Chapter 9
Dynamic String-Averaging Proximal Point Algorithm

In a Hilbert space, we study the convergence of a dynamic string-averaging proximal point method to a common zero of a finite family of maximal monotone operators under the presence of computational errors. We show that the algorithm generates a good approximate solution, if the sequence of computational errors is bounded from above by a constant. Moreover, for a known computational error, we find out what an approximate solution can be obtained and how many iterates one needs for this.

9.1 Preliminaries and Main Results

Let $(X, \langle \cdot, \cdot \rangle)$ be a Hilbert space equipped with an inner product $\langle \cdot, \cdot \rangle$ which induces the complete norm $\| \cdot \|$.

For each $x \in X$ and each nonempty set $A \subset X$ put

$$d(x, A) = \inf\{\|x - y\| : \ y \in A\}.$$

For each $x \in X$ and each $r > 0$ set

$$B(x, r) := \{y \in X : \ \|x - y\| \le r\}.$$

Denote by Card(A) the cardinality of a set A. The sum over an empty set is assumed to be zero.

Recall (see Sect. 8.1) that a multifunction $T : X \to 2^X$ is called a monotone operator if

$$\langle z - z', w - w' \rangle \ \ge 0 \quad \forall z, z', w, w' \in X$$
$$\text{such that } w \in T(z) \text{ and } w' \in T(z').$$

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It is called maximal monotone if, in addition, the graph

$$\{(z, w) \in X \times X : \ w \in T(z)\}$$

is not properly contained in the graph of any other monotone operator $T' : X \to 2^X$.

Let $T : X \to 2^X$ be a maximal monotone operator. Then (see Sect. 8.1) for each $z \in X$ and each $c > 0$, there is a unique $u \in X$ such that

$$z \in (I + cT)(u),$$

where $I : X \to X$ is the identity operator ($Ix = x$ for all $x \in X$).

The operator

$$P_{c,T} := (I + cT)^{-1} \tag{9.1}$$

is therefore single-valued from all of X onto X (where c is any positive number). It is also nonexpansive:

$$\|P_{c,T}(z) - P_{c,T}(z')\| \le \|z - z'\| \text{ for all } z, z' \in X \tag{9.2}$$

and

$$P_{c,T}(z) = z \text{ if and only if } 0 \in T(z) \tag{9.3}$$

(see Sect. 8.1).

Set

$$F(T) = \{z \in X : \ 0 \in T(z)\}. \tag{9.4}$$

Let $\mathcal{L}_1$ be a finite set of maximal monotone operators $T : X \to 2^X$ and $\mathcal{L}_2$ be a finite set of mappings $T : X \to X$. We suppose that the set $\mathcal{L}_1 \cup \mathcal{L}_2$ is nonempty. (Note that one of the sets $\mathcal{L}_1$ or $\mathcal{L}_2$ may be empty.)

Let $\bar{c} \in (0, 1]$ and let $\bar{c} = 1$, if $\mathcal{L}_2 = \emptyset$.

We suppose that

$$F(T) \neq \emptyset \text{ for any } T \in \mathcal{L}_1 \tag{9.5}$$

and that for each $T \in \mathcal{L}_2$,

$$\mathrm{Fix}(T) := \{z \in X : \ T(z) = z\} \neq \emptyset, \tag{9.6}$$

$$\|z - x\|^2 \ge \|z - T(x)\|^2 + \bar{c}\|x - T(x)\|^2 \tag{9.7}$$

$$\text{for all } x \in X \text{ and all } z \in \mathrm{Fix}(T).$$

Suppose that

$$F := (\cap_{T\in\mathcal{L}_1} F(T)) \cap (\cap_{Q\in\mathcal{L}_2}\mathrm{Fix}(Q)) \neq \emptyset. \tag{9.8}$$

Let $\epsilon > 0$. For any $T \in \mathcal{L}_1$ set

$$F_\epsilon(T) = \{x \in X :\ T(x) \cap B(0,\epsilon) \neq \emptyset\} \tag{9.9}$$

and for any $T \in \mathcal{L}_2$ put

$$\mathrm{Fix}_\epsilon(T) = \{x \in X :\ \|T(x) - x\| \leq \epsilon\}. \tag{9.10}$$

Set

$$\tilde{F}_\epsilon = (\cap_{T\in\mathcal{L}_1}\{x \in X :\ d(x, F_\epsilon(T)) \leq \epsilon\})$$
$$\cap(\cap_{Q\in\mathcal{L}_2}\{x \in X :\ d(x, \mathrm{Fix}_\epsilon(Q)) \leq \epsilon\}). \tag{9.11}$$

Let $\bar{\lambda} > 0$ and let $\bar{\lambda} = \infty$ and $\bar{\lambda}^{-1} = 0$, if $\mathcal{L}_1 = \emptyset$. Set

$$\mathcal{L} = \mathcal{L}_2 \cup \{P_{c,T} :\ T \in \mathcal{L}_1,\ c \in [\bar{\lambda}, \infty)\}. \tag{9.12}$$

Next we describe the dynamic string-averaging method with variable strings and weights.

By a mapping vector, we a mean a vector $T = (T_1, \dots, T_p)$ such that $T_i \in \mathcal{L}$ for all $i = 1, \dots, p$.

For a mapping vector $T = (T_1, \dots, T_q)$ set

$$p(T) = q,\ P[T] = T_q \cdots T_1. \tag{9.13}$$

It is easy to see that for each mapping vector $T = (T_1, \dots, T_p)$,

$$P[T](x) = x \text{ for all } x \in F, \tag{9.14}$$

$$\|P[T](x) - P[T](y)\| = \|(x) - P[T](y)\| \leq \|x - y\| \tag{9.15}$$

for every $x \in F$ and every $y \in X$.

Denote by $\mathcal{M}$ the collection of all pairs (Ω, w), where Ω is a finite set of mapping vectors and

$$w : \Omega \to (0, \infty) \text{ be such that } \sum_{T\in\Omega} w(T) = 1. \tag{9.16}$$

Let $(\Omega, w) \in \mathcal{M}$. Define

$$P_{\Omega,w}(x) = \sum_{T\in\Omega} w(T)P[T](x),\ x \in X. \tag{9.17}$$

It is not difficult to see that

$$P_{\Omega,w}(x) = x \text{ for all } x \in F, \tag{9.18}$$

$$\|P_{\Omega,w}(x) - P_{\Omega,w}(y)\| = \|x - P_{\Omega,w}(y)\| \le \|x - y\| \tag{9.19}$$

for all $x \in F$ and all $y \in X$.

The dynamic string-averaging method with variable strings and variable weights can now be described by the following algorithm.

Initialization: select an arbitrary $x_0 \in X$.

