Springer Optimization and Its Applications 112

Alexander J. Zaslavski

Approximate Solutions of Common Fixed-Point Problems

Springer

Springer Optimization and Its Applications

VOLUME 112

Aims and Scope
Optimization has been expanding in all directions at an astonishing rate during the last few decades. New algorithmic and theoretical techniques have been developed, the diffusion into other disciplines has proceeded at a rapid pace, and our knowledge of all aspects of the field has grown even more profound. At the same time, one of the most striking trends in optimization is the constantly increasing emphasis on the interdisciplinary nature of the field. Optimization has been a basic tool in all areas of applied mathematics, engineering, medicine, economics, and other sciences.

The series *Springer Optimization and Its Applications* publishes undergraduate and graduate textbooks, monographs and state-of-the-art expository work that focus on algorithms for solving optimization problems and also study applications involving such problems. Some of the topics covered include nonlinear optimization (convex and nonconvex), network flow problems, stochastic optimization, optimal control, discrete optimization, multi-objective programming, description of software packages, approximation techniques and heuristic approaches.

More information about this series at http://www.springer.com/series/7393

Alexander J. Zaslavski

Approximate Solutions of Common Fixed-Point Problems

Springer

Alexander J. Zaslavski
Department of Mathematics
The Technion – Israel Institute
of Technology
Rishon LeZion, Israel

ISSN 1931-6828 ISSN 1931-6836 (electronic)
Springer Optimization and Its Applications
ISBN 978-3-319-33253-6 ISBN 978-3-319-33255-0 (eBook)
DOI 10.1007/978-3-319-33255-0

Library of Congress Control Number: 2016942566

Mathematics Subject Classification (2010): 47H05, 47H09, 47H10, 65J15

Printed on acid-free paper

This Springer imprint is published by Springer Nature
The registered company is Springer International Publishing AG Switzerland

Preface

The book is devoted to the study of approximate solutions of common fixed point problems and convex feasibility problems in the presence of computational errors. A convex feasibility problem seeks to find a point which belongs to the intersection of a given finite family of subsets of a Hilbert space. This problem is a special case of a common fixed point problem which examines how to find a common fixed point of a finite family of self-mappings of a Hilbert space. The study of these problems has recently become a rapidly growing area of research. This is not only due to theoretical achievements in this area but also because of numerous applications to engineering and, in particular, to computed tomography and radiation therapy planning.

We present a number of results on the convergence behavior of algorithms, which are known as important tools for solving convex feasibility problems and common fixed point problems. According to the results known in the literature, these algorithms should converge to a solution. In this book, we study these algorithms taking into account computational errors which always present in practice. In this case, the convergence to a solution does not take place.

Moreover, we show that our algorithms generate a good approximate solution, if computational errors are bounded from above by a small positive constant. Clearly, in practice, it is sufficient to find a good approximate solution instead of constructing a minimizing sequence. On the other hand, practice, computations induce numerical errors, and if one uses methods in order to solve minimization problems, these methods usually provide only approximate solutions of the problems. Our main goal is, for a known computational error, to find out what an approximate solution can be obtained and how many iterates one needs for this.

The monograph contains twelve chapters. Chapter 1 is an introduction. In Chap. 2, we study dynamic string-averaging methods for common fixed point problems in a Hilbert space. Its results are a generalization of the results for the convex feasibility problems obtained in our recent paper in the journal *Journal of Nonlinear and Convex Analysis*. In Chap. 3, using iterative methods, we study common fixed point problems in metric spaces. In Chap. 4, approximate solutions of these problems are obtained by dynamic string-averaging methods in normed spaces. Dynamic string methods, for common fixed point problems in a metric space, are introduced and studied in Chap. 5. Common fixed point problems, in the spaces with distances of the Bregman type, are analyzed in Chap. 6. The results of Chaps. 3–6 are new. Chapter 7 is devoted to the study of the convergence of an abstract version of the algorithm which is called in the literature as component-averaged row projections or CARP. In Chap. 8, which is based on our recent paper published in the journal *Nonlinear Analysis*, we study a proximal algorithm for finding a common zero of a family of maximal monotone operators. In Chap. 9, we extend the results of Chap. 8 for a dynamic string-averaging version of the proximal algorithm. The results of Chaps. 7 and 9 are new. In Chaps. 10–12, subgradient projection algorithms for convex feasibility problems are studied for finite and infinite Hilbert spaces. The results of these chapters concerning iterative methods were obtained in our recent papers published in the *Journal on Optimization Theory and Applications* and in the *Journal of Approximation Theory*, while the results on their dynamic string-averaging versions are new.

Rishon LeZion, Israel — Alexander J. Zaslavski
November 16, 2015

Contents

Chapter 1
Introduction

In this book we study approximate solutions of common fixed point problems and of convex feasibility problems in the presence of computational errors. A convex feasibility problem is to find a point which belongs to the intersection of a given finite family of subsets of a Hilbert space. This problem is a special case of a common fixed point problem which is to find a common fixed point of a finite family of nonlinear mappings in a Hilbert space. Our goal is to obtain a good approximate solution of the problem in the presence of computational errors. We show that the algorithm generates a good approximate solution, if the sequence of computational errors is bounded from above by a constant. In this section we discuss several algorithms which are studied in the book.

1.1 Common Fixed Point Problems in a Hilbert Space

In Chap. 2 we study the convergence of dynamic string-averaging methods which were introduced for solving a convex feasibility problem, when a given collection of sets is divided into blocks and the algorithms operate in such a manner that all the blocks are processed in parallel. Iterative methods for solving common fixed point problems is a special case of dynamic string-averaging methods with only one block. Iterative methods and dynamic string-averaging methods are important tools for solving common fixed point problems in a Hilbert space [1, 3, 5–7, 10, 12, 13, 15, 16, 22, 23, 26, 27, 30–35, 37–41, 43, 45–49, 52–54, 67, 74, 75, 84, 85, 89, 95].

Let $(X, \langle \cdot, \cdot \rangle)$ be a Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ which induces a complete norm $\| \cdot \|$.

For each $x \in X$ and each nonempty set $E \subset X$ put

$$d(x, E) = \inf\{\|x - y\| : \ y \in E\}.$$

A. Zaslavski, *Approximate Solutions of Common Fixed-Point Problems*, Springer Optimization and Its Applications 112, DOI 10.1007/978-3-319-33255-0_1

For every point $x \in X$ and every positive number $r > 0$ set

$$B(x, r) = \{y \in X : \ \|x - y\| \le r\}.$$

Suppose that m is a natural number, $\bar{c} \in (0, 1)$, $P_i : X \to X$, $i = 1, \dots, m$, for every integer $i \in \{1, \dots, m\}$,

$$\mathrm{Fix}(P_i) := \{z \in X : \ P_i(z) = z\} \neq \emptyset$$

and that the inequality

$$\|z - x\|^2 \ge \|z - P_i(x)\|^2 + \bar{c}\|x - P_i(x)\|^2$$

holds for every integer $i \in \{1, \dots, m\}$, every point $x \in X$, and every point $z \in \mathrm{Fix}(P_i)$. Set

$$F = \cap_{i=1}^{m} \mathrm{Fix}(P_i).$$

For every positive number ϵ and every integer $i \in \{1, \dots, m\}$ set

$$F_\epsilon(P_i) = \{x \in X : \ \|x - P_i(x)\| \le \epsilon\},$$

$$\tilde{F}_\epsilon(P_i) = F_\epsilon(P_i) + B(0, \epsilon),$$

$$F_\epsilon = \cap_{i=1}^{m} F_\epsilon(P_i)$$

and

$$\tilde{F}_\epsilon = \cap_{i=1}^{m} \tilde{F}_\epsilon(P_i)$$

A point belonging to the set F is a solution of our common fixed point problem while a point which belongs to the set $\tilde{F}_\epsilon$ is its ϵ-approximate solution.

In Chap. 2 we obtain a good approximative solution of the common fixed point problem applying a dynamic string-averaging method with variable strings and weights which is described below.

By an index vector, we a mean a vector $t = (t_1, \dots, t_p)$ such that $t_i \in \{1, \dots, m\}$ for all $i = 1, \dots, p$.

For an index vector $t = (t_1, \dots, t_q)$ set

$$p(t) = q, \ P[t] = P_{t_q} \cdots P_{t_1}.$$

It is not difficult to see that for each index vector t

$$P[t](x) = x \text{ for all } x \in F,$$

$$\|P[t](x) - P[t](y)\| = \|x - P[t](y)\| \le \|x - y\|$$

for every point $x \in F$ and every point $y \in X$.

Denote by $\mathcal{M}$ the collection of all pairs (Ω, w), where Ω is a finite set of index vectors and

$$w : \Omega \to (0, \infty) \text{ satisfies } \sum_{t \in \Omega} w(t) = 1.$$

Let $(\Omega, w) \in \mathcal{M}$. Define

$$P_{\Omega,w}(x) = \sum_{t \in \Omega} w(t) P[t](x), \;\; x \in X.$$

It is easy to see that

$$P_{\Omega,w}(x) = x \text{ for all } x \in F,$$
$$\|P_{\Omega,w}(x) - P_{\Omega,w}(y)\| = \|x - P_{\Omega,w}(y)\| \le \|x - y\|$$

for every point $x \in F$ and every point $y \in X$.

The dynamic string-averaging method with variable strings and variable weights can now be described by the following algorithm.

Initialization: select an arbitrary point $x_0 \in X$.

Iterative step: given a current iteration vector x_k pick a pair

$$(\Omega_{k+1}, w_{k+1}) \in \mathcal{M}$$

and calculate the next iteration vector x_{k+1} by

$$x_{k+1} = P_{\Omega_{k+1}, w_{k+1}}(x_k).$$

Fix a number

$$\Delta \in (0, m^{-1}]$$

and an integer

$$\bar{q} \ge m.$$

Denote by $\mathcal{M}_*$ the set of all $(\Omega, w) \in \mathcal{M}$ such that

$$p(t) \le \bar{q} \;\text{ for all } t \in \Omega,$$
$$w(t) \ge \Delta \text{ for all } t \in \Omega.$$

Fix a natural number $\bar{N}$.

In the studies of the common fixed point problem the goal is to find a point $x \in F$. In order to meet this goal we apply an algorithm generated by

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$$

such that for each natural number j,

$$\{1,\dots,m\} \subset \cup_{i=j}^{j+\bar{N}-1}(\cup_{t\in\Omega_i}\{t_1,\dots,t_{p(t)}\}).$$

This algorithm generates, for any starting point $x_0 \in X$, a sequence $\{x_k\}_{k=0}^{\infty} \subset X$, where

$$x_{k+1} = P_{\Omega_{k+1},w_{k+1}}(x_k).$$

According to the results known in the literature, this sequence should converge to an element of F. In Chap. 2, we study the behavior of the sequences generated by $\{(\Omega_i, w_i)\}_{i=1}^{\infty}$ taking into account computational errors which always present in practice. These computational errors are bounded from above by a small constant depending only on our computer system which is denoted by δ. This computational error δ presents in all calculations which we do using our computer system. For example, if $x \in X$ and $i \in \{1,\dots,m\}$ and we need to calculate $P_i(x)$, then using our computer system we obtain a point $y \in X$ satisfying

$$\|y - P_i(x)\| \le \delta.$$

If k is a natural number, $y_i \in X$, $i = 1,\dots,k$, $\alpha_i > 0$, $i = 1,\dots,k$ satisfying $\sum_{i=1}^{k}\alpha_i = 1$ and if need to calculate $\sum_{i=1}^{k}\alpha_i y_i$, then by using our computer system we obtain a point $y \in X$ satisfying

$$\|y - \sum_{i=1}^{k}\alpha_i y_i\| \le \delta.$$

Surely, in this situation one cannot expect that the sequence of iterates generated by our algorithm converges to the set F. Our goal is to understand what approximate solutions of the common fixed point problem can be obtained.

In Chap. 2 we prove Theorem 2.1, which shows that in the presence of computational errors bounded from above by a constant δ, an ϵ-approximate solution can be obtained after $n\bar{N}$ iterations of the algorithm. Note that $\epsilon = c_1\delta^{1/2}$ and $n = \lfloor c_2\delta^{-1}\rfloor$, where c_1 and c_2 are positive constants which do not depend on δ and $\lfloor u\rfloor$ denotes the integer part of u.

1.2 Proximal Point Algorithm

Proximal point method is an important tool in solving optimization problems [4, 42, 44, 56, 59, 61, 68, 69, 78, 88]. It is also used for solving variational inequalities with monotone operators [2, 8, 11, 17–21, 25, 57, 60, 62–64, 79, 82, 83, 91, 92] which is an important topic of nonlinear analysis and optimization [9, 14, 28, 29, 36, 51, 55, 58, 65, 66, 80, 81, 86, 87, 90]. In Chap. 8 we study the convergence of an iterative proximal point method to a common zero of a finite family of maximal monotone operators in a Hilbert space, under the presence of computational errors. Most results known in the literature establish the convergence of proximal point methods, when computational errors are summable. In Chap. 8, the convergence of the method is proved for nonsummable computational errors. We show that the proximal point method generates a good approximate solution, if the sequence of computational errors is bounded from above by a constant. Moreover, for a known computational error, we find out what an approximate solution can be obtained and how many iterates one needs for this.

Let $(X, \langle\cdot,\cdot\rangle)$ be a Hilbert space equipped with an inner product $\langle\cdot,\cdot\rangle$ which induces the norm $\|\cdot\|$.

A multifunction $T : X \to 2^X$ is called a monotone operator if and only if

$$\langle z - z', w - w'\rangle \ \geq 0 \quad \forall z, z', w, w' \in X$$
$$\text{such that } w \in T(z) \text{ and } w' \in T(z').$$

It is called maximal monotone if, in addition, the graph

$$\{(z, w) \in X \times X : \ w \in T(z)\}$$

is not properly contained in the graph of any other monotone operator $T' : X \to 2^X$. A fundamental problem consists in determining an element z such that $0 \in T(z)$. For example, if T is the subdifferential ∂f of a lower semicontinuous convex function $f : X \to (-\infty, \infty]$, which is not identically infinity, then T is maximal monotone (see [71, 73]), and the relation $0 \in T(z)$ means that z is a minimizer of f.

Let $T : X \to 2^X$ be a maximal monotone operator. The proximal point algorithm generates, for any given sequence of positive real numbers and any starting point in the space, a sequence of points and the goal is to show the convergence of this sequence. Note that in a general infinite-dimensional Hilbert space this convergence is usually weak. The proximal algorithm for solving the inclusion $0 \in T(z)$ is based on the fact established by Minty [70], who showed that, for each $z \in X$ and each $c > 0$, there is a unique $u \in X$ such that

$$z \in (I + cT)(u),$$

where $I : X \to X$ is the identity operator ($Ix = x$ for all $x \in X$).

The operator

$$P_{c,T} := (I + cT)^{-1}$$

is therefore single-valued from all of X onto X (where c is any positive number). It is also nonexpansive:

$$\|P_{c,T}(z) - P_{c,T}(z')\| \leq \|z - z'\| \text{ for all } z, z' \in X$$

and

$$P_{c,T}(z) = z \text{ if and only if } 0 \in T(z).$$

Following the terminology of Moreau [73] $P_{c,T}$ is called the proximal mapping associated with cT.

The proximal point algorithm generates, for any given sequence $\{c_k\}_{k=0}^{\infty}$ of positive real numbers and any starting point $z^0 \in X$, a sequence $\{z^k\}_{k=0}^{\infty} \subset X$, where

$$z^{k+1} := P_{c_k,T}(z^k), \; k = 0, 1, \ldots$$

It is not difficult to see that the

$$\text{graph}(T) := \{(x, w) \in X \times X : \; w \in T(x)\}$$

is closed in the norm topology of $X \times X$.

Set

$$F(T) = \{z \in X : \; 0 \in T(z)\}.$$

Usually algorithms considering in the literature generate sequences which converge weakly to an element of $F(T)$. In Chap. 8, for a given $\epsilon > 0$, we are interested to find a point x for which there is $y \in T(x)$ such that $\|y\| \le \epsilon$. This point x is considered as an ϵ-approximate solution.

For every point $x \in X$ and every nonempty set $A \subset X$ define

$$d(x, A) := \inf\{\|x - y\| : \; y \in A\}.$$

For every point $x \in X$ and every positive number r put

$$B(x, r) = \{y \in X : \; \|x - y\| \le r\}.$$

We denote by $\text{Card}(A)$ the cardinality of the set A.

We apply the proximal point algorithm in order to obtain a good approximation of a point which is a common zero of a finite family of maximal monotone operators and a common fixed point of a finite family of quasi-nonexpansive operators.

Let $\mathcal{L}_1$ be a finite set of maximal monotone operators $T : X \to 2^X$ and $\mathcal{L}_2$ be a finite set of mappings $T : X \to X$. We suppose that the set $\mathcal{L}_1 \cup \mathcal{L}_2$ is nonempty. (Note that one of the sets $\mathcal{L}_1$ or $\mathcal{L}_2$ may be empty.)

Let $\bar{c} \in (0, 1]$ and let $\bar{c} = 1$, if $\mathcal{L}_2 = \emptyset$.

We suppose that

$$F(T) \neq \emptyset \text{ for any } T \in \mathcal{L}_1$$

and that for every mapping $T \in \mathcal{L}_2$,

$$\mathrm{Fix}(T) := \{z \in X : \ T(z) = z\} \neq \emptyset,$$
$$\|z - x\|^2 \geq \|z - T(x)\|^2 + \bar{c}\|x - T(x)\|^2$$
$$\text{for all } x \in X \text{ and all } z \in \mathrm{Fix}(T).$$

Let $\bar{\lambda} > 0$ and let $\bar{\lambda} = \infty$ and $\bar{\lambda}^{-1} = 0$, if $\mathcal{L}_1 = \emptyset$. Let a natural number

$$l \geq \mathrm{Card}(\mathcal{L}_1 \cup \mathcal{L}_2).$$

Denote by $\mathcal{R}$ the set of all mappings

$$S : \{0, 1, 2, \dots\} \to \mathcal{L}_2 \cup \{P_{c,T} : \ T \in \mathcal{L}_1, \ c \in [\bar{\lambda}, \infty)\}$$

such that the following properties hold:

(P1) for every nonnegative integer p and every mapping $T \in \mathcal{L}_2$ there exists an integer $i \in \{p, \dots, p + l - 1\}$ satisfying $S(i) = T$;

(P2) for every nonnegative integer p and every monotone operator $T \in \mathcal{L}_1$ there exist an integer $i \in \{p, \dots, p+l-1\}$ and a number $c \geq \bar{\lambda}$ satisfying that $S(i) = P_{c,T}$.

Suppose that

$$F := (\cap_{T \in \mathcal{L}_1} F(T)) \cap (\cap_{Q \in \mathcal{L}_2} \mathrm{Fix}(Q)) \neq \emptyset.$$

Let $\epsilon > 0$. For every monotone operator $T \in \mathcal{L}_1$ define

$$F_\epsilon(T) = \{x \in X : \ T(x) \cap B(0, \epsilon) \neq \emptyset\}$$

and for every mapping $T \in \mathcal{L}_2$ set

$$\mathrm{Fix}_\epsilon(T) = \{x \in X : \ \|T(x) - x\| \leq \epsilon\}.$$

Define

$$F_\epsilon = (\cap_{T \in \mathcal{L}_1} F_\epsilon(T)) \cap (\cap_{Q \in \mathcal{L}_2} \mathrm{Fix}_\epsilon(Q)),$$
$$\tilde{F}_\epsilon = (\cap_{T \in \mathcal{L}_1} \{x \in X : \ d(x, F_\epsilon(T)) \leq \epsilon\})$$
$$\cap (\cap_{Q \in \mathcal{L}_2} \{x \in X : \ d(x, \mathrm{Fix}_\epsilon(Q)) \leq \epsilon\}).$$

We are interested to find solutions of the inclusion $x \in F$. In order to meet this goal we apply algorithms generated by mappings $S \in \mathcal{R}$. More precisely, we associate with every mapping $S \in \mathcal{R}$ the algorithm which generates, for every starting point $x_0 \in X$, a sequence of points $\{x_k\}_{k=0}^{\infty} \subset X$ such that

$$x_{k+1} := [S(k)](x_k), \ k = 0, 1, \dots.$$

According to the results known in the literature, this sequence should converge weakly to a point of the set F. In Chap. 8, we study the behavior of the sequences generated by mappings $S \in \mathcal{R}$ taking into account computational errors which are always present in practice. Namely, in practice the algorithm associate with a mapping $S \in \mathcal{R}$ generates a sequence of points $\{x_k\}_{k=0}^{\infty}$ such that for every nonnegative integer k the inequality

$$\|x_{k+1} - [S(k)](x_k)\| \leq \delta$$

holds with a positive constant δ which depends only on our computer system. Surely, in this situation one cannot expect that the sequence $\{x_k\}_{k=0}^{\infty}$ converges to the set F. Our goal is to understand what subset of X attracts all sequences $\{x_k\}_{k=0}^{\infty}$ generated by algorithms associated with mappings $S \in \mathcal{R}$. The main result of Chap. 8 (Theorem 8.1) shows that this subset of X is the set $\tilde{F}_\epsilon$ with some $\epsilon > 0$ depending on δ.

In this result δ is the computational error made by our computer system, we obtain a point of the set $\tilde{F}_\epsilon$ and in order to obtain this point we need $n_0 l$ iterations. Note that $\epsilon = c_1\delta^{1/2}$ and $n_0 = \lfloor c_2\delta^{-1}\rfloor$, where c_1 and c_2 are positive constants which do not depend on δ.

1.3 Subgradient Projection Algorithms

In Chap. 10 we use subgradient projection algorithms for solving convex feasibility problems. We show that almost all iterates, generated by a subgradient projection algorithm in a Hilbert space, are approximate solutions. Moreover, we obtain an estimate of the number of iterates which are not approximate solutions. In a finite-dimensional case, we study the behavior of the subgradient projection algorithm in the presence of computational errors. Provided computational errors are bounded, we prove that our subgradient projection algorithm generates a good approximate solution after a certain number of iterates.

Let $(X, \langle\cdot,\cdot\rangle)$ be a Hilbert space with an inner product $\langle\cdot,\cdot\rangle$, which induces a complete norm $\|\cdot\|$. For each $x \in X$ and each nonempty set $A \subset X$ put

$$d(x, A) := \inf\{\|x - y\| : \ y \in A\}.$$

For each $x \in X$ and each $r > 0$ set

$$B(x, r) = \{y \in X : \ \|x - y\| \leq r\}.$$

It is well known (see Fact 1.5 and Lemma 2.4 of [7]) that for each nonempty, closed, and convex subset C of X and for each $x \in X$, there is a unique point $P_C(x) \in C$ satisfying

$$\|x - P_C(x)\| = d(x, C).$$

Let $f : X \to R^1$ be a continuous and convex function such that

$$\{x \in X : f(x) \leq 0\} \neq \emptyset.$$

Let $y_0 \in X$. Then the set

$$\partial f(y_0) := \{l \in X : f(y) - f(y_0) \geq \langle l, y - y_0 \rangle \text{ for all } y \in X\}$$

is the subdifferential of f at the point y_0 [72, 77]. It is not difficult to see that for any $l \in \partial f(y_0)$,

$$\{x \in X : f(x) \leq 0\} \subset \{x \in X : f(y_0) + \langle l, x - y_0 \rangle \leq 0\}.$$

It is well known that the following lemma holds (see Lemma 7.3 of [7]).

Lemma 1.1. *Let $y_0 \in X$, $f(y_0) > 0$, $l \in \partial f(y_0)$ and let*

$$D := \{x \in X : f(y_0) + \langle l, x - y_0 \rangle \leq 0\}.$$

Then $l \neq 0$ and $P_D(y_0) = y_0 - f(y_0)\|l\|^{-2} l$.

Denote by $\mathcal{N}$ the set of all nonnegative integers. Let m be a natural number, $\mathbb{I} = \{1, \ldots, m\}$ and $f_i : X \to R^1$, $i \in \mathbb{I}$, be convex and continuous functions. For each $i \in \mathbb{I}$ set

$$C_i := \{x \in X : f_i(x) \leq 0\},$$
$$C := \cap_{i \in \mathbb{I}} C_i = \cap_{i \in \mathbb{I}} \{x \in X : f_i(x) \leq 0\}.$$

Suppose that

$$C \neq \emptyset.$$

A point $x \in C$ is called a solution of our feasibility problem. For a given $\epsilon > 0$, a point $x \in X$ is called an ϵ-approximate solution of the feasibility problem if $f_i(x) \leq \epsilon$ for all $i \in \mathbb{I}$. We apply the subgradient projection method in order to obtain a good approximative solution of the feasibility problem.

Consider a natural number $\bar{p} \geq m$. Denote by $\mathbb{S}$ the set of all mappings $S : \mathcal{N} \to \mathbb{I}$ such that the following property holds:

(P1) For each integer $N \in \mathcal{N}$ and each $i \in \mathbb{I}$, there is $n \in \{N, \ldots, N + \bar{p} - 1\}$ such that $S(n) = i$.

We want to find approximate solutions of the inclusion $x \in C$. In order to meet this goal we apply algorithms generated by $S \in \mathbb{S}$.

For each $x \in X$, each number $\epsilon \geq 0$ and each $i \in \mathbb{I}$ set

$$A_i(x, \epsilon) := \{x\} \text{ if } f_i(x) \leq \epsilon$$

and

$$A_i(x,\epsilon) := x - f_i(x)\{\|l\|^{-2}l : \ l \in \partial f_i(x)\} \text{ if } f_i(x) > \epsilon.$$

We associate with any $S \in \mathbb{S}$ the algorithm which generates, for any starting point $x_0 \in X$, a sequence $\{x_n\}_{n=0}^{\infty} \subset X$ such that, for each integer $n \geq 0$,

$$x_{n+1} \in A_{S(n)}(x_n, 0).$$

It is not difficult to see that the sequence $\{x_n\}_{n=0}^{\infty}$ is well defined, and that for each integer $n \geq 0$, if $f_{S(n)}(x_n) > 0$, then $x_{n+1} = P_{D_n}(x_n)$, where

$$D_n = \{x \in X : \ f(x_n) + \langle l_n, x - x_n \rangle \leq 0\} \text{ and } l_n \in \partial f_{S(n)}(x_n).$$

In Chap. 10 we prove the following result which shows that, for the subgradient projection method considered in the chapter, almost all iterates are good approximate solutions. Denote by Card(A) the cardinality of the set A.

Theorem 1.2. *Let*

$$b > 0, \ \epsilon \in (0,1], \ \Lambda > 0, \ \gamma \in [0,\epsilon],$$
$$c \in B(0,b) \cap C,$$
$$|f_i(u) - f_i(v)| \leq \Lambda \|u - v\|, \ u, v \in B(0, 3b+1), \ i \in \mathbb{I},$$

let a positive number ϵ_0 *satisfy*

$$\epsilon_0 \leq \epsilon \Lambda^{-1}$$

and let a natural number n_0 *satisfy*

$$4\bar{p}\epsilon_0^{-2}b^2 \leq n_0.$$

Assume that

$$S \in \mathbb{S}, \ x_0 \in B(0,b),$$

and that for each integer $n \geq 0$,

$$x_{n+1} \in A_{S(n)}(x_n, \gamma).$$

Then

$$\|x_n\| \leq 3b \textit{ for all integers } n \geq 0$$

and

$$Card(\{N \in \mathcal{N} :\ \max\{\|x_{n+1} - x_n\| :\ n = N, \dots, N + \bar{p} - 1\} > \epsilon_0\}) \le n_0.$$

Moreover, if an integer $N \ge 0$ *satisfies*

$$\|x_{n+1} - x_n\| \le \epsilon_0,\ n = N, \dots, N + \bar{p} - 1,$$

then, for all integers $n, m \in \{N, \dots, N + \bar{p}\}$, $\|x_n - x_m\| \le \bar{p}\epsilon_0$ *and for all integers* $n = N, \dots, N + \bar{p}$ *and each* $i \in \mathbb{I}$, $f_i(x_n) \le \epsilon(\bar{p} + 1)$.

Chapter 2
Dynamic String-Averaging Methods in Hilbert Spaces

In this chapter we study the convergence of dynamic string-averaging methods for solving common fixed point problems in a Hilbert space. Our main goal is to obtain an approximate solution of the problem in the presence of computational errors. We show that our dynamic string-averaging algorithm generates a good approximate solution, if the sequence of computational errors is bounded from above by a constant. Moreover, for a known computational error, we find out what an approximate solution can be obtained and how many iterates one needs for this.

2.1 Preliminaries and the Main Result

Let $(X, \langle \cdot, \cdot \rangle)$ be a Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ which induces a complete norm $\| \cdot \|$.

For each $x \in X$ and each nonempty set $E \subset X$ put

$$d(x, E) = \inf\{\|x - y\| : \ y \in E\}.$$

For every point $x \in X$ and every positive number $r > 0$ set

$$B(x, r) = \{y \in X : \ \|x - y\| \le r\}.$$

Suppose that m is a natural number, $\bar{c} \in (0, 1)$, $P_i : X \to X$, $i = 1, \dots, m$, for every integer $i \in \{1, \dots, m\}$,

$$\mathrm{Fix}(P_i) := \{z \in X : \ P_i(z) = z\} \neq \emptyset \tag{2.1}$$

and that the inequality

$$\|z - x\|^2 \ge \|z - P_i(x)\|^2 + \bar{c}\|x - P_i(x)\|^2 \tag{2.2}$$

A. Zaslavski, *Approximate Solutions of Common Fixed-Point Problems*, Springer Optimization and Its Applications 112, DOI 10.1007/978-3-319-33255-0_2

holds for every integer $i \in \{1, \dots, m\}$, every point $x \in X$ and every point $z \in \mathrm{Fix}(P_i)$. Set

$$F = \cap_{i=1}^{m} \mathrm{Fix}(P_i). \tag{2.3}$$

For every positive number ϵ and every integer $i \in \{1, \dots, m\}$ set

$$F_\epsilon(P_i) = \{x \in X : \ \|x - P_i(x)\| \le \epsilon\}, \tag{2.4}$$

$$\tilde{F}_\epsilon(P_i) = F_\epsilon(P_i) + B(0, \epsilon), \tag{2.5}$$

$$F_\epsilon = \cap_{i=1}^{m} F_\epsilon(P_i) \tag{2.6}$$

and

$$\tilde{F}_\epsilon = \cap_{i=1}^{m} \tilde{F}_\epsilon(P_i) \tag{2.7}$$

A point belonging to the set F is a solution of our common fixed point problem while a point which belongs to the set $\tilde{F}_\epsilon$ is its ϵ-approximate solution.

We apply a dynamic string-averaging method with variable strings and weights in order to obtain a good approximative solution of the common fixed point problem.

Next we describe the dynamic string-averaging method with variable strings and weights.

By an index vector, we mean a vector $t = (t_1, \dots, t_p)$ such that $t_i \in \{1, \dots, m\}$ for all $i = 1, \dots, p$.

For an index vector $t = (t_1, \dots, t_q)$ set

$$p(t) = q, \ P[t] = P_{t_q} \cdots P_{t_1}. \tag{2.8}$$

It is not difficult to see that for each index vector t

$$P[t](x) = x \text{ for all } x \in F, \tag{2.9}$$

$$\|P[t](x) - P[t](y)\| = \|x - P[t](y)\| \le \|x - y\| \tag{2.10}$$

for every point $x \in F$ and every point $y \in X$.

Denote by $\mathcal{M}$ the collection of all pairs (Ω, w), where Ω is a finite set of index vectors and

$$w : \Omega \to (0, \infty) \text{ satisfies } \sum_{t \in \Omega} w(t) = 1. \tag{2.11}$$

Let $(\Omega, w) \in \mathcal{M}$. Define

$$P_{\Omega,w}(x) = \sum_{t \in \Omega} w(t) P[t](x), \ x \in X. \tag{2.12}$$

It is easy to see that

$$P_{\Omega,w}(x) = x \text{ for all } x \in F, \tag{2.13}$$

$$\|P_{\Omega,w}(x) - P_{\Omega,w}(y)\| = \|x - P_{\Omega,w}(y)\| \le \|x - y\| \tag{2.14}$$

for every point $x \in F$ and every point $y \in X$.

The dynamic string-averaging method with variable strings and variable weights can now be described by the following algorithm.

Initialization: select an arbitrary point $x_0 \in X$.

Iterative step: given a current iteration vector x_k pick a pair

$$(\Omega_{k+1}, w_{k+1}) \in \mathcal{M}$$

and calculate the next iteration vector x_{k+1} by

$$x_{k+1} = P_{\Omega_{k+1}, w_{k+1}}(x_k).$$

Fix a number

$$\Delta \in (0, m^{-1}] \tag{2.15}$$

and an integer

$$\bar{q} \ge m. \tag{2.16}$$

Denote by $\mathcal{M}_*$ the set of all $(\Omega, w) \in \mathcal{M}$ such that

$$p(t) \le \bar{q} \ \text{ for all } t \in \Omega, \tag{2.17}$$

$$w(t) \ge \Delta \text{ for all } t \in \Omega. \tag{2.18}$$

Fix a natural number $\bar{N}$.

In the studies of the common fixed point problem the goal is to find a point $x \in F$. In order to meet this goal we apply an algorithm generated by

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$$

such that for each natural number j,

$$\{1, \dots, m\} \subset \cup_{i=j}^{j+\bar{N}-1} (\cup_{t \in \Omega_i} \{t_1, \dots, t_{p(t)}\}).$$

This algorithm generates, for any starting point $x_0 \in X$, a sequence $\{x_k\}_{k=0}^{\infty} \subset X$, where

$$x_{k+1} = P_{\Omega_{k+1}, w_{k+1}}(x_k).$$

According to the results known in the literature, this sequence should converge to an element of F. In this chapter, we study the behavior of the sequences generated by $\{(\Omega_i, w_i)\}_{i=1}^{\infty}$ taking into account computational errors which always present in practice. These computational errors are bounded from above by a small constant depending only on our computer system which is denoted by δ. This computational error δ presents in all calculations which we do using our computer system. For example, if $x \in X$ and $i \in \{1, \dots, m\}$ and we need to calculate $P_i(x)$, then using our computer system we obtain a point $y \in X$ satisfying

$$\|y - P_i(x)\| \le \delta.$$

If k is a natural number, $y_i \in X$, $i = 1, \dots, k$, $\alpha_i > 0$, $i = 1, \dots, k$ satisfying $\sum_{i=1}^{k} \alpha_i = 1$ and if need to calculate $\sum_{i=1}^{k} \alpha_i y_i$, then by using our computer system we obtain a point $y \in X$ satisfying

$$\|y - \sum_{i=1}^{k} \alpha_i y_i\| \le \delta.$$

Surely, in this situation one cannot expect that the sequence of iterates generated by our algorithm converges to the set F. Our goal is to understand what approximate solutions of the common fixed point problem can be obtained.

We prove the following result (Theorem 2.1), which shows that in the presence of computational errors bounded from above by a constant δ, an ϵ_1-approximate solution can be obtained after $(n_0 - 1)\bar{N}$ iterations of the algorithm, where ϵ_1 and n_0 are constants depending on δ (see (2.23) and (2.24)).

In order to state Theorem 2.1 we need the following definitions.

Let $\delta \ge 0$, $x \in X$ and let $t = (t_1, \dots, t_{p(t)})$ be an index vector. Define

$$\begin{aligned} A_0(x, t, \delta) = \{(y, \lambda) \in X \times R^1 : \ & \text{there is a sequence } \{y_i\}_{i=0}^{p(t)} \subset X \text{ such that} \\ & y_0 = x \text{ and for all } i = 1, \dots, p(t), \\ & \|y_i - P_{t_i}(y_{i-1})\| \le \delta, \\ & y = y_{p(t)}, \\ & \lambda = \max\{\|y_i - y_{i-1}\| : \ i = 1, \dots, p(t)\}\}. \end{aligned} \tag{2.19}$$

Let $\delta \ge 0$, $x \in X$ and let $(\Omega, w) \in \mathcal{M}$. Define

$$\begin{aligned} A(x, (\Omega, w), \delta) = \{(y, \lambda) \in X \times R^1 : \ & \text{there exist} \\ & (y_t, \lambda_t) \in A_0(x, t, \delta), \ t \in \Omega \text{ such that} \\ & \|y - \sum_{t \in \Omega} w(t) y_t\| \le \delta, \ \lambda = \max\{\lambda_t : \ t \in \Omega\}\}. \end{aligned} \tag{2.20}$$

Denote by Card(A) the cardinality of a set A. Suppose that the sum over empty set is zero.

Theorem 2.1. *Let $M > 0$ satisfy*

$$B(0, M) \cap F \neq \emptyset, \tag{2.21}$$

$\delta > 0$ satisfy

$$\delta \leq (2\bar{q}\bar{N})^{-1}, \tag{2.22}$$

a natural number n_0 satisfy

$$n_0 \geq 1 + 4M^2\delta^{-1}(\bar{q}+1)^{-1}(2M+4)^{-1}(4\bar{N})^{-1} \tag{2.23}$$

and let

$$\epsilon_1 = \bar{c}^{-1/2}(\bar{q}+1)(\bar{N}+2)(64\Delta^{-1}\delta(\bar{q}+1)(2M+4)4\bar{N})^{1/2}. \tag{2.24}$$

Assume that

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$$

satisfies for each natural number j

$$\{1, \dots, m\} \subset \cup_{i=j}^{j+\bar{N}-1}(\cup_{t \in \Omega_i}\{t_1, \dots, t_{p(t)}\}), \tag{2.25}$$

$$x_0 \in B(0, M) \text{ and } \{x_i\}_{i=1}^{\infty} \subset X,\ \{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty) \tag{2.26}$$

satisfy for each natural number i,

$$(x_i, \lambda_i) \in A(x_{i-1}, (\Omega_i, w_i), \delta). \tag{2.27}$$

Then there exists an integer $q \in [0, n_0 - 1]$ such that

$$\|x_i\| \leq 3M + 1,\ i = 0, \dots, q\bar{N}, \tag{2.28}$$

$$\lambda_i \leq (64\Delta^{-1}\delta(\bar{q}+1)(2M+4)4\bar{N}\bar{c}^{-1})^{1/2}, \tag{2.29}$$
$$i = q\bar{N}+1, \dots, (q+1)\bar{N}.$$

Moreover, if an integer $q \in [0, n_0 - 1]$ satisfies (2.29), then for each $i = q\bar{N}, \dots, (q+1)\bar{N}$,

$$x_i \in \tilde{F}_{\epsilon_1}$$

and

$$\|x_i - x_j\| \leq (\bar{q}+1)\bar{N}(64\Delta^{-1}\delta(\bar{q}+1)(2M+4)4\bar{N}\bar{c}^{-1})^{1/2} \tag{2.30}$$

for each $i, j \in \{q\bar{N}, \dots, (q+1)\bar{N}\}$.

