Inference Rules for Fuzzy Functional Dependencies in Possibilistic Databases

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Abstract. We consider fuzzy functional dependencies (FFDs) which can exist between attributes in possibilistic databases. The degree of FFD is evaluated by two numbers from the unit interval which correspond to possibility and necessity measures. The notion of FFD is defined with the use of the extended Gödel implication operator. For such dependencies we present inference rules as a fuzzy extension of Armstrong's axioms. We show that they form a sound and complete system.

Keywords: Possibilistic databases · Possibility distribution · Possibility measure · Necessity measure · Fuzzy implicator · Fuzzy functional dependencies · Inference rules

1 Introduction

Conventional database systems are designed with the assumption of precision of information collected in them. The problem becomes more complex if our knowledge of the fragment of reality to be modeled is imperfect. In such cases one has to apply tools for describing uncertain or imprecise information [7,8]. One of them is the theory of possibility [1,3]. In the possibilistic database framework attribute values are represented by means of possibility distributions. Each value x of an attribute X is assigned with a number $\pi_X(x)$ from the unit interval which expresses the possibility degree of its occurrence. Different ways of determination of the possibility degree have been described in [4].

One of the most important notions of the database theory is the concept of functional dependency (FD). The classical definition of functional dependency $X{\rightarrow}Y$ between attributes X and Y of a relation scheme R is based on the assumption that the equality of attribute values may be evaluated formally with the use of two-valued logic. The existence of $X{\rightarrow}Y$ means that X-values uniquely determine Y-values. If attribute values are imprecise one can say about a certain degree of the dependency $X{\rightarrow}Y$. It contains the information to what extent X determines Y. In possibilistic databases closeness of compared values can be evaluated by means of possibility and necessity measures.

Since the notion of FD plays an important role in the design process [5], its fuzzy extension has attracted a lot of attention. Hence, different approaches

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concerning fuzzy functional dependencies (FFDs) have been described in professional literature. A number of different definitions emerged [2,6,9,10]. In the paper we extend the definition given by Chen [2]. According to [2] the degree of the fuzzy functional dependency is evaluated by means of the possibility measure. The necessity measure is not used. The equality degree equals the maximum value 1 when the two possibility distributions have the maximum degree 1 at the same element. Thus applying of the possibility measure for evaluation of the equality of two imprecise values expressed by possibility distributions is not sufficient. In the paper evaluation of closeness of imprecise values is made by means of both possibility and necessity measures. The notion of FFD is defined with the use of the extended Gödel implication operator [9]. For such dependencies we will present inference system based on the well known set of Armstrong's axioms which is an important property of FDs in classical databases.

The paper is organized as follows. In the next section we discuss the basic notions dealing with fuzzy functional dependencies in possibilistic databases and formulate the extended inference rules. Section 3 discusses properties of the extended Gödel implication operator. In Sect. 4 we proved the soundness and completeness of the inference rules.

2 Fuzzy Functional Dependencies in Possiblistic Databases

Let r be a relation of the scheme $R(U)$ where U denotes a set of attributes, $U = \{X_1, X_2, ..., X_n\}$. Let $DOM(X_i)$ denotes a domain of X_i. Let us assume that attribute values are given by means of normal possibility distributions:

$$t(X) = \{\pi_{t(X_i)}(x)/x : x \in DOM(X_i)\}\ , \quad \sup_{x \in DOM(X_i)} \pi_{t(X_i)}(x) = 1, \tag{1}$$

where t is a tuple of r and $\pi_{t(X_i)}(x)$ is a possibility degree of $t(X_i) = x$. The possibility distribution takes the form: $\{\pi_X(x_1)/x_1, \pi_X(x_2)/x_2, ..., \pi_X(x_n)/x_n\}$, where $x_i \in \mathrm{DOM}(X)$. At least one value must be completely possible i.e. its possibility degree equals 1. This requirement is referred to as the normalization condition. Let t_1 and t_2 be tuples of r. The degrees of possibility and necessity that $t_1(X_i) = t_2(X_i)$, denoted by Pos and Nec, respectively, are as follows:

$$\begin{aligned} &Pos(\Pi_{t_1(X_i)} = \Pi_{t_2(X_i)}) = \sup_x \min(\pi_{t_1(X_i)}(x), \pi_{t_2(X_i)}(x)), \\ &Nec(\Pi_{t_1(X_i)} = \Pi_{t_2(X_i)}) = 1 - \sup_{x \neq y} \min(\pi_{t_1(X_i)}(x), \pi_{t_2(X_i)}(y)). \end{aligned} \tag{2}$$

The closeness degree of $t_1(X_i)$ and $t_2(X_i)$, denoted by $\approx(t_1(X_i), t_2(X_i))$, is expressed by two numbers $\approx(t_1(X_i), t_2(X_i))_N$ and $\approx(t_1(X_i), t_2(X_i))_\Pi$ from the unit interval which correspond to necessity and possibility measures. Thus $\approx (t_1(X_i), t_2(X_i)) = (\approx (t_1(X_i), t_2(X_i))_N, \approx (t_1(X_i), t_2(X_i))_\Pi)$. For identical values of $t_1(X_i)$ and $t_2(X_i)$ we have $\approx(t_1(X_i), t_2(X_i)) = (1,1)$. Otherwise,

$$\begin{aligned} &\approx (t_1(X_i), t_2(X_i))_N = Nec(\Pi_{t_1(X_i)} = \Pi_{t_2(X_i)}), \\ &\approx (t_1(X_i), t_2(X_i))_\Pi = Pos(\Pi_{t_1(X_i)} = \Pi_{t_2(X_i)}). \end{aligned} \tag{3}$$

For estimation of tuple closeness, denoted by $=_c(t_1(X), t_2(X)) = (=_c(t_1(X), t_2(X))_N, =_c(t_1(X), t_2(X))_\Pi)$, one must consider all the components X_i of X ($X_i \in X$) and apply the operation *min*:

$$=_c (t_1(X), t_2(X))_N = \min_i \approx ((t_1(X_i), t_2(X_i))_N,$$
$$=_c (t_1(X), t_2(X))_\Pi = \min_i \approx ((t_1(X_i), t_2(X_i))_\Pi, \quad (4)$$

In order to evaluate the degree of a fuzzy functional dependency by means of both possibility and necessity measures we will apply the following extension of the Gödel implication operator $I_G(a,b) = (I_G(a,b)_N, I_G(a,b)_\Pi)$, $a = (a_N, a_\Pi)$, $b = (b_N, b_\Pi)$, a_N, a_Π, b_N, $b_\Pi \in [0,1]$ where

$$I_G(a,b)_\Pi = \begin{cases} 1 & \text{if } a_\Pi \leq b_\Pi \\ b_\Pi & \text{otherwise,} \end{cases} \quad (5)$$

$$I_G(a,b)_N = \begin{cases} 1 & \text{if } a_N \leq b_N \textit{ and } a_\Pi \leq b_\Pi \\ b_\Pi & \text{if } a_N \leq b_N \text{ and } a_\Pi > b_\Pi \\ b_N & \text{otherwise.} \end{cases} \quad (6)$$

