Chapter 1
Introduction

In this book we study approximate solutions of common fixed point problems and of convex feasibility problems in the presence of computational errors. A convex feasibility problem is to find a point which belongs to the intersection of a given finite family of subsets of a Hilbert space. This problem is a special case of a common fixed point problem which is to find a common fixed point of a finite family of nonlinear mappings in a Hilbert space. Our goal is to obtain a good approximate solution of the problem in the presence of computational errors. We show that the algorithm generates a good approximate solution, if the sequence of computational errors is bounded from above by a constant. In this section we discuss several algorithms which are studied in the book.

1.1 Common Fixed Point Problems in a Hilbert Space

In Chap. 2 we study the convergence of dynamic string-averaging methods which were introduced for solving a convex feasibility problem, when a given collection of sets is divided into blocks and the algorithms operate in such a manner that all the blocks are processed in parallel. Iterative methods for solving common fixed point problems is a special case of dynamic string-averaging methods with only one block. Iterative methods and dynamic string-averaging methods are important tools for solving common fixed point problems in a Hilbert space [1, 3, 5–7, 10, 12, 13, 15, 16, 22, 23, 26, 27, 30–35, 37–41, 43, 45–49, 52–54, 67, 74, 75, 84, 85, 89, 95].

Let $(X, \langle \cdot, \cdot \rangle)$ be a Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ which induces a complete norm $\| \cdot \|$.

For each $x \in X$ and each nonempty set $E \subset X$ put

$$d(x, E) = \inf\{\|x - y\| : y \in E\}.$$

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For every point $x \in X$ and every positive number $r > 0$ set

$$B(x, r) = \{y \in X : \ \|x - y\| \le r\}.$$

Suppose that m is a natural number, $\bar{c} \in (0, 1)$, $P_i : X \to X$, $i = 1, \dots, m$, for every integer $i \in \{1, \dots, m\}$,

$$\mathrm{Fix}(P_i) := \{z \in X : \ P_i(z) = z\} \neq \emptyset$$

and that the inequality

$$\|z - x\|^2 \ge \|z - P_i(x)\|^2 + \bar{c}\|x - P_i(x)\|^2$$

holds for every integer $i \in \{1, \dots, m\}$, every point $x \in X$, and every point $z \in \mathrm{Fix}(P_i)$. Set

$$F = \cap_{i=1}^m \mathrm{Fix}(P_i).$$

For every positive number ϵ and every integer $i \in \{1, \dots, m\}$ set

$$F_\epsilon(P_i) = \{x \in X : \ \|x - P_i(x)\| \le \epsilon\},$$
$$\tilde{F}_\epsilon(P_i) = F_\epsilon(P_i) + B(0, \epsilon),$$
$$F_\epsilon = \cap_{i=1}^m F_\epsilon(P_i)$$

and

$$\tilde{F}_\epsilon = \cap_{i=1}^m \tilde{F}_\epsilon(P_i)$$

A point belonging to the set F is a solution of our common fixed point problem while a point which belongs to the set $\tilde{F}_\epsilon$ is its ϵ-approximate solution.

In Chap. 2 we obtain a good approximative solution of the common fixed point problem applying a dynamic string-averaging method with variable strings and weights which is described below.

By an index vector, we a mean a vector $t = (t_1, \dots, t_p)$ such that $t_i \in \{1, \dots, m\}$ for all $i = 1, \dots, p$.

For an index vector $t = (t_1, \dots, t_q)$ set

$$p(t) = q, \ P[t] = P_{t_q} \cdots P_{t_1}.$$

It is not difficult to see that for each index vector t

$$P[t](x) = x \text{ for all } x \in F,$$
$$\|P[t](x) - P[t](y)\| = \|x - P[t](y)\| \le \|x - y\|$$

for every point $x \in F$ and every point $y \in X$.

