Chapter 10
Convex Feasibility Problems

We use subgradient projection algorithms for solving convex feasibility problems. We show that almost all iterates, generated by a subgradient projection algorithm in a Hilbert space, are approximate solutions. Moreover, we obtain an estimate of the number of iterates which are not approximate solutions. In a finite-dimensional case, we study the behavior of the subgradient projection algorithm in the presence of computational errors. Provided computational errors are bounded, we prove that our subgradient projection algorithm generates a good approximate solution after a certain number of iterates.

10.1 Iterative Methods in Infinite-Dimensional Spaces

Let $(X, \langle \cdot, \cdot \rangle)$ be a Hilbert space with an inner product $\langle \cdot, \cdot \rangle$, which induces a complete norm $\| \cdot \|$. For each $x \in X$ and each nonempty set $A \subset X$ put

$$d(x, A) := \inf\{||x - y|| : \ y \in A\}.$$

For each $x \in X$ and each $r > 0$ set

$$B(x, r) = \{y \in X : \ ||x - y|| \leq r\}.$$

It is well known that the following proposition holds (see Fact 1.5 and Lemma 2.4 of [7]).

Proposition 10.1. *Let C be a nonempty, closed and convex subset of X. Then, for each $x \in X$, there is a unique point $P_C(x) \in C$ satisfying*

$$\|x - P_C(x)\| = d(x, C).$$

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Moreover, $\|P_C(x) - P_C(y)\| \le \|x - y\|$ *for all* $x, y \in X$ *and, for each* $x \in X$ *and each* $z \in C$*,*

$$\langle z - P_C(x), x - P_C(x)\rangle \le 0,$$
$$\|z - P_C(x)\|^2 + \|x - P_C(x)\|^2 \le \|z - x\|^2. \tag{10.1}$$

Let $f : X \to R^1$ be a continuous and convex function such that

$$\{x \in X :\ f(x) \le 0\} \ne \emptyset. \tag{10.2}$$

Let $y_0 \in X$. Then the set

$$\partial f(y_0) := \{l \in X :\ f(y) - f(y_0) \ge \langle l, y - y_0\rangle \text{ for all } y \in X\} \tag{10.3}$$

is the subdifferential of f at the point y_0 [72, 77]. For any $l \in \partial f(y_0)$, in view of (10.3),

$$\{x \in X :\ f(x) \le 0\} \subset \{x \in X :\ f(y_0) + \langle l, x - y_0\rangle \le 0\}. \tag{10.4}$$

It is well known that the following lemma holds (see Lemma 7.3 of [7]).

Lemma 10.2. *Let* $y_0 \in X$*,* $f(y_0) > 0$*,* $l \in \partial f(y_0)$ *and let*

$$D := \{x \in X :\ f(y_0) + \langle l, x - y_0\rangle \le 0\}.$$

Then $l \ne 0$ *and* $P_D(y_0) = y_0 - f(y_0)\|l\|^{-2}l$*.*

Denote by $\mathcal{N}$ the set of all nonnegative integers. Let m be a natural number, $\mathbb{I} = \{1, \ldots, m\}$ and $f_i : X \to R^1$, $i \in \mathbb{I}$, be convex and continuous functions. For each $i \in \mathbb{I}$ set

$$C_i := \{x \in X :\ f_i(x) \le 0\},$$
$$C := \cap_{i \in \mathbb{I}} C_i = \cap_{i \in \mathbb{I}} \{x \in X :\ f_i(x) \le 0\}.$$

Suppose that

$$C \ne \emptyset.$$

A point $x \in C$ is called a solution of our feasibility problem. For a given $\epsilon > 0$, a point $x \in X$ is called an ϵ-approximate solution of the feasibility problem if $f_i(x) \le \epsilon$ for all $i \in \mathbb{I}$. We apply the subgradient projection method in order to obtain a good approximative solution of the feasibility problem.

Consider a natural number $\bar{p} \ge m$. Denote by $\mathbb{S}$ the set of all mappings $S : \mathcal{N} \to \mathbb{I}$ such that the following property holds:

(P1) For each integer $N \in \mathcal{N}$ and each $i \in \mathbb{I}$, there is $n \in \{N, \dots, N + \bar{p} - 1\}$ such that $S(n) = i$.

We want to find approximate solutions of the inclusion $x \in C$. In order to meet this goal we apply algorithms generated by $S \in \mathbb{S}$.

For each $x \in X$, each number $\epsilon \geq 0$ and each $i \in \mathbb{I}$ set

$$A_i(x, \epsilon) := \{x\} \text{ if } f_i(x) \leq \epsilon \tag{10.5}$$

and, in view of Lemma 10.2,

$$A_i(x, \epsilon) := x - f_i(x)\{\|l\|^{-2}l : \ l \in \partial f_i(x)\} \text{ if } f_i(x) > \epsilon. \tag{10.6}$$

We associate with any $S \in \mathbb{S}$ the algorithm which generates, for any starting point $x_0 \in X$, a sequence $\{x_n\}_{n=0}^{\infty} \subset X$ such that, for each integer $n \geq 0$,

$$x_{n+1} \in A_{S(n)}(x_n, 0).$$

Note that by Lemma 10.2 the sequence $\{x_n\}_{n=0}^{\infty}$ is well defined, and that for each integer $n \geq 0$, if $f_{S(n)}(x_n) > 0$, then $x_{n+1} = P_{D_n}(x_n)$, where

$$D_n = \{x \in X : \ f(x_n) + \langle l_n, x - x_n \rangle \leq 0\} \text{ and } l_n \in \partial f_{S(n)}(x_n).$$

We will prove the following result (Theorem 10.3) which shows that, for the subgradient projection method considered in the chapter, almost all iterates are good approximate solutions. Denote by Card(A) the cardinality of the set A.

Theorem 10.3. *Let*

$$b > 0, \ \epsilon \in (0, 1], \ \Lambda > 0, \ \gamma \in [0, \epsilon], \tag{10.7}$$

$$c \in B(0, b) \cap C, \tag{10.8}$$

$$|f_i(u) - f_i(v)| \leq \Lambda \|u - v\|, \ u, v \in B(0, 3b + 1), \ i \in \mathbb{I}, \tag{10.9}$$

let a positive number ϵ_0 satisfy

$$\epsilon_0 \leq \epsilon \Lambda^{-1} \tag{10.10}$$

and let a natural number n_0 satisfy

$$4\bar{p}\epsilon_0^{-2}b^2 \leq n_0. \tag{10.11}$$

Assume that

$$S \in \mathbb{S}, \ x_0 \in B(0, b), \tag{10.12}$$

and that for each integer $n \geq 0$,

$$x_{n+1} \in A_{S(n)}(x_n, \gamma). \tag{10.13}$$

Then

$$\|x_n\| \leq 3b \text{ for all integers } n \geq 0 \tag{10.14}$$

and

$$Card(\{N \in \mathcal{N} : \max\{\|x_{n+1}-x_n\| : n = N, \ldots, N+\bar{p}-1\} > \epsilon_0\}) \leq n_0. \tag{10.15}$$

Moreover, if an integer $N \geq 0$ *satisfies*

$$\|x_{n+1} - x_n\| \leq \epsilon_0, \; n = N, \ldots, N + \bar{p} - 1,$$

then, for all integers $n, m \in \{N, \ldots, N + \bar{p}\}$, $\|x_n - x_m\| \leq \bar{p}\epsilon_0$ *and for all integers* $n = N, \ldots, N + \bar{p}$ *and each* $i \in \mathbb{I}$, $f_i(x_n) \leq \epsilon(\bar{p} + 1)$.

Theorem 10.3 was obtained in [96].

10.2 Proof of Theorem 10.3

By (10.5), (10.6), and (10.13), there exists a sequence $\{l_n\}_{n=0}^{\infty} \subset X$ such that

$$x_{n+1} = x_n \text{ if } f_{S(n)}(x_n) \leq \gamma \tag{10.16}$$

and

$$\text{if } f_{S(n)}(x_n) > \gamma, \text{ then } l_n \in \partial f_{S(n)}(x_n) \text{ and}$$
$$x_{n+1} = x_n - f_{S(n)}(x_n)\|l_n\|^{-2} l_n. \tag{10.17}$$

By (10.16), (10.17), (10.8), (10.12), (10.4), Lemma 10.2, and Proposition 10.1 for all integers $n \geq 0$,

$$\|c - x_{n+1}\| \leq \|c - x_n\| \leq 2b, \tag{10.18}$$

$$\|x_n\| \leq 3b \text{ for all integers } n \geq 0. \tag{10.19}$$

Assume that an integer $N \geq 0$ and that

$$\|x_{n+1} - x_n\| \leq \epsilon_0 \text{ for } n = N, \ldots, N + \bar{p} - 1. \tag{10.20}$$

This implies that for all $n, m \in \{N, \dots, N+\bar{p}\}$,

$$\|x_n - x_m\| \le \bar{p}\epsilon_0. \tag{10.21}$$

Let $i \in \mathbb{I}$. By (P1), there is $m \in \{N, \dots, N+\bar{p}-1\}$ such that

$$S(m) = i. \tag{10.22}$$

We show that

$$f_i(x_m) = f_{S(m)}(x_m) \le \epsilon. \tag{10.23}$$

Assume the contrary. Then

$$f_i(x_m) > \epsilon. \tag{10.24}$$

By (10.20), (10.24), (10.7), (10.16), (10.17), and (10.22),

$$\epsilon_0 \ge \|x_{m+1} - x_m\| = \|f_{S(m)}(x_m)\|l_m\|^{-2}l_m\| > \epsilon\|l_m\|^{-1}. \tag{10.25}$$

By (10.24), (10.22), (10.16), (10.17), and (10.7), $l_m \in \partial f_{S(m)}(x_m)$. Combined with (10.19) and (10.9) this implies that

$$\|l_m\| \le \Lambda. \tag{10.26}$$

In view of (10.25) and (10.26), $\epsilon_0 > \epsilon\Lambda^{-1}$. This inequality contradicts (10.10). This contradiction proves (10.23).

Let $n \in \{N, \dots, N+\bar{p}\}$. It follows from (10.22), (10.23), (10.19), (10.9), (10.21), and (10.10) that

$$\begin{aligned} f_i(x_n) &\le f_{S(m)}(x_m) + |f_{S(m)}(x_n) - f_{S(m)}(x_m)| \\ &\le \epsilon + \Lambda\|x_n - x_m\| \le \epsilon + \Lambda\bar{p}\epsilon_0 \le \epsilon(\bar{p}+1) \end{aligned}$$

and

$$f_i(x_n) \le \epsilon(\bar{p}+1) \text{ for } n = N, \dots, N+\bar{p}, \tag{10.27}$$

for all integers $i \in \mathbb{I}$.

Thus we have shown that the following property holds:

(P2) if an integer $N \ge 0$ and (10.20) holds, then (10.27) is valid for all $i \in \mathbb{I}$.

Set

$$E_1 = \{n \in \mathcal{N} : \|x_n - x_{n+1}\| \le \epsilon_0\}, \tag{10.28}$$

$$E_2 = \mathcal{N} \setminus E_1, \tag{10.29}$$

$$E_3 = \{n \in \mathcal{N} : \{n, \dots, n+\bar{p}-1\} \cap E_2 \ne \emptyset\}. \tag{10.30}$$

By (10.18), (10.29), (10.28), (10.16), (10.17), (10.8), Lemma 10.2, and Proposition 10.1 (see (10.1)), for any natural number n,

$$4b^2 \geq \|c - x_0\|^2 \geq \|c - x_0\|^2 - \|c - x_n\|^2$$

$$= \sum_{m=0}^{n-1} [\|c - x_m\|^2 - \|c - x_{m+1}\|^2] \geq \sum_{m \in E_2 \cap [0, n-1]} [\|c - x_m\|^2 - \|c - x_{m+1}\|^2]$$

$$\geq \sum_{m \in E_2 \cap [0, n-1]} \|x_m - x_{m+1}\|^2 \geq \epsilon_0^2 \operatorname{Card}(E_2 \cap [0, n-1])$$

and

$$\operatorname{Card}(E_2 \cap [0, n-1]) \leq 4\epsilon_0^{-2} b^2.$$

Since the inequality above holds for any natural number n, we conclude that

$$\operatorname{Card}(E_2) \leq 4\epsilon_0^{-2} b^2. \tag{10.31}$$

By (10.31), (10.30), and (10.11),

$$\operatorname{Card}(E_3) \leq \operatorname{Card}(E_2)\bar{p} \leq 4\epsilon_0^{-2} b^2 \bar{p} \leq n_0.$$

This completes the proof of Theorem 10.3. □

10.3 Iterative Methods in Finite-Dimensional Spaces

We use all the notation and the definitions introduced in Sect. 10.1 and suppose that all the assumptions made in Sect. 10.1 hold. In this section, we suppose that the space X is finite-dimensional. The results presented in the section were obtained in [96].

We prove the following result, which describes the asymptotic behavior of the subgradient projection method without computational errors.