Definition 1. *Let R(U) be a relation scheme where* $U = \{X_1, X_2, \ldots, X_n\}$*. Let* X *and* Y *be subsets of* U*:* $X, Y \subseteq U$*.* Y *is functionally dependent on* X *in* $\theta = (\theta_N, \theta_\Pi)$ *degree,*$\theta_N, \theta_\Pi \in [0,1]$*, denoted by* $X \to_\theta Y$*, if and only if for every relation* r *of* R *the following conditions are met:*

$$min_{t_1,t_2 \in r} I(t_1(X) =_c t_2(X), t_1(Y) =_c t_2(Y))_N \geq \theta_N,$$
$$min_{t_1,t_2 \in r} I(t_1(X) =_c t_2(X), t_1(Y) =_c t_2(Y))_\Pi \geq \theta_\Pi, \quad (7)$$

where $=_c$ *is the closeness measure* (4) *and* I *is the following implicator:*

$$I(a,b) = \begin{cases} I_c & \textit{if } t_1(X) \textit{ and } t_2(X) \textit{ are identical} \\ I_G & \textit{otherwise,} \end{cases} \quad (8)$$

where I_c *is the classical implication operator and* I_G *is the extended Gödel implicator.*

Like in classical relational databases one can formulate the following inference rules known as extended Armstrong's axioms:

A1: $Y \subseteq X \Rightarrow X \to_\theta Y$ for all θ
A2: $X \to_\theta Y \Rightarrow XZ \to_\theta YZ$
A3: $X \to_\alpha Y \wedge Y \to_\beta Z \Rightarrow X \to_\gamma Z, \quad \gamma = (\min(\alpha_N, \beta_N), \min(\alpha_\Pi, \beta_\Pi))$

where $\theta = (\theta_N, \theta_\Pi)$, $\alpha = (\alpha_N, \alpha_\Pi)$, $\beta = (\beta_N, \beta_\Pi)$ and $\gamma = (\gamma_N, \gamma_\Pi)$ are pairs of numbers belonging to the unit interval $[0, 1]$.

3 Properties of the Extended Gödel Implicator

In order to prove the correctness of the inference rules for fuzzy functional dependencies (7) we will first show certain properties of the implicator I_G.

Theorem 1. *Let* $a = (a_N, a_\Pi)$, $a' = (a'_N, a'_\Pi)$, $b = (b_N, b_\Pi)$, $b' = (b'_N, b'_\Pi)$, $c = (c_N, c_\Pi)$, $\alpha = (\alpha_N, \alpha_\Pi)$, $\beta = (\beta_N, \beta_\Pi)$, $\gamma = (\gamma_N, \gamma_\Pi)$ *and* $\theta = (\theta_N, \theta_\Pi)$ *be pairs of numbers belonging to the unit interval* $[0, 1]$. *The implicator* I_G *satisfies the following conditions:*

P1: $a_N \leq b_N \wedge a_\Pi \leq b_\Pi \Rightarrow I_G(a, b) = (1, 1)$,
P2: $I_G(a,b)_N \geq \theta_N \wedge I_G(a,b)_\Pi \geq \theta_\Pi \Rightarrow I_G(a',b')_N \geq \theta_N \wedge I_G(a',b')_\Pi \geq \theta_\Pi$ *for* $a'_N = \min(a_N,c_N)$, $a'_\Pi = \min(a_\Pi,c_\Pi)$, $b'_N = \min(b_N,c_N)$, $b'_\Pi = \min(b_\Pi,c_\Pi)$,
P3: $I_G(a,b)_N \geq \alpha_N \wedge I_G(a,b)_\Pi \geq \alpha_\Pi \wedge I_G(b,c)_N \geq \beta_N \wedge I_G(b,c)_\Pi \geq \beta_\Pi \Rightarrow I_G(a,c)_N \geq \gamma_N \wedge I_G(a,c)_\Pi \geq \gamma_\Pi$ *for* $\gamma_N = \min(\alpha_N,\beta_N)$, $\gamma_\Pi = \min(\alpha_\Pi,\beta_\Pi)$.

Proof.
P1: This condition directly follows from the definition of I_G.
P2: Let $I_G(a,b)_N \geq \theta_N$ and $I_G(a,b)_\Pi \geq \theta_\Pi$. If $I_G(a,b) = (1,1)$ then $a_N \leq b_N$ and $a_\Pi \leq b_\Pi$. It follows that $a'_N \leq b'_N$ and $a'_\Pi \leq b'_\Pi$ and so $I_G(a',b') = (1,1)$. If $I_G(a,b) \neq (1,1)$ we must prove P2 for different cases of a, b and c. If $c_N < a_N$, $c_\Pi < a_\Pi$, $c_N < b_N$ and $c_\Pi < b_\Pi$ then $a' = b' = c \Rightarrow I_G(a',b') = (1,1)$. If $c_N \geq a_N$, $c_\Pi \geq a_\Pi$, $c_N \geq b_N$ and $c_\Pi \geq b_\Pi$ then ($a' = a$ and $b' = b$) $\Rightarrow I_G(a',b') = I_G(a,b)$.