Denote by $\mathcal{M}$ the collection of all pairs (Ω, w), where Ω is a finite set of index vectors and

$$w : \Omega \to (0, \infty) \text{ satisfies } \sum_{t \in \Omega} w(t) = 1.$$

Let $(\Omega, w) \in \mathcal{M}$. Define

$$P_{\Omega,w}(x) = \sum_{t \in \Omega} w(t) P[t](x), \ x \in X.$$

It is easy to see that

$$P_{\Omega,w}(x) = x \text{ for all } x \in F,$$
$$\|P_{\Omega,w}(x) - P_{\Omega,w}(y)\| = \|x - P_{\Omega,w}(y)\| \le \|x - y\|$$

for every point $x \in F$ and every point $y \in X$.

The dynamic string-averaging method with variable strings and variable weights can now be described by the following algorithm.

Initialization: select an arbitrary point $x_0 \in X$.

Iterative step: given a current iteration vector x_k pick a pair

$$(\Omega_{k+1}, w_{k+1}) \in \mathcal{M}$$

and calculate the next iteration vector x_{k+1} by

$$x_{k+1} = P_{\Omega_{k+1}, w_{k+1}}(x_k).$$

Fix a number

$$\Delta \in (0, m^{-1}]$$

and an integer

$$\bar{q} \ge m.$$

Denote by $\mathcal{M}_*$ the set of all $(\Omega, w) \in \mathcal{M}$ such that

$$p(t) \le \bar{q} \ \text{ for all } t \in \Omega,$$
$$w(t) \ge \Delta \text{ for all } t \in \Omega.$$

Fix a natural number $\bar{N}$.

In the studies of the common fixed point problem the goal is to find a point $x \in F$. In order to meet this goal we apply an algorithm generated by

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$$

such that for each natural number j,

$$\{1,\dots,m\} \subset \cup_{i=j}^{j+\bar{N}-1}(\cup_{t\in\Omega_i}\{t_1,\dots,t_{p(t)}\}).$$

This algorithm generates, for any starting point $x_0 \in X$, a sequence $\{x_k\}_{k=0}^{\infty} \subset X$, where

$$x_{k+1} = P_{\Omega_{k+1},w_{k+1}}(x_k).$$

According to the results known in the literature, this sequence should converge to an element of F. In Chap. 2, we study the behavior of the sequences generated by $\{(\Omega_i, w_i)\}_{i=1}^{\infty}$ taking into account computational errors which always present in practice. These computational errors are bounded from above by a small constant depending only on our computer system which is denoted by δ. This computational error δ presents in all calculations which we do using our computer system. For example, if $x \in X$ and $i \in \{1,\dots,m\}$ and we need to calculate $P_i(x)$, then using our computer system we obtain a point $y \in X$ satisfying

$$\|y - P_i(x)\| \le \delta.$$

If k is a natural number, $y_i \in X$, $i = 1,\dots,k$, $\alpha_i > 0$, $i = 1,\dots,k$ satisfying $\sum_{i=1}^{k}\alpha_i = 1$ and if need to calculate $\sum_{i=1}^{k}\alpha_i y_i$, then by using our computer system we obtain a point $y \in X$ satisfying

$$\|y - \sum_{i=1}^{k}\alpha_i y_i\| \le \delta.$$

Surely, in this situation one cannot expect that the sequence of iterates generated by our algorithm converges to the set F. Our goal is to understand what approximate solutions of the common fixed point problem can be obtained.

In Chap. 2 we prove Theorem 2.1, which shows that in the presence of computational errors bounded from above by a constant δ, an ϵ-approximate solution can be obtained after $n\bar{N}$ iterations of the algorithm. Note that $\epsilon = c_1\delta^{1/2}$ and $n = \lfloor c_2\delta^{-1}\rfloor$, where c_1 and c_2 are positive constants which do not depend on δ and $\lfloor u\rfloor$ denotes the integer part of u.

