

# Chapter 4

## Robust $\mathcal{H}_\infty$ Switching Control of Polytopic Parameter-Varying Systems via Dynamic Output Feedback

Chengzhi Yuan, Chang Duan, and Fen Wu

**Abstract** The problem of designing a globally optimal robust output-feedback controller for time-varying polytopic uncertain systems is a well-known non-convex optimization problem. In this paper, new sufficient conditions for robust  $\mathcal{H}_\infty$  output-feedback control synthesis are proposed in terms of a special type of bilinear matrix inequalities (BMIs), which can be solved effectively using linear matrix inequality (LMI) optimization plus a line search. In order to reduce the conservatism of robust output-feedback control methods based on single quadratic Lyapunov function, we utilize multiple Lyapunov functions. The associated robust output-feedback controller is constructed as a switching-type full-order dynamic output-feedback controller, consisting of a family of linear subcontrollers and a min-switching logic. The proposed approach features the important property of computational efficiency with stringent performance. Its effectiveness and advantages have been demonstrated through numerical studies.

**Keywords** Parameter-varying system • Switched control • Robust control

### 4.1 Introduction

During the past decades, a great deal of attention was devoted to the study of systems with time-varying parametric uncertainties, due to their theoretical importance in control theory and widespread applications in practical engineering problems. Both issues of stability and control design for these types of systems have been examined extensively in the literature (see, e.g., [1–5] and the references therein). A typical robust control strategy is to construct a single linear time-invariant (LTI) controller

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C. Yuan (✉) • F. Wu  
Department of Mechanical and Aerospace Engineering, North Carolina State University,  
Raleigh, NC, USA  
e-mail: [cyuan2@ncsu.edu](mailto:cyuan2@ncsu.edu); [fwu@eos.ncsu.edu](mailto:fwu@eos.ncsu.edu)

C. Duan  
Department of Mechanical Engineering, Prairie View A&M University, Prairie View, TX, USA  
e-mail: [chduan@pvamu.edu](mailto:chduan@pvamu.edu)

for norm-bounded uncertain systems using a single quadratic Lyapunov function [4, 6]. Some classical works along this line are worth to be mentioned. Different tools for both robustness analysis and controller design for systems subject to structured uncertainties can be found in [4, 7–9], while [2, 10–12] considered similar problems for systems with unstructured uncertainties. One potential drawback of these classical methods lies in the conservatism due to the use of a single quadratic Lyapunov function. In recent years, more advanced robust control approaches were proposed to achieve better controlled performance. In particular, originated from the pioneering works [13, 14], different switching-type robust controllers have been proposed for various systems with different types of uncertainties, such as [15, 16] on linear fractional transformation (LFT) systems and [7, 17] for polytopic uncertain systems, both of which utilized the multiple Lyapunov function technique from the switching control context [18]. A comprehensive review of the literature on robust control of uncertain systems, including some recent results from either deterministic or probabilistic perspective, can be found in [5].

Different from the state-feedback control case, the problem of designing a robust output-feedback controller for linear uncertain systems is known to be difficult. The main source of difficulty stems from the non-convex nature of the problem itself. Specifically, the associated robust control synthesis problem is typically represented as a bilinear matrix inequality (BMI) optimization problem for most design objectives. This type of non-convex optimization problems is NP-hard, even under the single quadratic Lyapunov function framework (see, e.g., [4, 6, 19–21]). Various approaches have been reported to tackle the non-convexity issue. Some rely on heuristic optimization algorithms to attain a locally optimal solution [19, 22], which could be very involved and time-consuming; some resort to certain mathematical relaxations to arrive at a convex synthesis condition but of more conservatism [20]. As such, developing an effective robust output-feedback control synthesis framework that simultaneously renders stringent performance and computational efficacy is urgently desirable but still remains as an open problem.

In this paper, we propose a new robust switching output-feedback (RSOF) control scheme for a class of polytopic parameter-varying uncertain systems. Different from the classical robust output-feedback control techniques, the proposed RSOF controller consists of a family of full-order dynamic LTI subcontrollers and a min-switching logic that governs the switching among them, which therefore results in a switched closed-loop system with time-varying polytopic uncertainties. The basic idea applied here for switching stability analysis and controller design is borrowed from the switching control literature (see, for instance, [17, 18, 23–28]). In particular, motivated by the methodologies from [17] on switched state-feedback control of polytopic uncertain systems, [29] on asynchronous switching output-feedback controller synthesis, and [30] on stabilization of switched linear systems via min-switching control, we will first derive the analysis conditions for robust  $\mathcal{H}_\infty$  stability of the resulting switched closed loop by using piecewise switched Lyapunov functions. Then, based on the analysis conditions, the associated robust switching control synthesis problem is formulated as a special type of BMIs, which can be solved effectively in terms of LMIs plus a line search. The proposed

switching control design scheme advances existing methods for robust output-feedback control synthesis in two important ways: better achievable controlled performance in terms of  $\mathcal{H}_\infty$  criterion due to the adoption of piecewise switched Lyapunov functions; reduced computational complexity by solving a convex LMI-based optimization coupled with a single line search. Numerical examples are given to illustrate the effectiveness and advantages of the proposed design approach.

The rest of the paper is organized as follows. The problem statement and the form of RSOF controller are presented in Sect. 4.2. Sections 4.3 and 4.4 contain the main results of this paper including the robust analysis and control synthesis conditions, respectively. Simulation results are provided in Sect. 4.5. Conclusions are finally drawn in Sect. 4.6.

**Notation**  $\mathbb{R}$  stands for the set of real numbers and  $\mathbb{R}_+$  for the positive real numbers. The set of non-negative integers is denoted by  $\mathbb{N}_+$ .  $\mathbb{R}^{m \times n}$  is the set of real  $m \times n$  matrices, and  $\mathbb{R}^n$  represents the set of real  $n \times 1$  vectors. The transpose of a real matrix  $M$  is denoted by  $M^T$ . The Hermitian operator  $He\{\cdot\}$  is defined as  $He\{M\} = M + M^T$  for real matrices. The identity matrix of any dimension is denoted by  $I$ .  $\mathbb{S}^n$  and  $\mathbb{S}_+^n$  are used to denote the set of real symmetric  $n \times n$  matrices and positive definite matrices, respectively. If  $M \in \mathbb{S}^n$ , then  $M > 0$  ( $M \geq 0$ ) indicates that  $M$  is a positive definite (positive semi-definite) matrix and  $M < 0$  ( $M \leq 0$ ) denotes a negative definite (negative semi-definite) matrix. A block diagonal matrix with matrices  $X_1, X_2, \dots, X_p$  on its main diagonal is denoted by  $diag\{X_1, X_2, \dots, X_p\}$ . Furthermore, we use the symbol  $\star$  in LMIs to denote entries that follow from symmetry. For  $x \in \mathbb{R}^n$ , its norm is defined as  $\|x\| := (x^T x)^{1/2}$ . The space of square integrable functions is denoted by  $\mathcal{L}_2$ , that is, for any  $u \in \mathcal{L}_2$ ,  $\|u\|_2 := (\int_0^\infty u^T(t)u(t)dt)^{1/2} < \infty$ . For two integers  $k_1 < k_2$ , we denote  $\mathbf{I}[k_1, k_2] = \{k_1, k_1 + 1, \dots, k_2\}$ . The set of Metzler matrices  $\mathcal{M}$  consists of all matrices  $\Pi \in \mathbb{R}^{N \times N}$  with elements  $\pi_{ji}$ , such that  $\pi_{ji} \geq 0$  for all  $i, j \in \mathbf{I}[1, N]$  with  $i \neq j$  and  $\sum_{j=1}^N \pi_{ji} = 0$  for all  $i \in \mathbf{I}[1, N]$ .