I. Let $a_N > b_N$ and $a_\Pi > b_\Pi$. Thus $I_G(a,b) = b$.

1. $a_N > b_N \geq c_N$ and ($a_\Pi > c_\Pi \geq b_\Pi$ or $a_\Pi \geq c_\Pi > b_\Pi$). a'_N=c_N, a'_Π=c_Π, b'_N= c_N, b'_Π=b_Π. If $b_\Pi = c_\Pi$ then $I_G(a',b') = (1,1)$, otherwise $I_G(a',b') = (b_\Pi,b_\Pi)$.
2. $a_N > b_N \geq c_N$ and $c_\Pi \geq a_\Pi > b_\Pi$
   $a'_N = c_N$, $a'_\Pi = a_\Pi$, $b'_N = c_N$, $b'_\Pi = b_\Pi \Rightarrow I_G(a',b') = (b_\Pi,b_\Pi)$.
3. ($a_N > c_N \geq b_N$ or $a_N \geq c_N > b_N$) and $a_\Pi > b_\Pi \geq c_\Pi$. a'_N=c_N, a'_Π=c_Π, b'_N= b_N, b'_Π=c_Π. If $c_N = b_N$ then $I_G(a',b') = (1,1)$, otherwise $I_G(a',b') = (b_N,1)$.
4. ($a_N > c_N \geq b_N$ or $a_N \geq c_N > b_N$) and ($a_\Pi > c_\Pi \geq b_\Pi$ or $a_\Pi \geq c_\Pi > b_\Pi$)
   $a'_N = c_N$, $a'_\Pi = c_\Pi$, $b'_N = b_N$, $b'_\Pi = b_\Pi$. If ($c_N > b_N$ and $c_\Pi > b_\Pi$) then $I_G(a',b') = b$. If ($c_N > b_N$ and $c_\Pi = b_\Pi$) then $I_G(a',b') = (b_N,1)$. If ($c_N = b_N$ and $c_\Pi > b_\Pi$) then $I_G(a',b') = (b_\Pi,b_\Pi)$. If ($c_N = b_N$ and $c_\Pi = b_\Pi$) then $I_G(a',b') = (1,1)$.
5. ($a_N > c_N \geq b_N$ or $a_N \geq c_N > b_N$) and $c_\Pi \geq a_\Pi > b_\Pi$. a'_N=c_N, a'_Π=a_Π, b'_N= b_N, b'_Π=b_Π. If $c_N > b_N$ then $I_G(a',b') = b$, otherwise $I_G(a',b') = (b_\Pi,b_\Pi)$.
6. $c_N \geq a_N > b_N$ and $a_\Pi > b_\Pi \geq c_\Pi$
   $a'_N = a_N$, $a'_\Pi = c_\Pi$, $b'_N = b_N$, $b'_\Pi = c_\Pi \Rightarrow I_G(a',b') = (b_N,1)$.
7. $c_N \geq a_N > b_N$ and ($a_\Pi > c_\Pi \geq b_\Pi$ or $a_\Pi \geq c_\Pi > b_\Pi$). a'_N=a_N, a'_Π=c_Π, b'_N= b_N, b'_Π=b_Π. If $c_\Pi > b_\Pi$ then $I_G(a',b') = b$, otherwise $I_G(a', b') = (b_N,1)$.

II. Let $a_N > b_N$ and $a_\Pi \leq b_\Pi$. Thus $I_G(a,b) = (b_N,1)$.

1. $a_N > b_N \geq c_N$ and $b_\Pi \geq c_\Pi \geq a_\Pi$
   $a'_N = c_N$, $a'_\Pi = a_\Pi$, $b'_N = c_N$, $b'_\Pi = c_\Pi \Rightarrow I_G(a',b') = (1,1)$.

2. $a_N > b_N \geq c_N$ and $c_\Pi \geq b_\Pi \geq a_\Pi$
$a'_N = c_N$, $a'_\Pi = a_\Pi$, $b'_N = c_N$, $b'_\Pi = b_\Pi \Rightarrow I_G(a',b') = (1,1)$.
3. $(a_N > c_N \geq b_N$ or $a_N \geq c_N > b_N)$ and $b_\Pi \geq a_\Pi \geq c_\Pi$. $a'_N{=}c_N$, $a'_\Pi{=}c_\Pi$, $b'_N{=}$ b_N, $b'_\Pi{=}c_\Pi$. If $c_N = b_N$ then $I_G(a',b') = (1,1)$, otherwise $I_G(a',b') = (b_N,1)$.
4. $(a_N > c_N \geq b_N$ or $a_N \geq c_N > b_N)$ and $b_\Pi \geq c_\Pi \geq a_\Pi$. $a'_N{=}c_N$, a'_Π-a_Π, $b'_N{=}$ b_N, $b'_\Pi{=}c_\Pi$. If $c_N = b_N$ then $I_G(a',b') = (1,1)$, otherwise $I_G(a',b') = (b_N,1)$.
5. $(a_N > c_N \geq b_N$ or $a_N \geq c_N > b_N)$ and $c_\Pi \geq b_\Pi \geq a_\Pi$. $a'_N{=}c_N$, $a'_\Pi{=}a_\Pi$, $b'_N{=}$ b_N, $b'_\Pi{=}b_\Pi$. If $c_N = b_N$ then $I_G(a',b') = (1,1)$, otherwise $I_G(a',b') = (b_N,1)$.
6. $c_N \geq a_N > b_N$ and $b_\Pi \geq a_\Pi \geq c_\Pi$
$a'_N = a_N$, $a'_\Pi = c_\Pi$, $b'_N = b_N$, $b'_\Pi = c_\Pi \Rightarrow I_G(a',b') = (b_N,1)$.
7. $c_N \geq a_N > b_N$ and $b_\Pi \geq c_\Pi \geq a_\Pi$
$a'_N = a_N$, $a'_\Pi = a_\Pi$, $b'_N = b_N$, $b'_\Pi = c_\Pi \Rightarrow I_G(a',b') = (b_N,1)$.

III. Let $a_N \leq b_N$ and $a_\Pi > b_\Pi$. Thus $I_G(a,b) = (b_\Pi,b_\Pi)$.

1. $b_N \geq a_N \geq c_N$ and $(a_\Pi > c_\Pi \geq b_\Pi$ or $a_\Pi \geq c_\Pi > b_\Pi)$. $a'_N{=}c_N$, $a'_\Pi{=}c_\Pi$, $b'_N{=}$ c_N, $b'_\Pi{=}b_\Pi$. If $c_\Pi = b_\Pi$ then $I_G(a',b') = (1,1)$, otherwise $I_G(a',b') = (b_\Pi,b_\Pi)$.
2. $b_N \geq a_N \geq c_N$ and $c_\Pi \geq a_\Pi > b_\Pi$
$a'_N = c_N$, $a'_\Pi = a_\Pi$, $b'_N = c_N$, $b'_\Pi = b_\Pi \Rightarrow I_G(a',b') = (b_\Pi,b_\Pi)$.
3. $b_N \geq c_N \geq a_N$ and $a_\Pi > b_\Pi \geq c_\Pi$
$a'_N = a_N$, $a'_\Pi = c_\Pi$, $b'_N = c_N$, $b'_\Pi = c_\Pi \Rightarrow I_G(a',b') = (1,1)$.
4. $b_N \geq c_N \geq a_N$ and $(a_\Pi > c_\Pi \geq b_\Pi$ or $a_\Pi \geq c_\Pi > b_\Pi)$. $a'_N{=}a_N$, $a'_\Pi{=}c_\Pi$, $b'_N{=}$ c_N, $b'_\Pi{=}b_\Pi$. If $c_\Pi = b_\Pi$ then $I_G(a',b') = (1,1)$, otherwise $I_G(a',b') = (b_\Pi,b_\Pi)$.
5. $b_N \geq c_N \geq a_N$ and $c_\Pi \geq a_\Pi > b_\Pi$
$a'_N = a_N$, $a'_\Pi = a_\Pi$, $b'_N = c_N$, $b'_\Pi = b_\Pi \Rightarrow I_G(a',b') = (b_\Pi,b_\Pi)$.
6. $c_N \geq b_N \geq a_N$ and $a_\Pi > b_\Pi \geq c_\Pi$
$a'_N = a_N$, $a'_\Pi = c_\Pi$, $b'_N = b_N$, $b'_\Pi = c_\Pi \Rightarrow I_G(a',b') = (1,1)$.
7. $c_N \geq b_N \geq a_N$ and $(a_\Pi > c_\Pi \geq b_\Pi$ or $a_\Pi \geq c_\Pi > b_\Pi)$. $a'_N{=}a_N$, $a'_\Pi{=}c_\Pi$, $b'_N{=}$ b_N, $b'_\Pi{=}b_\Pi$. If $c_\Pi = b_\Pi$ then $I_G(a',b') = (1,1)$, otherwise $I_G(a',b') = (b_\Pi,b_\Pi)$.

