Semidefinite and Linear Programming Integrality Gaps for Scheduling Identical Machines

Adam Kurpisz[1], Monaldo Mastrolilli[1], Claire Mathieu[2], Tobias Mömke[3(✉)], Victor Verdugo[2,4], and Andreas Wiese[5]

[1] Dalle Molle Institute for Artificial Intelligence Research, Manno, Switzerland
[2] Department of Computer Science, CNRS UMR 8548, École normale supérieure, Paris, France
[3] Department of Computer Science, Saarland University, Saarbrücken, Germany
moemke@cs.uni-saarland.de
[4] Department of Industrial Engineering, Universidad de Chile, Santiago, Chile
[5] Max Planck Institute for Informatics, Saarbrücken, Germany

Abstract. Sherali-Adams [25] and Lovász-Schrijver [21] developed systematic procedures to strengthen a relaxation known as *lift-and-project* methods. They have been proven to be a strong tool for developing approximation algorithms, matching the best relaxations known for problems like Max-Cut and Sparsest-Cut. In this work we provide lower bounds for these hierarchies when applied over the configuration LP for the problem of scheduling identical machines to minimize the makespan. First we show that the configuration LP has an integrality gap of at least 1024/1023 by providing a family of instances with 15 different job sizes. Then we show that for any integer n there is an instance with n jobs in this family such that after $\Omega(n)$ rounds of the Sherali-Adams (SA) or the Lovász-Schrijver (LS_+) hierarchy the integrality gap remains at least 1024/1023.

1 Introduction

Scheduling

Machine scheduling is a classical family of problems in combinatorial optimization. In this paper we study the problem, known as $P||C_{\max}$, of scheduling a set J of n jobs on a set M of identical machines to minimize the *makespan*, i. e., the maximum completion time of a job, where each job $j \in J$ has a *processing time* p_j. A job cannot be preempted nor migrated to a different machine, and every job is released at time zero. This problem admits a polynomial-time approximation

Supported by the Swiss National Science Foundation project 200020-144491/1 "Approximation Algorithms for Machine Scheduling Through Theory and Experiments," by Sciex Project 12.311, by DFG grant MO 2889/1-1, and by CONICYT-PCHA/Doctorado Nacional/2014-21140930.

Q. Louveaux and M. Skutella (Eds.): IPCO 2016, LNCS 9682, pp. 152–163, 2016.
DOI: 10.1007/978-3-319-33461-5_13

scheme (PTAS) [16] and even an EPTAS [2], which is the best possible approximation result since the problem is strongly NP-hard [13]. The convex relaxations studied for the problem are weaker than those algorithmic results.

Assignment LP. A straightforward way to model $P||C_{\max}$ with a linear program (LP) is the *assignment LP* which has a variable x_{ij} for each combination of a machine $i \in M$ and a job $j \in J$, modeling whether job j is assigned to machine i. The goal is to minimize a variable T (modeling the makespan) for which we require that $\sum_{j \in J} x_{ij} \cdot p_j \leq T$ for each machine i.

$$
\begin{aligned}
\text{[Assign]}:\ \min T & \\
\sum_{i \in M} x_{ij} \geq 1 &\quad \text{for every } j \in J \\
\sum_{j \in J} x_{ij} p_j \leq T &\quad \text{for every } i \in M \\
T \geq p_j &\quad \text{for every } j \in J \\
x_{ij} \geq 0 &\quad \text{for every } i \in M, j \in J.
\end{aligned}
$$

Configuration LP. The assignment LP is dominated by the *configuration LP* which is, to the best of our knowledge, the strongest relaxation for the problem studied in the literature. Suppose we are given a value $T > 0$ that is an estimate on the optimal makespan, e. g., given by a binary search framework. A *configuration* corresponds to a multiset of processing times $C \subseteq \{p_j : j \in J\}$ such that $\sum_{p \in C} p \leq T$, i. e., it is a feasible assignment for a machine when the time availability is equal to T. Let, for given T, $\mathcal{C}$ denotes the set of all feasible configurations. The multiplicity function $m(p, C)$ indicates the number of times that the processing time p appears in the multiset C. For each combination of a machine i and a configuration C the configuration LP has a variable y_{iC} that models whether we want to assign exactly jobs with processing times in configuration C to machine i. Letting n_p denote the number of jobs $j \in J$ with processing time $p_j = p$, we can write:

$$
\begin{aligned}
[\mathrm{clp}(T)]: & \\
\sum_{C \in \mathcal{C}} y_{iC} = 1 &\quad \text{for every } i \in M, \\
\sum_{i \in M} \sum_{C \in \mathcal{C}} m(p, C) y_{iC} = n_p &\quad \text{for every } p \in \{p_j : j \in J\}, \\
y_{iC} \geq 0 &\quad \text{for every } \ i \in M, C \in \mathcal{C}.
\end{aligned}
$$

We remark that in another common definition [26], a configuration is a subset, not of processing times but of jobs. We can solve that LP to a $(1 + \epsilon)$-accuracy in polynomial time [26] and similarly our LP above. The definition in terms of multisets makes sense since we are working in a setting of identical machines.