## 4.2 Problem Statement

Consider the following linear system with uncertain time-varying parameters:

$$\begin{bmatrix} \dot{x}_p(t) \\ e(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A_p(\xi(t)) & B_{p1}(\xi(t)) & B_{p2}(\xi(t)) \\ C_{p1}(\xi(t)) & D_{p11}(\xi(t)) & D_{p12}(\xi(t)) \\ C_{p2}(\xi(t)) & D_{p21}(\xi(t)) & D_{p22}(\xi(t)) \end{bmatrix} \begin{bmatrix} x_p(t) \\ d(t) \\ u(t) \end{bmatrix} \quad (4.1)$$

where the vectors  $x_p \in \mathbb{R}^{n_x}$ ,  $u \in \mathbb{R}^{n_u}$ ,  $d \in \mathbb{R}^{n_d}$ ,  $y \in \mathbb{R}^{n_y}$ , and  $e \in \mathbb{R}^{n_e}$  denote the plant state, control input, exogenous disturbance, measurement output, and error (performance) output, respectively. The system matrices are uncertain and time-varying, they are given by the convex combination

$$\begin{bmatrix} A_p(\xi(t)) & B_{p1}(\xi(t)) & B_{p2}(\xi(t)) \\ C_{p1}(\xi(t)) & D_{p11}(\xi(t)) & D_{p12}(\xi(t)) \\ C_{p2}(\xi(t)) & D_{p21}(\xi(t)) & D_{p22}(\xi(t)) \end{bmatrix} = \sum_{i=1}^{N_p} \xi_i(t) \begin{bmatrix} A_{p,i} & B_{p1,i} & B_{p2,i} \\ C_{p1,i} & D_{p11,i} & D_{p12,i} \\ C_{p2,i} & D_{p21,i} & D_{p22,i} \end{bmatrix} \quad (4.2)$$

where the constant matrices at the polytope vertex, i.e.,  $(A_{p,i}, B_{p1,i}, B_{p2,i}, C_{p1,i}, D_{p11,i}, D_{p12,i}, C_{p2,i}, D_{p21,i}, D_{p22,i})$  for all  $i \in \mathbf{I}[1, N_p]$ , are known for controller design. The vector  $\xi(t) := [\xi_1(t) \dots \xi_{N_p}(t)]^T \in \mathbb{R}^{N_p}$  represents the time-varying parametric uncertainty which is not measurable in real time, and belongs to the unitary simplex  $\Lambda$  defined by

$$\Lambda = \left\{ \lambda \in \mathbb{R}^{N_p} : \sum_{i=1}^{N_p} \lambda_i = 1, \lambda_i \geq 0, \forall i \in \mathbf{I}[1, N_p] \right\} \quad (4.3)$$

To ease the notation and whenever the context is clear, the explicit time dependence of vector  $\xi(t) \in \Lambda$  will be dropped. Furthermore, for simplicity of presentation, we have the following assumptions regarding system (4.1):

**Assumption 1.**  $(A_{p,i}, B_{p2,i}, C_{p2,i})$  is stabilizable and detectable for any  $i \in \mathbf{I}[1, N_p]$ .

**Assumption 2.** Matrices  $(B_{p2,i}, C_{p2,i}, D_{p12,i}) = (B_{p2}, C_{p2}, D_{p12})$  are constant matrices, and  $D_{p22,i} = 0$  for all  $i \in \mathbf{I}[1, N_p]$ .

We stress that these two assumptions are made without losing any generality. Assumption 1 is necessary to guarantee the existence of an output-feedback stabilizing controller from  $y$  to  $u$  for each subsystem of (4.1) on the polytope vertices. In the second assumption,  $D_{p22,i} = 0$  can be relaxed using the well-known loop transformation technique [4], while the assumptions on matrices  $(B_{p2,i}, C_{p2,i}, D_{p12,i})$  can also be satisfied by adding stable pre- and post-filters to the input and output channels, respectively [31]. An illustrative example will be given in Sect. 4.5 (Example 1) to show how to satisfy this assumption.

Keeping in mind that the time-varying uncertainty  $\xi \in \Lambda$  is not available for feedback control use, the objective of this work is to design an RSOF control law such that the overall closed-loop system is asymptotically stable and achieves certain performance level from the disturbance  $d$  to the error output  $e$  for all uncertain parameter  $\xi \in \Lambda$ .

To fulfill this objective, we will construct the following dynamic RSOF controller:

$$\begin{bmatrix} \dot{x}_c(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} A_{c,\sigma(x_c(t))} & B_{c,\sigma(x_c(t))} \\ C_{c,\sigma(x_c(t))} & D_{c,\sigma(x_c(t))} \end{bmatrix} \begin{bmatrix} x_c(t) \\ y(t) \end{bmatrix} \quad (4.4)$$

where  $x_c \in \mathbb{R}^{n_c}$  is the controller state with its dimension  $n_c$  to be determined.  $\sigma(x_c(t))$  is a switching rule of controller that selects a particular sequence of LTI subcontrollers among  $N_p$  available ones defined by  $(A_{c,j}, B_{c,j}, C_{c,j}, D_{c,j})$  with  $j \in \mathbf{I}[1, N_p]$ . Its value is determined by the min-switching strategy as shown in

Fig. 4.1, where  $j_q$  is the current active controller index, and  $x_{cl} := [x_p^T \ x_c^T]^T$ . Matrices  $P_{j_q} \in \mathbb{S}_+^{n_x+n_c}$  are positive definite. The matrices  $P_j$  together with matrices  $(A_{c,j}, B_{c,j}, C_{c,j}, D_{c,j})$  ( $\forall j \in \mathbf{I}[1, N_p]$ ) of compatible dimensions are subject to design.

The closed-loop system formed by interconnecting the controlled plant (4.1) and the RSOF controller (4.4) can be written in the following switched polytopic form:

$$\begin{bmatrix} \dot{x}_{cl}(t) \\ e(t) \end{bmatrix} = \begin{bmatrix} A_{cl,\xi\sigma} & B_{cl,\xi\sigma} \\ C_{cl,\xi\sigma} & D_{cl,\xi\sigma} \end{bmatrix} \begin{bmatrix} x_{cl}(t) \\ d(t) \end{bmatrix} \quad (4.5)$$

where  $x_{cl} = [x_p^T \ x_c^T]^T$  and

$$\begin{aligned} A_{cl,\xi\sigma} &= \begin{bmatrix} A_p(\xi) + B_{p2}D_{c,\sigma}C_{p2} & B_{p2}C_{c,\sigma} \\ B_{c,\sigma}C_{p2} & A_{c,\sigma} \end{bmatrix}, \\ B_{cl,\xi\sigma} &= \begin{bmatrix} B_{p1}(\xi) + B_{p2}D_{c,\sigma}D_{p21}(\xi) \\ B_{c,\sigma}D_{p21}(\xi) \end{bmatrix}, \\ C_{cl,\xi\sigma} &= [C_{p1}(\xi) + D_{p12}D_{c,\sigma}C_{p2} \ D_{p12}C_{c,\sigma}], \\ D_{cl,\xi\sigma} &= D_{p11}(\xi) + D_{p12}D_{c,\sigma}D_{p21}(\xi). \end{aligned}$$