P3: Let $\theta = (\min(I_G(a,b)_N, I_G(b,c)_N), \min(I_G(a,b)_\Pi, I_G(b,c)_\Pi))$.
If $a_N \leq c_N$ and $a_\Pi \leq c_\Pi$ then $I_G(a,c) = (1,1)$. If $a_N \leq b_N$ and $a_\Pi \leq b_\Pi$ then $I_G(a,b) = (1,1) \Rightarrow \theta = I_G(b,c)$. Since the components of I_G are decreasing in the first argument [9], we obtain $I_G(a,c)_N \geq \theta_N$ and $I_G(a,c)_\Pi \geq \theta_\Pi$. If $b_N \leq c_N$ and $b_\Pi \leq c_\Pi$ then $I_G(b,c) = (1,1) \Rightarrow \theta = I_G(a,b)$. Since the components of I_G are increasing in the second argument [9], we obtain $I_G(a,c)_N \geq \theta_N$ and $I_G(a,c)_\Pi$ $\geq \theta_\Pi$. Otherwise, we must prove P3 for different cases of a, b and c.

I. Let $a_N > b_N$ and $a_\Pi > b_\Pi$. Thus $I_G(a,b) = b$.

1. $a_N > b_N \geq c_N$ and $a_\Pi > b_\Pi \geq c_\Pi$. $I_G(a,c) = c$.
If $(b_N > c_N$ and $b_\Pi > c_\Pi)$ then $I_G(b,c) = c$. If $(b_N > c_N$ and $b_\Pi = c_\Pi)$ then $I_G(b,c) = (c_N,1)$. If $(b_N = c_N$ and $b_\Pi > c_\Pi)$ then $I_G(b,c) = (c_\Pi,c_\Pi)$. If $(b_N = c_N$ and $b_\Pi = c_\Pi)$ then $I_G(b,c) = (1,1)$. Thus in all cases $\theta = c \Rightarrow$ $I_G(a,c) = \theta$.

2. $a_N > b_N \geq c_N$ and ($a_\Pi > c_\Pi \geq b_\Pi$ or $a_\Pi \geq c_\Pi > b_\Pi$).
   $I_G(a,c)_N = c_N$ and $I_G(a,c)_\Pi \geq c_\Pi$. $I_G(b,c)_N \geq c_N$ and $I_G(b,c)_\Pi = 1$.
   Thus $\theta = (c_N, b_\Pi) \Rightarrow (I_G(a,c)_N = \theta_N$ and $I_G(a,c)_\Pi \geq \theta_\Pi)$.
3. $a_N > b_N \geq c_N$ and $c_\Pi \geq a_\Pi > b_\Pi$. $I_G(a,c) = (c_N, 1)$. $I_G(b,c)_N \geq c_N$. $I_G(b,c)_\Pi = 1$. Thus $\theta = (c_N, b_\Pi) \Rightarrow (I_G(a,c)_N = \theta_N$ and $I_G(a,c)_\Pi > \theta_\Pi)$.
4. ($a_N > c_N \geq b_N$ or $a_N \geq c_N > b_N$) and $a_\Pi > b_\Pi \geq c_\Pi$.
   $I_G(a,c)_N \geq c_N$. $I_G(a,c)_\Pi = c_\Pi$. If $b_\Pi > c_\Pi$ then $I_G(b,c) = (c_\Pi, c_\Pi)$. If $b_\Pi = c_\Pi$ then $I_G(b,c) = (1,1)$. Thus $\theta = (b_N, c_\Pi) \Rightarrow (I_G(a,c)_N \geq \theta_N$ and $I_G(a,c)_\Pi = \theta_\Pi)$.
5. $c_N \geq a_N > b_N$ and $a_\Pi > b_\Pi \geq c_\Pi$. $I_G(a,c) = (c_\Pi, c_\Pi)$.
   If $b_\Pi > c_\Pi$ then $I_G(b,c) = (c_\Pi, c_\Pi)$ and if $b_\Pi = c_\Pi$ then $I_G(b,c) = (1,1)$. Thus $\theta = (b_N, c_\Pi) \Rightarrow (I_G(a,c)_N > \theta_N$ and $I_G(a,c)_\Pi = \theta_\Pi)$.