Iterative step: given a current iteration vector x_k pick a pair

$$(\Omega_{k+1}, w_{k+1}) \in \mathcal{M}$$

and calculate the next iteration vector x_{k+1} by

$$x_{k+1} = P_{\Omega_{k+1},w_{k+1}}(x_k).$$

Fix a number

$$\Delta \in (0, \mathrm{Card}(\mathcal{L}_1 \cup \mathcal{L}_2)^{-1}) \tag{9.20}$$

and natural numbers $\bar{N}$ and $\bar{q}$ satisfying

$$\bar{q} \ge \mathrm{Card}(\mathcal{L}_1 \cup \mathcal{L}_2). \tag{9.21}$$

Denote by $\mathcal{M}_*$ the set of all $(\Omega, w) \in \mathcal{M}$ such that

$$p(T) \le \bar{q} \text{ for all } T \in \Omega, \tag{9.22}$$

$$w(T) \ge \Delta \text{ for all } T \in \Omega. \tag{9.23}$$

Denote by $\mathcal{R}$ the set of all sequences

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$$

such that the following properties hold:

(P1) for each integer $j \ge 1$ and each $S \in \mathcal{L}_2$ there exist $k \in \{j, \dots, j + \bar{N} - 1\}$, $T = (T_1, \dots, T_{p(T)}) \in \Omega_k$ such that

$$S \in \{T_1, \dots, T_{p(T)}\};$$

(P2) for each integer $j \ge 1$ and each $S \in \mathcal{L}_1$ there exist $k \in \{j, \dots, j + \bar{N} - 1\}$, $T = (T_1, \dots, T_{p(T)}) \in \Omega_k$ and $c \ge \bar{\lambda}$ such that

$$P_{c,S} \in \{T_1, \dots, T_{p(T)}\}.$$

In order to state our main results we need the following definitions.

Let $\delta \geq 0$, $x \in X$ and let $T = (T_1, \dots, T_{p(t)})$ be a mapping vector. Define

$$A_0(x, T, \delta) = \{(y, \lambda) \in X \times R^1 : \text{ there is a sequence } \{y_i\}_{i=0}^{p(t)} \subset X \text{ such that}$$

$$y_0 = x \text{ and for all } i = 1, \dots, p(t),$$

$$\|y_i - T_i(y_{i-1})\| \leq \delta,$$

$$y = y_{p(T)},$$

$$\lambda = \max\{\|y_i - y_{i-1}\| : i = 1, \dots, p(T)\}\}. \tag{9.24}$$

Let $\delta \geq 0$, $x \in X$ and let $(\Omega, w) \in \mathcal{M}$. Define

$$A(x, (\Omega, w), \delta) = \{(y, \lambda) \in X \times R^1 : \text{ there exist}$$

$$(y_T, \lambda_T) \in A_0(x, T, \delta), \ T \in \Omega \text{ such that}$$

$$\|y - \sum_{T \in \Omega} w(T) y_T\| \leq \delta, \ \lambda = \max\{\lambda_T : T \in \Omega\}\}. \tag{9.25}$$

In this chapter we prove the following two results.

Theorem 9.1. *Let $M > 0$ satisfy*

$$B(0, M) \cap F \neq \emptyset, \tag{9.26}$$

$\delta > 0$ satisfy

$$\delta \leq (2\bar{q}\bar{N})^{-1}, \tag{9.27}$$

a natural number n_0 satisfy

$$n_0 \geq M^2 \delta^{-1} (\bar{q} + 1)^{-1} (2M + 4)^{-1} (4\bar{N})^{-1}, \tag{9.28}$$

$$\epsilon_0 = (64 \Delta^{-1} \delta (\bar{q} + 1)(2M + 4) 4\bar{N})^{1/2} \bar{c}^{-1/2}, \tag{9.29}$$

and let

$$\epsilon_1 = (\bar{q} + 1)(\bar{N} + 4)\epsilon_0 \max\{\bar{\lambda}^{-1}, 1\}. \tag{9.30}$$

Assume that

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \in \mathcal{R}, \tag{9.31}$$

$$x_0 \in B(0, M) \text{ and } \{x_i\}_{i=1}^{\infty} \subset X, \ \{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty) \tag{9.32}$$

satisfy for each natural number i,

$$(x_i, \lambda_i) \in A(x_{i-1}, (\Omega_i, w_i), \delta). \tag{9.33}$$

Then there exists an integer $q \in [0, n_0]$ *such that*

$$\|x_i\| \le 3M + 1,\ i = 0, \dots, q\bar{N}, \tag{9.34}$$

$$\lambda_i \le \epsilon_0,\ i = q\bar{N} + 1, \dots, (q+1)\bar{N}. \tag{9.35}$$

Moreover, if an integer $q \ge 0$ *satisfies (9.35), then for each* $i = q\bar{N}, \dots, (q+1)\bar{N}$,

$$x_i \in \tilde{F}_{\epsilon_1}$$

and

$$\|x_i - x_j\| \le (\bar{q} + 1)\bar{N}\epsilon_0$$

for each $i, j \in \{q\bar{N}, \dots, (q+1)\bar{N}\}$.

Note that in Theorem 9.1 δ is the computational error made by our computer system, we obtain a point of the set $\tilde{F}_{\epsilon_1}$ and in order to obtain this point we need $n_0\bar{N}$ iterations. It is not difficult to see that $\epsilon_1 = c_1\delta^{1/2}$ and $n_0 = \lfloor c_2(\delta^{-1} \rfloor$, where c_1 and c_2 are positive constants depending on M.

Theorem 9.2. *Let* $M, \epsilon > 0$ *satisfy*

$$B(0, M) \cap F \ne \emptyset. \tag{9.36}$$

Assume that

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \in \mathcal{R}, \tag{9.37}$$

$$x_0 \in B(0, M), \{x_i\}_{i=1}^{\infty} \subset X,\ \{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty) \tag{9.38}$$

satisfy for each natural number i,

$$(x_i, \lambda_i) \in A(x_{i-1}, (\Omega_i, w_i), 0). \tag{9.39}$$

Then

$$Card(\{i \in \{0, 1, \dots\} :\ x_i \notin \tilde{F}_\epsilon\}) \le 4\bar{N}M^2\bar{c}^{-1}\Delta^{-1}\epsilon^{-2}(\bar{N} + 1)^2\bar{q}^2.$$

9.2 Proof of Theorem 9.1

By (9.26) there exists

$$z \in B(0, M) \cap F. \tag{9.40}$$

Let $k \geq 0$ be an integer. By (9.33),

$$(x_{k+1}, \lambda_{k+1}) \in A(x_k, (\Omega_{k+1}, w_{k+1}), \delta). \tag{9.41}$$

