Chapter 5

A Digression: Semigroups

We have seen that the Markov kernel $p_t(x, B)$ of a Lévy or Markov process induces a semigroup of linear operators $(P_t)_{t\geqslant 0}$. In this chapter we collect a few tools from functional analysis for the study of operator semigroups. By $\mathcal{B}_b(\mathbb{R}^d)$ we denote the bounded Borel functions $f : \mathbb{R}^d \to \mathbb{R}$, and $\mathcal{C}_\infty(\mathbb{R}^d)$ are the continuous functions vanishing at infinity, i.e., $\lim_{|x|\to\infty} f(x) = 0$; when equipped with the uniform norm $\|\cdot\|_\infty$ both sets become Banach spaces.

Definition 5.1. A **Feller semigroup** is a family of linear operators

$$P_t : \mathcal{B}_b(\mathbb{R}^d) \to \mathcal{B}_b(\mathbb{R}^d)$$

satisfying the properties a)–g) of Definition 4.7: $(P_t)_{t\geqslant 0}$ is a semigroup of conservative, sub-Markovian operators which enjoy the **Feller property** $P_t(\mathcal{C}_\infty(\mathbb{R}^d)) \subset \mathcal{C}_\infty(\mathbb{R}^d)$ and which are strongly continuous on $\mathcal{C}_\infty(\mathbb{R}^d)$.

Notice that $(t, x) \mapsto P_t f(x)$ is for every $f \in \mathcal{C}_\infty(\mathbb{R}^d)$ continuous. This follows from

$$\begin{aligned}|P_t f(x) - P_s f(y)| &\leqslant |P_t f(x) - P_t f(y)| + |P_t f(y) - P_s f(y)|\\ &\leqslant |P_t f(x) - P_t f(y)| + \|P_{|t-s|} f - f\|_\infty,\end{aligned}$$

the Feller property 4.7.f) and the strong continuity 4.7.g).

Lemma 5.2. *If $(P_t)_{t\geqslant 0}$ is a Feller semigroup, then there exists a Markov transition function $p_t(x, dy)$ (Definition 4.1) such that $P_t f(x) = \int f(y)\, p_t(x, dy)$.*

Proof. By the Riesz representation theorem we see that the operators P_t are of the form $P_t f(x) = \int f(y)\, p_t(x, dy)$ where $p_t(x, dy)$ is a Markov kernel. The tricky part is to show the joint measurability $(t, x) \mapsto p_t(x, B)$ and the Chapman–Kolmogorov identities (4.2).

For every compact set $K \subset \mathbb{R}^d$ the functions defined by

$$f_n(x) := \frac{d(x, U_n^c)}{d(x, K) + d(x, U_n^c)}, \quad d(x, A) := \inf_{a\in A} |x - a|, \quad U_n := \{y : d(y, K) < 1/n\},$$

D. Khoshnevisan, R. Schilling, *From Lévy-Type Processes to Parabolic SPDEs*,
Advanced Courses in Mathematics - CRM Barcelona, DOI 10.1007/978-3-319-34120-0_5

are in $\mathcal{C}_\infty(\mathbb{R}^d)$ and $f_n \downarrow \mathbb{1}_K$. By monotone convergence, $p_t(x,K) = \inf_{n\in\mathbb{N}} P_t f_n(x)$ which proves the joint measurability in (t,x) for all compact sets.

By the same, the semigroup property $P_{t+s}f_n = P_s P_t f_n$ entails the Chapman–Kolmogorov identities for compact sets: $p_{t+s}(x,K) = \int p_t(y,K)\,p_s(x,dy)$. Since

$$\mathscr{D} := \left\{ B \in \mathscr{B}(\mathbb{R}^d) \;\middle|\; \begin{array}{l} (t,x) \mapsto p_t(x,B) \quad \text{is measurable \&} \\ p_{t+s}(x,B) = \displaystyle\int p_t(y,B)\,p_s(x,dy) \end{array} \right\}$$

is a Dynkin system containing the compact sets, we have $\mathscr{D} = \mathscr{B}(\mathbb{R}^d)$. □

To get an intuition for semigroups it is a good idea to view the semigroup property

$$P_{t+s} = P_s \circ P_t \quad \text{and} \quad P_0 = \mathrm{id}$$

as an operator-valued Cauchy functional equation. If $t \mapsto P_t$ is – in a suitable sense – continuous, the unique solution will be of the form $P_t = \mathrm{e}^{tA}$ for some **operator** A. This can be easily made rigorous for matrices $A, P_t \in \mathbb{R}^{n\times n}$ since the matrix exponential is well-defined by the uniformly convergent series

$$P_t = \exp(tA) := \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} \quad \text{and} \quad A = \frac{d}{dt} P_t\Big|_{t=0}$$

with $A^0 := \mathrm{id}$ and $A^k = A \circ A \circ \cdots \circ A$ (k times). With a bit more care, this can be made to work also in general settings.