1.2 Proximal Point Algorithm

Proximal point method is an important tool in solving optimization problems [4, 42, 44, 56, 59, 61, 68, 69, 78, 88]. It is also used for solving variational inequalities with monotone operators [2, 8, 11, 17–21, 25, 57, 60, 62–64, 79, 82, 83, 91, 92] which is an important topic of nonlinear analysis and optimization [9, 14, 28, 29, 36, 51, 55, 58, 65, 66, 80, 81, 86, 87, 90]. In Chap. 8 we study the convergence of an iterative proximal point method to a common zero of a finite family of maximal monotone operators in a Hilbert space, under the presence of computational errors. Most results known in the literature establish the convergence of proximal point methods, when computational errors are summable. In Chap. 8, the convergence of the method is proved for nonsummable computational errors. We show that the proximal point method generates a good approximate solution, if the sequence of computational errors is bounded from above by a constant. Moreover, for a known computational error, we find out what an approximate solution can be obtained and how many iterates one needs for this.

Let $(X, \langle\cdot,\cdot\rangle)$ be a Hilbert space equipped with an inner product $\langle\cdot,\cdot\rangle$ which induces the norm $\|\cdot\|$.

A multifunction $T : X \to 2^X$ is called a monotone operator if and only if

$$\langle z - z', w - w'\rangle \ \geq 0 \quad \forall z, z', w, w' \in X$$
$$\text{such that } w \in T(z) \text{ and } w' \in T(z').$$

It is called maximal monotone if, in addition, the graph

$$\{(z, w) \in X \times X : \ w \in T(z)\}$$

is not properly contained in the graph of any other monotone operator $T' : X \to 2^X$. A fundamental problem consists in determining an element z such that $0 \in T(z)$. For example, if T is the subdifferential ∂f of a lower semicontinuous convex function $f : X \to (-\infty, \infty]$, which is not identically infinity, then T is maximal monotone (see [71, 73]), and the relation $0 \in T(z)$ means that z is a minimizer of f.

Let $T : X \to 2^X$ be a maximal monotone operator. The proximal point algorithm generates, for any given sequence of positive real numbers and any starting point in the space, a sequence of points and the goal is to show the convergence of this sequence. Note that in a general infinite-dimensional Hilbert space this convergence is usually weak. The proximal algorithm for solving the inclusion $0 \in T(z)$ is based on the fact established by Minty [70], who showed that, for each $z \in X$ and each $c > 0$, there is a unique $u \in X$ such that

$$z \in (I + cT)(u),$$

where $I : X \to X$ is the identity operator ($Ix = x$ for all $x \in X$).

The operator

$$P_{c,T} := (I + cT)^{-1}$$

is therefore single-valued from all of X onto X (where c is any positive number). It is also nonexpansive:

$$\|P_{c,T}(z) - P_{c,T}(z')\| \leq \|z - z'\| \text{ for all } z, z' \in X$$

and

$$P_{c,T}(z) = z \text{ if and only if } 0 \in T(z).$$

Following the terminology of Moreau [73] $P_{c,T}$ is called the proximal mapping associated with cT.

The proximal point algorithm generates, for any given sequence $\{c_k\}_{k=0}^{\infty}$ of positive real numbers and any starting point $z^0 \in X$, a sequence $\{z^k\}_{k=0}^{\infty} \subset X$, where

$$z^{k+1} := P_{c_k,T}(z^k), \; k = 0, 1, \ldots$$

It is not difficult to see that the

$$\mathrm{graph}(T) := \{(x, w) \in X \times X : \; w \in T(x)\}$$

is closed in the norm topology of $X \times X$.

Set

$$F(T) = \{z \in X : \; 0 \in T(z)\}.$$

Usually algorithms considering in the literature generate sequences which converge weakly to an element of $F(T)$. In Chap. 8, for a given $\epsilon > 0$, we are interested to find a point x for which there is $y \in T(x)$ such that $\|y\| \leq \epsilon$. This point x is considered as an ϵ-approximate solution.

For every point $x \in X$ and every nonempty set $A \subset X$ define

$$d(x, A) := \inf\{\|x - y\| : \; y \in A\}.$$

For every point $x \in X$ and every positive number r put

$$B(x, r) = \{y \in X : \; \|x - y\| \leq r\}.$$

We denote by $\mathrm{Card}(A)$ the cardinality of the set A.