By (9.25) and (9.41) there exist

$$(y_{k,T}, \lambda_{k,T}) \in A_0(x_k, T, \delta),\ T \in \Omega_{k+1} \tag{9.42}$$

such that

$$\|x_{k+1} - \sum_{T \in \Omega_{k+1}} w_{k+1}(T) y_{k,T}\| \leq \delta, \tag{9.43}$$

$$\lambda_{k+1} = \max\{\lambda_{k,T} :\ T \in \Omega_{k+1}\}. \tag{9.44}$$

Let

$$T = (T_1, \dots, T_{p(T)}) \in \Omega_{k+1}.$$

It follows from (9.24) and (9.42) that there exists a finite sequence

$$\{y_i^{(k,T)}\}_{i=0}^{p(T)} \subset X$$

such that

$$y_0^{(k,T)} = x_k,\ y_{p(T)}^{(k,T)} = y_{k,T}, \tag{9.45}$$

$$\|y_i^{(k,T)} - T_i(y_{i-1}^{(k,T)})\| \leq \delta \text{ for each integer } i = 1, \dots, p(T), \tag{9.46}$$

$$\lambda_{k,T} = \max\{\|y_i^{(k,T)} - y_{i-1}^{(k,T)}\| :\ i = 1, \dots, p(T)\}. \tag{9.47}$$

Let

$$T = (T_1, \dots, T_{p(T)}) \in \Omega_{k+1},$$
$$i \in \{1, \dots, p(T)\}. \tag{9.48}$$

By (9.12), (9.37), (9.40), (9.48) and Lemma 8.19,

$$\|z - y_{i-1}^{(k,T)}\|^2 \geq \|z - T_i(y_{i-1}^{(k,T)})\|^2 + \bar{c}\|y_{i-1}^{(k,T)} - T_i(y_{i-1}^{(k,T)})\|^2. \tag{9.49}$$

Relations (9.46) and (9.49) imply that

$$\|z - y_i^{(k,T)}\| \le \|z - T_i(y_{i-1}^{(k,T)})\| + \|T_i(y_{i-1}^{(k,T)}) - y_i^{(k,T)}\| \\ \le \|z - y_{i-1}^{(k,T)}\| + \delta. \tag{9.50}$$

In view of (9.22), (9.45), and (9.50), for all $i = 1, \dots, p(T)$,

$$\|z - y_i^{(k,T)}\| \le \|z - x_k\| + i\delta \le \|z - x_k\| + \bar{q}\delta. \tag{9.51}$$

It follows from (9.45) and (9.51) that

$$\|z - y_{k,T}\| \le \|z - x_k\| + \bar{q}\delta. \tag{9.52}$$

By (9.16), (9.43), (9.52) and the convexity of the norm $\|\cdot\|$,

$$\|z - x_{k+1}\| \le \|z - \sum_{T \in \Omega_{k+1}} w_{k+1}(T) y_{k,T}\| + \|\sum_{T \in \Omega_{k+1}} w_{k+1}(T) y_{k,T} - x_{k+1}\| \\ \le \sum_{T \in \Omega_{k+1}} w_{k+1}(T)\|z - y_{k,T}\| + \delta \le \|z - x_k\| + (\bar{q}+1)\delta. \tag{9.53}$$

In view (9.32) and (9.40),

$$\|x_0 - z\| \le 2M. \tag{9.54}$$

Assume that a nonnegative integer s satisfies for each integer $k \in [0, s]$,

$$\max\{\lambda_i : \ i = k\bar{N}+1, \dots, (k+1)\bar{N}\} > \epsilon_0. \tag{9.55}$$

We prove the following auxiliary result.

Lemma 9.3. *Assume that an integer $k \in [0, s]$ satisfies*

$$\|x_{k\bar{N}} - z\| \le 2M \tag{9.56}$$

and that

$$i \in [0, \bar{N} - 1]. \tag{9.57}$$

Then

$$\|x_{k\bar{N}+i+1} - z\|^2 \le \|x_{k\bar{N}+i} - z\|^2 + \delta(\bar{q}+1)(4M+2) \tag{9.58}$$

and if $\lambda_{k\bar{N}+i+1} > \epsilon_0$, then

$$\|x_{k\bar{N}+i+1} - z\|^2 - \|x_{k\bar{N}+i} - z\|^2 \le -8^{-1}\Delta\epsilon_0^2\bar{c}. \tag{9.59}$$

Proof. In view of (9.27), (9.53), (9.56), and (9.57),

$$\|x_{k\bar{N}+i+1} - z\|,\ \|x_{k\bar{N}+i} - z\| \le \|x_{k\bar{N}} - z\| + (i+1)(\bar{q}+1)\delta$$
$$\le 2M + \bar{N}(\bar{q}+1)\delta \le 2M + 1. \tag{9.60}$$

By (9.53) and (9.60),

$$\|x_{k\bar{N}+i+1} - z\|^2 - \|x_{k\bar{N}+i} - z\|^2$$
$$(\|x_{k\bar{N}+i+1} - z\| - \|x_{k\bar{N}+i} - z\|)(\|x_{k\bar{N}+i+1} - z\| + \|x_{k\bar{N}+i} - z\|)$$
$$\le \delta(\bar{q}+1)(4M+2)$$

and (9.58) holds.

Assume that

$$\lambda_{k\bar{N}+i+1} > \epsilon_0. \tag{9.61}$$

In view of (9.53),

$$\|x_{k\bar{N}+i} - z\| \le \|x_{k\bar{N}} - z\| + i(\bar{q}+1)\delta. \tag{9.62}$$

Relations (9.52) and (9.62) imply that for each $T \in \Omega_{k\bar{N}+i+1}$,

$$\|z - y_{k\bar{N}+i,T}\| \le \|z - x_{k\bar{N}+i}\| + \bar{q}\delta. \tag{9.63}$$

It follows from (9.27), (9.56), (9.57), (9.62), and (9.63) that for each $T \in \Omega_{k\bar{N}+i+1}$,

$$\|z - y_{k\bar{N}+i,T}\|^2 - \|z - x_{k\bar{N}+i}\|^2$$
$$= (\|z - y_{k\bar{N}+i,T}\| - \|z - x_{k\bar{N}+i}\|)(\|z - y_{k\bar{N}+i,T}\| + \|z - x_{k\bar{N}+i}\|)$$
$$\le \bar{q}\delta(2\|z - x_{k\bar{N}+i}\| + \bar{q}\delta)$$
$$\le \bar{q}\delta(2\|z - x_{k\bar{N}}\| + 2i(\bar{q}+1)\delta + \bar{q}\delta)$$
$$\le \bar{q}\delta(4M + (\bar{q}+1)\delta(2i+1))$$
$$\le \bar{q}\delta(4M + 2(\bar{q}+1)\delta\bar{N}) \le \bar{q}\delta(4M+2). \tag{9.64}$$