Definition 5.3. Let $(P_t)_{t\geqslant 0}$ be a Feller semigroup. The (infinitesimal) **generator** is a linear operator defined by

$$\mathcal{D}(A) := \left\{ f \in \mathcal{C}_\infty(\mathbb{R}^d) \;\middle|\; \exists g \in \mathcal{C}_\infty(\mathbb{R}^d) \,:\, \lim_{t\to 0} \left\| \frac{P_t f - f}{t} - g \right\|_\infty = 0 \right\} \tag{5.1}$$

$$Af := \lim_{t\to 0} \frac{P_t f - f}{t}, \quad f \in \mathcal{D}(A). \tag{5.2}$$

The next lemma is the rigorous version for the symbolic notation '$P_t = \mathrm{e}^{tA}$'.

Lemma 5.4. *Let $(P_t)_{t\geqslant 0}$ be a Feller semigroup with infinitesimal generator $(A, \mathcal{D}(A))$. Then $P_t(\mathcal{D}(A)) \subset \mathcal{D}(A)$ and*

$$\frac{d}{dt} P_t f = A P_t f = P_t A f \quad \text{for all} \;\; f \in \mathcal{D}(A),\; t \geqslant 0. \tag{5.3}$$

Moreover, $\int_0^t P_s f\,ds \in \mathcal{D}(A)$ for any $f \in \mathcal{C}_\infty(\mathbb{R}^d)$, and

$$P_t f - f = A \int_0^t P_s f\,ds, \quad f \in \mathcal{C}_\infty(\mathbb{R}^d),\; t > 0 \tag{5.4}$$

$$= \int_0^t P_s A f\,ds, \quad f \in \mathcal{D}(A),\; t > 0. \tag{5.5}$$

Proof. Let $0 < \epsilon < t$ and $f \in \mathcal{D}(A)$. The semigroup and contraction properties give

$$\begin{aligned}\left\|\frac{P_t f - P_{t-\epsilon} f}{\epsilon} - P_t A f\right\|_\infty &\leqslant \left\|P_{t-\epsilon}\frac{P_\epsilon f - f}{\epsilon} - P_{t-\epsilon}Af\right\|_\infty + \left\|P_{t-\epsilon}Af - P_{t-\epsilon}P_\epsilon A f\right\|_\infty \\ &\leqslant \left\|\frac{P_\epsilon f - f}{\epsilon} - Af\right\|_\infty + \left\|Af - P_\epsilon Af\right\|_\infty \xrightarrow[\epsilon\to 0]{} 0\end{aligned}$$

where we use the strong continuity in the last step. This shows $\frac{d^-}{dt}P_t f = AP_t f = P_t Af$; a similar (but simpler) calculation proves this also for $\frac{d^+}{dt}P_t f$.

Let $f \in \mathcal{C}_\infty(\mathbb{R}^d)$ and $t, \epsilon > 0$. By Fubini's theorem and the representation of P_t with a Markov transition function (Lemma 5.2) we get

$$P_\epsilon \int_0^t P_s f(x)\,ds = \int_0^t P_\epsilon P_s f(x)\,ds,$$

and so,

$$\begin{aligned}\frac{P_\epsilon - \mathrm{id}}{\epsilon}\int_0^t P_s f(x)\,ds &= \frac{1}{\epsilon}\int_0^t \big(P_{s+\epsilon}f(x) - P_s f(x)\big)\,ds \\ &= \frac{1}{\epsilon}\int_t^{t+\epsilon} P_s f(x)\,ds - \frac{1}{\epsilon}\int_0^\epsilon P_s f(x)\,ds.\end{aligned}$$

Since $t \mapsto P_t f(x)$ is continuous, the fundamental theorem of calculus applies, and we get $\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_r^{r+\epsilon} P_s f(x)\,ds = P_r f(x)$ for $r \geqslant 0$. This shows that $\int_0^t P_s f\,ds \in \mathcal{D}(A)$ as well as (5.4). If $f \in \mathcal{D}(A)$, then we deduce (5.5) from

$$\int_0^t P_s Af(x)\,ds \overset{(5.3)}{=} \int_0^t \frac{d}{ds}P_s f(x)\,ds = P_t f(x) - f(x) \overset{(5.4)}{=} A\int_0^t P_s f(x)\,ds. \quad \square$$