Moreover, we define for all  $i, j \in \mathbf{I}[1, N_p]$ ,

$$\begin{aligned} A_{cl,ij} &= \begin{bmatrix} A_{p,i} + B_{p2}D_{c,j}C_{p2} & B_{p2}C_{c,j} \\ B_{c,j}C_{p2} & A_{c,j} \end{bmatrix}, \quad B_{cl,ij} = \begin{bmatrix} B_{p1,i} + B_{p2}D_{c,j}D_{p21,i} \\ B_{c,j}D_{p21,i} \end{bmatrix} \\ C_{cl,ij} &= [C_{p1,i} + D_{p12}D_{c,j}C_{p2} \ D_{p12}C_{c,j}], \quad D_{cl,ij} = D_{p11,i} + D_{p12}D_{c,j}D_{p21,i}. \end{aligned}$$

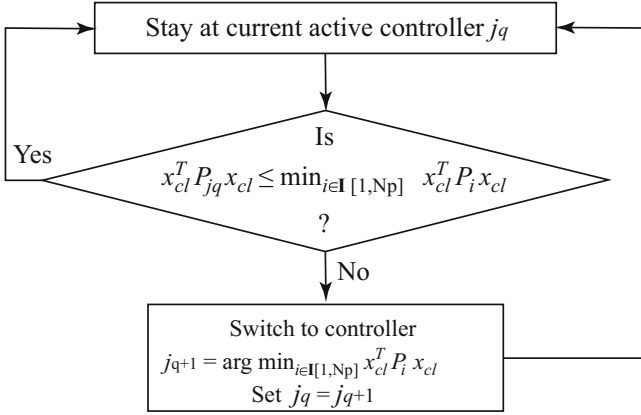
Then, we have

$$\begin{aligned} A_{cl,\xi j} &= \sum_{i=1}^{N_p} \xi_i A_{cl,ij}, \quad B_{cl,\xi j} = \sum_{i=1}^{N_p} \xi_i B_{cl,ij}, \\ C_{cl,\xi j} &= \sum_{i=1}^{N_p} \xi_i C_{cl,ij}, \quad D_{cl,\xi j} = \sum_{i=1}^{N_p} \xi_i D_{cl,ij}. \end{aligned}$$

for all  $j \in \mathbf{I}[1, N_p]$ .

In this paper, the robust  $\mathcal{H}_\infty$  control problem will be considered. More precise descriptions about this problem is given as follows:

**Problem 4.1.** Given the uncertain system (4.1). The robust  $\mathcal{H}_\infty$  control design objective is to determine matrices  $(A_{c,j}, B_{c,j}, C_{c,j}, D_{c,j}, P_j)$  subject to (4.4) and the switching strategy in Fig. 4.1, such that the switched closed-loop system in (4.5) is robustly asymptotically stable and achieves a minimal worst-case  $\mathcal{H}_\infty$  norm  $\gamma_\infty$  defined by



**Fig. 4.1** Min-switching strategy

$$\max_{\xi \in \Lambda} \sup_{\|d\|_2 \neq 0} \frac{\|e\|_2}{\|d\|_2} < \gamma_\infty. \quad (4.6)$$

With respect to the  $\mathcal{H}_\infty$  control problem, the following sections will be devoted to studying the robust stability property of the switched closed-loop system (4.5) under the min-switching logic in Fig. 4.1, and subsequently deriving computationally tractable conditions for the RSOF controller synthesis.

### 4.3 Robust Analysis via Min-Switching

In this section, we will first present the analysis conditions for robust  $\mathcal{H}_\infty$  performance of the time-varying switched polytopic system (4.5) by using multiple quadratic Lyapunov functions and parameter-dependent Metzler matrix [23]. Specifically, we will utilize the parameter-dependent Metzler matrix  $\Pi(\xi) : \Lambda \rightarrow \mathbb{R}^{N_p \times N_p}$  with elements given by

$$\pi_{ij}(\xi) := \begin{cases} \tau_j \xi_i, & i \neq j \\ \tau_j(\xi_j - 1), & i = j \end{cases} \quad (4.7)$$

where  $\tau_j \geq 0$  for all  $j \in \mathbf{I}[1, N_p]$ . It can be easily verified through the same arguments as in [23] that  $\Pi(\xi) \in \mathcal{M}$  for all  $\xi \in \Lambda$ .

Then, we have the following theorem summarize the  $\mathcal{H}_\infty$  analysis conditions:

**Theorem 4.1.** *Given a scalar  $\gamma_\infty \in \mathbb{R}_+$ , the RSOF controller (4.4) with the min-switching strategy as shown in Fig. 4.1 globally asymptotically stabilizes the time-varying polytopic uncertain system (4.1) and renders an  $\mathcal{H}_\infty$  performance level less*

than  $\gamma_\infty$ , if there exist matrices  $P_j \in \mathbb{S}_+^{n_x+n_c}$ , and scalars  $\tau_j \geq 0$  such that

$$\begin{bmatrix} He\{P_j A_{cl,ij}\} + \tau_j(P_i - P_j) & \star & \star \\ B_{cl,ij}^T P_j & -\gamma_\infty^2 I & \star \\ C_{cl,ij} & D_{cl,ij} & -I \end{bmatrix} < 0 \quad (4.8)$$

hold for all  $i, j \in \mathbf{I}[1, N_p]$ .

*Proof.* Consider the closed-loop system (4.5), we define the following piecewise Lyapunov function:

$$V(x_{cl}) := x_{cl}^T P_{j_q} x_{cl} \quad (4.9)$$

where  $P_{j_q} \in \mathbb{S}_+^{n_x+n_c}$  and  $j_q \in \mathbf{I}[1, N_p]$  are the current active subcontroller index determined by the min-switching strategy in Fig. 4.1. Then, multiplying  $\xi_i$  to both sides of inequality (4.8) and summing up from  $i = 1$  to  $i = N_p$ , it yields

$$\begin{bmatrix} He\{P_j A_{cl,\xi j}\} + \tau_j \sum_{i=1}^{N_p} \xi_i(P_i - P_j) & \star & \star \\ B_{cl,\xi j}^T P_j & -\gamma_\infty^2 I & \star \\ C_{cl,\xi j} & D_{cl,\xi j} & -I \end{bmatrix} < 0 \quad (4.10)$$

We first examine the stability property for the closed-loop system (4.5) with  $d \equiv 0$ . In light of the definition in (4.7), the min-switching strategy in Fig. 4.1, and since  $\tau_j, \xi_i \geq 0$  ( $\forall i, j \in \mathbf{I}[1, N_p]$ ), we have

$$\tau_{j_q} \sum_{i=1}^{N_p} \xi_i x_{cl}^T (P_i - P_{j_q}) x_{cl} = \sum_{i=1}^{N_p} \pi_{ij_q}(\xi) x_{cl}^T P_i x_{cl} \geq \sum_{i=1}^{N_p} \pi_{ij_q}(\xi) x_{cl}^T P_{j_q} x_{cl} = 0 \quad (4.11)$$

Therefore, the (1, 1) element of condition (4.10) ensures

$$\dot{V}(x_{cl}) = x_{cl}^T (A_{cl,\xi j_q}^T P_{j_q} + P_{j_q} A_{cl,\xi j_q}) x_{cl} < 0$$

for  $x_{cl} \neq 0$ . Let  $t_q^-$  denote the time when the controller switched out from  $j_q$ th subcontroller and  $t_q^+$  be the time when the controller switched to the next subcontroller. Then, at  $t_q^-$ , condition  $x_{cl}^T P_{j_q} x_{cl} \leq \min_{i \in \mathbf{I}[1, N_p]} x_{cl}^T P_i x_{cl}$  must be violated, that is,

$$\min_{i \in \mathbf{I}[1, N_p]} x_{cl}^T(t_q^-) P_i x_{cl}(t_q^-) < x_{cl}^T(t_q^-) P_{j_q} x_{cl}(t_q^-)$$