Theorem 10.4. *Let* $b > 0$, $\epsilon \in (0, 1]$ *and*

$$c \in B(0, b) \cap C. \tag{10.32}$$

Then there exist a natural number n_0 *and* $\gamma_0 \in (0, \epsilon]$ *such that the following assertion holds.*

Assume that

$$\gamma \in [0, \gamma_0],\ S \in \mathbb{S},\ x_0 \in B(0, b) \tag{10.33}$$

and that, for each integer $n \geq 0$,

$$x_{n+1} \in A_{S(n)}(x_n, \gamma). \tag{10.34}$$

Then $\|x_n\| \leq 3b$ *for all integers* $n \geq 0$,

$$f_i(x_n) \leq \epsilon \text{ for all } i \in \mathbb{I} \text{ and all integers } n \geq n_0$$

and $d(x_n, C) \leq \epsilon$ *for all integers* $n \geq n_0$.

Theorem 10.4 is proved in Sect. 10.5.

For each $x \in X$, each $\delta \geq 0$, each $\tilde{\delta} \geq 0$ and each $i \in \mathbb{I}$ set

$$A_i(x, \tilde{\delta}, \delta) := \{x\} \text{ if } f_i(x) \leq \tilde{\delta}, \tag{10.35}$$

and, if $f_i(x) > \tilde{\delta}$, then set

$$A_i(x, \tilde{\delta}, \delta) := \{x - f_i(x)\|l\|^{-2}l : \ l \in \partial f_i(x) + B(0, \delta), \ l \neq 0\} + B(0, \delta). \tag{10.36}$$

The following theorem is one of our main results of this chapter. It describes the behavior of iterates under the presence of computational errors which occur in the calculations of subgradients as well in the calculations of iterates themselves.

Theorem 10.5. *Let* $b > 0$, $\epsilon \in (0, 1]$, *(10.32) hold and let*

$$c_i \in B(0, b) \text{ and } f_i(c_i) < 0, \ i \in \mathbb{I}. \tag{10.37}$$

Then, there exist a natural number n_0 *and* $\delta > 0$ *such that the following assertion holds.*

Assume that

$$\tilde{\delta}_n \in [0, \delta], \ n \in \mathcal{N}, \ S \in \mathbb{S}, \ x_0 \in B(0, b), \tag{10.38}$$

and that, for each integer $n \geq 0$,

$$x_{n+1} \in A_{S(n)}(x_n, \tilde{\delta}_n, \delta). \tag{10.39}$$

Then $\|x_n\| \leq 3b + 1, \ n = 0, \ldots, n_0$, $d(x_{n_0}, C) \leq \epsilon$ *and* $f_i(x_{n_0}) \leq \epsilon$ *for all* $i \in \mathbb{I}$.

This result is proved in Sect. 10.6. Theorem 10.5 easily implies the following result.

Theorem 10.6. *Let* $b > 0$, $\epsilon \in (0, 1]$, *(10.32), and (10.37) hold and let a natural number* n_0 *and* $\delta > 0$ *be given, as guaranteed by Theorem 10.5.*

Assume that (10.38) holds, for each integer $n \geq 0$, *(10.39) holds and that a sequence* $\{x_n\}_{n=0}^{\infty} \subset B(0, b)$. *Then, for all integers* $n \geq n_0$,

$$d(x_n, C) \leq \epsilon \text{ and } f_i(x_n) \leq \epsilon \text{ for all } i \in \mathbb{I}.$$

Theorem 10.6 easily implies the following result.

Theorem 10.7. *Let $b > 0$, (10.32) and (10.37) hold, $\{\delta_n\}_{n=0}^{\infty}$ be a sequence of positive numbers such that $\lim_{n\to\infty}\delta_n = 0$ and let $\epsilon \in (0, 1]$. Then there exists a natural number n_ϵ such that the following assertion holds.*

Assume that $\tilde{\delta}_n \in [0, \delta_n]$ for all $n \in \mathcal{N}$, $S \in \mathbb{S}$, $\{x_n\}_{n=0}^{\infty} \subset B(0, b)$ and that, for each integer $n \geq 0$,

$$x_{n+1} \in A_{S(n)}(x_n, \tilde{\delta}_n, \delta_n).$$

Then, for all integers $n \geq n_\epsilon$, $d(x_n, C) \leq \epsilon$ and $f_i(x_n) \leq \epsilon$ for all $i \in \mathbb{I}$.

In the last two theorems we consider the case when the set C is bounded.

Theorem 10.8. *Suppose that the set C is bounded, (10.32) and (10.37) hold with $b > 0$ and $b_0, \epsilon > 0$. Then there exist a natural number n_0 and $\delta > 0$ such that the following assertion holds.*

Assume that

$$\tilde{\delta}_n \in [0, \delta],\ n \in \mathcal{N},\ S \in \mathbb{S}, \tag{10.40}$$

$$x_0 \in B(0, b_0) \tag{10.41}$$

and that, for each integer $n \geq 0$,

$$x_{n+1} \in A_{S(n)}(x_n, \tilde{\delta}_n, \delta). \tag{10.42}$$

Then, for all integers $n \geq n_0$, $d(x_n, C) \leq \epsilon$ and $f_i(x_n) \leq \epsilon$ for all $i \in \mathbb{I}$.

Proof. We may assume without any loss of generality that

$$b_0 > \sup\{\|z\| :\ z \in C\} + 4,\ b_0 > \|c_i\|,\ i \in I,\ b > 3b_0 + 1 \text{ and } \epsilon < 1. \tag{10.43}$$

By Theorem 10.5, there exist a natural number n_1 and $\delta_1 > 0$ such that the following property holds:

(P3) for each $\tilde{\delta}_n \in [0, \delta_1]$, $n \in \mathcal{N}$, each $S \in \mathbb{S}$, each $\{x_n\}_{n=0}^{\infty} \subset X$ such that $\|x_0\| \leq b_0$ and that, for each integer $n \geq 0$,

$$x_{n+1} \in A_{S(n)}(x_n, \tilde{\delta}_n, \delta_1),$$

we have $\|x_n\| \leq 3b_0 + 1$, $n = 0, \ldots, n_1$ and $d(x_{n_1}, C) \leq \epsilon$.

By Theorem 10.6, there exist a natural number n_0 and $\delta \in (0, \delta_1)$ such that the following property holds:

(P4) if (10.38) holds and if for each integer $n \geq 0$, (10.39) holds and if $\{x_n\}_{n=0}^{\infty} \subset B(0, b)$, then for all integers $n \geq n_0$,

$$d(x_n, C) \leq \epsilon \text{ and } f_i(x_n) \leq \epsilon \text{ for all } i \in \mathbb{I}. \tag{10.44}$$

Assume that $\tilde{\delta}_n \in [0, \delta]$, $n \in \mathcal{N}$, $S \in \mathbb{S}$, $\{x_n\}_{n=0}^{\infty} \subset X$, (10.41) holds and (10.42) holds for each integer $n \geq 0$. By (P3), (10.41)–(10.43) and the inequality $\delta < \delta_1$,

$$\|x_{nn_1}\| \leq b_0, \ n \in \mathcal{N} \text{ and } \|x_n\| \leq 3b_0 + 1, \ n \in \mathcal{N}. \tag{10.45}$$

By (10.45), (10.43), (10.41), (10.42), and (P4), (10.44) holds for all integers $n \geq n_0$. This completes the proof of Theorem 10.8. □

Theorems 10.7 and 10.8 easily imply the following result.

Theorem 10.9. *Let (10.32) and (10.37) hold with $b > 0$ and the set C be bounded. Then there exists $\delta > 0$ such that the following assertion holds.*

Assume that a sequence $\{\delta_n\}_{n=0}^{\infty} \subset [0, \delta]$ satisfy $\lim_{n\to\infty} \delta_n = 0$ and let $\epsilon > 0$. Then there exists a natural number n_ϵ such that, for each $\tilde{\delta}_n \in [0, \delta_n]$, $n \in \mathcal{N}$, each $S \in \mathbb{S}$ and each $\{x_n\}_{n=0}^{\infty} \subset X$ which satisfies $\|x_0\| \leq b$ and

$$x_{n+1} \in A_{S(n)}(x_n, \tilde{\delta}_n, \delta_n) \text{ for each integer } n \geq 0,$$

the following relations hold:

$$d(x_n, C) \leq \epsilon \text{ and } f_i(x_n) \leq \epsilon \text{ for all } i \in \mathbb{I} \text{ and all integers } n \geq n_\epsilon.$$

10.4 Auxiliary Results

We use the notation and the definitions introduced in Sect. 10.3 and suppose that all the assumptions made there hold.

Lemma 10.10. *Let $M > 0$, $\gamma_1 > 0$. Then there exists $\gamma_2 > 0$ such that, for each $x \in B(0, M)$ satisfying $f_i(x) \leq \gamma_2$, $i \in \mathbb{I}$, the inequality $d(x, C) \leq \gamma_1$ holds.*

Proof. Assume the contrary. Then, for any natural number n, there is $x_n \in B(0, M)$ such that

$$f_i(x_n) \leq 1/n, \ i \in \mathbb{I} \text{ and } d(x_n, C) > \gamma_1. \tag{10.46}$$

Extracting a subsequence and re-indexing, if necessary, we may assume without any loss of generality that there is $x = \lim_{n\to\infty} x_n$. It is easy to see that $x \in B(0, M)$, $f_i(x) \leq 0$ for all $i \in \mathbb{I}$ and $x \in C$. Clearly,

$$d(x_n, C) \leq \|x_n - x\| < \gamma_1/2$$

for all sufficiently large natural numbers n. This contradicts (10.46). The contradiction we have reached completes the proof of Lemma 10.10. □

10.5 Proof of Theorem 10.4

Since the functions $f_i,\ i \in \mathbb{I}$ are convex [50], there exists $\Lambda > 0$ such that

$$|f_i(u) - f_i(v)| \le \Lambda \|u - v\| \text{ for all } u, v \in B(0, 3b + 1),\ i \in \mathbb{I}. \tag{10.47}$$

Choose a positive number $\gamma_1 < \epsilon$ such that

$$\Lambda \gamma_1 < \epsilon. \tag{10.48}$$

By Lemma 10.10, there exists $\gamma_2 \in (0, \epsilon)$ such that the following property holds:

(P5) for each $y \in B(0, 3b + 1)$ satisfying $f_i(y) \le \gamma_2$, $i \in \mathbb{I}$ we have $d(y, C) < \gamma_1$.

Choose a positive number γ_0 such that

$$\gamma_0 < \gamma_1 \text{ and } \gamma_0(\bar{p} + 1) < \gamma_2. \tag{10.49}$$

By (10.47) and Theorem 10.3 (with $\epsilon = \gamma_0$), there exists a natural number n_0 such that the following property holds:

(P6) Let $\gamma \in [0, \gamma_0]$, $S \in \mathbb{S}$, $x_0 \in B(0, b)$ and let for each integer $n \ge 0$,

$$x_{n+1} \in A_{S(n)}(x_n, \gamma).$$

Then

$$\|x_n\| \le 3b \text{ for all integers } n \ge 0, \tag{10.50}$$

and there is an integer $q \in [0, n_0]$ such that

$$f_i(x_q) \le \gamma_0(\bar{p} + 1),\ i \in \mathbb{I}. \tag{10.51}$$

Assume that (10.33) holds and that (10.34) holds for each integer $n \ge 0$. Together with (P6) this implies that (10.50) holds and that there is an integer $q \in [0, n_0]$ such that (10.51) holds. By (10.49) and (10.51), $f_i(x_q) \le \gamma_2$ for all $i \in \mathbb{I}$. Together with (P5) and (10.51), this implies that $d(x_q, C) < \gamma_1$ and that there is $\tilde{z} \in X$ such that

$$\tilde{z} \in C \text{ and } \|x_q - \tilde{z}\| < \gamma_1. \tag{10.52}$$

By (10.52), (10.33), (10.34), (10.5), (10.6), (10.4). Lemma 10.2, and Proposition 10.1 (see (10.1)),

$$\|x_n - \tilde{z}\| < \gamma_1 < \epsilon \text{ for all integers } n \ge q. \tag{10.53}$$

In view of (10.53) and (10.50), $\|\tilde{z}\| \le 3b+1$. Together with (10.52), (10.47), (10.53), (10.50), and (10.48), this implies that for all integers $n \ge n_0$ and all $i \in \mathbb{I}$,

$$f_i(x_n) \le f_i(\tilde{z}) + |f_i(x_n) - f_i(\tilde{z})| \le \Lambda \|x_n - \tilde{z}\| < \Lambda \gamma_1 < \epsilon.$$