Theorem 2.1 is proved in Sect. 2.2. It provides the estimations for the constants ϵ_1 and n_0, which follow from (2.23) and (2.24). Note that $\epsilon_1 = c_1\delta^{1/2}$ and $n_0 = \lfloor c_2\delta^{-1}\rfloor + 1$, where c_1 and c_2 are positive constants depending on M and $\lfloor u \rfloor$ denotes the integer part of u.

Let $\delta > 0$ satisfy (2.22) and a natural number n_0 satisfy (2.23). Assume that we apply an algorithm associated with

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$$

which satisfies (2.25) for each natural number j, under the presence of computational errors bounded from above by a constant δ and that our goal is to find an ϵ_1-approximate solution with ϵ_1 defined by (2.24). Theorem 2.1 also answers an important question: how we can find an iteration number k for which x_k is an ϵ_1-approximate solution of the common fixed point problem. By Theorem 2.1 we need just to find the smallest integer $q \in [0, \dots, n_0 - 1]$ satisfying (2.29).

Note that Theorem 2.1 is a generalization of the main result of [98] obtained for the convex feasibility problem.

2.2 Proof of Theorem 2.1

By (2.21) there exists a point

$$z \in B(0, M) \cap F. \tag{2.31}$$

Fix a positive number

$$\epsilon_0 = (64\Delta^{-1}\delta(\bar{q}+1)(2M+4)4\bar{N}\bar{c}^{-1})^{1/2}. \tag{2.32}$$

Assume that a nonnegative integer s satisfies for each integer $k \in [0, s]$,

$$\max\{\lambda_i : \ i = k\bar{N}+1, \dots, (k+1)\bar{N}\} > \epsilon_0. \tag{2.33}$$

By (2.26) and (2.31),

$$\|x_0 - z\| \le 2M. \tag{2.34}$$

Assume that an integer $k \in [0, s]$ satisfies

$$\|x_{k\bar{N}} - z\| \le 2M. \tag{2.35}$$

We prove the following auxiliary result.

Lemma 2.2. *Assume that an integer*

$$i \in [0, \bar{N} - 1] \tag{2.36}$$

satisfies

$$\|x_{k\bar{N}+i} - z\| \le 2M + i\delta(\bar{q} + 1). \tag{2.37}$$

Then

$$\|x_{k\bar{N}+i+1} - z\| \le \delta(\bar{q} + 1) + \|x_{k\bar{N}+i} - z\| \tag{2.38}$$

and

$$\|x_{k\bar{N}+i+1} - z\|^2 \le \|x_{k\bar{N}+i} - z\|^2 + \delta(\bar{q} + 1)(4M + 3). \tag{2.39}$$

If $\lambda_{k\bar{N}+i+1} > \epsilon_0$, *then*

$$\|x_{k\bar{N}+i+1} - z\|^2 - \|x_{k\bar{N}+i} - z\|^2 \le -32^{-1}\Delta\epsilon_0^2\bar{c}. \tag{2.40}$$

Proof. In view of (2.37),

$$(x_{k\bar{N}+i+1}, \lambda_{k\bar{N}+i+1}) \in A(x_{k\bar{N}+i}, (\Omega_{k\bar{N}+i+1}, w_{k\bar{N}+i+1}), \delta). \tag{2.41}$$

By (2.20) and (2.41) there exists vectors

$$(y_t, \alpha_t) \in A_0(x_{k\bar{N}+i}, t, \delta),\ t \in \Omega_{k\bar{N}+i+1} \tag{2.42}$$

such that

$$\|x_{k\bar{N}+i+1} - \sum_{t\in\Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(t)y_t\| \le \delta, \tag{2.43}$$

$$\lambda_{k\bar{N}+i+1} = \max\{\alpha_t :\ t \in \Omega_{k\bar{N}+i+1}\}. \tag{2.44}$$

It follows from (2.19) and (2.42) that for each index vector $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k\bar{N}+i+1}$ there exists a finite sequence $\{y_i^{(t)}\}_{i=0}^{p(t)} \subset X$ such that

$$y_0^{(t)} = x_{k\bar{N}+i},\ y_{p(t)}^{(t)} = y_t, \tag{2.45}$$

$$\|y_j^{(t)} - P_{t_j}(y_{j-1}^{(t)})\| \le \delta \text{ for each integer } j = 1, \dots, p(t), \tag{2.46}$$

$$\alpha_t = \max\{\|y_{i+1}^{(t)} - y_i^{(t)}\| :\ i = 0, \dots, p(t) - 1\}. \tag{2.47}$$

Let

$$t = (t_1, \dots, t_{p(t)}) \in \Omega_{k\bar{N}+i+1}$$

be an index vector and let

$$j \in \{1, \dots, p(t)\}. \tag{2.48}$$

By (2.2), (2.3), and (2.31),

$$\|z - P_{t_j}(y_{j-1}^{(t)})\|^2 + \bar{c}\|P_{t_j}(y_{j-1}^{(t)}) - y_{j-1}^{(t)}\|^2 \leq \|z - y_{j-1}^{(t)}\|^2. \tag{2.49}$$

It follows from (2.1), (2.3), (2.31), (2.46), and (2.48) that

$$\begin{aligned}\|z - y_j^{(t)}\|^2 &= \|z - P_{t_j}(y_{j-1}^{(t)}) + P_{t_j}(y_{j-1}^{(t)}) - y_j^{(t)}\|^2\\ &\leq \|z - P_{t_j}(y_{j-1}^{(t)})\|^2 + \|P_{t_j}(y_{j-1}^{(t)}) - y_j^{(t)}\|^2\\ &+2\|z - P_{t_j}(y_{j-1}^{(t)})\|\|P_{t_j}(y_{j-1}^{(t)}) - y_j^{(t)}\|\\ &\leq \|z - P_{t_j}(y_{j-1}^{(t)})\|^2 + \delta^2 + 2\delta\|z - P_{t_j}(y_{j-1}^{(t)})\|\\ &\leq \|z - P_{t_j}(y_{j-1}^{(t)})\|^2 + \delta^2 + 2\delta\|z - y_{j-1}^{(t)}\|. \end{aligned}\tag{2.50}$$

By (2.49) and (2.50),

$$\begin{aligned}\|z - y_j^{(t)}\|^2 \leq \|z - y_{j-1}^{(t)}\|^2 - \|P_{t_j}(y_{j-1}^{(t)}) - y_{j-1}^{(t)}\|^2\bar{c}\\ +\delta^2 + 2\delta\|z - y_{j-1}^{(t)}\|.\end{aligned}\tag{2.51}$$

In view of (2.51),

$$\|z - y_j^{(t)}\| \leq \|z - y_{j-1}^{(t)}\| + \delta. \tag{2.52}$$

Thus we have shown that the following property holds:

(P1) for each index vector $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{k\bar{N}+i+1}$ and each integer $j \in \{1, \ldots, p(t)\}$ relations (2.51) and (2.52) hold.

By property (P1), (2.17), (2.45), and (2.52), for each index vector $t \in \Omega_{k\bar{N}+i+1}$ and each integer $j \in \{1, \ldots, p(t)\}$,

$$\begin{aligned}\|z - y_j^{(t)}\| \leq \|z - y_0^{(t)}\| + \delta j = \|z - x_{k\bar{N}+i}\| + \delta j\\ \leq \|z - x_{k\bar{N}+i}\| + \delta\bar{q}.\end{aligned}\tag{2.53}$$

It follows from (2.37) and (2.53) that for every index vector $t \in \Omega_{k\bar{N}+i+1}$ and every integer $j \in \{1, \ldots, p(t)\}$,

$$\|z - y_j^{(t)}\| \leq 2M + (1 + \bar{q})i\delta + \delta\bar{q} \leq 2M + \delta(\bar{q}(i + 1) + i). \tag{2.54}$$

By (2.22), (2.36), (2.37), (2.45), and (2.54) the following property holds:

(P2) for every index vector $t \in \Omega_{k\bar{N}+i+1}$ and every $j \in \{0, 1, \ldots, p(t)\}$,

$$\|z - y_j^{(t)}\| \leq 2M + 2\delta\bar{q}\bar{N} \leq 2M + 1. \tag{2.55}$$

In view of (2.45) and (2.53) for every index vector $t \in \Omega_{k\bar{N}+i+1}$,

$$\|z - y_t\| = \|z - y^{(t)}_{p(t)}\| \le \|z - x_{k\bar{N}+i}\| + \delta\bar{q}. \tag{2.56}$$

By (2.11), (2.43), and (2.56),

$$\|x_{k\bar{N}+i+1} - z\| \le \|x_{k\bar{N}+i+1} - \sum_{t\in\Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(t)y_t\|$$

$$+\|\sum_{t\in\Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(t)y_t - z\|$$

$$\le \delta + \sum_{t\in\Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(t)\|y_t - z\| \le \delta + \|x_{k\bar{N}+i} - z\| + \delta\bar{q},$$

$$\|x_{k\bar{N}+i+1} - z\| \le \delta(\bar{q}+1) + \|x_{k\bar{N}+i} - z\|$$

and (2.38) is true.

It follows from (2.22), (2.36), (2.37), and (2.38) that

$$\|x_{k\bar{N}+i+1} - z\|^2 \le \|x_{k\bar{N}+i} - z\|^2 + \delta^2(\bar{q}+1)^2 + 2\delta(\bar{q}+1)\|x_{k\bar{N}+i} - z\|$$
$$\le \|x_{k\bar{N}+i} - z\|^2 + \delta^2(\bar{q}+1)^2 + 2\delta(\bar{q}+1)(2M+1)$$
$$\le \|x_{k\bar{N}+i} - z\|^2 + \delta(\bar{q}+1)(4M+3).$$

Thus (2.39) is true.

Assume that

$$\lambda_{k\bar{N}+i+1} > \epsilon_0. \tag{2.57}$$

In view of (2.44) there exists an index vector

$$s = (s_1, \dots, s_{p(s)}) \in \Omega_{k\bar{N}+i+1} \tag{2.58}$$

such that

$$\alpha_s = \lambda_{k\bar{N}+i+1} > \epsilon_0. \tag{2.59}$$

By (2.47), (2.58), and (2.59), there exists an integer

$$j_0 \in \{1, \dots, p(s)\} \tag{2.60}$$

such that

$$\|y^{(s)}_{j_0} - y^{(s)}_{j_0-1}\| = \alpha_s > \epsilon_0. \tag{2.61}$$

By properties (P1), (P2), (2.36), (2.37), and (2.51) applied with $t = s$, $j = j_0$ we have

$$\|z - y_{j_0}^{(s)}\|^2 \le \|z - y_{j_0-1}^{(s)}\|^2 - \|P_{s_{j_0}}(y_{j_0-1}^{(s)}) - y_{j_0-1}^{(s)}\|^2 \bar{c} + \delta^2 + 2\delta(2M+1)$$
$$\le \|z - y_{j_0-1}^{(s)}\|^2 - \|P_{s_{j_0}}(y_{j_0-1}^{(s)}) - y_{j_0-1}^{(s)}\|^2 \bar{c} + 2\delta(2M+2). \tag{2.62}$$

In view of (2.46) and (2.61),

$$\|P_{s_{j_0}}(y_{j_0-1}^{(s)}) - y_{j_0-1}^{(s)}\| \ge \|y_{j_0}^{(s)} - y_{j_0-1}^{(s)}\| - \|y_{j_0}^{(s)} - P_{s_{j_0}}(y_{j_0-1}^{(s)})\| > \epsilon_0 - \delta. \tag{2.63}$$

By (2.62) and (2.63),

$$\|z - y_{j_0}^{(s)}\|^2 \le \|z - y_{j_0-1}^{(s)}\|^2 - \bar{c}(\epsilon_0 - \delta)^2 + 2\delta(2M+2). \tag{2.64}$$

In view of property (P2), applied with $t = s$ for all integers $j \in \{0, 1, \ldots, p(s)\}$ we have

$$\|z - y_j^{(s)}\| \le 2M + 1. \tag{2.65}$$

It follows from property (P1), (2.52) with $t = s$ and (2.65) that for all integers $j \in \{1, \ldots, p(s)\}$ we have

$$\|z - y_j^{(s)}\|^2 \le \|z - y_{j-1}^{(s)}\|^2 + \delta^2 + 2\delta\|z - y_{j-1}^{(s)}\|$$
$$\le \|z - y_{j-1}^{(s)}\|^2 + 2\delta(2M+2). \tag{2.66}$$

By (2.17), (2.45), (2.60), (2.64), and (2.66),

$$\|z - x_{k\bar{N}+i}\|^2 - \|z - y_s\|^2$$
$$= \sum_{i=1}^{p(s)} [\|z - y_{i-1}^{(s)}\|^2 - \|z - y_i^{(s)}\|^2]$$
$$\ge \bar{c}(\epsilon_0 - \delta)^2 - 2\delta(2M+2) - 2\delta(2M+2)\bar{q}$$
$$\ge \bar{c}(\epsilon_0 - \delta)^2 - 2\delta(2M+2)(\bar{q}+1). \tag{2.67}$$

By properties (P1), (P2), and (2.52), for every index vector $t \in \Omega_{k\bar{N}+i+1}$ and every integer $j \in \{1, \ldots, p(t)\}$ we have

$$\|z - y_j^{(t)}\|^2 \le \|z - y_{j-1}^{(t)}\|^2 + \delta^2 + 2\delta\|z - y_{j-1}^{(t)}\|$$
$$\le \|z - y_{j-1}^{(t)}\|^2 + 2\delta(2M+2),$$
$$\|z - y_{j-1}^{(t)}\|^2 - \|z - y_j^{(t)}\|^2 \ge -2\delta(2M+2). \tag{2.68}$$

In view of (2.17), (2.45), and (2.68), for every index vector $t \in \Omega_{k\bar{N}+i+1}$,

$$\|z - x_{k\bar{N}+i}\|^2 - \|z - y_t\|^2$$
$$= \sum_{i=1}^{p(t)} [\|z - y_{i-1}^{(t)}\|^2 - \|z - y_i^{(t)}\|^2] \geq -2\bar{q}\delta(2M+2). \quad (2.69)$$

Since the function $u \to \|u - z\|^2$, $u \in X$ is convex it follows from (2.11) and (2.58) that

$$\|\sum_{t\in\Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(t)y_t - z\|^2$$
$$\leq \sum_{t\in\Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(t)\|y_t - z\|^2$$
$$= \|z - x_{k\bar{N}+i}\|^2 + \sum_{t\in\Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(t)[\|y_t - z\|^2 - \|z - x_{k\bar{N}+i}\|^2]$$
$$\leq \|z - x_{k\bar{N}+i}\|^2 + w_{k\bar{N}+i+1}(s)[\|y_s - z\|^2 - \|z - x_{k\bar{N}+i}\|^2]$$
$$+ \sum \{w_{k\bar{N}+i+1}(t)[\|y_t - z\|^2 - \|z - x_{k\bar{N}+i}\|^2] : \ t \in \Omega_{k\bar{N}+i+1} \setminus \{s\}\}. \quad (2.70)$$

It follows from (2.11), (2.18), (2.32), (2.67), (2.69), and (2.70) that

$$\|\sum_{t\in\Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(t)y_t - z\|^2$$
$$\leq w_{k\bar{N}+i+1}(s)[-(\epsilon_0 - \delta)^2\bar{c} + 2\delta(2M+2)(\bar{q}+1)]$$
$$+2\bar{q}\delta(2M+2) + \|z - x_{k\bar{N}+i}\|^2$$
$$\leq \|z - x_{k\bar{N}+i}\|^2 + 2\bar{q}\delta(2M+2)$$
$$-w_{k\bar{N}+i+1}(s)[4^{-1}\epsilon_0^2\bar{c} - 2\delta(2M+2)(\bar{q}+1)]$$
$$\leq \|z - x_{k\bar{N}+i}\|^2 + 2\delta\bar{q}(2M+2) - \Delta(4^{-1}\epsilon_0^2\bar{c} - 2\delta(M+2)(\bar{q}+1)). \quad (2.71)$$

In view of (2.32) and (2.71) we have

$$\|\sum_{t\in\Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(t)y_t - z\|^2 - \|x_{k\bar{N}+i} - z\|^2$$
$$\leq 2\delta(2M+2)\bar{q} - 8^{-1}\Delta\epsilon_0^2\bar{c} \leq -16^{-1}\Delta\epsilon_0^2\bar{c}. \quad (2.72)$$

In view of (2.11), (2.43), and (2.72),

$$\begin{aligned}
&\|x_{k\bar{N}+i+1} - z\|^2 \\
&= \|x_{k\bar{N}+i+1} - \sum_{t\in\Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(t)y_t + \sum_{t\in\Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(t)y_t - z\|^2 \\
&\le \|x_{k\bar{N}+i+1} - \sum_{t\in\Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(t)y_t\|^2 + \|\sum_{t\in\Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(t)y_t - z\|^2 \\
&+2\|x_{k\bar{N}+i+1} - \sum_{t\in\Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(t)y_t\|\|\sum_{t\in\Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(t)y_t - z\| \\
&\le \delta^2 - 16^{-1}\bar{c}\Delta\epsilon_0^2 + \|x_{k\bar{N}+i} - z\|^2 \\
&+2\delta\|\sum_{t\in\Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(t)y_t - z\| \\
&\le \delta^2 - 16^{-1}\Delta\bar{c}\epsilon_0^2 + \|x_{k\bar{N}+i} - z\|^2 \\
&+2\delta \max\{\|y_t - z\| :\ t \in \Omega_{k\bar{N}+i+1}\}. \qquad (2.73)
\end{aligned}$$

In view of (2.32), (2.45), (2.73), and property (P2) we have

$$\begin{aligned}
&\|x_{k\bar{N}+i+1} - z\|^2 - \|x_{k\bar{N}+i} - z\|^2 \\
&\le \delta^2 - 16^{-1}\Delta\epsilon_0^2\bar{c} + 2\delta(2M+1) \\
&\le -16\Delta\epsilon_0^2\bar{c} + 2\delta(2M+2) \le -32^{-1}\Delta\epsilon_0^2\bar{c}.
\end{aligned}$$

This completes the proof of Lemma 2.2. □

It follows from (2.35), Lemma 2.2 applied by induction and (2.22) that for all integers $i = 0, \dots, \bar{N}-1$,

$$\|x_{k\bar{N}+i+1} - z\| \le \|x_{k\bar{N}+i} - z\| + \delta(\bar{q}+1),$$

$$\|x_{k\bar{N}+i+1} - z\| \le 2M + \delta(\bar{q}+1)(i+1) \le 2M + \delta(\bar{q}+1)\bar{N} \le 2M+1, \qquad (2.74)$$

$$\|x_{k\bar{N}+i} - z\| \le 2M+1,\ i = 0, \dots, \bar{N}. \qquad (2.75)$$

By (2.32), (2.33), (2.35), (2.74), and Lemma 2.2 we have

$$\begin{aligned}
&\|x_{(k+1)\bar{N}} - z\|^2 - \|x_{k\bar{N}} - z\|^2 \\
&= \sum_{i=0}^{\bar{N}-1}[\|x_{k\bar{N}+i+1} - z\|^2 - \|x_{k\bar{N}+i} - z\|^2] \\
&\le -32^{-1}\Delta\epsilon_0^2\bar{c} + \bar{N}\delta(\bar{q}+1)(4M+3) \le -64^{-1}\Delta\epsilon_0^2\bar{c}.
\end{aligned}$$

Thus we have shown that the following property holds:

(P3) if an integer $k \in [0, s]$ satisfies $\|x_{k\bar{N}} - z\| \le 2M$, then

$$\|x_j - z\| \le 2M + 1,\ j = k\bar{N}, \dots, (k+1)\bar{N},$$
$$\|x_{(k+1)\bar{N}} - z\|^2 - \|x_{k\bar{N}} - z\|^2 \le -64^{-1}\Delta\epsilon_0^2\bar{c}. \tag{2.76}$$

In view of (2.34) and property (P3) we have

$$\|x_j - z\| \le 2M + 1,\ j = 0, \dots, (s+1)\bar{N} \tag{2.77}$$

and (2.76) is true for every integer $k = 0, \dots, s$.

By (2.34) and (2.76),

$$64^{-1}\bar{c}\Delta\epsilon_0^2(s+1) \le \sum_{k=0}^{s}[\|x_{k\bar{N}} - z\|^2 - \|x_{(k+1)\bar{N}} - z\|^2]$$
$$= \|x_0 - z\|^2 - \|x_{(s+1)\bar{N}} - z\|^2 \le \|x_0 - z\|^2 \le 4M^2,$$
$$s + 1 \le 256M^2\Delta^{-1}\epsilon_0^{-2}\bar{c}^{-1}.$$

Thus we have shown that the following property holds:

(P4) If an integer $s \ge 0$ and for every integer $k \in [0, s]$ relation (2.33) holds, then

$$s \le 256M^2\Delta^{-1}\epsilon_0^{-2}\bar{c}^{-1} - 1,$$
$$\|x_j - z\| \le 2M + 1,\ j = 0, \dots, (s+1)\bar{N},$$
$$\|x_{k\bar{N}} - z\| \le 2M,\ k = 0, \dots, s+1.$$

By property (P4), (2.23), and (2.32), there exists an integer $q \in [0, n_0 - 1]$ such that for every integer k satisfying $0 \le k < q$,

$$\max\{\lambda_i :\ i = k\bar{N} + 1, \dots, (k+1)\bar{N}\} > \epsilon_0,$$
$$\max\{\lambda_i :\ i = q\bar{N} + 1, \dots, (q+1)\bar{N}\} \le \epsilon_0.$$

In view of (2.31), (2.34), property (P4), and the choice of q we have

$$\|x_{q\bar{N}} - z\| \le 2M,$$
$$\|x_j - z\| \le 2M + 1,\ j = 0, \dots, q\bar{N},$$
$$\|x_j\| \le 3M + 1,\ j = 0, \dots, q\bar{N}.$$

Assume that an integer $q \in [0, n_0 - 1]$ satisfies

$$\lambda_i \le \epsilon_0,\ i = q\bar{N} + 1, \dots, (q+1)\bar{N}. \tag{2.78}$$

Let

$$j \in \{q\bar{N}, \dots, (q+1)\bar{N} - 1\}. \tag{2.79}$$

It follows from (2.27) and (2.79),

$$(x_{j+1}, \lambda_{j+1}) \in A(x_j, (\Omega_{j+1}, w_{j+1}), \delta). \tag{2.80}$$

By (2.20), (2.78), and (2.80), there exist vectors

$$(y_t^{(j)}, \alpha_t^{(j)}) \in A_0(x_j, t, \delta),\ t \in \Omega_{j+1} \tag{2.81}$$

such that

$$\|x_{j+1} - \sum_{t \in \Omega_{j+1}} w_{j+1}(t) y_t^{(j)}\| \le \delta, \tag{2.82}$$

$$\max\{\alpha_t^{(j)} : t \in \Omega_{j+1}\} \le \epsilon_0. \tag{2.83}$$

It follows from (2.19), (2.81), and (2.83) that for every index vector $t = (t_1, \dots, t_{p(t)}) \in \Omega_{j+1}$ there exists a finite sequence $\{y_i^{(t,j)}\}_{i=0}^{p(t)} \subset X$ such that

$$y_0^{(t,j)} = x_j, \tag{2.84}$$

for every integer $i = 1, \dots, p(t)$,

$$\|y_i^{(t,j)} - P_{t_i}(y_{i-1}^{(t,j)})\| \le \delta, \tag{2.85}$$

$$y_{p(t)}^{(t,j)} = y_t^{(j)}, \tag{2.86}$$

$$\epsilon_0 \ge \alpha_t^{(j)} = \max\{\|y_i^{(t,j)} - y_{i-1}^{(t,j)}\| : i = 1, \dots, p(t)\}. \tag{2.87}$$

By (2.17), (2.84), (2.86), and (2.87), for every index vector $t \in \Omega_{j+1}$ and every integer $i = 1, \dots, p(t)$ we have

$$\|x_j - y_i^{(t,j)}\| \le i\epsilon_0 \le \epsilon_0 \bar{q}, \tag{2.88}$$

$$\|x_j - y_t^{(j)}\| \le \epsilon_0 \bar{q}. \tag{2.89}$$

In view of (2.85) and (2.88) for every index vector $t = (t_1, \dots, t_{p(t)}) \in \Omega_{j+1}$ and every integer $i = 1, \dots, p(t)$,

$$\begin{aligned} &\|x_j - P_{t_i}(y_{i-1}^{(t,j)})\| \\ &\le \|x_j - y_i^{(t,j)}\| + \|y_i^{(t,j)} - P_{t_i}(y_{i-1}^{(t,j)})\| \le \epsilon_0 \bar{q} + \delta. \end{aligned} \tag{2.90}$$

It follows from (2.11), (2.82), and (2.89) that

$$\|x_{j+1} - x_j\| \le \|x_{j+1} - \sum_{t\in\Omega_{j+1}} w_{j+1}(t)y_t^{(j)}\|$$
$$+\|\sum_{t\in\Omega_{j+1}} w_{j+1}(t)y_t^{(j)} - x_j\|$$
$$\le \delta + \sum_{t\in\Omega_{j+1}} w_{j+1}(t)\|y_t^{(j)} - x_j\| \le \delta + \epsilon_0\bar{q}.$$

Combined with (2.32) this implies that

$$\|x_{j+1} - x_j\| \le \epsilon_0(\bar{q} + 1). \tag{2.91}$$

By (2.32), (2.84), (2.88), and (2.90), for every index vector $t = (t_1, \dots, t_{p(t)}) \in \Omega_{j+1}$ and every integer $i = 1, \dots, p(t)$ we have

$$\|y_{i-1}^{(t,j)} - P_{t_i}(y_{i-1}^{(t,j)})\| \ge \|y_{i-1}^{(t,j)} - x_j\| + \|x_j - P_{t_i}(y_{i-1}^{(t,j)})\| \le 2\epsilon_0\bar{q} + \delta \le \epsilon_0(2\bar{q}+1). \tag{2.92}$$

In view of (2.84), (2.88), and (2.92), for every index vector $t = (t_1, \dots, t_{p(t)}) \in \Omega_{j+1}$ and every integer $i = 1, \dots, p(t)$,

$$x_j \in \tilde{F}_{\epsilon_0(2\bar{q}+1)}(P_{t_i}).$$

Therefore

$$x_j \in \cap\{\cap_{i=1}^{p(t)} \tilde{F}_{\epsilon_0(2\bar{q}+1)}(P_{t_i}) : \ t = (t_1, \dots, t_{p(t)}) \in \Omega_{j+1}\}. \tag{2.93}$$

It is clear that (2.91) and (2.93) are true for all integers $j = q\bar{N}, \dots, (q+1)\bar{N} - 1$. In view of (2.91), for every pair of integers $j_1, j_2 \in \{q\bar{N}, \dots, (q+1)\bar{N}\}$,

$$\|x_{j_1} - x_{j_2}\| \le \epsilon_0(\bar{q} + 1)\bar{N}. \tag{2.94}$$

Let $s \in \{1, \dots, m\}$. By (2.25), there exist an integer $j \in \{q\bar{N}, \dots, (q+1)\bar{N} - 1\}$ and an index vector $t = (t_1, \dots, t_{p(t)}) \in \Omega_{j+1}$ such that

$$s \in \{t_1, \dots, t_{p(t)}\}.$$

Together with (2.93) this implies that

$$x_j \in \tilde{F}_{\epsilon_0(2\bar{q}+1)}(P_s). \tag{2.95}$$

It follows from (2.94) and (2.95) that for every integer $i \in \{q\bar{N}, \dots, (q+1)\bar{N}\}$ we have

$$x_i \in \tilde{F}_{\epsilon_0(\bar{q}+1)(\bar{N}+2)}(P_s).$$

Since the inclusion above holds for every integer $s \in \{1, \dots, m\}$ we conclude that for each $i \in \{q\bar{N}, \dots, (q+1)\bar{N}\}$,

$$x_i \in \tilde{F}_{\epsilon_0(\bar{q}+1)(\bar{N}+2)} = \tilde{F}_{\epsilon_1}.$$

This completes the proof of Theorem 2.1. □

2.3 Asymptotic Behavior of Inexact Iterates

We use all the notation, definitions, and assumptions introduced in Sect. 2.1. It is not difficult to see that the following result holds.

Proposition 2.3. *Assume that for every $i \in \{1, \dots, m\}$,*

$$P_i(X) = Fix(P_i).$$

Then for every $i \in \{1, \dots, m\}$ and every $\epsilon > 0$,

$$F_\epsilon(P_i) \subset Fix(P_i) + B(0, \epsilon),$$

$$\tilde{F}_\epsilon(P_i) \subset F_{2\epsilon}(P_i),$$

$$\tilde{F}_\epsilon \subset F_{2\epsilon}.$$

Remark 2.4. If $P_i(X) = \mathrm{Fix}(P_i)$ for every $i \in \{1, \dots, m\}$, then in view of Proposition 2.3, we can easily obtain a version of Theorem 2.1, where in its conclusion the relation $x_i \in \tilde{F}_{\epsilon_1}$ is replaced by the inclusion $x_i \in F_{2\epsilon_1}$.

Proposition 2.5. *Assume that for every $i \in \{1, \dots, m\}$, every $x \in X$ and every $y \in X$,*

$$\|P_i(x) - P_i(y)\| \leq \|x - y\|. \tag{2.96}$$

Then for every $i \in \{1, \dots, m\}$ and every $\epsilon > 0$,

$$\tilde{F}_\epsilon(P_i) \subset F_{3\epsilon}(P_i),$$

$$\tilde{F}_\epsilon \subset F_{3\epsilon}.$$

Proof. Let $i \in \{1, \dots, m\}$, $\epsilon > 0$ and $x \in \tilde{F}_\epsilon(P_i)$. Then there exists

$$y \in F_\epsilon(P_i) \tag{2.97}$$

such that

$$\|y - x\| \leq \epsilon. \tag{2.98}$$

By (2.96)–(2.98),

$$\|x - P_i(x)\| \le \|x - y\| + \|y - P_i(y)\| + \|P_i(y) - P_i(x)\|$$
$$\le \epsilon + \epsilon + \|y - x\| \le 3\epsilon$$

and $x \in F_{3\epsilon}(P_i)$. Proposition 2.5 is proved. □

Remark 2.6. If (2.96) holds for all $x, y \in X$ and all $i \in \{1, \dots, m\}$, then we can easily obtain a version of Theorem 2.1, where in its conclusion the relation $x_i \in \tilde{F}_{\epsilon_1}$ is replaced by the inclusion $x_i \in F_{3\epsilon_1}$.

For each $z \in R^1$ set

$$\lfloor z \rfloor = \max\{i : \ i \text{ is an integer and } i \le z\}.$$

For each $M, \delta > 0$ set

$$\epsilon(\delta, M) = \bar{c}^{-1/2}(\bar{q}+1)(\bar{N}+2)(64\Delta^{-1}\delta(\bar{q}+1)(2M+4)4\bar{N})^{1/2}, \tag{2.99}$$

$$n(\delta, M) = \lfloor 2 + 4M^2\delta^{-1}(\bar{q}+1)^{-1}(2M+4)^{-1}(4\bar{N})^{-1} \rfloor. \tag{2.100}$$

Theorem 2.7. *Suppose that* $\bar{\epsilon} \in (0, 1)$, $\bar{M} > 0$,

$$\tilde{F}_{\bar{\epsilon}} \subset B(0, \bar{M}) \text{ and } F \ne \emptyset. \tag{2.101}$$

Let $M > \bar{M}$ *and* $\delta > 0$ *satisfy*

$$\delta \le (2\bar{q}\bar{N})^{-1} \text{ and } \epsilon(\delta, M) < \bar{\epsilon}. \tag{2.102}$$

Assume that

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$$

satisfies for each natural number j

$$\{1, \dots, m\} \subset \cup_{i=j}^{j+\bar{N}-1}(\cup_{t \in \Omega_i}\{t_1, \dots, t_{p(t)}\}),$$
$$x_0 \in B(0, M) \text{ and } \{x_i\}_{i=1}^{\infty} \subset X, \ \{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty)$$

satisfy for each natural number i,

$$(x_i, \lambda_i) \in A(x_{i-1}, (\Omega_i, w_i), \delta).$$

Then

$$\|x_i\| \le 3M + 1 \text{ for all integers } i \ge 0$$

and there exists a strictly increasing sequence of integers $\{q_p\}_{p=0}^{\infty}$ *such that*

$$0 \le q_0 \le n(\delta, M) - 1,$$

$$1 \le q_{p+1} - q_p \le n(\delta, M) \textit{ for all integers } p \ge 0$$

and that for each integer $p \ge 0$ *and each* $i = q_p\bar{N}, \dots, (q_p + 1)\bar{N}$,

$$x_i \in \tilde{F}_{\epsilon(\delta,M)}.$$

We can prove Theorem 2.7 applying by induction Theorem 2.1 and using (2.101) and (2.102).

Remark 2.8. Note that the set $\tilde{F}_{\bar{\epsilon}}$ is bounded if there exists an integer $j \in \{1, \dots, m\}$ such that the set $F_j(X)$ is bounded.

Assume that $C_1, \dots, C_m \subset X$ and $\cap_{i=1}^m C_i \neq \emptyset$. We say that the family of sets $\{C_1, \dots, C_m\}$ has a bounded regularity property [7] if for each $\epsilon > 0$ and each $M > 0$ there exists $\delta > 0$ such that if $x \in B(0, M)$ satisfies $d(x, C_i) \le \delta$ for all $i = 1, \dots, m$, then $d(x, \cap_{i=1}^m C_i) \le \epsilon$.

Theorem 2.9. *Suppose that*

$$P_i(X) = Fix(P_i),\ i = 1, \dots, m,$$

the family of sets $\{Fix(P_i),\ i = 1, \dots, m\}$ *has the bounded regularity property,* $M > 0$ *satisfies*

$$B(0, M) \cap F \neq \emptyset$$

and that $\epsilon_0 \in (0, 1)$. *Let* $\epsilon_1 \in (0, \epsilon_0)$ *be such that the following property holds:*

(i) if $z \in B(0, 3M + 2)$ *satisfies* $d(x, Fix(P_i)) \le 2\epsilon_1$ *for all* $i = 1, \dots, m$, *then* $d(z, F) \le \epsilon_0$.

Let $\delta > 0$ *satisfy*

$$\delta \le (2\bar{q}\bar{N})^{-1} \textit{ and } \epsilon(\delta, M) \le \epsilon_1.$$

Assume that

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$$

satisfies for each natural number j

$$\{1, \dots, m\} \subset \cup_{i=j}^{j+\bar{N}-1}(\cup_{t \in \Omega_i}\{t_1, \dots, t_{p(t)}\}),$$

$$x_0 \in B(0, M) \textit{ and } \{x_i\}_{i=1}^{\infty} \subset X,\ \{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty)$$

satisfy for each natural number i,

$$(x_i, \lambda_i) \in A(x_{i-1}, (\Omega_i, w_i), \delta).$$

Then there exists an integer $q \in [0, n(\delta, M) - 1]$ *such that*

$$\|x_i\| \leq 3M + 1, \ i = 0, \ldots, q\bar{N}, \tag{2.103}$$

$$\lambda_i \leq (64\Delta^{-1}\delta(\bar{q} + 1)(2M + 4)4\bar{N}\bar{c}^{-1})^{1/2}, \tag{2.104}$$
$$i = q\bar{N} + 1, \ldots, (q + 1)\bar{N}.$$

Moreover, if an integer $q \in [0, n(\delta, M) - 1]$ *satisfies (2.103) and (2.104), then for each* $i = q\bar{N}, \ldots, (q + 1)\bar{N}$,

$$d(x_i, F) \leq \epsilon_0 \tag{2.105}$$

and

$$\|x_i - x_j\| \leq \epsilon(\delta, M) \text{ for each } i, j \in \{q\bar{N}, \ldots, (q + 1)\bar{N}\}. \tag{2.106}$$

Proof. By Theorem 2.1, there exists an integer $q \in [0, n(\delta, M) - 1]$ such that (2.103) and (2.104) hold.

Assume that an integer $q \in [0, n(\delta, M) - 1]$ satisfies (2.103) and (2.104). By Theorem 2.1, (2.106) holds and

$$x_i \in \tilde{F}_{\epsilon_1}, \ i = q\bar{N}, \ldots, (q + 1)\bar{N}.$$

Together with Proposition 2.3 this implies that for all $i = q\bar{N}, \ldots, (q + 1)\bar{N}$,

$$x_i \in \tilde{F}_{\epsilon_1} \subset F_{2\epsilon_1} \subset \cap_{j=1}^{m}(\mathrm{Fix}(P_i) + B(0, 2\epsilon_1)). \tag{2.107}$$

In view of (2.103) and (2.106), for all $i = q\bar{N}, \ldots, (q + 1)\bar{N}$,

$$\|x_i\| \leq \epsilon(\delta, M) + \|x_{q\bar{N}}\| \leq 3M + 1 + \epsilon(\delta, M) \leq 3M + 2. \tag{2.108}$$

By (2.107), (2.108), property (i), and the choice of ϵ_1, for all $i = q\bar{N}, \ldots, (q + 1)\bar{N}$,

$$d(x_i, F) \leq \epsilon_0.$$

Theorem 2.9 is proved. □

Applying by induction Theorem 2.9 we obtain the following result.