II. Let $a_N > b_N$ and $a_\Pi \leq b_\Pi$. Thus $I_G(a,b) = (b_N, 1)$.

1. $a_N > b_N \geq c_N$ and $b_\Pi \geq a_\Pi \geq c_\Pi$. $I_G(a,c)_N = c_N$ and $I_G(a,c)_\Pi \geq c_\Pi$.
   If ($b_N > c_N$ and $b_\Pi > c_\Pi$) then $I_G(b,c) = c \Rightarrow \theta = c \Rightarrow (I_G(a,c)_N = \theta_N$ and $I_G(a,c)_\Pi \geq \theta_\Pi)$. If ($b_N > c_N$ and $b_\Pi = c_\Pi$) then $I_G(b,c) = (c_N, 1) \Rightarrow \theta = (c_N, 1)$. If $b_\Pi = c_\Pi$ then $a_\Pi = c_\Pi \Rightarrow I_G(a,c) = (c_N, 1) = \theta$. If ($b_N = c_N$ and $b_\Pi > c_\Pi$) then $I_G(b,c) = (c_\Pi, c_\Pi) \Rightarrow \theta = c \Rightarrow (I_G(a,c)_N = \theta_N$ and $I_G(a,c)_\Pi \geq \theta_\Pi)$. If $b = c$ then ($I_G(b,c) = (1,1)$ and $I_G(a,c) = (c_N, 1)) \Rightarrow \theta = (c_N, 1) \Rightarrow I_G(a,c) = \theta$.
2. $a_N > b_N \geq c_N$ and $b_\Pi \geq c_\Pi \geq a_\Pi$. $I_G(a,c) = (c_N, 1)$.
   $I_G(b,c)_N \geq c_N \Rightarrow \theta_N = c_N \Rightarrow I_G(a,c)_N = \theta_N$.
3. $a_N > b_N \geq c_N$ and $c_\Pi \geq b_\Pi \geq a_\Pi$. $I_G(a,c) = (c_N, 1)$.
   If $b_N > c_N$ then $I_G(b,c) = (c_N, 1) \Rightarrow \theta = (c_N, 1) \Rightarrow I_G(a,c) = \theta$.
   If $b_N = c_N$ then $I_G(b,c) = (1,1) \Rightarrow \theta = (c_N, 1) \Rightarrow I_G(a,c) = \theta$.
4. ($a_N > c_N \geq b_N$ or $a_N \geq c_N > b_N$) and $b_\Pi \geq a_\Pi \geq c_\Pi$
   $I_G(a,c)_N \geq c_N$ and $I_G(a,c)_\Pi \geq c_\Pi$. $I_G(b,c)_N = 1 \Rightarrow \theta_N = b_N \Rightarrow I_G(a,c)_N \geq \theta_N$. If $b_\Pi > c_\Pi$ then $I_G(b,c)_\Pi = c_\Pi \Rightarrow \theta_\Pi = c_\Pi \Rightarrow I_G(a,c)_\Pi \geq \theta_\Pi$. If $b_\Pi = c_\Pi$ then $I_G(b,c)_\Pi = 1 \Rightarrow \theta_\Pi = 1$. If $b_\Pi = c_\Pi$ then $a_\Pi = c_\Pi \Rightarrow I_G(a,c)_\Pi = 1 = \theta_\Pi$.
5. ($a_N > c_N \geq b_N$ or $a_N \geq c_N > b_N$) and $b_\Pi \geq c_\Pi \geq a_\Pi$
   $I_G(a,c)_N \geq c_N$ and $I_G(a,c)_\Pi = 1$. $I_G(b,c)_N = 1 \Rightarrow \theta_N = b_N \Rightarrow I_G(a,c)_N \geq \theta_N$.
6. $c_N \geq a_N > b_N$ and $b_\Pi \geq a_\Pi \geq c_\Pi$. $I(a,c)_N \geq c_\Pi$ and $I(a,c)_\Pi \geq c_\Pi$.
   $I(b,c)_N \geq c_\Pi \Rightarrow \theta_N = b_N \Rightarrow I(a,c)_N \geq \theta_N$. If $b_\Pi > c_\Pi$ then $I_G(b,c)_\Pi = c_\Pi \Rightarrow \theta_\Pi = c_\Pi \Rightarrow I_G(a,c)_\Pi \geq \theta_\Pi$. If $b_\Pi = c_\Pi$ then $I_G(b,c)_\Pi = 1 \Rightarrow \theta_\Pi = 1$. If $b_\Pi = c_\Pi$ then $a_\Pi = c_\Pi \Rightarrow I_G(a,c)_\Pi = 1 = \theta_\Pi$.

III. Let $a_N \leq b_N$ and $a_\Pi > b_\Pi$. Thus $I_G(a,b) = (b_\Pi, b_\Pi)$.

1. $b_N \geq a_N \geq c_N$ and $a_\Pi > b_\Pi \geq c_\Pi$
   If $a_N > c_N$ then $I_G(a,c) = c$, otherwise $I_G(a,c) = (c_\Pi, c_\Pi)$.
   If ($b_N > c_N$ and $b_\Pi > c_\Pi$) then $I_G(b,c) = c \Rightarrow \theta = c \Rightarrow (I_G(a,c)_N = \theta_N$ and $I_G(a,c)_\Pi = \theta_\Pi)$. If ($b_N > c_N$ and $b_\Pi = c_\Pi$) then $I_G(b,c) = (c_N, 1) \Rightarrow \theta = c \Rightarrow (I_G(a,c)_N \geq \theta_N$ and $I_G(a,c)_\Pi = \theta_\Pi)$. If ($b_N = c_N$ and $b_\Pi > c_\Pi$) then

$I_G(b,c) = (c_\Pi, c_\Pi) \Rightarrow \theta = (c_\Pi, c_\Pi)$. If $b_N = c_N$ then $a_N = c_N \Rightarrow I_G(a,c) = (c_\Pi, c_\Pi) = \theta$. If $b = c$ then $I_G(b,c) = (1,1) \Rightarrow \theta = (c_\Pi, c_\Pi) \Rightarrow I_G(a,c) = \theta$.

2. $b_N \geq a_N \geq c_N$ and $(a_\Pi > c_\Pi \geq b_\Pi$ or $a_\Pi \geq c_\Pi > b_\Pi)$
$I_G(a,c)_N \geq c_N$ and $I_G(a,c)_\Pi \geq c_\Pi$. $I_G(b,c)_\Pi = 1$.
If $b_N > c_N$ then $I_G(b,c) = (c_N,1) \Rightarrow \theta = (c_N, b_\Pi) \Rightarrow (I_G(a,c)_N \geq \theta_N$ and $I_G(a,c)_\Pi \geq \theta_\Pi)$. If $b_N{=}c_N$ then $I_G(b,c) = (1,1) \Rightarrow \theta = (b_\Pi, b_\Pi)$. If $b_N{=}c_N$ then $a_N{=}c_N \Rightarrow I_G(a,c)_N \geq c_\Pi \Rightarrow (I_G(a,c)_N \geq \theta_N$ and $I_G(a,c)_\Pi \geq \theta_\Pi)$.
3. $b_N \geq a_N \geq c_N$ and $c_\Pi \geq a_\Pi > b_\Pi$. $I_G(a,c)_N \geq c_N$ and $I_G(a,c)_\Pi = 1$.
$I_G(b,c)_\Pi = 1$. If $b_N > c_N$ then $I_G(b,c) = (c_N,1) \Rightarrow \theta = (c_N, b_\Pi) \Rightarrow (I_G(a,c)_N \geq \theta_N$ and $I_G(a,c)_\Pi > \theta_\Pi)$. If $b_N{=}c_N$ then $I_G(b,c) = (1,1) \Rightarrow \theta = (b_\Pi, b_\Pi)$. If $b_N{=}c_N$ then $a_N{=}c_N \Rightarrow I_G(a,c)_N = 1 \Rightarrow (I_G(a,c)_N > \theta_N$ and $I_G(a,c)_\Pi > \theta_\Pi)$.
4. $b_N \geq c_N \geq a_N$ and $a_\Pi > b_\Pi \geq c_\Pi$. $I_G(a,c) = (c_\Pi, c_\Pi)$.
If $(b_N > c_N$ and $b_\Pi > c_\Pi)$ then $I_G(b,c) = c \Rightarrow \theta = c \Rightarrow (I_G(a,c)_N \geq \theta_N$ and $I_G(a,c)_\Pi = \theta_\Pi)$. If $(b_N > c_N$ and $b_\Pi = c_\Pi)$ then $I_G(b,c) = (c_N,1) \Rightarrow \theta = c \Rightarrow (I_G(a,c)_N \geq \theta_N$ and $I_G(a,c)_\Pi = \theta_\Pi)$. If $(b_N = c_N$ and $b_\Pi > c_\Pi)$ then $I_G(b,c) = (c_\Pi, c_\Pi) \Rightarrow \theta = (c_\Pi, c_\Pi) \Rightarrow I_G(a,c) = \theta$. If $b = c$ then $I_G(b,c) = (1,1) \Rightarrow \theta = (c_\Pi, c_\Pi) \Rightarrow I_G(a,c) = \theta$.
5. $b_N \geq c_N \geq a_N$ and $(a_\Pi > c_\Pi \geq b_\Pi$ or $a_\Pi \geq c_\Pi > b_\Pi)$ If $a_\Pi > c_\Pi$ then $I_G(a,c){=}(c_\Pi, c_\Pi)$. If $a_\Pi{=}c_\Pi$ then $I_G(a,c){=}(1,1)$. If $b_N > c_N$ then $I_G(b,c){=}(c_N,1) \Rightarrow \theta{=}(c_N, b_\Pi) \Rightarrow (I_G(a,c)_N \geq \theta_N$ and $I_G(a,c)_\Pi \geq \theta_\Pi)$. If $b_N{=}c_N$ then $I_G(b,c){=}(1,1) \Rightarrow \theta{=}(b_\Pi, b_\Pi) \Rightarrow (I_G(a,c)_N \geq \theta_N$ and $I_G(a,c)_\Pi \geq \theta_\Pi)$.
6. $c_N \geq b_N \geq a_N$ and $a_\Pi > b_\Pi \geq c_\Pi$. $I_G(a,c) = (c_\Pi, c_\Pi)$.