This completes the proof of Theorem 10.4. □

10.6 Proof of Theorem 10.5

Put

$$r = \min\{-f_i(c_i) : \ i \in \mathbb{I}\}. \tag{10.54}$$

By (10.54) and (10.37),

$$r > 0. \tag{10.55}$$

Since the functions $f_i, \ i \in \mathbb{I}$ are convex [50], there exists $\Lambda > 0$ such that

$$|f_i(u) - f_i(v)| \le \Lambda \|u - v\| \text{ for all } u, v \in B(0, 3b + 2), \ i \in \mathbb{I}, \tag{10.56}$$

$$|f_i(u)| \le \Lambda \text{ for all } u \in B(0, 3b + 2), \ i \in \mathbb{I}. \tag{10.57}$$

By Theorem 10.4, there exist a natural number n_0 and $\bar{\gamma}_0 \in (0, \epsilon]$ such that the following property holds:

(P7) If $\gamma \in [0, \bar{\gamma}_0], \ S \in \mathbb{S}, \ x_0 \in B(0, b)$ and, if for each integer $n \ge 0$,

$$x_{n+1} \in A_{S(n)}(x_n, \gamma),$$

then

$$\|x_n\| \le 3b \text{ for all integers } n \ge 0, \tag{10.58}$$

$$f_i(x_n) \le \epsilon/4 \text{ for all } i \in \mathbb{I} \text{ and all integers } n \ge n_0, \tag{10.59}$$

$$d(x_n, C) \le \epsilon/4 \text{ for all integers } n \ge n_0. \tag{10.60}$$

By (10.56), for each $u \in B(0, 3b + 1)$, all $i \in \mathbb{I}$ and each $g \in \partial f_i(u)$,

$$\|g\| \le \Lambda. \tag{10.61}$$

Let

$$u \in B(0, 3b + 1), \ i \in \mathbb{I}, \ f_i(u) > 0, \ g \in \partial f_i(u). \tag{10.62}$$

By (10.62), (10.54), (10.55), and (10.37),

$$-r \ge f_i(c_i) > f_i(c_i) - f_i(u) \ge \langle g, c_i - u \rangle \ge -\|g\|(4b + 1)$$

and

$$\|g\| > r(4b + 1)^{-1}. \tag{10.63}$$

We have shown that the following property holds:

(P8) if $u \in B(0, 3b + 1)$, $i \in \mathbb{I}$, $f_i(u) > 0$ and if $g \in X$ satisfies

$$d(g, \partial f_i(u)) \le r(4b + 1)^{-1}4^{-1},$$

then $\|g\| > r(4b + 1)^{-1}2^{-1}$.

For each $\gamma \ge 0$ denote by $\mathcal{M}_\gamma$ the set of all sequences $\{x_n\}_{n=0}^\infty \subset X$ for which $\|x_0\| \le b$, and there exist $\tilde{\gamma}_n \in [0, \gamma]$, $n \in \mathcal{N}$, $S \in \mathbb{S}$ such that, for each integer $n \ge 0$,

$$x_{n+1} \in A_{S(n)}(x_n, \tilde{\gamma}_n, \gamma).$$

By induction we show that for all $m = 0, \dots, n_0$ the following assertion holds.

(A) For each $\gamma > 0$ there exists $\delta > 0$ such that, for each $\{x_n\}_{n=0}^\infty \in \mathcal{M}_\delta$, there is $\{y_n\}_{n=0}^\infty \in \mathcal{M}_0$ such that $\|y_n - x_n\| \le \gamma$, $n = 0, \dots, m$.

Clearly, for $m = 0$ this assertion holds. Assume that assertion (A) holds for $m = q$ where $q \in [0, n_0 - 1]$ is an integer. We show that (A) holds for $m = q + 1$. Since (A) holds for $m = q$, it follows from (P7) and (10.58) that there is $\gamma_0 > 0$ such that

$$\gamma_0 < 2^{-1} \text{ and } \gamma_0 < 4^{-1}r(4b + 1)^{-1} \tag{10.64}$$

and that, for each $\{y_n\}_{n=0}^\infty \in \mathcal{M}_{\gamma_0}$,

$$\|y_n\| \le 3b + 1/2, \ n = 0, \dots, q. \tag{10.65}$$

Assume that assertion (A) does not hold for $m = q + 1$. Then there exists $\gamma > 0$ such that for each natural number j there is

$$\{x_n^{(j)}\}_{n=0}^\infty \in \mathcal{M}_{\gamma_0/j} \tag{10.66}$$

such that

$$\max\{\|y_n - x_n^{(j)}\| : \ n = 0, \dots, q + 1\} > \gamma \text{ for each } \{y_n\}_{n=0}^\infty \in \mathcal{M}_0. \tag{10.67}$$

By (10.66) and the choice of γ_0 (see (10.65)), for all natural numbers j,

$$\|x_n^{(j)}\| \le 3b + 1/2, \ n = 0, \dots, q. \tag{10.68}$$

By the definition of $\mathcal{M}_\gamma$, $\gamma \ge 0$, (10.36), (10.37), and (10.66), for each integer $j \ge 1$ there is

$$\tilde{\gamma}_{j,n} \in [0, \gamma_0/j], \ n \in \mathcal{N}, \ S_j \in \mathbb{S}, \ \{g_n^{(j)}\}_{n=0}^\infty \subset X \tag{10.69}$$

such that, for each integer $n \in \{0, \ldots, q+1\}$ satisfying $f_{S_j(n)}(x_n^{(j)}) > \tilde{\gamma}_{j,n}$,

$$g_n^{(j)} \neq 0 \text{ and } d(g_n^{(j)}, \partial f_{S_j(n)}(x_n^{(j)})) \leq \gamma_0/j, \tag{10.70}$$

$$\|x_{n+1}^{(j)} - (x_n^{(j)} - f_{S_j(n)}(x_n^{(j)})\|g_n^{(j)}\|^{-2} g_n^{(j)})\| \leq \gamma_0/j \tag{10.71}$$

and that, for each integer $n \in \{0, \ldots, q+1\}$ satisfying $f_{S_j(n)}(x_n^{(j)}) \leq \tilde{\gamma}_{j,n}$,

$$g_n^{(j)} = 0, \; x_{n+1}^{(j)} = x_n^{(j)}. \tag{10.72}$$

Extracting a subsequence and re-indexing, if necessary, we may assume that

$$S_j(n) = S_1(n) \text{ for all natural numbers } j \text{ and all } n \in \mathcal{N}. \tag{10.73}$$

Put

$$S(n) = S_1(n), \; n \in \mathcal{N}. \tag{10.74}$$

By (10.74), (10.73), (10.70), (10.68), (10.64), (P8), and (10.61), for all $j = 1, 2, \ldots$ and all $n = 0, \ldots, q$,

$$\text{if } f_{S(n)}(x_n^{(j)}) > \tilde{\gamma}_{j,n}, \text{ then } 2^{-1}(4b+1)^{-1} r \leq \|g_n^{(j)}\| \leq \Lambda + 1. \tag{10.75}$$

Assume that j is a natural number. We estimate $\|x_{q+1}^{(j)}\|$. If $f_{S(q)}(x_q^{(j)}) \leq \tilde{\gamma}_{j,q}$, then, in view of (10.74), (10.72), and (10.68),

$$\|x_{q+1}^{(j)}\| = \|x_q^{(j)}\| \leq 3b + 2^{-1}. \tag{10.76}$$

If $f_{S(q)}(x_q^{(j)}) > \tilde{\gamma}_{j,q}$, then by (10.70), (10.71), (10.74), (10.64), (10.68), (10.57), and (10.75),

$$\|x_{q+1}^{(j)}\| \leq \gamma_0 j^{-1} + \|x_q^{(j)} - f_{S(q)}(x_q^{(j)})\|g_q^{(j)}\|^{-2} g_q^{(j)}\|$$
$$\leq 1 + 3b + 2^{-1} + \Lambda\|g_q^{(j)}\|^{-1} \leq 3/2 + 3b + 2\Lambda(4b+1)r^{-1}.$$

Thus for all $j = 1, 2, \ldots,$

$$\|x_{q+1}^{(j)}\| \leq 3/2 + 3b + 2\Lambda(4b+1)r^{-1}. \tag{10.77}$$

By (10.77) and (10.68), extracting a subsequence and re-indexing, if necessary, we may assume without any loss of generality that for any $n \in \{0, \ldots, q+1\}$ there is

$$y_n = \lim_{j \to \infty} x_n^{(j)} \tag{10.78}$$

and that for any $n \in \{0, \dots, q+1\}$ one of the following cases holds:

$$f_{S(n)}(x_n^{(j)}) \le \tilde{\gamma}_{j,n} \text{ for all natural numbers } j, \tag{10.79}$$

$$f_{S(n)}(x_n^{(j)}) > \tilde{\gamma}_{j,n} \text{ for all natural numbers } j. \tag{10.80}$$

For all $n = 0, 1, 2, \dots, q+1$ and all $j = 1, 2, \dots$ choose $\tilde{g}_n^{(j)} \in X$ as follows:

$$\tilde{g}_n^{(j)} = 0 \text{ if } f_{S(n)}(x_n^{(j)}) \le \tilde{\gamma}_{j,n}, \tag{10.81}$$

$$\text{if } f_{S(n)}(x_n^{(j)}) > \tilde{\gamma}_{j,n}, \text{ then } \tilde{g}_n^{(j)} \in \partial f_{S(n)}(x_n^{(j)}),\ \|\tilde{g}_n^{(j)} - g_n^{(j)}\| \le 2\gamma_0 j^{-1}. \tag{10.82}$$

In view (10.69)–(10.74), $\tilde{g}_n^{(j)}$, $n = 0, 1, 2, \dots, q+1, j = 1, 2, \dots$ are well-defined. Set

$$E = \{n \in \{0, \dots, q\} : f_{S(n)}(x_n^{(j)}) > \tilde{\gamma}_{j,n} \text{ for all } j = 1, 2, \dots\}. \tag{10.83}$$

By (10.75), (10.83), (10.79), and (10.80), extracting a subsequence and re-indexing, if necessary, we may assume without any loss of generality that, for each $n \in E$ there is

$$g_n = \lim_{j \to \infty} g_n^{(j)}. \tag{10.84}$$

For each $n \in \{0, \dots, q+1\} \setminus E$ set

$$g_n = 0. \tag{10.85}$$

Let $n \in \{0, \dots, q\}$. There are two cases:

$$f_{S(n)}(y_n) > 0; \tag{10.86}$$

$$f_{S(n)}(y_n) \le 0. \tag{10.87}$$

Consider the case (10.86). By (10.78), we may assume without any loss of generality that

$$f_{S(n)}(x_n^{(j)}) > 2^{-1} f_{S(n)}(y_n) > 0 \text{ for all natural numbers } j. \tag{10.88}$$

Then, in view of (10.69)–(10.72) and (10.88) for all sufficiently large natural numbers j,

$$\|x_{n+1}^{(j)} - (x_n^{(j)} - f_{S(n)}(x_n^{(j)})\|g_n^{(j)}\|^{-2} g_n^{(j)})\| \le \gamma_0/j. \tag{10.89}$$

By (10.78)–(10.80), (10.88), (10.82), and (10.84) for each $u \in X$,

$$f_{S(n)}(u) - f_{S(n)}(y_n) = \lim_{j\to\infty} (f_{S(n)}(u) - f_{S(n)}(x_n^{(j)}))$$
$$\geq \lim_{j\to\infty} \langle \tilde{g}_n^{(j)}, u - x_n^{(j)} \rangle = \langle g_n, u - y_n \rangle$$

and

$$g_n \in \partial f_{S(n)}(y_n). \tag{10.90}$$

By (10.78), (10.89), (10.84), (10.75), (10.88), (10.79), and (10.80),

$$y_{n+1} = \lim_{j\to\infty} x_{n+1}^{(j)}$$
$$= \lim_{j\to\infty} [x_n^{(j)} - f_{S(n)}(x_n^{(j)}) \|g_n^{(j)}\|^{-2} g_n^{(j)}]$$
$$= y_n - f_{S(n)}(y_n) \|g_n\|^{-2} g_n. \tag{10.91}$$

Consider the case (10.87). If $n \notin E$, then by (10.78)–(10.80), (10.83), and (10.69)–(10.72), $x_{n+1}^{(j)} = x_n^{(j)}$ for all natural numbers j and

$$y_{n+1} = \lim_{j\to\infty} x_{n+1}^{(j)} = \lim_{j\to\infty} x_n^{(j)} = y_n. \tag{10.92}$$

Assume that

$$n \in E. \tag{10.93}$$

By (10.83) and (10.93), for each natural numbers j,

$$f_{S(n)}(x_n^{(j)}) > \tilde{\gamma}_{j,n}. \tag{10.94}$$

By (10.87), (10.94), (10.78), and (10.69),

$$f_{S(n)}(y_n) = 0, \ \lim_{j\to\infty} f_{S(n)}(x_n^{(j)}) = 0. \tag{10.95}$$

By (10.78), (10.94), (10.69)–(10.72), (10.74), (10.95), and (10.75),

$$y_{n+1} = \lim_{j\to\infty} x_{n+1}^{(j)} = \lim_{j\to\infty} [x_n^{(j)} - f_{S(n)}(x_n^{(j)}) \|g_n^{(j)}\|^{-2} g_n^{(j)}]$$
$$= \lim_{j\to\infty} x_n^{(j)} - [\lim_{j\to\infty} f_{S(n)}(x_n^{(j)}) \|g_n^{(j)}\|^{-2} g_n^{(j)})] = \lim_{j\to\infty} x_n^{(j)} = y_n.$$

Thus in both cases (10.87) implies that

$$y_{n+1} = y_n. \tag{10.96}$$

Thus (10.86) implies (10.90), (10.91), and (10.87) implies (10.96). Clearly, there are $y_n \in X$ for all integers $n \geq q+1$ such that $\{y_n\}_{n=0}^{\infty} \in \mathcal{M}_0$. By (10.78), for all sufficiently large natural numbers j, we have $\|x_n^{(j)} - y_n\| < \gamma/2$, $n = 0, 1, \dots, q+1$. This contradicts (10.67). The contradiction we have reached proves that assertion (A) holds with $m = q+1$. Thus by induction we have shown that assertion (A) holds with $m = n_0$.