Since the min-switching strategy determines  $j_{q+1} = \arg \min_{i \in \mathbf{I}[1, N_p]} x_{cl}^T P_i x_{cl}$ , that is,

$$x_{cl}^T(t_q^+) P_{j_{q+1}} x_{cl}(t_q^+) = \min_{i \in \mathbf{I}[1, N_p]} x_{cl}^T(t_q^-) P_i x_{cl}(t_q^-)$$

Then, we have  $x_{cl}^T(t_q^+)P_{j_{q+1}}x_{cl}(t_q^+) < x_{cl}^T(t_q^-)P_{j_q}x_{cl}^T(t_q^-)$ , which implies  $V(x_{cl}(t_q^+)) < V(x_{cl}(t_q^-))$ , and  $V(x_{cl})$  thus satisfies the monotonically non-increasing condition. According to the Theorem 2.3 in [32], the switched system (4.5) is globally asymptotically stable.

Now, we examine the closed-loop  $\mathcal{H}_\infty$  performance. Through Schur complement, condition (4.10) with (4.11) gives

$$\begin{bmatrix} He\{P_j A_{cl,\xi j}\} & \star \\ B_{cl,\xi j}^T P_j & -\gamma_\infty^2 I \end{bmatrix} + \begin{bmatrix} C_{cl,\xi j}^T \\ D_{cl,\xi j}^T \end{bmatrix} [C_{cl,\xi j} \ D_{cl,\xi j}] < 0$$

Multiplying  $[x_{cl}^T \ d^T]$  from the left of the above condition and its transpose to the right, it yields

$$\dot{V}(x_{cl}) + e^T e - \gamma_\infty^2 d^T d < 0$$

Integrating both sides of the above condition from  $t = 0$  to  $\infty$  and taking into account that under zero initial condition  $V(x_{cl}(0)) = 0$  and  $V(x_{cl}(\infty)) \geq 0$ , we can conclude that  $\|e\|_2 < \gamma_\infty \|d\|_2$ .

*Remark 4.1.* Compared with classical results on robust stability analysis of linear parameter-varying (LPV) systems [4], we have adopted a piecewise switched Lyapunov function instead of using a single quadratic Lyapunov function, which is motivated from the context of switching control [18, 23]. The resulting conditions Theorem 4.1 for  $\mathcal{H}_\infty$  control improve classical results in the sense that quadratic stability of each system matrix  $A_{cl,ij}$  with  $i, j \in \mathbf{I}[1, N_p]$  and  $i \neq j$  is not necessarily required to guarantee feasibility.

## 4.4 RSOF Controller Synthesis

Based on the analysis results in the previous section, we are in the position to study the  $\mathcal{H}_\infty$  control synthesis problem for the RSOF controller (4.4). The RSOF control synthesis problem requires the determination of the coefficient matrices  $(A_{c,j}, B_{c,j}, C_{c,j}, D_{c,j})$  in the controller dynamics (4.4) and  $P_j$  with respect to the min-switching strategy in Fig. 4.1, for all  $j \in \mathbf{I}[1, N_p]$ . However, since the plant state  $x_p$  is not always available for feedback control use, and in order to make the switching logic in Fig. 4.1 implementable, we will specify the Lyapunov matrices with a prescribed structure so as to structurally incorporate switching rules that depend only on available information, i.e.,

$$P_j = \begin{bmatrix} S & N \\ N^T & X_j \end{bmatrix}, \quad (4.12)$$



where  $S \in \mathbb{S}_+^{n_x}$ ,  $N \in \mathbb{R}^{n_x \times n_c}$ , and  $X_j \in \mathbb{S}_+^{n_c}$ , for all  $j \in \mathbf{I}[1, N_p]$ . We aim to derive computationally tractable conditions, such that all these controller coefficient matrices can be jointly synthesized through convex optimization. To this end, we first introduce the following lemma, which is useful in the subsequent derivation for our main results.

**Lemma 4.1.** *Given a symmetric matrix  $\Upsilon_0$  and matrices  $\Upsilon_1, \Upsilon_2$  with compatible dimensions, condition  $\Upsilon_0 + \begin{bmatrix} 0 & \star \\ \Upsilon_2^T \Upsilon_1 & 0 \end{bmatrix} < 0$  holds if and only if the following condition holds for some positive number  $\epsilon$ .*

$$\begin{bmatrix} \Upsilon_0 - \begin{bmatrix} \epsilon \Upsilon_1^T \Upsilon_1 & \star \\ 0 & \frac{1}{\epsilon} \Upsilon_2^T \Upsilon_2 \end{bmatrix} & \star \\ \begin{bmatrix} \epsilon \Upsilon_1 & \Upsilon_2 \end{bmatrix} & -\epsilon I \end{bmatrix} < 0 \quad (4.13)$$

*Proof.* Through Schur complement, condition (4.13) is equivalent to

$$\Upsilon_0 - \begin{bmatrix} \epsilon \Upsilon_1^T \Upsilon_1 & \star \\ 0 & \frac{1}{\epsilon} \Upsilon_2^T \Upsilon_2 \end{bmatrix} + \begin{bmatrix} \epsilon \Upsilon_1^T \Upsilon_1 & \star \\ \Upsilon_2^T \Upsilon_1 & \frac{1}{\epsilon} \Upsilon_2^T \Upsilon_2 \end{bmatrix} = \Upsilon_0 + \begin{bmatrix} 0 & \star \\ \Upsilon_2^T \Upsilon_1 & 0 \end{bmatrix} < 0.$$

Then, using this lemma and the analysis results in Theorem 4.1, we have the following theorem solve the robust  $\mathcal{H}_\infty$  control synthesis problem in terms of matrix inequalities.

**Theorem 4.2.** *Given tunable scalars  $\tau_j \geq 0$ , if there exist positive definite matrices  $R_j, \hat{S} \in \mathbb{S}_+^{n_x}$ , symmetric matrices  $T_{ij} \in \mathbb{S}^{n_x}$ , rectangular matrices  $\hat{A}_{c,j} \in \mathbb{R}^{n_x \times n_x}$ ,  $\hat{B}_{c,j} \in \mathbb{R}^{n_x \times n_y}$ ,  $\hat{C}_{c,j} \in \mathbb{R}^{n_u \times n_x}$ ,  $\hat{D}_{c,j} \in \mathbb{R}^{n_u \times n_y}$ , and positive scalars  $\hat{\epsilon}, \hat{\gamma}_\infty \in \mathbb{R}_+$  such that for all  $i, j \in \mathbf{I}[1, N_p]$ , the following conditions hold:*

$$\begin{bmatrix} \text{He}\{A_{p,i}R_j + B_{p2}\hat{C}_{c,j}\} + \tau_j T_{ij} - 2R_j + \hat{\epsilon}I & \star \\ \hat{A}_{c,j} + \hat{\epsilon}A_{p,i}^T + C_{p2}^T \hat{D}_{c,j}^T B_{p2}^T & \text{He}\{\hat{S}A_{p,j} + \hat{B}_{c,j}C_{p2}\} + \hat{\epsilon}I \\ \hat{\epsilon}B_{p1,i}^T + D_{p21,i}^T \hat{D}_{c,j}^T B_{p2}^T & B_{p1,i}^T \hat{S} + D_{p21,i}^T \hat{B}_{c,j}^T \\ C_{p1,i}R_j + D_{p12}\hat{C}_{c,j} & \hat{\epsilon}C_{p1,i} + D_{p12}\hat{D}_{c,j}C_{p2} \\ R_j & (A_{p,i} - A_{p,j})^T \hat{S} \\ & \star & \star & \star \\ & \star & \star & \star \\ & -\hat{\gamma}_\infty^2 I & \star & \star \\ \hat{\epsilon}D_{p11,i} + D_{p12}\hat{D}_{c,j}D_{p21,i} & -I & \star \\ & 0 & 0 & -\hat{\epsilon}I \end{bmatrix} < 0, \quad (4.14)$$

$$\begin{bmatrix} T_{ij} + R_j & \star & \star \\ R_j & R_i & \star \\ \hat{\epsilon}I & \hat{\epsilon}I & \hat{S} \end{bmatrix} \geq 0, \quad \begin{bmatrix} R_j & \star \\ \hat{\epsilon}I & \hat{S} \end{bmatrix} > 0. \quad (4.15)$$