Integrality Gap. The configuration LP clp(T) does not have an objective function and instead we seek to determine the smallest value T for which it is feasible. In this context, for a convex relaxation $K(T)$ we define the *integrality gap* to be the supremum value $T_{\text{opt}}(I)/T^*(I)$ over all problem instances I, where $T_{\text{opt}}(I)$ is the optimal value and $T^*(I)$ is the minimum value T for which $K(T)$ is feasible. With the additional constraint that $T \geq \max_{j\in J} p_j$, the Assignment LP relaxation has an integrality gap of 2 (which can be shown using the analysis of the list scheduling algorithm, see e. g., [27]). Here we prove that the configuration LP has an integrality gap of at least 1024/1023 (Theorem 1(i)).

Linear Programming and Semi-definite Programming Hierarchies

Hierarchies. An interesting question is whether other convex relaxations have better integrality gaps. Convex hierarchies, parametrized by a number of *levels* or *rounds*, are systematic approaches to design improved approximation algorithms by gradually tightening the integrality gap between the integer formulation and corresponding relaxation, at the cost of increased running time. Popular among these methods are the Sherali-Adams (SA) hierarchy [25] (Definition 1), the Lovász-Schrijver (LS_+) semi-definite programming hierarchy [21] (Definition 4) and the Lasserre/Sum-Of-Squares hierarchy [18,22], which is the strongest of the three. For a comparison between them and their algorithmic implications we refer to [10,19,23]. In some settings, for example the Independent Set problem in sparse graphs [4], a *mixed* SA has also been considered.

Positive Results. For many problems the approximation factors of the best known algorithms match the integrality gap after performing a constant number of rounds of this hierarchies. Examples of such problems are: Max-Cut [1] and Sparsest-Cut [1,9], dense Max-Cut [11], Knapsack and Set-Cover [8]. In the scheduling context, for minimizing the makespan on two machines in the setting of unit size jobs and precedence constraints, Svensson solves the problem optimally with only one level of the linear LS hierarchy [23] (Sect. 3.1, personal communication between Svensson and the author of [23]). Furthermore, for a constant number of machines, Levey and Rothvoss give a $(1+\varepsilon)$-approximation algorithm using $(\log(n))^{\Theta(\log\log n)}$ rounds of SA hierarchy [20]. For minimizing weighted completion time on unrelated machines, one round of LS_+ leads to the current best algorithm [5]. Thus, hierarchies are a strong tool for approximation algorithms.

Negative Results. Nevertheless, there are known limitations on these hierarchies. Lower bounds on the integrality gap of LS_+ are known for Independent Set [12], Vertex Cover [3,7,14,24], Max-3-Sat and Hypergraph Vertex Cover [1], and k-Sat [6]. For the Max-Cut problem, there are lower bounds for the SA [11] and LS_+ [24]. For the Min-Sum scheduling problem (i. e., scheduling with job dependent cost functions on one machine) the integrality gap is unbounded even after $O(\sqrt{n})$ rounds of Lasserre [17]. In particular, that holds for the problem of minimizing the number of tardy jobs even though that problem is solvable in polynomial time, thus SDP hierarchies sometimes fail to reduce the integrality gap even on easy problems.

Our Results

Our key question in this paper is: is it possible to obtain a polynomial time $(1+\epsilon)$-approximation algorithm based on applying the SA or the LS_+ hierarchy to one of the known LP-formulations of our problem? This would match the best known (polynomial time) approximation factor we know [2,16].

We answer this question in the negative. We prove that even after $\Omega(n)$ rounds of SA or LS_+ to the configuration LP the integrality gap of the resulting relaxation is still at least $1 + 1/1023$. Since the configuration LP dominates[1] the assignment LP, our result also holds if we apply $\Omega(n)$ rounds of SA or LS_+ to the assignment LP.

Theorem 1. *Consider the problem of scheduling identical machines to minimize the makespan, $P||C_{\max}$. For each $n \in \mathbb{N}$ there exists an instance with n jobs such that:*

(i) the configuration LP has an integrality gap of at least $1024/1023$.
(ii) after applying $r = \Omega(n)$ rounds of the SA hierarchy to the configuration LP the obtained relaxation has an integrality gap of at least $1024/1023$.
(iii) after applying $r = \Omega(n)$ rounds of the LS_+ hierarchy to the configuration LP the obtained relaxation has an integrality gap of at least $1024/1023$.

Since polynomial time approximations schemes are known [2,16] for $P||C_{\max}$, Theorem 1 implies that the SA and the LS_+ hierarchies do *not* yield the best possible approximation algorithms. We remark that for the hierarchies studied in Theorem 1, n rounds suffice to bring the integrality gap down to exactly 1, so results (*ii*) and (*iii*) are almost tight in terms of number of levels.