Fix a positive number γ_1 such that

$$\gamma_1 < \epsilon/4, \ \gamma_1 < 1/2, \ \gamma_1 < (\epsilon/2)\Lambda^{-1}. \tag{10.97}$$

By (A) with $m = n_0$, there is $\delta > 0$ such that, for each $\{x_n\}_{n=0}^{\infty} \in \mathcal{M}_\delta$ there is $\{y_n\}_{n=0}^{\infty} \subset \mathcal{M}_0$ for which

$$\|y_n - x_n\| \leq \gamma_1, \ n = 0, \dots, n_0. \tag{10.98}$$

Let

$$\{x_n\}_{n=0}^{\infty} \in \mathcal{M}_\delta. \tag{10.99}$$

By (10.99) and the choice of δ there is

$$\{y_n\}_{n=0}^{\infty} \in \mathcal{M}_0 \tag{10.100}$$

such that (10.98) holds. By (10.100), (P7), and the definition of $\mathcal{M}_0$,

$$\|y_n\| \leq 3b \text{ for all integers } n \geq 0, \tag{10.101}$$

$$f_i(y_{n_0}) \leq \epsilon/4, \ i \in \mathbb{I}, \tag{10.102}$$

$$d(y_{n_0}, C) \leq \epsilon/4. \tag{10.103}$$

By (10.98), (10.103), and (10.97),

$$d(x_{n_0}, C) \leq \|x_{n_0} - y_{n_0}\| + d(y_{n_0}, C) < \epsilon/2. \tag{10.104}$$

By (10.101), (10.98), and (10.97),

$$\|x_n\| \leq 3b + 1/2, \ n = 0, \dots, n_0. \tag{10.105}$$

By (10.102), (10.101), (10.105), (10.56), (10.98), and (10.97), for any $i \in \mathbb{I}$,

$$\begin{aligned} f_i(x_{n_0}) &\leq f_i(y_{n_0}) + |f_i(x_{n_0}) - f_i(y_{n_0})| \leq \epsilon/4 + \Lambda\|x_{n_0} - y_{n_0}\| \\ &\leq \epsilon/4 + \gamma_1\Lambda < \epsilon/2 + \epsilon/4. \end{aligned}$$

Theorem 10.5 is proved. □

10.7 Dynamic String-Averaging Methods in Infinite-Dimensional Spaces

Let $(X, \langle \cdot, \cdot \rangle)$ be a Hilbert space with an inner product $\langle \cdot, \cdot \rangle$, which induces a complete norm $\| \cdot \|$. For each $x \in X$ and each nonempty set $A \subset X$ put

$$d(x, A) := \inf\{\| |x - y\| : \ y \in A\}.$$

For each $x \in X$ and each $r > 0$ set

$$B(x, r) = \{y \in X : \ \|x - y\| \le r\}.$$

We use the notation, definitions, and assumptions introduced in Sect. 10.1.

Let $f : X \to R^1$ be a continuous and convex function such that

$$\{x \in X : \ f(x) \le 0\} \ne \emptyset. \tag{10.106}$$

Let $y_0 \in X$. Recall that the set

$$\partial f(y_0) := \{l \in X : \ f(y) - f(y_0) \ge \langle l, y - y_0 \rangle \text{ for all } y \in X\} \tag{10.107}$$

is the subdifferential of f at the point y_0 [72, 77]. For any $g \in \partial f(y_0)$, in view of (10.107),

$$\{x \in X : \ f(x) \le 0\} \subset \{x \in X : \ f(y_0) + \langle g, x - y_0 \rangle \le 0\}. \tag{10.108}$$

Let m be a natural number and $f_i : X \to R^1$, $i = 1, \dots, m$, be convex and continuous functions. For each $i \in \{1, \dots, m\}$ set

$$C_i := \{x \in X : \ f_i(x) \le 0\}, \tag{10.109}$$

$$C := \cap_{i=1}^m C_i = \cap_{i=1}^m \{x \in X : \ f_i(x) \le 0\}. \tag{10.110}$$

Suppose that

$$C \ne \emptyset.$$

A point $x \in C$ is called a solution of our feasibility problem. For a given $\epsilon > 0$, a point $x \in X$ is called an ϵ-approximate solution of the feasibility problem if $f_i(x) \le \epsilon$ for all $i = 1, \dots .m$. We apply the dynamic string-averaging subgradient projection method in order to obtain a good approximative solution of the feasibility problem.

Denote by $\mathcal{N}$ the set of all nonnegative integers.

By an index vector, we mean a vector $t = (t_1, \dots, t_p)$ such that $t_i \in \{1, \dots, m\}$ for all $i = 1, \dots, p$.

For an index vector $t = (t_1, \dots, t_q)$ set

$$p(t) = q. \tag{10.111}$$

Denote by $\mathcal{M}$ the collection of all pairs (Ω, w), where Ω is a finite set of index vectors and

$$w : \Omega \to (0, \infty) \text{ be such that } \sum_{t \in \Omega} w(t) = 1. \tag{10.112}$$

Fix a number

$$\Delta \in (0, m^{-1}] \tag{10.113}$$

and an integer

$$\bar{q} \geq m. \tag{10.114}$$

Denote by $\mathcal{M}_*$ the set of all $(\Omega, w) \in \mathcal{M}$ such that

$$p(t) \leq \bar{q} \ \text{ for all } t \in \Omega, \tag{10.115}$$

$$w(t) \geq \Delta \text{ for all } t \in \Omega. \tag{10.116}$$

Fix a natural number $\bar{N}$.

For each $x \in X$, each number $\epsilon \geq 0$ and each $i \in \{1, \dots, m\}$ set

$$A_i(x, \epsilon) := \{x\} \text{ if } f_i(x) \leq \epsilon \tag{10.117}$$

and, in view of Lemma 10.2,

$$A_i(x, \epsilon) = x - f_i(x)\{\|g\|^{-2} g : \ g \in \partial f_i(x)\} \text{ if } f_i(x) > \epsilon. \tag{10.118}$$

Let $\epsilon \geq 0$, $x \in X$ and let $t = (t_1, \dots, t_{p(t)})$ be an index vector. Define

$$A_0(t, x, \epsilon) = \{(y, \lambda) \in X \times R^1 : \ \text{there is a sequence } \{y_i\}_{i=0}^{p(t)} \subset X \text{ such that}$$

$$y_0 = x, \tag{10.119}$$

for each $i = 1, \dots, p(t)$,

$$y_i \in A_{t_i}(y_{i-1}, \epsilon), \tag{10.120}$$

$$y = y_{p(t)}, \tag{10.121}$$

$$\lambda = \max\{\|y_i - y_{i-1}\| : \ i = 1, \dots, p(t)\}\}. \tag{10.122}$$

Let $\epsilon \geq 0$, $x \in X$ and let $(\Omega, w) \in \mathcal{M}$. Define

$$A(x, (\Omega, w), \epsilon) = \{(y, \lambda) \in X \times R^1 : \text{ there exist}$$

$$(y_t, \lambda_t) \in A_0(t, x, \epsilon), \; t \in \Omega \text{ such that}$$

$$y = \sum_{t \in \Omega} w(t) y_t, \tag{10.123}$$

$$\lambda = \max\{\lambda_t : \; t \in \Omega\}\}. \tag{10.124}$$

Denote by Card(A) the cardinality of a set A. Suppose that the sum over empty set is zero.

Theorem 10.11. *Let*

$$M_0 > 0, \; \epsilon \in (0, 1), \; M_1 > 0, \; \gamma \in [0, \epsilon], \tag{10.125}$$

$$B(0, M_0) \cap C \neq \emptyset, \tag{10.126}$$

$$|f_i(u) - f_i(v)| \leq M_1 \|u - v\|, \; u, v \in B(0, 3M_0 + 1), \; i \in \{1, \dots, m\}, \tag{10.127}$$

$$\epsilon_0 \in (0, \epsilon M_1^{-1}] \tag{10.128}$$

and let a natural number n_0 *satisfy*

$$n_0 \geq 4M_0^2 \epsilon_0^{-2} \Delta^{-1} \bar{N}. \tag{10.129}$$

Assume that

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*, \tag{10.130}$$

satisfies for each natural number j,

$$\{1, \dots, m\} \subset \cup_{i=j}^{j+\bar{N}-1} (\cup_{t \in \Omega_i} \{t_1, \dots, t_{p(t)}\}), \tag{10.131}$$

$$x_0 \in B(0, M_0), \tag{10.132}$$

$$\{x_i\}_{i=1}^{\infty} \subset X, \; \{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty) \tag{10.133}$$

satisfy for each natural number i,

$$(x_i, \lambda_i) \in A(x_{i-1}, (\Omega_i, w_i), \gamma). \tag{10.134}$$

Then

$$\|x_i\| \leq 3M_0 \text{ for all integers } i \geq 0 \tag{10.135}$$

and

$$Card(\{n \in \mathcal{N} : \max\{\lambda_i : i = n+1, \dots, n+\bar{N}\} > \epsilon_0\}) \le n_0. \quad (10.136)$$

Moreover, if an integer $n \ge 0$ satisfies

$$\lambda_i \le \epsilon_0, \ i = n+1, \dots, n+\bar{N}, \quad (10.137)$$

then, for all integers $i, j \in \{n, \dots, n+\bar{N}\}$, $\|x_i - x_j\| \le \bar{N}\bar{q}\epsilon_0$ and for all integers $j \in \{n, \dots, n+\bar{N}\}$ and each $s \in \{1, \dots, m\}$,

$$f_s(x_j) \le \epsilon(\bar{q}(\bar{N}+1)+1).$$

10.8 Proof of Theorem 10.11

Let n be a natural number. In view of (10.134),

$$(x_n, \lambda_n) \in A(x_{n-1}, (\Omega_n, w_n), \gamma). \quad (10.138)$$

By (10.124) and (10.124), for any $t \in \Omega_n$ there exists

$$(y_{n,t}, \lambda_{n,t}) \in A_0(t, x_{n-1}, \gamma) \quad (10.139)$$

such that

$$x_n = \sum_{t \in \Omega_n} w_n(t) y_{n,t}, \quad (10.140)$$

$$\lambda_n = \max\{\lambda_{n,t} : t \in \Omega_n\}. \quad (10.141)$$

Let

$$t = (t_1, \dots, t_{p(t)}) \in \Omega_n. \quad (10.142)$$

By (10.119), (10.139), and (10.142), there is a sequence $\{y_{n,t,i}\}_{i=0}^{p(t)} \subset X$ such that

$$y_{n,t,0} = x_{n-1}, \quad (10.143)$$

$$y_{n,t,i} \in A_{t_i}(y_{n,t,i-1}, \gamma), \ i = 1, \dots, p(t), \quad (10.144)$$

$$y_{n,t} = y_{n,t,p(t)}, \quad (10.145)$$

$$\lambda_{n,t} = \max\{\|y_{n,t,i} - y_{n,t,i-1}\| : i = 1, \dots, p(t)\}. \quad (10.146)$$

In view of (10.126), there exists

$$z \in B(0, M_0) \cap C. \tag{10.147}$$

Let n be a natural number,

$$t = (t_1, \dots, t_{p(t)}) \in \Omega_n, \ i = 1, \dots, p(t). \tag{10.148}$$

By (10.117), (10.118), and (10.144), the following properties hold:
(P1) if $f_{t_i}(y_{n,t,i-1}) \le \gamma$, then $y_{n,t,i} = y_{n,t,i-1}$;
(P2) if $f_{t_i}(y_{n,t,i-1}) > \gamma$, then there is

$$g_{n,t,i} \in \partial f_{t_i}(y_{n,t,i-1}) \tag{10.149}$$

such that

$$y_{n,t,i} = y_{n,t,i-1} - f_{t_i}(y_{n,t,i-1}) \|g_{n,t,i}\|^{-2} g_{n,t,i}. \tag{10.150}$$

If (P1) holds, the we set $g_{n,t,i} = 0$. Set

$$D_{n,t,i} = \{x \in X : \ f_{t_i}(y_{n,t,i-1}) + \langle g_{n,t,i}, x - y_{n,t,i-1} \rangle \le 0\}. \tag{10.151}$$

Clearly, if $f_{t_i}(y_{n,t,i-1}) \le \gamma$, then

$$\|z - y_{n,t,i}\| = \|z - y_{n,t,i-1}\|. \tag{10.152}$$

Assume that

$$f_{t_i}(y_{n,t,i-1}) > \gamma.$$

Property (P2), Lemma 10.2 and (10.149)–(10.151) imply that

$$g_{n,t,i} \ne 0,$$
$$y_{n,t,i} = P_{D_{n,t,i}}(y_{n,t,i-1}). \tag{10.153}$$

It follows from (10.108), (10.110), (10.147), (10.149), and (10.151) that

$$z \in C \subset \{x \in X : \ f_{t_i}(x) \le 0\} \subset D_{n,t,i}. \tag{10.154}$$

Proposition 10.1, (10.151), and (10.153) imply that

$$\|z - y_{n,t,i}\|^2 + \|y_{n,t,i} - y_{n,t,i-1}\|^2 \le \|z - y_{n,t,i-1}\|^2. \tag{10.155}$$

Thus we have shown that the following property holds:

(P3) if $f_{t_i}(y_{n,t,i-1}) > \gamma$, then (10.155) holds.