Then, the time-varying polytopic uncertain system (4.1) is globally asymptotically stabilized by the RSOF controller (4.4) of order  $n_c = n_x$ , and the closed-loop  $\mathcal{H}_\infty$  performance level is less than  $\gamma_\infty = \frac{\hat{\gamma}_\infty}{\hat{\epsilon}}$  under the min-switching strategy with the condition in Fig. 4.1 replaced by

$$x_c^T X_j x_c \leq \min_{i \in \mathbf{I}[1, N_p]} x_c^T X_i x_c, \quad (4.16)$$

where  $X_j = -N^T R_j M_j^{-T}$ ,  $M_j N^T = I - R_j S$ , and  $S = \frac{1}{\hat{\epsilon}^2} \hat{S}$ . Furthermore, the coefficient matrices of the RSOF controller are given by

$$\begin{bmatrix} A_{c,j} & B_{c,j} \\ C_{c,j} & D_{c,j} \end{bmatrix} = \begin{bmatrix} \hat{\epsilon} N & \hat{\epsilon} S B_{p2} \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} \hat{A}_{c,j} - \hat{\epsilon} S A_{p,j} R_j & \hat{B}_{c,j} \\ \hat{C}_{c,j} & \hat{D}_{c,j} \end{bmatrix} \begin{bmatrix} M_j^T & 0 \\ C_{p2} R_j & \hat{\epsilon} I \end{bmatrix}^{-1}. \quad (4.17)$$

for all  $j \in \mathbf{I}[1, N_p]$ .

*Proof.* According to Theorem 4.1, and using the partitions in (4.12), we define for all  $j \in \mathbf{I}[1, N_p]$ ,

$$Z_{1,j} = \begin{bmatrix} R_j & I \\ M_j^T & 0 \end{bmatrix}, \quad Z_2 = \begin{bmatrix} I & S \\ 0 & N^T \end{bmatrix}, \quad (4.18)$$

such that  $P_j Z_{1,j} = Z_2$  and  $M_j N^T = I - R_j S$ , which implies  $X_j = -N^T R_j M_j^{-T}$ . Moreover, we specify

$$\tilde{Z}_{1,j} = \begin{bmatrix} R_j & \hat{\epsilon} I \\ M_j^T & 0 \end{bmatrix}, \quad \hat{Z}_2 = \begin{bmatrix} I & \hat{\epsilon} S \\ 0 & \hat{\epsilon} N^T \end{bmatrix}, \quad (4.19)$$

which gives  $P_j \tilde{Z}_{1,j} = \hat{Z}_2$ . The definition of  $\hat{\epsilon} > 0$  will be given later. Based on condition (4.15), it can be verified that  $\tilde{Z}_{1,j}^T P_j \tilde{Z}_{1,j} = \begin{bmatrix} R_j & \hat{\epsilon} I \\ \hat{\epsilon} I & \hat{S} \end{bmatrix} > 0$ , in turn,  $P_j > 0$  as  $\tilde{Z}_{1,j}$  is nonsingular.

Then, by performing congruence transformation with matrix  $\text{diag}\{\tilde{Z}_{1,j}, \hat{\epsilon} I, I\}$  on condition (4.8), we obtain the following results:

$$\begin{aligned} \tilde{Z}_{1,j}^T P_j A_{cl,ij} \tilde{Z}_{1,j} &= \tilde{Z}_2^T A_{cl,ij} \tilde{Z}_{1,j} = \begin{bmatrix} A_{p,i} R_j + B_{p2} \hat{C}_{c,j} & \hat{\epsilon} A_{p,i} + B_{p2} \hat{D}_{c,j} C_{p2} \\ \hat{A}_{c,j} + \hat{\epsilon} S (A_{p,i} - A_{p,j}) R_j & \hat{S} A_{p,i} + \hat{B}_{c,j} C_{p2} \end{bmatrix}, \\ \hat{\epsilon} B_{cl,ij}^T P_j \tilde{Z}_{1,j} &= \hat{\epsilon} B_{cl,ij}^T \tilde{Z}_2 = \begin{bmatrix} \hat{\epsilon} B_{p1,i}^T + D_{p21,i}^T \hat{D}_{c,j}^T B_{p2}^T & B_{p1,i}^T \hat{S} + D_{p21,i}^T \hat{B}_{c,j}^T \end{bmatrix}, \\ C_{cl,ij} \tilde{Z}_{1,j} &= \begin{bmatrix} C_{p1,i} R_j + D_{p12} \hat{C}_{c,j} & \hat{\epsilon} C_{p1} + D_{p12} \hat{D}_{c,j} C_{p2} \end{bmatrix}, \\ \hat{\epsilon}^2 \gamma_\infty^2 &= \hat{\gamma}_\infty^2, \end{aligned} \quad (4.20)$$

where

$$\begin{aligned}
\hat{A}_{c,j} &= \hat{\epsilon}SA_{p,j}R_j + \hat{\epsilon}SB_{p2}D_{c,j}C_{p2}R_j + \hat{\epsilon}NB_{c,j}C_{p2}R_j + \hat{\epsilon}SB_{p2}C_{c,j}M_j^T + \hat{\epsilon}NA_{c,j}M_j^T, \\
\hat{B}_{c,j} &= \hat{S}B_{p2}D_{c,j} + \hat{\epsilon}^2NB_{c,j}, \\
\hat{C}_{c,j} &= D_{c,j}C_{p2}R_j + C_{c,j}M_j^T, \\
\hat{D}_{c,j} &= \hat{\epsilon}D_{c,j}, \quad \hat{S} = \hat{\epsilon}^2S.
\end{aligned} \tag{4.21}$$

On the other hand, we have

$$\tilde{Z}_{1,j}^T(P_i - P_j)\tilde{Z}_{1,j} = \tilde{Z}_{1,j}^T \begin{bmatrix} 0 & 0 \\ 0 & X_i - X_j \end{bmatrix} \tilde{Z}_{1,j} = \begin{bmatrix} M_j(X_i - X_j)M_j^T & 0 \\ 0 & 0 \end{bmatrix}$$

Since  $M_jN^T = I - R_jS$ , it can be shown that  $X_j = N^T(S - R_j^{-1})^{-1}N > 0$ . Using the matrix inversion lemma [4], and through algebraic manipulations, we obtain

$$M_j(X_i - X_j)M_j^T = S^{-1} - R_j + (R_j - S^{-1})(R_i - S^{-1})^{-1}(R_j - S^{-1}). \tag{4.22}$$