In view of (9.44) and (9.61) there exists

$$S = (S_1, \dots, S_{p(S)}) \in \Omega_{k\bar{N}+i+1} \tag{9.65}$$

such that

$$\epsilon_0 < \lambda_{k\bar{N}+i+1} = \lambda_{k\bar{N}+i,S}. \tag{9.66}$$

By (9.47) and (9.66), there exists

$$j_0 \in \{1, \dots, p(S)\}$$

such that

$$\epsilon_0 < \lambda_{k\bar{N}+i,S} = \|y_{j_0}^{(k\bar{N}+i,S)} - y_{j_0-1}^{(k\bar{N}+i,S)}\|. \tag{9.67}$$

By (9.27), (9.45), (9.51), (9.53), (9.56), and (9.57), for each $j \in \{1, \dots, p(S)\}$,

$$\|z - y_j^{(k\bar{N}+i,S)}\|,\ \|z - y_{j-1}^{(k\bar{N}+i,S)}\| \le \|z - x_{k\bar{N}+i}\| + \bar{q}\delta, \tag{9.68}$$

$$\|z - y_j^{(k\bar{N}+i,S)}\| \le \|z - y_{j-1}^{(k\bar{N}+i,S)}\| + \delta, \tag{9.69}$$

$$\begin{aligned}
&\|z - y_j^{(k\bar{N}+i,S)}\|^2 - \|z - y_{j-1}^{(k\bar{N}+i,S)}\|^2 \\
&(\|z - y_j^{(k\bar{N}+i,S)}\| - \|z - y_{j-1}^{(k\bar{N}+i,S)}\|)(\|z - y_j^{(k\bar{N}+i,S)}\| + \|z - y_{j-1}^{(k\bar{N}+i,S)}\|) \\
&\le \delta(2\|z - x_{k\bar{N}+i}\| + 2\bar{q}\delta) \\
&\le \delta(2\|z - x_{k\bar{N}}\| + 2(\bar{q}+1)\delta i + 2\bar{q}\delta) \\
&\le \delta(4M + 2(\bar{q}+1)\delta(i+1)) \le \delta(4M+2).
\end{aligned} \tag{9.70}$$

In view of (9.49),

$$\begin{aligned}
&\|z - y_{j_0-1}^{(k\bar{N}+i,S)}\|^2 \\
&\ge \|z - S_{j_0}(y_{j_0-1}^{(k\bar{N}+i,S)})\|^2 + \bar{c}\|y_{j_0-1}^{(k\bar{N}+i,S)} - S_{j_0}(y_{j_0-1}^{(k\bar{N}+i,S)})\|^2.
\end{aligned} \tag{9.71}$$

It follows from (9.29), (9.46), (9.67) that

$$\begin{aligned}
&\|y_{j_0-1}^{(k\bar{N}+i,S)} - S_{j_0}(y_{j_0-1}^{(k\bar{N}+i,S)})\| \\
&\ge \|y_{j_0-1}^{(k\bar{N}+i,S)} - y_{j_0}^{(k\bar{N}+i,S)}\| - \|y_{j_0}^{(k\bar{N}+i,S)} - S_{j_0}(y_{j_0-1}^{(k\bar{N}+i,S)})\| \\
&> \epsilon_0 - \delta > \epsilon_0/2.
\end{aligned} \tag{9.72}$$

Relations (9.71) and (9.72) imply that

$$\|z - y_{j_0-1}^{(k\bar{N}+i,S)}\|^2 - \bar{c}\epsilon_0^2/4 \ge \|z - S_{j_0}(y_{j_0-1}^{(k\bar{N}+i,S)})\|^2. \tag{9.73}$$

In view of (9.73),

$$\|z - S_{j_0}(y_{j_0-1}^{(k\bar{N}+i,S)})\| \le \|z - y_{j_0-1}^{(k\bar{N}+i,S)}\|. \tag{9.74}$$

By (9.46), (9.53), (9.56), (9.68), and (9.74),

$$\begin{aligned}
&\|z - y_{j_0}^{(k\bar{N}+i,S)}\|^2 - \|z - S_{j_0}(y_{j_0-1}^{(k\bar{N}+i,S)})\|^2 \\
&= (\|z - y_{j_0}^{(k\bar{N}+i,S)}\| - \|z - S_{j_0}(y_{j_0-1}^{(k\bar{N}+i,S)})\|) \\
&\times(\|z - y_{j_0}^{(k\bar{N}+i,S)}\| + \|z - S_{j_0}(y_{j_0-1}^{(k\bar{N}+i,S)})\|) \\
&\leq \|y_{j_0}^{(k\bar{N}+i,S)} - S_{j_0}(y_{j_0-1}^{(k\bar{N}+i,S)})\|(2\|z - x_{k\bar{N}+i}\| + 2\bar{q}\delta) \\
&\leq \delta(2\|z - x_{k\bar{N}}\| + 2i(\bar{q}+1)\delta + 2\bar{q}\delta) \\
&\leq \delta(4M + 2\bar{N}(\bar{q}+1)\delta) \leq \delta(4M+2). \qquad (9.75)
\end{aligned}$$

In view of (9.73) and (9.75),

$$\begin{aligned}
\|z - y_{j_0}^{(k\bar{N}+i,S)}\|^2 &\leq \|z - S_{j_0}(y_{j_0-1}^{(k\bar{N}+i,S)})\|^2 + \delta(4M+2) \\
&\leq \|z - y_{j_0-1}^{(k\bar{N}+i,S)}\|^2 - \bar{c}\epsilon_0^2/4 + \delta(4M+2). \qquad (9.76)
\end{aligned}$$

It follows from (9.22), (9.45), (9.70), and (9.76) that

$$\begin{aligned}
&\|z - x_{k\bar{N}+i}\|^2 - \|z - y_{k\bar{N}+i,S}\|^2 \\
&= \|z - y_0^{(k\bar{N}+i,S)}\|^2 - \|z - y_{p(S)}^{(k\bar{N}+i,S)}\|^2 \\
&= \sum_{j=1}^{p(S)}(\|z - y_{j-1}^{(k\bar{N}+i,S)}\|^2 - \|z - y_j^{(k\bar{N}+i,S)}\|^2) \\
&= \sum\{\|z - y_{j-1}^{(k\bar{N}+i,S)}\|^2 - \|z - y_j^{(k\bar{N}+i,S)}\|^2 :\ j \in \{1,\dots,p(S)\} \setminus \{j_0\}\} \\
&+\|z - y_{j_0-1}^{(k\bar{N}+i,S)}\|^2 - \|z - y_{j_0}^{(k\bar{N}+i,S)}\|^2 \\
&\geq -\delta(4M+2)(p(S)-1) + \bar{c}\epsilon_0^2/4 - \delta(4M+2) \\
&\geq \bar{c}\epsilon_0^2/4 - \delta(4M+2)\bar{q}. \qquad (9.77)
\end{aligned}$$