In view of (10.152) and (10.153),

$$\|z - y_{n,t,i}\| \le \|z - y_{n,t,i-1}\| \text{ for all } t = 1, \ldots, p(t). \tag{10.156}$$

By (10.143), (10.147), and (10.156), for all $i = 1, \ldots, p(t)$,

$$\|y_{n,t,i}\| \le \|y_{n,t,i} - z\| + \|z\|$$
$$\le \|y_{n,t,0} - z\| + M_0 \le \|x_{n-1} - z\| + M_0. \tag{10.157}$$

It follows from (10.143), (10.145), and (10.156) that

$$\|z - x_{n-1}\| = \|z - y_{n,t,0}\| \ge \|z - y_{n,t,p(t)}\| = \|z - y_{n,t}\| \tag{10.158}$$

for all $t \in \Omega_n$. By (10.112), (10.140), (10.158) and the convexity of the norm,

$$\|z - x_n\| = \|z - \sum_{t \in \Omega_n} w_n(t) y_{n,t}\|$$
$$\le \sum_{t \in \Omega_n} w_n(t) \|z - y_{n,t}\| \le \|z - x_{n-1}\|.$$

Thus

$$\|z - x_n\| \le \|z - x_{n-1}\| \text{ for all integers } n \ge 1. \tag{10.159}$$

In view of (10.132) and (10.147),

$$\|z - x_0\| \le 2M_0. \tag{10.160}$$

It follows from (10.159) and (10.160) that

$$\|z - x_n\| \le 2M_0 \text{ for all integers } n \ge 0. \tag{10.161}$$

By (10.157) and (10.161), for all natural numbers n, all $t \in \Omega_n$ and all $i \in \{1, \ldots, p(t)\}$,

$$\|y_{n,t,i}\| \le M_0 + \|x_{n-1} - z\| \le 3M_0. \tag{10.162}$$

Relations (10.132) and (10.161) that

$$\|x_n\| \le 3M_0 \text{ for all } n \in \mathcal{N}. \tag{10.163}$$

Assume that $n \ge 0$ is an integer such that

$$\lambda_k \le \epsilon_0, \; k = n + 1, \ldots, n + \bar{N}. \tag{10.164}$$

In view of (10.141), (10.146), and (10.164), for each $k \in \{n+1, \dots, n+\bar{N}\}$, each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}$ and each $i \in \{1, \dots, p(t)\}$,

$$\|y_{k,t,i} - y_{k,t,i-1}\| \le \epsilon_0. \tag{10.165}$$

By (10.115), (10.143), and (10.165), for each $k \in \{n+1, \dots, n+\bar{N}\}$, each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}$ and each $i \in \{1, \dots, p(t)\}$,

$$\|x_{k-1} - y_{k,t,i}\| \le \bar{q}\epsilon_0. \tag{10.166}$$

Relations (10.112), (10.140), (10.145), and (10.166) imply that

$$\|x_{k-1} - x_k\| \le \bar{q}\epsilon_0. \tag{10.167}$$

It follows from (10.164) and (10.167) that for all $k, m \in \{n, \dots, n+\bar{N}\}$

$$\|x_k - x_m\| \le \bar{N}\bar{q}\epsilon_0. \tag{10.168}$$

Let

$$s \in \{1, \dots, m\}.$$

By (10.131), there exist

$$k \in \{n+1, \dots, n+\bar{N}\},\ t = (t_1, \dots, t_{p(t)}) \in \Omega_k \tag{10.169}$$

such that

$$s \in \{t_1, \dots, t_{p(t)}\}. \tag{10.170}$$

In view of (10.170), there is $i \in \{1, \dots, p(t)\}$ such that

$$s = t_i.$$

We show that

$$f_s(y_{k,t,i-1}) \le \epsilon. \tag{10.171}$$

Assume the contrary. Then by (10.125),

$$f_{t_i}(y_{k,t,i-1}) > \epsilon \ge \gamma. \tag{10.172}$$

Property (P2), (10.150), (10.165), (10.169), and (10.172) imply that

$$\begin{aligned} \epsilon_0 &\ge \|y_{k,t,i} - y_{k,t,i-1}\| \\ &= \|f_{t_i}(y_{k,t,i-1})\|g_{k,t,i}\|^{-2}g_{k,t,i}\| > \|g_{k,t,i}\|^{-1}\epsilon. \end{aligned} \tag{10.173}$$

It follows from (10.143), (10.162), (10.163), and (10.169) that

$$\|y_{k,t,i-1}\| \le 3M_0. \tag{10.174}$$

By (10.127), (10.149), and (10.174),

$$\|g_{k,t,i}\| \le M_1. \tag{10.175}$$

In view of (10.173) and (10.175),

$$\epsilon_0 > \epsilon M_1^{-1}.$$

This contradicts (10.128). The contradiction we have reached proves (10.171). Relations (10.143), (10.162), (10.163), (10.166), and (10.169) imply that

$$\|y_{k,t,i-1}\| \le 3M_0, \ \|x_{k-1}\| \le 3M_0, \tag{10.176}$$

$$\|x_{k-1} - y_{k,t,i-1}\| \le \bar{q}\epsilon_0. \tag{10.177}$$

By (10.127), (10.128), (10.170), (10.171), (10.176), and (10.177),

$$|f_s(x_{k-1}) - f_s(y_{k,t,i-1})| \le M_1\bar{q}\epsilon_0,$$

$$f_s(x_{k-1}) \le f_s(y_{k,t,i-1}) + M_1\bar{q}\epsilon_0 \le \epsilon(\bar{q}+1). \tag{10.178}$$

Let $j \in \{n, \dots, n+\bar{N}\}$. In view of (10.168) and (10.169),

$$\|x_j - x_{k-1}\| \le \bar{N}\bar{q}\epsilon_0. \tag{10.179}$$

It follows from (10.127), (10.128), (10.163), (10.178), and (10.179) that

$$f_s(x_j) \le f_s(x_{k-1}) + M_1\bar{N}\bar{q}\epsilon_0$$

$$\le \epsilon(\bar{q}+1) + \bar{N}\bar{q}\epsilon \le \epsilon(\bar{q}(\bar{N}+1)+1) \tag{10.180}$$

for all $s \in \{1, \dots, m\}$.

Set

$$E_1 = \{n \in \mathcal{N} : \ \lambda_{n+1} \le \epsilon_0\}, \tag{10.181}$$

$$E_2 = \mathcal{N} \setminus E_1, \tag{10.182}$$

$$E_3 = \{n \in \mathcal{N} : \ \{n, \dots, n+\bar{N}-1\} \cap E_2\} \ne \emptyset. \tag{10.183}$$

Let

$$n \in E_2. \tag{10.184}$$

In view of (10.181), (10.182), and (10.184),

$$\lambda_{n+1} > \epsilon_0. \tag{10.185}$$

By (10.112), (10.140) and the convexity of the function $\|\cdot\|^2$,

$$\begin{aligned}
&\|z - x_n\|^2 - \|z - x_{n+1}\|^2 \\
&= \|z - x_n\|^2 - \|z - \sum_{t \in \Omega_{n+1}} w_{n+1}(t) y_{n+1,t}\|^2 \\
&\geq \|z - x_n\|^2 - \sum_{t \in \Omega_{n+1}} w_{n+1}(t) \|z - y_{n+1,t}\|^2.
\end{aligned} \tag{10.186}$$

In view of (10.141) and (10.185), there exists

$$\hat{t} \in \Omega_{n+1} \tag{10.187}$$

such that

$$\epsilon_0 < \lambda_{n+1} = \lambda_{n+1,\hat{t}}. \tag{10.188}$$

It follows from (10.112), (10.143), (10.145), (10.156), (10.186), and (10.187) that

$$\begin{aligned}
&\|z - x_n\|^2 - \|z - x_{n+1}\|^2 \\
&\geq \sum_{t \in \Omega_{n+1}} w_{n+1}(t)[\|z - x_n\|^2 - \|z - y_{n+1,t}\|^2] \\
&= \sum_{t \in \Omega_{n+1}} w_{n+1}(t)[\|z - y_{n+1,t,0}\|^2 - \|z - y_{n+1,t,p(t)}\|^2] \\
&\geq w_{n+1}(\hat{t})[\|z - y_{n+1,\hat{t},0}\|^2 - \|z - y_{n+1,\hat{t},p(\hat{t})}\|^2].
\end{aligned} \tag{10.189}$$

Property (P3), (10.116), (10.152), (10.155), (10.156), and (10.189) imply that

$$\begin{aligned}
&\|z - x_n\|^2 - \|z - x_{n+1}\|^2 \\
&\geq \Delta[\|z - y_{n+1,\hat{t},0}\|^2 - \|z - y_{n+1,\hat{t},p(\hat{t})}\|^2] \\
&\geq \Delta \sum_{i=1}^{p(\hat{t})} [\|z - y_{n+1,\hat{t},i-1}\|^2 - \|z - y_{n+1,\hat{t},i}\|^2] \\
&\geq \Delta \sum_{i=1}^{p(\hat{t})} \|y_{n+1,\hat{t},i-1} - y_{n+1,\hat{t},i}\|^2.
\end{aligned} \tag{10.190}$$

By (10.146), (10.188), and (10.190),

$$\|z - x_n\|^2 - \|z - x_{n+1}\|^2 \geq \Delta\epsilon_0^2 \text{ for all } t \in E_2. \tag{10.191}$$

In view of (10.159), (10.160), and (10.191), for any natural number n,

$$4M^2 \geq \|z - x_0\|^2 \geq \|z - x_0\|^2 - \|z - x_{n+1}\|^2$$

$$\sum_{i=0}^{n}[\|z - x_i\|^2 - \|z - x_{i+1}\|^2]$$

$$\geq \sum\{\|z - x_i\|^2 - \|z - x_{i+1}\|^2 : i \in [0, n] \cap E_2\}$$

$$\geq \text{Card}([0, n] \cap E_2)\Delta\epsilon_0^2,$$

$$\text{Card}([0, n] \cap E_2) \leq 4M_0^2(\Delta\epsilon_0^2)^{-1}.$$

Since the inequality above holds for any natural number n,

$$\text{Card}(E_2) \leq 4M_0^2(\Delta\epsilon_0^2)^{-1}.$$

In view of the relation above, (10.129) and (10.183),

$$\text{Card}(E_3) \leq \bar{N}\text{Card}(E_2) \leq 4M_0^2(\Delta\epsilon_0^2)^{-1}\bar{N} \leq n_0.$$

This completes the proof of Theorem 10.11. □

10.9 Dynamic String-Averaging Methods in Finite-Dimensional Spaces

We use the notation, definitions, and assumptions introduced in Sects. 10.1 and 10.7.

Suppose that the space X is finite-dimensional. We prove the following result.

Theorem 10.12. *Let $M_0 > 0$, $\epsilon \in (0, 1)$,*

$$B(0, M_0) \cap C \neq \emptyset, \tag{10.192}$$

Then there exist a natural number n_0 and $\gamma_0 \in (0, \epsilon)$ such that the following assertion holds.