   If $b_\Pi > c_\Pi$ then $I_G(b,c) = (c_\Pi, c_\Pi) \Rightarrow \theta = (c_\Pi, c_\Pi) \Rightarrow I_G(a,c) = \theta$.
   If $b_\Pi = c_\Pi$ then $I_G(b,c) = (1,1) \Rightarrow \theta = (c_\Pi, c_\Pi) \Rightarrow I_G(a,c) = \theta$. □

Moreover, the extended Gödel implicator has the following properties [9]:
P4: $a_N \leq a'_N \wedge a_\Pi \leq a'_\Pi \Rightarrow I_G(a,b)_N \geq I_G(a',b)_N \wedge I_G(a,b)_\Pi \geq I_G(a',b)_\Pi$,
P5: $b_N \geq b'_N \wedge b_\Pi \geq b'_\Pi \Rightarrow I_G(a,b)_N \geq I_G(a,b')_N \wedge I_G(a,b)_\Pi \geq I_G(a,b')_\Pi$,
P6: $I_G(1,b)_N = b_N$ and $I_G(1,b)_\Pi = b_\Pi$,
P7: $I_G(a,b)_N \geq b_N$ and $I_G(a,b)_\Pi \geq b_\Pi$,
P8: $I_G(a, I_G(b,c))_N = I_G(b, I_G(a,c))_N$ and $I_G(a, I_G(b,c))_\Pi = I_G(b, I_G(a,c))_\Pi$.

4 Soundness and Completeness of the Inference Rules

The set of extended Armstrong's axioms can be used to derive new fuzzy functional dependencies implied by a given set of FFDs. Let F be a set of FFDs (7) with respect to the relation scheme R (U). Let us denote by F^+ the set of all FFDs which can be derived from F by means of the extended Armstrong's axioms:

$$F^+ = \{X \rightarrow_\theta Y, \theta = (\theta_N, \theta_\Pi) : F \models X \rightarrow_\theta Y\}. \tag{9}$$

Theorem 2. *The extended Armstrong's axioms are sound.*

Proof. Let F be a set of FFDs for the relation scheme $R(U)$. Let t_1 and t_2 be tuples of r, where r is relation of $R(U)$. Let X, Y, $Z \subseteq U$.
Let $a = =_c (t_1(X), t_2(X))$, $b = =_c (t_1(Y), t_2(Y))$, $c = =_c (t_1(Z), t_2(Z))$.
A1: If $Y \subseteq X$ then by (4) $a_N \leq b_N$ and $a_\Pi \leq b_\Pi$. Since I_G satisfies P1, we get $I_G(a, b) = (1, 1)$ and so

$$I_G(=_c (t_1(X), t_2(X)), =_c (t_1(Y), t_2(Y)))_N = 1 \geq \theta_N,$$
$$I_G(=_c (t_1(X), t_2(X)), =_c (t_1(Y), t_2(Y)))_\Pi = 1 \geq \theta_\Pi.$$

A2: Since $X \rightarrow_\theta Y$ holds, we have $I_G(a, b)_N \geq \theta_N$ and $I_G(a, b)_\Pi \geq \theta_\Pi$. Let $a' = =_c (t_1(XZ), t_2(XZ))$ and $b' = =_c (t_1(YZ), t_2(YZ))$. From (4) we get $a'_N = \min(a_N, c_N)$, $a'_\Pi = \min(a_\Pi, c_\Pi)$ and $b'_N = \min(b_N, c_N)$, $b'_\Pi = \min(b_\Pi, c_\Pi)$. Since I_G satisfies P2, we obtain $I_G(a', b')_N \geq \theta_N$ and $I_G(a', b')_\Pi \geq \theta_\Pi$ and so

$$I_G(=_c (t_1(XZ), t_2(XZ)), =_c (t_1(YZ), t_2(YZ)))_N \geq \theta_N,$$
$$I_G(=_c (t_1(XZ), t_2(XZ)), =_c (t_1(YZ), t_2(YZ)))_\Pi \geq \theta_\Pi.$$

Thus, if $X \rightarrow_\theta Y \in F^+$ then $XZ \rightarrow_\theta YZ \in F^+$.
A3: Since $X \rightarrow_\alpha Y$ and $Y \rightarrow_\beta Z$ hold, we have: $I_G(a, b)_N \geq \alpha_N$, $I_G(a, b)_\Pi \geq \alpha_\Pi$ and $I_G(b, c)_N \geq \beta_N$, $I_G(b, c)_\Pi \geq \beta_\Pi$. By P3 we obtain $I_G(a, c)_N \geq \gamma_N$ and $I_G(a, c)_\Pi \geq \gamma_\Pi$, where $\gamma_N = \min(\alpha_N, \beta_N)$ and $\gamma_\Pi = \min(\alpha_\Pi, \beta_\Pi)$ and so

$$I_G(=_c (t_1(X), t_2(X)), =_c (t_1(Z), t_2(Z)))_N \geq \gamma_N,$$
$$I_G(=_c (t_1(X), t_2(X)), =_c (t_1(Z), t_2(Z)))_\Pi \geq \gamma_\Pi.$$

Thus, if $X \rightarrow_\alpha Y \in F^+$ and $Y \rightarrow_\beta Z \in F^+$ then $X \rightarrow_\gamma Z \in F^+$ for γ=(min(α_N, β_N), min(α_Π, β_Π)). □

The following rules result from Armstrong's axioms:
D1:$X \rightarrow_\alpha Y \wedge X \rightarrow_\beta Z \Rightarrow X \rightarrow_\lambda YZ$ for $\gamma = (\min(\alpha_N, \beta_N), \min(\alpha_\Pi, \beta_\Pi))$

