let R be an order in a simple Artinian ring such that each ideal of R is projective. Then each maximal invertible ideal of R is the intersection of a cycle.*

Theorem 6.4 ([32, Theorem 2.9]) *Let R be an order in a simple Artinian ring such that each ideal of R is projective. Then the invertible ideals of R generate an Abelian group.*

An ideal A is called *eventually idempotent* if A^k is idempotent for some $k \geq 1$.

Proposition 6.5 ([32, Proposition 4.5]) *Let R be an HNP ring and A be an ideal of R which is not contained in any invertible ideal. Then A is eventually idempotent. More precisely, there are only a finite number of idempotent ideals* $M_1, \ldots, M_k$ *containing A and* $A^k = (M_1 \cap \cdots \cap M_k)^k$ *is idempotent. (see [37] for more detail results on eventually idempotent.)*

Theorem 6.6 ([32, Theorem 4.2]) *Let R be an HNP ring and I an ideal of R. Then* $I = XA$, *where X is an invertible ideal and A is an eventually idempotent ideal.*

Let A be a right ideal of R. The subring $\mathbf{I}(A) = \{r \in R | rA \subseteq A\}$ of R is called the *idealizer* of A in R. A is said to be *generative* if $RA = R$. The idealizer is one of the powerful tools to study HNP rings.

Theorem 6.7 ([74, Theorem 5.3]) *Let R be an HNP ring and A be an essential right ideal which is generative. Then* $\mathbf{I}(A)$ *is an HNP ring if and only if A is semimaximal, that is, A is a finite intersection of maximal right ideals.*

Theorem 6.8 ([74, Theorem 6.3]) *The following conditions on an HNP ring R are equivalent.*

(1) R is contained in and is equivalent to a Dedekind prime ring.
(2) R has finitely many idempotent ideals.
(3) R has finitely many idempotent maximal ideals.
(4) R is obtained as an iterated idealizer from a Dedekind prime ring.

It was an interesting question that any HNP ring has only finitely many idempotent ideals or not. In [78], they obtained examples of HNP rings in which there are infinite many idempotent maximal ideals.

A right ideal A is called *isomaximal* if R/A is a finite direct sum of isomorphic simple modules. In case A is isomaximal and generative, we have the following correspondence between Spec(R) and Spec(S), where $S = \mathbf{I}(A)$.

Theorem 6.9 ([68, Theorem 5.6.11]) *Let R be an HNP ring, A be a generative isomaximal right ideal and* $S = \mathbf{I}(A)$. *Then there is a set embedding* ϕ: $\{P \in Spec(R) | P \nsubseteq A\} \to \mathrm{Spec}(S)$ *given by* $P \to P \cap S$. *This preserves idempotence and invertibility. Further:*
(1) If there is no nonzero prime ideal P of R with $P \subseteq A$, *then there is only one nonzero prime of S not in the image of* ϕ, *that is, A, which is idempotent.*
(2) If there is a (necessarily unique) nonzero prime ideal $P \subseteq A$, *then there are exactly two nonzero primes of S not in the image of* ϕ, *A and, say* A'. *Both are idempotent and* A' *is an isomaximal generative left ideal of R containing P.*

We refer the readers to [57, 68] for more information about ideal theory in HNP rings.

Finally we discuss the ideal theory of polynomial rings over an HNP ring and propose a generalization of HNP rings. Let R be an HNP ring and $S = R[x]$ be the polynomial ring. Then S is not necessarily an HNP ring. In fact S is an HNP ring if and only if $R = Q$.

Note: any one sided reflexive ideal of S is projective since $\text{gl.dim}(S) \leq 2$.

Let A be a nonzero ideal of S such that $A = A^{*+}$ or $A = A^{+*}$, equivalently, A is right projective or A is left projective. Then we have the following [5]:
(a) If $\mathfrak{a} = A \cap R \neq (0)$, then $A = \mathfrak{a}[x]$.
(b) If $A \cap R = (0)$, then $A = B\mathfrak{a}[x]$ for an invertible ideal B of S and an ideal $\mathfrak{a}$ of R.

In both cases, A is left and right projective.

These properties suggest us to define the following which are, in some sense, a polynomial-type generalization of HNP rings.

Definition 6.10 ([6]) (1) A τ-Noetherian prime Goldie ring R is called a generalized HNP ring (a G-HNP ring for short) if each ideal A with $A = A^{*+}$ or $A = A^{+*}$ is left and right projective.
(2) A G-HNP ring is said to be a strongly G-HNP ring if each essential right (left) ideal $I(J)$ with $I = I^{*+}(J = J^{+*})$ is right (left) projective, respectively.

If R is an HNP ring, then $R[x]$ is a strongly G-HNP ring. The following is a structure theorem for G-HNP rings (compare with Theorem 3.3).

Theorem 6.11 (Structure theorem for G-HNP rings, [6]) *Let R be a G-HNP ring. Then*

(1) any maximal invertible ideal P is localizable and R_P is a semi-local HNP ring.
(2) $R = \cap R_P \cap S(R)$, where P ranges over all maximal invertible ideals of R and $S(R)$ is a G-HNP ring with no invertible ideals.
(3) R has a finite character property.

We end the paper with the following questions.

Let σ be an automorphism of R and δ be a left σ-derivation on R.

Question 6.12 *(1) What are necessary and sufficient conditions for $R[x; \sigma, \delta]$ to be a G-HNP ring and describe all projective ideals of $R[x; \sigma, \delta]$.*
(2) Let I be an invertible ideal of R. What are necessary and sufficient conditions for $R[Ix; \sigma, \delta]$ to be a G-HNP ring and describe all projective ideals of $R[Ix; \sigma, \delta]$.

Let H be a monoid with quotient group Q. By adopting dual basis lemma for projective modules [68, (3.5.2)], we can define the concept of right hereditary monoids as follows: H is *right hereditary* if $II^* = O_l(I)$ for any right ideal I of H, where $I^* = \{q \in Q | qI \subseteq H\}$. Similarly we can define left hereditary monoids.

Question 6.13 *Is it possible to obtain ideal theories (as ones in HNP rings) in left and right hereditary monoids?*

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