Assume that

$$\gamma \in [0, \gamma_0], \tag{10.193}$$

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*, \tag{10.194}$$

satisfies for each natural number j,

$$\{1, \ldots, m\} \subset \cup_{i=j}^{j+\bar{N}-1} (\cup_{t \in \Omega_i} \{t_1, \ldots, t_{p(t)}\}), \tag{10.195}$$

$$x_0 \in B(0, M_0), \tag{10.196}$$

$$\{x_i\}_{i=1}^{\infty} \subset X, \ \{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty) \tag{10.197}$$

satisfy for each natural number i,

$$(x_i, \lambda_i) \in A(x_{i-1}, (\Omega_i, w_i), \gamma). \tag{10.198}$$

Then

$$\|x_i\| \le 3M_0 \text{ for all integers } i \ge 0,$$

$$f_i(x_n) \le \epsilon \text{ for all } i \in \{1, \ldots, m\} \text{ and all integers } n \ge n_0,$$

$$d(x_n, C) \le \epsilon \text{ for all integers } n \ge n_0.$$

10.10 Proof of Theorem 10.12

Since the functions f_i, $i \in \{1, \ldots, m\}$ are convex [50], there exists $M_1 > 0$ such that

$$|f_i(u) - f_i(v)| \le M_1 \|u - v\| \text{ for all } u, v \in B(0, 3M_0 + 1), \ i \in \{1, \ldots, m\}. \tag{10.199}$$

Choose a positive number

$$\gamma_1 < \min\{\epsilon, \ M_1^{-1}\epsilon\}. \tag{10.200}$$

By Lemma 10.10, there exists $\gamma_2 \in (0, \epsilon)$ such that the following property holds:

(P4) for each $y \in B(0, 3M_0 + 1)$ satisfying $f_i(y) \le \gamma_2$, $i \in \{1, \ldots, m\}$, the inequality $d(y, C) \le \gamma_1/2$ holds.

Choose a positive number γ_0 such that

$$\gamma_0 < \gamma_1 \text{ and } (\bar{N} + 1)\gamma_0(\bar{q} + 1) < \gamma_2. \tag{10.201}$$

By (10.199) and Theorem 10.11 (with $\epsilon = \gamma_0$), there exists a natural number n_0 such that the following property holds:

(P5) let $\gamma \in [0, \gamma_0]$,

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*,$$

for each natural number j, (10.195) hold,

$$x_0 \in B(0, M_0),$$

$$\{x_i\}_{i=1}^{\infty} \subset X, \ \{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty),$$

for each natural number i (10.198) hold. Then

$$\|x_n\| \le 3M_0 \text{ for all integers } n \ge 0 \tag{10.202}$$

and there exists an integer $q \in [0, n_0]$ such that

$$f_i(x_q) \le \gamma_0(\bar{q}(\bar{N}+1)+1),\ i \in \{1,\dots,m\}. \tag{10.203}$$

Let

$$\gamma \in [0, \gamma_0],\ \{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*, \tag{10.204}$$

for each natural number j, (10.195) holds, (10.196) is true,

$$\{x_i\}_{i=1}^{\infty} \subset X,\ \{\lambda_i\}_{i=1}^{\infty} \subset [0,\infty)$$

and for each natural number i (10.198) holds. Property (P5), (10.196), (10.198), and (10.204) imply that (10.202) holds and there exists an integer $q \in [0, n_0]$ such that (10.203) holds. It follows from (10.201) to (10.203) and (P4) that

$$d(x_q, C) \le 2^{-1}\gamma_1. \tag{10.205}$$

In view of (10.205), there exists $\tilde{z} \in X$ such that

$$\tilde{z} \in C,\ \|x_q - \tilde{z}\| < \gamma_1. \tag{10.206}$$

Proposition 10.1, Lemma 10.2, (10.119), (10.124), (10.198), (10.200), and (10.206) imply that

$$\|x_n - \tilde{z}\| \le \|x_q - \tilde{z}\| < \gamma_1 < \epsilon. \tag{10.207}$$

By (10.200), (10.202), and (10.206),

$$\|\tilde{z}\| \le 3M_0 + 1. \tag{10.208}$$

In view of (10.124), (10.199), (10.200), (10.202), (10.206), and (10.208), for all integers $n \ge n_0$ and all $i \in \{1,\dots,m\}$,

$$\begin{aligned} f_i(x_n) &\le f_i(\tilde{z}) + |f_i(x_n) - f_i(\tilde{z})| \\ &\le M_1\|x_n - \tilde{z}\| < M_1\gamma_1 < \epsilon. \end{aligned}$$

Theorem 10.12 is proved. □

10.11 Problems in Finite-Dimensional Spaces with Computational Errors

We use all the notation, definitions, and assumptions introduced in Sects. 10.7 and 10.9. In particular, we assume that the space X is finite-dimensional.

For each $x \in X$, each $\epsilon \geq 0$, each $\bar{\epsilon} \geq 0$ and each $i \in \{1, \dots, m\}$ set

$$A_i(x, \bar{\epsilon}, \epsilon) := \{x\} \text{ if } f_i(x) \leq \bar{\epsilon} \tag{10.209}$$

and if $f_i(x) > \bar{\epsilon}$, then set

$$A_i(x, \bar{\epsilon}, \epsilon) = \{x - f_i(x)\|g\|^{-2}g : \ g \in \partial f_i(x) + B(0, \epsilon), \ g \neq 0\} + B(0, \epsilon). \tag{10.210}$$

Let $x \in X$ and let $t = (t_1, \dots, t_{p(t)})$ be an index vector, $\epsilon \geq 0$, $\bar{\epsilon} \geq 0$. Define

$$A_0(t, x, \bar{\epsilon}, \epsilon) = \{(y, \lambda) \in X \times R^1 : \ \text{there is a sequence } \{y_i\}_{i=0}^{p(t)} \subset X \text{ such that}$$

$$y_0 = x, \tag{10.211}$$

for each $i = 1, \dots, p(t)$,

$$y_i \in A_{t_i}(y_{i-1}, \bar{\epsilon}, \epsilon), \tag{10.212}$$

$$y = y_{p(t)}, \tag{10.213}$$

$$\lambda = \max\{\|y_i - y_{i-1}\| : \ i = 1, \dots, p(t)\}\}. \tag{10.214}$$

Let $x \in X$, $(\Omega, w) \in \mathcal{M}$, $\epsilon \geq 0$, $\bar{\epsilon} \geq 0$. Define

$$A(x, (\Omega, w), \bar{\epsilon}, \epsilon) = \{(y, \lambda) \in X \times R^1 : \ \text{there exist}$$

$$(y_t, \lambda_t) \in A_0(t, x, \bar{\epsilon}, \epsilon), \ t \in \Omega \text{ such that}$$

$$\|y - \sum_{t \in \Omega} w(t) y_t\| \leq \epsilon, \ \lambda = \max\{\lambda_t : \ t \in \Omega\}\}. \tag{10.215}$$

We prove the following result.

Theorem 10.13. *Let $M_0 > 0$, $\epsilon \in (0, 1)$,*

$$B(0, M_0) \cap C \neq \emptyset, \tag{10.216}$$

$$\{z \in X : \ f_i(z) < 0\} \cap B(0, M_0) \neq \emptyset \textit{ for all } i = 1, \dots, m. \tag{10.217}$$

Then there exist a natural number n_0 and $\delta > 0$ such that the following assertion holds.

Assume that

$$\tilde{\delta}_n \in [0, \delta],\ n \in \mathcal{N} \setminus \{0\}, \tag{10.218}$$

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*, \tag{10.219}$$

satisfies for each natural number j,

$$\{1, \dots, m\} \subset \cup_{i=j}^{j+\bar{N}-1}(\cup_{t \in \Omega_i}\{t_1, \dots, t_{p(t)}\}), \tag{10.220}$$

$$x_0 \in B(0, M_0), \tag{10.221}$$

$$\{x_i\}_{i=1}^{\infty} \subset X,\ \{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty) \tag{10.222}$$

and that for each natural number i,

$$(x_i, \lambda_i) \in A(x_{i-1}, (\Omega_i, w_i), \tilde{\delta}_i, \delta). \tag{10.223}$$

Then

$$\|x_n\| \le 3M_0 + 1 \textit{ for all integers } n = 0, \dots, n_0,$$

$$d(x_{n_0}, C) \le \epsilon,$$

$$f_i(x_{n_0}) \le \epsilon,\ i \in \{1, \dots, m\}.$$

10.12 Proof of Theorem 10.13

In view of (10.217), for each $i \in \{1, \dots, m\}$, there exists

$$z_i \in B(0, M_0) \tag{10.224}$$

such that

$$f_i(z_i) < 0. \tag{10.225}$$

Set

$$r = \min\{-f_i(z_i) :\ i \in \{1, \dots, m\}\}. \tag{10.226}$$

By (10.225) and (10.226),

$$r > 0. \tag{10.227}$$

Since the functions $f_i,\ i = 1, \dots, m$ are convex [50], there exists $\Lambda > 0$ such that

$$|f_i(u) - f_i(v)| \le \Lambda \|u - v\| \text{ for all } u, v \in B(0, 3M_0 + 2),\ i = 1, \dots, m, \tag{10.228}$$

$$|f_i(u)| \le \Lambda \text{ for all } u \in B(0, 3M_0 + 2),\ i = 1, \dots, m. \tag{10.229}$$

By Theorem 10.12, there exist a natural number n_0 and $\bar{\gamma}_0 \in (0, \epsilon)$ such that the following property holds:

(P6) For each $\gamma \in [0, \tilde{\gamma}]$, each

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$$

satisfying (10.220) for all integers $j \ge 1$, each pair of sequences

$$\{x_i\}_{i=1}^{\infty} \subset X,\ \{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty)$$

which satisfy

$$x_0 \in B(0, M_0), \tag{10.230}$$

$$(x_i, \lambda_i) \in A(x_{i-1}, (\Omega_i, w_i), \gamma) \text{ for all natural numbers } i \tag{10.231}$$

we have

$$\|x_n\| \le 3M_0 \text{ for all integers } n \ge 0, \tag{10.232}$$

$$f_i(x_n) \le \epsilon/4 \text{ for all } i \in \{1, \dots, m\} \text{ and all integers } n \ge n_0, \tag{10.233}$$

$$d(x_n, C) \le \epsilon/4 \text{ for all integers } n \ge n_0. \tag{10.234}$$

By the choice of Λ (see (10.228)), for each $u \in B(0, 3M_0 + 1)$, all $i \in \{1, \dots, m\}$,

$$\partial f_i(u) \subset B(0, \Lambda). \tag{10.235}$$

We show that the following property holds:

(P7) for each $M \ge M_0$, each $i \in \{1, \dots, m\}$, each $u \in B(0, 3M + 1)$ satisfying $f_i(u) > 0$ and each $g \in X$ which satisfies

$$d(g, \partial f_i(u)) \le r(4M + 1)^{-1}4^{-1},$$

we have

$$\|g\| > r(M + 1)^{-1}2^{-1}.$$

Let

$$M \ge M_0,\ u \in B(0, 3M + 1),\ i \in \{1, \dots, m\},\ f_i(u) > 0 \tag{10.236}$$

and let $g \in X$ satisfy (10.235). Let

$$\xi \in \partial f_i(u). \tag{10.237}$$

By (10.224), (10.226), (10.236), and (10.237),

$$\begin{aligned} -r \geq f_i(z_i) > f_i(z_i) - f_i(u) \geq \langle \xi, z_i - u \rangle \\ \geq -\|\xi\| \|z_i - u\| \geq -\|\xi\|(4M + 1), \\ \|\xi\| \geq r(4M + 1)^{-1} \end{aligned}$$

and

$$\partial f_i(u) \subset \{\xi \in X : \ \|\xi\| \geq r(4M + 1)^{-1}\}.$$

Together with (10.235) this implies that

$$\|g\| > 2^{-1} r(4M + 1)^{-1}.$$

Thus (P7) holds.

Property (P7) implies that the following property holds:

(P8) let $M \geq M_0$, $i \in \{1, \ldots, m\}$, $u \in B(0, 3M + 1)$ satisfy $f_i(u) > 0$, $g \in X$ satisfy

$$d(g, \partial f_i(u)) \leq r(4M + 1)^{-1} 4^{-1}$$

and

$$u' \in u - f_i(u)\|g\|^{-2} g + B(0, 1).$$

Then

$$\|u'\| \leq 3M + 2 + 2(4M + 1) r^{-1} f_i(u).$$

For each $\gamma \geq 0$ denote by $\mathcal{K}_\gamma$ the set of all sequences $\{x_n\}_{n=1}^\infty \subset X$ such that

$$\|x_0\| \leq M_0$$

and there exist $\{\lambda_i\}_{i=1}^\infty \subset [0, \infty)$,

$$\tilde{\gamma}_n \in [0, \gamma], \ n \in \mathcal{N} \setminus \{0\}, \tag{10.238}$$

$$\{(\Omega_i, w_i)\}_{i=1}^\infty \subset \mathcal{M}_*, \tag{10.239}$$

satisfying (10.220) such that

$$(x_i, \lambda_i) \in A(x_{i-1}, (\Omega_i, w_i), \tilde{\gamma}_i, \gamma) \text{ for all integers } i \geq 1. \tag{10.240}$$

By induction we show that for all $m = 0, \dots, n_0$ the following assertion holds.