Remark 5.5 (Consequences of Lemma 5.4)**.** Write $\mathcal{C}_\infty := \mathcal{C}_\infty(\mathbb{R}^d)$.

a) (5.4) shows that $\mathcal{D}(A)$ is dense in $\mathcal{C}_\infty$, since $\mathcal{D}(A) \ni t^{-1}\int_0^t P_s f\,ds \xrightarrow[t\to 0]{} f$ for any $f \in \mathcal{C}_\infty$.

b) (5.5) shows that A is a **closed operator**, i.e.,

$$f_n \in \mathcal{D}(A), (f_n, Af_n) \xrightarrow[n\to\infty]{\text{uniformly}} (f,g) \in \mathcal{C}_\infty \times \mathcal{C}_\infty \implies f \in \mathcal{D}(A)\ \&\ Af = g.$$

c) (5.3) means that A determines $(P_t)_{t\geqslant 0}$ uniquely.

Let us now consider the Laplace transform of $(P_t)_{t\geqslant 0}$.

Definition 5.6. Let $(P_t)_{t\geqslant 0}$ be a Feller semigroup. The **resolvent** is a linear operator on $\mathcal{B}_b(\mathbb{R}^d)$ given by

$$R_\lambda f(x) := \int_0^\infty \mathrm{e}^{-\lambda t} P_t f(x)\,dt, \quad f \in \mathcal{B}_b(\mathbb{R}^d),\ x \in \mathbb{R}^d,\ \lambda > 0. \tag{5.6}$$

The following formal calculation can easily be made rigorous. Let $f \in \mathcal{D}(A)$ and $(\lambda - A) := (\lambda \operatorname{id} - A)$ for $\lambda > 0$. Then

$$
\begin{aligned}
(\lambda - A)R_\lambda f &= (\lambda - A)\int_0^\infty e^{-\lambda t} P_t f \, dt \\
&\overset{(5.4),(5.5)}{=} \int_0^\infty e^{-\lambda t} (\lambda - A) P_t f \, dt \\
&= \lambda \int_0^\infty e^{-\lambda t} P_t f \, dt - \int_0^\infty e^{-\lambda t} \left(\frac{d}{dt} P_t f \right) dt \\
&\overset{\text{parts}}{=} \lambda \int_0^\infty e^{-\lambda t} P_t f \, dt - \lambda \int_0^\infty e^{-\lambda t} P_t f \, dt - [e^{-\lambda t} P_t f]_{t=0}^\infty = f.
\end{aligned}
$$

A similar calculation for $R_\lambda(\lambda - A)$ gives

Theorem 5.7. *Let $(A, \mathcal{D}(A))$ and $(R_\lambda)_{\lambda>0}$ be the generator and the resolvent of a Feller semigroup. Then*

$$R_\lambda = (\lambda - A)^{-1} \quad \textit{for all} \quad \lambda > 0.$$

Since R_λ is the Laplace transform of $(P_t)_{t\geqslant 0}$, the properties of $(R_\lambda)_{\lambda>0}$ can be found from $(P_t)_{t\geqslant 0}$ and vice versa. With some effort one can even invert the (operator-valued) Laplace transform which leads to the familiar expression for e^x:

$$\left(\frac{n}{t} R_{\frac{n}{t}} \right)^n = \left(\operatorname{id} - \frac{t}{n} A \right)^{-n} \xrightarrow[n\to\infty]{\text{strongly}} e^{tA} = P_t \tag{5.7}$$

(the notation $e^{tA} = P_t$ is, for unbounded operators A, formal), see Pazy [42, Chapter 1.8].