We apply the proximal point algorithm in order to obtain a good approximation of a point which is a common zero of a finite family of maximal monotone operators and a common fixed point of a finite family of quasi-nonexpansive operators.

Let $\mathcal{L}_1$ be a finite set of maximal monotone operators $T : X \to 2^X$ and $\mathcal{L}_2$ be a finite set of mappings $T : X \to X$. We suppose that the set $\mathcal{L}_1 \cup \mathcal{L}_2$ is nonempty. (Note that one of the sets $\mathcal{L}_1$ or $\mathcal{L}_2$ may be empty.)

Let $\bar{c} \in (0, 1]$ and let $\bar{c} = 1$, if $\mathcal{L}_2 = \emptyset$.

We suppose that

$$F(T) \neq \emptyset \text{ for any } T \in \mathcal{L}_1$$

and that for every mapping $T \in \mathcal{L}_2$,

$$\mathrm{Fix}(T) := \{z \in X : \ T(z) = z\} \neq \emptyset,$$
$$\|z - x\|^2 \geq \|z - T(x)\|^2 + \bar{c}\|x - T(x)\|^2$$
$$\text{for all } x \in X \text{ and all } z \in \mathrm{Fix}(T).$$

Let $\bar{\lambda} > 0$ and let $\bar{\lambda} = \infty$ and $\bar{\lambda}^{-1} = 0$, if $\mathcal{L}_1 = \emptyset$. Let a natural number

$$l \geq \mathrm{Card}(\mathcal{L}_1 \cup \mathcal{L}_2).$$

Denote by $\mathcal{R}$ the set of all mappings

$$S : \{0, 1, 2, \dots\} \to \mathcal{L}_2 \cup \{P_{c,T} : \ T \in \mathcal{L}_1, \ c \in [\bar{\lambda}, \infty)\}$$

such that the following properties hold:

(P1) for every nonnegative integer p and every mapping $T \in \mathcal{L}_2$ there exists an integer $i \in \{p, \dots, p + l - 1\}$ satisfying $S(i) = T$;

(P2) for every nonnegative integer p and every monotone operator $T \in \mathcal{L}_1$ there exist an integer $i \in \{p, \dots, p+l-1\}$ and a number $c \geq \bar{\lambda}$ satisfying that $S(i) = P_{c,T}$.

Suppose that

$$F := (\cap_{T\in\mathcal{L}_1} F(T)) \cap (\cap_{Q\in\mathcal{L}_2} \mathrm{Fix}(Q)) \neq \emptyset.$$

Let $\epsilon > 0$. For every monotone operator $T \in \mathcal{L}_1$ define

$$F_\epsilon(T) = \{x \in X : \ T(x) \cap B(0, \epsilon) \neq \emptyset\}$$

and for every mapping $T \in \mathcal{L}_2$ set

$$\mathrm{Fix}_\epsilon(T) = \{x \in X : \ \|T(x) - x\| \leq \epsilon\}.$$

Define

$$F_\epsilon = (\cap_{T\in\mathcal{L}_1} F_\epsilon(T)) \cap (\cap_{Q\in\mathcal{L}_2} \mathrm{Fix}_\epsilon(Q)),$$
$$\tilde{F}_\epsilon = (\cap_{T\in\mathcal{L}_1} \{x \in X : \ d(x, F_\epsilon(T)) \leq \epsilon\})$$
$$\cap(\cap_{Q\in\mathcal{L}_2} \{x \in X : \ d(x, \mathrm{Fix}_\epsilon(Q)) \leq \epsilon\}).$$