Theorem 2.10. *Suppose that*

$$P_i(X) = Fix(P_i), \ i = 1, \ldots, m,$$

the family of sets $\{Fix(P_i),\ i = 1,\dots,m\}$ *has the bounded regularity property,* $\bar{M} > 0$ *satisfies*

$$F \subset B(0,\bar{M}),$$

$M > \bar{M} + 1$ *and that* $\epsilon_0 \in (0,1)$*. Let* $\epsilon_1 \in (0,\epsilon_0)$ *be such that the following property holds:*

if $z \in B(0,3M+2)$ *satisfies* $d(z, Fix(P_i)) \le 2\epsilon_1$ *for all* $i = 1,\dots,m$*, then* $d(z,F) \le \epsilon_0$.

Let $\delta > 0$ *satisfy*

$$\delta \le (2\bar{q}\bar{N})^{-1} \text{ and } \epsilon(\delta,M) \le \epsilon_1.$$

Assume that

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$$

satisfies for each natural number j

$$\{1,\dots,m\} \subset \cup_{i=j}^{j+\bar{N}-1}(\cup_{t\in\Omega_i}\{t_1,\dots,t_{p(t)}\}),$$

$$x_0 \in B(0,M) \text{ and } \{x_i\}_{i=1}^{\infty} \subset X,\ \{\lambda_i\}_{i=1}^{\infty} \subset [0,\infty)$$

satisfy for each natural number i,

$$(x_i,\lambda_i) \in A(x_{i-1},(\Omega_i,w_i),\delta).$$

Then

$$\|x_i\| \le 3M+1 \text{ for all integers } i \ge 0$$

and there exists a strictly increasing sequence of integers $\{q_p\}_{p=0}^{\infty}$ *such that*

$$0 \le q_0 \le n(\delta,M) - 1,$$

$$1 \le q_{p+1} - q_p \le n(\delta,M) \text{ for all integers } p \ge 0$$

and that for each integer $p \ge 0$ *and each* $i = q_p\bar{N},\dots,(q_p+1)\bar{N}$,

$$d(x_i,F) \le \epsilon_0.$$

The following result is proved in Sect. 2.4.

Theorem 2.11. *Suppose that*

$$P_i(X) = Fix(P_i),\ i = 1,\dots,m,$$

the family of sets $\{Fix(P_i),\ i = 1, \dots, m\}$ has the bounded regularity property, $F \neq \emptyset$, $\bar{M} > 0$ satisfies

$$F \subset B(0, \bar{M}),$$

$M > \bar{M} + 1$ and that $\epsilon_0 \in (0, 1)$. Let $\epsilon_1 \in (0, \epsilon_0/2)$ be such that the following property holds:

(ii) if $z \in B(0, 3M + 2)$ satisfies $d(z, Fix(P_i)) \le 2\epsilon_1$ for all $i = 1, \dots, m$, then $d(z, F) \le \epsilon_0/2$.

Let $\delta_0 > 0$ satisfy

$$\delta_0 \le (2\bar{q}\bar{N})^{-1} \text{ and } \epsilon(\delta_0, M) \le \epsilon_1 \tag{2.109}$$

and let a positive number δ satisfy

$$\delta < \delta_0 \text{ and } \delta n(\delta_0, M)\bar{N}(\bar{q} + 1) < \epsilon_0/2. \tag{2.110}$$

Assume that

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$$

satisfies for each natural number j

$$\{1, \dots, m\} \subset \cup_{i=j}^{j+\bar{N}-1}(\cup_{t\in\Omega_i}\{t_1, \dots, t_{p(t)}\}),$$

$$x_0 \in B(0, M) \text{ and } \{x_i\}_{i=1}^{\infty} \subset X,\ \{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty)$$

satisfy for each natural number i,

$$(x_i, \lambda_i) \in A(x_{i-1}, (\Omega_i, w_i), \delta).$$

Then

$$\|x_i\| \le 3M + 1 \text{ for all integers } i \ge 0$$

and for each integer $i \ge (n(\delta_0, M) - 1)\bar{N}$,

$$d(x_i, F) \le \epsilon_0.$$

2.4 Proof of Theorem 2.11

By Theorem 2.10,

$$\|x_i\| \le 3M + 1 \text{ for all integers } i \ge 0$$

and there exists a strictly increasing sequence of integers $\{q_p\}_{p=0}^{\infty}$ such that

$$0 \le q_0 \le n(\delta_0, M) - 1, \tag{2.111}$$

$$1 \le q_{p+1} - q_p \le n(\delta_0, M) \text{ for all integers } p \ge 0 \tag{2.112}$$

and that for each integer $p \ge 0$ and each $i = q_p\bar{N}, \dots, (q_p + 1)\bar{N}$,

$$d(x_i, F) \le \epsilon_0/2. \tag{2.113}$$

Assume that an integer $p \ge 0$ and that an integer i satisfies

$$(q_p + 1)\bar{N} \le i < q_{p+1}\bar{N} \tag{2.114}$$

and

$$d(x_i, F) \le \epsilon_0/2 + (i - (q_p + 1)\bar{N})\delta(\bar{q} + 1). \tag{2.115}$$

(Note that in view of (2.113), inequality (2.115) is true for $i = (q_p + 1)\bar{N}$.)

Let $\gamma > 0$. By (2.115), there exists $z \in X$ such that

$$z \in F,$$

$$\|x_i - z\| < \epsilon_0/2 + (i - (q_p + 1)\bar{N})\delta(\bar{q} + 1) + \gamma. \tag{2.116}$$

Set

$$\gamma_1 = \epsilon_0/2 + (i - (q_p + 1)\bar{N})\delta(\bar{q} + 1) + \gamma. \tag{2.117}$$

The inclusion

$$(x_{i+1}, \lambda_{i+1}) \in A(x_i, (\Omega_{i+1}, w_{i+1}), \delta) \tag{2.118}$$

is true. By (2.20) and (2.118) there exists

$$(y_t, \alpha_t) \in A_0(x_i, t, \delta),\ t \in \Omega_{i+1} \tag{2.119}$$

such that

$$\|x_{i+1} - \sum_{t \in \Omega_{i+1}} w_{i+1}(t) y_t\| \le \delta, \tag{2.120}$$

$$\lambda_{i+1} = \max\{\alpha_t :\ t \in \Omega_{i+1}\}. \tag{2.121}$$

It follows from (2.19) and (2.119) that for each $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{i+1}$ there exists a finite sequence $\{y_j^{(t)}\}_{j=0}^{p(t)} \subset X$ such that

$$y_0^{(t)} = x_i, \; y_{p(t)}^{(t)} = y_t, \tag{2.122}$$

$$\|y_j^{(t)} - P_{t_j}(y_{j-1}^{(t)})\| \le \delta \text{ for each integer } j = 1, \ldots, p(t), \tag{2.123}$$

$$\alpha_t = \max\{\|y_{j+1}^{(t)} - y_j^{(t)}\| : \; j = 0, \ldots, p(t) - 1\}. \tag{2.124}$$

Let

$$t = (t_1, \ldots, t_{p(t)}) \in \Omega_{i+1}.$$

By (2.116), (2.117), and (2.122),

$$\|z - y_0^{(t)}\| = \|z - x_i\| \le \gamma_1. \tag{2.125}$$

Assume that an integer i satisfies $0 \le i < p(t)$ and

$$\|z - y_i^{(t)}\| \le \gamma_1 + i\delta. \tag{2.126}$$

(Note that in view of (2.125), inequality (2.126) is true for $i = 0$.) By (2.2), (2.116), (2.123), and (2.126),

$$\begin{aligned} \|z - y_{i+1}^{(t)}\| &\le \|z - P_{t_{i+1}}(y_i^{(t)})\| + \|P_{t_{i+1}}(y_i^{(t)}) - y_{i+1}^{(t)}\| \\ &\le \|z - y_i^{(t)}\| + \delta \le \gamma_1 + (i+1)\delta. \end{aligned}$$

Thus we have shown by induction that (2.126) holds for all $i = 0, \ldots, p(t)$. Combined with (2.17) and (2.122) this implies that

$$\|z - y_t\| = \|z - y_{p(t)}^{(t)}\| \le \gamma_1 + p(t)\delta \le \gamma_1 + \bar{q}\delta \tag{2.127}$$

for every $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{i+1}$.

It follows from (2.11), (2.117), (2.120), and (2.127) that

$$\begin{aligned} \|z - x_{i+1}\| &\le \|z - \sum_{t \in \Omega_{i+1}} w_{i+1}(t) y_t\| + \|\sum_{t \in \Omega_{i+1}} w_{i+1}(t) y_t - x_{i+1}\| \\ &\le \sum_{t \in \Omega_{i+1}} w_{i+1}(t)\|z - y_t\| + \delta \le \gamma_1 + \bar{q}\delta + \delta \\ &\le \epsilon_0/2 + (i - (q_p + 1)\bar{N})\delta(\bar{q} + 1) + (\bar{q} + 1)\delta + \gamma. \end{aligned}$$

Since γ is any positive number we conclude that

$$\|z - x_{i+1}\| \le \epsilon_0/2 + (i + 1 - (q_p + 1)\bar{N})\delta(\bar{q} + 1).$$

Thus we have shown by induction that (2.115) holds for all $i = (q_p + 1)\bar{N}, \dots, q_{p+1}\bar{N}$. Combined with (2.110), (2.111), (2.112), and (2.113) this implies that for each integer $i \ge (n(\delta_0, M) - 1)\bar{N}$,

$$d(x_i, F) \le \epsilon_0/2 + n(\delta_0, M)\bar{N}\delta(\bar{q} + 1) < \epsilon_0.$$

Theorem 2.11 is proved. □

2.5 Auxiliary Results

We use the notation, definitions, and assumptions introduced in Sects. 2.1 and 2.3.

Proposition 2.12. *Let*

$$P_i(X) = Fix(P_i),\ i = 1, \dots, m,$$

$M > 0$ *satisfy*

$$F \cap B(0, M) \neq \emptyset,$$

$r > 0$ *and* k *be a natural number. Assume that*

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$$

satisfies for each natural number j

$$\{1, \dots, m\} \subset \cup_{i=j}^{j+\bar{N}-1}(\cup_{t\in\Omega_i}\{t_1, \dots, t_{p(t)}\}),$$

$$x_0 \in B(0, M) \tag{2.128}$$

and

$$\{x_i\}_{i=1}^{\infty} \subset X,\ \{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty)$$

satisfy for each natural number i,

$$(x_i, \lambda_i) \in A(x_{i-1}, (\Omega_i, w_i), r).$$

Then for all integers $i = 0, \dots, k$,

$$\|x_i\| \le 3M + k(\bar{q} + 1)r.$$

Proof. Fix

$$z \in F \cap B(0, M). \tag{2.129}$$

By (2.128) and (2.129),

$$\|z - x_0\| \leq 2M. \tag{2.130}$$

We show that for all $i = 0, \ldots, k$,

$$\|z - x_i\| \leq 2M + i(\bar{q} + 1)r. \tag{2.131}$$

In view of (2.130), inequality (2.131) holds for $i = 0$. Assume that an integer i satisfies $0 \leq i < k$ and that (2.131) holds.

The inclusion

$$(x_{i+1}, \lambda_{i+1}) \in A(x_i, (\Omega_{i+1}, w_{i+1}), r) \tag{2.132}$$

is true. By (2.20) and (2.132) there exist

$$(y_t, \alpha_t) \in A_0(x_i, t, r),\ t \in \Omega_{i+1} \tag{2.133}$$

such that

$$\|x_{i+1} - \sum_{t \in \Omega_{i+1}} w_{i+1}(t) y_t\| \leq r, \tag{2.134}$$

$$\lambda_{i+1} = \max\{\alpha_t :\ t \in \Omega_{i+1}\}. \tag{2.135}$$

It follows from (2.19) and (2.133) that for each $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{i+1}$ there exists a finite sequence $\{y_j^{(t)}\}_{j=0}^{p(t)} \subset X$ such that

$$y_0^{(t)} = x_i,\ y_{p(t)}^{(t)} = y_t, \tag{2.136}$$

$$\|y_j^{(t)} - P_{t_j}(y_{j-1}^{(t)})\| \leq r \text{ for each integer } j = 1, \ldots, p(t), \tag{2.137}$$

$$\alpha_t = \max\{\|y_{j+1}^{(t)} - y_j^{(t)}\| :\ j = 0, \ldots, p(t) - 1\}. \tag{2.138}$$

Let

$$t = (t_1, \ldots, t_{p(t)}) \in \Omega_{i+1}.$$

We show that for all $j = 0, \ldots, p(t)$,

$$\|z - y_j^{(t)}\| \leq \|z - x_i\| + jr. \tag{2.139}$$

Note that in view of (2.136), inequality (2.139) is true for $j = 0$.

Assume that an integer j satisfies $0 \leq j < p(t)$ and that (2.139) holds. By (2.2), (2.129), (2.137), and (2.139),

$$\|z - y_{j+1}^{(t)}\| \leq \|z - P_{t_{j+1}}(y_j^{(t)})\| + \|P_{t_{j+1}}(y_j^{(t)}) - y_{j+1}^{(t)}\|$$
$$\leq \|z - y_j^{(t)}\| + r \leq \|z - x_i\| + (j+1)r.$$

Thus we have shown by induction that (2.139) holds for all $j = 0, \ldots, p(t)$. Combined with (2.17) and (2.136) this implies that

$$\|z - y_t\| = \|z - x_i\| + p(t)r \leq \|z - x_i\| + \bar{q}r \tag{2.140}$$

for all $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{i+1}$. It follows from (2.11), (2.134), and (2.140) that

$$\|z - x_{i+1}\| \leq \|z - \sum_{t \in \Omega_{i+1}} w_{i+1}(t)y_t\| + \|\sum_{t \in \Omega_{i+1}} w_{i+1}(t)y_t - x_{i+1}\|$$
$$\leq \sum_{t \in \Omega_{i+1}} w_{i+1}(t)\|z - y_t\| + r \leq \|z - x_i\| + \bar{q}r + r.$$

Combined with (2.31) this implies that

$$\|z - x_{i+1}\| \leq \|z - x_i\| + \bar{q}r + r \leq 2M + r(\bar{q}+1)(i+1).$$

Thus we have shown by induction that (2.131) holds for all $i = 0, \ldots, k$. Together with (2.129) this implies that for all $i = 0, \ldots, k$,

$$\|x_i\| \leq 3M + (\bar{q}+1)kr.$$

Proposition 2.12 is proved. □

Proposition 2.13. *Let*

$$P_i(X) = Fix(P_i),\ i = 1, \ldots, m,$$

$F \neq \emptyset$, $r > 0$ and $\bar{M} > 0$ satisfy

$$F \subset B(0, \bar{M}).$$

Suppose that

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$$

satisfies for each natural number j the following properties:

(a)

$$\{1, \dots, m\} \subset \cup_{i=j}^{j+\bar{N}-1} (\cup_{t \in \Omega_i} \{t_1, \dots, t_{p(t)}\});$$

(b) there exists $i(j) \in \{j, \dots, j + \bar{N} - 1\}$ *such that for each*

$$t = (t_1, \dots, t_{p(t)}) \in \Omega_{i(j)}$$

there exists $s \in \{t_1, \dots, t_{p(t)}\}$ *for which*

$$P_s(X) \subset B(0, \bar{M}).$$

Assume that

$$\{x_i\}_{i=0}^{\infty} \subset X, \; \{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty)$$

satisfy for each natural number i,

$$(x_i, \lambda_i) \in A(x_{i-1}, (\Omega_i, w_i), r).$$

Then for all integers $i \geq \bar{N}$,

$$\|x_i\| \leq 6\bar{M} + r(\bar{q} + 1)(\bar{N} + 3) + 3r.$$

Proof. Fix

$$z \in F \cap B(0, \bar{M}). \tag{2.141}$$

Assume that $p \geq 0$ is an integer. By property (b) there is $\tilde{i} \in \{p + 1, \dots, p + \bar{N}\}$ such that the following property holds:

(c) for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{\tilde{i}}$ there exists $s \in \{t_1, \dots, t_{p(t)}\}$ for which

$$P_s(X) \subset B(0, \bar{M}).$$

The inclusion

$$(x_{\tilde{i}}, \lambda_{\tilde{i}}) \in A(x_{\tilde{i}-1}, (\Omega_{\tilde{i}}, w_{\tilde{i}}), r). \tag{2.142}$$

is true. By (2.20) and (2.142) there exist

$$(y_t, \alpha_t) \in A_0(x_{\tilde{i}-1}, t, r), \; t \in \Omega_{\tilde{i}} \tag{2.143}$$

such that

$$\|x_{\tilde{i}} - \sum_{t \in \Omega_{\tilde{i}}} w_{\tilde{i}}(t) y_t\| \leq r, \tag{2.144}$$

$$\lambda_{\tilde{i}} = \max\{\alpha_t : t \in \Omega_{\tilde{i}}\}. \tag{2.145}$$

It follows from (2.19) and (2.143) that for each $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{\tilde{i}}$ there exists a finite sequence $\{y_j^{(t)}\}_{j=0}^{p(t)} \subset X$ such that

$$y_0^{(t)} = x_{\tilde{i}-1},\ y_{p(t)}^{(t)} = y_t, \tag{2.146}$$

$$\|y_j^{(t)} - P_{t_j}(y_{j-1}^{(t)})\| \leq r \text{ for each integer } j = 1, \ldots, p(t), \tag{2.147}$$

$$\alpha_t = \max\{\|y_{j+1}^{(t)} - y_j^{(t)}\| : j = 0, \ldots, p(t) - 1\}. \tag{2.148}$$

Let

$$t = (t_1, \ldots, t_{p(t)}) \in \Omega_{\tilde{i}}.$$

In view of property (c), there exists $\tilde{j} \in \{t_1, \ldots, t_{p(t)}\}$ such that

$$P_{t_{\tilde{j}}}(X) \subset B(0, \bar{M}). \tag{2.149}$$

By (2.147),

$$\|y_{\tilde{j}}^{(t)} - P_{t_{\tilde{j}}}(y_{\tilde{j}-1}^{(t)})\| \leq r. \tag{2.150}$$

Relations (2.149) and (2.150) imply that

$$\|y_{\tilde{j}}^{(t)}\| \leq \bar{M} + r. \tag{2.151}$$

It follows from (2.141) and (2.51) that

$$\|z - y_{\tilde{j}}^{(t)}\| \leq 2\bar{M} + r. \tag{2.152}$$

Assume that an integer j satisfies

$$\tilde{j} \leq j < p(t)$$

and

$$\|z - y_j^{(t)}\| \leq 2\bar{M} + r + (j - \tilde{j})r. \tag{2.153}$$

(Note that in view of (2.152), inequality (2.153) is true for $j = \tilde{j}$.) By (2.2), (2.14), (2.147), and (2.153),

$$\|z - y_{j+1}^{(t)}\| \le \|z - P_{t_{j+1}}(y_j^{(t)})\| + \|P_{t_{j+1}}(y_j^{(t)}) - y_{j+1}^{(t)}\|$$
$$\le \|z - y_j^{(t)}\| + r \le 2\bar{M} + r + (j + 1 - \tilde{j})r.$$

Thus we have shown by induction that (2.153) holds for all $j = \tilde{j}, \dots, p(t)$. Together with (2.17) and (2.146) this implies that

$$\|z - y_t\| = \|z - y_{p(t)}^{(t)}\| \le 2\bar{M} + r + (p(t) - \tilde{j})r \le 2\bar{M} + r + r\bar{q},$$
$$\|z - y_t\| \le 2\bar{M} + r(\bar{q} + 1) \text{ for all } t \in \Omega_{\tilde{i}}. \tag{2.154}$$

By (2.11), (2.144), and (2.154),

$$\|x_{\tilde{i}} - z\| \le \|x_{\tilde{i}} - \sum_{t \in \Omega_{\tilde{i}}} w_{\tilde{i}}(t) y_t\| + \|\sum_{t \in \Omega_{\tilde{i}}} w_{\tilde{i}}(t) y_t - z\|$$
$$\le r + \sum_{t \in \Omega_{\tilde{i}}} w_{\tilde{i}}(t)\|y_t - z\| \le r + 2\bar{M} + r(\bar{q} + 1).$$

Thus we have shown that the following property holds:
for each integer $p \ge 0$ there exists $\tilde{i} \in \{p + 1, \dots, p + \bar{N}\}$ such that

$$\|x_{\tilde{i}} - z\| \le 2\bar{M} + r(\bar{q} + 2).$$

This property implies that there exists a strictly increasing sequence of natural numbers $\{p_i\}_{i=1}^{\infty}$ such that

$$1 \le p_1 \le \bar{N},$$
$$1 \le p_{i+1} - p_i \le \bar{N} \text{ for all integers } i \ge 1$$

and that

$$\|x_{p_i} - z\| \le 2\bar{M} + r(\bar{q} + 2) \text{ for all integers } i \ge 1.$$

Applying Proposition 2.12 we obtain that for all integers $i \ge \bar{N}$,

$$\|x_i\| \le 6\bar{M} + r(\bar{q} + 1)(\bar{N} + 3)r + 3r.$$

Proposition 2.13 is proved. □

2.6 A Convergence Result

We use the notation, definitions, and assumptions introduced in Sects. 2.1 and 2.3. We prove the following convergence result under the assumption that the computation errors tend to zero.

Theorem 2.14. *Suppose that*

$$P_i(X) = Fix(P_i),\ i = 1, \dots, m,$$

$F \neq \emptyset$, *the family* $\{Fix(P_i) :\ i = 1, \dots, m\}$ *has the bounded regularity property,* $\epsilon > 0$, $\bar{M} > 0$ *satisfy*

$$F \subset B(0, \bar{M})$$

and that a sequence $\{\delta_i\}_{i=1}^{\infty} \subset (0, \infty)$ *satisfies*

$$\lim_{i \to \infty} \delta_i = 0. \tag{2.155}$$

Then there exist a natural number k_1 *such that the following assertion holds.*

Let

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$$

satisfy for each natural number j the following properties:

(P5)

$$\{1, \dots, m\} \subset \cup_{i=j}^{j+\bar{N}-1} (\cup_{t \in \Omega_i} \{t_1, \dots, t_{p(t)}\});$$

(P6) there exists $i(j) \in \{j, \dots, j + \bar{N} - 1\}$ *such that for each* $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i(j)}$ *there exists* $s \in \{t_1, \dots, t_{p(t)}\}$ *for which*

$$P_s(X) \subset B(0, \bar{M}).$$

Assume that

$$\{x_i\}_{i=0}^{\infty} \subset X,\ \{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty)$$

satisfy for each natural number i,

$$(x_i, \lambda_i) \in A(x_{i-1}, (\Omega_i, w_i), \delta_i).$$

Then for all integers $i \geq k_1$,

$$d(x_i, F) \leq \epsilon.$$

Proof. Set

$$r = \max\{\delta_i :\ i = 1, 2, \dots\}. \tag{2.156}$$

By Theorem 2.11, there exists $\bar{\delta} \in (0, 1)$ such that the following property holds:

(P7) for each

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$$

which satisfies for each natural number j

$$\{1, \ldots, m\} \subset \cup_{i=j}^{j+\bar{N}-1} (\cup_{t \in \Omega_i} \{t_1, \ldots, t_{p(t)}\}),$$

each

$$x_0 \in B(0, 6\bar{M} + r(\bar{q} + 1)(\bar{N} + 3) + 3r)$$

and each pair of sequences $\{x_i\}_{i=1}^{\infty} \subset X$, $\{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty)$ which satisfies for each natural number i,

$$(x_i, \lambda_i) \in A(x_{i-1}, (\Omega_i, w_i), \bar{\delta})$$

we have

$$d(x_i, F) \leq \epsilon$$

for each integer $i \geq \bar{N} n(\bar{\delta}, 6\bar{M} + r(\bar{q} + 1)(\bar{N} + 3) + 3r)$. By (2.155), there is an integer $k_0 \geq 1$ such that

$$\delta_i \leq \bar{\delta} \text{ for all integers } i \geq k_0. \tag{2.157}$$

Assume that

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$$

satisfies for each natural number j properties (P5) and (P6) and that

$$\{x_i\}_{i=0}^{\infty} \subset X, \ \{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty)$$

satisfy for each natural number i,

$$(x_i, \lambda_i) \in A(x_{i-1}, (\Omega_i, w_i), \delta_i). \tag{2.158}$$

By (2.156), (2.158), properties (P5) and (P6), and Proposition 2.13,

$$\|x_i\| \leq 6\bar{M} + r(\bar{q} + 1)(\bar{N} + 3) + 3r \text{ for all integers } i \geq \bar{N}. \tag{2.159}$$

It follows from (2.157), (2.159), and property (P7) that

$$d(x_i, F) \leq \epsilon$$

for all integers $i \geq \bar{N} + k_0 + \bar{N} n(\bar{\delta}, 6\bar{M} + r(\bar{q} + 1)(\bar{N} + 3) + 3r)$. Theorem 2.14 is proved. □

2.7 Asymptotic Behavior of Exact Iterates

We use the notation, definitions, and assumptions introduced in Sects. 2.1 and 2.3.

Theorem 2.15. *Let $M > 0$ satisfy*

$$B(0, M) \cap F \neq \emptyset$$

and let $\epsilon > 0$. Assume that

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_* \tag{2.160}$$

satisfies for every natural number j

$$\{1, \ldots, m\} \subset \cup_{i=j}^{j+\bar{N}-1}(\cup_{t\in\Omega_i}\{t_1, \ldots, t_{p(t)}\}), \tag{2.161}$$

$$x_0 \in B(0, M) \tag{2.162}$$

and $\{x_i\}_{i=1}^{\infty} \subset X,\ \{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty)$ satisfy for every natural number i,

$$(x_i, \lambda_i) \in A(x_{i-1}, (\Omega_i, w_i), 0).$$

Then

$$Card(\{i \in \{0, 1, \ldots\} :\ x_i \notin \tilde{F}_\epsilon\}) \le \bar{N}(4M^2\bar{c}^{-1}\Delta^{-1}\epsilon^{-2}(\bar{N}+1)^2\bar{q}^2 + 1) + 1.$$

Proof. Fix a point

$$z \in B(0, M) \cap F. \tag{2.163}$$

Set

$$\gamma_0 = \epsilon(\bar{N}+1)^{-1}\bar{q}^{-1}. \tag{2.164}$$

Let $i \ge 0$ be an integer. The inclusion

$$(x_{i+1}, \lambda_{i+1}) \in A(x_i, (\Omega_{i+1}, w_{i+1}), 0) \tag{2.165}$$

is true. By (2.20) and (2.165) there exist vectors

$$(y_t, \alpha_t) \in A_0(x_i, t, 0),\ t \in \Omega_{i+1} \tag{2.166}$$

such that

$$x_{i+1} = \sum_{t\in\Omega_{i+1}} w_{i+1}(t)y_t, \tag{2.167}$$

$$\lambda_{i+1} = \max\{\alpha_t :\ t \in \Omega_{i+1}\}. \tag{2.168}$$

It follows from (2.19) and (2.166) that for every $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}$ there exists a finite sequence $\{y_j^{(t)}\}_{j=0}^{p(t)} \subset X$ such that

$$y_0^{(t)} = x_i, \; y_{p(t)}^{(t)} = y_t, \tag{2.169}$$

$$y_j^{(t)} = P_{t_j}(y_{j-1}^{(t)}) \text{ for each integer } j = 1, \dots, p(t), \tag{2.170}$$

$$\alpha_t = \max\{\|y_{j+1}^{(t)} - y_j^{(t)}\| : j = 0, \dots, p(t) - 1\}. \tag{2.171}$$

Let

$$t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}.$$

By (2.2), (2.163) and (2.170), for every integer j satisfying $0 \le j < p(t)$, we have

$$\begin{aligned}\|z - y_j^{(t)}\|^2 - \|z - y_{j+1}^{(t)}\|^2 &= \|z - y_j^{(t)}\|^2 - \|z - P_{t_{j+1}}(y_j^{(t)})\|^2 \\ &\ge \bar{c}\|y_j^{(t)} - y_{j+1}^{(t)}\|^2.\end{aligned} \tag{2.172}$$

In view of (2.17), (2.169) and (2.172),

$$\begin{aligned}\|z - x_i\|^2 - \|z - y_t\|^2 &= \|z - y_0^{(t)}\|^2 - \|z - y_{p(t)}^{(t)}\|^2 \\ &= \sum_{j=0}^{p(t)-1} (\|z - y_j^{(t)}\|^2 - \|z - y_{j+1}^{(t)}\|^2) \\ &\ge \bar{c} \sum_{j=0}^{p(t)-1} \|y_j^{(t)} - y_{j+1}^{(t)}\|^2 \ge \bar{c}\alpha_t^2.\end{aligned} \tag{2.173}$$

It follows from (2.11), (2.18), (2.167), (2.168), and (2.173) that

$$\begin{aligned}\|z - x_{i+1}\|^2 &= \|z - \sum_{t \in \Omega_{i+1}} w_{i+1}(t) y_t\|^2 \le \sum_{t \in \Omega_{i+1}} \|z - y_t\|^2 w_{i+1}(t) \\ &\le \sum_{t \in \Omega_{i+1}} w_{i+1}(t)(\|z - x_i\|^2 - \bar{c}\alpha_t^2) \\ &\le \|z - x_i\|^2 - \bar{c}\Delta \sum_{t \in \Omega_{i+1}} \alpha_t^2 \le \|z - x_i\|^2 - \bar{c}\Delta\lambda_{i+1}^2.\end{aligned}$$

Thus

$$\|z - x_{i+1}\|^2 \le \|z - x_i\|^2 - \bar{c}\Delta\lambda_{i+1}^2 \text{ for all integers } i \ge 0. \tag{2.174}$$

By (2.162), (2.163), and (2.174), for each natural number n,

$$4M^2 \geq \|z - x_0\|^2 \geq \|z - x_0\|^2 - \|z - x_n\|^2 = \sum_{i=0}^{n-1}(\|z - x_i\|^2 - \|z - x_{i+1}\|^2)$$

$$\geq \sum_{i=0}^{n-1} \bar{c}\Delta\lambda_{i+1}^2 \geq \bar{c}\Delta\gamma_0^2 \mathrm{Card}(\{i \in \{1, \dots, n\} : \lambda_i \geq \gamma_0\}).$$

Since the relation above holds for every natural number n we conclude that

$$\mathrm{Card}(\{i \in \{1, 2, \dots, \} : \lambda_i \geq \gamma_0\}) \leq 4M^2\bar{c}^{-1}\Delta^{-1}\gamma_0^{-2} + 1. \tag{2.175}$$

Assume that an integer $i \geq 1$ and $\lambda_i < \gamma_0$. The inclusion

$$(x_i, \lambda_i) \in A(x_{i-1}, (\Omega_i, w_i), 0) \tag{2.176}$$

is true. By (2.20) and (2.176) there exist vectors

$$(y_t, \alpha_t) \in A_0(x_{i-1}, t, 0), \ t \in \Omega_i \tag{2.177}$$

such that

$$x_i = \sum_{t \in \Omega_i} w_i(t) y_t, \tag{2.178}$$

$$\lambda_i = \max\{\alpha_t : t \in \Omega_i\}. \tag{2.179}$$

It follows from (2.19) and (2.177) that for every $t = (t_1, \dots, t_{p(t)}) \in \Omega_i$ there exists a finite sequence $\{y_j^{(t)}\}_{j=0}^{p(t)} \subset X$ such that

$$y_0^{(t)} = x_{i-1}, \ y_{p(t)}^{(t)} = y_t, \tag{2.180}$$

$$y_j^{(t)} = P_{t_j}(y_{j-1}^{(t)}) \text{ for each integer } j = 1, \dots, p(t), \tag{2.181}$$

$$\alpha_t = \max\{\|y_{j+1}^{(t)} - y_j^{(t)}\| : j = 0, \dots, p(t) - 1\}. \tag{2.182}$$

Let

$$t = (t_1, \dots, t_{p(t)}) \in \Omega_i$$

be an index vector. By (2.179), (2.181), (2.182) and the inequality $\lambda_i < \gamma_0$, for every $j = 0, \dots, p(t) - 1$,

$$y_j^{(t)} \in F_{\gamma_0}(P_{t_{j+1}}). \tag{2.183}$$

It follows from (2.17), (2.179), (2.180), (2.182), (2.183), and the inequality $\lambda_i < \gamma_0$ that for every integer $j = 0, \dots, p(t)$ we have

$$\|x_{i-1} - y_j^{(t)}\| \le j\lambda_i \le \bar{q}\gamma_0$$

and if $j < p(t)$, then

$$x_{i-1} \in \tilde{F}_{\bar{q}\gamma_0}(P_{t_j+1}).$$

Therefore

$$x_{i-1} \in \tilde{F}_{\bar{q}\gamma_0}(P_s) \text{ for all } s = 1, \dots, p(t) \tag{2.184}$$

and

$$\|x_{i-1} - y_t\| \le \bar{q}\gamma_0 \tag{2.185}$$

for all $t \in \Omega_i$. In view of (2.185),

$$x_{i-1} \in \cap\{\tilde{F}_{\bar{q}\gamma_0}(P_s) : \ s \in \cup_{t\in\Omega_i}\{t_1, \dots, t_{p(t)}\}\}. \tag{2.186}$$

It follows from (2.11), (2.178), and (2.185) that

$$\|x_{i-1} - x_i\| = \|x_{i-1} - \sum_{t\in\Omega_i} w_i(t)y_t\|$$
$$\le \sum_{t\in\Omega_i} w_i(t)\|x_{i-1} - y_t\| \le \gamma_0\bar{q}. \tag{2.187}$$

Set

$$E_0 = \{i \in \{1, 2, \dots\} : \ \lambda_i \ge \gamma_0\}. \tag{2.188}$$

By (2.175), (2.186), (2.187), and (2.188),

$$\mathrm{Card}(E_0) \le 4M^2\bar{c}^{-1}\Delta^{-1}\gamma_0^{-2} + 1 \tag{2.189}$$

and the following property holds:

(P8) if an integer $i \ge 1$ satisfies the inequality $\lambda_i < \gamma_0$, then

$$x_{i-1} \in \tilde{F}_{\bar{q}\gamma_0}(P_s), \ s \in \cup_{t\in\Omega_i}\{t_1, \dots, t_{p(t)}\}, \tag{2.190}$$
$$\|x_{i-1} - x_i\| \le \gamma_0\bar{q}. \tag{2.191}$$

Set

$$E_1 = \{i \in \{1, 2, \dots\} : \ [i, i + \bar{N} - 1] \cap E_0 \ne \emptyset\}. \tag{2.192}$$

By (2.164), (2.189), and (2.191) we have

$$\mathrm{Card}(E_1) \le \bar{N}(4M^2\bar{c}^{-1}\Delta^{-1}\gamma_0^{-2} + 1)$$
$$\le \bar{N}(4M^2\bar{c}^{-1}\Delta^{-1}\bar{q}^2(\bar{N}+1)^2\epsilon^{-2} + 1). \tag{2.193}$$

Assume that a natural number $j \notin E_1$. In view of (2.192),

$$\{j, \dots, j + \bar{N} - 1\} \cap E_0 = \emptyset.$$

Together with (2.188) this implies that for every integer $i \in \{j, \dots, j + \bar{N} - 1\}$, the inequality $\lambda_i < \gamma_0$ is true and (2.190) and (2.191) hold. In view of (2.191) which holds for every integer $i \in \{j, \dots, j + \bar{N} - 1\}$ and for every pair of integers $i_1, i_2 \in \{j - 1, \dots, j + \bar{N} - 1\}$ we have

$$\|x_{i_1} - x_{i_2}\| \le \gamma_0 \bar{N} \bar{q}. \tag{2.194}$$

By (2.161), (2.194), and (2.190) which holds for each $i \in \{j, \dots, j + \bar{N} - 1\}$,

$$x_j \in \tilde{F}_{\bar{q}\gamma_0(\bar{N}+1)}(P_s),\ s \in \cup_{i=j}^{j+\bar{N}-1}(\cup_{t \in \Omega_i}\{t_1, \dots, t_{p(t)}\}) = \{1, \dots, m\}.$$

Together with (2.164) this implies that

$$x_j \in \tilde{F}_\epsilon$$

for all $j \in \{1, 2, \dots, \} \setminus E_1$. Theorem 2.15 is proved. □

Note that Theorem 2.15 is a generalization of the main result of [93] obtained for the convex feasibility problem.

Chapter 3
Iterative Methods in Metric Spaces

In this chapter we study the convergence of iterative methods for solving common fixed point problems in a metric space. Our main goal is to obtain an approximate solution of the problem in the presence of computational errors. We show that the iterative method generates a good approximate solution, if the sequence of computational errors is bounded from above by a constant. Moreover, for a known computational error, we find out what an approximate solution can be obtained and how many iterates one needs for this.

3.1 The First Problem

Let (X, d) be a metric space. For each $x \in X$ and each nonempty set $E \subset X$ put

$$d(x, E) = \inf\{d(x, y) : \; y \in E\}.$$

For each $x \in X$ and each $r > 0$ set

$$B(x, r) = \{y \in X : \; d(x, y) \le r\}.$$

Suppose that m is a natural number, $\bar{c} \in (0, 1)$, $P_i : X \to X$, $i = 1, \dots, m$, for every $i \in \{1, \dots, m\}$,

$$\mathrm{Fix}(P_i) := \{z \in X : \; P_i(z) = z\} \neq \emptyset \tag{3.1}$$

and that the inequality

$$d(z, x)^2 \ge d(z, P_i(x))^2 + \bar{c} d(x, P_i(x))^2 \tag{3.2}$$

A. Zaslavski, *Approximate Solutions of Common Fixed-Point Problems*, Springer Optimization and Its Applications 112, DOI 10.1007/978-3-319-33255-0_3

holds for every $i \in \{1, \dots, m\}$, every $x \in X$ and every $z \in \mathrm{Fix}(P_i)$. Set

$$F = \cap_{i=1}^{m} \mathrm{Fix}(P_i). \tag{3.3}$$

For every $\epsilon > 0$ and every $i \in \{1, \dots, m\}$ put

$$F_\epsilon(P_i) = \{x \in X : \ d(x, P_i(x)) \le \epsilon\}, \tag{3.4}$$

$$\tilde{F}_\epsilon(P_i) = \{y \in X : \ d(y, F_\epsilon(P_i)) \le \epsilon\}, \tag{3.5}$$

$$F_\epsilon = \cap_{i=1}^{m} F_\epsilon(P_i) \tag{3.6}$$

and

$$\tilde{F}_\epsilon = \cap_{i=1}^{m} \tilde{F}_\epsilon(P_i). \tag{3.7}$$

A point belonging to the set F is a solution of our common fixed point problem while a point which belongs to the set $\tilde{F}_\epsilon$ is its ϵ-approximate solution.