Lemma 5.8. *Let $(R_\lambda)_{\lambda>0}$ be the resolvent of a Feller*[1] *semigroup $(P_t)_{t\geqslant 0}$. Then*

$$\frac{d^n}{d\lambda^n} R_\lambda = n!(-1)^n R_\lambda^{n+1} \quad n \in \mathbb{N}_0. \tag{5.8}$$

Proof. Using a symmetry argument we see

$$t^n = \int_0^t \dots \int_0^t dt_1 \dots dt_n = n! \int_0^t \int_0^{t_n} \dots \int_0^{t_2} dt_1 \dots dt_n.$$

Let $f \in \mathcal{C}_\infty(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$. Then

$$(-1)^n \frac{d^n}{d\lambda^n} R_\lambda f(x) = \int_0^\infty (-1)^n \frac{d^n}{d\lambda^n} e^{-\lambda t} P_t f(x) \, dt = \int_0^\infty t^n e^{-\lambda t} P_t f(x) \, dt$$

[1] This lemma only needs that the operators P_t are strongly continuous and contractive, Definition 4.7.g), d).

$$
\begin{aligned}
&= n! \int_0^\infty \int_0^t \int_0^{t_n} \cdots \int_0^{t_2} \mathrm{e}^{-\lambda t} P_t f(x)\, dt_1 \dots dt_n\, dt \\
&= n! \int_0^\infty \int_{t_n}^\infty \cdots \int_{t_1}^\infty \mathrm{e}^{-\lambda t} P_t f(x)\, dt\, dt_1 \dots dt_n \\
&= n! \int_0^\infty \int_0^\infty \cdots \int_0^\infty \mathrm{e}^{-\lambda(t+t_1+\dots+t_n)} P_{t+t_1+\dots+t_n} f(x)\, dt\, dt_1 \dots dt_n \\
&= n!\, R_\lambda^{n+1} f(x).
\end{aligned}
$$ □

The key result identifying the generators of Feller semigroups is the following theorem due to Hille, Yosida and Ray, a proof can be found in Pazy [42, Chapter 1.4] or Ethier & Kurtz [17, Chapter 4.2]; a probabilistic approach is due to Itô [25].

Theorem 5.9 (Hille–Yosida–Ray)**.** *A linear operator* $(A, \mathcal{D}(A))$ *on* $\mathcal{C}_\infty(\mathbb{R}^d)$ *generates a Feller semigroup* $(P_t)_{t\geqslant 0}$ *if, and only if,*

a) $\mathcal{D}(A) \subset \mathcal{C}_\infty(\mathbb{R}^d)$ *dense.*
b) A *is* **dissipative***, i.e.,* $\|\lambda f - Af\|_\infty \geqslant \lambda \|f\|_\infty$ *for some (or all)* $\lambda > 0$*.*
c) $(\lambda - A)(\mathcal{D}(A)) = \mathcal{C}_\infty(\mathbb{R}^d)$ *for some (or all)* $\lambda > 0$*.*
d) A *satisfies the* **positive maximum principle***:*

$$
f \in \mathcal{D}(A),\ f(x_0) = \sup_{x\in\mathbb{R}^d} f(x) \geqslant 0 \implies Af(x_0) \leqslant 0. \tag{PMP}
$$

This variant of the Hille–Yosida theorem is not the standard version from functional analysis since we are interested in positivity preserving (sub-Markov) semigroups. Let us briefly discuss the role of the positive maximum principle.

Remark 5.10. Let $(P_t)_{t\geqslant 0}$ be a strongly continuous contraction semigroup on $\mathcal{C}_\infty(\mathbb{R}^d)$, i.e., $\|P_t f\|_\infty \leqslant \|f\|_\infty$ and $\lim_{t\to 0} \|P_t f - f\|_\infty = 0$, cf. Definition 4.7.d),g).[2]

1° Sub-Markov ⇒ (PMP). Assume that $f \in \mathcal{D}(A)$ is such that $f(x_0) = \sup f \geqslant 0$. Then

$$
P_t f(x_0) - f(x_0) \overset{f\leqslant f^+}{\leqslant} P_t^+ f(x_0) - f^+(x_0) \leqslant \|f^+\|_\infty - f^+(x_0) = 0.
$$

$$
\implies Af(x_0) = \lim_{t\to 0} \frac{P_t f(x_0) - f(x_0)}{t} \leqslant 0.
$$

Thus, (PMP) holds.

2° (PMP) ⇒ dissipativity. Assume that (PMP) holds and let $f \in \mathcal{D}(A)$. Since $f \in \mathcal{C}_\infty(\mathbb{R}^d)$, we may assume that $f(x_0) = |f(x_0)| = \sup |f|$ (otherwise $f \rightsquigarrow -f$). Then

$$
\|\lambda f - Af\|_\infty \geqslant \lambda f(x_0) - \underbrace{Af(x_0)}_{\leqslant 0} \geqslant \lambda f(x_0) = \lambda \|f\|_\infty.
$$

[2] These properties are essential for the existence of a generator and the resolvent on $\mathcal{C}_\infty(\mathbb{R}^d)$.