By (9.16), (9.23), (9.64), (9.65), (9.77) and the convexity of the function $\|\cdot\|^2$,

$$\begin{aligned}
&\|z - \sum_{T\in\Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(T) y_{k\bar{N}+i,T}\|^2 \\
&\leq \sum_{T\in\Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(T)\|z - y_{k\bar{N}+i,T}\|^2 \\
&= \sum\{w_{k\bar{N}+i+1}(T)\|z - y_{k\bar{N}+i,T}\|^2 :\ T \in \Omega_{k\bar{N}+i+1} \setminus \{S\}\}
\end{aligned}$$

$$+w_{k\bar{N}+i+1}(S)\|z-y_{k\bar{N}+i,S}\|^2$$
$$\le \sum\{w_{k\bar{N}+i+1}(T)(\|z-x_{k\bar{N}+i}\|^2$$
$$+\bar{q}\delta(4M+2)) : \ T \in \Omega_{k\bar{N}+i+1} \setminus \{S\}\}$$
$$+w_{k\bar{N}+i+1}(S)(\|z-x_{k\bar{N}+i}\|^2 - \bar{c}\epsilon_0^2/4 + \delta(4M+2)\bar{q})$$
$$\le \|z-x_{k\bar{N}+i}\|^2 + \bar{q}\delta(4M+2) - \Delta\bar{c}\epsilon_0^2/4. \tag{9.78}$$

By (9.29), (9.53), (9.56), (9.57), and (9.78),

$$\|z - \sum_{T\in\Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(T)y_{k\bar{N}+i,T}\| \le \|z-x_{k\bar{N}+i}\|$$
$$\le \|z-x_{k\bar{N}}\| + i(\bar{q}+1)\delta \le 2M + \delta(\bar{q}+1)(\bar{N}-1). \tag{9.79}$$

In view of (9.79),

$$\|z-x_{k\bar{N}+i+1}\|$$
$$\le \|z - \sum_{T\in\Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(T)y_{k\bar{N}+i,T}\|$$
$$+\|\sum_{T\in\Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(T)y_{k\bar{N}+i,T} - x_{k\bar{N}+i+1}\|$$
$$\le 2M + \delta(\bar{q}+1)(\bar{N}-1) + \delta. \tag{9.80}$$

It follows from (9.27), (9.43), (9.79), and (9.80) that

$$\|z-x_{k\bar{N}+i+1}\|^2 - \|z - \sum_{T\in\Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(T)y_{k\bar{N}+i,T}\|^2$$
$$= (\|z-x_{k\bar{N}+i+1}\| - \|z - \sum_{T\in\Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(T)y_{k\bar{N}+i,T}\|)$$
$$\times(\|z-x_{k\bar{N}+i+1}\| + \|z - \sum_{T\in\Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(T)y_{k\bar{N}+i,T}\|)$$
$$\le \|x_{k\bar{N}+i+1} - \sum_{T\in\Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(T)y_{k\bar{N}+i,T}\|(4M+2) \le \delta(4M+2). \tag{9.81}$$

By (9.29), (9.78), and (9.81),

$$\|z-x_{k\bar{N}+i+1}\|^2 \le \|z - \sum_{T\in\Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(T)y_{k\bar{N}+i,T}\| + \delta(4M+2)$$

$$\leq \|z - x_{k\bar{N}+i}\|^2 + (\bar{q}+1)\delta(4M+2) - \Delta\bar{c}\epsilon_0^2/4$$
$$\leq \|z - x_{k\bar{N}+i}\|^2 - \Delta\bar{c}\epsilon_0^2/8.$$

Lemma 9.3 is proved. □

Lemma 9.4. *Assume that an integer* $k \in [0, s]$ *satisfies*

$$\|x_{k\bar{N}} - z\| \leq 2M.$$

Then

$$\|x_{k\bar{N}} - z\|^2 - \|x_{(k+1)\bar{N}} - z\|^2 \geq \Delta\bar{c}\epsilon_0^2/16.$$

Proof. By Lemma 9.3, for all $i \in [0, \bar{N}-1]$,

$$\|x_{k\bar{N}+i+1} - z\|^2 \leq \|x_{k\bar{N}+i} - z\|^2 + \delta(\bar{q}+1)(4M+2). \tag{9.82}$$

In view of (9.55), there exists

$$j_0 \in \{0, \dots, \bar{N}-1\} \tag{9.83}$$

such that

$$\lambda_{k\bar{N}+j_0+1} > \epsilon_0. \tag{9.84}$$

Lemma 9.3, (9.29), (9.82), (9.83), and (9.84) imply that

$$\|x_{k\bar{N}} - z\|^2 - \|x_{(k+1)\bar{N}} - z\|^2$$
$$= \sum_{i=0}^{\bar{N}-1} (\|x_{k\bar{N}+i} - z\|^2 - \|x_{k\bar{N}+i+1} - z\|^2)$$
$$\geq -(\bar{N}-1)\delta(\bar{q}+1)(4M+2) + 8^{-1}\Delta\epsilon_0^2\bar{c} \geq 16^{-1}\Delta\epsilon_0^2\bar{c}.$$

Lemma 9.4 is proved. □

By (9.54) and Lemma 9.4 applied by induction, for all integers $k = 0, \dots, s$,

$$\|x_{k\bar{N}} - z\|^2 - \|x_{(k+1)\bar{N}} - z\|^2 \geq 16^{-1}\Delta\epsilon_0^2\bar{c}, \tag{9.85}$$

$$\|x_{(k+1)\bar{N}} - z\| \leq \|x_{k\bar{N}} - z\| \leq 2M. \tag{9.86}$$

It follows from (9.27), (9.53), and (9.86) that for all integers $k = 0, \dots, s$ and all $i = 0, \dots, \bar{N}$,

$$\|x_{k\bar{N}+i} - z\| \leq \|x_{k\bar{N}} - z\| + i\delta(\bar{q}+1) \leq \|x_{k\bar{N}} - z\| + \bar{N}\delta(\bar{q}+1) \leq 2M + 1.$$

Together with (9.40) this implies that for all $i = 0, \dots, (s+1)\bar{N}$,

$$\|x_i - z\| \le 2M + 1, \ \|x_i\| \le 3M + 1. \tag{9.87}$$

Relations (9.54) and (9.85) imply that

$$4M^2 \ge \|x_0 - z\|^2 \ge \|x_0 - z\|^2 - \|x_{(s+1)\bar{N}} - z\|^2$$

$$= \sum_{k=0}^{s} (\|x_{k\bar{N}} - z\|^2 - \|x_{(k+1)\bar{N}} - z\|^2) \ge 16^{-1}(s+1)\Delta\epsilon_0^2\bar{c},$$

$$s + 1 \le 64M^2\Delta^{-1}\bar{c}^{-1}\epsilon_0^{-2}.$$

Thus we have shown that the following property holds:

(P3) if an integer $s \ge 0$ and for each integer $k \in [0, s]$, (9.55) holds, then (see (9.87))

$$s \le 64M^2\Delta^{-1}\bar{c}^{-1}\epsilon_0^{-2} - 1,$$

$$\|x_k\| \le 3M + 1, \ k = 0, \dots, (s+1)\bar{N}.$$

Property (P3), (9.28), (9.29), and (9.55) imply that there exists an integer $q \in \{0, \dots, n_0\}$ such that for each integer k satisfying $0 \le k < q$,

$$\max\{\lambda_i : \ i = k\bar{N} + 1, \dots, (k+1)\bar{N}\} > \epsilon_0,$$

$$\max\{\lambda_i : \ i = q\bar{N} + 1, \dots, (q+1)\bar{N}\} \le \epsilon_0.$$

By property (P3), (9.40), (9.54), (9.55), the choice of q and the inequalities above,

$$\|x_k\| \le 3M + 1, \ k = 0, \dots, q\bar{N}.$$

Assume that $q \ge 0$ is an integer and that

$$\lambda_i \le \epsilon_0, \ i = q\bar{N} + 1, \dots, (q+1)\bar{N}. \tag{9.88}$$

In view of (9.44), (9.47), and (9.88), for each $j \in \{q\bar{N}, \dots, (q+1)\bar{N} - 1\}$, each $T = (T_1, \dots, T_{p(T)}) \in \Omega_{j+1}$ and each $i \in \{1, \dots, p(T)\}$,

$$\lambda_{j+1} \le \epsilon_0, \ \lambda_{j,T} \le \epsilon_0, \tag{9.89}$$

$$\|y_{i-1}^{(j,T)} - y_i^{(j,T)}\| \le \epsilon_0. \tag{9.90}$$

By (9.22), (9.45), and (9.90), for each $j \in \{q\bar{N}, \dots, (q+1)\bar{N} - 1\}$, each $T = (T_1, \dots, T_{p(T)}) \in \Omega_{j+1}$ and each $i \in \{1, \dots, p(T)\}$,

$$\|x_j - y_i^{(j,T)}\| \le \epsilon_0 i \le \epsilon_0 \bar{q}, \tag{9.91}$$

$$\|x_j - y_{j,T}\| \le \epsilon_0 \bar{q}. \tag{9.92}$$

It follows from (9.29), (9.46), and (9.91) that for each $j \in \{q\bar{N}, \dots, (q+1)\bar{N} - 1\}$, each $T = (T_1, \dots, T_{p(T)}) \in \Omega_{j+1}$ and each $i \in \{1, \dots, p(T)\}$ that

$$\|x_j - T_i(y_{i-1}^{(j,T)})\| \le \|x_j - y_i^{(j,T)}\| + \|y_i^{(j,T)} - T_i(y_{i-1}^{(j,T)})\| \le \epsilon_0 \bar{q} + \delta \le \epsilon_0(\bar{q}+1). \tag{9.93}$$

By (9.16), (9.29), (9.43), (9.92) and the convexity of the norm, for each $j \in \{q\bar{N}, \dots, (q+1)\bar{N} - 1\}$,

$$\|x_{j+1} - x_j\| \le \|x_{j+1} - \sum_{T \in \Omega_{j+1}} w_{j+1}(T) y_{j,T}\| + \|\sum_{T \in \Omega_{j+1}} w_{j+1}(T) y_{j,T} - x_j\|$$

$$\le \delta + \sum_{T \in \Omega_{j+1}} w_{j+1}(T) \|y_{j,T} - x_j\| \le \delta + \epsilon_0 \bar{q} \le \epsilon_0(\bar{q} + 1). \tag{9.94}$$

In view of (9.91) and (9.93), for each $j \in \{q\bar{N}, \dots, (q+1)\bar{N} - 1\}$, each $T = (T_1, \dots, T_{p(T)}) \in \Omega_{j+1}$ and each $i \in \{1, \dots, p(T)\}$,

$$\|y_{i-1}^{(j,T)} - T_i(y_{i-1}^{(j,T)})\| \le \|y_{i-1}^{(j,T)} - x_j\| + \|x_j - T_i(y_{i-1}^{(j,T)})\| \le \epsilon_0(2\bar{q} + 1). \tag{9.95}$$

Relation (9.94) implies that for all $j_1, j_2 \in \{q\bar{N}, \dots, (q+1)\bar{N}\}$,

$$\|x_{j_1} - x_{j_2}\| \le \epsilon_0 \bar{N}(\bar{q} + 1). \tag{9.96}$$

Let

$$Q \in \mathcal{L}_2. \tag{9.97}$$

Property (P1), (9.31) and (9.97) imply that there exist $j \in \{q\bar{N}, \dots, (q+1)\bar{N} - 1\}$, $T = (T_1, \dots, T_{p(T)}) \in \Omega_{j+1}$ and $s \in \{1, \dots, p(T)\}$ such that

$$Q = T_s. \tag{9.98}$$

In view of (9.95) and (9.98),

$$y_{s-1}^{(j,T)} \in \mathrm{Fix}_{\epsilon_0(2\bar{q}+1)}(Q). \tag{9.99}$$

By (9.91) and (9.99),

$$d(x_j, \mathrm{Fix}_{\epsilon_0(2\bar{q}+1)}(Q)) \le \|x_j - y_{s-1}^{(j,T)}\| \le \epsilon_0 \bar{q}.$$

Together with (9.30) and (9.96) this implies that for all $i \in \{q\bar{N}, \dots, (q+1)\bar{N}\}$,

$$d(x_i, \mathrm{Fix}_{\epsilon_0(2\bar{q}+1)}(Q)) \le \epsilon_0(\bar{q}+1)(\bar{N}+1) \le \epsilon_1 \tag{9.100}$$

for all $Q \in \mathcal{L}_2$.