We prove Theorem 1 by defining a family of instances $\{I_k\}_{k\in\mathbb{N}}$ constructed from the Petersen graph (see Fig. 1). In Sect. 2 we prove that the configuration LP is feasible for $T = 1023$ while the integral optimum has a makespan of at least 1024. In Sect. 3, we show for each instance I_k that using the hypergeometric distribution we can define a fractional solution that is feasible for the polytope obtained by applying $\Omega(k)$ rounds of SA to the configuration LP parametrized by $T = 1023$. In Sect. 4 we prove the same for the semidefinite relaxations obtained with the LS_+ hierarchy, and we study the protection matrices used in the lower bound proof. In this part we work with *covariances* matrices by applying *Schur's complement* and a posterior analysis for block-symmetry matrices.

The Hard Instances

To prove the integrality gaps of 1024/1023, for each odd $k \in \mathbb{N}$ we define an instance I_k that is inspired by the Petersen graph G (see Fig. 1) with vertex set $V = \{0, 1, \ldots, 9\}$. For each edge $e = \{u, v\}$ of G, in I_k we introduce k copies of a job $j_{\{uv\}}$ of size $2^u + 2^v$. Thus I_k has $15k$ jobs. (If n is not an odd multiple of 15, let $n = 15k + \ell$ where k is the greatest odd integer such that $15k < n$. In this case we simply add to the instance ℓ jobs that each have processing time

[1] The projection of the configuration LP onto the assignment space is a contained inside the polytope of the assignment LP [26].

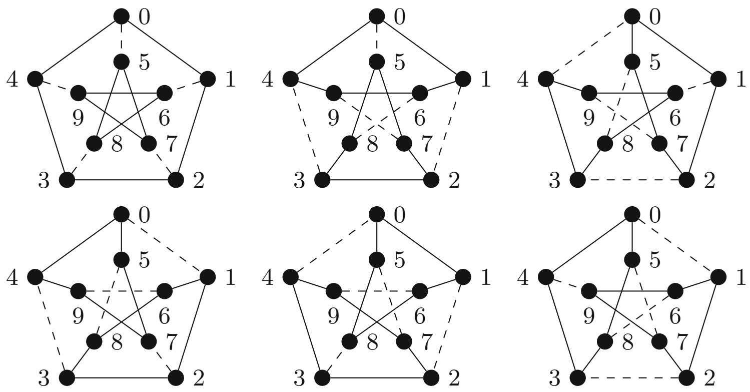

Fig. 1. The Petersen graph and its six perfect matchings (dashed lines)

equal to zero.) We define the number of machines for I_k to be $3k$. For simplicity, in the following we do not distinguish between jobs and their sizes. The graph G has exactly six perfect matchings $\bar{M}_1, \bar{M}_2, \ldots, \bar{M}_6$. Since the sum of the job sizes in a perfect matching $\bar{M}_\ell$ is

$$\sum_{e \in \bar{M}_\ell} p_{j_e} = \sum_{0 \leq u \leq 9} 2^u = 1023,$$

$\bar{M}_\ell$ corresponds to a configuration C_ℓ that contains one job corresponding to each edge in $\bar{M}_\ell$ and has makespan 1023. The configurations $C_1, \ldots, C_6$ are called *matching configurations* and we denote them by $\mathcal{D} = \{C_1, \ldots, C_6\}$.

2 Integrality Gap of the Configuration LP (Theorem 1(i))

Lemma 1. *clp*[1023] *is feasible for* I_k.

Proof. To define the fractional solution, for every machine i and each $\ell \in \{1, 2, \ldots, 6\}$ we set $y_{iC_\ell} = 1/6$. For all other configurations C we set $y_{iC} = 0$.

The first set of constraints in *clp*(T) (for the machines) is clearly satisfied. For the second set of constraints (for the job sizes), let p be such a job size and let e be the corresponding edge in G. The Petersen graph is such that there are exactly two perfect matchings $\bar{M}_\ell, \bar{M}_{\ell'}$ containing e, thus we get $\sum_{i=1}^{3k}(y_{iC_\ell} + y_{iC_{\ell'}}) = k$ and y is feasible. $\square$

Lemma 2. *The optimal makespan for* I_k *is at least 1024.*

Proof. Assume, for a contradiction, that clp[1023] for I_k has an integer solution y. Since the total size of jobs is $k \cdot 3 \cdot 1023$ and there are $3k$ machines, only configurations C with makespan exacly equal to 1023 may have $y_{iC} \neq 0$.

Consider such a configuration C. Since $1023 = \sum_{u=0}^{9} 2^u$, considering the binary representation of 1023, by induction on u it must be that for every u, configuration C contains an odd number of jobs corresponding to edges adjacent to vertex u in G. Furthermore, since the sum does not exceed 1023, that odd number must be exactly 1. Thus C exactly corresponds to a perfect matching of G, and so the integer solution y corresponds to a 1-factorization of the multigraph G_k obtained by taking k copies of each edge in the Petersen graph.