3° (PMP) ⇒ sub-Markov. Since P_t is contractive, we have $P_t f(x) \leqslant \|P_t f\|_\infty \leqslant \|f\|_\infty \leqslant 1$ for all $f \in \mathcal{C}_\infty(\mathbb{R}^d)$ such that $|f| \leqslant 1$. In order to see positivity, let $f \in \mathcal{C}_\infty(\mathbb{R}^d)$ be non-negative. We distinguish between two cases:

a) $R_\lambda f$ does not attain its infimum.] Since $R_\lambda f \in \mathcal{C}_\infty(\mathbb{R}^d)$ vanishes at infinity, we have necessarily $R_\lambda f \geqslant 0$.

b) $\exists x_0 \,:\, R_\lambda f(x_0) = \inf R_\lambda f$. Because of the (PMP) we find

$$\begin{aligned} \lambda R_\lambda f(x_0) - f(x_0) &= A R_\lambda f(x_0) \geqslant 0 \\ \implies \lambda R_\lambda f(x) &\geqslant \inf \lambda R_\lambda f = \lambda R_\lambda f(x_0) \geqslant f(x_0) \geqslant 0. \end{aligned}$$

This proves that $f \geqslant 0 \implies \lambda R_\lambda f \geqslant 0$. From (5.8) we see that $\lambda \mapsto R_\lambda f(x)$ is completely monotone, hence it is the Laplace transform of a positive measure. Since $R_\lambda f(x)$ has the integral representation (5.6), we see that $P_t f(x) \geqslant 0$ (for all $t \geqslant 0$ as $t \mapsto P_t f$ is continuous).

Using the Riesz representation theorem (as in Lemma 5.2) we can extend P_t as a sub-Markov operator onto $\mathcal{B}_b(\mathbb{R}^d)$.

In order to determine the domain $\mathcal{D}(A)$ of the generator the following 'maximal dissipativity' result is handy.

Lemma 5.11 (Dynkin, Reuter)**.** *Assume that* $(A, \mathcal{D}(A))$ *generates a Feller semigroup and that* $(\mathfrak{A}, \mathcal{D}(\mathfrak{A}))$ *extends* A, *i.e.,* $\mathcal{D}(A) \subset \mathcal{D}(\mathfrak{A})$ *and* $\mathfrak{A}|_{\mathcal{D}(A)} = A$. *If*

$$u \in \mathcal{D}(\mathfrak{A}),\ u - \mathfrak{A}u = 0 \implies u = 0, \tag{5.9}$$

then $(A, \mathcal{D}(A)) = (\mathfrak{A}, \mathcal{D}(\mathfrak{A}))$.

Proof. Since A is a generator, $(\mathrm{id} - A) : \mathcal{D}(A) \to \mathcal{C}_\infty(\mathbb{R}^d)$ is bijective. On the other hand, the relation (5.9) means that $(\mathrm{id} - \mathfrak{A})$ is injective, but $(\mathrm{id} - A)$ cannot have a proper injective extension. □

Theorem 5.12. *Let* $(P_t)_{t\geqslant 0}$ *be a Feller semigroup with generator* $(A, \mathcal{D}(A))$. *Then*

$$\mathcal{D}(A) = \left\{ f \in \mathcal{C}_\infty(\mathbb{R}^d) \;\middle|\; \exists g \in \mathcal{C}_\infty(\mathbb{R}^d)\ \forall x \,:\, \lim_{t\to 0} \frac{P_t f(x) - f(x)}{t} = g(x) \right\}. \tag{5.10}$$

Proof. Denote by $\mathcal{D}(\mathfrak{A})$ the right-hand side of (5.10) and define

$$\mathfrak{A} f(x) := \lim_{t\to 0} \frac{P_t f(x) - f(x)}{t} \quad \text{for all} \quad f \in \mathcal{D}(\mathfrak{A}),\ x \in \mathbb{R}^d.$$

Obviously, $(\mathfrak{A}, \mathcal{D}(\mathfrak{A}))$ is a linear operator which extends $(A, \mathcal{D}(A))$. Since (PMP) is, essentially, a pointwise assertion (see Remark 5.10, 1°), $\mathfrak{A}$ inherits (PMP); in particular, $\mathfrak{A}$ is dissipative (see Remark 5.10, 2°):

$$\|\mathfrak{A} f - \lambda f\|_\infty \geqslant \lambda \|f\|_\infty.$$

This implies (5.9), and the claim follows from Lemma 5.11. □