Let

$$Q \in \mathcal{L}_1. \tag{9.101}$$

Property (P2), (9.31) and (9.101) imply that there exist $j \in \{q\bar{N}, \dots, (q+1)\bar{N}-1\}$, $T = (T_1, \dots, T_{p(T)}) \in \Omega_{j+1}$,

$$s \in \{1, \dots, p(T)\},\ c \ge \bar{\lambda} \tag{9.102}$$

such that

$$P_{Q,c} = T_s. \tag{9.103}$$

By (9.95) and (9.101)–(9.103),

$$\|y_{s-1}^{(j,T)} - P_{Q,c}(y_{s-1}^{(j,T)})\| \le \epsilon_0(2\bar{q}+1). \tag{9.104}$$

Set

$$\xi = P_{Q,c}(y_{s-1}^{(j,T)}). \tag{9.105}$$

In view of (9.1) and (9.105),

$$\begin{aligned} y_{s-1}^{(j,T)} &\in (I + cQ)(\xi), \\ y_{s-1}^{(j,T)} - \xi &\in cQ(\xi), \\ c^{-1}(y_{s-1}^{(j,T)} - \xi) &\in Q(\xi). \end{aligned} \tag{9.106}$$

It follows from (9.29), (9.30), (9.46), (9.90), (9.102), (9.103), and (9.105) that

$$\begin{aligned} &\|c^{-1}(y_{s-1}^{(j,T)} - \xi)\| \le \bar{\lambda}^{-1}(\|y_{s-1}^{(j,T)} - y_s^{(j,T)}\| + \|y_s^{(j,T)} - \xi)\|) \\ &= \bar{\lambda}^{-1}(\epsilon_0 + \|y_s^{(j,T)} - T_s(y_{s-1}^{(j,T)})\|) \le \bar{\lambda}^{-1}(\epsilon_0 + \delta) \le 2\bar{\lambda}^{-1}\epsilon_0 \le \epsilon_1. \end{aligned} \tag{9.107}$$

By (9.106) and (9.107),

$$\xi \in F_{\epsilon_1}(Q). \tag{9.108}$$

In view of (9.104), (9.105) and (9.108),

$$d(y_{s-1}^{(j,T)}, F_{\epsilon_1}(Q)) \le \epsilon_0(2\bar{q}+1). \tag{9.109}$$

It follows from (9.91) and (9.109) that

$$d(x_j, F_{\epsilon_1}(Q)) \le \|x_j - y_{s-1}^{(j,T)}\| + d(y_{s-1}^{(j,T)}, F_{\epsilon_1}(Q))$$
$$\le \epsilon_0 \bar{q} + \epsilon_0(2\bar{q}+1) = \epsilon_0(3\bar{q}+1).$$

Together with (9.30) and (9.96) this implies that for all $i \in \{q\bar{N}, \dots, (q+1)\bar{N}\}$,

$$d(x_i, F_{\epsilon_1}(Q)) \le \epsilon_0(\bar{q}+1)(\bar{N}+4) \le \epsilon_1$$

for all $Q \in \mathcal{L}_1$. This implies that

$$x_i \in \tilde{F}_{\epsilon_1} \text{ for all } i \in \{q\bar{N}, \dots, (q+1)\bar{N}\}.$$

Theorem 9.1 is proved. □

9.3 Proof of Theorem 9.2

Set

$$\gamma_0 = \epsilon(\bar{N}+1)^{-1}\bar{q}^{-1}\min\{1, \bar{\lambda}\}. \tag{9.110}$$

By (9.36) there exists

$$z \in B(0, M) \cap F. \tag{9.111}$$

Let $k \ge 0$ be an integer. By (9.39),

$$(x_{k+1}, \lambda_{k+1}) \in A(x_k, (\Omega_{k+1}, w_{k+1}), 0). \tag{9.112}$$

By (9.25) and (9.112) there exists

$$(y_{k,T}, \lambda_{k,T}) \in A_0(x_k, T, 0),\ T \in \Omega_{k+1} \tag{9.113}$$

such that

$$x_{k+1} = \sum_{T \in \Omega_{k+1}} w_{k+1}(T) y_{k,T}, \tag{9.114}$$

$$\lambda_{k+1} = \max\{\lambda_{k,T} :\ T \in \Omega_{k+1}\}. \tag{9.115}$$

Let

$$T = (T_1, \dots, T_{p(T)}) \in \Omega_{k+1}.$$

It follows from (9.24) and (9.113) that there exists $\{y_i^{(k,T)}\}_{i=0}^{p(T)} \subset X$ such that

$$y_0^{(k,T)} = x_k,\ y_{p(T)}^{(k,T)} = y_{k,T}, \tag{9.116}$$

$$y_i^{(k,T)} = T_i(y_{i-1}^{(k,T)}) \text{ for each integer } i = 1, \dots, p(T), \tag{9.117}$$

$$\lambda_{k,T} = \max\{\|y_i^{(k,T)} - y_{i-1}^{(k,T)}\| :\ i = 0, \dots, p(T)\}. \tag{9.118}$$

Let

$$T = (T_1, \dots, T_{p(T)}) \in \Omega_{k+1},$$
$$i \in \{1, \dots, p(T)\}. \tag{9.119}$$

By (9.12), (9.37), (9.111), (9.119), (9.117) and Lemma 8.19,

$$\|z - y_{i-1}^{(k,T)}\|^2 \geq \|z - T_i(y_{i-1}^{(k,T)})\|^2 + \bar{c}\|y_{i-1}^{(k,T)} - T_i(y_{i-1}^{(k,T)})\|^2$$
$$= \|z - y_i^{(k,T)}\|^2 + \bar{c}\|y_{i-1}^{(k,T)} - y_i^{(k,T)})\|^2. \tag{9.120}$$

It follows from (9.116), (9.118), and (9.120) that

$$\|z - x_k\|^2 - \|z - y_{k,T}\|^2 = \|z - y_0^{(k,T)}\|^2 - \|z - y_{p(T)}^{(k,T)}\|^2$$
$$= \sum_{j=1}^{p(T)} (\|z - y_{j-1}^{(k,T)}\|^2 - \|z - y_j^{(k,T)}\|^2)$$
$$\geq \bar{c} \sum_{j=1}^{p(T)} \|y_{j-1}^{(k,T)} - y_j^{(k,T)}\|^2 \geq \bar{c}\lambda_{k,T}^2. \tag{9.121}$$

By (9.16), (9.23), (9.114), (9.121), (9.145) and the convexity of the function $\|\cdot\|^2$,

$$\|z - x_{k+1}\|^2 = \|z - \sum_{T \in \Omega_{k+1}} w_{k+1}(T) y_{k,T}\|^2$$
$$\leq \sum_{T \in \Omega_{k+1}} w_{k+1}(T)\|z - y_{k,T}\|^2$$
$$\leq \sum_{T \in \Omega_{k+1}} w_{k+1}(T)(\|z - x_k\|^2 - \bar{c}\lambda_{k,T}^2)$$
$$\leq \|z - x_k\|^2 - \bar{c}\Delta \sum_{T \in \Omega_{k+1}} \lambda_{k,T}^2 \leq \|z - x_k\|^2 - \bar{c}\Delta\lambda_{k+1}^2. \tag{9.122}$$