Fix $\theta \in X$ and a natural number $\bar{N}$.

Denote by $\mathcal{R}$ the set of all mappings $r : \{1, 2, \dots, \} \to \{1, \dots, m\}$ such that for each integer j,

$$\{1, \dots, m\} \subset \{r(j), \dots, r(j + \bar{N} - 1)\}. \tag{3.8}$$

Every $r \in \mathcal{R}$ generates the following algorithm:

Initialization: select an arbitrary $x_0 \in X$.

Iterative step: given a current iteration point x_k calculate the next iteration point x_{k+1} by

$$x_{k+1} = P_{r(k+1)}(x_k).$$

Denote by $\mathrm{Card}(A)$ the cardinality of a set A. Suppose that the sum over empty set is zero.

Recall that for each $z \in R^1$,

$$\lfloor z \rfloor = \max\{i : \ i \text{ is an integer and } i \le z\}.$$

In Sect. 3.2 we prove the following result.

Theorem 3.1. *Let* $M > 0$ *satisfy*

$$B(\theta, M) \cap F \ne \emptyset, \tag{3.9}$$

$\delta > 0$ *satisfy*

$$\delta \le \bar{N}^{-1}, \tag{3.10}$$

a natural number n_0 satisfy

$$n_0 \geq 1 + \lfloor 4M^2\delta^{-1}(2M+1)^{-1}\bar{N}^{-1}\rfloor \tag{3.11}$$

and let

$$\epsilon_1 = (\bar{N}+1)(32(2M+1)\bar{N}\bar{c}^{-1}\delta)^{1/2}. \tag{3.12}$$

Assume that

$$r \in \mathcal{R}, \tag{3.13}$$

$$x_0 \in B(\theta, M) \text{ and } \{x_i\}_{i=1}^{\infty} \subset X \tag{3.14}$$

satisfy for each natural number i,

$$d(x_i, P_{r(i)}(x_{i-1})) \leq \delta. \tag{3.15}$$

Then there exists an integer $q \in [0, n_0 - 1]$ such that

$$x_i \in B(\theta, 3M+1),\ i = 0, \dots, q\bar{N}, \tag{3.16}$$

$$d(x_i, x_{i-1}) \leq (32(2M+1)\bar{N}\bar{c}^{-1}\delta)^{1/2},$$
$$i = q\bar{N}+1, \dots, (q+1)\bar{N}. \tag{3.17}$$

Moreover, if an integer $q \in [0, n_0 - 1]$ satisfies (3.1), then for each $i = q\bar{N}, \dots, (q+1)\bar{N}$,

$$x_i \in \tilde{F}_{\epsilon_1}. \tag{3.18}$$

Note that in Theorem 3.1 δ is the computational error made by our computer system, we obtain a point of the set $\tilde{F}_{\epsilon_1}$ and in order to obtain this point we need $(n_0 - 1)\bar{N}$ iterations. It is not difficult to see that $\epsilon_1 = c_1\delta^{1/2}$ and $n_0 = \lfloor c_2\delta^{-1}\rfloor + 1$, where c_1 and c_2 are positive constants depending on M.

Applying by induction Theorem 3.1 and using (3.19) and (3.20) we can prove the following result.

Theorem 3.2. *Suppose that $\bar{\epsilon} \in (0,1)$, $\bar{M} > 0$,*

$$\tilde{F}_{\bar{\epsilon}} \subset B(\theta, \bar{M}) \text{ and } F \neq \emptyset. \tag{3.19}$$

Let $\epsilon \in (0, \bar{\epsilon})$, $M > \bar{M}$, $\delta > 0$ satisfy

$$\delta \leq \bar{N}^{-1} \text{ and } (\bar{N}+1)(16(2M+1)\bar{N}\bar{c}^{-1}\delta)^{1/2} < \epsilon \tag{3.20}$$

and let

$$n_0 \geq 1 + \lfloor 2M^2\delta^{-1}(2M+1)^{-1}\bar{N}^{-1} \rfloor.$$

Assume that

$$r \in \mathcal{R},$$
$$x_0 \in B(\theta, M) \text{ and } \{x_i\}_{i=1}^{\infty} \subset X$$

satisfy for each natural number i,

$$d(x_i, P_{r(i)}(x_{i-1})) \leq \delta.$$

Then

$$x_i \in B(\theta, 3M+1) \text{ for all integers } i \geq 0$$

and there exists a strictly increasing sequence of integers $\{q_p\}_{p=0}^{\infty}$ *such that*

$$0 \leq q_0 \leq n_0,$$
$$1 \leq q_{p+1} - q_p \leq n_0 \text{ for all integers } p \geq 0$$

and that for each integer $p \geq 0$ *and each* $i = q_p\bar{N}, \dots, (q_p+1)\bar{N}$,

$$x_i \in \tilde{F}_\epsilon.$$

The next theorem was obtained in [99].

Theorem 3.3. *Let* $M > 0$ *satisfy*

$$B(\theta, M) \cap F \neq \emptyset,$$

$\epsilon > 0$,

$$r \in \mathcal{R}, \tag{3.21}$$
$$x_0 \in B(\theta, M)$$

and let $\{x_i\}_{i=1}^{\infty} \subset X$ *satisfy for each natural number i,*

$$x_i = P_{r(i)}(x_{i-1}). \tag{3.22}$$

Then

$$Card(\{i \in \{0, 1, \dots\} : \ x_i \notin \tilde{F}_\epsilon\}) \leq 4\bar{N}^3 M^2 \bar{c}^{-1} \epsilon^{-2}.$$

3.2 Proof of Theorem 3.1

By (3.9) there exists

$$z \in B(\theta, M) \cap F. \tag{3.23}$$

Fix a positive number

$$\epsilon_0 = (32(2M+1)\bar{N}\bar{c}^{-1}\delta)^{1/2}. \tag{3.24}$$

Assume that a nonnegative integer s satisfies for each integer $k \in [0, s]$,

$$\max\{d(x_i, x_{i-1}) : \; i = k\bar{N}+1, \dots, (k+1)\bar{N}\} > \epsilon_0. \tag{3.25}$$

By (3.14) and (3.23),

$$d(x_0, z) \le 2M. \tag{3.26}$$

Assume that an integer $k \in [0, s]$ satisfies

$$d(x_{k\bar{N}}, z) \le 2M. \tag{3.27}$$

We prove the following auxiliary result.

Lemma 3.4. *Assume that an integer*

$$i \in [0, \bar{N}-1] \tag{3.28}$$

satisfies

$$d(x_{k\bar{N}+i}, z) \le 2M + i\delta. \tag{3.29}$$

Then

$$d(x_{k\bar{N}+i+1}, z) \le d(x_{k\bar{N}+i}, z) + \delta \tag{3.30}$$

and

$$d(x_{k\bar{N}+i+1}, z)^2 \le d(x_{k\bar{N}+i}, z)^2 + 2\delta(2M+1). \tag{3.31}$$

If $d(x_{k\bar{N}+i+1}, x_{k\bar{N}+i}) > \epsilon_0$, *then*

$$d(x_{k\bar{N}+i+1}, z)^2 - d(x_{k\bar{N}+i}, z)^2 \le -8^{-1}\epsilon_0^2\bar{c}. \tag{3.32}$$

Proof. In view of (3.15),

$$d(x_{k\bar{N}+i+1}, P_{r(k\bar{N}+i+1)}(x_{k\bar{N}+i})) \le \delta. \tag{3.33}$$

By (3.2), (3.23), and (3.33),

$$\begin{aligned} & d(z, x_{k\bar{N}+i+1})^2 \\ & \le (d(z, P_{r(k\bar{N}+i+1)}(x_{k\bar{N}+i})) + d(P_{r(k\bar{N}+i+1)}(x_{k\bar{N}+i}), x_{k\bar{N}+i+1}))^2 \\ & \le (d(z, x_{k\bar{N}+i}) + \delta)^2. \end{aligned} \tag{3.34}$$

Clearly, (3.34) implies (3.30). It follows from (3.10), (3.28), (3.29), and (3.34) that

$$\begin{aligned} d(z, x_{k\bar{N}+i+1})^2 & \le d(z, x_{k\bar{N}+i})^2 + \delta^2 + 2\delta d(z, x_{k\bar{N}+i}) \\ & \le d(z, x_{k\bar{N}+i})^2 + 2\delta(d(z, x_{k\bar{N}+i}) + \delta) \\ & \le d(z, x_{k\bar{N}+i})^2 + 2\delta(2M + (i+1)\delta) \\ & \le d(z, x_{k\bar{N}+i})^2 + 2\delta(2M + \bar{N}\delta) \\ & \le d(z, x_{k\bar{N}+i})^2 + 2\delta(2M + 1). \end{aligned}$$

Thus (3.31) is true.

Assume that

$$d(x_{k\bar{N}+i+1}, x_{k\bar{N}+i}) > \epsilon_0. \tag{3.35}$$

By (3.2), (3.23), and (3.33),

$$\begin{aligned} & d(z, x_{k\bar{N}+i+1})^2 \\ & \le (d(z, P_{r(k\bar{N}+i+1)}(x_{k\bar{N}+i})) + d(P_{r(k\bar{N}+i+1)}(x_{k\bar{N}+i}), x_{k\bar{N}+i+1}))^2 \\ & \le (d(z, P_{r(k\bar{N}+i+1)}(x_{k\bar{N}+i})))^2 + \delta^2 + 2\delta d(z, P_{r(k\bar{N}+i+1)}(x_{k\bar{N}+i})) \\ & \le d(z, x_{k\bar{N}+i})^2 - \bar{c} d(x_{k\bar{N}+i}, P_{r(k\bar{N}+i+1)}(x_{k\bar{N}+i}))^2 + 2\delta d(z, x_{k\bar{N}+i}) + \delta^2. \end{aligned} \tag{3.36}$$

Relations (3.33) and (3.35) imply that

$$\begin{aligned} & d(x_{k\bar{N}+i}, P_{r(k\bar{N}+i+1)}(x_{k\bar{N}+i})) \\ & \ge d(x_{k\bar{N}+i}, x_{k\bar{N}+i+1}) - d(x_{k\bar{N}+i+1}, P_{r(k\bar{N}+i+1)}(x_{k\bar{N}+i})) > \epsilon_0 - \delta. \end{aligned}$$

By (3.10), (3.24), (3.28), (3.29), (3.36) and the equation above,

$$d(z, x_{k\bar{N}+i+1})^2 \le d(z, x_{k\bar{N}+i})^2 - \bar{c}(\epsilon_0 - \delta)^2 + 2\delta(2M + \delta(i+1))$$
$$\le d(z, x_{k\bar{N}+i})^2 - \bar{c}(\epsilon_0 - \delta)^2 + 2\delta(2M + 1)$$
$$\le d(z, x_{k\bar{N}+i})^2 - 4^{-1}\bar{c}\epsilon_0^2 + 2\delta(2M + 1)$$
$$\le d(z, x_{k\bar{N}+i})^2 - 8^{-1}\bar{c}\epsilon_0^2.$$

Thus (3.32) holds. This completes the proof of Lemma 3.4. □

It follows from (3.10), (3.27), and Lemma 3.4 applied by induction that for all $i = 0, \dots, \bar{N} - 1$,

$$d(z, x_{k\bar{N}+i+1}) \le d(z, x_{k\bar{N}+i}) + \delta,$$
$$d(z, x_{k\bar{N}+i+1}) \le 2M + \delta(i+1) \le 2M + \delta\bar{N} \le 2M + 1, \tag{3.37}$$
$$d(z, x_{k\bar{N}+i}) \le 2M + 1, \; i = 0, \dots, \bar{N}.$$

Relations (3.25), (3.27), (3.31), (3.37) and Lemma 3.4 imply that

$$d(z, x_{(k+1)\bar{N}})^2 - d(z, x_{k\bar{N}})^2 = \sum_{i=0}^{\bar{N}-1} (d(z, x_{k\bar{N}+i+1})^2 - d(z, x_{k\bar{N}+i})^2)$$
$$\le -8^{-1}\bar{c}\epsilon_0^2 + 2\delta\bar{N}(2M + 1) \le -16^{-1}\bar{c}\epsilon_0^2.$$

Thus we have shown that the following property holds:
(P1) if an integer $k \in [0, s]$ satisfies $d(z, x_{k\bar{N}}) \le 2M$, then

$$d(z, x_j) \le 2M + 1, \; i = k\bar{N}, \dots, (k+1)\bar{N}, \tag{3.38}$$
$$d(x_{(k+1)\bar{N}}, z)^2 - d(x_{k\bar{N}}, z)^2 \le -16^{-1}\bar{c}\epsilon_0^2. \tag{3.39}$$

By (3.26) and property (P1),

$$d(x_j, z) \le 2M + 1, \; j = 0, \dots, (s+1)\bar{N}, \tag{3.40}$$

and (3.39) holds for all $k = 0, \dots, s$. In view (3.26) and (3.39),

$$16^{-1}\bar{c}\epsilon_0^2(s+1) \le \sum_{k=0}^{s} (d(x_{k\bar{N}}, z)^2 - d(x_{(k+1)\bar{N}}, z)^2)$$
$$= d(x_0, z)^2 - d(x_{(s+1)\bar{N}}, z)^2 \le d(x_0, z)^2 \le 4M^2,$$
$$s + 1 \le 64M^2\bar{c}^{-1}\epsilon_0^{-2}.$$

Thus we have shown that the following property holds:

(P2) If an integer $s \geq 0$ and for each integer $k \in [0, s]$ (3.25) holds, then

$$s \leq 64M^2 \bar{c}^{-1} \epsilon_0^{-2} - 1,$$
$$d(x_j, z) \leq 2M + 1, \ j = 0, \ldots, (s+1)\bar{N},$$
$$d(x_{k\bar{N}}, z) \leq 2M, \ k = 0, \ldots, (s+1).$$

Property (P2), (3.11), and (3.24) imply that there exists an integer $q \in [0, n_0 - 1]$ such that for each integer k satisfying $0 \leq k < q$,

$$\max\{d(x_i, x_{i-1}) : \ i = k\bar{N} + 1, \ldots, (k+1)\bar{N}\} > \epsilon_0;$$
$$\max\{d(x_i, x_{i-1}) : \ i = q\bar{N} + 1, \ldots, (q+1)\bar{N}\} \leq \epsilon_0.$$

By (3.16), (3.23), property (P2), and the choice of q,

$$d(x_{q\bar{N}}, z) \leq 2M,$$
$$d(x_j, z) \leq 2M + 1, \ j = 0, \ldots, q\bar{N},$$
$$d(x_j, \theta) \leq 3M + 1, \ j = 0, \ldots, q\bar{N}.$$

Assume that an integer $q \in [0, n_0 - 1]$ satisfies

$$d(x_i, x_{i-1}) \leq \epsilon_0, \ i = q\bar{N} + 1, \ldots, (q+1)\bar{N}. \tag{3.41}$$

In view of (3.15), (3.24), and (3.41), for all $i = q\bar{N} + 1, \ldots, (q+1)\bar{N}$,

$$d(x_{i-1}, P_{r(i)}(x_{i-1})) \leq d(x_{i-1}, x_i) + d(x_i, P_{r(i)}(x_{i-1})) \leq \epsilon_0 + \delta < 2\epsilon_0. \tag{3.42}$$

Let

$$i \in \{q\bar{N}, \ldots, (q+1)\bar{N}\}, \tag{3.43}$$
$$s \in \{1, \ldots, m\}. \tag{3.44}$$

Relations (3.8) and (3.44) imply that there exists

$$j \in \{q\bar{N} + 1, \ldots, (q+1)\bar{N}\} \tag{3.45}$$

such that

$$s = r(j). \tag{3.46}$$

By (3.42), (3.45), and (3.46),

$$d(x_{j-1}, P_s(x_{j-1})) = d(x_{j-1}, P_{r(j)}(x_{j-1})) < 2\epsilon_0. \tag{3.47}$$

In view of (3.41), (3.43), and (3.45),

$$d(x_i, x_{j-1}) \le \bar{N}\epsilon_0.$$

Together with (3.47) this implies that

$$x_i \in \tilde{F}_{(\bar{N}+1)\epsilon_0}(P_s).$$

Since the inclusion above holds for all $s \in \{1, \dots, m\}$ we conclude (see (3.12), (3.24)) that

$$x_i \in \tilde{F}_{(\bar{N}+1)\epsilon_0} = \tilde{F}_{\epsilon_1}$$

for all $i \in \{q\bar{N}, \dots, (q+1)\bar{N}\}$. Theorem 3.1 is proved. □

3.3 Proof of Theorem 3.3

Fix

$$z \in B(\theta, M) \cap F. \tag{3.48}$$

Set

$$\gamma_0 = \epsilon\bar{N}^{-1}. \tag{3.49}$$

Let $i \ge 0$ be an integer. By (3.22),

$$x_{i+1} = P_{r(i+1)}(x_i). \tag{3.50}$$

Relations (3.2), (3.48), and (3.50) imply that

$$d(z, x_i)^2 - d(z, x_{i+1})^2 \ge \bar{c}d(x_i, x_{i+1})^2. \tag{3.51}$$

It follows from (3.14), (3.48), and (3.51) that for each natural number n,

$$\begin{aligned} 4M^2 \ge d(z, x_0)^2 &\ge d(z, x_0)^2 - d(z, x_n)^2 \\ &= \sum_{i=0}^{n-1}[d(z, x_i)^2 - d(z, x_{i+1})^2] \\ &\ge \sum_{i=0}^{n-1}\bar{c}d(x_i, x_{i+1})^2 \\ &\ge \bar{c}\gamma_0^2\mathrm{Card}(\{i \in \{0, \dots, n-1\} : \ d(x_i, x_{i+1}) \ge \gamma_0\}). \end{aligned}$$

Since the relation above holds for every natural number n we conclude that

$$\mathrm{Card}(\{i \in \{0, 1, \dots\} :\ d(x_i, x_{i+1}) \geq \gamma_0\}) \leq 4M^2\bar{c}^{-1}\gamma_0^{-2}. \tag{3.52}$$

Set

$$E_0 = \{i \in \{0, 1, \dots\} :\ d(x_i, x_{i+1}) \geq \gamma_0\}. \tag{3.53}$$

In view of (3.52) and (3.53),

$$\mathrm{Card}(E_0) \leq 4M^2\bar{c}^{-1}\gamma_0^{-2}. \tag{3.54}$$

Set

$$E_1 = \{i \in \{0, 1, \dots\} :\ [i, i + \bar{N} - 1] \cap E_0 \neq \emptyset\}. \tag{3.55}$$

By (3.54) and (3.55),

$$\mathrm{Card}(E_1) \leq \bar{N}\mathrm{Card}(E_0) \leq 4\bar{N}M^2\bar{c}^{-1}\gamma_0^{-2}. \tag{3.56}$$

Let a nonnegative integer j satisfies

$$j \notin E_1.$$

Then in view of (3.22), (3.53), and (3.55),

$$[j, j + \bar{N} - 1] \cap E_0 = \emptyset$$

and for each $i \in \{j, \dots, j + \bar{N} - 1\}$,

$$d(x_i, P_{r(i+1)}(x_i)) = d(x_i, x_{i+1}) < \gamma_0. \tag{3.57}$$

This implies that for each pair of integers $i_1, i_2 \in \{j, \dots, j + \bar{N}\}$,

$$d(x_{i_1}, x_{i_2}) < \bar{N}\gamma_0. \tag{3.58}$$

Let

$$i \in \{j, \dots, j + \bar{N}\},\ s \in \{1, \dots, m\}. \tag{3.59}$$

By (3.8), there is

$$j_0 \in \{j + 1, \dots, j + \bar{N}\} \tag{3.60}$$

such that

$$s = r(j_0). \tag{3.61}$$

It follows from (3.57), (3.60), and (3.61) that

$$d(x_{j_0-1}, P_s(x_{j_0-1})) = d(x_{j_0-1}, P_{r(j_0)}(x_{j_0-1})) < \gamma_0.$$

Together with (3.49) and (3.58)–(3.60) this implies that

$$x_j \in \tilde{F}_{\bar{N}\gamma_0}(P_s) = \tilde{F}_\epsilon(P_s).$$

Since the inclusion above holds for every $s \in \{1, \dots, m\}$ we conclude that

$$x_j \in \tilde{F}_\epsilon$$

for all nonnegative integers j satisfying $j \notin E_1$. This completes the proof of Theorem 3.3. □

3.4 The Second Problem

Let (X, d) be a metric space. Recall that for each $x \in X$ and each $r > 0$,

$$B(x, r) = \{y \in X : \ d(x, y) \le r\}.$$

For each $x \in X$ and each nonempty set $E \subset X$,

$$d(x, E) = \inf\{d(x, y) : \ y \in E\}.$$

Fix $\theta \in X$. Suppose that m is a natural number, $P_i : X \to X$, $i = 1, \dots, m$, for every $i \in \{1, \dots, m\}$,

$$\mathrm{Fix}(P_i) := \{z \in X : \ P_i(z) = z\} \ne \emptyset.$$

Set

$$F = \cap_{i=1}^m \mathrm{Fix}(P_i). \tag{3.62}$$

For every $\epsilon > 0$ and every $i \in \{1, \dots, m\}$ put

$$F_\epsilon(P_i) = \{x \in X : \ d(x, P_i(x)) \le \epsilon\},$$

$$\tilde{F}_\epsilon(P_i) = \{y \in X : \ d(y, F_\epsilon(P_i)) \le \epsilon\}, \tag{3.63}$$

$$F_\epsilon = \cap_{i=1}^m F_\epsilon(P_i), \ \tilde{F}_\epsilon = \cap_{i=1}^m \tilde{F}_\epsilon(P_i). \tag{3.64}$$

Suppose that

$$F \ne \emptyset. \tag{3.65}$$

and that the following assumption holds:

(A1) For each $M > 0$ and each $\gamma > 0$ there exists $\delta > 0$ such that for each $i \in \{1, \dots, m\}$, each

$$z \in B(\theta, M) \cap \mathrm{Fix}(P_i)$$

and each $x \in B(\theta, M)$ satisfying $d(x, P_i(x)) \geq \gamma$,

$$d(z, P_i(x)) \leq d(z, x) - \delta.$$

Fix a natural number $\bar{N}$.

Denote by $\mathcal{R}$ the set of all mappings $r : \{1, 2, \dots, \} \to \{1, \dots, m\}$ such that for each integer j,

$$\{1, \dots, m\} \subset \{r(j), \dots, r(j + \bar{N} - 1)\}. \tag{3.66}$$

In Sect. 3.5 we prove the following result.

Theorem 3.5. *Let* $M > 0$ *satisfy*

$$B(\theta, M) \cap F \neq \emptyset,\ \epsilon \in (0, 1),\ \epsilon_0 = \epsilon(\bar{N} + 1)^{-1}, \tag{3.67}$$

$\gamma \in (0, 1)$ *and let the following property hold:*

(P1) for each $i \in \{1, \dots, m\}$, *each*

$$z \in B(\theta, 3M + 1) \cap \mathit{Fix}(P_i)$$

and each $x \in B(\theta, 3M + 1)$ *satisfying* $d(x, P_i(x)) \geq \epsilon_0/2$,

$$d(z, P_i(x)) \leq d(z, x) - \gamma.$$

Let

$$n_0 = \lfloor 4\gamma^{-1} M \rfloor + 1, \tag{3.68}$$

$$0 < \delta < \min\{\epsilon_0/2, \gamma(2\bar{N})^{-1}\}. \tag{3.69}$$

Assume that

$$r \in \mathcal{R}, \tag{3.70}$$

$$x_0 \in B(\theta, M) \text{ and } \{x_i\}_{i=1}^{\infty} \subset X \tag{3.71}$$

satisfy for each natural number i,

$$d(x_i, P_{r(i)}(x_{i-1})) \leq \delta. \tag{3.72}$$

Then there exists an integer $q \in [0, n_0 - 1]$ *such that*

$$x_i \in B(\theta, 3M + 1),\ i = 0, \dots, q\bar{N}, \tag{3.73}$$

$$d(x_i, x_{i-1}) \le \epsilon_0,\ i = q\bar{N} + 1, \dots, (q + 1)\bar{N}. \tag{3.74}$$

Moreover, if an integer $q \in [0, n_0 - 1]$ *satisfies* (3.74)*, then for each* $i = q\bar{N}, \dots, (q + 1)\bar{N}$,

$$x_i \in \tilde{F}_\epsilon.$$

Note that the existence of $\gamma \in (0, 1)$ in the statement of Theorem 3.5 follows from assumption (A1).

Applying by induction Theorem 3.5 and using (3.75) we can prove the following result.

Theorem 3.6. *Suppose that* $\bar{\epsilon} \in (0, 1)$, $\bar{M} > 0$,

$$\tilde{F}_{\bar{\epsilon}} \subset B(\theta, \bar{M}),\ \epsilon \in (0, \bar{\epsilon}),\ M > \bar{M},\ \epsilon_0 = \epsilon(\bar{N} + 1)^{-1}, \tag{3.75}$$

$\gamma \in (0, 1)$ *and let property (P1) (see Theorem 3.5) hold. Let*

$$n_0 = \lfloor 4\gamma^{-1} M \rfloor + 1,$$

$$0 < \delta < \min\{\epsilon_0/2, \gamma(2\bar{N})^{-1}\}.$$

Assume that

$$r \in \mathcal{R},$$

$$x_0 \in B(\theta, M) \text{ and } \{x_i\}_{i=1}^{\infty} \subset X$$

satisfy for each natural number i,

$$d(x_i, P_{r(i)}(x_{i-1})) \le \delta.$$

Then

$$x_i \in B(\theta, 3M + 1) \text{ for all integers } i \ge 0$$

and there exists a strictly increasing sequence of integers $\{q_p\}_{p=0}^{\infty}$ *such that*

$$0 \le q_0 \le n_0,$$

$$1 \le q_{p+1} - q_p \le n_0 \text{ for all integers } p \ge 0$$

and that for each integer $p \ge 0$ *and each* $i = q_p\bar{N}, \dots, (q_p + 1)\bar{N}$,

$$x_i \in \tilde{F}_\epsilon.$$

The following theorem is proved in Sect. 3.6.

Theorem 3.7. *Let $M > 0$ satisfy*

$$B(\theta, M) \cap F \neq \emptyset,$$

$\epsilon > 0$. *Then there exists a constant $Q > 0$ such that for each $r \in \mathcal{R}$, each sequence $\{x_i\}_{i=0}^{\infty} \subset X$ which satisfies*

$$x_0 \in B(\theta, M)$$

and

$$x_i = P_{r(i)}(x_{i-1})$$

for all natural numbers i, the following inequality holds:

$$Card(\{i \in \{0, 1, \dots\} : x_i \notin \tilde{F}_\epsilon\}) \leq Q.$$

Assume that $C_1, \dots, C_m \subset X$ and $\cap_{i=1}^m C_i \neq \emptyset$. We say that the family of sets $\{C_1, \dots, C_m\}$ has a bounded regularity property [7] if for each $\epsilon > 0$ and each $M > 0$ there exists $\delta > 0$ such that if $x \in B(\theta, M)$ satisfies $d(x, C_i) \leq \delta$ for all $i = 1, \dots, m$, then $d(x, \cap_{i=1}^m C_i) \leq \epsilon$.

Theorem 3.5 implies the following result.

Theorem 3.8. *Suppose that*

$$P_i(X) = Fix(P_i), \ i = 1, \dots, m,$$

the family of sets $\{Fix(P_i), \ i = 1, \dots, m\}$ has the bounded regularity property, $M > 0$ satisfies

$$B(\theta, M) \cap F \neq \emptyset.$$

Then there exist $\delta > 0$ and a natural number n_0 such that for each $r \in \mathcal{R}$, each $x_0 \in B(\theta, M)$ and each sequence $\{x_i\}_{i=1}^{\infty} \subset X$ which satisfies for each natural number i,

$$d(x_i, P_{r(i)}(x_{i-1})) \leq \delta,$$

there exists an integer $q \in [0, n_0 - 1]$ such that

$$x_i \in B(\theta, 3M + 2), \ i = 0, \dots, (q + 1)\bar{N},$$

$$d(x_i, F) \leq \epsilon, \ i = q\bar{N}, \dots, (q + 1)\bar{N}.$$

Applying by induction Theorem 3.8 we can prove the following result.

Proposition 3.9. *Suppose that*

$$P_i(X) = Fix(P_i),\ i = 1, \dots, m,$$

the family of sets $\{Fix(P_i),\ i = 1, \dots, m\}$ *has the bounded regularity property,* $\bar{M} > 0$ *satisfies*

$$F \subset B(\theta, \bar{M}),\ M > \bar{M} + 1,\ \epsilon \in (0, 1).$$

Then there exist $\delta > 0$ *and a natural number* n_0 *such that for each* $r \in \mathcal{R}$*, each* $x_0 \in B(\theta, M)$ *and each sequence* $\{x_i\}_{i=1}^{\infty} \subset X$ *which satisfies for each natural number i,*

$$d(x_i, P_{r(i)}(x_{i-1})) \le \delta,$$

the inclusion $x_i \in B(\theta, 3M + 2)$ *holds for all integers* $i \ge 0$ *and there exists a strictly increasing sequence of integers* $\{q_p\}_{p=0}^{\infty}$ *such that*

$$0 \le q_0 \le n_0,$$
$$1 \le q_{p+1} - q_p \le n_0 \text{ for all integers } p \ge 0$$

and that for each integer $p \ge 0$ *and each* $i = (q_p - 1)\bar{N}, \dots, q_p\bar{N}$*,*

$$d(x_i, F) \le \epsilon.$$

The following two results are deduced from Proposition 3.9.

Theorem 3.10. *Suppose that*

$$P_i(X) = Fix(P_i),\ i = 1, \dots, m,$$

the family of sets $\{Fix(P_i),\ i = 1, \dots, m\}$ *has the bounded regularity property,* $\bar{M} > 0$ *satisfies*

$$F \subset B(\theta, \bar{M}),\ M > \bar{M} + 1,\ \epsilon \in (0, 1).$$

Then there exist $\delta > 0$ *and a natural number* n_0 *such that for each* $r \in \mathcal{R}$*, each* $x_0 \in B(\theta, M)$ *and each sequence* $\{x_i\}_{i=1}^{\infty} \subset X$ *which satisfies for each natural number i,*

$$d(x_i, P_{r(i)}(x_{i-1})) \le \delta,$$

the inclusion $x_i \in B(\theta, 3M + 2)$ *holds for all integers* $i \ge 0$ *and*

$$d(x_i, F) \le \epsilon \text{ for all integers } i \ge n_0\bar{N}.$$

Theorem 3.11. *Suppose that*

$$P_i(X) = Fix(P_i),\ i = 1, \dots, m,$$

the family of sets $\{Fix(P_i),\ i = 1, \dots, m\}$ *has the bounded regularity property, there exists* $s_0 \in \{1, \dots, m\}$ *such that the set* $P_{s_0}(X)$ *is bounded,* $\epsilon > 0$ *and that a sequence* $\{\delta_i\}_{i=1}^{\infty} \subset (0, \infty)$ *satisfies*

$$\lim_{i \to \infty} \delta_i = 0. \tag{3.76}$$

Then there exists a natural number n_ϵ *such that for each* $r \in \mathcal{R}$ *and each sequence* $\{x_i\}_{i=0}^{\infty} \subset X$ *which satisfies for each natural number i,*

$$d(x_i, P_{r(i)}(x_{i-1})) \le \delta_i$$

the inequality

$$d(x_i, F) \le \epsilon \text{ holds for all integers } i \ge n_\epsilon.$$

Note that Theorems 3.21 and 3.22 are generalizations of Theorems 3.10 and 3.11, respectively. They are stated in Sect. 3.8 and proved in Sects. 3.11 and 3.12 respectively.

Example 3.12. Let $\bar{c} \in (0, 1)$ and for every $i \in \{1, \dots, m\}$,

$$d(z, x)^2 \ge d(z, P_i(x))^2 + \bar{c} d(x, P_i(x))^2$$
$$\text{for all } x \in X \text{ and all } z \in \mathrm{Fix}(P_i). \tag{3.77}$$

We claim that (A1) holds.

Let $M > 0, \gamma > 0$ and $i \in \{1, \dots, m\}$,

$$z \in B(\theta, M) \cap \mathrm{Fix}(P_i), \tag{3.78}$$

$$x \in B(\theta, M),\ d(x, P_i(x)) \ge \gamma. \tag{3.79}$$

By (3.77)–(3.79),

$$\bar{c}\gamma^2 \le \bar{c} d(x, P_i(x))^2 \le d(z, x)^2 - d(z, P_i(x))^2$$
$$= (d(z, x) - d(z, P_i(x)))(d(z, x) + d(z, P_i(x)))$$
$$\le 2(d(z, x) - d(z, P_i(x)))d(z, x)$$
$$\le 4M(d(z, x) - d(z, P_i(x))).$$
$$d(z, x) - d(z, P_i(x)) \ge (4M)^{-1}\bar{c}\gamma^2$$

and (A1) holds with $\delta = (4M)^{-1}\bar{c}\gamma^2$.

Example 3.13. Assume that every bounded closed subset of X is compact, mappings P_i, $i = 1, \dots, m$ are continuous and that for every $i \in \{1, \dots, m\}$, every $z \in \mathrm{Fix}(P_i)$ and every $x \in X \setminus \mathrm{Fix}(P_i)$,

$$d(z, P_i(x)) < d(z, x). \tag{3.80}$$

We claim that (A1) hods.

Let $M > 0, \gamma > 0$ and $j \in \{1, \dots, m\}$. In order to meet our goal it is sufficient to show that there exists $\delta > 0$ such that

$$d(z, P_j(x)) \le d(z, x) - \delta$$

for each

$$z \in B(\theta, M) \cap \mathrm{Fix}(P_j)$$

and each $x \in B(\theta, M)$ satisfying $d(x, P_j(x)) \ge \gamma$.

Assume the contrary. Then there exist sequences

$$\{z_k\}_{k=1}^{\infty} \subset B(\theta, M) \cap \mathrm{Fix}(P_j) \text{ and } \{x_k\}_{k=1}^{\infty} \subset B(\theta, M)$$

such that

$$d(x_k, P_j(x_k)) \ge \gamma \text{ for all natural numbers } k,$$

$$\lim_{k\to\infty} (d(z_k, x_k) - d(z_k, P_j(x_k)) = 0.$$

Extracting a subsequence and re-indexing, if necessary, we may assume without loss of generality that there exist

$$x = \lim_{k\to\infty} x_k, \ z = \lim_{k\to\infty} z_k.$$

It is not difficult to see that

$$d(x, P_j(x)) = \lim_{k\to\infty} d(x_k, P_j(x_k)) \ge \gamma,$$

$$d(z, P_j(z)) = \lim_{k\to\infty} d(z_k, P_j(z_k)) = 0,$$

$$d(z, x) - d(z, P_j(x)) = \lim_{k\to\infty} (d(z_k, x_k) - d(z_k, P_j(x_k))) = 0.$$

Thus

$$z \in \mathrm{Fix}(P_j), \ x \notin \mathrm{Fix}(P_j), \ d(z, x) = d(z, P_j(x)).$$

This contradicts (3.80). The contradiction we have reached that (A1) holds.

3.5 Proof of Theorem 3.5

By (3.67) there exists

$$z \in B(\theta, M) \cap F. \tag{3.81}$$

Assume that a nonnegative integer s satisfies for each integer $k \in [0, s]$,

$$\max\{d(x_i, x_{i-1}) : \ i = k\bar{N} + 1, \dots, (k+1)\bar{N}\} > \epsilon_0. \tag{3.82}$$

By (3.71) and (3.81),

$$d(x_0, z) \le 2M. \tag{3.83}$$

Assume that an integer $k \in [0, s]$ satisfies

$$d(x_{k\bar{N}}, z) \le 2M. \tag{3.84}$$

We prove the following auxiliary result.

Lemma 3.14. *Assume that an integer*

$$i \in [0, \bar{N} - 1] \tag{3.85}$$

satisfies

$$d(x_{k\bar{N}+i}, z) \le 2M + i\delta. \tag{3.86}$$

Then

$$d(x_{k\bar{N}+i+1}, z) \le d(x_{k\bar{N}+i}, z) + \delta \tag{3.87}$$

and if

$$d(x_{k\bar{N}+i+1}, x_{k\bar{N}+i}) > \epsilon_0, \tag{3.88}$$

then

$$d(x_{k\bar{N}+i+1}, z) \le d(x_{k\bar{N}+i}, z) - \gamma + \delta. \tag{3.89}$$

Proof. In view of (3.72), (3.81), and (A1),

$$\begin{aligned} d(x_{k\bar{N}+i+1}, z) &\le d(x_{k\bar{N}+i+1}, P_{r(k\bar{N}+i+1)}(x_{k\bar{N}+i})) + d(P_{r(k\bar{N}+i+1)}(x_{k\bar{N}+i}), z) \\ &\le \delta + d(x_{k\bar{N}+i}, z) \end{aligned}$$

and (3.87) holds.