Proof. By A2 we have $X \rightarrow_\alpha Y \Rightarrow X \rightarrow_\alpha XY$ and $X \rightarrow_\beta Z \Rightarrow XY \rightarrow_\beta ZY$. Then by A3 we obtain $X \rightarrow_\alpha XY \wedge XY \rightarrow_\beta YZ \Rightarrow X \rightarrow_\gamma YZ$ for γ = (min(α_N,β_N), min(α_Π,β_Π)). □

D2:$X \rightarrow_\alpha Y \wedge WY \rightarrow_\beta Z \Rightarrow XW \rightarrow_\lambda Z$ for $\gamma = (\min(\alpha_N, \beta_N), \min(\alpha_\Pi, \beta_\Pi))$

Proof. By A2 we have $X \rightarrow_\alpha Y \Rightarrow XW \rightarrow_\alpha YW$. Then by A3 we obtain $XW \rightarrow_\alpha YW \wedge WY \rightarrow_\beta Z \Rightarrow XW \rightarrow_\gamma Z$ for γ = (min(α_N,β_N), min(α_Π,β_Π)). □

D3: $X \rightarrow_\alpha Y \wedge Z \subseteq Y \Rightarrow X \rightarrow_\alpha Z$

Proof. By A1 we have $Z \subseteq Y \Rightarrow Y \rightarrow_\alpha Z$ for every $\alpha = (\alpha_N, \alpha_\Pi)$, $\alpha_N, \alpha_\Pi \in$ [0,1]. Then by A3 we obtain $X \rightarrow_\alpha Y \wedge Y \rightarrow_\alpha Z \Rightarrow X \rightarrow_\alpha Z$. □

D4: $X \rightarrow_\alpha Y \Rightarrow X \rightarrow_\beta Y$ for $\beta_N \leq \alpha_N$ and $\beta_\Pi \leq \alpha_\Pi$

Proof. By A1 we have $Y \rightarrow_\theta Y$ for every θ=(θ_N, θ_Π), $\theta_N, \theta_\Pi \in$ [0,1]. Then by A3 we obtain $X \rightarrow_\alpha Y \wedge Y \rightarrow_\theta Y \Rightarrow X \rightarrow_\beta Y$ for β=(min(α_N,θ_N),min(α_Π,θ_Π)). □

The closure of a set of attributes $X \subseteq U$ with respect to the set F of FFDs, denoted by X_F^+, is defined as a set of triples $(A, \theta_N, \theta_\Pi)$, where $A \in U$, $\theta_N = \sup\left\{\alpha : X \rightarrow_{(\alpha,\beta)} A \in F^+\right\}$ and $\theta_\Pi = \sup\left\{\beta : X \rightarrow_{(\alpha,\beta)} A \in F^+\right\}$. The set of attributes occurring in X_F^+ will be denoted by DOM(X_F^+):

$$DOM(X_F^+) = \left\{A : (A, \theta_N, \theta_\Pi) \in X_F^+\right\}.$$

Lemma 1. *Let F be a set of FFDs defined over a relation scheme $R(U)$. Let $X, Y \subseteq U$ and $Y = \{A_1, A_2, ..., A_k\}$, $A_i \in U$. Then $X \rightarrow_{(\theta_N,\theta_\Pi)} Y$ is deduced by means of the extended Armstrong's axioms if and only if $\forall i (A_i, \theta_{i,N}, \theta_{i,\Pi}) \in X_F^+$, where $\theta_{i,N} \geq \theta_N$ and $\theta_{i,\Pi} \geq \theta_\Pi$.*

Proof.
Necessity: If $X \rightarrow_{(\theta_N,\theta_\Pi)} Y$ is deduced by means of the extended Armstrong's axioms then by D3 we obtain $X \rightarrow_{(\theta_N,\theta_\Pi)} A_i$ for i = 1, 2,...,k. Thus there exist $\theta_{i,N} \geq \theta_N$ and $\theta_{i,\Pi} \geq \theta_\Pi$ such that $(A_i, \theta_{i,N}, \theta_{i,\Pi}) \in X_F^+$ (definition of X_F^+).
Sufficiency: If $(A_i, \theta_{i,N}, \theta_{i,\Pi}) \in X_F^+$ where $\theta_{i,N} \geq \theta_N$ and $\theta_{i,\Pi} \geq \theta_\Pi$ for i = 1, 2,...,k, then $X \rightarrow_{(\theta_{i,N},\theta_{i\Pi})} A_i \in F^+$. By D1 we obtain $X \rightarrow_{(\theta_N,\theta_\Pi)} Y$, where $\theta_N = \min_i(\theta_{i,N})$ and $\theta_\Pi = \min_i(\theta_{i,\Pi})$. □

Theorem 3. *The extended Armstrong's axioms are complete.*

Proof. In order to prove the theorem we will show that if $X \rightarrow_{(\theta_N,\theta_\Pi)} Y \notin F^+$, then it is possible to construct a relation where all FFDs in F are satisfied and $X \rightarrow_{(\theta_N,\theta_\Pi)} Y$ does not hold, which means that $X \rightarrow_{(\theta_N,\theta_\Pi)} Y$ cannot be derived from F. Let F be a set of FFDs for relation scheme $R(U)$. Suppose that $X \rightarrow_{(\theta_N,\theta_\Pi)} Y \notin F^+$. Let $X = \{X_1, X_2, ..., X_k\}$ and DOM(X_F^+) = $\{X_1, X_2, ..., X_k, A_1, A_2, ..., A_l\}$. Let us construct a relation r of the scheme $R(U)$, U = {DOM(X_F^+), $B_1, B_2, ..., B_m$}, consisting of two tuples t_1 and t_2 such that:

$$t_1(X) = t_2(X) = 1,$$
$$t_1(A_i) = c_i, \quad t_2(A_i) = d_i \quad \text{for} \quad A_i \in DOM(X_F^+) - X, \quad i = 1, 2, ..., l$$
$$t_1(B_i) = 0, \quad t_2(B_i) = 1 \quad \text{for} \quad B_i \in U - DOM(X_F^+), \quad i = 1, 2, ..., m$$

where c_i and d_i are possibility distributions with degrees of closeness $\phi_{i,N}$ and $\phi_{i,\Pi}$. Let $\phi_{0,N} = \min_i \phi_{i,N}$ and $\phi_{0,\Pi} = \min_i \phi_{i,\Pi}$.