Let $\bar{M}_1$ be the perfect matching of the Petersen graph consisting of the five edges $\{0,5\},\{1,6\},\{2,7\},\{3,8\}\{4,9\}$, called *spokes*. Let $\ell = \sum_{\underline{i}} y_{iC_1}$. Since each spoke, which appears in exactly one other perfect matching $\bar{M}_j$ $(j > 1)$, must be contained in k matchings in total, we must have $\sum_i y_{iC_j} = k - \ell$ for each $j \in [2,6]$. Thus $\sum_{i,C} y_{iC} = 5(k-\ell)+\ell = 5k-4\ell$. However, that sum equals $3k$, the total number of machines, and so $\ell = k/2$. Since k is odd and ℓ an integer, the contradiction follows. □

3 Integrality Gap for SA (Theorem 1(ii))

We show that for the family of instances $\{I_k\}_{k\in\mathbb{N}}$ defined in Sect. 2, if we apply $O(k)$ rounds of SA to the configuration LP for $T = 1023$, then the resulting relaxation is feasible. Thus, after $\Omega(k)$ rounds of SA the configuration LP still has an integrality gap of at least 1024/1023 on an instance with $O(k)$ jobs and machines. First, we define the polytope $\mathrm{SA}^r(P)$ obtained after r rounds of SA to a polytope P that is defined via equality constraints[2].

Definition 1 (Polytope SA$^r(P)$). *Consider a polytope $P \subseteq [0,1]^E$ defined by equality constraints. For every constraint $\sum_{i\in E} a_{i,\ell} y_i = b_\ell$ and every $H \subseteq E$ such that $|H| \le r$, the constraint $\sum_{i\in E} a_{i,\ell} y_{H\cup\{i\}} = b_\ell y_H$ is included in $SA^r(P)$, the level r of the Sherali-Adams hierarchy applied to P. The polytope $SA^r(P)$ lives in $\mathbb{R}^{\mathcal{P}_{r+1}(E)}$, where $\mathcal{P}_{r+1}(E) = \{A \subseteq E : |A| \le r+1\}$.*

For the configuration LP clp(T) the variables set is $E = M \times \mathcal{C}$. Since it is defined by equality constraints, the polytope $\mathrm{SA}^r(\mathrm{clp}(T))$ corresponds to

$$
\begin{aligned}
&[\mathrm{SA}^r(\mathrm{clp}(T))]: \\
&\sum_{C\in\mathcal{C}} y_{H\cup\{(i,C)\}} = y_H \qquad \forall\, i \in M, \forall\, H \subseteq E:\ |H| \le r, \\
\sum_i \sum_{C\in\mathcal{C}} m(p,C) &y_{H\cup\{(i,C)\}} = n_p y_H \quad \forall\, p \in \{p_j : j \in J\}, \forall\, H \subseteq E : |H| \le r, \\
&y_H \ge 0 \qquad \forall\, H \subseteq E : |H| \le r+1, \\
&y_\emptyset = 1.
\end{aligned}
$$

Intuitively, the configuration LP computes a set of edges in a complete bipartite graph with vertex sets U, V where U is the set of machines and V is the set of configurations. The edges are selected such that they form a U-matching, i.e., such that each node in U is incident to at most one selected edge.

[2] This definition is slightly different from the one in Sherali & Adams [25]; for simplicity we give a definition that, in the case of equality constraints, is equivalent.

Definition 2. *Given two sets U, V and $F \subseteq U \times V$, the F-degree of $u \in U$ is $\delta_F(u) = |\{v : (u,v) \in F\}|$, and $\delta_F(v) = |\{u : (u,v) \in F\}|$ if $v \in V$. We say that F is an U-matching if $\delta_F(u) \leq 1$ for every $u \in U$. An element $u \in U$ is incident to F if $\delta_F(u) = 1$.*

In the following we consider the same family of instances $\{I_k \colon k \in \mathbb{N}, k \text{ is odd}\}$ as in Sect. 2 and $T = 1023$. For any set S we define $\mathcal{P}(S)$ to be the power set of S. We want to define a solution to $\mathrm{SA}^r(\mathrm{clp}(T))$ for $T = 1023$. To this end, we need to define a value y_A for each set $A \in \mathcal{P}_{r+1}(M \times \mathcal{C})$. In particular, for $A \in \mathcal{P}_r(M \times \mathcal{D})$, we define this value according to the *hypergeometric distribution.*

Definition 3. *Let $\phi : \mathcal{P}(M \times \mathcal{D}) \to \mathbb{R}$ be such that*

$$\phi(A) = \frac{1}{(3k)_{|A|}} \prod_{j \in [6]} (k/2)_{\delta_A(C_j)}$$

if A is an M-matching, and zero otherwise, where $(x)_a = x(x-1)\cdots(x-a+1)$, for integer $a \geq 1$, is the lower factorial function.

To get some understanding about how the distribution ϕ works, the following lemma intuitively shows the following: suppose that we know that a set A is chosen (i.e., we condition on this), then the conditional probability that also a pair (i, C_j) is chosen equals $\frac{k/2-\delta_A(C_j)}{3k-|A|}$, assuming that $A \cup \{(i, C_j)\}$ forms an M-matching.