(A) For each $\gamma > 0$ there exists $\delta > 0$ such that, for each $\{x_n\}_{n=0}^{\infty} \in \mathcal{K}_\delta$, there is $\{y_n\}_{n=0}^{\infty} \in \mathcal{K}_0$ such that $\|y_n - x_n\| \le \gamma,\ n = 0, \dots, m$.

Clearly, for $m = 0$ this assertion holds. Assume that assertion (A) holds for $m = q$ where $q \in [0, n_0 - 1]$ is an integer. We show that (A) holds for $m = q + 1$. Set

$$M_1 = (M_0 + 1)(1 + r^{-1}) \tag{10.241}$$

and for each integer $i \ge 1$ set

$$\begin{aligned} M_{i+1} = 18(M_i + 1)(1 + r^{-1})(1 + \sup\{|f_s(h)| :\\ h \in B(0, 3M_i + 1),\ s = 1, \dots, m\}). \end{aligned} \tag{10.242}$$

Since (A) holds for $m = q$, it follows from (P8) that there is $\gamma_0 > 0$ such that

$$\gamma_0 < 2^{-1} \text{ and } \gamma_0 < 4^{-1} r(4M_q + 1)^{-1} \tag{10.243}$$

and that, for each $\{y_n\}_{n=0}^{\infty} \in \mathcal{K}_{\gamma_0}$,

$$\|y_n\| \le 3M_0 + 1/2,\ n = 0, \dots, q. \tag{10.244}$$

Assume that assertion (A) does not hold for $m = q + 1$. Then there exists $\gamma > 0$ such that for each natural number j there is

$$\{x_n^{(j)}\}_{n=0}^{\infty} \in \mathcal{K}_{\gamma_0/j} \tag{10.245}$$

such that

$$\max\{\|y_n - x_n^{(j)}\| :\ n = 0, \dots, q + 1\} > \gamma \text{ for each } \{y_n\}_{n=0}^{\infty} \in \mathcal{K}_0. \tag{10.246}$$

By (10.245) and the choice of γ_0 (see (10.244)), for all natural numbers j,

$$\|x_n^{(j)}\| \le 3M_0 + 1/2,\ n = 0, \dots, q. \tag{10.247}$$

By the definition of $\mathcal{K}_\gamma$, $\gamma \ge 0$, for each integer $j \ge 1$ there exist

$$\{\lambda_i^{(j)}\}_{i=1}^{\infty} \subset [0, \infty), \tag{10.248}$$

$$\tilde{\gamma}_{j,n} \in [0, \gamma_0/j],\ n \in \mathcal{N} \setminus \{0\}, \tag{10.249}$$

$$\{(\Omega_i^{(j)}, w_i^{(j)})\}_{i=1}^{\infty} \subset \mathcal{M}_* \tag{10.250}$$

such that for each natural number s,

$$\{1, \dots, m\} \subset \cup_{i=s}^{s+\bar{N}-1}(\cup_{t \in \Omega_i^{(j)}}\{t_1, \dots, t_{p(t)}\}), \tag{10.251}$$

$$(x_i^{(j)}, \lambda_i^{(j)}) \in A(x_{i-1}^{(j)}, (\Omega_i^{(j)}, w_i^{(j)}), \tilde{\gamma}_{j,i}, \gamma_0/j) \tag{10.252}$$

for all integers $i \geq 1$.

In view of (10.115) and (10.150), extracting a subsequence and re-indexing if necessary, we may assume that

$$\Omega_i^{(j)} = \Omega_i^{(1)} \text{ for all pairs of natural numbers } i, j. \tag{10.253}$$

Set

$$\Omega_i = \Omega_i^{(1)} \text{ for all natural numbers } i. \tag{10.254}$$

Let j be a natural number. By (10.215) and (10.252), for each natural number s there exist

$$(y_t^{(j,s)}, \lambda_t^{(j,s)}) \in A_0(t, x_{s-1}^{(j)}, \tilde{\gamma}_{j,s}, \gamma_0/j),\ t \in \Omega_s \tag{10.255}$$

such that

$$\lambda_s^{(j)} = \max\{\lambda_t^{(j,s)} :\ t \in \Omega_s\}, \tag{10.256}$$

$$\|x_s^{(j)} - \sum_{t \in \Omega_s} w_s^{(j)}(t) y_t^{(j,s)}\| \leq \gamma_0/j. \tag{10.257}$$

By (10.211)–(10.214) and (10.255), for each natural number s and each

$$t \in \Omega_s, \tag{10.258}$$

there exists finite a sequence

$$\{y_{t,i}^{(j,s)}\}_{i=0}^{p(t)} \subset X \tag{10.259}$$

such that

$$y_{t,0}^{(j,s)} = x_{s-1}^{(j)}, \tag{10.260}$$

for each $i = 1, \dots, p(t)$,

$$y_{t,i}^{(j,s)} \in A_{t_i}(y_{t,i-1}^{(j,s)}, \tilde{\gamma}_{j,s}, \gamma_0/j), \tag{10.261}$$

$$y_t^{(j,s)} = y_{t,p(t)}^{(j,s)}, \tag{10.262}$$

$$\lambda_t^{(j,s)} = \max\{\|y_{t,i}^{(j,s)} - y_{t,i-1}^{(j,s)}\| :\ i = 1, \dots, p(t)\}. \tag{10.263}$$

By (10.209), (10.210), and (10.261), for each natural number s, each $t = (t_1, \dots, t_{p(t)}) \in \Omega_s$ and each $i \in \{1, \dots, p(t)\}$, if

$$f_{t_i}(y_{t,i-1}^{(j,s)}) \le \tilde{\gamma}_{j,s}, \tag{10.264}$$

then

$$y_{t,i}^{(j,s)} = y_{t,i-1}^{(j,s)}; \tag{10.265}$$

if

$$f_{t_i}(y_{t,i-1}^{(j,s)}) > \tilde{\gamma}_{j,s},$$

then there exists

$$g_{t,i}^{(j,s)} \in \partial f_{t_i}(y_{t,i-1}^{(j,s)}) + B(0, \gamma_0/j) \setminus \{0\} \tag{10.266}$$

such that

$$y_{t,i}^{(j,s)} \in y_{t,i-1}^{(j,s)} - f_{t_i}(y_{t,i-1}^{(j,s)}) \|g_{t,i}^{(j,s)}\|^{-2} g_{t,i}^{(j,s)} + B(0, \gamma_0/j). \tag{10.267}$$

For each natural number s, each $t = (t_1, \dots, t_{p(t)} \in \Omega_s$ and each $i \in \{1, \dots, p(t)\}$ satisfying (10.264) set

$$g_{t,i}^{(j,s)} = 0. \tag{10.268}$$

Let s be a natural number such that

$$s \le q + 1 \text{ and } t = (t_1, \dots, t_{p(t)}) \in \Omega_s. \tag{10.269}$$

By induction we show that for all $i = 1, \dots, p(t)$,

$$\|y_{t,i}^{(j,s)}\| \le 3M_i + 1. \tag{10.270}$$

In view of (10.247) and (10.260), (10.270) holds for $i = 0$.

Assume that an integer $i \in \{1, \dots, p(t)\}$ satisfies

$$\|y_{t,i-1}^{(j,s)}\| \le 3M_{i-1} + 1. \tag{10.271}$$

If (10.264) is true, then by (10.242), (10.265), and (10.271),

$$\|y_{t,i}^{(j,s)}\| = \|y_{t,i-1}^{(j,s)}\| \le 3M_{i-1} + 1 \le 3M_i + 1. \tag{10.272}$$

Assume that

$$f_{t_i}(y_{t,i-1}^{(j,s)}) > \tilde{\gamma}_{j,s}. \tag{10.273}$$

By (10.273), there exists $g_{t,i}^{(j,s)}$ satisfying (10.266) such that $y_{t,i}^{(j,s)}$ satisfies (10.267). It follows from (10.241), (10.242), (10.243), (10.267), (10.271), (10.273), and (P8) that

$$\|y_{t,i}^{(j,s)}\| \leq 3M_{i-1} + 2 + 2(4M_{i-1} + 1)r^{-1} \sup\{|f_{t_i}(\eta)| :$$
$$\eta \in B(0, 3M_{i-1} + 1)\} \leq M_i.$$

Thus (10.270) holds for all $i = 0, \dots, p(t)$. Hence

$$\|y_{t,i}^{(j,s)}\| \leq 3M_i + 1 \leq 3M_q + 1 \tag{10.274}$$

for all natural numbers j, each natural number $s \leq q+1$, each $t = (t_1, \dots, t_{p(t)}) \in \Omega_s$ and all $i = 0, \dots, p(t)$.

Since the functions f_i, $i = 1, \dots, m$ are Lipschitz on bounded subsets of R^n it follows from (10.214), (10.266), and (10.268) and property (P8) that there exists a constant $\tilde{M} > 0$ such that

$$\|g_{t,i}^{(j,s)}\| \leq \tilde{M} \tag{10.275}$$

for all natural numbers j, each natural number $s \leq q+1$, each $t = (t_1, \dots, t_{p(t)}) \in \Omega_s$ and all $i = 1, \dots, p(t)$.

By (10.247), (10.274), and (10.275), extracting a subsequence and re-indexing, if necessary, we may assume without any loss of generality that for any $s \in \{0, \dots, q\}$ there is

$$x_s = \lim_{j \to \infty} x_s^{(j)} \tag{10.276}$$

and that for every natural number $s \leq q + 1$ and every $t = (t_1, \dots, t_{p(t)}) \in \Omega_s$ there exist

$$y_{t,i}^{(s)} = \lim_{j \to \infty} y_{t,i}^{(j,s)} \text{ for all } i = 0, \dots, p(t), \tag{10.277}$$

$$g_{t,i}^{(s)} = \lim_{j \to \infty} g_{t,i}^{(j,s)}, \; i = 1, \dots, p(t), \tag{10.278}$$

for all integers $s \geq 1$ and all $t \in \Omega_s$ there exists

$$w_s(t) = \lim_{j \to \infty} w_s^{(j)}(t). \tag{10.279}$$

For each natural number $s \leq q + 1$ and each $t \in \Omega_s$ set

$$y_t^{(s)} = y_{t,p(t)}^{(s)}. \tag{10.280}$$

In view of (10.112), (10.116), and (10.279), for each integer $s \geq 1$,

$$\sum_{t \in \Omega_s} w_s(t) = 1, \tag{10.281}$$

$$w_s(t) \geq \Delta,\ t \in \Omega_s, \tag{10.282}$$

$$\{(\Omega_s, w_s) :\ s = 1, 2, \dots\} \subset \mathcal{M}_*. \tag{10.283}$$

By (10.245) and (10.276),

$$\|x_0\| \leq M_0. \tag{10.284}$$

Assume that

$$s \in \{0, \dots, q\}. \tag{10.285}$$

It follows from (10.257), (10.276), (10.278), (10.279), and (10.280) that

$$\begin{aligned} & x_s - \sum_{t \in \Omega_s} w_s(t) y_t^{(s)} \\ &= \lim_{j \to \infty} x_s^{(j)} - \sum_{t \in \Omega_s} \lim_{j \to \infty} w_s^{(j)}(t) \lim_{j \to \infty} y_{t,p(t)}^{(j,s)} \\ &= \lim_{j \to \infty} [x_s^{(j)} - \sum_{t \in \Omega_s} w_s^{(j)}(t) y_t^{(j,s)}] = 0 \end{aligned} \tag{10.286}$$

for all $s \in \{0, \dots, q\}$.