Moreover, through Schur complement, condition (4.15) implies

$$T_{ij} \geq S^{-1} - R_j + (R_j - S^{-1})(R_i - S^{-1})^{-1}(R_j - S^{-1})$$

which together with (4.22) concludes that  $\tau_j \left( M_j(X_i - X_j)M_j^T \right) \leq \tau_j T_{ij}$ . Therefore, after the congruence transformation, condition (4.15) can be deduced. Moreover, condition (4.8) becomes

$$\Upsilon_0 + \begin{bmatrix} 0 & 0 \\ \left[ \hat{\epsilon}S(A_{p,i} - A_{p,j}) \right] R_j & 0 \end{bmatrix} < 0. \tag{4.23}$$

where

$$\Upsilon_0 :=$$

$$\begin{bmatrix} He\{A_{p,i}R_j + B_{p2}\hat{C}_{c,j}\} + \tau_j T_{ij} & \star & \star & \star \\ \hat{A}_{c,j} + \hat{\epsilon}A_{p,i}^T + C_{p2}^T \hat{D}_{c,j}^T B_{p2}^T & He\{\hat{S}A_{p,i} + \hat{B}_{c,j}C_{p2}\} & \star & \star \\ \hat{\epsilon}B_{p1,i}^T + D_{p21,i}^T \hat{D}_{c,j}^T B_{p2}^T & B_{p1,i}^T \hat{S} + D_{p21,i}^T \hat{B}_{c,j}^T & -\hat{\gamma}_\infty^2 I & \star \\ C_{p1,i}R_j + D_{p12}\hat{C}_{c,j} & \hat{\epsilon}C_{p1,i} + D_{p12}\hat{D}_{c,j}C_{p2} & \hat{\epsilon}D_{p11,i} + D_{p12}\hat{D}_{c,j}D_{p21,i} & -I \end{bmatrix}$$

Then, by setting  $\Upsilon_1 = R_j$ ,  $\Upsilon_2 = [\hat{\epsilon}(A_{p,i} - A_{p,j})^T S 0]$ , invoking Lemma 4.1, to guarantee the satisfaction of condition (4.23), it is equivalent to have the following

condition for some positive number  $\epsilon = \frac{1}{\epsilon}$ ,

$$\begin{bmatrix} \left\{ \begin{array}{l} He\{A_{p,i}R_j + B_{p2}\hat{C}_{c,j}\} \\ + \tau_j T_{ij} - \epsilon R_j R_j \end{array} \right\} & & \star \\ \hat{A}_{c,j} + \hat{\epsilon}A_{p,i}^T + C_{p2}^T \hat{D}_{c,j}^T B_{p2}^T & \left\{ \begin{array}{l} He\{\hat{S}A_{p,i} + \hat{B}_{c,j}C_{p2}\} \\ -\hat{\epsilon}^3 S(A_{p,i} - A_{p,j})(A_{p,i} - A_{p,j})^T S \end{array} \right\} & \\ \hat{\epsilon}B_{p1,i}^T + D_{p21,i}^T \hat{D}_{c,j}^T B_{p2}^T & B_{p1,i}^T \hat{S} + D_{p21,i}^T \hat{B}_{c,j}^T & \\ C_{p1,i}R_j + D_{p12}\hat{C}_{c,j} & \hat{\epsilon}C_{p1,i} + D_{p12}\hat{D}_{c,j}C_{p2} & \\ \epsilon R_j & \hat{\epsilon}(A_{p,i} - A_{p,j})^T S & \\ & \star & \star \quad \star \\ & \star & \star \quad \star \\ & -\hat{\gamma}_\infty^2 I & \star \quad \star \\ \hat{\epsilon}D_{p11,i} + D_{p12}\hat{D}_{c,j}D_{p21,i} & -I & \star \\ 0 & 0 & -\epsilon I \end{bmatrix} < 0,$$

Consequently, perform congruence transformation with matrix  $diag\{I, I, I, I, \hat{\epsilon}I\}$  on the above condition, and based on the fact that  $-Z^T W^{-1} Z \leq -Z^T - Z + W$  holds for any pair of  $W > 0$  and  $Z$ , we have  $-\epsilon R_j R_j \leq -2R_j + \hat{\epsilon}I$  and  $-\hat{\epsilon}^3 S(A_{p,i} - A_{p,j})(A_{p,i} - A_{p,j})^T S \leq -He\{\hat{S}(A_{p,i} - A_{p,j})\} + \hat{\epsilon}I$ . This yields exactly condition (4.14). Moreover, the controller formula (4.17) can be verified by inverting the relations in (4.21).

Due to the product of scalar variables  $\tau_j$  and matrix variables  $T_{ij}$ , condition (4.14) in Theorem 4.2 is non-convex by nature. For this special type of BMIs, one can resort to LMI optimization technique coupled with a multi-dimensional search over the scalar variables. When the number of  $N_p$  is large, a possible way to reduce computational cost of the synthesis problem is by enforcing  $\tau_j = \tau \geq 0$  for all  $j \in [1, N_p]$ . Although the resulting conditions are more conservative, they can be solved relatively easier via LMI optimization with a single line search parameter. The following corollary formally presents this result for the robust  $\mathcal{H}_\infty$  control problem.

**Corollary 4.1.** *Given a tunable scalar  $\tau \geq 0$ , the result of Theorem 4.2 remains valid whenever inequalities (4.14) are replaced by*

$$\left[ \begin{array}{l} He\{A_{p,i}R_j + B_{p2}\hat{C}_{c,j}\} + \tau T_{ij} - 2R_j + \hat{\epsilon}I \\ \hat{A}_{c,j} + \hat{\epsilon}A_{p,i}^T + C_{p2}^T \hat{D}_{c,j}^T B_{p2}^T \\ \hat{\epsilon}B_{p1,i}^T + D_{p21,i}^T \hat{D}_{c,j}^T B_{p2}^T \\ C_{p1,i}R_j + D_{p12}\hat{C}_{c,j} \\ R_j \end{array} \quad \begin{array}{l} \star \\ He\{\hat{S}A_{p,j} + \hat{B}_{c,j}C_{p2}\} + \hat{\epsilon}I \\ B_{p1,i}^T \hat{S} + D_{p21,i}^T \hat{B}_{c,j}^T \\ \hat{\epsilon}C_{p1,i} + D_{p12}\hat{D}_{c,j}C_{p2} \\ (A_{p,i} - A_{p,j})^T \hat{S} \end{array} \right]$$

$$\begin{bmatrix} \star & \star & \star \\ \star & \star & \star \\ -\hat{\gamma}_\infty^2 I & \star & \star \\ \hat{e}D_{p11,i} + D_{p12}\hat{D}_{c,j}D_{p21,i} & -I & \star \\ 0 & 0 & -\hat{e}I \end{bmatrix} < 0. \quad (4.24)$$

for all  $i, j \in \mathbf{I}[1, N_p]$ .

The results of Theorem 4.2 and Corollary 4.1 then can be used to pose the following optimization problem for the robust  $\mathcal{H}_\infty$  control problem, such that the RSOF controller that renders the closed loop a suboptimal  $\mathcal{H}_\infty$  performance level can be designed. As mentioned above, this type of optimization problem can be solved through a line search over  $\tau$  with LMIs.

$$\begin{aligned} \min_{R_j, \hat{S}, T_{ij}, \hat{A}_{c,j}, \hat{B}_{c,j}, \hat{C}_{c,j}, \hat{D}_{c,j}, \hat{e}, \tau, \forall i, j \in \mathbf{I}[1, N_p]} \quad & \hat{\gamma}_\infty^2 \\ \text{s.t.} \quad & (4.15) \text{ and } (4.24). \end{aligned} \quad (4.25)$$

## 4.5 Numerical Examples

In this section, two examples will be used to illustrate the design procedure and effectiveness of the proposed RSOF control scheme. The first example aims to design a robust output-feedback  $\mathcal{H}_\infty$  controller for a system with sensor outages. Moreover, it will be demonstrated via the second example that the proposed design scheme based on using a piecewise switched Lyapunov function is indeed capable of rendering a better  $\mathcal{L}_2$ -gain performance for the closed-loop system than that obtained under the single quadratic Lyapunov function framework.