We are interested to find solutions of the inclusion $x \in F$. In order to meet this goal we apply algorithms generated by mappings $S \in \mathcal{R}$. More precisely, we associate with every mapping $S \in \mathcal{R}$ the algorithm which generates, for every starting point $x_0 \in X$, a sequence of points $\{x_k\}_{k=0}^{\infty} \subset X$ such that

$$x_{k+1} := [S(k)](x_k), \ k = 0, 1, \dots.$$

According to the results known in the literature, this sequence should converge weakly to a point of the set F. In Chap. 8, we study the behavior of the sequences generated by mappings $S \in \mathcal{R}$ taking into account computational errors which are always present in practice. Namely, in practice the algorithm associate with a mapping $S \in \mathcal{R}$ generates a sequence of points $\{x_k\}_{k=0}^{\infty}$ such that for every nonnegative integer k the inequality

$$\|x_{k+1} - [S(k)](x_k)\| \le \delta$$

holds with a positive constant δ which depends only on our computer system. Surely, in this situation one cannot expect that the sequence $\{x_k\}_{k=0}^{\infty}$ converges to the set F. Our goal is to understand what subset of X attracts all sequences $\{x_k\}_{k=0}^{\infty}$ generated by algorithms associated with mappings $S \in \mathcal{R}$. The main result of Chap. 8 (Theorem 8.1) shows that this subset of X is the set $\tilde{F}_\epsilon$ with some $\epsilon > 0$ depending on δ.

In this result δ is the computational error made by our computer system, we obtain a point of the set $\tilde{F}_\epsilon$ and in order to obtain this point we need $n_0 l$ iterations. Note that $\epsilon = c_1\delta^{1/2}$ and $n_0 = \lfloor c_2\delta^{-1}\rfloor$, where c_1 and c_2 are positive constants which do not depend on δ.

1.3 Subgradient Projection Algorithms

In Chap. 10 we use subgradient projection algorithms for solving convex feasibility problems. We show that almost all iterates, generated by a subgradient projection algorithm in a Hilbert space, are approximate solutions. Moreover, we obtain an estimate of the number of iterates which are not approximate solutions. In a finite-dimensional case, we study the behavior of the subgradient projection algorithm in the presence of computational errors. Provided computational errors are bounded, we prove that our subgradient projection algorithm generates a good approximate solution after a certain number of iterates.

Let $(X, \langle\cdot,\cdot\rangle)$ be a Hilbert space with an inner product $\langle\cdot,\cdot\rangle$, which induces a complete norm $\|\cdot\|$. For each $x \in X$ and each nonempty set $A \subset X$ put

$$d(x, A) := \inf\{\|x - y\| : \ y \in A\}.$$

For each $x \in X$ and each $r > 0$ set

$$B(x, r) = \{y \in X : \ \|x - y\| \le r\}.$$

It is well known (see Fact 1.5 and Lemma 2.4 of [7]) that for each nonempty, closed, and convex subset C of X and for each $x \in X$, there is a unique point $P_C(x) \in C$ satisfying

$$\|x - P_C(x)\| = d(x, C).$$

Let $f : X \to R^1$ be a continuous and convex function such that

$$\{x \in X : f(x) \le 0\} \neq \emptyset.$$

Let $y_0 \in X$. Then the set

$$\partial f(y_0) := \{l \in X : f(y) - f(y_0) \ge \langle l, y - y_0\rangle \text{ for all } y \in X\}$$

is the subdifferential of f at the point y_0 [72, 77]. It is not difficult to see that for any $l \in \partial f(y_0)$,

$$\{x \in X : f(x) \le 0\} \subset \{x \in X : f(y_0) + \langle l, x - y_0\rangle \le 0\}.$$

It is well known that the following lemma holds (see Lemma 7.3 of [7]).

Lemma 1.1. *Let $y_0 \in X$, $f(y_0) > 0$, $l \in \partial f(y_0)$ and let*

$$D := \{x \in X : f(y_0) + \langle l, x - y_0\rangle \le 0\}.$$

Then $l \neq 0$ and $P_D(y_0) = y_0 - f(y_0)\|l\|^{-2} l$.