Assume that (3.88) holds. By (3.72),

$$d(x_{k\bar{N}+i+1}, z) \leq d(x_{k\bar{N}+i+1}, P_{r(k\bar{N}+i+1)}(x_{k\bar{N}+i})) + d(P_{r(k\bar{N}+i+1)}(x_{k\bar{N}+i}), z)$$
$$\leq \delta + d(P_{r(k\bar{N}+i+1)}(x_{k\bar{N}+i}), z). \tag{3.90}$$

It follows from (3.70), (3.72), and (3.88) that

$$d(x_{k\bar{N}+i}, P_{r(k\bar{N}+i+1)}(x_{k\bar{N}+i}))$$
$$\geq d(x_{k\bar{N}+i}, x_{k\bar{N}+i+1}) - d(P_{r(k\bar{N}+i+1)}(x_{k\bar{N}+i}), x_{k\bar{N}+i+1})$$
$$\geq \epsilon_0 - \delta \geq 2^{-1}\epsilon_0. \tag{3.91}$$

In view of (3.81), (3.85), (3.86), and (3.96),

$$d(x_{k\bar{N}+i}, z) \leq 2M + 1, \tag{3.92}$$
$$d(\theta, x_{k\bar{N}+i}) \leq 3M + 1. \tag{3.93}$$

By (3.81), (3.91), (3.93), and property (P1),

$$d(z, P_{r(k\bar{N}+i+1)}(x_{k\bar{N}+i})) \leq d(z, x_{k\bar{N}+i}) - \gamma. \tag{3.94}$$

Relations (3.72) and (3.94) imply that

$$d(z, x_{k\bar{N}+i+1}) \leq \delta + d(P_{r(k\bar{N}+i+1)}(x_{k\bar{N}+i}), z) \leq d(z, x_{k\bar{N}+i}) - \gamma + \delta.$$

This completes the proof of Lemma 3.14. □

It follows from (3.69), (3.84), (3.85), and Lemma 3.14 applied by induction that for all $i = 0, \ldots, \bar{N} - 1$,

$$d(z, x_{k\bar{N}+i+1}) \leq d(z, x_{k\bar{N}+i}) + \delta, \tag{3.95}$$
$$d(z, x_{k\bar{N}+i+1}) \leq 2M + \delta(i+1) \leq 2M + \delta\bar{N} \leq 2M + 1, \tag{3.96}$$
$$d(z, x_{k\bar{N}+i}) \leq 2M + 1, \ i = 0, \ldots, \bar{N}. \tag{3.97}$$

Relations (3.69), (3.82), (3.84), (3.96), and Lemma 3.14 imply that

$$d(z, x_{(k+1)\bar{N}}) - d(z, x_{k\bar{N}}) = \sum_{i=0}^{\bar{N}-1} (d(z, x_{k\bar{N}+i+1}) - d(z, x_{k\bar{N}+i}))$$
$$\leq \delta(\bar{N} - 1) - \gamma + \delta \leq -\gamma + \delta\bar{N} \leq -\gamma/2.$$

Thus we have shown that the following property holds:

(P2) if an integer $k \in [0, s]$ satisfies $d(z, x_{k\bar{N}}) \leq 2M$, then

$$d(z, x_j) \leq 2M + 1,\ i = k\bar{N}, \dots, (k+1)\bar{N},$$
$$d(x_{(k+1)\bar{N}}, z) - d(x_{k\bar{N}}, z) \leq -\gamma/2. \tag{3.98}$$

By (3.83) and property (P2),

$$d(x_j, z) \leq 2M + 1,\ j = 0, \dots, (s+1)\bar{N}$$

and (3.98) holds for all $k = 0, \dots, s$. In view (3.71), (3.81), and (3.98) holding for all $k = 0, \dots, s$,

$$-(\gamma/2)(s+1) \leq \sum_{k=0}^{s} (d(x_{k\bar{N}}, z) - d(x_{(k+1)\bar{N}}, z))$$
$$= d(x_0, z) - d(x_{(s+1)\bar{N}}, z) \leq d(x_0, z) \leq 2M,$$
$$s + 1 \leq 4\gamma^{-1}M.$$

Thus we have shown that the following property holds:

(P3) If an integer $s \geq 0$ and for each integer $k \in [0, s]$, (3.82) holds, then

$$s \leq 4\gamma^{-1}M - 1,$$
$$d(x_j, z) \leq 2M + 1,\ j = 0, \dots, (s+1)\bar{N},$$
$$d(x_{k\bar{N}}, z) \leq 2M,\ k = 0, \dots, (s+1).$$

Property (P3) imply that there exists an integer $q \in [0, n_0 - 1]$ such that for each integer k satisfying $0 \leq k < q$,

$$\max\{d(x_i, x_{i-1}) :\ i = k\bar{N} + 1, \dots, (k+1)\bar{N}\} > \epsilon_0;$$
$$\max\{d(x_i, x_{i-1}) :\ i = q\bar{N} + 1, \dots, (q+1)\bar{N}\} \leq \epsilon_0.$$

By (3.81), property (P3), and the choice of q,

$$d(x_{q\bar{N}}, z) \leq 2M,$$
$$d(x_j, z) \leq 2M + 1,\ j = 0, \dots, q\bar{N},$$
$$d(x_j, \theta) \leq 3M + 1,\ j = 0, \dots, q\bar{N}.$$

Assume that an integer $q \in [0, n_0 - 1]$ satisfies

$$d(x_i, x_{i-1}) \leq \epsilon_0,\ i = q\bar{N} + 1, \dots, (q+1)\bar{N}. \tag{3.99}$$

In view of (3.69), (3.72), and (3.99), for all $i = q\bar{N} + 1, \dots, (q+1)\bar{N}$,

$$d(x_{i-1}, P_{r(i)}(x_{i-1})) \le d(x_{i-1}, x_i) + d(x_i, P_{r(i)}(x_{i-1})) \le \epsilon_0 + \delta < 2\epsilon_0. \quad (3.100)$$

Let

$$i \in \{q\bar{N}, \dots, (q+1)\bar{N}\}, \quad (3.101)$$

$$s \in \{1, \dots, m\}. \quad (3.102)$$

Relations (3.66) and (3.102) imply that there exists

$$j \in \{q\bar{N} + 1, \dots, (q+1)\bar{N}\} \quad (3.103)$$

such that

$$s = r(j). \quad (3.104)$$

By (3.100), (3.103), and (3.104),

$$d(x_{j-1}, P_s(x_{j-1})) = d(x_{j-1}, P_{r(j)}(x_{j-1})) < 2\epsilon_0. \quad (3.105)$$

In view of (3.67), (3.99), (3.103), and (3.104),

$$d(x_i, x_{j-1}) \le \bar{N}\epsilon_0 \le \epsilon.$$

Together with (3.67) and (3.105) this implies that

$$x_i \in \tilde{F}_\epsilon(P_s).$$

Since the inclusion above holds for all $s \in \{1, \dots, m\}$ we conclude that

$$x_i \in \tilde{F}_\epsilon$$

for all $i \in \{q\bar{N}, \dots, (q+1)\bar{N}\}$. Theorem 3.5 is proved. □

3.6 Proof of Theorem 3.7

Fix

$$z \in B(\theta, M) \cap F. \quad (3.106)$$

Set

$$\epsilon_0 = \epsilon\bar{N}^{-1}. \quad (3.107)$$

By (A1), there exists $\gamma \in (0, 1)$ such that the following property holds:

(P4) for each $i \in \{1, \dots, m\}$, each

$$\xi \in B(\theta, 3M + 1) \cap \mathrm{Fix}(P_i)$$

and each $x \in B(\theta, 3M + 1)$ satisfying $d(x, P_i(x)) \ge \epsilon_0/2$,

$$d(\xi, P_i(x)) \le d(\xi, x) - \gamma.$$

Set

$$Q = 1 + \lfloor 2\bar{N}M\gamma^{-1} \rfloor. \tag{3.108}$$

Assume that

$$r \in \mathcal{R}, \tag{3.109}$$

$$x_0 \in B(\theta, M) \tag{3.110}$$

and let $\{x_i\}_{i=1}^{\infty} \subset X$ satisfy for each natural number i,

$$x_i = P_{r(i)}(x_{i-1}). \tag{3.111}$$

In view of (3.106) and (3.110),

$$d(z, x_0) \le 2M. \tag{3.112}$$

By (3.106), (3.111), and (A1),

$$d(z, x_{i+1}) \le d(z, x_i),\ i = 0, 1, \dots. \tag{3.113}$$

Relations (3.106) and (3.113) imply that

$$d(z, x_i) \le 2M,\ i = 0, 1, \dots. \tag{3.114}$$

Set

$$E_0 = \{i \in \{0, 1, \dots\} :\ d(x_i, x_{i+1}) \ge \epsilon_0/2\}. \tag{3.115}$$

Let n be a natural number. By (3.112) and (A1),

$$2M \ge d(z, x_0) \ge d(z, x_0) - d(z, x_n)$$

$$= \sum_{i=0}^{n-1} (d(z, x_i) - d(z, x_{i+1}))$$

$$\ge \sum \{d(z, x_i) - d(z, x_{i+1}) :\ i \in \{0, \dots, n-1\},\ d(x_i, x_{i+1}) \ge 2^{-1}\epsilon_0\}. \tag{3.116}$$

If $i \in \{0, \dots, n-1\}$ satisfies $d(x_i, x_{i+1}) \geq 2^{-1}\epsilon_0$, then in view of (3.111),

$$d(x_i, P_{r(i+1)}(x_i)) \geq 2^{-1}\epsilon_0$$

and in view of (3.106), (3.111), (3.114), and (P4),

$$d(z, x_i) - \gamma \geq d(z, P_{r(i+1)}(x_i)) = d(z, x_{i+1}).$$

Together with (3.116) this implies that

$$2M \geq \sum \{d(z, x_i) - d(z, x_{i+1}) : \ i \in \{0, \dots, n-1\}, \ d(x_i, x_{i+1}) \geq 2^{-1}\epsilon_0\}$$
$$\geq \gamma \mathrm{Card}(E_0 \cap [0, n-1]).$$

Since the relation above holds for every natural number n we conclude that

$$\mathrm{Card}(E_0) \leq 2M\gamma^{-1}. \tag{3.117}$$

Set

$$E_1 = \{i \in \{0, 1, \dots\} : \ [i, i + \bar{N} - 1] \cap E_0 \neq \emptyset\}. \tag{3.118}$$

By (3.108), (3.117), and (3.118),

$$\mathrm{Card}(E_1) \leq \bar{N}\mathrm{Card}(E_0) \leq 2\bar{N}M\gamma^{-1} \leq Q. \tag{3.119}$$

Let a nonnegative integer j satisfies

$$j \notin E_1.$$

Then in view of (3.111), (3.115), and (3.118),

$$[j, j + \bar{N} - 1] \cap E_0 = \emptyset$$

and for each $i \in \{j, \dots, j + \bar{N} - 1\}$,

$$d(x_i, P_{r(i+1)}(x_i)) = d(x_i, x_{i+1}) < \epsilon_0/2. \tag{3.120}$$

In view of (3.120), for each pair of integers $i_1, i_2 \in \{j, \dots, j + \bar{N}\}$,

$$d(x_{i_1}, x_{i_2}) < \bar{N}\epsilon_0/2. \tag{3.121}$$

Let

$$i \in \{j, \dots, j + \bar{N}\}, \ s \in \{1, \dots, m\}. \tag{3.122}$$

By (3.66) and (3.122), there is

$$j_0 \in \{j+1, \dots, j+\bar{N}\} \tag{3.123}$$

such that

$$s = r(j_0). \tag{3.124}$$

It follows from (3.120), (3.123), and (3.124) that

$$d(x_{j_0-1}, P_s(x_{j_0-1})) = d(x_{j_0-1}, P_{r(j_0)}(x_{j_0-1})) < \epsilon_0/2.$$

Together with (3.107) and (3.121) and (3.123) this implies that

$$x_j \in \tilde{F}_{\bar{N}\epsilon_0/2}(P_s) \subset \tilde{F}_\epsilon(P_s).$$

Since the inclusion above holds for every $s \in \{1, \dots, m\}$ we conclude that

$$x_j \in \tilde{F}_\epsilon$$

for all nonnegative integers j satisfying $j \notin E_1$. This completes the proof of Theorem 3.7. □

3.7 The Third Problem

Let (X, d) be a metric space. Recall that for each $x \in X$ and each $r > 0$,

$$B(x, r) = \{y \in X : \ d(x, y) \le r\}.$$

For each $x \in X$ and each nonempty set $E \subset X$,

$$d(x, E) = \inf\{d(x, y) : \ y \in E\}.$$

Fix $\theta \in X$. Suppose that m is a natural number, $C_i \subset X$, $i = 1, \dots, m$ be nonempty closed sets and that

$$C := \cap_{i=1}^m C_i \ne \emptyset. \tag{3.125}$$

Assume that mappings $P_i : X \to X$, $i = 1, \dots, m$ and that for every $i \in \{1, \dots, m\}$,

$$P_i(x) = x \text{ for all } x \in C_i. \tag{3.126}$$

Suppose that the following assumption holds:

(A2) For each $M > 0$ and each $\gamma > 0$ there exists $\delta > 0$ such that for each $i \in \{1, \dots, m\}$, each $x \in B(\theta, M)$ satisfying $d(x, C_i) \geq \gamma$ and each

$$z \in B(\theta, M) \cap C_i,$$

we have

$$d(P_i(x), z) \leq d(x, z) - \delta.$$

Fix a natural number $\bar{N}$.

Denote by $\mathcal{R}$ the set of all mappings $r : \{1, 2, \dots, \} \to \{1, \dots, m\}$ such that for each integer j,

$$\{1, \dots, m\} \subset \{r(j), \dots, r(j + \bar{N} - 1)\}. \tag{3.127}$$

In Sect. 3.8 we prove the following result.

Theorem 3.15. *Let* $M > 0$ *satisfy*

$$B(\theta, M) \cap C \neq \emptyset, \tag{3.128}$$

$\epsilon \in (0, 1)$,

$$\epsilon_0 = \epsilon(3\bar{N} + 1)^{-1}, \tag{3.129}$$

$\gamma \in (0, 1)$ *and let the following property hold:*

(P5) for each $i \in \{1, \dots, m\}$, *each*

$$z \in B(\theta, 3M + 1) \cap C_i$$

and each $x \in B(\theta, 3M + 1)$ *satisfying* $d(x, C_i) \geq \epsilon_0/2$,

$$d(z, P_i(x)) \leq d(z, x) - \gamma.$$

Let

$$n_0 = \lfloor 4\gamma^{-1} M \rfloor + 1, \tag{3.130}$$

$$0 < \delta \leq \min\{\epsilon_0/4, \gamma(2\bar{N})^{-1}\}. \tag{3.131}$$

Assume that

$$r \in \mathcal{R}, \tag{3.132}$$

$$x_0 \in B(\theta, M) \text{ and } \{x_i\}_{i=1}^{\infty} \subset X \tag{3.133}$$

satisfy for each natural number i,

$$d(x_i, P_{r(i)}(x_{i-1})) \leq \delta. \tag{3.134}$$

Then there exists an integer $q \in [0, n_0 - 1]$ such that

$$x_i \in B(\theta, 3M + 1), \ i = 0, \ldots, q\bar{N}, \tag{3.135}$$

$$d(x_{i-1}, C_{r(i)}) \leq \epsilon_0, \ i = q\bar{N} + 1, \ldots, (q+1)\bar{N}. \tag{3.136}$$

Moreover, if an integer $q \in [0, n_0 - 1]$ satisfies (3.136), then for each $i = q\bar{N}, \ldots, (q+1)\bar{N}$,

$$d(x_i, C_s) \leq \epsilon, \ s = 1, \ldots, m.$$

Note that the existence of $\gamma \in (0, 1)$ in the statement of Theorem 3.15 follows from assumption (A2).

In Sect. 3.9 we prove the following result.

Theorem 3.16. *Let $M > 0$ satisfy*

$$B(\theta, M) \cap C \neq \emptyset, \tag{3.137}$$

$\epsilon \in (0, 1)$,

$$\epsilon_1 \in (0, \epsilon(2\bar{N})^{-1}) \tag{3.138}$$

and let the following property hold:

(P6) for each $i \in \{1, \ldots, m\}$, each

$$z \in B(\theta, 3M + 1) \cap C_i$$

and each $x \in B(\theta, 3M + 2)$ satisfying $d(x, C_i) \geq \epsilon_0/2$,

$$d(z, P_i(x)) \leq d(z, x) - 2\epsilon_1.$$

Let $\gamma \in (0, \epsilon_1)$ and the following property hold:

(P7) for each $i \in \{1, \ldots, m\}$, each

$$z \in B(\theta, 3M + 1) \cap C_i$$

and each $x \in B(\theta, 3M + 1)$ satisfying $d(x, C_i) \geq \epsilon_1/4$,

$$d(z, P_i(x)) \leq d(z, x) - \gamma. \tag{3.139}$$

Let

$$n_0 = \lfloor 4\gamma^{-1}M \rfloor + 1, \tag{3.140}$$

$$0 < \delta \le \min\{\epsilon_1/4, \gamma(2\bar{N})^{-1}\}. \tag{3.141}$$

Assume that

$$r \in \mathcal{R}, \tag{3.142}$$

$$x_0 \in B(\theta, M) \text{ and } \{x_i\}_{i=1}^{\infty} \subset X \tag{3.143}$$

satisfy for each natural number i,

$$d(x_i, P_{r(i)}(x_{i-1})) \le \delta. \tag{3.144}$$

Then there exists an integer $q \in [0, n_0 - 1]$ *such that*

$$x_i \in B(\theta, 3M + 1),\ i = 0, \dots, q\bar{N}, \tag{3.145}$$

$$d(x_{i-1}, x_i) \le \epsilon_1,\ i = q\bar{N} + 1, \dots, (q + 1)\bar{N}. \tag{3.146}$$

Moreover, if an integer $q \in [0, n_0 - 1]$ *satisfies* (3.145) *and* (3.146)*, then for each* $i = q\bar{N}, \dots, (q + 1)\bar{N}$,

$$d(x_i, C_s) \le \epsilon,\ s = 1, \dots, m.$$

Note that the existence of $\epsilon_1, \gamma \in (0, 1)$ in the statement of Theorem 3.16 follows from assumption (A2).

Applying by induction Theorem 3.15 we can prove the following result.

Theorem 3.17. *Suppose that* $\bar{\epsilon} \in (0, 1)$*,* $\bar{M} > 0$*,*

$$\{x \in X :\ d(x, C_s) \le \bar{\epsilon},\ s = 1, \dots, m\} \subset B(\theta, \bar{M}),\ \epsilon \in (0, \bar{\epsilon}),\ \epsilon_0 = \epsilon(3\bar{N} + 1)^{-1},$$

$\gamma \in (0, 1)$ *and let property (P5) (see Theorem 3.15) hold. Let*

$$n_0 = \lfloor 4\gamma^{-1}M \rfloor + 1,$$

$$0 < \delta \le \min\{\epsilon_0/4, \gamma(2\bar{N})^{-1}\}.$$

Assume that

$$r \in \mathcal{R},$$

$$x_0 \in B(\theta, M) \text{ and } \{x_i\}_{i=1}^{\infty} \subset X$$

satisfy for each natural number i,

$$d(x_i, P_{r(i)}(x_{i-1})) \le \delta.$$

Then

$$x_i \in B(\theta, 3M + 1) \text{ for all integers } i \ge 0$$

and there exists a strictly increasing sequence of integers $\{q_p\}_{p=0}^{\infty}$ *such that*

$$0 \le q_0 \le n_0,$$
$$1 \le q_{p+1} - q_p \le n_0 \text{ for all integers } p \ge 0$$

and that for each integer $p \ge 0$ *and each* $i = q_p\bar{N}, \dots, (q_p + 1)\bar{N}$,

$$d(x_i, C_s) \le \epsilon, \ s = 1, \dots, m.$$

The next result is proved in Sect. 3.10..

Theorem 3.18. *Let* $M > 0$ *satisfy*

$$B(\theta, M) \cap C \ne \emptyset,$$

$\epsilon > 0$. *Then there exists a constant* $Q > 0$ *such that for each* $r \in \mathcal{R}$, *each sequence* $\{x_i\}_{i=0}^{\infty} \subset X$ *which satisfies*

$$x_0 \in B(\theta, M)$$

and

$$x_i = P_{r(i)}(x_{i-1})$$

for all natural numbers i,

$$Card(\{i \in \{0, 1, \dots\} : \ \max\{d(x_i, C_s) : \ s = 1, \dots, m\} > \epsilon\}) \le Q.$$

Theorem 3.16 implies the following result.

Theorem 3.19. *Suppose that the family of sets* $\{C_i, \ i = 1, \dots, m\}$ *has the bounded regularity property,* $\epsilon > 0$ *and that* $M > 0$ *satisfies*

$$B(\theta, M) \cap C \ne \emptyset.$$

Then there exist $\delta > 0$ *and a natural number* n_0 *such that for each* $r \in \mathcal{R}$, *each* $x_0 \in B(\theta, M)$ *and each sequence* $\{x_i\}_{i=1}^{\infty} \subset X$ *which satisfies for each natural number i,*

$$d(x_i, P_{r(i)}(x_{i-1})) \le \delta,$$

there exists an integer $q \in [0, n_0 - 1]$ *such that*

$$x_i \in B(\theta, 3M + 2),\ i = 0, \dots, (q + 1)\bar{N},$$

$$d(x_i, C) \le \epsilon,\ i = q\bar{N}, \dots, (q + 1)\bar{N}.$$

Applying by induction Theorem 3.19 we can prove the following result.

Proposition 3.20. *Suppose that the family of sets* $\{C_i,\ i = 1, \dots, m\}$ *has the bounded regularity property,* $\bar{M} > 0$ *satisfies*

$$C \subset B(\theta, \bar{M}),\ M > \bar{M} + 1,\ \epsilon \in (0, 1).$$

Then there exist $\delta > 0$ *and a natural number* n_0 *such that for each* $r \in \mathcal{R}$*, each* $x_0 \in B(\theta, M)$ *and each sequence* $\{x_i\}_{i=1}^{\infty} \subset X$ *which satisfies for each natural number i,*

$$d(x_i, P_{r(i)}(x_{i-1})) \le \delta$$

the inclusion $x_i \in B(\theta, 3M + 2)$ *holds for all integers* $i \ge 0$ *and there exists a strictly increasing sequence of integers* $\{q_p\}_{p=0}^{\infty}$ *such that*

$$0 \le q_0 \le n_0,$$

$$1 \le q_{p+1} - q_p \le n_0 \text{ for all integers } p \ge 0$$

and that for each integer $p \ge 0$ *and each* $i = (q_p - 1)\bar{N}, \dots, q_p\bar{N}$*,*

$$d(x_i, C) \le \epsilon.$$

The following result is proved in Sect. 3.11.

Theorem 3.21. *Suppose that the family of sets* $\{C_i,\ i = 1, \dots, m\}$ *has the bounded regularity property,* $\bar{M} > 0$ *satisfies*

$$C \subset B(\theta, \bar{M}),\ M > \bar{M} + 1,\ \epsilon \in (0, 1).$$

Then there exist $\delta > 0$ *and a natural number* n_0 *such that for each* $r \in \mathcal{R}$*, each* $x_0 \in B(\theta, M)$ *and each sequence* $\{x_i\}_{i=1}^{\infty} \subset X$ *which satisfies for each natural number i,*

$$d(x_i, P_{r(i)}(x_{i-1})) \le \delta,$$

the inclusion $x_i \in B(\theta, 3M + 2)$ *holds for all integers* $i \ge 0$ *and*

$$d(x_i, C) \le \epsilon \text{ for all integers } i \ge n_0\bar{N}.$$

The following result is proved in Sect. 3.12.

Theorem 3.22. *Suppose that the family of sets* $\{C_i,\ i = 1, \dots, m\}$ *has the bounded regularity property, there exists* $s_0 \in \{1, \dots, m\}$ *such that the set* $P_{s_0}(X)$ *is bounded,* $\epsilon > 0$ *and that a sequence* $\{\delta_i\}_{i=1}^{\infty} \subset (0, \infty)$ *satisfies*

$$\lim_{i\to\infty} \delta_i = 0. \tag{3.147}$$

Then there exists a natural number n_ϵ *such that for each* $r \in \mathcal{R}$ *and each sequence* $\{x_i\}_{i=0}^{\infty} \subset X$ *which satisfies for each natural number i,*

$$d(x_i, P_{r(i)}(x_{i-1})) \le \delta_i,$$

the inequality

$$d(x_i, C) \le \epsilon$$

holds for all integers $i \ge n_\epsilon$*.*

Example 3.23. Assume that $P_i(X) = C_i$, $i \in \{1, \dots, m\}$. We show that in this case (A2) follows from (A1).

Assume that (A1) holds, $M > 0$ $\gamma > 0$ and that $\delta > 0$ is as guaranteed by (A1). Let

$$i \in \{1, \dots, m\}\ , z \in B(\theta, M) \cap C_i,\ x \in B(\theta, M),$$
$$d(x, C_i) \ge \gamma. \tag{3.148}$$

Since $P_i(x) \in C_i$, it follows from (3.148) that

$$d(x, P_i(x)) \ge \gamma. \tag{3.149}$$

By (3.149), (A1) and the choice of δ,

$$d(z, P_i(x)) \le d(z, x) - \delta.$$

Thus (A2) holds.

Assume that $P_i(X) = C_i$, $i \in \{1, \dots, m\}$, $\bar{c} \in (0, 1)$ and that for every $i \in \{1, \dots, m\}$,

$$d(z, x)^2 \ge d(z, P_i(x))^2 + \bar{c} d(x, P_i(x))^2$$
$$\text{for all } x \in X \text{ and all } z \in C_i.$$

In view of Example 3.12, (A1) holds. Therefore (A2) holds too.

Example 3.24. Assume that every bounded closed subset of X is compact, mappings P_i, $i = 1, \dots, m$ are continuous and that for every $i \in \{1, \dots, m\}$, every $z \in C_i$ and every $x \in X \setminus C_i$,

$$d(z, P_i(x))) < d(z, x). \tag{3.150}$$

We claim that (A2) hods.

Let $M > 0, \gamma > 0$ and $j \in \{1, \dots, m\}$. In order to meet our goal it is sufficient to show that there exists $\delta > 0$ such that

$$d(z, P_j(x)) \le d(z, x) - \delta$$

for each

$$z \in B(\theta, M) \cap C_j$$

and each $x \in B(\theta, M)$ satisfying $d(x, C_j) \ge \gamma$.

Assume the contrary. Then there exist sequences

$$\{z_k\}_{k=1}^{\infty} \subset B(\theta, M) \cap C_j \text{ and } \{x_k\}_{k=1}^{\infty} \subset B(\theta, M)$$

such that for all natural numbers k,

$$d(x_k, C_j) \ge \gamma,$$
$$\lim_{k\to\infty} (d(z_k, x_k) - d(z_k, P_j(x_k))) = 0.$$

Extracting a subsequence and re-indexing, if necessary, we may assume without loss of generality that there exist

$$x = \lim_{k\to\infty} x_k, \; z = \lim_{k\to\infty} z_k.$$

It is not difficult to see that

$$z \in C_j,$$
$$d(x, C_j) \ge \liminf_{k\to\infty} d(x_k, C_j) \ge \gamma,$$
$$d(z, x) = \lim_{k\to\infty} d(z_k, x_k),$$
$$d(z, P_j(z)) = \lim_{k\to\infty} d(z_k, P_j(x_k)),$$
$$d(z, x) - d(z, P_j(x)) = \lim_{k\to\infty} (d(z_k, x_k) - d(z_k, P_j(x_k))) = 0.$$

This contradicts (3.150). The contradiction we have reached proves that (A2) holds.

In view of Example 3.23, Theorem 3.10 is a particular case of Theorem 3.21 and Theorem 3.11 is a particular case of Theorem 3.22.

3.8 Proof of Theorem 3.5

By (3.128) there exists

$$z \in B(\theta, M) \cap C. \tag{3.151}$$

Assume that a nonnegative integer s satisfies for each integer $k \in [0, s]$,

$$\max\{d(x_{i-1}, C_{r(i)}) : \ i = k\bar{N} + 1, \ldots, (k+1)\bar{N}\} > \epsilon_0. \tag{3.152}$$

By (3.143) and (3.151),

$$d(x_0, z) \le 2M. \tag{3.153}$$

Assume that an integer $k \in [0, s]$ satisfies

$$d(x_{k\bar{N}}, z) \le 2M. \tag{3.154}$$

We prove the following auxiliary result.

Lemma 3.25. *Assume that an integer*

$$i \in [0, \bar{N} - 1] \tag{3.155}$$

satisfies

$$d(x_{k\bar{N}+i}, z) \le 2M + i\delta. \tag{3.156}$$

Then

$$d(x_{k\bar{N}+i+1}, z) \le d(x_{k\bar{N}+i}, z) + \delta \tag{3.157}$$

and if

$$d(x_{k\bar{N}+i}, C_{r(k\bar{N}+i+1)}) > \epsilon_0, \tag{3.158}$$

then

$$d(x_{k\bar{N}+i+1}, z) \le d(x_{k\bar{N}+i}, z) - \gamma + \delta. \tag{3.159}$$

Proof. In view of (3.144), (3.151), and (A2),

$$\begin{aligned} d(x_{k\bar{N}+i+1}, z) &\le d(x_{k\bar{N}+i+1}, P_{r(k\bar{N}+i+1)}(x_{k\bar{N}+i})) + d(P_{r(k\bar{N}+i+1)}(x_{k\bar{N}+i}), z) \\ &\le \delta + d(x_{k\bar{N}+i}, z) \end{aligned}$$

and (3.157) holds.

Assume that (3.158) holds. It follows from (3.132), (3.151), (3.155), and (3.156) that

$$d(x_{k\bar{N}+i}, \theta) \le d(x_{k\bar{N}+i}, z) + d(z, \theta) \le 2M + \bar{N}\delta + M \le 3M + 1. \tag{3.160}$$

By (3.151), (3.158), (3.160), and property (P5),

$$d(z, P_{r(k\bar{N}+i+1)}(x_{k\bar{N}+i})) \le d(z, x_{k\bar{N}+i}) - \gamma. \tag{3.161}$$

Relations (3.134) and (3.161) imply that

$$\begin{aligned} d(z, x_{k\bar{N}+i+1}) &\le d(z, P_{r(k\bar{N}+i+1)}(x_{k\bar{N}+i})) + d(P_{r(k\bar{N}+i+1)}(x_{k\bar{N}+i}), x_{k\bar{N}+i+1}) \\ &\le d(z, x_{k\bar{N}+i}) - \gamma + \delta. \end{aligned}$$

Thus (3.159) holds. This completes the proof of Lemma 3.25. □

It follows from (3.131), (3.132), (3.154), and Lemma 3.25 applied by induction that for all $i = 0, \dots, \bar{N} - 1$,

$$d(z, x_{k\bar{N}+i+1}) \le d(z, x_{k\bar{N}+i}) + \delta,$$

$$d(z, x_{k\bar{N}+i+1}) \le 2M + \delta(i + 1) \le 2M + \delta\bar{N} \le 2M + 1, \tag{3.162}$$

$$d(z, x_{k\bar{N}+i}) \le 2M + 1, \ i = 0, \dots, \bar{N}. \tag{3.163}$$

Relations (3.131), (3.152), (3.154), (3.162), and Lemma 3.25 imply that

$$\begin{aligned} d(z, x_{(k+1)\bar{N}}) - d(z, x_{k\bar{N}}) &= \sum_{i=0}^{\bar{N}-1} (d(z, x_{k\bar{N}+i+1}) - d(z, x_{k\bar{N}} + i)) \\ &\le \delta(\bar{N} - 1) - \gamma + \delta \le -\gamma + \delta\bar{N} \le -\gamma/2. \end{aligned}$$

Thus we have shown that the following property holds:

(P8) if an integer $k \in [0, s]$ satisfies $d(z, x_{k\bar{N}}) \le 2M$, then

$$d(z, x_j) \le 2M + 1, \ j = k\bar{N}, \dots, (k + 1)\bar{N},$$

$$d(z, x_{(k+1)\bar{N}}) - d(z, x_{k\bar{N}}) \le -\gamma/2. \tag{3.164}$$

By (3.153) and property (P8),

$$d(x_j, z) \le 2M + 1, \ j = 0, \dots, (s + 1)\bar{N}$$

and (3.164) holds for all $k = 0, \dots, s$. In view (3.153) and (3.164) holding for all $k = 0, \dots, s$,

$$(s+1)\gamma/2 \le \sum_{k=0}^{s}(d(x_{k\bar{N}}, z) - d(x_{(k+1)\bar{N}}, z))$$
$$= d(x_0, z) - d(x_{(s+1)\bar{N}}, z) \le d(x_0, z) \le 2M,$$
$$s + 1 \le 4\gamma^{-1}M.$$

Thus we have shown that the following property holds:

(P9) If an integer $s \ge 0$ and for each integer $k \in [0, s]$, (3.152) holds, then

$$s \le 4\gamma^{-1}M - 1,$$
$$d(x_j, z) \le 2M + 1,\ j = 0, \dots, (s+1)\bar{N},$$
$$d(x_{k\bar{N}}, z) \le 2M,\ k = 0, \dots, (s+1).$$

Property (P9) implies that there exists an integer $q \in [0, n_0 - 1]$ such that for each integer k satisfying $0 \le k < q$,

$$\max\{d(x_{i-1}, C_{r(i)}) :\ i = k\bar{N} + 1, \dots, (k+1)\bar{N}\} > \epsilon_0;$$
$$\max\{d(x_{i-1}, C_{r(i)}) :\ i = q\bar{N} + 1, \dots, (q+1)\bar{N}\} \le \epsilon_0.$$

By (3.151), property (P9), and the choice of q,

$$d(x_{q\bar{N}}, z) \le 2M,$$
$$d(x_j, z) \le 2M + 1,\ j = 0, \dots, q\bar{N},$$
$$d(x_j, \theta) \le 3M + 1,\ j = 0, \dots, q\bar{N}.$$

Assume that an integer $q \in [0, n_0 - 1]$ satisfies

$$d(x_{i-1}, C_{r(i)}) \le \epsilon_0,\ i = q\bar{N} + 1, \dots, (q+1)\bar{N}. \tag{3.165}$$

Let $i \in \{q\bar{N} + 1, \dots, (q+1)\bar{N}\},\ \alpha > 0$. In view of (3.165), there exists

$$\xi \in C_{r(i)} \tag{3.166}$$

such that

$$d(x_{i-1}, \xi) < d(x_{i-1}, C_{r(i)}) + \alpha < \epsilon_0 + \alpha. \tag{3.167}$$

By (3.134), (3.166), (3.167), and (A2),

$$d(x_i, \xi) \le d(x_i, P_{r(i)}(x_{i-1})) + d(P_{r(i)}(x_{i-1}), \xi)$$
$$\le \delta + d(x_{i-1}, \xi) \le \delta + \epsilon_0 + \alpha.$$

Together with (3.131) and (3.167) this implies that

$$d(x_{i-1}, x_i) \le 2\epsilon_0 + 2\alpha + \delta.$$

Since α is any positive number we conclude that

$$d(x_{i-1}, x_i) \le 3\epsilon_0 \tag{3.168}$$

for every $i \in \{q\bar{N} + 1, \dots, (q + 1)\bar{N}\}$. By (3.168), for each $i_1, i_2 \in \{q\bar{N}, \dots, (q + 1)\bar{N}\}$,

$$d(x_{i_1}, x_{i_2}) \le 3\bar{N}\epsilon_0. \tag{3.169}$$

Let

$$s \in \{1, \dots, m\},\ i \in \{q\bar{N}, \dots, (q + 1)\bar{N}\}, \tag{3.170}$$

Relations (3.127) and (3.170) imply that there exists

$$j \in \{q\bar{N} + 1, \dots, (q + 1)\bar{N}\} \tag{3.171}$$

such that

$$s = r(j). \tag{3.172}$$

By (3.165), (3.171), and (3.172),

$$d(x_{j-1}, C_s) \le \epsilon_0.$$

Together with (3.129) and (3.169)–(3.171) this implies that

$$d(x_i, C_s) \le d(x_i, x_{j-1}) + d(x_{j-1}, C_s) \le \epsilon_0(3\bar{N} + 1) \le \epsilon$$

for all $s \in \{1, \dots, m\}$ and all $i \in \{q\bar{N}, \dots, (q + 1)\bar{N}\}$. Theorem 3.15 is proved. □

3.9 Proof of Theorem 3.16

By (3.137) there exists

$$z \in B(\theta, M) \cap C. \tag{3.173}$$

Assume that a nonnegative integer s satisfies for each integer $k \in [0, s]$,

$$\max\{d(x_{i-1}, x_i) : \ i = k\bar{N} + 1, \dots, (k + 1)\bar{N}\} > \epsilon_1. \tag{3.174}$$

By (3.143) and (3.173),

$$d(x_0, z) \le 2M. \tag{3.175}$$

Assume that an integer $k \in [0, s]$ satisfies

$$d(x_{k\bar{N}}, z) \le 2M. \tag{3.176}$$

We prove the following auxiliary result.

Lemma 3.26. *Assume that an integer*

$$i \in [0, \bar{N} - 1] \tag{3.177}$$

satisfies

$$d(x_{k\bar{N}+i}, z) \le 2M + i\delta. \tag{3.178}$$

Then

$$d(x_{k\bar{N}+i+1}, z) \le d(x_{k\bar{N}+i}, z) + \delta \tag{3.179}$$

and if

$$d(x_{k\bar{N}+i}, x_{k\bar{N}+i+1}) > \epsilon_1, \tag{3.180}$$

then

$$d(z, x_{k\bar{N}+i+1}) \le d(z, x_{k\bar{N}+i}) - \gamma + \delta. \tag{3.181}$$

Proof. In view of (3.144), (3.173), and (A2),

$$\begin{aligned} d(x_{k\bar{N}+i+1}, z) &\le d(x_{k\bar{N}+i+1}, P_{r(k\bar{N}+i+1)}(x_{k\bar{N}+i})) + d(P_{r(k\bar{N}+i+1)}(x_{k\bar{N}+i}), z) \\ &\le \delta + d(x_{k\bar{N}+i}, z) \end{aligned}$$

and (3.179) holds.