Lemma 3. *Let $A \subseteq M \times \mathcal{D}$ be an M-matching of size at most $3k-1$. If $i \in M$ is not incident to A, then $\phi(A \cup \{(i, C_j)\}) = \phi(A)\frac{k/2-\delta_A(C_j)}{3k-|A|}$.*

Proof. Given that i is not incident to A, we have $|A \cup \{(i, C_j)\}| = |A| + 1$. Furthermore, for $\ell \neq j$ we have that $\delta_{A\cup\{(i,C_j)\}}(C_\ell) = \delta_A(C_\ell)$ and $\delta_{A\cup\{(i,C_j)\}}(C_j) = \delta_A(C_j) + 1$. Therefore, $\frac{\phi(A\cup\{(i,C_j)\})}{\phi(A)} = \frac{k/2-\delta_A(C_j)}{3k-|A|}$. □

The Feasible Solution. We are ready now to define our solution to $\mathrm{SA}^r(\mathrm{clp}(T))$. It is the vector $y^\phi \in \mathbb{R}^{\mathcal{P}_{r+1}(E)}$ defined such that $y^\phi_A = \phi(A)$ if A is an M-matching in $M \times \mathcal{D}$, and zero otherwise.

Lemma 4. *For every odd k, y^ϕ is a feasible solution for $SA^r(clp(T))$ for the instance I_k when $r = \lfloor k/2 \rfloor$ and $T = 1023$.*

Proof. We first prove that $y^\phi \geq 0$. Consider some $H \subseteq E$. Since $y^\phi_H = \phi(H)$, using Definition 3, it is easy to check that the lower factorial stays non-negative for $r = \lfloor k/2 \rfloor$.

We next prove that y^ϕ satisfies the machine constraints in $\mathrm{SA}^r(\mathrm{clp})$. If i is a machine incident to H, then all terms in the left-hand summation are 0 except for the unique pair (i, C) that belongs to H, so the sum equals y^ϕ_H. If i is not incident to H, then by Lemma 3 we have

$$\sum_{C} y^{\phi}_{H\cup\{(i,C)\}} = \frac{\phi(H)}{3k-|H|}\sum_{j\in[6]}(k/2-\delta_H(C_j)) = \phi(H) = y^{\phi}_H,$$

since $6\cdot k/2 = 3k$ and $\sum_{j\in[6]}\delta_H(C_j) = |H|$.

Finally we prove that y^{ϕ} satisfies the set of constraints for every processing time. Fix p and H. Since y^{ϕ} is supported by six configurations, we have

$$\sum_{i\in M}\sum_{C\in\mathcal{C}} m(p,C)y^{\phi}_{H\cup\{(i,C)\}} = \sum_{i\in M}\sum_{j\in[6]} m(p,C_j)\phi(H\cup\{(i,C_j)\}).$$

There are exactly two configurations $C_1^p, C_2^p \in \mathcal{D}$ such that $m(p,C_1^p) = m(p,C_2^p) = 1$, and for the others it is zero, so

$$\sum_{j\in[6]} m(p,C_j)\phi(H\cup\{(i,C_j)\}) = \phi(H\cup\{(i,C_1^p)\}) + \phi(H\cup\{(i,C_2^p)\}).$$

Let $\pi_M(H) = \{i\in M : \delta_H(i) = 1\}$ be the subset of machines incident to H. We split the sum over $i\in M$ into two parts, $i\in\pi_M(H)$ and $i\notin\pi_M(H)$. For the first part,

$$\sum_{i\in\pi_M(H)}(\phi(H\cup\{(i,C_1^p)\}) + \phi(H\cup\{(i,C_2^p)\})) = \phi(H)(\delta_H(C_1^p)+\delta_H(C_2^p))$$

since $\phi(H\cup\{(i,C_1^p)\})$ is either $\phi(H)$ or 0 depending on whether $(i,C_1^p)\in H$, and the same holds for C_2^p.

For the second part, using Lemma 3 we have that

$$\begin{aligned}
&\sum_{i\notin\pi_M(H)}(\phi(H\cup\{(i,C_1^p)\}) + \phi(H\cup\{(i,C_2^p)\}))\\
&= \frac{\phi(H)}{3k-|H|}\sum_{i\notin\pi_M(H)}(k/2-\delta_H(C_1^p)+k/2-\delta_H(C_2^p))\\
&= \phi(H)(k/2-\delta_H(C_1^p)+k/2-\delta_H(C_2^p)),
\end{aligned}$$

since $|H\setminus\pi_M(H)| = 3k-|H|$. Adding, thanks to cancellations we get precisely what we want:

$$\sum_{i\in M}\sum_{C\in\mathcal{C}} m(p,C)y^{\phi}_{H\cup\{(i,C)\}} = k\phi(H) = n_p y^{\phi}_H.$$

□

Proof (of Theorem 1(ii)). Consider instance I_k as defined before, $T = 1023$ and $r = \lfloor k/2\rfloor$. By Lemma 4 the vector $y^{\phi}\in SA^r(clp(T))$. □

We note that in the above proof, the projection of y^{ϕ} onto the space of the configuration LP is the fractional solution from the proof of Lemma 1.