Let

$$s \in \{0, \dots, q+1\} \tag{10.287}$$

and

$$t = (t_1, \dots, t_{p(t)}) \in \Omega_s. \tag{10.288}$$

In view of (10.260) and (10.276), if

$$s \geq 1, \tag{10.289}$$

then

$$x_{s-1} = \lim_{j \to \infty} x_{s-1}^{(j)} = \lim_{j \to \infty} y_{t,0}^{(j,s)} = y_{t,0}^{(s)}. \tag{10.290}$$

Let

$$i \in \{1, \dots, p(t)\}. \tag{10.291}$$

There are two cases:

$$f_{t_i}(y_{t,i-1}^{(s)}) > 0; \tag{10.292}$$

$$f_{t_i}(y_{t_i-1}^{(s)}) \le 0. \tag{10.293}$$

Assume that (10.292) holds. By (10.249), (10.266), (10.267), (10.277), and (10.292), for all sufficiently large natural numbers j,

$$f_{t_i}(y_{t,i-1}^{(j,s)}) > 2^{-1} f_{t_i}(y_{t,i-1}^{(s)}) > \gamma_0/j \ge \tilde{\gamma}_{j,s} \tag{10.294}$$

and

$$\|y_{t,i}^{(j,s)} - y_{t,i-1}^{(j,s)} + f_{t_i}(y_{t,i-1}^{(j,s)})\|g_{t,i}^{(j,s)}\|^{-2} g_{t,i}^{(j,s)}\| \le \gamma_0/j. \tag{10.295}$$

By (10.266), (10.277), (10.278), and (10.294), for each $u \in X$,

$$f_{t_i}(u) - f_{t_i}(y_{t,i-1}^{(s)}) = \lim_{j\to\infty} (f_{t_i}(u) - f_{t_i}(y_{t,i-1}^{(j,s)}))$$

$$\ge \lim_{j\to\infty} \langle g_{t,i}^{(j,s)}, u - y_{t,i-1}^{(j,s)} \rangle \ge \langle g_{t,i}^{(s)}, u - y_{t,i-1}^{(s)} \rangle,$$

$$g_{t,i}^{(s)} \in \partial f_{t_i}(y_{t,i-1}^{(s)}). \tag{10.296}$$

It follows from (10.266), (10.274), (10.277), (10.278), (10.293)–(10.295), and (P7) that

$$y_{t,i}^{(s)} = \lim_{j\to\infty} y_{t,i}^{(j,s)}$$

$$= \lim_{j\to\infty} [y_{t,i-1}^{(j,s)} + f_{t_i}(y_{t,i-1}^{(j,s)})\|g_{t,i}^{(j,s)}\|^{-2} g_{t,i}^{(j,s)}]$$

$$= y_{t,i-1}^{(s)} + \|g_{t,i}^{(s)}\|^{-2} g_{t,i}^{(s)} f_{t_i}(y_{t,i-1}^{(s)}). \tag{10.297}$$

Assume that (10.293) holds. There are two cases:

$$f_{t_i}(y_{t,i-1}^{(j,s)}) < \tilde{\gamma}_{j,i-1} \text{ for infinitely many integers } j \ge 1, \tag{10.298}$$

$$f_{t_i}(y_{t,i-1}^{(j,s)}) < \tilde{\gamma}_{j,i-1} \text{ only for finite numbers of integers } j \ge 1. \tag{10.299}$$

If (10.298) holds, then by (10.264), (10.265), and (10.277),

$$y_{t,i}^{(s)} = \lim_{j\to\infty} y_{t,i}^{(j,s)} = \lim_{j\to\infty} y_{t,i-1}^{(j,s)} = y_{t,i-1}^{(s)}.$$

Assume that (10.299) holds. Then there exists $j_0 \in \mathcal{N} \setminus \{0\}$ such that

$$f_{t_i}(y_{t,i-1}^{(j,s)}) \geq \tilde{\gamma}_{j,i-1} \text{ for all integers } j \geq j_0. \tag{10.300}$$

By (10.277), (10.293), and (10.300),

$$f_{t_i}(y_{t,i-1}^{(s)}) = 0. \tag{10.301}$$

It follows from (10.267), (10.277), (10.300), and (10.301) that

$$y_{t,i}^{(s)} = \lim_{j\to\infty} y_{t,i}^{(j,s)}$$

$$= \lim_{j\to\infty} [y_{t,i-1}^{(j,s)} + \|g_{t,i}^{(j,s)}\|^{-2} g_{t,i}^{(j,s)} f_{t_i}(y_{t,i-1}^{(j,s)})] = y_{t,i-1}^{(j,s)}.$$

Thus

$$y_{t,i}^{(s)} = y_{t,i-1}^{(s)} \tag{10.302}$$

in both cases.

Set

$$x_{q+1} = \sum_{t\in\Omega_{q+1}} w_{q+1}(t) y_t^{(q+1)}. \tag{10.303}$$

In view of (10.257), (10.262), (10.277), (10.279), (10.280), and (10.303),

$$x_{q+1} = \lim_{j\to\infty} x_{q+1}^{(j)}.$$

Clearly, there exist $x_s \in X$ for all integers $s > q+1$ such that $\{x_i\}_{i=0}^{\infty} \in \mathcal{K}_0$. For all sufficiently large natural numbers j,

$$\|x_n^{(j)} - x_n\| < \gamma/2, \ n = 0, \ldots, q+1.$$

This contradicts (10.246). The contradiction we have reached proves that (A) holds for $m = q+1$. By induction we showed that (A) holds for $m = n_0$.

Fix a positive number γ_1 such that

$$\gamma_1 \leq \epsilon/4, \ \gamma_1 \leq \Lambda^{-1}\epsilon/4.$$

By (A) with $m = n_0$ there is $\delta > 0$ such that for each $\{x_n\}_{n=0}^{\infty} \in \mathcal{K}_\delta$, there is $\{y_n\}_{n=0}^{\infty} \in \mathcal{K}_0$ such that

$$\|y_n - x_n\| \leq \gamma_1, \ n = 0, \ldots, n_0. \tag{10.304}$$

Let $\{x_n\}_{n=0}^{\infty} \in \mathcal{K}_\delta$. By the choice of δ, there is

$$\{y_n\}_{n=0}^{\infty} \in \mathcal{K}_0 \tag{10.305}$$

such that (10.304) hold. Property (P6) and (10.305) imply that

$$\|y_n\| \le 3M_0 \text{ for all integers } n \ge 0, \tag{10.306}$$

$$f_i(y_n) \le \epsilon/4 \text{ for all } i \in \{1, \dots, m\} \text{ and all integers } n \ge n_0, \tag{10.307}$$

$$d(x_n, C) \le \epsilon/4 \text{ for all integers } n \ge n_0. \tag{10.308}$$

In view of (10.304)–(10.308),

$$d(x_{n_0}, C) \le \|x_{n_0} - y_{n_0}\| + d(y_{n_0}, C) \le \epsilon/2,$$

$$\|x_n\| \le 3M_0 + 2^{-1},\ n = 0, \dots, n_0. \tag{10.309}$$

By (10.228), (10.304), (10.306), (10.307), (10.309) and the inequalities

$$\gamma_1 \le \epsilon/4,\ \gamma_1 \le \Lambda^{-1}\epsilon/4,$$

for all $i \in \{1, \dots, m\}$,

$$f_i(x_{n_0}) \le f_i(y_{n_0}) + |f_i(x_{n_0}) - f_i(y_{n_0})| \le \epsilon/4 + \Lambda\|x_{n_0} - y_{n_0}\| < \epsilon.$$

Theorem 10.13 is proved. □

10.13 Extensions

Theorem 10.13 implies the following result.

Theorem 10.14. *Let* $M_0 > 0,\ \epsilon \in (0, 1)$,

$$B(0, M_0) \cap C \neq \emptyset,$$

$$\{z \in X :\ f_i(z) < 0\} \cap B(0, M_0) \neq \emptyset \textit{ for all } i = 1, \dots, m.$$

Let a natural number n_0 *and* $\delta > 0$ *be as guaranteed by Theorem 10.13. Assume that (10.218)–(10.223) hold and* $\{x_n\}_{i=1}^{\infty} \subset B(0, M_0)$. *Then for all integers* $n \ge n_0$,

$$d(x_n, C) \le \epsilon,$$

$$f_i(x_n) \le \epsilon,\ i \in \{1, \dots, m\}.$$

Theorem 10.14 easily implies the following result.

Theorem 10.15. *Let* $M_0 > 0$,

$$B(0, M_0) \cap C \neq \emptyset,$$
$$\{z \in X : f_i(z) < 0\} \cap B(0, M_0) \neq \emptyset \text{ for all } i = 1, \dots, m,$$

$\{\delta_n\}_{n=0}^{\infty} \subset (0, \infty)$, $\lim_{n\to\infty} \delta_n = 0$, $\epsilon \in (0, 1)$. *Then there exist a natural number* n_ϵ *such that the following assertion holds.*

Assume that

$$\tilde{\delta}_n \in [0, \delta_n],\ n \in \mathcal{N} \setminus \{0\},$$

(10.219)–(10.222) hold,

$$\{x_i\}_{i=0}^{\infty} \subset B(0, M_0)$$

for each natural number i,

$$(x_i, \lambda_i) \in A(x_{i-1}, (\Omega_i, w_i), \tilde{\delta}_i, \delta_i).$$

Then for all integers $n \geq n_\epsilon$,

$$d(x_n, C) \leq \epsilon,$$
$$f_i(x_n) \leq \epsilon,\ i \in \{1, \dots, m\}.$$

Theorem 10.16. *Suppose that the set C is bounded,* $M_0 > 0$,

$$B(0, M_0) \cap C \neq \emptyset,$$
$$\{z \in X : f_i(z) < 0\} \cap B(0, M_0) \neq \emptyset \text{ for all } i = 1, \dots, m,$$

$M_1 > 0$, $\epsilon \in (0, 1)$. *Then there exist a natural number* n_0 *and* $\delta > 0$ *such that the following assertion holds.*

Assume that

$$\tilde{\delta}_n \in [0, \delta],\ n \in \mathcal{N} \setminus \{0\}, \tag{10.310}$$
$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*, \tag{10.311}$$

satisfies (10.220) for each natural number j,

$$x_0 \in B(0, M_1), \tag{10.312}$$

$$\{x_i\}_{i=1}^{\infty} \subset X,\ \{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty) \tag{10.313}$$

satisfy for each natural number i,

$$(x_i, \lambda_i) \in A(x_{i-1}, (\Omega_i, w_i), \tilde{\delta}_i, \delta). \tag{10.314}$$

Then for all integers $n \geq n_0$,

$$d(x_n, C) \leq \epsilon,$$
$$f_i(x_{n_0}) \leq \epsilon, \ i \in \{1, \ldots, m\}.$$

Proof. For each $i \in \{1, \ldots, m\}$, there exists $z_i \in X$ such that

$$z_i \in B(0, M_0), \ f_i(z_i) < 0.$$

We may assume without loss of generality that

$$M_1 > \sup\{\|z\| : \ z \in C\} + 4, \tag{10.315}$$
$$M_0 > 3M_1 + 1. \tag{10.316}$$

By Theorem 10.13, there exist a natural number n_1 and $\gamma_1 > 0$ such that the following assertion holds.

(i) for each

$$\tilde{\delta}_n \in [0, \gamma_1], \ n \in \mathcal{N} \setminus \{0\},$$

each

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*,$$

satisfying (10.220) for each natural number j, each

$$x_0 \in B(0, M_1),$$

each pair of sequences

$$\{x_i\}_{i=1}^{\infty} \subset X, \ \{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty)$$

satisfying for each natural number i,

$$(x_i, \lambda_i) \in A(x_{i-1}, (\Omega_i, w_i), \tilde{\delta}_i, \gamma_1)$$

we have

$$\|x_i\| \leq 3M_1 + 1 \text{ for all integers } i = 0, \ldots, n_1,$$
$$d(x_{n_1}, C) \leq \epsilon.$$

By Theorem 10.14, there exist a natural number n_0 and $\delta \in (0, \gamma_1)$ such that the following property hold:

(ii) for each

$$\tilde{\delta}_n \in [0, \delta], \; n \in \mathcal{N} \setminus \{0\},$$

each

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$$

satisfying (10.220) for each natural number j and (10.221)–(10.223), and each

$$\{x_n\}_{n=0}^{\infty} \subset B(0, M_0),$$

for all integers $n \geq n_0$ we have

$$d(x_n, C) \leq \epsilon,$$
$$f_i(x_n) \leq \epsilon, \; i \in \{1, \ldots, m\}.$$

Assume that (10.310) and (10.311) hold, for all integers $j \geq 1$ (10.220) is true and that (10.312), (10.313) hold. By (i) and (10.315), (10.314) is true for all integers $i \geq 1$,

$$\|x_{nn_1}\| \leq M_1, \; n \in \mathcal{N}, \; \|x_n\| \leq 3M_1 + 1, \; n \in \mathcal{N}. \tag{10.317}$$

In view of property (ii), (10.316), and (10.317), for all integers $n \geq n_0$,

$$d(x_n, C) \leq \epsilon,$$
$$f_i(x_n) \leq \epsilon, \; i \in \{1, \ldots, m\}.$$

This completes the proof of Theorem 10.16. □

Theorem 10.16 implies the following result.

Theorem 10.17. *Let $M_0 > 0$, the set C be bounded,*

$$\{z \in X : \; f_i(z) < 0\} \neq \emptyset \text{ for all } i = 1, \ldots, m.$$

Then there exists $\delta > 0$ such that the following assertion holds.

Assume that

$$\{\delta_n\}_{n=0}^{\infty} \subset (0, \delta), \; \lim_{n \to \infty} \delta_n = 0, \; \epsilon > 0.$$

Then there exists a natural number n_ϵ such that for each $\tilde{\delta}_n \in [0, \delta_n]$, $n \in \mathcal{N} \setminus \{0\}$, each

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$$

satisfying (10.220) for each natural number j, each

$$x_0 \in B(0, M_0),$$

each $\{x_i\}_{i=1}^{\infty} \subset X$*, each* $\{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty)$ *satisfying for each natural number i,*

$$(x_i, \lambda_i) \in A(x_{i-1}, (\Omega_i, w_i), \tilde{\delta}_i, \delta_i),$$

the inequalities

$$d(x_n, C) \leq \epsilon,$$

$$f_i(x_n) \leq \epsilon, \ i \in \{1, \ldots, m\}$$

hold for all integers $n \geq n_\epsilon$.