Assume that (3.180) holds. It follows from (3.141) and (3.177)–(3.179) that

$$d(x_{k\bar{N}+i+1}, z),\ d(x_{k\bar{N}+i}, z) \le 2M + 1. \tag{3.182}$$

Relations (3.173) and (3.182) imply that

$$x_{k\bar{N}+i},\ x_{k\bar{N}+i+1} \in B(\theta, 3M + 1). \tag{3.183}$$

We show that

$$d(x_{k\bar{N}+i}, C_{r(k\bar{N}+i+1)}) \ge \epsilon_1/4. \tag{3.184}$$

Assume the contrary. Then there exists

$$\xi \in C_{r(k\bar{N}+i+1)} \tag{3.185}$$

such that

$$d(x_{k\bar{N}+i}, \xi) < \epsilon_1/4. \tag{3.186}$$

By (3.144), (3.185), (3.186), and (A2),

$$\begin{aligned} d(x_{k\bar{N}+i+1}, \xi) &\leq d(x_{k\bar{N}+i+1}, P_{r(k\bar{N}+i+1)}(x_{k\bar{N}+i})) + d(P_{r(k\bar{N}+i+1)}(x_{k\bar{N}+i}), \xi) \\ &\leq \delta + d(x_{k\bar{N}+i}, \xi) < \epsilon_1/4 + \delta. \end{aligned} \tag{3.187}$$

Relations (3.141), (3.186), and (3.187) imply that

$$d(x_{k\bar{N}+i}, x_{k\bar{N}+i+1}) \leq d(x_{k\bar{N}+i}, \xi) + d(\xi, x_{k\bar{N}+i+1}) < \epsilon_1/2 + \delta < \epsilon_1.$$

This contradicts (3.180). The contradiction we have reached proves (3.184). By (3.173), (3.183), (3.184), and property (P7),

$$d(z, P_{r(k\bar{N}+i+1)}(x_{k\bar{N}+i})) \leq d(z, x_{k\bar{N}+i}) - \gamma. \tag{3.188}$$

In view of (3.144) and (3.188),

$$\begin{aligned} d(z, x_{k\bar{N}+i+1}) &\leq d(z, P_{r(k\bar{N}+i+1)}(x_{k\bar{N}+i})) + d(P_{r(k\bar{N}+i+1)}(x_{k\bar{N}+i}), x_{k\bar{N}+i+1}) \\ &\leq d(z, x_{k\bar{N}+i}) - \gamma + \delta. \end{aligned}$$

Thus (3.181) holds. This completes the proof of Lemma 3.26. □

It follows from (3.141), (3.170), and Lemma 3.26 applied by induction that for all $i = 0, \ldots, \bar{N} - 1$,

$$d(z, x_{k\bar{N}+i+1}) \leq d(z, x_{k\bar{N}+i}) + \delta,$$

$$d(z, x_{k\bar{N}+i+1}) \leq 2M + \delta(i+1) \leq 2M + \delta\bar{N} \leq 2M + 1, \tag{3.189}$$

$$d(z, x_{k\bar{N}+i}) \leq 2M + 1, \ \ i = 0, \ldots, \bar{N}. \tag{3.190}$$

Relations (3.141), (3.174), (3.176), (3.189), and Lemma 3.26 imply that

$$\begin{aligned} d(z, x_{(k+1)\bar{N}}) - d(z, x_{k\bar{N}}) &= \sum_{i=0}^{\bar{N}-1} (d(z, x_{k\bar{N}+i+1}) - d(z, x_{k\bar{N}} + i)) \\ &\leq \delta(\bar{N} - 1) - \gamma + \delta \leq -\gamma + \delta\bar{N} \leq -\gamma/2. \end{aligned}$$

Thus we have shown that the following property holds:
(P10) if an integer $k \in [0, s]$ satisfies $d(z, x_{k\bar{N}}) \le 2M$, then

$$d(z, x_j) \le 2M + 1,\ j = k\bar{N}, \dots, (k+1)\bar{N},$$
$$d(z, x_{(k+1)\bar{N}}) - d(z, x_{k\bar{N}}) \le -\gamma/2. \tag{3.191}$$

By (3.175) and property (P10),

$$d(x_j, z) \le 2M + 1,\ j = 0, \dots, (s+1)\bar{N}$$

and (3.191) holds for all $k = 0, \dots, s$. In view (3.175) and (3.191) holding for all $k = 0, \dots, s$,

$$(s+1)\gamma/2 \le \sum_{k=0}^{s} (d(x_{k\bar{N}}, z) - d(x_{(k+1)\bar{N}}, z))$$
$$= d(x_0, z) - d(x_{(s+1)\bar{N}}, z) \le d(x_0, z) \le 2M,$$
$$s + 1 \le 4\gamma^{-1} M.$$

Thus we have shown that the following property holds:
(P11) If an integer $s \ge 0$ and for each integer $k \in [0, s]$, (3.174) holds, then

$$s \le 4\gamma^{-1} M - 1,$$
$$d(x_j, z) \le 2M + 1,\ j = 0, \dots, (s+1)\bar{N},$$
$$d(x_{k\bar{N}}, z) \le 2M,\ k = 0, \dots, (s+1).$$

Property (P11) and (3.140) imply that there exists an integer $q \in [0, n_0 - 1]$ such that for each integer k satisfying $0 \le k < q$,

$$\max\{d(x_{i-1}, x_i) :\ i = k\bar{N} + 1, \dots, (k+1)\bar{N}\} > \epsilon_1,$$
$$\max\{d(x_{i-1}, x_i) :\ i = q\bar{N} + 1, \dots, (q+1)\bar{N}\} \le \epsilon_1.$$

By (3.173), property (P11), and the choice of q,

$$d(x_{q\bar{N}}, z) \le 2M,$$
$$d(x_j, z) \le 2M + 1,\ j = 0, \dots, q\bar{N},$$
$$d(x_j, \theta) \le 3M + 1,\ j = 0, \dots, q\bar{N}. \tag{3.192}$$

Assume that an integer $q \in [0, n_0 - 1]$ satisfies (3.192) and

$$d(x_{i-1}, x_i) \le \epsilon_1,\ i = q\bar{N} + 1, \dots, (q+1)\bar{N}. \tag{3.193}$$

By (3.193), for each $i_1, i_2 \in \{q\bar{N}, \dots, (q+1)\bar{N}\}$,

$$d(x_{i_1}, x_{i_2}) \le \bar{N}\epsilon_1. \tag{3.194}$$

We show that for all $i \in \{q\bar{N}+1, \dots, (q+1)\bar{N}\}$,

$$d(x_{i-1}, C_{r(i)}) \le \epsilon/2. \tag{3.195}$$

Assume the contrary. Then there exists

$$i \in \{q\bar{N}+1, \dots, (q+1)\bar{N}\} \tag{3.196}$$

such that

$$d(x_{i-1}, C_{r(i)}) > \epsilon/2. \tag{3.197}$$

Relations (3.194) and (3.196) imply that

$$d(x_{i-1}, x_{q\bar{N}}) \le \bar{N}\epsilon_1. \tag{3.198}$$

By (3.138), (3.192), and (3.198),

$$d(x_{i-1}, \theta) \le 3M + 2. \tag{3.199}$$

By (3.173), (3.197), (3.199), and property (P6),

$$d(P_{r(i)}(x_{i-1}), z) \le d(x_{i-1}, z) - 2\epsilon_1. \tag{3.200}$$

It follows from (3.144), (3.193), (3.196), and (3.200) that

$$\begin{gathered} d(x_i, z) \le d(x_i, P_{r(i)}(x_{i-1})) + d(P_{r(i)}(x_{i-1}), z) \\ \le \delta + d(x_{i-1}, z) - 2\epsilon_1, \\ 2\epsilon_1 - \delta \le d(x_{i-1}, z) - d(x_i, z) \le d(x_{i-1}, x_i) \le \epsilon_1, \\ \epsilon_1 \le \delta. \end{gathered}$$

This contradicts (3.141). The contradiction we have reached proves that

$$d(x_{i-1}, C_{r(i)}) \le \epsilon/2, \ i \in \{q\bar{N}+1, \dots, (q+1)\bar{N}\}. \tag{3.201}$$

Let

$$s \in \{1, \dots, m\}, \ i \in \{q\bar{N}, \dots, (q+1)\bar{N}\}, \tag{3.202}$$

Relations (3.127) and (3.202) imply that there exists

$$j \in \{q\bar{N} + 1, \dots, (q+1)\bar{N}\} \tag{3.203}$$

such that

$$s = r(j). \tag{3.204}$$

By (3.201), (3.203), and (3.204),

$$d(x_{j-1}, C_s) \le \epsilon/2.$$

Together with (3.138), (3.194), (3.202), and (3.203) this implies that

$$d(x_i, C_s) \le d(x_i, x_{j-1}) + d(x_{j-1}, C_s) \le \epsilon/2 + \bar{N}\epsilon_1 \le \epsilon$$

for all $s \in \{1, \dots, m\}$ and all $i \in \{q\bar{N}, \dots, (q+1)\bar{N}\}$. Theorem 3.16 is proved. □

3.10 Proof of Theorem 3.18

Fix

$$z \in B(\theta, M) \cap C. \tag{3.205}$$

Set

$$\epsilon_0 = \epsilon(\bar{N} + 1)^{-1}. \tag{3.206}$$

By (3.205) and (A2), there exists $\gamma \in (0, 1)$ such that the following property holds:

(P12) for each $i \in \{1, \dots, m\}$, each

$$z \in B(\theta, 3M + 1) \cap C_i$$

and each $x \in B(\theta, 3M + 1)$ satisfying $d(x, C_i) \ge \epsilon_0/2$,

$$d(z, P_i(x)) \le d(z, x) - \gamma.$$

Set

$$Q = 1 + \lfloor 2\bar{N}M\gamma^{-1} \rfloor. \tag{3.207}$$

Assume that

$$r \in \mathcal{R}, \tag{3.208}$$

$$x_0 \in B(\theta, M) \tag{3.209}$$

and let $\{x_i\}_{i=1}^{\infty} \subset X$ satisfy for each natural number i,

$$x_i = P_{r(i)}(x_{i-1}). \tag{3.210}$$

In view of (3.205) and (3.209),

$$d(z, x_0) \le d(z, \theta) + d(\theta, x_0) \le 2M. \tag{3.211}$$

By (3.205), (3.210), and (A2), for all integers $i \ge 0$,

$$d(z, x_{i+1}) = d(z, P_{r(i+1)}(x_i)) \le d(z, x_i). \tag{3.212}$$

Relations (3.205), (3.211), and (3.212) imply that

$$d(z, x_i) \le 2M,\ d(x_i, \theta) \le 3M,\ i = 0, 1, \dots. \tag{3.213}$$

Set

$$E_0 = \{i \in \{0, 1, \dots\} :\ d(x_i, C_{r(i+1)}) \ge \epsilon_0/2\}. \tag{3.214}$$

Relations (3.205), (3.213), (3.214), and property (P12) imply that for each $i \in E_0$,

$$d(z, P_{r(i+1)}(x_i)) \le d(z, x_i) - \gamma. \tag{3.215}$$

It follows from (3.210) and (3.215) that for all $i \in E_0$,

$$d(z, x_{i+1}) = d(z, P_{r(i+1)}(x_i)) \le d(z, x_i) - \gamma. \tag{3.216}$$

Let n be a natural number. By (3.211), (3.212), (3.216), and (A2),

$$\begin{aligned} 2M \ge d(z, x_0) &\ge d(z, x_0) - d(z, x_n) \\ &= \sum_{i=0}^{n-1} (d(z, x_i) - d(z, x_{i+1})) \end{aligned}$$

$$\ge \sum \{d(z, x_i) - d(z, x_{i+1}) :\ i \in E_0 \cap \{0, \dots, n-1\}\} \ge \gamma \mathrm{Card}(E_0 \cap [0, n-1]).$$

Since the relation above holds for every natural number n we conclude that

$$\mathrm{Card}(E_0) \le 2M\gamma^{-1}. \tag{3.217}$$

Set

$$E_1 = \{i \in \{0, 1, \dots\} :\ [i, i + \bar{N} - 1] \cap E_0 \ne \emptyset\}. \tag{3.218}$$

By (3.207), (3.217), and (3.218),

$$\mathrm{Card}(E_1) \le \bar{N}\mathrm{Card}(E_0) \le 2\bar{N}M\gamma^{-1} \le Q. \tag{3.219}$$

Let a nonnegative integer j satisfies

$$j \notin E_1.$$

Then in view of (3.214) and (3.218),

$$[j, j + \bar{N} - 1] \cap E_0 = \emptyset. \tag{3.220}$$

By (3.220), for each $i \in \{j, \dots, j + \bar{N} - 1\}$,

$$d(x_i, C_{r(i+1)}) < \epsilon_0/2. \tag{3.221}$$

Let

$$i \in \{j, \dots, j + \bar{N} - 1\}. \tag{3.222}$$

In view of (3.221) and (3.222), there exists

$$\xi \in C_{r(i+1)} \tag{3.223}$$

such that

$$d(x_i, \xi) < \epsilon_0/2. \tag{3.224}$$

By (3.205), (3.210), (3.224), and (A2),

$$d(x_{i+1}, \xi) \le d(P_{r(i+1)}(x_i), \xi) \le d(x_i, \xi) < \epsilon_0/2. \tag{3.225}$$

Relations (3.224) and (3.225) imply that

$$d(x_i, x_{i+1}) < \epsilon_0. \tag{3.226}$$

This implies that for each pair of integers $i_1, i_2 \in \{j, \dots, j + \bar{N}\}$,

$$d(x_{i_1}, x_{i_2}) < \bar{N}\epsilon_0. \tag{3.227}$$

Let

$$i \in \{j, \dots, j + \bar{N}\}, \ s \in \{1, \dots, m\}. \tag{3.228}$$

In view of (3.127), there is

$$j_0 \in \{j + 1, \dots, j + \bar{N}\} \tag{3.229}$$

such that

$$s = r(j_0). \tag{3.230}$$

It follows from (3.221), (3.229), and (3.230) that

$$d(x_{j_0-1}, C_s) = d(x_{j_0-1}, C_{r(j_0)}) < \epsilon_0/2. \tag{3.231}$$

By (3.206), (3.227), (3.229), and (3.231) this implies that

$$d(x_j, C_s) < (\bar{N} + 1)\epsilon_0 \le \epsilon$$

for every $s \in \{1, \ldots, m\}$ and all nonnegative integers $j \neq E_1$. This completes the proof of Theorem 3.18. □

3.11 Proof of Theorem 3.21

By Proposition 3.20, there exist $\delta_0 \in (0, \epsilon)$ and a natural number n_0 such that the following property holds:

(P13) for each $r \in \mathcal{R}$, each $x_0 \in B(\theta, M)$ and each sequence $\{x_i\}_{i=1}^{\infty} \subset X$ which satisfies for each natural number i,

$$d(x_i, P_{r(i)}(x_{i-1})) \le \delta_0$$

we have

$$x_i \in B(\theta, 3M + 2) \text{ for all integers } i \ge 0$$

and there exists a strictly increasing sequence of integers $\{q_p\}_{p=0}^{\infty}$ such that

$$0 \le q_0 \le n_0,$$
$$1 \le q_{p+1} - q_p \le n_0 \text{ for all integers } p \ge 0 \tag{3.232}$$

and that for each integer $p \ge 0$ and each $i = (q_p - 1)\bar{N}, \ldots, q_p\bar{N}$,

$$d(x_i, C) \le \epsilon/2. \tag{3.233}$$

Set

$$\delta = \delta_0(4n_0\bar{N})^{-1}. \tag{3.234}$$

Assume that

$$r \in \mathcal{R}, \; x_0 \in B(\theta, M), \; \{x_i\}_{i=1}^{\infty} \subset X \tag{3.235}$$

satisfy for each natural number i,

$$d(x_i, P_{r(i)}(x_{i-1})) \le \delta. \tag{3.236}$$

In view of (3.234) and (3.235),

$$x_i \in B(\theta, 3M + 2) \text{ for all integers } i \ge 0$$

and there exists a strictly increasing sequence of integers $\{q_p\}_{p=0}^{\infty}$ such that (3.232) is true and (3.233) holds for all integers $p \ge 0$ and all $i = (q_p - 1)\bar{N}, \dots, q_p\bar{N}$.

Let $p \ge 0$ be an integer. By (3.233),

$$d(x_{q_p\bar{N}}, C) \le \epsilon/2.$$

There exists a point z such that

$$z \in C \text{ and } d(x_{q_p\bar{N}}, z) \le 3\epsilon/4. \tag{3.237}$$

We show that for all integers $i \in \{0, \dots, (q_{p+1} - q_p)\bar{N}\}$,

$$d(x_{q_p\bar{N}+i}, z) \le 3\epsilon/4 + i\delta. \tag{3.238}$$

In view of (3.237), (3.238) is true for $i = 0$.

Assume that

$$i \in \{0, \dots, (q_{p+1} - q_p)\bar{N}\},\ i < (q_{p+1} - q_p)\bar{N}$$

and that (3.238) holds. By (3.236)–(3.238) and (A2),

$$\begin{aligned} &d(z, x_{q_p\bar{N}+i+1}) \\ &\le d(z, P_{r(q_p\bar{N}+i+1)}(x_{q_p\bar{N}+i})) + d(P_{r(q_p\bar{N}+i+1)}(x_{q_p\bar{N}+i}), x_{q_p\bar{N}+i+1}) \\ &\le d(z, x_{q_p\bar{N}+i}) + \delta \le (3/4)\epsilon + (i+1)\delta. \end{aligned}$$

Thus we have shown by induction that (3.238) holds for all $i = 0, \dots, (q_{p+1} - q_p)\bar{N}$. Together with (3.232) and (3.234) this implies that for all integers $i \in \{0, \dots, (q_{p+1} - q_p)\bar{N}\}$,

$$d(x_{q_p\bar{N}+i}, z) \le 3\epsilon/4 + \bar{N}n_0\delta \le \epsilon.$$

Combined with (3.232) this implies that

$$d(x_i, C) \le \epsilon$$

for integers $i \ge n_0\bar{N}$. This completes the proof of Theorem 3.21. □

3.12 Proof of Theorem 3.22

There exists $\bar{M} > 1$ such that

$$P_{s_0}(X) \subset B(\theta, \bar{M} - 1). \tag{3.239}$$

By Theorem 3.21, there exist $\delta \in (0, 1)$ and a natural number n_0 such that the following property holds:

(P14) for each $r \in \mathcal{R}$, each $x_0 \in B(\theta, \bar{M})$ and each sequence $\{x_i\}_{i=1}^{\infty} \subset X$ which satisfies for each natural number i,

$$d(x_i, P_{r(i)}(x_{i-1})) \leq \delta$$

we have

$$d(x_i, C) \leq \epsilon \text{ for all integers } i \geq n_0.$$

By (3.147), there exists a natural number k_1 such that

$$\delta_i \leq \delta \text{ for each integer } i \geq k_1. \tag{3.240}$$

Set

$$n_\epsilon = k_1 + \bar{N} + n_0. \tag{3.241}$$

Let

$$r \in \mathcal{R} \tag{3.242}$$

and $\{x_i\}_{i=0}^{\infty} \subset X$ satisfy for each natural number i,

$$d(x_i, P_{r(i)}(x_{i-1})) \leq \delta_i. \tag{3.243}$$

By (3.127), there exists

$$j_0 \in \{k_1, \ldots, k_1 + \bar{N}\} \tag{3.244}$$

such that

$$s_0 = r(j_0). \tag{3.245}$$

Relations (3.240), (3.243), and (3.244) imply that

$$d(x_{j_0}, P_{r(j_0)}(x_{j_0-1})) \leq \delta_{j_0} \leq \delta < 1. \tag{3.246}$$

It follows from (3.245) and (3.246) that

$$P_{r(j_0)}(x_{j_0-1}) \in B(\theta, \bar{M}-1).$$

Together with (3.246) this implies that

$$x_{j_0} \in B(\theta, \bar{M}).$$

By the inclusion above, (3.240), (3.242), (3.243), and property (P14), for all integers $i \geq j_0 + n_0$,

$$d(x_i, C) \leq \epsilon. \tag{3.247}$$

In view of (3.241) and (3.244), inequality (3.247) holds for all integers $i \geq n_\epsilon$. Theorem 3.22 is proved. □

3.13 Generic Properties

Let X be a normed space equipped with the norm $\|\cdot\|$ and $d(x,y) = \|x-y\|$, $x, y \in X$. We use the notation and definitions of Sect. 3.7.

Let $D \subset X$ be a nonempty set. Denote by $\mathcal{M}_D$ the set of all mappings $T : X \to X$ such that

$$T(z) = z \text{ for all } z \in D, \tag{3.248}$$

$$\|T(x) - z\| \leq \|x - z\| \text{ for all } x \in X \text{ and all } z \in D. \tag{3.249}$$

Denote by $\mathcal{M}_{D,c}$ the set of all continuous mappings $T \in \mathcal{M}_D$ and denote by $\mathcal{M}_{D,n}$ the set of all mappings $T \in \mathcal{M}_D$

$$\|T(x) - T(y)\| \leq \|x - y\| \text{ for all } x, y \in X. \tag{3.250}$$

The set $\mathcal{M}_D$ is equipped with the uniformity which has the base

$$\mathcal{E}(M,\epsilon) = \{(T_1, T_2) \in \mathcal{M}_D : \ \|T_1(x) - T_2(x)\| \leq \epsilon \text{ for all } x \in B(0, M)\},$$

where $M, \epsilon > 0$. It is not difficult to see that the uniform space $\mathcal{M}_D$ is metrizable and complete and that $\mathcal{M}_{D,c}$ and $\mathcal{M}_{D,n}$ are its closed subsets. We consider the topological (uniform) subspaces $\mathcal{M}_{D,c}$, $\mathcal{M}_{D,n} \subset \mathcal{M}_D$ equipped with the relative topology (uniformity).

A mapping $T : X \to X$ is called (D)-quasi-contractive if (3.248) is true and the following property holds:

(P15) For each $M > 0$ and each $r > 0$ there exists $\delta > 0$ such that for each $x \in B(0, M)$ satisfying $d(x, D) \geq r$ and each $z \in B(0, M) \cap D$, we have

$$\|T(x) - z\| \leq \|x - z\| - \delta.$$

Theorem 3.27. *Let a uniform space $\mathcal{A}$ be either $\mathcal{M}_D$ or $\mathcal{M}_{D,c}$ or $\mathcal{M}_{D,n}$. Suppose that there exists a (D)-quasi-contractive mapping $T_* \in \mathcal{A}$. Then there exists a set $\mathcal{F} \subset \mathcal{A}$ which is a countable intersection of open everywhere dense subsets of $\mathcal{A}$ such that every element of $\mathcal{F}$ is (D)-quasi-contractive.*

Proof. Let $T \in \mathcal{A}$ and $\gamma \in (0, 1)$. Define

$$T_\gamma(x) = (1 - \gamma)T(x) + \gamma T_*(x), \ x \in X. \tag{3.251}$$

Clearly,

$$T_\gamma(z) = z \text{ for all } z \in D. \tag{3.252}$$

By (3.249) and (3.251), for each $x \in X$ and each $z \in D$,

$$\begin{aligned} \|T_\gamma(x) - z\| &= \|(1 - \gamma)T(x) + \gamma T_*(x) - z\| \\ &\leq (1 - \gamma)\|T(x) - z\| + \gamma\|T_*(x) - z\| \leq \|x - z\|. \end{aligned} \tag{3.253}$$

Thus $T_\gamma \in \mathcal{M}_D$. It is not difficult to see that $T_\gamma \in \mathcal{A}$ and that for each $T \in \mathcal{A}$,

$$T_\gamma \to T \text{ as } \gamma \to 0^+.$$

Thus

$$\{T_\gamma : \ T \in \mathcal{A}, \ \gamma \in (0, 1)\}$$

is an everywhere dense subset of $\mathcal{A}$.

Let $T \in \mathcal{A}$, $\gamma \in (0, 1)$ and $k \geq 1$ be an integer. Since the mapping T_* is (D)-quasi-contractive there exists $\delta_k > 0$ such that the following property holds:

(P16) for each $x \in B(0, k)$ satisfying $d(x, D) \geq 1/k$ and each $z \in B(0, k) \cap D$, we have

$$\|T_*(x) - z\| \leq \|x - z\| - \delta_k. \tag{3.254}$$

Assume that

$$x \in B(0, k), \ d(x, D) \geq k^{-1}, \ z \in B(0, k) \cap D. \tag{3.255}$$

In view of (3.255) and property (P16), relation (3.254) holds. By (3.249), (3.251), and (3.254),

$$\begin{aligned}\|T_\gamma(x) - z\| &= \|(1-\gamma)T(x) + \gamma T_*(x) - z\| \\ &\le (1-\gamma)\|T(x) - z\| + \gamma\|T_*(x) - z\| \\ \le (1-\gamma)\|x - z\| + \gamma(\|x-z\| - \delta_k) &= \|x - z\| - \gamma\delta_k. \end{aligned} \tag{3.256}$$

Denote by $\mathcal{U}(T,\gamma,k)$ an open neighborhood of T_γ in $\mathcal{A}$ such that

$$\|S(x) - T_\gamma(x)\| \le 2^{-1}\gamma\delta_k \text{ for all } x \in B(0,k). \tag{3.257}$$

Assume that $x, z \in X$ satisfy (3.255). We have shown that (3.256) holds. By (3.255)–(3.257), for each $S \in \mathcal{U}(T,\gamma,k)$,

$$\|S(x) - z\| \le \|S(x) - T_\gamma(x)\| + \|T_\gamma(x) - z\| \le \|z - x\| - \gamma\delta_k + 2^{-1}\gamma\delta_k.$$

Thus we have shown that the following property holds:

(P17) for each $S \in \mathcal{U}(T,\gamma,k)$, each $x, z \in X$ satisfying (3.255),

$$\|S(x) - z\| \le \|z - x\| - \gamma\delta_k/2.$$

Set

$$\mathcal{F} = \cap_{k=1}^{\infty} \cup \{\mathcal{U}(T,\gamma,k) : \ T \in \mathcal{A},\ \gamma \in (0,1)\}. \tag{3.258}$$

It is not difficult to see that $\mathcal{F}$ is a countable intersection of open everywhere dense subsets of $\mathcal{A}$.

Assume that

$$S \in \mathcal{F}. \tag{3.259}$$

We show that the mapping S is (D)-quasi-contractive.

Let $r, M > 0$. Choose a natural number k such that

$$k > M,\ k > r^{-1}. \tag{3.260}$$

By (3.258) and (3.259), there exist $T_k \in \mathcal{A}$ and $\gamma_k \in (0,1)$ such that

$$S \in \mathcal{U}(T_k, \gamma_k, k). \tag{3.261}$$

Let

$$x \in B(0,M),\ d(x,D) \ge r,\ z \in B(0,M) \cap D. \tag{3.262}$$

In view of (3.260) and (3.262),

$$x \in B(0,k),\ d(x,D) \ge k^{-1},\ z \in B(0,k) \cap D. \tag{3.263}$$

By (3.261), (3.263), property (P17),

$$\|S(x) - z\| \le \|z - x\| - \gamma_k \delta_k / 2.$$

Thus the mapping S is (D)-quasi-contractive. Theorem 3.27 is proved. □

Assume that $C_1, \dots, C_m$ are nonempty subsets of X, $\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_m$, where for each $i \in \{1, \dots, m\}$, $\mathcal{A}_i$ is either $\mathcal{M}_{C_i}$ or $\mathcal{M}_{C_i,c}$ or $\mathcal{M}_{C_i,n}$. The space $\mathcal{A}$ is equipped with the product topology.

Assume that for every $i \in \{1, \dots, m\}$, the space $\mathcal{A}_i$ contains a (C_i)-quasi-contractive mapping. Theorem 3.27 implies that there exists a set $\mathcal{F} \subset \mathcal{A}$ which is a countable intersection of open everywhere dense subsets of $\mathcal{A}$ such that for every $(T_1, \dots, T_m) \in \mathcal{F}$, T_i is (C_i)-quasi-contractive for all $i = 1, \dots, m$, and therefore $(T_1, \dots, T_m)$ satisfies (A2).

Example 3.28. Let X be a Hilbert space, $D \subset X$ be a nonempty closed convex set. Then for each $x \in X$ there exists a unique point $P_D(x) \in D$ which satisfies

$$\|x - P_D(x)\| = \inf\{\|x - y\| : y \in D\}$$

and P_D is (D)-quasi-contractive.

Chapter 4
Dynamic String-Averaging Methods in Normed Spaces

In this chapter we study the convergence of dynamic string-averaging methods for solving common fixed point problems in a normed space. Our main goal is to obtain an approximate solution of the problem in the presence of computational errors. We show that our dynamic string-averaging algorithm generates a good approximate solution, if the sequence of computational errors is bounded from above by a constant.

4.1 Preliminaries and the First Problem

Let $(X, \|\cdot\|)$ be a normed space. For each $x \in X$ and each nonempty set $E \subset X$ put

$$d(x, E) = \inf\{\|x - y\| : \ y \in E\}.$$

For each $x \in X$ and each $r > 0$ set

$$B(x, r) = \{y \in X : \ \|x - y\| \le r\}.$$

Suppose that m is a natural number, $P_i : X \to X$, $i = 1, \dots, m$, for every $i \in \{1, \dots, m\}$,

$$\mathrm{Fix}(P_i) := \{z \in X : \ P_i(z) = z\} \neq \emptyset.$$

Set

$$F = \cap_{i=1}^{m} \mathrm{Fix}(P_i). \tag{4.1}$$

A. Zaslavski, *Approximate Solutions of Common Fixed-Point Problems*, Springer Optimization and Its Applications 112, DOI 10.1007/978-3-319-33255-0_4

For every $\epsilon > 0$ and every $i \in \{1, \dots, m\}$ put

$$F_\epsilon(P_i) = \{x \in X : \ \|x - P_i(x)\| \le \epsilon\}, \tag{4.2}$$

$$\tilde{F}_\epsilon(P_i) = \{y \in X : \ d(y, F_\epsilon(P_i)) \le \epsilon\},$$

$$F_\epsilon = \cap_{i=1}^m F_\epsilon(P_i), \ \tilde{F}_\epsilon = \cap_{i=1}^m \tilde{F}_\epsilon(P_i). \tag{4.3}$$

A point belonging to the set F is a solution of our common fixed point problem while a point which belongs to the set $\tilde{F}_\epsilon$ is its ϵ-approximate solution.

We apply a dynamic string-averaging method with variable strings and weights in order to obtain a good approximative solution of the common fixed point problem.

Suppose that

$$F \neq \emptyset. \tag{4.4}$$

and that the following assumption holds.

(A1) For each $M > 0$ and each $\gamma > 0$ there exists $\delta > 0$ such that for each $i \in \{1, \dots, m\}$, each

$$z \in B(0, M) \cap \mathrm{Fix}(P_i)$$

and each $x \in B(0, M)$ satisfying $\|x - P_i(x)\| \ge \gamma$,

$$\|z - P_i(x)\| \le \|z - x\| - \delta.$$

The results of this section are a generalization of results of Sect. 3.4 for dynamic string-averaging methods.

Next we describe the dynamic string-averaging method with variable strings and weights.

By an index vector, we mean a vector $t = (t_1, \dots, t_p)$ such that $t_i \in \{1, \dots, m\}$ for all $i = 1, \dots, p$.

For an index vector $t = (t_1, \dots, t_q)$ set

$$p(t) = q, \ P[t] = P_{t_q} \cdots P_{t_1}. \tag{4.5}$$

It is easy to see that for each index vector t

$$P[t](x) = x \text{ for all } x \in F, \tag{4.6}$$

$$\|P[t](x) - P[t](y)\| = \|(x) - P[t](y)\| \le \|x - y\| \tag{4.7}$$

for every $x \in F$ and every $y \in X$.

Denote by $\mathcal{M}$ the collection of all pairs (Ω, w), where Ω is a finite set of index vectors and

$$w : \Omega \to (0, \infty) \text{ be such that } \sum_{t \in \Omega} w(t) = 1. \tag{4.8}$$

Let $(\Omega, w) \in \mathcal{M}$. Define

$$P_{\Omega,w}(x) = \sum_{t\in\Omega} w(t)P[t](x),\ x \in X. \tag{4.9}$$

It is not difficult to see that

$$P_{\Omega,w}(x) = x \text{ for all } x \in F, \tag{4.10}$$

$$\|P_{\Omega,w}(x) - P_{\Omega,w}(y)\| = \|x - P_{\Omega,w}(y)\| \le \|x - y\| \tag{4.11}$$

for all $x \in F$ and all $y \in X$.

The dynamic string-averaging method with variable strings and variable weights can now be described by the following algorithm (see Chap. 2).

Initialization: select an arbitrary $x_0 \in X$.

Iterative step: given a current iteration vector x_k pick a pair

$$(\Omega_{k+1}, w_{k+1}) \in \mathcal{M}$$

and calculate the next iteration vector x_{k+1} by

$$x_{k+1} = P_{\Omega_{k+1},w_{k+1}}(x_k).$$

Fix a number

$$\Delta \in (0, m^{-1}] \tag{4.12}$$

and an integer

$$\bar{q} \ge m. \tag{4.13}$$

Denote by $\mathcal{M}_*$ the set of all $(\Omega, w) \in \mathcal{M}$ such that

$$p(t) \le \bar{q} \text{ for all } t \in \Omega, \tag{4.14}$$

$$w(t) \ge \Delta \text{ for all } t \in \Omega. \tag{4.15}$$

Fix a natural number $\bar{N}$.

In the studies of the common fixed point problem the goal is to find a point $x \in F$. In order to meet this goal we apply an algorithm generated by

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$$

such that for each natural number j,

$$\{1, \dots, m\} \subset \cup_{i=j}^{j+\bar{N}-1}(\cup_{t\in\Omega_i}\{t_1, \dots, t_{p(t)}\}).$$

This algorithm generates, for any starting point $x_0 \in X$, a sequence $\{x_k\}_{k=0}^{\infty} \subset X$, where

$$x_{k+1} = P_{\Omega_{k+1},w_{k+1}}(x_k),\ k = 0, 1, \ldots .$$

In order to state the main result of this section we need the following definitions. Let $\delta \geq 0$, $x \in X$ and let $t = (t_1, \ldots, t_{p(t)})$ be an index vector. Define

$$\begin{aligned} A_0(x,t,\delta) = \{(y,\lambda) \in X \times R^1 :\ & \text{there is a sequence } \{y_i\}_{i=0}^{p(t)} \subset X \text{ such that} \\ & y_0 = x \text{ and for all } i = 1,\ldots,p(t), \\ & \|y_i - P_{t_i}(y_{i-1})\| \leq \delta, \\ & y = y_{p(t)}, \\ & \lambda = \max\{\|y_i - y_{i-1}\| :\ i = 1,\ldots,p(t)\}\}. \end{aligned} \tag{4.16}$$

Let $\delta \geq 0$, $x \in X$ and let $(\Omega, w) \in \mathcal{M}$. Define

$$\begin{aligned} A(x,(\Omega,w),\delta) = \{(y,\lambda) \in X \times R^1 :\ & \text{there exist} \\ & (y_t,\lambda_t) \in A_0(x,t,\delta),\ t \in \Omega \text{ such that} \\ & \|y - \sum_{t\in\Omega} w(t)y_t\| \leq \delta,\ \lambda = \max\{\lambda_t :\ t \in \Omega\}\}. \end{aligned} \tag{4.17}$$

Denote by Card(A) the cardinality of a set A. Suppose that the sum over empty set is zero. The following result is proved in Sect. 4.2.

Theorem 4.1. *Let $M > 0$ satisfy*

$$B(0,M) \cap F \neq \emptyset, \tag{4.18}$$

$\epsilon \in (0,1)$,

$$\epsilon_0 = \epsilon(\bar{N}+2)^{-1}(\bar{q}+1)^{-1} \tag{4.19}$$

and let $\gamma \in (0,\epsilon_0]$ be such that the following property holds:

(P1) for each $i \in \{1,\ldots,m\}$, each

$$z \in B(0,3M+1) \cap Fix(P_i)$$

and each $x \in B(0,3M+1)$ satisfying $\|x - P_i(x)\| \geq \epsilon_0/2$,

$$\|z - P_i(x)\| \leq \|z - x\| - \gamma.$$

Let

$$n_0 = \lfloor 4M(\Delta\gamma)^{-1} \rfloor + 1, \tag{4.20}$$

and $\delta > 0$ *satisfy*

$$\delta \le (2(\bar{q}+1)\bar{N})^{-1}\Delta\gamma. \tag{4.21}$$

Assume that

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_* \tag{4.22}$$

satisfies for each natural number j

$$\{1,\dots,m\} \subset \cup_{i=j}^{j+\bar{N}-1}(\cup_{t\in\Omega_i}\{t_1,\dots,t_{p(t)}\}), \tag{4.23}$$

$$x_0 \in B(0,M) \text{ and } \{x_i\}_{i=1}^{\infty} \subset X,\ \{\lambda_i\}_{i=1}^{\infty} \subset [0,\infty) \tag{4.24}$$

satisfy for each natural number i,

$$(x_i,\lambda_i) \in A(x_{i-1},(\Omega_i,w_i),\delta). \tag{4.25}$$

Then there exists an integer $q \in [0, n_0-1]$ *such that*

$$\|x_i\| \le 3M+1,\ i=0,\dots,q\bar{N}, \tag{4.26}$$

$$\lambda_i \le \epsilon_0,\ i=q\bar{N}+1,\dots,(q+1)\bar{N}. \tag{4.27}$$

Moreover, if an integer $q \in [0,n_0-1]$ *satisfies* (4.27)*, then for each* $i = q\bar{N},\dots,(q+1)\bar{N}$,

$$x_i \in \tilde{F}_\epsilon$$

and

$$\|x_i - x_j\| \le \epsilon_0(\bar{q}+1)\bar{N} \le \epsilon$$

for each $i,j \in \{q\bar{N},\dots,(q+1)\bar{N}\}$.