*Example 1.* Consider a fourth order two-input two-output system subject to sensor outages, which is borrowed from [33] and also considered in [20]. The system can be described as the following polytopic uncertain system:

$$G : \begin{cases} \dot{x}(t) = Ax(t) + B_1d(t) + B_2u(t) \\ e(t) = C_1x(t) + D_{11}d(t) + D_{12}u(t) \\ y(t) = C_2(\xi(t))x(t) + D_{21}d(t) \end{cases} \quad (4.26)$$

where

$$\begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2(\xi) & D_{21} & 0 \end{bmatrix} = \left[ \begin{array}{cccc|ccc} -2 & 1 & 1 & 1 & 1 & 0 & 0 \\ 3 & 0 & 0 & 2 & 0 & 1 & 0 \\ -1 & 0 & -2 & -3 & 1 & 0 & 0 \\ -2 & -1 & 2 & -1 & 0 & 0 & 1 \\ \hline 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ c_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_2 & 0 & 0 & 0 & 0 \end{array} \right]$$

where two unknown parameters  $c_1$  and  $c_2$  both take values from  $\{0, 1\}$ . Specifically,  $c_i = 0$  with  $i = 1, 2$  indicates the  $i$ th measurement experiences an outage. We assume as in [33] and [20] that there always exists at least one measurement working for feedback control use, i.e.,  $c_1$  and  $c_2$  will not be simultaneously equal to zero. This will result in a polytope of  $N_p = 3$  vertices with

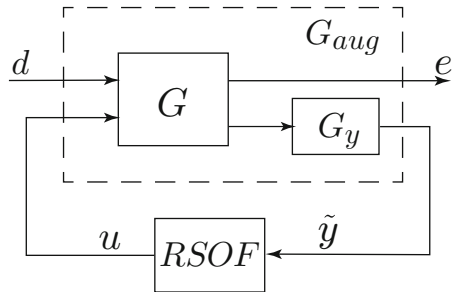
$$C_{2,1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad C_{2,2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad C_{2,3} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \quad (4.27)$$

To apply the proposed RSOF control scheme to solve the  $\mathcal{H}_\infty$  control problem, we observe that the output matrix  $C_2(\xi)$  does not satisfy Assumption 2. Nevertheless, following the methodology from [31], the original plant can be transformed to a new system that fits into the proposed design framework by concatenating a stable LTI filter to the measurement channel (as depicted in Fig. 4.2). The state-space model of the LTI filter is chosen as

$$G_y : \begin{bmatrix} \dot{x}_y \\ \tilde{y} \end{bmatrix} = \begin{bmatrix} A_y & B_y \\ C_y & 0 \end{bmatrix} \begin{bmatrix} x_y \\ y \end{bmatrix},$$

where  $x_y \in \mathbb{R}^{n_a}$  is the filter state, and  $\tilde{y} \in \mathbb{R}^{n_y}$  is the filtered output that will serve as the controller input. Then, the resulting augmented system can be written in the form of (4.1) with

**Fig. 4.2** Augmented system structure (Example 1)



$$G_{aug} : \begin{cases} A_{p,i} = \begin{bmatrix} A & 0 \\ B_y C_{2,i} & A_y \end{bmatrix}, & B_{p1} = \begin{bmatrix} B_1 \\ B_y D_{21} \end{bmatrix}, & B_{p2} = \begin{bmatrix} B_2 \\ 0 \end{bmatrix}, \\ C_{p1} = [C_1 \ 0], & D_{p11} = D_{11}, & D_{p12} = D_{12}, \\ C_{p2} = [0 \ C_y], & D_{p21} = 0, & D_{p22} = 0. \end{cases} \quad (4.28)$$

For controller synthesis, we specify the filter with

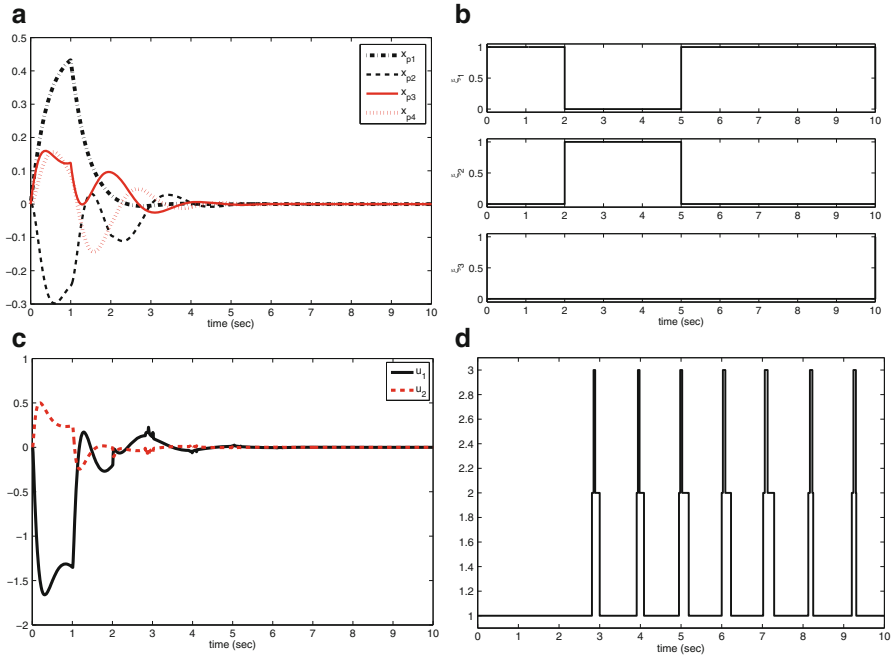
$$\begin{bmatrix} A_y & B_y \\ C_y & 0 \end{bmatrix} = \begin{bmatrix} -100I_2 & I_2 \\ 100I_2 & 0 \end{bmatrix}$$

Therefore, based on the augmented system data, we solve the optimization problem (4.25) to yield a suboptimal value of  $\gamma_\infty = 1.7415$ , which significantly improves those obtained by using the methods in [33] and [20] for all scenarios discussed therein (see Tables 1 and 2 in [20]). Furthermore, the corresponding RSOF controller in the form of (4.4) contains three subcontrollers with the order  $n_c = 4 + 2 = 6$ .