In view of (9.38), (9.111), and (9.122), for each natural number n,

$$4M^2 \geq \|z - x_0\|^2 \geq \|z - x_0\|^2 - \|z - x_n\|^2$$
$$= \sum_{i=0}^{n-1} [\|z - x_i\|^2 - \|z - x_{i+1}\|^2]$$
$$\geq \sum_{i=0}^{n-1} \bar{c}\Delta\lambda_{i+1}^2 \geq \bar{c}\Delta\gamma_0^2 \mathrm{Card}(\{i \in \{0, \ldots, n-1\} : \lambda_{i+1} \geq \gamma_0\}).$$

Since the relation above holds for any natural number n we conclude that

$$\mathrm{Card}(\{i \in \{0, 1, \ldots\} : \lambda_{i+1} \geq \gamma_0\}) \leq 4M^2 \bar{c}^{-1} \Delta^{-1} \gamma_0^{-2}. \tag{9.123}$$

Set

$$E = \{k \in \{0, 1, \ldots\} : \text{ there is an integer } i \in [k, k + \bar{N} - 1] \text{ such that } \lambda_{i+1} \geq \gamma_0\}. \tag{9.124}$$

By (9.123) and (9.124),

$$\mathrm{Card}(E) \leq 4M^2 \bar{N} \bar{c}^{-1} \Delta^{-1} \gamma_0^{-2}$$
$$\leq 4M^2 \bar{N} \bar{c}^{-1} \Delta^{-1} \epsilon^{-2} \bar{q}^2 (\bar{N} + 1)^2 (\min\{1, \bar{\lambda}\})^{-2}. \tag{9.125}$$

Assume that an integer $q \geq 0$ satisfies

$$q \notin E. \tag{9.126}$$

In view of (9.124) and (9.126),

$$\lambda_{k+1} < \gamma_0 \text{ for all } k \in \{q, \ldots, q + \bar{N} - 1\}. \tag{9.127}$$

It follows from (9.22), (9.115)–(9.118), (9.127), and the convexity of the norm that for each $k \in \{q, \ldots, q + \bar{N} - 1\}$, each $T = (T_1, \ldots, T_{p(T)}) \in \Omega_{k+1}$ and each $j \in \{1, \ldots, p(T)\}$,

$$\gamma_0 > \|y_j^{(k,T)} - y_{j-1}^{(k,T)}\| = \|y_{j-1}^{(k,T)} - T_j(y_{j-1}^{(k,T)})\|, \tag{9.128}$$

$$\|x_k - y_j^{(k,T)}\|, \ \|x_k - y_{j-1}^{(k,T)}\| \leq \gamma_0 j \leq \bar{q}\gamma_0, \tag{9.129}$$

$$\|x_k - x_{k+1}\| \leq \bar{q}\gamma_0. \tag{9.130}$$

Relation (9.130) implies that for each $k_1, k_2 \in \{q, \ldots, q + \bar{N}\}$,

$$\|x_{k_1} - x_{k_2}\| \leq \gamma_0 \bar{N} \bar{q}. \tag{9.131}$$

Let

$$Q \in \mathcal{L}_2. \tag{9.132}$$

Property (P1) and (9.132) imply that there exist

$$k \in \{q, \dots, q+\bar{N}-1\},\ T = (T_1, \dots, T_{p(T)}) \in \Omega_{k+1},\ s \in \{1, \dots, p(T)\} \tag{9.133}$$

such that

$$Q = T_s. \tag{9.134}$$

In view of (9.128), (9.133), and (9.134),

$$y_{s-1}^{(k,T)} \in \mathrm{Fix}_{\gamma_0}(Q). \tag{9.135}$$

By (9.129), (9.133), and (9.135),

$$d(x_k, \mathrm{Fix}_{\gamma_0}(Q)) \le \|x_k - y_{s-1}^{(k,T)}\| \le \gamma_0 \bar{q}. \tag{9.136}$$

It follows from (9.110), (9.131), (9.133), and (9.136) that

$$d(x_q, \mathrm{Fix}_{\gamma_0}(Q)) \le \|x_q - x_k\| + d(x_k, \mathrm{Fix}_{\gamma_0}(Q)) \le \bar{N}\bar{q}\gamma_0 + \bar{q}\gamma_0$$

and

$$d(x_q, \mathrm{Fix}_{\epsilon}(Q)) \le \epsilon \text{ for all } Q \in \mathcal{L}_2. \tag{9.137}$$

Let

$$Q \in \mathcal{L}_1. \tag{9.138}$$

Property (P2), (9.37), and (9.138) imply that there exist

$$k \in \{q, \dots, q + \bar{N} - 1\},\ T = (T_1, \dots, T_{p(T)}) \in \Omega_{k+1},\ s \in \{1, \dots, p(T)\},\ c \ge \bar{\lambda} \tag{9.139}$$

such that

$$P_{Q,c} = T_s. \tag{9.140}$$

By (9.117), (9.128), (9.139), and (9.140),

$$\gamma_0 > \|y_{s-1}^{(k,T)} - T_s(y_{s-1}^{(k,T)})\| = \|y_{s-1}^{(k,T)} - P_{Q,c}(y_{s-1}^{(k,T)})\|. \tag{9.141}$$

By (9.1), (9.117), (9.139), and (9.140),

$$y_{s-1}^{(k,T)} \in (I + cQ)(y_s^{(k,T)}). \tag{9.142}$$

In view of (9.142),

$$y_{s-1}^{(k,T)} - y_s^{(k,T)} \in cQ(y_s^{(k,T)}),$$

$$c^{-1}(y_{s-1}^{(j,T)} - y_s^{(k,T)}) \in Q(y_s^{(k,T)}). \tag{9.143}$$

It follows from (9.110), (9.128), (9.139), and (9.143) that

$$\|c^{-1}(y_{s-1}^{(k,T)} - y_s^{(k,T)})\| \le \bar{\lambda}^{-1}\gamma_0,$$

$$y_s^{(k,T)} \in F_{\bar{\lambda}^{-1}\gamma_0}(Q) \subset F_\epsilon(Q). \tag{9.144}$$

In view of (9.129), (9.139), and (9.144),

$$d(x_k, F_\epsilon(Q)) \le \|x_k - y_s^{(k,T)}\| \le \bar{q}\gamma_0. \tag{9.145}$$

By (9.111), (9.131), and (9.145),

$$d(x_q, F_\epsilon(Q)) \le \|x_q - x_k\| + d(x_k, F_\epsilon(Q))$$

$$\le \gamma_0\bar{q}\bar{N} + \gamma_0\bar{q},$$

$$d(x_q, F_\epsilon(Q)) \le \epsilon \text{ for all } Q \in \mathcal{L}_1.$$

Together with (9.137) this implies that

$$x_q \in \tilde{F}_\epsilon.$$

Theorem 9.2 is proved. □