4 Integrality Gap for LS$_+$ (Theorem 1(iii))

Given a polytope $P \subseteq \mathbb{R}^d$, we consider the convex cone $Q = \{(a, x) \in \mathbb{R}^* \times P : x/a \in P\}$. We define an operator N_+ on convex cones $R \subseteq \mathbb{R}^{d+1}$ as follows: $y \in N_+(R)$ if and only if there exists a symmetric matrix $Y \subseteq \mathbb{R}^{(d+1)\times(d+1)}$, called the *protection matrix* of y, such that

1. $y = Ye_\emptyset = \mathrm{diag}(Y)$,
2. for all i, $Ye_i, Y(e_\emptyset - e_i) \in R$,
3. Y is positive semidefinite,

where e_i denotes the vector with a 1 in the ith coordinate and 0's elsewhere.

Definition 4. *For any $r \geq 0$ and polytope $P \subseteq \mathbb{R}^d$, level r of the LS$_+$ hierarchy, $N_+^r(Q) \subseteq \mathbb{R}^{d+1}$, is defined recursively by: $N_+^0(Q) = Q$ and $N_+^r(Q) = N_+(N_+^{r-1}(Q))$.*

To prove the integrality gap for LS$_+$ we follow an inductive argument. We start from $P = \mathrm{clp}(T)$. Along the proof, we use a special type of vectors that are integral in a subset of coordinates and fractional in the others.

The Feasible Solution. Let A be an M-matching in $M \times \mathcal{D}$. The *partial schedule* $y(A) \in \mathbb{R}^{M\times\mathcal{C}}$ is the vector such that for every $i \in M$ and $j \in \{1, 2, \ldots, 6\}$, $y(A)_{iC_j} = \phi(A \cup \{(i, C_j)\})/\phi(A)$, and zero otherwise. Here is the key Lemma.

Lemma 5. *Let k be an odd integer and $r \leq \lfloor k/2 \rfloor$. Let Q_k be the convex cone of clp(T) for instance I_k and $T = 1023$. Then, for every M-matching A of cardinality $\lfloor k/2 \rfloor - r$ in $M \times \mathcal{D}$, we have $y(A) \in N_+^r(Q_k)$.*

Before proving Lemma 5, let us see how it implies the Theorem.

Proof (of Theorem 1(iii)). Consider instance I_k defined in Sect. 2, $T = 1023$ and $r = \lfloor k/2 \rfloor$. By Lemma 5 for $A = \emptyset$ we have $y(\emptyset) \in N_+^r(Q_k)$. □

In the following two helper lemmas we describe structural properties of every partial schedule.

Lemma 6. *Let A be an M-matching in $M \times \mathcal{D}$, and let i be a machine incident to A. Then, $y(A)_{iC} \in \{0, 1\}$ for all configuration C.*

Proof. If $C \notin \mathcal{D}$ then $y(A)_{iC} = 0$ by definition. If $(i, C_j) \in A$ then $y(A)_{iC_j} = \phi(A \cup \{(i, C_j)\})/\phi(A) = \phi(A)/\phi(A) = 1$. For $\ell \neq j$, the set $A \cup \{(i, C_\ell)\}$ is not an M-matching and thus $y(A)_{iC_k} = 0$. □

Lemma 7. *Let A be an M-matching in $M \times \mathcal{D}$ of cardinality at most $\lfloor k/2 \rfloor$. Then, $y(A) \in$ clp(T).*

Proof. We note that $y(A)_{iC} = y^\phi_{A\cup\{(i,C)\}}/y^\phi_A$, and then the feasibility of $y(A)$ in clp(T) is implied by the feasibility of y^ϕ in SA^r(clp(T)), for $r = \lfloor k/2 \rfloor$. □

Given a partial schedule $y(A)$, let $Y(A) \in \mathbb{R}^{(|M\times\mathcal{C}|+1)\times(|M\times\mathcal{C}|+1)}$ be the matrix such that its principal submatrix indexed by $\{\emptyset\} \cup (M \times \mathcal{D})$ equals

$$\begin{pmatrix} 1 & y(A)^\top \\ y(A) & Z(A) \end{pmatrix},$$

where $Z(A)_{iC_j,\ell C_h} = \phi(A \cup \{(i, C_j), (\ell, C_h)\})/\phi(A)$. All the other entries of the matrix $Y(A)$ have value equal to zero. The matrix $Y(A)$ provides the *protection matrix* we need in the proof of the key Lemma.

Theorem 2. *For every M-matching A in $M \times \mathcal{D}$ such that $|A| \leq \lfloor k/2 \rfloor$, the matrix $Y(A)$ is positive semidefinite.*

Proof (Sketch). We prove that $Y(A)$ is positive semidefinite by performing several transformations that preserve this property. First, we remove all those zero columns and rows. Then, $Y(A)$ is positive semidefinite if and only if its principal submatrix indexed by $\{\emptyset\} \cup (M \times \mathcal{D})$ is positive semidefinite. We then construct the covariance matrix $\mathrm{Cov}(A)$ by taking the Schur's Complement of $Y(A)$ respect to the entry $(\{\emptyset\}, \{\emptyset\})$. The resulting matrix is positive semidefinite if and only if $Y(A)$ is positive semidefinite. After removing null rows and columns in $\mathrm{Cov}(A)$ we obtain a new matrix, $\mathrm{Cov}_+(A)$, which can be written using Kronecker products as $I \otimes Q + (J - I) \otimes W$, with $Q, W \in \mathbb{R}^{6\times6}$, $Q = \alpha W$ for some $\alpha \in (-1, 0)$ and I, J being the identity and the all-ones matrix, respectively. By applying a lemma about block matrices in [15], $Y(A)$ is positive semidefinite if and only if W is positive semidefinite. The matrix W is of the form $D_u - uu^\top$, with $u \in \mathbb{R}^6$ and D_u is a diagonal matrix such that $\mathrm{diag}(D_u) = u$. By Jensen's inequality with the function $t(y) = y^2$ it follows that W is positive semidefinite. A complete proof of the theorem can be found in the Appendix. □