We will show that each FFD $V \rightarrow_{(\gamma_N,\gamma_\Pi)} W \in F$ holds in r. One should consider only the case when $V \subseteq$ DOM(X_F^+) and $W \subseteq$ DOM(X_F^+)$-X$. (If $V \nsubseteq$ DOM(X_F^+) then $t_1(V) \neq t_2(V)$ and so $V \rightarrow_{(1,1)} W$. Similarly, if $W \subseteq X$ then $t_1(W) = t_2(W)$ and so $V \rightarrow_{(1,1)} W$. Suppose that $V \subseteq$ DOM(X_F^+) and $W \subseteq U$ $-$ DOM(X_F^+). Degrees of closeness of $t_1(W)$ and $t_2(W)$ are equal to 0. By Lemma 1 we obtain $X \rightarrow_{(\phi_{0,N},\phi_{0,\Pi})} V \in F$. Since $V \rightarrow_{(\gamma_N,\gamma_\Pi)} W \in F$ we have $X \rightarrow_{(\psi_N,\psi_\Pi)} W \in F$, where $\psi_N = \min(\phi_{0,N}, \gamma_N)$ and $\psi_\Pi = \min(\phi_{0,\Pi}, \gamma_\Pi)$. Thus $W \subseteq$ DOM(X_F^+): a contradiction.)

Suppose that there exists $V \rightarrow_{(\gamma_N,\gamma_\Pi)} W \in F$, which does not hold in r. Let $V \subseteq \mathrm{DOM}(X_F^+)$, $V - X = \{A_{p_1}, A_{p_2}, ..., A_{p_r}\}$, $W \subseteq \mathrm{DOM}(X_F^+) - X$ and $W = \{A_{s_1}, A_{s_2}, ..., A_{s_t}\}$, where $p_1, p_2, ..., p_r, s_1, s_2, ..., s_t \in \{1, 2, ..., l\}$. Let $\phi_{V,N} = \min_i \phi_{p_i,N}$, $\phi_{V,\Pi} = \min_i \phi_{p_i,\Pi}$ and $\phi_{W,N} = \min_i \phi_{s_i,N}$, $\phi_{W,\Pi} = \min_i \phi_{s_i,\Pi}$. Thus, $=_c (t_1(V), t_2(V)) = (\phi_{V,N}, \phi_{V,\Pi})$ and $=_c (t_1(W), t_2(W)) = (\phi_{W,N}, \phi_{W,\Pi})$. If $V \rightarrow_{(\gamma_N,\gamma_\Pi)} W$ does not hold in r, then $\phi_{W,N} < \min(\gamma_N, \phi_{V,N})$ or $\phi_{W,\Pi} < \min(\gamma_\Pi, \phi_{V,\Pi})$. Thus, $\phi_{p_i,N} > \phi_{W,N}$ or $\phi_{p_i,\Pi} > \phi_{W,\Pi}$ for every p_i. By Lemma 1 we obtain $X \rightarrow_{(\phi_{V,N},\phi_{V,\Pi})} V \in F^+$. Since $V \rightarrow_{(\gamma_N,\gamma_\Pi)} W \in F$ we have $X \rightarrow_{(\psi_N,\psi_\Pi)} W \in F^+$, where $\psi_N = \min(\phi_{V,N}, \gamma_N)$ and $\psi_\Pi = \min(\phi_{V,\Pi}, \gamma_\Pi)$. Since $W \subseteq \mathrm{DOM}(X_F^+) - X$ then $(A_{s_i}, \phi_{s_i,N}, \phi_{s_i,\Pi}) \in X_F^+$ for every s_i. According to the definition of X_F^+, $\phi_{s_i,N}$ and $\phi_{s_i,\Pi}$ are upper bounds. Thus, conditions $\phi_{p_i,N} > \phi_{W,N}$ or $\phi_{p_i,\Pi} > \phi_{W,\Pi}$ for every p_i are not satisfied. We obtained a contradiction. Thus, $V \rightarrow_{(\gamma_N,\gamma_\Pi)} W$ holds in r.

Now we prove that $X \rightarrow_{(\theta_N,\theta_\Pi)} Y \notin F^+$ does not hold in r. We should consider only the case when $Y \subseteq \mathrm{DOM}(X_F^+)$. (If $Y \nsubseteq \mathrm{DOM}(X_F^+)$ then $=_c (t_1(Y), t_2(Y))_N = \, =_c (t_1(Y), t_2(Y))_\Pi = 0$ and so $X \rightarrow_{(\theta_N,\theta_\Pi)} Y$ does not hold). Let $Y \subseteq \{A_1, A_2, ..., A_l\}$. Let $\phi_{Y,N} = \min_i \phi_{i,N}$ and $\phi_{Y,\Pi} = \min_i \phi_{i,\Pi}$. By Lemma 1, it follows that $X \rightarrow_{(\theta_N,\theta_\Pi)} Y$ holds in r for $\phi_{Y,N} \geq \theta_N$ and $\phi_{Y,\Pi} \geq \theta_\Pi$ and $X \rightarrow_{(\theta_N,\theta_\Pi)} Y \in F^+$ which is a contradiction to the assumption. □

Example 1. Let us consider relation scheme $R(A,B,C,D)$ with the following set of FFDs: $F = \{ ABC \rightarrow_{(0,0.8)} D, BCD \rightarrow_{(0.5,1)} A, ACD \rightarrow_{(1,1)} B, ABD \rightarrow_{(0.8,1)} C, A \rightarrow_{(0,0.7)} C, A \rightarrow_{(1,1)} D, B \rightarrow_{(0,0.6)} AC \}$. By D2 we obtain: $AB \rightarrow_{(0,0.7)} D$, $AD \rightarrow_{(0,0.7)} B$, $AC \rightarrow_{(1,1)} B$, $AB \rightarrow_{(0.8,1)} C$ and $B \rightarrow_{(0,0.6)} D$. Since $A \rightarrow_{(1,1)} D \Rightarrow AB \rightarrow_{(1,1)} D$ (by A2 and D3) and $AB \rightarrow_{(1,1)} D \Rightarrow AB \rightarrow_{(0,0.7)} D$ (by D4) we conclude that $AB \rightarrow_{(0,0.7)} D$ is redundant. Similarly $ABC \rightarrow_{(0,0.8)} D$, $ACD \rightarrow_{(1,1)} B$, $ABD \rightarrow_{(0.8,1)} C$ are also redundant. By D3 we have $B \rightarrow_{(0,0.6)} A$ and $B \rightarrow_{(0,0.6)} C$. Thus $F_m = \{BCD \rightarrow_{(0.5,1)} A, AD \rightarrow_{(0,0.7)} B, AC \rightarrow_{(1,1)} B, AB \rightarrow_{(0.8,1)} C, A \rightarrow_{(0,0.7)} C, A \rightarrow_{(1,1)} D, B \rightarrow_{(0,0.6)} A, B \rightarrow_{(0,0.6)} C, B \rightarrow_{(0,0.6)} D\}$ is a minimal set of FFDs for the scheme R.

5 Conclusions

The paper deals with data dependencies in possibilistic databases. We applied and extended the definition of fuzzy functional dependency which was formulated by Chen [2]. Its level is evaluated by measures of possibility and necessity. For FFDs we have established inference rules which are an extension of Armstrong's axioms for conventional databases and showed that they form a sound and complete system. Similar results may be expected for other approaches. The obtained results could be generalized when using t-norms. Another line of future work is an extension of the presented considerations by taking into account unknown and inapplicable (missing) values.

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