Theorem 4.2. *Suppose that* $\bar{\epsilon} \in (0,1)$, $\bar{M} > 0$,

$$\tilde{F}_{\bar{\epsilon}} \subset B(0,\bar{M}). \tag{4.28}$$

Let $\epsilon \in (0,1)$, $M > \bar{M}$, ϵ_0 *satisfies* (4.19)*,* $\gamma \in (0,\epsilon_0]$ *be such that property (P1) (see Theorem 4.1) holds, a natural number* n_0 *satisfy* (4.20) *and a positive number* δ *satisfy* (4.21).

Assume that

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$$

satisfies for each natural number j

$$\{1,\dots,m\} \subset \cup_{i=j}^{j+\bar{N}-1}(\cup_{t\in\Omega_i}\{t_1,\dots,t_{p(t)}\}),$$

$$x_0 \in B(0,M) \text{ and } \{x_i\}_{i=1}^{\infty} \subset X,\ \{\lambda_i\}_{i=1}^{\infty} \subset [0,\infty)$$

satisfy for each natural number i,

$$(x_i,\lambda_i) \in A(x_{i-1},(\Omega_i,w_i),\delta).$$

Then

$$\|x_i\| \le 3M+1 \text{ for all integers } i \ge 0$$

and there exists a strictly increasing sequence of integers $\{q_p\}_{p=0}^{\infty}$ *such that*

$$0 \le q_0 \le n_0,$$

$$1 \le q_{p+1}-q_p \le n_0 \text{ for all integers } p \ge 0$$

and that for each integer $p \ge 0$ *and each* $i = (q_p-1)\bar{N},\dots,q_p\bar{N}$*,*

$$x_i \in \tilde{F}_\epsilon.$$

Theorem 4.2 is proved applying by induction Theorem 4.1 and using (4.28). The next result is proved in Sect. 4.3.

Theorem 4.3. *Let* $M>0$ *satisfy*

$$B(0,M)\cap F \ne \emptyset$$

and let $\epsilon \in (0,1)$. *Then there exists a constant* $Q>0$ *such that for each*

$$\{(\Omega_i,w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$$

which satisfies for each natural number j,

$$\{1,\dots,m\} \subset \cup_{i=j}^{j+\bar{N}-1}(\cup_{t\in\Omega_i}\{t_1,\dots,t_{p(t)}\}),$$

each $x_0 \in B(0,M)$ *and each pair of sequences* $\{x_i\}_{i=1}^{\infty} \subset X,\ \{\lambda_i\}_{i=1}^{\infty} \subset [0,\infty)$ *satisfying for each natural number i,*

$$(x_i,\lambda_i) \in A(x_{i-1},(\Omega_i,w_i),0)$$

the inequality

$$Card(\{i \in \{0,1,\dots\} :\ x_i \notin \tilde{F}_\epsilon\}) \le Q$$

holds.

4.2 Proof of Theorem 4.1

By (4.18) there exists

$$z \in B(0, M) \cap F. \tag{4.29}$$

Assume that a nonnegative integer s satisfies for each integer $k \in [0, s]$,

$$\max\{\lambda_i : \ i = k\bar{N} + 1, \dots, (k+1)\bar{N}\} > \epsilon_0. \tag{4.30}$$

By (4.24) and (4.29),

$$\|x_0 - z\| \le 2M. \tag{4.31}$$

Assume that an integer $k \in [0, s]$ satisfies

$$\|x_{k\bar{N}} - z\| \le 2M. \tag{4.32}$$

We prove the following auxiliary result.

Lemma 4.4. *Assume that an integer*

$$i \in [0, \bar{N} - 1] \tag{4.33}$$

satisfies

$$\|x_{k\bar{N}+i} - z\| \le 2M + i\delta(\bar{q} + 1). \tag{4.34}$$

Then

$$\|x_{k\bar{N}+i+1} - z\| \le \|x_{k\bar{N}+i} - z\| + \delta(\bar{q} + 1) \tag{4.35}$$

and if $\lambda_{k\bar{N}+i+1} > \epsilon_0$*, then*

$$\|x_{k\bar{N}+i+1} - z\| - \|x_{k\bar{N}+i} - z\| \le -\Delta\gamma + \delta(\bar{q} + 1). \tag{4.36}$$

Proof. In view of (4.25),

$$(x_{k\bar{N}+i+1}, \lambda_{k\bar{N}+i+1}) \in A(x_{k\bar{N}+i}, (\Omega_{k\bar{N}+i+1}, w_{k\bar{N}+i+1}), \delta). \tag{4.37}$$

By (4.17) and (4.37) there exist

$$(y_t, \alpha_t) \in A_0(x_{k\bar{N}+i}, t, \delta), \ \ t \in \Omega_{k\bar{N}+i+1} \tag{4.38}$$

such that

$$\|x_{k\bar{N}+i+1} - \sum_{t \in \Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(t) y_t\| \le \delta, \tag{4.39}$$

$$\lambda_{k\bar{N}+i+1} = \max\{\alpha_t : \ t \in \Omega_{k\bar{N}+i+1}\}. \tag{4.40}$$

It follows from (4.16) and (4.38) that for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k\bar{N}+i+1}$ there exists a finite sequence $\{y_i^{(t)}\}_{i=0}^{p(t)} \subset X$ such that

$$y_0^{(t)} = x_{k\bar{N}+i},\ y_{p(t)}^{(t)} = y_t, \tag{4.41}$$

$$\|y_j^{(t)} - P_{t_j}(y_{j-1}^{(t)})\| \le \delta \text{ for each integer } j = 1, \dots, p(t), \tag{4.42}$$

$$\alpha_t = \max\{\|y_{i+1}^{(t)} - y_i^{(t)}\| :\ i = 0, \dots, p(t) - 1\}. \tag{4.43}$$

Let

$$t = (t_1, \dots, t_{p(t)}) \in \Omega_{k\bar{N}+i+1}$$

and

$$j \in \{1, \dots, p(t)\}. \tag{4.44}$$

It follows from (4.29), (4.42) and (A1) that

$$\|z - y_j^{(t)}\| \le \|z - P_{t_j}(y_{j-1}^{(t)})\| + \|P_{t_j}(y_{j-1}^{(t)}) - y_j^{(t)}\|$$
$$\le \|z - y_{j-1}^{(t)}\| + \delta.$$

Thus we have shown that the following property holds:

(P2) for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k\bar{N}+i+1}$ and each $j \in \{1, \dots, p(t)\}$,

$$\|z - y_j^{(t)}\| \le \|z - y_{j-1}^{(t)}\| + \delta.$$

By (P2), (4.14), and (4.41), the following property holds:

(P3) for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k\bar{N}+i+1}$,

$$\|z - y_t\| = \|z - y_{p(t)}^{(t)}\| \le \|z - y_0^{(t)}\| + p(t)\delta \le \|x_{k\bar{N}+i} - z\| + \bar{q}\delta.$$

It follows from (4.8), (4.39), and property (P3) that

$$\|z - x_{k\bar{N}+i+1}\| \le \|z - \sum_{t \in \Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(t) y_t\|$$
$$+ \|\sum_{t \in \Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(t) y_t - x_{k\bar{N}+i+1}\|$$
$$\le \sum_{t \in \Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(t) \|z - y_t\| + \delta$$
$$\le \|z - x_{k\bar{N}+i}\| + (\bar{q} + 1)\delta$$

and (4.35) holds.

Assume that

$$\lambda_{k\bar{N}+i+1} > \epsilon_0. \tag{4.45}$$

In view of (4.40) and (4.45) there exists

$$s = (s_1, \ldots, s_{p(s)}) \in \Omega_{k\bar{N}+i+1} \tag{4.46}$$

such that

$$\alpha_s = \lambda_{k\bar{N}+i+1} > \epsilon_0. \tag{4.47}$$

By (4.43) and (4.47), there exists

$$j_0 \in \{1, \ldots, p(s)\} \tag{4.48}$$

such that

$$\|y_{j_0}^{(s)} - y_{j_0-1}^{(s)}\| = \alpha_s > \epsilon_0. \tag{4.49}$$

In view of (4.21), (4.42), and (4.49),

$$\|y_{j_0-1}^{(s)} - P_{s_{j_0}}(y_{j_0-1}^{(s)})\| \geq \|y_{j_0-1}^{(s)} - y_{j_0}^{(s)}\| - \|y_{j_0}^{(s)} - P_{s_{j_0}}(y_{j_0-1}^{(s)})\| > \epsilon_0 - \delta \geq \epsilon_0/2. \tag{4.50}$$

By (4.14), (4.21), (4.29), (4.32), (4.33), (4.41), and property (P2), for each $i \in \{0, 1, \ldots, p(s)\}$,

$$\begin{aligned} \|y_i^{(s)}\| &\leq \|y_i^{(s)} - z\| + \|z\| \leq M + \|y_i^{(s)} - z\| \\ &\leq M + \|y_0^{(s)} - z\| + i\delta \leq M + \|x_{k\bar{N}+i} - z\| + \bar{q}\delta \\ &\leq 3M + (\bar{q}+1)\delta(i+1) \leq 3M + \delta(\bar{q}+1)\bar{N} \leq 3M + 1. \end{aligned}$$

Thus

$$\|y_{j_0-1}^{(s)}\| \leq 3M + 1. \tag{4.51}$$

It follows from (4.29), (4.50), (4.51), and property (P1) that

$$\|z - P_{s_{j_0}}(y_{j_0-1}^{(s)})\| \leq \|z - y_{j_0-1}^{(s)}\| - \gamma. \tag{4.52}$$

Relations (4.42) and (4.52) imply that

$$\begin{aligned} \|z - y_{j_0}^{(s)}\| &\leq \|z - P_{s_{j_0}}(y_{j_0-1}^{(s)})\| + \|P_{s_{j_0}}(y_{j_0-1}^{(s)}) - y_{j_0}^{(s)}\| \\ &\leq \|z - y_{j_0-1}^{(s)}\| - \gamma + \delta. \end{aligned} \tag{4.53}$$

By (4.14), (4.41), (4.48), (4.53), and property (P2),

$$\begin{aligned}\|z-y_s\|-\|z-x_{k\bar{N}+i}\| &= \|z-y^{(s)}_{p(s)}\|-\|z-y^{(s)}_0\| \\ &= \sum_{i=0}^{p(s)-1}[\|z-y^{(s)}_{i+1}\|-\|z-y^{(s)}_i\|] \\ &\le \delta(p(s)-1)-\gamma+\delta=-\gamma+\delta p(s)\le \gamma+\delta\bar{q}.\end{aligned}$$

Thus

$$\|z-y_s\|\le\|z-x_{k\bar{N}+i}\|-\gamma+\delta\bar{q}. \tag{4.54}$$

It follows from (4.8), (4.15), (4.39), (4.46), (4.54), and property (P3) that

$$\begin{aligned}\|z-x_{k\bar{N}+i+1}\| &\le \|z-\sum_{t\in\Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(t)y_t\| \\ &+\|\sum_{t\in\Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(t)y_t-x_{k\bar{N}+i+1}\| \\ &\le \sum_{t\in\Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(t)\|z-y_t\|+\delta \\ =\delta+\sum\{w_{k\bar{N}+i+1}(t)\|z-y_t\| &: \ t\in\Omega_{k\bar{N}+i+1}\setminus\{s\}\}+w_{k\bar{N}+i+1}(s)\|z-y_s\| \\ \le \delta+\sum\{w_{k\bar{N}+i+1}(t)(\|x_{k\bar{N}+i}-z\| &+\bar{q}\delta): \ t\in\Omega_{k\bar{N}+i+1}\setminus\{s\}\} \\ &+w_{k\bar{N}+i+1}(s)(\|z-x_{k\bar{N}+i}\|-\gamma+\delta\bar{q}) \\ &\le \|z-x_{k\bar{N}+i}\|+(\bar{q}+1)\delta-w_{k\bar{N}+i+1}(s)\gamma \\ &\le \|z-x_{k\bar{N}+i}\|+(\bar{q}+1)\delta-\Delta\gamma.\end{aligned}$$

Thus (4.36) holds. Lemma 4.4 is proved. □

It follows from (4.21), (4.32), and Lemma 4.4 applied by induction that for all $i=0,\dots,\bar{N}-1$,

$$\|x_{k\bar{N}+i+1}-z\|\le\|x_{k\bar{N}+i}-z\|+\delta(\bar{q}+1),$$

$$\|x_{k\bar{N}+i+1}-z\|\le 2M+\delta(\bar{q}+1)(i+1)\le 2M+\delta(\bar{q}+1)\bar{N}\le 2M+1, \tag{4.55}$$

$$\|x_{k\bar{N}+i}-z\|\le 2M+1,\ i=0,\dots,\bar{N}. \tag{4.56}$$

By (4.21), (4.30), and Lemma 4.4,

$$\begin{aligned}&\|x_{(k+1)\bar{N}}-z\|-\|x_{k\bar{N}}-z\| \\ &= \sum_{i=0}^{\bar{N}-1}[\|x_{k\bar{N}+i+1}-z\|-\|x_{k\bar{N}+i}-z\|] \\ &\le \delta(\bar{q}+1)(\bar{N}-1)-\Delta\gamma+\delta(\bar{q}+1)=\bar{N}\delta(\bar{q}+1)-\Delta\gamma\le-\Delta\gamma/2.\end{aligned}$$

Thus we have shown that the following property holds:
(P4) if an integer $k \in [0, s]$ satisfies $\|x_{k\bar{N}} - z\| \le 2M$, then

$$\|x_j - z\| \le 2M + 1,\ j = k\bar{N}, \dots, (k+1)\bar{N},$$
$$\|x_{(k+1)\bar{N}} - z\| - \|x_{k\bar{N}} - z\| \le -\Delta\gamma/2. \tag{4.57}$$

In view of (4.31) and property (P4),

$$\|x_j - z\| \le 2M + 1,\ j = 0, \dots, (s+1)\bar{N} \tag{4.58}$$

and (4.57) holds for all $k = 0, \dots, s$.
By (4.31) and (4.57) holding for all $k = 0, \dots, s$,

$$2^{-1}\Delta\gamma(s+1) \le \sum_{k=0}^{s}[\|x_{k\bar{N}} - z\| - \|x_{(k+1)\bar{N}} - z\|]$$
$$= \|x_0 - z\| - \|x_{(s+1)\bar{N}} - z\| \le \|x_0 - z\| \le 2M,$$
$$s + 1 \le 4M(\Delta\gamma)^{-1}.$$

Thus we have shown that the following property holds:
(P5) If an integer $s \ge 0$ and for each integer $k \in [0, s]$ relation (4.30) holds, then

$$s \le 4M(\Delta\gamma)^{-1} - 1,$$
$$\|x_j - z\| \le 2M + 1,\ j = 0, \dots, (s+1)\bar{N},$$
$$\|x_{k\bar{N}} - z\| \le 2M,\ k = 0, \dots, s+1.$$

By (P5), (4.20), and (4.30), there exists an integer $q \in [0, n_0 - 1]$ such that for each integer k satisfying $0 \le k < q$,

$$\max\{\lambda_i :\ i = k\bar{N} + 1, \dots, (k+1)\bar{N}\} > \epsilon_0,$$
$$\max\{\lambda_i :\ i = q\bar{N} + 1, \dots, (q+1)\bar{N}\} \le \epsilon_0.$$

In view of (4.29), (P5) and the choice of q,

$$\|x_{q\bar{N}} - z\| \le 2M,$$
$$\|x_j - z\| \le 2M + 1,\ j = 0, \dots, q\bar{N},$$
$$\|x_j\| \le 3M + 1,\ j = 0, \dots, q\bar{N}.$$

Assume that an integer $q \in [0, n_0 - 1]$ satisfies

$$\lambda_i \le \epsilon_0,\ i = q\bar{N} + 1, \dots, (q+1)\bar{N}. \tag{4.59}$$

Let

$$j \in \{q\bar{N}, \ldots, (q+1)\bar{N} - 1\}. \tag{4.60}$$

It follows from (4.25) and (4.60) that

$$(x_{j+1}, \lambda_{j+1}) \in A(x_j, (\Omega_{j+1}, w_{j+1}), \delta). \tag{4.61}$$

By (4.17), (4.59), (4.60), and (4.61), there exist

$$(y_t^{(j)}, \alpha_t^{(j)}) \in A_0(x_j, t, \delta),\ t \in \Omega_{j+1} \tag{4.62}$$

such that

$$\|x_{j+1} - \sum_{t \in \Omega_{j+1}} w_{j+1}(t) y_t^{(j)}\| \le \delta, \tag{4.63}$$

$$\epsilon_0 \ge \lambda_{j+1} = \max\{\alpha_t^{(j)} :\ t \in \Omega_{j+1}\}. \tag{4.64}$$

It follows from (4.16), (4.62), and (4.64) that for each $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{j+1}$ there exists a finite sequence $\{y_i^{(t,j)}\}_{i=0}^{p(t)} \subset X$ such that

$$y_0^{(t,j)} = x_j, \tag{4.65}$$

for each integer $i = 1, \ldots, p(t)$,

$$\|y_i^{(t,j)} - P_{t_i}(y_{i-1}^{(t,j)})\| \le \delta, \tag{4.66}$$

$$y_{p(t)}^{(t,j)} = y_t^{(j)}, \tag{4.67}$$

$$\epsilon_0 \ge \alpha_t^{(j)} = \max\{\|y_i^{(t,j)} - y_{i-1}^{(t,j)}\| :\ i = 1, \ldots, p(t)\}. \tag{4.68}$$

By (4.14), (4.65), (4.67), and (4.68), for each $t \in \Omega_{j+1}$ and each integer $i = 1, \ldots, p(t)$,

$$\|x_j - y_i^{(t,j)}\| \le i\epsilon_0 \le \epsilon_0 \bar{q}, \tag{4.69}$$

$$\|x_j - y_t^{(j)}\| \le \epsilon_0 \bar{q}. \tag{4.70}$$

In view of (4.66) and (4.69), for each $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{j+1}$ and each $i = 1, \ldots, p(t)$,

$$\begin{aligned} &\|x_j - P_{t_i}(y_{i-1}^{(t,j)})\| \\ &\le \|x_j - y_i^{(t,j)}\| + \|y_i^{(t,j)} - P_{t_i}(y_{i-1}^{(t,j)})\| \le \epsilon_0 \bar{q} + \delta. \end{aligned} \tag{4.71}$$

It follows from (4.8), (4.63), and (4.70) that

$$\|x_{j+1} - x_j\| \le \|x_{j+1} - \sum_{t \in \Omega_{j+1}} w_{j+1}(t) y_t^{(j)}\|$$

$$+ \|\sum_{t \in \Omega_{j+1}} w_{j+1}(t) y_t^{(j)} - x_j\|$$

$$\le \delta + \sum_{t \in \Omega_{j+1}} w_{j+1}(t) \|y_t^{(j)} - x_j\| \le \delta + \epsilon_0 \bar{q}.$$

Combined with (4.21) this implies that

$$\|x_{j+1} - x_j\| \le \epsilon_0(\bar{q} + 1). \tag{4.72}$$

By (4.19), (4.21), (4.65), (4.69), and (4.71), for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{j+1}$ and each $i = 1, \dots, p(t)$,

$$\|y_{i-1}^{(t,j)} - P_{t_i}(y_{i-1}^{(t,j)})\| \ge \|y_{i-1}^{(t,j)} - x_j\| + \|x_j - P_{t_i}(y_{i-1}^{(t,j)})\| \le 2\epsilon_0 \bar{q} + \delta \le \epsilon_0(2\bar{q} + 1). \tag{4.73}$$

In view of (4.69) and (4.73), for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{j+1}$ and each $i = 1, \dots, p(t)$,

$$x_j \in \tilde{F}_{\epsilon_0(2\bar{q}+1)}(P_{t_i}).$$

Therefore

$$x_j \in \cap\{\cap_{i=1}^{p(t)} \tilde{F}_{\epsilon_0(2\bar{q}+1)}(P_{t_i}) : \ t = (t_1, \dots, t_{p(t)}) \in \Omega_{j+1}\}. \tag{4.74}$$

Clearly, (4.72) and (4.74) hold for all $j = q\bar{N}, \dots, (q+1)\bar{N} - 1$. In view of (4.72), for all $j_1, j_2 \in \{q\bar{N}, \dots, (q+1)\bar{N}\}$,

$$\|x_{j_1} - x_{j_2}\| \le \epsilon_0(\bar{q} + 1)\bar{N}. \tag{4.75}$$

Let $s \in \{1, \dots, m\}$. By (4.23), there exist $j \in \{q\bar{N}, \dots, (q+1)\bar{N} - 1\}$ and $t = (t_1, \dots, t_{p(t)}) \in \Omega_{j+1}$ such that

$$s \in \{t_1, \dots, t_{p(t)}\}.$$

Together with (4.74) this implies that

$$x_j \in \tilde{F}_{\epsilon_0(2\bar{q}+1)}(P_s). \tag{4.76}$$

It follows from (4.75) and (4.76) that for each $i \in \{q\bar{N}, \dots, (q+1)\bar{N}\}$,

$$x_i \in \tilde{F}_{\epsilon_0(\bar{q}+1)(\bar{N}+2)}(P_s).$$

Since the inclusion above holds for every $s \in \{1, \dots, m\}$ we conclude that for each $i \in \{q\bar{N}, \dots, (q+1)\bar{N}\}$,

$$x_i \in \tilde{F}_{\epsilon_0(\bar{q}+1)(\bar{N}+2)} = \tilde{F}_\epsilon$$

(see (4.19)). This completes the proof of Theorem 4.1. □

4.3 Proof of Theorem 4.3

Fix

$$z \in B(0, M) \cap F. \tag{4.77}$$

Set

$$\epsilon_0 = \epsilon(\bar{N}\bar{q})^{-1}. \tag{4.78}$$

By (A1), there exists $\gamma \in (0, 1)$ such that the following property holds:

(P6) for each $i \in \{1, \dots, m\}$, each

$$\xi \in B(0, 3M+1) \cap \mathrm{Fix}(P_i)$$

and each $x \in B(0, 3M+1)$ satisfying $\|x - P_i(x)\| \geq \epsilon_0/2$,

$$\|z - P_i(x)\| \leq \|z - x\| - \gamma.$$

Set

$$Q = 1 + \lfloor 2\bar{N}M(\Delta\gamma)^{-1} \rfloor. \tag{4.79}$$

Assume that

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_* \tag{4.80}$$

satisfies for each natural number j

$$\{1, \dots, m\} \subset \cup_{i=j}^{j+\bar{N}-1}(\cup_{t \in \Omega_i}\{t_1, \dots, t_{p(t)}\}), \tag{4.81}$$

$$x_0 \in B(0, M) \tag{4.82}$$

and $\{x_i\}_{i=1}^{\infty} \subset X,\ \{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty)$ satisfy for each natural number i,

$$(x_i, \lambda_i) \in A(x_{i-1}, (\Omega_i, w_i), 0). \tag{4.83}$$

By (4.16), (4.17) and the relation above, for each integer $i \geq 0$,

$$x_{i+1} = P_{\Omega_{i+1}, w_{i+1}}(x_i). \tag{4.84}$$

In view of (4.77) and (4.82),

$$\|x_0 - z\| \leq 2M.$$

It follows from (4.11), (4.77), (4.84) and the relation above that for every integer $i \geq 0$,

$$\|x_{i+1} - z\| \leq \|x_i - z\| \leq \|x_0 - z\| \leq 2M,\ \|x_{i+1}\| \leq 3M. \tag{4.85}$$

Set

$$E_0 = \{i \in \{0, 1, 2, \dots\} :\ \lambda_{i+1} \geq \epsilon_0/2\}. \tag{4.86}$$

Let $i \geq 0$ be an integer. By (4.17) and (4.83) there exist

$$(y_t, \alpha_t) \in A_0(x_i, t, 0),\ t \in \Omega_{i+1} \tag{4.87}$$

such that

$$x_{i+1} = \sum_{t \in \Omega_{i+1}} w_{i+1}(t) y_t, \tag{4.88}$$

$$\lambda_{i+1} = \max\{\alpha_t :\ t \in \Omega_{i+1}\}. \tag{4.89}$$

It follows from (4.16) and (4.87) that for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}$ there exists a finite sequence $\{y_j^{(t)}\}_{j=0}^{p(t)} \subset X$ such that

$$y_0^{(t)} = x_i,\ y_{p(t)}^{(t)} = y_t,$$
$$y_j^{(t)} = P_{t_j}(y_{j-1}^{(t)}) \text{ for each integer } j = 1, \dots, p(t),$$
$$\alpha_t = \max\{\|y_{j+1}^{(t)} - y_j^{(t)}\| :\ j = 0, \dots, p(t) - 1\}. \tag{4.90}$$

Let

$$t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}.$$

By (4.77), (4.90), and (A1), for each integer $j = 1, \dots, p(t)$,

$$\|y_j^{(t)} - z\| = \|P_{t_j}(y_{j-1}^{(t)}) - z\| \le \|y_{j-1}^{(t)} - z\|,$$
$$\|y_j^{(t)} - z\| \le \|y_0^{(t)} - z\| = \|x_i - z\|, \ \|y_t - z\| \le \|x_i - z\|. \tag{4.91}$$

Assume that

$$\alpha_t \ge \epsilon_0/2.$$

By (4.90) and the relation above, there exists $j_0 \in \{1, \dots, p(t)\}$ such that

$$\|y_{j_0}^{(t)} - y_{j_0-1}^{(t)}\| \ge \epsilon_0/2.$$

Relations (4.90) and the inequality above imply that

$$\|P_{t_{j_0}}(y_{j_0-1}^{(t)}) - y_{j_0-1}^{(t)}\| \ge \epsilon_0/2.$$

In view of (4.77), (4.83), (4.90), (4.91),

$$\|y_{j_0-1}^{(t)}\| \le 3M.$$

It follows from (4.77), (4.90), the relations above and property (P6) that

$$\|z - y_{j_0}^{(t)}\| = \|z - P_{t_{j_0}}(y_{j_0-1}^{(t)})\| \le \|z - y_{j_0-1}^{(t)}\| - \gamma. \tag{4.92}$$

By (4.90) and (4.91),

$$\begin{aligned}\|z - x_i\| - \|z - y_t\| &= \|z - y_0^{(t)}\| - \|z - y_{p(t)}^{(t)}\| \\ &= \sum_{i=0}^{p(t)-1} (\|z - y_i^{(t)}\| - \|z - y_{i+1}^{(t)}\|) \\ &\ge \|z - y_{j_0-1}^{(t)}\| - \|z - y_{j_0}^{(t)}\| \ge \gamma.\end{aligned}$$

Thus we have shown that for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}$,

$$\|y_t - z\| \le \|x_i - z\|, \tag{4.93}$$
$$\text{if } \alpha_t \ge \epsilon_0/2, \text{ then } \|z - x_i\| \ge \|z - y_t\| + \gamma. \tag{4.94}$$

Assume that an integer

$$i \in E_0. \tag{4.95}$$

In view of (4.86) and (4.95),

$$\lambda_{i+1} \geq \epsilon_0/2. \tag{4.96}$$

It follows from (4.89) and (4.96) that there exists

$$s = (s_1, \dots, s_{p(s)}) \in \Omega_{i+1} \tag{4.97}$$

such that

$$\alpha_s = \lambda_{i+1} \geq \epsilon_0/2. \tag{4.98}$$

Relations (4.94), (4.97), and (4.98) imply that

$$\|z - x_i\| - \gamma \geq \|z - y_s\|. \tag{4.99}$$

By (4.8), (4.15), (4.88), (4.93), and (4.99),

$$\begin{aligned}
\|z - x_{i+1}\| &\leq \|z - \sum_{t \in \Omega_{i+1}} w_{i+1}(t) y_t\| \\
&\leq \sum_{t \in \Omega_{i+1}} w_{i+1}(t)\|z - y_t\| \\
&+ \sum \{w_{i+1}(t)\|z - y_t\| : \ t \in \Omega_{i+1} \setminus \{s\}\} + w_{i+1}(s)\|z - y_s\| \\
&\leq \sum \{w_{i+1}(t)\|x_i - z\| : \ t \in \Omega_{i+1} \setminus \{s\}\} + w_{i+1}(s)(\|z - x_i\| - \gamma) \\
&\leq \|z - x_i\| - \Delta\gamma.
\end{aligned}$$

Thus we have shown that the following property holds:

(P7) for each integer $i \in E_0$,

$$\|z - x_{i+1}\| \leq \|z - x_i\| - \Delta\gamma.$$

Let n be a natural number. By (4.83), (4.84), and property (P7),

$$\begin{aligned}
2M \geq \|z - x_0\| \geq \|z - x_0\| - \|z - x_n\| &= \sum_{i=0}^{n-1} (\|z - x_i\| - \|z - x_{i+1}\|) \\
&\geq \sum \{\|z - x_i\| - \|z - x_{i+1}\| : \ i \in \{0, \dots, n-1\} \cap E_0\} \\
&\geq \mathrm{Card}(E_0 \cap \{0, \dots, n-1\})\Delta\gamma, \\
\mathrm{Card}(E_0 \cap \{0, \dots, n-1\}) &\leq 2M(\Delta\gamma)^{-1}.
\end{aligned}$$

Since the relation above holds for every natural number n we conclude that

$$\mathrm{Card}(E_0) \le 2M(\Delta\gamma)^{-1}. \tag{4.100}$$

Assume that an integer $i \ge 1$ and

$$\lambda_{i+1} < \epsilon_0/2. \tag{4.101}$$

In view of (4.83), the inclusion

$$(x_{i+1}, \lambda_{i+1}) \in A(x_i, (\Omega_{i+1}, w_{i+1}), 0) \tag{4.102}$$

is true. By (4.17) and (4.102) there exist

$$(y_t, \alpha_t) \in A_0(x_i, t, 0),\ t \in \Omega_{i+1} \tag{4.103}$$

such that

$$x_{i+1} = \sum_{t\in\Omega_{i+1}} w_{i+1}(t)y_t, \tag{4.104}$$

$$\lambda_{i+1} = \max\{\alpha_t :\ t \in \Omega_{i+1}\}. \tag{4.105}$$

It follows from (4.16) and (4.103) that for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}$ there exists a finite sequence $\{y_j^{(t)}\}_{j=0}^{p(t)} \subset X$ such that

$$y_0^{(t)} = x_i,\ y_{p(t)}^{(t)} = y_t, \tag{4.106}$$

$$y_j^{(t)} = P_{t_j}(y_{j-1}^{(t)}) \text{ for each integer } j = 1, \dots, p(t), \tag{4.107}$$

$$\alpha_t = \max\{\|y_{j+1}^{(t)} - y_j^{(t)}\| :\ j = 0, \dots, p(t) - 1\}. \tag{4.108}$$

Let

$$t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}. \tag{4.109}$$

By (4.101), (4.105), (4.107), (4.108), and (4.109), for each $j = 0, \dots, p(t) - 1$,

$$y_j^{(t)} \in F_{\epsilon_0/2}(P_{t_{j+1}}). \tag{4.110}$$

It follows from (4.101), (4.105), (4.106), (4.108), (4.109), and (4.114) that for each $j = 0, \dots, p(t)$,

$$\|x_i - y_j^{(t)}\| \le j\lambda_{i+1} \le \bar{q}\epsilon_0/2,\ \|x_i - y_t\| \le \bar{q}\epsilon_0/2. \tag{4.111}$$

In view of (4.110) and (4.111), for each $j = 0, \dots, p(t)$ satisfying $j < p(t)$,

$$x_i \in \tilde{F}_{\bar{q}\epsilon_0/2}(P_{t_{j+1}}).$$

Therefore

$$x_i \in \tilde{F}_{\bar{q}\epsilon_0/2}(P_{t_s}) \text{ for all } s = 1, \dots, p(t) \tag{4.112}$$

and

$$\|x_i - y_t\| \le \bar{q}\epsilon_0/2 \tag{4.113}$$

for all $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}$. In view of (4.112),

$$x_i \in \cap\{\tilde{F}_{\bar{q}\epsilon_0/2}(P_s) : \ s \in \cup_{t\in\Omega_{i+1}}\{t_1, \dots, t_{p(t)}\}\}. \tag{4.114}$$

It follows from (4.8), (4.104), and (4.113) that

$$\|x_i - x_{i+1}\| = \|x_i - \sum_{t\in\Omega_{i+1}} w_{i+1}(t)y_t\|$$

$$\le \sum_{t\in\Omega_{i+1}} w_{i+1}(t)\|x_i - y_t\| \le \bar{q}\epsilon_0/2.$$

Thus we have shown that the following property holds:

(P8) if an integer $i \ge 0$ satisfies $\lambda_{i+1} < \epsilon_0/2$, then (4.114) holds and

$$\|x_{i+1} - x_i\| \le \bar{q}\epsilon_0/2. \tag{4.115}$$

Set

$$E_1 = \{i \in \{0, 1, 2, \dots\} : \ [i, i + \bar{N} - 1] \cap E_0 \ne \emptyset\}. \tag{4.116}$$

By (4.79), (4.100), and (4.116),

$$\mathrm{Card}(E_1) \le \bar{N}\mathrm{Card}(E_0) \le 2M\bar{N}(\Delta\gamma)^{-1} \le Q. \tag{4.117}$$

Assume that a nonnegative integer $j \notin E_1$. In view of (4.116),

$$\{j, \dots, j + \bar{N} - 1\} \cap E_0 = \emptyset.$$

Together with (4.86) and property (P8) this implies that for each $i \in \{j, \dots, j + \bar{N} - 1\}$, the inequality $\lambda_{i+1} < \epsilon_0/2$ is true and (4.114) and (4.115) hold. In view of (4.115) which holds for each $i \in \{j, \dots, j + \bar{N} - 1\}$, for every pair of integers $i_1, i_2 \in \{j, \dots, j + \bar{N}\}$,

$$\|x_{i_1} - x_{i_2}\| \le \bar{N}\bar{q}\epsilon_0/2. \tag{4.118}$$

By (4.81), (4.114), and (4.118),

$$x_j \in \tilde{F}_{\bar{q}\bar{N}\epsilon_0}(P_s),\ s \in \cup_{i=j}^{j+\bar{N}-1}(\cup_{t\in\Omega_{i+1}}\{t_1, \dots, t_{p(t)}\}) = \{1, \dots, m\},$$
$$x_j \in \tilde{F}_\epsilon$$

for all $j \in \{1, 2, \dots, \} \setminus E_1$. Theorem 4.3 is proved. □

4.4 The Second Problem

We use the notation of Sect. 4.1. Let $(X, \|\cdot\|)$ be a normed space. Suppose that m is a natural number, $C_i \subset X$, $i = 1, \dots, m$ be nonempty closed sets,

$$C := \cap_{i=1}^m C_i \ne \emptyset. \tag{4.119}$$

Assume that $P_i : X \to X$, $i = 1, \dots, m$, for every $i \in \{1, \dots, m\}$,

$$P_i(x) = x \text{ for all } x \in C_i. \tag{4.120}$$

Suppose that the following assumptions hold.

(A2) For each $M > 0$ and each $\gamma > 0$ there exists $\delta > 0$ such that for each $i \in \{1, \dots, m\}$, each $x \in B(0, M)$ satisfying $d(x, C_i) \ge \gamma$ and each $z \in B(0, M) \cap C_i$,

$$\|z - P_i(x)\| \le \|z - x\| - \delta.$$

The results of this section are a generalization of results of Sect. 3.7 for string-averaging methods.

Next we describe the dynamic string-averaging method with variable strings and weights.

By an index vector, we a mean a vector $t = (t_1, \dots, t_p)$ such that $t_i \in \{1, \dots, m\}$ for all $i = 1, \dots, p$.

For an index vector $t = (t_1, \dots, t_q)$ set

$$p(t) = q,\ P[t] = P_{t_q} \cdots P_{t_1}. \tag{4.121}$$

It is easy to see that for each index vector t

$$P[t](x) = x \text{ for all } x \in C, \tag{4.122}$$
$$\|P[t](x) - P[t](y)\| = \|(x) - P[t](y)\| \le \|x - y\| \tag{4.123}$$

for every $x \in C$ and every $y \in X$.

Denote by $\mathcal{M}$ the collection of all pairs (Ω, w), where Ω is a finite set of index vectors and

$$w : \Omega \to (0, \infty) \text{ be such that } \sum_{t \in \Omega} w(t) = 1. \tag{4.124}$$

Let $(\Omega, w) \in \mathcal{M}$. Define

$$P_{\Omega,w}(x) = \sum_{t \in \Omega} w(t)P[t](x), \ x \in X. \tag{4.125}$$

It is not difficult to see that

$$P_{\Omega,w}(x) = x \text{ for all } x \in C, \tag{4.126}$$

$$\|P_{\Omega,w}(x) - P_{\Omega,w}(y)\| = \|x - P_{\Omega,w}(y)\| \le \|x - y\| \tag{4.127}$$

for all $x \in C$ and all $y \in X$.

The dynamic string-averaging method with variable strings and variable weights can now be described by the following algorithm (see Chap. 2).

Initialization: select an arbitrary $x_0 \in X$.

Iterative step: given a current iteration vector x_k pick a pair

$$(\Omega_{k+1}, w_{k+1}) \in \mathcal{M}$$

and calculate the next iteration vector x_{k+1} by

$$x_{k+1} = P_{\Omega_{k+1}, w_{k+1}}(x_k).$$

Fix a number

$$\Delta \in (0, m^{-1}] \tag{4.128}$$

and an integer

$$\bar{q} \ge m. \tag{4.129}$$

Denote by $\mathcal{M}_*$ the set of all $(\Omega, w) \in \mathcal{M}$ such that

$$p(t) \le \bar{q} \ \text{ for all } t \in \Omega, \tag{4.130}$$

$$w(t) \ge \Delta \text{ for all } t \in \Omega. \tag{4.131}$$

Fix a natural number $\bar{N}$.