Lemma 8. *Let A be an M-matching in $M \times \mathcal{D}$ and i a non-incident machine to A. Then, $\sum_{j\in[6]} Y(A)e_{iC_j} = Y(A)e_\emptyset$.*

Proof. Let S be the index of a row of $Y(A)$. If $S \notin \{0\} \cup (M \times \mathcal{D})$ then that row is identically zero, so the equality is satisfied. Otherwise,

$$e_S^\top \sum_{j\in[6]} Y(A)e_{iC_j} = \sum_{j\in[6]} \frac{\phi(A \cup \{(i, C_j)\} \cup S)}{\phi(A)}.$$

If $A\cup S$ is not an M-matching then $\phi(A\cup S\cup\{i, C_j\}) = 0$ for all i and $j \in [6]$, and $e_S^\top Y(A)e_\emptyset = \phi(A\cup S) = 0$, so the equality is satisfied. If $A\cup S$ is an M-matching, then

$$\begin{aligned}\sum_{j\in[6]} \frac{\phi(A \cup \{(i, C_j)\} \cup S)}{\phi(A)} &= \frac{\phi(A \cup S)}{\phi(A)} \sum_{j\in[6]} \frac{\phi(A \cup S \cup \{(i, C_j)\})}{\phi(A \cup S)} \\ &= e_S^\top Y(A)e_\emptyset \sum_{j\in[6]} \frac{y^\phi_{A\cup S\cup\{(i,C_j)\}}}{y^\phi_{A\cup S}} \\ &= e_S^\top Y(A)e_\emptyset,\end{aligned}$$

since y^ϕ is a feasible solution for the SA hierarchy. □

Having previous two results we are ready to prove the key Lemma.

Proof (of Lemma 5). We proceed by induction in r. The base case $r = 0$ is implied by Lemma 7, and now suppose that it is true for $r = t$. Let $y(A)$ be a partial schedule of A of cardinality $\lfloor k/2 \rfloor - t - 1$. We prove that the matrix $Y(A)$ is a protection matrix for $y(A)$. It is symmetric by definition, $y(A)e_\emptyset = \text{diag}(y(A)) = y(A)$ and thanks to Theorem 2 the matrix $Y(A)$ is positive semidefinite. Let (i, C) be such that $y(A)_{iC} \in (0, 1)$. In particular, by Lemma 6 we have $(i, C) \notin A$ and $C \in \mathcal{D}$. We claim that $Y(A)e_{iC}/y(A)_{iC}$ is equal to the partial schedule $(1, y(A \cup \{(i, C)\}))$. If S indexes a row not in $M \times \mathcal{D}$ then the respective entry in both vectors is zero, so the equality is satisfied. Otherwise,

$$\frac{e_S^\top Y(A)e_{iC}}{y(A)_{iC}} = \frac{\phi(A \cup \{(i, C)\} \cup S)}{\phi(A \cup \{(i, C)\})} = y(A \cup \{(i, C)\})_S.$$

The cardinality of the M-matching $A \cup \{(i, C)\}$ is equal to $|A| + 1 = \lfloor k/2 \rfloor - t$, and therefore by induction we have that $Y(A)e_{iC}/y(A)_{iC} = (1, y(A \cup \{(i, C)\})) \in N_+^t(Q_k)$. Now we have to prove that the vectors $Y(A)(e_\emptyset - e_{iC})/(1 - y(A)_{iC})$ are feasible for $N_+^t(Q_k)$. By Lemma 8 we have that for every $\ell \in \{1, 2, \ldots, 6\}$,

$$\frac{Y(A)(e_\emptyset - e_{iC_\ell})}{1 - y(A)_{iC_\ell}} = \sum_{j \in [6] \setminus \{\ell\}} \left(\frac{y(A)_{iC_j}}{\sum_{j \in [6] \setminus \{\ell\}} y(A)_{iC_j}} \right) y(A \cup \{(i, C_j)\}),$$

and then $Y(A)(e_\emptyset - e_{iC_\ell})/(1 - y(A)_{iC_\ell})$ is a convex combination of the partial schedules $\{y(A \cup \{(i, C_j)\}) : j \in \{1, 2, \ldots, 6\} \setminus \ell\} \subset N_+^t(Q_k)$, concluding the induction. □