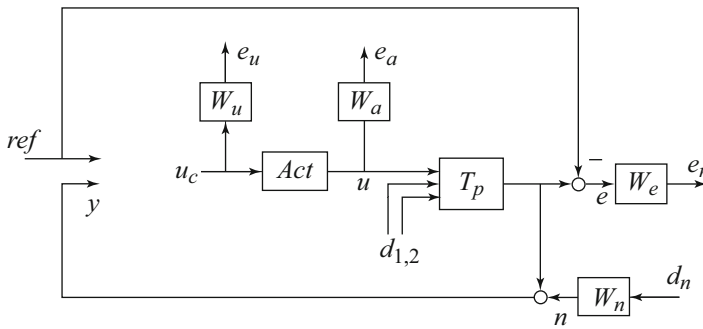
With the synthesized RSOF controller, we run the time-domain simulation by applying a pulse disturbance input of magnitude 1 starting from  $t = 0$  and ending at  $t = 1$  sec. The closed-loop responses, including four plant states (Fig. 4.3a), the uncertain time-varying vector  $\xi(t)$  (Fig. 4.3b), two control inputs (Fig. 4.3c), and the controller switching signals (Fig. 4.3d), are presented. According to (4.27), in Fig. 4.3b,  $\xi(t) = [1 \ 0 \ 0]^T$  corresponds to the case of sensor failure on  $y_1$ , while  $\xi(t) = [0 \ 1 \ 0]^T$  is with respect to the case when the second output  $y_2$  fails. As can be seen, for this simulation study, only one output measurement is available at each time instant. Nevertheless, from Fig. 4.3a, c, it is observed that in spite of the sensor outages, the designed RSOF controller is still capable of stabilizing the overall closed-loop system with reasonable control input efforts.

*Example 2.* This example aims to further demonstrate the effectiveness and advantages of the proposed switching control scheme based on piecewise switched Lyapunov functions. We consider a two-disk  $\mathcal{H}_\infty$  control problem as discussed in [34]. The uncertain dynamics of the two-disk model is given in the following form:

$$T_\rho : \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \rho_1(t) - \frac{k}{M_1} & -\frac{k}{M_1} & -\frac{b}{M_1} & 0 & \frac{1}{M_1} & \frac{0.1}{M_1} & 0 \\ -\frac{k}{M_2} & \rho_2(t) - \frac{k}{M_2} & 0 & -\frac{b}{M_2} & 0 & 0 & \frac{0.1}{M_2} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ u(t) \\ d_1(t) \\ d_2(t) \end{bmatrix}$$



**Fig. 4.3**  $\mathcal{H}_\infty$  RSOF control (Example 1). (a) Plant states, (b) uncertain parameter  $\xi(t)$ , (c) control input, (d) switching signal



**Fig. 4.4** Weighted open-loop plant interconnection of the two-disk problem [34] (Example 2)

with  $M_1 = 1, M_2 = 0.5, b = 1, k = 200$  and two uncertain parameters  $\rho_1(t) \in [0, 9], \rho_2(t) \in [0, 25]$  yielding a polytope of  $N_p = 4$  vertices. For robust  $\mathcal{H}_\infty$  control synthesis, we adopt the same performance weighting functions as in [34] to form a weighted open-loop plant as depicted in Fig. 4.4, where the weighting functions are specified as



$$W_e(s) = \frac{0.3s + 1.2}{s + 0.04}, \quad W_u(s) = \frac{s + 0.1}{0.01s + 125},$$

$$W_a(s) = 0.00001, \quad W_n(s) = \frac{s + 0.4}{0.01s + 400}.$$

The actuator dynamics is assumed to be  $Act(s) = \frac{1}{0.01s+1}$ .

Based on such a system setup, we solve the optimization problem (4.25) with  $\tau = 1$ . The RSOF control synthesis yields a suboptimal  $\mathcal{L}_2$  gain  $\gamma_\infty = 1.1139$ . To demonstrate the effectiveness of the proposed RSOF control approach, this result is compared with that obtained by using  $\mu$ -type synthesis method [4, 6]. Specifically, a robust controller consisting of a single LTI output-feedback control law is designed by using a single quadratic Lyapunov function. It should be pointed out that the robust output-feedback control synthesis problem is known to be non-convex. For fairness of comparison, we utilize a global optimization technique, namely the Branch and Bound algorithm [22], to yield a globally optimal solution. After extensive search over the solution space, we are able to obtain the corresponding global optima as  $\gamma_\infty = 1.55$ , which is larger than our result by 28.14%. The gain of performance can be attributed to the adoption of piecewise switched Lyapunov functions in the proposed design framework. Further comparisons are conducted through time-domain simulations. The closed-loop system responses with a step reference input by using, respectively, the single LTI controller and the proposed RSOF controller are plotted in Fig. 4.5. As can be seen from Fig. 4.5a, consistent with the calculated  $\mathcal{H}_\infty$  norm, the RSOF controller indeed outperforms the LTI controller with less overshoot, faster settling time, less steady-state error, as well as less control efforts (see Fig. 4.5b) during the transient period.

## 4.6 Conclusions

A new RSOF control scheme has been proposed for a class of linear systems with time-varying polytopic uncertainties. The proposed RSOF controller is constructed in a switching fashion, which consists of a set of linear dynamic output-feedback controllers and a switching rule (namely the min-switching strategy) that governs the switching among them. The novelty of the proposed control design scheme lies in that: (1) no online measurements of the uncertain time-varying parameters are required for controller implementation; (2) the robust control synthesis conditions are cast as a special type of BMIs, which can be solved via LMI optimization plus a line search; (3) owing to the use of piecewise switched Lyapunov functions, better controlled performance can be achieved comparing with those obtained by using a single quadratic Lyapunov function. The effectiveness and advantages of the proposed control design scheme have been demonstrated through numerical studies.

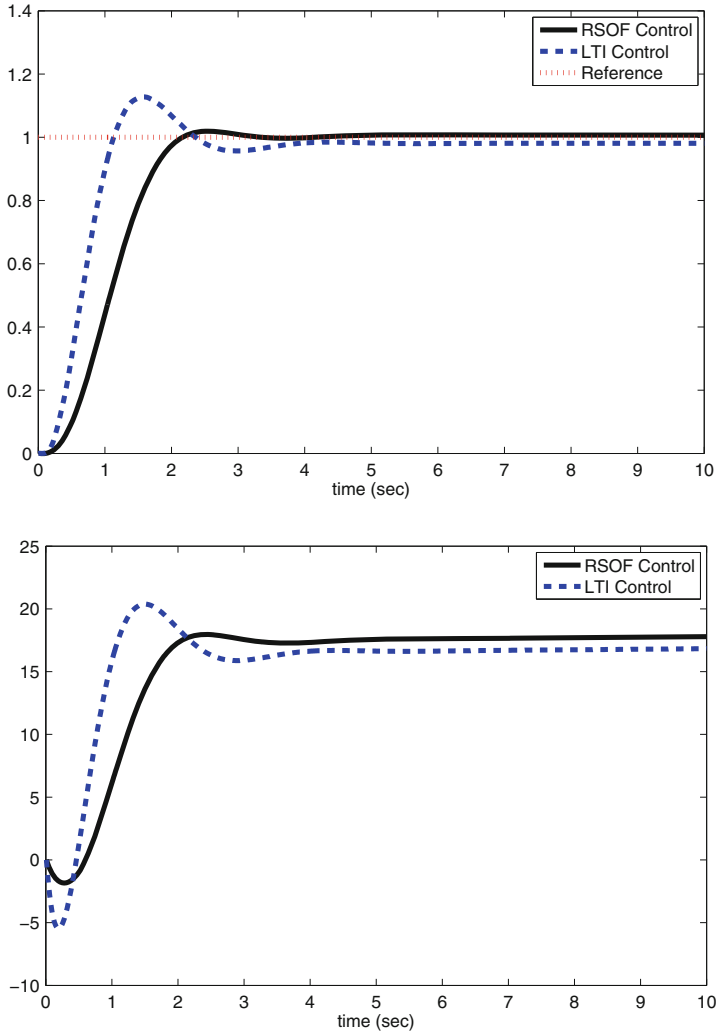


Fig. 4.5 Step response of the two-disk system (Example 2). (a) Tracking output, (b) control input

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