Denote by $\mathcal{N}$ the set of all nonnegative integers. Let m be a natural number, $\mathbb{I} = \{1, \dots, m\}$ and $f_i : X \to R^1$, $i \in \mathbb{I}$, be convex and continuous functions. For each $i \in \mathbb{I}$ set

$$C_i := \{x \in X : f_i(x) \le 0\},$$
$$C := \cap_{i \in \mathbb{I}} C_i = \cap_{i \in \mathbb{I}} \{x \in X : f_i(x) \le 0\}.$$

Suppose that

$$C \neq \emptyset.$$

A point $x \in C$ is called a solution of our feasibility problem. For a given $\epsilon > 0$, a point $x \in X$ is called an ϵ-approximate solution of the feasibility problem if $f_i(x) \le \epsilon$ for all $i \in \mathbb{I}$. We apply the subgradient projection method in order to obtain a good approximative solution of the feasibility problem.

Consider a natural number $\bar{p} \ge m$. Denote by $\mathbb{S}$ the set of all mappings $S : \mathcal{N} \to \mathbb{I}$ such that the following property holds:

(P1) For each integer $N \in \mathcal{N}$ and each $i \in \mathbb{I}$, there is $n \in \{N, \dots, N + \bar{p} - 1\}$ such that $S(n) = i$.

We want to find approximate solutions of the inclusion $x \in C$. In order to meet this goal we apply algorithms generated by $S \in \mathbb{S}$.

For each $x \in X$, each number $\epsilon \ge 0$ and each $i \in \mathbb{I}$ set

$$A_i(x, \epsilon) := \{x\} \text{ if } f_i(x) \le \epsilon$$

and

$$A_i(x,\epsilon) := x - f_i(x)\{\|l\|^{-2}l : \ l \in \partial f_i(x)\} \text{ if } f_i(x) > \epsilon.$$

We associate with any $S \in \mathbb{S}$ the algorithm which generates, for any starting point $x_0 \in X$, a sequence $\{x_n\}_{n=0}^{\infty} \subset X$ such that, for each integer $n \geq 0$,

$$x_{n+1} \in A_{S(n)}(x_n, 0).$$

It is not difficult to see that the sequence $\{x_n\}_{n=0}^{\infty}$ is well defined, and that for each integer $n \geq 0$, if $f_{S(n)}(x_n) > 0$, then $x_{n+1} = P_{D_n}(x_n)$, where

$$D_n = \{x \in X : \ f(x_n) + \langle l_n, x - x_n\rangle \leq 0\} \text{ and } l_n \in \partial f_{S(n)}(x_n).$$

In Chap. 10 we prove the following result which shows that, for the subgradient projection method considered in the chapter, almost all iterates are good approximate solutions. Denote by Card(A) the cardinality of the set A.

Theorem 1.2. *Let*

$$b > 0, \ \epsilon \in (0,1], \ \Lambda > 0, \ \gamma \in [0,\epsilon],$$
$$c \in B(0,b) \cap C,$$
$$|f_i(u) - f_i(v)| \leq \Lambda\|u - v\|, \ u, v \in B(0, 3b+1), \ i \in \mathbb{I},$$

let a positive number ϵ_0 satisfy

$$\epsilon_0 \leq \epsilon\Lambda^{-1}$$

and let a natural number n_0 satisfy

$$4\bar{p}\epsilon_0^{-2}b^2 \leq n_0.$$

Assume that

$$S \in \mathbb{S}, \ x_0 \in B(0,b),$$

and that for each integer $n \geq 0$,

$$x_{n+1} \in A_{S(n)}(x_n, \gamma).$$

Then

$$\|x_n\| \leq 3b \text{ for all integers } n \geq 0$$

and

$$Card(\{N \in \mathcal{N} : \max\{\|x_{n+1} - x_n\| : n = N, \dots, N + \bar{p} - 1\} > \epsilon_0\}) \le n_0.$$

Moreover, if an integer $N \ge 0$ *satisfies*

$$\|x_{n+1} - x_n\| \le \epsilon_0, \ n = N, \dots, N + \bar{p} - 1,$$

then, for all integers $n, m \in \{N, \dots, N + \bar{p}\}$, $\|x_n - x_m\| \le \bar{p}\epsilon_0$ *and for all integers* $n = N, \dots, N + \bar{p}$ *and each* $i \in \mathbb{I}$, $f_i(x_n) \le \epsilon(\bar{p} + 1)$.