References

1. Alekhnovich, M., Arora, S., Tourlakis, I.: Towards strong nonapproximability results in the Lovász-Schrijver hierarchy. In: STOC, pp. 294–303 (2005)
2. Alon, N., Azar, Y., Woeginger, G.J., Yadid, T.: Approximation schemes for scheduling. In: SODA, pp. 493–500 (1997)
3. Arora, S., Bollobás, B., Lovász, L., Tourlakis, I.: Proving integrality gaps without knowing the linear program. Theor. Comput. **2**, 19–51 (2006)
4. Bansal, N.: Approximating independent sets in sparse graphs. In: SODA, pp. 1–8 (2015)
5. Bansal, N., Srinivasan, A., Svensson, O.: Lift-and-round to improve weighted completion time on unrelated machines. CoRR, abs/1511.07826 (2015)
6. Buresh-Oppenheim, J., Galesi, N., Hoory, S., Magen, A., Pitassi, T.: Rank bounds and integrality gaps for cutting planes procedures. Theor. Comput. **2**, 65–90 (2006)
7. Charikar, M.: On semidefinite programming relaxations for graph coloring and vertex cover. In: SODA, pp. 616–620 (2002)
8. Chlamtáč, E., Friggstad, Z., Georgiou, K.: Lift-and-project methods for set cover and Knapsack. In: Dehne, F., Solis-Oba, R., Sack, J.-R. (eds.) WADS 2013. LNCS, vol. 8037, pp. 256–267. Springer, Heidelberg (2013)

9. Chlamtac, E., Krauthgamer, R., Raghavendra, P.: Approximating sparsest cut in graphs of bounded treewidth. In: Serna, M., Shaltiel, R., Jansen, K., Rolim, J. (eds.) APPROX 2010. LNCS, vol. 6302, pp. 124–137. Springer, Heidelberg (2010)
10. Chlamtac, E., Tulsiani, M.: Convex relaxations and integrality gaps. In: Anjos, M.F., Lasserre, J.B. (eds.) Handbook on Semidefinite, Conic and Polynomial Optimization, pp. 139–169. Springer, New York (2012)
11. Fernandez de la Vega, W., Mathieu, C.: Linear programming relaxations of maxcut. In: SODA, pp. 53–61 (2007)
12. Feige, U., Krauthgamer, R.: The probable value of the Lovász-Schrijver relaxations for maximum independent set. SIAM J. Comput. **32**, 345–370 (2003)
13. Garey, M.R., Johnson, D.S.: "Strong" NP-completeness results: motivation, examples, and implications. J. ACM **25**, 499–508 (1978)
14. Georgiou, K., Magen, A., Pitassi, T., Tourlakis, I.: Integrality gaps of 2-o(1) for vertex cover SDPs in the Lovász-Schrijver hierarchy. SIAM J. Comput. **39**, 3553–3570 (2010)
15. Gvozdenovic, N., Laurent, M.: The operator ψ for the chromatic number of a graph. SIAM J. Optim. **19**, 572–591 (2008)
16. Hochbaum, D., Shmoys, D.: Using dual approximation algorithms for scheduling problems theoretical and practical results. J. ACM **34**, 144–162 (1987)
17. Kurpisz, Adam, Leppänen, S., Mastrolilli, M.: A Lasserre lower bound for the min-sum single machine scheduling problem. In: Bansal, N., Finocchi, I. (eds.) ESA 2015. LNCS, vol. 9294, pp. 853–864. Springer, Heidelberg (2015). doi:10.1007/978-3-662-48350-3_71
18. Lasserre, J.: Global optimization with polynomials and the problem of moments. SIAM J. Optim. **11**, 796–817 (2001)
19. Laurent, M.: A comparison of the Sherali-Adams, Lovász-Schrijver, and Lasserre relaxations for 0–1 programming. Math. Oper. Res. **28**, 470–496 (2003)
20. Levey, E., Rothvoss, T.: A Lasserre-based $(1+\varepsilon)$-approximation for $\mathrm{P}m|p_j = 1, \mathrm{prec}|\mathrm{C}_{\max}$. CoRR, abs/1509.07808 (2015)
21. Lovász, L., Schrijver, A.: Cones of matrices and set-functions and 0–1 optimization. SIAM J. Optim. **1**, 166–190 (1991)
22. Parrilo, P.: Semidefinite programming relaxations for semialgebraic problems. Math. Program. **96**, 293–320 (2003)
23. Rothvoß, T.: The Lasserre hierarchy in approximation algorithms. Lecture notes for the MAPSP (2013)
24. Schoenebeck, G., Trevisan, L., Tulsiani, M.: Tight integrality gaps for Lovász-Schrijver LP relaxations of vertex cover and max cut. In: Proceedings of the Thirty-Ninth Annual ACM Symposium on Theory of Computing, pp. 302–310 (2007)
25. Sherali, H., Adams, W.: A hierarchy of relaxations between the continuous and convex hull representations for zero-one programming problems. SIAM J. Discrete Math. **3**, 411–430 (1990)
26. Verschae, J., Wiese, A.: On the configuration-LP for scheduling on unrelated machines. J. Sched. **17**, 371–383 (2014)
27. Williamson, D., Shmoys, D.: The Design of Approximation Algorithms. Cambridge University Press, New York (2011)