In the studies of the common fixed point problem the goal is to find a point $x \in F$. In order to meet this goal we apply an algorithm generated by

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$$

such that for each natural number j,

$$\{1, \ldots, m\} \subset \cup_{i=j}^{j+\bar{N}-1} (\cup_{t \in \Omega_i} \{t_1, \ldots, t_{p(t)}\}).$$

This algorithm generates, for any starting point $x_0 \in X$, a sequence $\{x_k\}_{k=0}^{\infty} \subset X$, where

$$x_{k+1} = P_{\Omega_{k+1}, w_{k+1}}(x_k).$$

In order to state the main result of this section we need the following definitions. Let $\delta \geq 0$, $x \in X$ and let $t = (t_1, \ldots, t_{p(t)})$ be an index vector. Define

$$\begin{aligned}
&\mathcal{A}_0(x, t, \delta) = \{(y, \lambda, \mu) \in X \times R^1 \times R^1 : \\
&\text{there is a sequence } \{y_i\}_{i=0}^{p(t)} \subset X \text{ such that} \\
&y_0 = x \text{ and for all } i = 1, \ldots, p(t), \\
&\|y_i - P_{t_i}(y_{i-1})\| \leq \delta, \\
&y = y_{p(t)}, \\
&\lambda = \max\{\|y_i - y_{i-1}\| : \ i = 1, \ldots, p(t)\}, \\
&\mu = \max\{d(y_{i-1}, C_{t_i}) : \ i = 1, \ldots, p(t)\}\}.
\end{aligned} \tag{4.132}$$

Let $\delta \geq 0$, $x \in X$ and let $(\Omega, w) \in \mathcal{M}_*$. Define

$$\begin{aligned}
&\mathcal{A}(x, (\Omega, w), \delta) = \{(y, \lambda, \mu) \in X \times R^1 \times R^1 : \ \text{ there exist} \\
&(y_t, \lambda_t, \mu_t) \in \mathcal{A}_0(x, t, \delta), \ t \in \Omega \text{ such that} \\
&\|y - \sum_{t \in \Omega} w(t) y_t\| \leq \delta, \\
&\lambda = \max\{\lambda_t : \ t \in \Omega\}, \\
&\mu = \max\{\mu_t : \ t \in \Omega\}\}.
\end{aligned} \tag{4.133}$$

Denote by Card(A) the cardinality of a set A. Suppose that the sum over empty set is zero. The following result is proved in Sect. 4.5.

Theorem 4.5. *Let $M > 0$ satisfy*

$$B(0, M) \cap C \neq \emptyset, \tag{4.134}$$

$\epsilon \in (0, 1)$,

$$\epsilon_0 = \epsilon(\bar{N} + 1)^{-1}(3\bar{q} + 1)^{-1} \tag{4.135}$$

and let $\gamma \in (0, \epsilon_0)$ *be such that the following property holds:*
(P9) for each $i \in \{1, \dots, m\}$*, each*

$$z \in B(0, 3M + 1) \cap C_i$$

and each $x \in B(0, 3M + 1)$ *satisfying* $d(x, C_i) \geq \epsilon_0/2$,

$$\|z - P_i(x)\| \leq \|z - x\| - \gamma.$$

Let

$$n_0 = \lfloor 4M(\Delta\gamma)^{-1} \rfloor + 1, \tag{4.136}$$

and $\delta > 0$ *satisfy*

$$\delta \leq (2(\bar{q} + 1)\bar{N})^{-1}\Delta\gamma. \tag{4.137}$$

Assume that

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_* \tag{4.138}$$

satisfies for each natural number j

$$\{1, \dots, m\} \subset \cup_{i=j}^{j+\bar{N}-1}(\cup_{t \in \Omega_i}\{t_1, \dots, t_{p(t)}\}), \tag{4.139}$$

$$x_0 \in B(0, M) \text{ and } \{x_i\}_{i=1}^{\infty} \subset X,\ \{\lambda_i\}_{i=1}^{\infty},\ \{\mu_i\}_{i=1}^{\infty} \subset [0, \infty) \tag{4.140}$$

satisfy for each natural number i,

$$(x_i, \lambda_i, \mu_i) \in \mathcal{A}(x_{i-1}, (\Omega_i, w_i), \delta). \tag{4.141}$$

Then there exists an integer $q \in [0, n_0 - 1]$ *such that*

$$\|x_i\| \leq 3M + 1,\ i = 0, \dots, q\bar{N}, \tag{4.142}$$

$$\mu_i \leq \epsilon_0,\ i = q\bar{N} + 1, \dots, (q + 1)\bar{N}. \tag{4.143}$$

Moreover, if an integer $q \in [0, n_0 - 1]$ *satisfies* (4.143)*, then for each* $i = q\bar{N}, \dots, (q + 1)\bar{N}$,

$$d(x_i, C_s) \leq \epsilon \text{ for all } s = 1, \dots, m$$

and

$$\|x_i - x_j\| \le \epsilon$$

for each $i, j \in \{q\bar{N}, \ldots, (q+1)\bar{N}\}$.

Note that γ in Theorem 4.5 exists by (A2).

The next theorem is proved in Sect. 4.6.

Theorem 4.6. *Let* $M > 0$ *satisfy*

$$B(0, M) \cap C \neq \emptyset, \tag{4.144}$$

$\epsilon \in (0, 1)$ *and*

$$\epsilon_1 \in (0, \epsilon(2\bar{N} + 2)^{-1}(\bar{q} + 1)^{-1}] \tag{4.145}$$

be such that the following property holds:

(P10) for each $i \in \{1, \ldots, m\}$*, each*

$$z \in B(0, 3M + 1) \cap C_i$$

and each $x \in B(0, 3M + 2)$ *satisfying* $d(x, C_i) \ge \epsilon/2$*,*

$$\|z - P_i(x)\| \le \|z - x\| - 2\epsilon_1.$$

Let $\gamma \in (0, \epsilon_1)$ *be such that the following property holds:*

(P11) for each $i \in \{1, \ldots, m\}$*, each*

$$z \in B(0, 3M + 1) \cap C_i$$

and each $x \in B(0, 3M + 1)$ *satisfying* $d(x, C_i) \ge \epsilon_1/4$*,*

$$\|z - P_i(x)\| \le \|z - x\| - \gamma.$$

Let

$$n_0 = \lfloor 4M(\Delta\gamma)^{-1} \rfloor + 1 \tag{4.146}$$

and $\delta > 0$ *satisfy*

$$\delta \le (2(\bar{q} + 1)\bar{N})^{-1}\Delta\gamma. \tag{4.147}$$

Assume that

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_* \tag{4.148}$$

satisfies for each natural number j

$$\{1, \dots, m\} \subset \cup_{i=j}^{j+\bar{N}-1}(\cup_{t \in \Omega_i}\{t_1, \dots, t_{p(t)}\}), \tag{4.149}$$

$$x_0 \in B(0, M) \text{ and } \{x_i\}_{i=1}^{\infty} \subset X,\ \{\lambda_i\}_{i=1}^{\infty},\ \{\mu_i\}_{i=1}^{\infty} \subset [0, \infty) \tag{4.150}$$

satisfy for each natural number i,

$$(x_i, \lambda_i, \mu_i) \in \mathcal{A}(x_{i-1}, (\Omega_i, w_i), \delta). \tag{4.151}$$

Then there exists an integer $q \in [0, n_0 - 1]$ *such that*

$$\|x_i\| \le 3M + 1,\ i = 0, \dots, q\bar{N}, \tag{4.152}$$

$$\lambda_i \le \epsilon_1,\ i = q\bar{N} + 1, \dots, (q+1)\bar{N}. \tag{4.153}$$

Moreover, if an integer $q \in [0, n_0 - 1]$ *satisfies* (4.152) *and* (4.153)*, then for each* $i = q\bar{N}, \dots, (q+1)\bar{N}$,

$$d(x_i, C_s) \le \epsilon \text{ for all } s = 1, \dots, m$$

and

$$\|x_i - x_j\| \le \epsilon$$

for each $i, j \in \{q\bar{N}, \dots, (q+1)\bar{N}\}$.

Note that ϵ_1, γ in Theorem 4.6 exists by (A2).

Applying by induction Theorem 4.5 we obtain the following result.

Theorem 4.7. *Suppose that* $\bar{\epsilon} \in (0, 1)$, $\bar{M} > 0$,

$$\{x \in X :\ d(x, C_s) \le \bar{\epsilon},\ s = 1, \dots, m\} \subset B(0, \bar{M}),$$

$\epsilon \in (0, \bar{\epsilon})$, $\epsilon_0 = \epsilon(N\bar{q})^{-1}$, *and let* $\gamma \in (0, \epsilon_0)$ *be such that property (P9) (see Theorem 4.5) holds,*

$$n_0 = \lfloor 4M(\Delta\gamma)^{-1} \rfloor + 1,$$

and $\delta > 0$ *satisfy*

$$\delta \le (2(\bar{q} + 1)\bar{N})^{-1}\Delta\gamma.$$

Assume that

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$$

satisfies for each natural number j

$$\{1, \dots, m\} \subset \cup_{i=j}^{j+\bar{N}-1} (\cup_{t \in \Omega_i} \{t_1, \dots, t_{p(t)}\}),$$

$$x_0 \in B(0, M) \text{ and } \{x_i\}_{i=1}^{\infty} \subset X, \ \{\lambda_i\}_{i=1}^{\infty}, \{\mu_i\}_{i=1}^{\infty} \subset [0, \infty)$$

satisfy for each natural number i,

$$(x_i, \lambda_i, \mu_i) \in \mathcal{A}(x_{i-1}, (\Omega_i, w_i), \delta).$$

Then

$$\|x_i\| \le 3M + 1 \text{ for all integers } i \ge 0$$

and there exists a strictly increasing sequence of integers $\{q_p\}_{p=0}^{\infty}$ such that

$$0 \le q_0 \le n_0,$$

$$1 \le q_{p+1} - q_p \le n_0 \text{ for all integers } p \ge 0$$

and that for each integer $p \ge 0$ and each $i = (q_p - 1)\bar{N}, \dots, q_p \bar{N}$,

$$d(x_i, C_s) \le \epsilon, \ s = 1, \dots, m.$$

The next result is proved in Sect. 4.7.

Theorem 4.8. *Let $M > 0$ satisfy*

$$B(0, M) \cap C \ne \emptyset$$

and let $\epsilon > 0$. Then there exists a constant $Q > 0$ such that for each

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$$

which satisfies for each natural number j,

$$\{1, \dots, m\} \subset \cup_{i=j}^{j+\bar{N}-1} (\cup_{t \in \Omega_i} \{t_1, \dots, t_{p(t)}\}),$$

each $x_0 \in B(0, M)$ and each triplet of sequences

$$\{x_i\}_{i=1}^{\infty} \subset X, \ \{\lambda_i\}_{i=1}^{\infty}, \ \{\mu_i\}_{i=1}^{\infty} \subset [0, \infty)$$

satisfying for each natural number i,

$$(x_i, \lambda_i, \mu_i) \in \mathcal{A}(x_{i-1}, (\Omega_i, w_i), 0)$$

the inequality

$$Card(\{i \in \{0, 1, \dots\} : \max\{d(x_i, C_s) : s = 1, \dots, m\} > \epsilon\}) \le Q$$

holds.

Theorem 4.6 implies the following result.

Theorem 4.9. *Suppose that the family of sets $\{C_i, \ i = 1, \dots, m\}$ has the bounded regularity property, $M > 0$ satisfies*

$$B(0, M) \cap C \neq \emptyset$$

and that $\epsilon \in (0, 1)$. Then there exist $\delta > 0$ and a natural number n_0 such that for each

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$$

satisfying for each natural number j

$$\{1, \dots, m\} \subset \cup_{i=j}^{j+\bar{N}-1}(\cup_{t\in\Omega_i}\{t_1, \dots, t_{p(t)}\}),$$

each $x_0 \in B(0, M)$ each $\{x_i\}_{i=1}^{\infty} \subset X$ and each pair of sequences

$$\{\lambda_i\}_{i=1}^{\infty}, \{\mu_i\}_{i=1}^{\infty} \subset [0, \infty)$$

satisfying for each natural number i,

$$(x_i, \lambda_i, \mu_i) \in \mathcal{A}(x_{i-1}, (\Omega_i, w_i), \delta)$$

there exists an integer $q \in [1, n_0]$ such that

$$\|x_i\| \le 3M + 1, \ i = 0, \dots, q\bar{N},$$

$$d(x_i, C) \le \epsilon$$

for each $i = (q-1)\bar{N}, \dots, q)\bar{N}$.

Applying by induction Theorem 4.9 we obtain the following result.

Proposition 4.10. *Suppose that the family of sets $\{C_i, \ i = 1, \dots, m\}$ has the bounded regularity property, $\bar{M} > 0$ satisfies*

$$C \subset B(0, \bar{M}),$$

$M > \bar{M} + 1$ and that $\epsilon \in (0, 1)$. Then there exist $\delta > 0$ and a natural number n_0 such that for each

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$$

satisfying for each natural number j

$$\{1,\dots,m\} \subset \cup_{i=j}^{j+\bar{N}-1}(\cup_{t\in\Omega_i}\{t_1,\dots,t_{p(t)}\}),$$

each $x_0 \in B(0,M)$, *each* $\{x_i\}_{i=1}^{\infty} \subset X$ *and each pair of sequences*

$$\{\lambda_i\}_{i=1}^{\infty}, \{\mu_i\}_{i=1}^{\infty} \subset [0,\infty)$$

satisfying for each natural number i,

$$(x_i,\lambda_i,\mu_i) \in \mathcal{A}(x_{i-1},(\Omega_i,w_i),\delta),$$

the inequality $\|x_i\| \le 3M+2$ *holds for all integers* $i \ge 0$ *and there exists a strictly increasing sequence of integers* $\{q_p\}_{p=0}^{\infty}$ *such that*

$$1 \le q_0 \le n_0,$$

$$1 \le q_{p+1} - q_p \le n_0 \text{ for all integers } p \ge 0$$

and that for each integer $p \ge 0$ *and each* $i = (q_p - 1)\bar{N},\dots,q_p\bar{N}$,

$$d(x_i, C) \le \epsilon.$$

The following result is proved analogously to Theorem 2.11, by using Proposition 4.10.

Theorem 4.11. *Suppose that the family of sets* $\{C_i,\ i=1,\dots,m\}$ *has the bounded regularity property,* $\bar{M} > 0$ *satisfies*

$$C \subset B(0,\bar{M}),$$

$M > \bar{M}+1$ *and that* $\epsilon \in (0,1)$. *Then there exist* $\delta > 0$ *and a natural number* n_0 *such that for each*

$$\{(\Omega_i,w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$$

satisfying for each natural number j

$$\{1,\dots,m\} \subset \cup_{i=j}^{j+\bar{N}-1}(\cup_{t\in\Omega_i}\{t_1,\dots,t_{p(t)}\}),$$

each $x_0 \in B(0,M)$, *each* $\{x_i\}_{i=1}^{\infty} \subset X$ *and each pair of sequences*

$$\{\lambda_i\}_{i=1}^{\infty}, \{\mu_i\}_{i=1}^{\infty} \subset [0,\infty)$$

satisfying for each natural number i,

$$(x_i, \lambda_i, \mu_i) \in \mathcal{A}(x_{i-1}, (\Omega_i, w_i), \delta),$$

the inequality $\|x_i\| \leq 3M + 2$ *holds for all integers* $i \geq 0$ *and*

$$d(x_i, C) \leq \epsilon$$

for all integers $i \geq n_0 \bar{N}$.

The next result is easily deduced from Theorem 4.11.

Theorem 4.12. *Suppose that the family* $\{C_i : i = 1, \dots, m\}$ *has the bounded regularity property,* $\bar{M} > 0$ *satisfy*

$$C \subset B(0, \bar{M})$$

and that a sequence $\{\delta_i\}_{i=1}^{\infty} \subset (0, \infty)$ *satisfies*

$$\lim_{i \to \infty} \delta_i = 0$$

and that $\epsilon > 0$. *Then there exist a natural number* k_1 *such that the following assertion holds.*

Assume that

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$$

satisfies for each natural number j the following properties:

$$\{1, \dots, m\} \subset \cup_{i=j}^{j+\bar{N}-1} (\cup_{t \in \Omega_i} \{t_1, \dots, t_{p(t)}\});$$

there exists $i(j) \in \{j, \dots, j + \bar{N} - 1\}$ *such that for each* $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i(j)}$ *there exists* $s \in \{t_1, \dots, t_{p(t)}\}$ *for which*

$$P_s(X) \subset B(0, \bar{M}).$$

Then for each each $\{x_i\}_{i=0}^{\infty} \subset X$ *and each pair of sequences*

$$\{\lambda_i\}_{i=1}^{\infty}, \{\mu_i\}_{i=1}^{\infty} \subset [0, \infty)$$

satisfying for each natural number i,

$$(x_i, \lambda_i, \mu_i) \in \mathcal{A}(x_{i-1}, (\Omega_i, w_i), \delta),$$

the inequality $d(x_i, C) \leq \epsilon$ *holds for all integers* $i \geq k_1$.

4.5 Proof of Theorem 4.5

By (4.134) there exists

$$z \in B(0, M) \cap C. \tag{4.154}$$

Assume that a nonnegative integer s satisfies for each integer $k \in [0, s]$,

$$\max\{\mu_i : \ i = k\bar{N} + 1, \ldots, (k+1)\bar{N}\} > \epsilon_0. \tag{4.155}$$

By (4.140) and (4.154),

$$\|x_0 - z\| \le 2M. \tag{4.156}$$

Assume that an integer $k \in [0, s]$ satisfies

$$\|x_{k\bar{N}} - z\| \le 2M. \tag{4.157}$$

We prove the following auxiliary result.

Lemma 4.13. *Assume that an integer*

$$i \in [0, \bar{N} - 1] \tag{4.158}$$

satisfies

$$\|x_{k\bar{N}+i} - z\| \le 2M + i\delta(\bar{q} + 1). \tag{4.159}$$

Then

$$\|x_{k\bar{N}+i+1} - z\| \le \|x_{k\bar{N}+i} - z\| + \delta(\bar{q} + 1) \tag{4.160}$$

and if

$$\mu_{k\bar{N}+i+1} > \epsilon_0, \tag{4.161}$$

then

$$\|x_{k\bar{N}+i+1} - z\| \le \|x_{k\bar{N}+i} - z\| - \Delta\gamma + \delta(\bar{q} + 1). \tag{4.162}$$

Proof. In view of (4.141),

$$(x_{k\bar{N}+i+1}, \lambda_{k\bar{N}+i+1}, \mu_{k\bar{N}+i+1}) \in \mathcal{A}(x_{k\bar{N}+i}, (\Omega_{k\bar{N}+i+1}, w_{k\bar{N}+i+1}), \delta). \tag{4.163}$$

By (4.133) and (4.163) there exist

$$(y_t, \alpha_t, \beta_t) \in \mathcal{A}_0(x_{k\bar{N}+i}, t, \delta),\ t \in \Omega_{k\bar{N}+i+1} \tag{4.164}$$

such that

$$\|x_{k\bar{N}+i+1} - \sum_{t \in \Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(t) y_t\| \le \delta, \tag{4.165}$$

$$\lambda_{k\bar{N}+i+1} = \max\{\alpha_t :\ t \in \Omega_{k\bar{N}+i+1}\}, \tag{4.166}$$

$$\mu_{k\bar{N}+i+1} = \max\{\beta_t :\ t \in \Omega_{k\bar{N}+i+1}\}. \tag{4.167}$$

It follows from (4.132) and (4.164) that for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k\bar{N}+i+1}$ there exists a finite sequence $\{y_i^{(t)}\}_{i=0}^{p(t)} \subset X$ such that

$$y_0^{(t)} = x_{k\bar{N}+i}, \tag{4.168}$$

$$y_{p(t)}^{(t)} = y_t, \tag{4.169}$$

$$\|y_j^{(t)} - P_{t_j}(y_{j-1}^{(t)})\| \le \delta \text{ for each integer } j = 1, \dots, p(t), \tag{4.170}$$

$$\alpha_t = \max\{\|y_{i+1}^{(t)} - y_i^{(t)}\| :\ i = 0, \dots, p(t) - 1\}, \tag{4.171}$$

$$\beta_t = \max\{d(y_i^{(t)}, C_{t_{i+1}}) :\ i = 0, \dots, p(t) - 1\}. \tag{4.172}$$

Let

$$t = (t_1, \dots, t_{p(t)}) \in \Omega_{k\bar{N}+i+1} \tag{4.173}$$

and

$$j \in \{1, \dots, p(t)\}. \tag{4.174}$$

It follows from (4.154), (4.170), and (A2) that

$$\begin{aligned}\|z - y_j^{(t)}\| &\le \|z - P_{t_j}(y_{j-1}^{(t)})\| + \|P_{t_j}(y_{j-1}^{(t)}) - y_j^{(t)}\| \\ &\le \|z - y_{j-1}^{(t)}\| + \delta.\end{aligned} \tag{4.175}$$

Thus we have shown that for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k\bar{N}+i+1}$ and each $j \in \{1, \dots, p(t)\}$,

$$\|z - y_j^{(t)}\| \le \|z - y_{j-1}^{(t)}\| + \delta. \tag{4.176}$$

By (4.130), (4.168), (4.16), and (4.176), for each

$$t = (t_1, \dots, t_{p(t)}) \in \Omega_{k\bar{N}+i+1}, \tag{4.177}$$

$$\|z - y_t\| = \|z - y_{p(t)}^{(t)}\| \le \|z - y_0^{(t)}\| + p(t)\delta \le \|x_{k\bar{N}+i+1} - z\| + \bar{q}\delta. \tag{4.178}$$

It follows from (4.124), (4.165), and (4.178) that

$$\begin{aligned}\|z - x_{k\bar{N}+i+1}\| &\le \|z - \sum_{t\in\Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(t)y_t\| \\ &+\| \sum_{t\in\Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(t)y_t - x_{k\bar{N}+i+1}\| \\ &\le \sum_{t\in\Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(t)\|z - y_t\| + \delta \\ &\le \|z - x_{k\bar{N}+i}\| + (\bar{q}+1)\delta\end{aligned}$$

and (4.160) holds.

Assume that

$$\mu_{k\bar{N}+i+1} > \epsilon_0. \tag{4.179}$$

In view of (4.167) and (4.179) there exists

$$s = (s_1, \ldots, s_{p(s)}) \in \Omega_{k\bar{N}+i+1} \tag{4.180}$$

such that

$$\beta_s = \mu_{k\bar{N}+i+1} > \epsilon_0. \tag{4.181}$$

By (4.172), (4.180), and (4.181), there exists

$$j_0 \in \{1, \ldots, p(s)\} \tag{4.182}$$

such that

$$d(y_{j_0-1}^{(s)}, C_{s_{j_0}}) = \beta_s > \epsilon_0. \tag{4.183}$$

In view of (4.130), (4.137), (4.154), (4.158), (4.159), (4.168), (4.169), (4.176), (4.180), and (4.182),

$$\begin{aligned}\|y_{j_0-1}^{(s)}\| &\le \|y_{j_0-1}^{(s)} - z\| + \|z\| \\ &\le M + \|y_0^{(s)} - z\| + p(s)\delta \le M + \|x_{k\bar{N}+i} - z\| + \bar{q}\delta \\ &\le 3M + (\bar{q}+1)\delta(i+1) \le 3M + \delta(\bar{q}+1)\bar{N} \le 3M + 1.\end{aligned} \tag{4.184}$$

By (4.154), (4.182), (4.183), (4.184), and property (P9),

$$\|z - P_{s_{j_0}}(y^{(s)}_{j_0-1})\| \le \|z - y^{(s)}_{j_0-1}\| - \gamma. \tag{4.185}$$

Relations (4.170) and (4.185) imply that

$$\begin{aligned}\|z - y^{(s)}_{j_0}\| &\le \|z - P_{s_{j_0}}(y^{(s)}_{j_0-1})\| + \|P_{s_{j_0}}(y^{(s)}_{j_0-1}) - y^{(s)}_{j_0}\| \\ &\le \|z - y^{(s)}_{j_0-1}\| - \gamma + \delta.\end{aligned} \tag{4.186}$$

By (4.130), (4.168), (4.169), (4.176), (4.180), (4.182), and (4.186),

$$\begin{aligned}\|z - y_s\| - \|z - x_{k\bar{N}+i}\| &= \|z - y^{(s)}_{p(s)}\| - \|z - y^{(s)}_0\| \\ &= \sum_{i=0}^{p(s-1)} [\|z - y^{(s)}_{i+1}\| - \|z - y^{(s)}_i\|] \\ &\le \delta(p(s) - 1) - \gamma + \delta = -\gamma + \delta p(s) \le -\gamma + \delta\bar{q}.\end{aligned}$$

Thus

$$\|z - y_s\| \le \|z - x_{k\bar{N}+i}\| - \gamma + \delta\bar{q}. \tag{4.187}$$

It follows from (4.124), (4.131), (4.165), (4.178), (4.180), and (4.187) that

$$\begin{aligned}\|z - x_{k\bar{N}+i+1}\| &\le \|z - \sum_{t\in\Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(t)y_t\| \\ &+\|\sum_{t\in\Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(t)y_t - x_{k\bar{N}+i+1}\| \\ &\le \sum_{t\in\Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(t)\|z - y_t\| + \delta \\ &= \delta + \sum\{w_{k\bar{N}+i+1}(t)\|z - y_t\| : t \in \Omega_{k\bar{N}+i+1} \setminus \{s\}\} + w_{k\bar{N}+i+1}(s)\|z - y_s\| \\ &\le \delta + \sum\{w_{k\bar{N}+i+1}(t)(\|x_{k\bar{N}+i} - z\| + \bar{q}\delta) : t \in \Omega_{k\bar{N}+i+1} \setminus \{s\}\} \\ &+w_{k\bar{N}+i+1}(s)(\|z - x_{k\bar{N}+i}\| - \gamma + \delta\bar{q}) \\ &\le \|z - x_{k\bar{N}+i}\| + (\bar{q}+1)\delta - w_{k\bar{N}+i+1}(s)\gamma \\ &\le \|z - x_{k\bar{N}+i}\| + (\bar{q}+1)\delta - \Delta\gamma.\end{aligned}$$

Thus (4.162) holds. Lemma 4.13 is proved. □

It follows from (4.137), (4.157), and Lemma 4.13 applied by induction that for all $i = 0, \dots, \bar{N} - 1$,

$$\|x_{k\bar{N}+i+1} - z\| \le \|x_{k\bar{N}+i} - z\| + \delta(\bar{q} + 1), \tag{4.188}$$

$$\|x_{k\bar{N}+i+1} - z\| \le 2M + \delta(\bar{q} + 1)(i + 1) \le 2M + \delta(\bar{q} + 1)\bar{N} \le 2M + 1, \tag{4.189}$$

$$\|x_{k\bar{N}+i} - z\| \le 2M + 1, \ i = 0, \dots, \bar{N}. \tag{4.190}$$

By (4.137), (4.155), (4.188), and Lemma 4.13,

$$\begin{aligned} &\|x_{(k+1)\bar{N}} - z\| - \|x_{k\bar{N}} - z\| \\ &= \sum_{i=0}^{\bar{N}-1} [\|x_{k\bar{N}+i+1} - z\| - \|x_{k\bar{N}+i} - z\|] \\ &\le \delta(\bar{q} + 1)(\bar{N} - 1) - \Delta\gamma + \delta(\bar{q} + 1) = \bar{N}\delta(\bar{q} + 1) - \Delta\gamma \le -\Delta\gamma/2. \end{aligned}$$

Thus we have shown that the following property holds:

(P12) if an integer $k \in [0, s]$ satisfies $\|x_{k\bar{N}} - z\| \le 2M$, then

$$\|x_j - z\| \le 2M + 1, \ j = k\bar{N}, \dots, (k + 1)\bar{N},$$

$$\|x_{(k+1)\bar{N}} - z\| - \|x_{k\bar{N}} - z\| \le -\Delta\gamma/2. \tag{4.191}$$

In view of (4.156) and property (P12),

$$\|x_j - z\| \le 2M + 1, \ j = 0, \dots, (s + 1)\bar{N} \tag{4.192}$$

and (4.191) holds for all $k = 0, \dots, s$.

By (4.156) and (4.191) holding for all $k = 0, \dots, s$,

$$\begin{aligned} 2^{-1}\Delta\gamma(s + 1) &\le \sum_{k=0}^{s} [\|x_{k\bar{N}} - z\| - \|x_{(k+1)\bar{N}} - z\|] \\ &= \|x_0 - z\| - \|x_{(s+1)\bar{N}} - z\| \le \|x_0 - z\| \le 2M, \\ s + 1 &\le 4M(\Delta\gamma)^{-1}. \end{aligned}$$

Thus we have shown that the following property holds:

(P13) If an integer $s \ge 0$ and for each integer $k \in [0, s]$ relation (4.155) holds, then

$$s \le 4M(\Delta\gamma)^{-1} - 1,$$

$$\|x_j - z\| \le 2M + 1, \ j = 0, \dots, (s + 1)\bar{N},$$

$$\|x_{k\bar{N}} - z\| \le 2M, \ k = 0, \dots, s + 1.$$

By (P13) and (4.136), there exists an integer $q \in [0, n_0 - 1]$ such that for each integer k satisfying $0 \le k < q$,

$$\max\{\mu_i : i = k\bar{N} + 1, \dots, (k+1)\bar{N}\} > \epsilon_0,$$

$$\max\{\mu_i : i = q\bar{N} + 1, \dots, (q+1)\bar{N}\} \le \epsilon_0.$$

In view of (4.154), (P13) and the choice of q,

$$\|x_{q\bar{N}} - z\| \le 2M,$$

$$\|x_j - z\| \le 2M + 1, \ j = 0, \dots, q\bar{N},$$

$$\|x_j\| \le 3M + 1, \ j = 0, \dots, q\bar{N}.$$

Assume that an integer $q \in [0, n_0 - 1]$ satisfies

$$\mu_i \le \epsilon_0, \ i = q\bar{N} + 1, \dots, (q+1)\bar{N}. \tag{4.193}$$

Let

$$j \in \{q\bar{N}, \dots, (q+1)\bar{N} - 1\}. \tag{4.194}$$

It follows from (4.141) and (4.194) that,

$$(x_{j+1}, \lambda_{j+1}, \mu_{j+1}) \in \mathcal{A}(x_j, (\Omega_{j+1}, w_{j+1}), \delta). \tag{4.195}$$

By (4.133), (4.194), and (4.195), there exist

$$(y_t^{(j)}, \alpha_t^{(j)}, \beta_t^{(j)}) \in \mathcal{A}_0(x_j, t, \delta), \ t \in \Omega_{j+1} \tag{4.196}$$

such that

$$\|x_{j+1} - \sum_{t \in \Omega_{j+1}} w_{j+1}(t) y_t^{(j)}\| \le \delta, \tag{4.197}$$

$$\lambda_{j+1} = \max\{\alpha_t^{(j)} : t \in \Omega_{j+1}\}, \ \mu_{j+1} = \max\{\beta_t^{(j)} : t \in \Omega_{j+1}\}. \tag{4.198}$$

In view of (4.193), (4.194), and (4.198),

$$\max\{\beta_t^{(j)} : t \in \Omega_{j+1}\} \le \epsilon_0. \tag{4.199}$$

It follows from (4.132) and (4.196) that for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{j+1}$ there exists a finite sequence $\{y_i^{(t,j)}\}_{i=0}^{p(t)} \subset X$ such that

$$y_0^{(t,j)} = x_j, \ y_{p(t)}^{(t,j)} = y_t^{(j)}, \tag{4.200}$$

for each integer $i = 1, \dots, p(t)$,

$$\|y_i^{(t,j)} - P_{t_i}(y_{i-1}^{(t,j)})\| \le \delta, \tag{4.201}$$

$$\alpha_t^{(j)} = \max\{\|y_{i+1}^{(t,j)} - y_i^{(t,j)}\| : i = 0, \dots, p(t) - 1\}, \tag{4.202}$$

$$\beta_t^{(j)} = \max\{d(y_i^{(t,j)}, C_{t_{i+1}}) : i = 0, \dots, p(t) - 1\}. \tag{4.203}$$

By (4.199) and (4.203), for each $t \in \Omega_{j+1}$,

$$\max\{d(y_i^{(t,j)}, C_{t_{i+1}}) : i = 0, \dots, p(t) - 1\} \le \epsilon_0. \tag{4.204}$$

Fix $\kappa > 0$. Let

$$t = (t_1, \dots, t_{p(t)}) \in \Omega_{j+1} \tag{4.205}$$

and

$$j \in \{0, \dots, p(t) - 1\}. \tag{4.206}$$

In view of (4.204), (4.205), and (4.206), there exists

$$\xi \in C_{t_{i+1}} \tag{4.207}$$

such that

$$\|y_i^{(t,j)} - \xi\| < \epsilon_0 + \kappa. \tag{4.208}$$

Relations (4.154), (4.164), and (A2) imply that

$$\|\xi - P_{t_{i+1}}(y_i^{(t,j)})\| < \epsilon_0 + \kappa. \tag{4.209}$$

It follows from (4.137), (4.201), (4.208), and (4.209) that

$$\|y_i^{(t,j)} - y_{i+1}^{(t,j)}\| \le \|y_i^{(t,j)} - \xi\| + \|\xi - P_{t_{i+1}}(y_i^{(t,j)})\|$$
$$+ \|P_{t_{i+1}}(y_i^{(t,j)}) - y_{i+1}^{(t,j)}\| \le 2\epsilon_0 + 2\kappa + \delta \le 3\epsilon_0 + 2\kappa.$$

Since κ is any positive number we conclude that

$$\|y_i^{(t,j)} - y_{i+1}^{(t,j)}\| \le 3\epsilon_0. \tag{4.210}$$

By (4.130), (4.206), and (4.210), for each $i_1, i_2 \in \{0, \dots, p(t)\}$,

$$\|y_{i_1}^{(t,j)} - y_{i_2}^{(t,j)}\| \le 3\epsilon_0(p(t) \le 3\epsilon_0\bar{q}. \tag{4.211}$$

It follows from (4.200), (4.204), and (4.211) that for each $i = 1, \dots, p(t)$,

$$d(x_j, C_{t_i}) \le \epsilon_0(3\bar{q} + 1), \tag{4.212}$$

$$d(y_t^{(j)}, C_{t_i}) \le \epsilon_0(3\bar{q} + 1), \tag{4.213}$$

$$\|x_j - y_t^{(j)}\| \le \epsilon_0(3\bar{q}). \tag{4.214}$$

In view of (4.13), (4.124), (4.137), (4.197), and (4.214),

$$\|x_j - x_{j+1}\| \le \|x_j - \sum_{t \in \Omega_{j+1}} w_{j+1}(t) y_t^{(j)}\|$$

$$+ \|\sum_{t \in \Omega_{j+1}} w_{j+1}(t) y_t^{(j)} - x_{j+1}\|$$

$$\le \sum_{t \in \Omega_{j+1}} w_{j+1}(t) \|x_j - y_t^{(j)}\| + \delta \le \epsilon_0(3\bar{q}) + \delta \le \epsilon_0(3\bar{q} + 1).$$

Thus we have shown that for each $j \in \{q\bar{N}, \dots, (q+1)\bar{N} - 1\}$,

$$\|x_j - x_{j+1}\| \le \epsilon_0(3\bar{q} + 1), \tag{4.215}$$

$$d(x_j, C_{t_i}) \le \epsilon_0(3\bar{q} + 1) \text{ for each } t \in \Omega_{j+1} \text{ and each } i \in \{1, \dots, p(t)\}. \tag{4.216}$$

Relations (4.135) and (4.215) imply that for all $j_1, j_2 \in \{q\bar{N}, \dots, (q+1)\bar{N}\}$,

$$\|x_{j_1} - x_{j_2}\| \le \epsilon_0(3\bar{q} + 1)\bar{N} \le \epsilon. \tag{4.217}$$

Let $s \in \{1, \dots, m\}$. By (4.139), there exist

$$j \in \{q\bar{N}, \dots, (q+1)\bar{N} - 1\} \tag{4.218}$$

and $t = (t_1, \dots, t_{p(t)}) \in \Omega_{j+1}$ such that

$$s \in \{t_1, \dots, t_{p(t)}\}.$$

In view of (4.216),

$$d(x_j, C_s) \le \epsilon_0(3\bar{q} + 1).$$

Together with (4.135), (4.217), and (4.218) this implies that for each $i \in \{q\bar{N}, \dots, (q+1)\bar{N}\}$ and each $s \in \{1, \dots, m\}$,

$$d(x_i, C_s) \le \epsilon_0(3\bar{q} + 1)(\bar{N} + 1) = \epsilon.$$

This completes the proof of Theorem 4.5. □

4.6 Proof of Theorem 4.6

By (4.144) there exists

$$z \in B(0, M) \cap C. \tag{4.219}$$

Assume that a nonnegative integer s satisfies for each integer $k \in [0, s]$,

$$\max\{\lambda_i : \; i = k\bar{N} + 1, \dots, (k+1)\bar{N}\} > \epsilon_1. \tag{4.220}$$

By (4.150) and (4.219),

$$\|x_0 - z\| \le 2M. \tag{4.221}$$

Assume that an integer $k \in [0, s]$ satisfies

$$\|x_{k\bar{N}} - z\| \le 2M. \tag{4.222}$$

We prove the following auxiliary result.

Lemma 4.14. *Assume that an integer*

$$i \in [0, \bar{N} - 1] \tag{4.223}$$

satisfies

$$\|x_{k\bar{N}+i} - z\| \le 2M + i\delta(\bar{q} + 1). \tag{4.224}$$

Then

$$\|x_{k\bar{N}+i+1} - z\| \le \|x_{k\bar{N}+i} - z\| + \delta(\bar{q} + 1) \tag{4.225}$$

and if $\lambda_{k\bar{N}+i+1} > \epsilon_1$, *then*

$$\|x_{k\bar{N}+i+1} - z\| \le \|x_{k\bar{N}+i} - z\| - \Delta\gamma + \delta(\bar{q} + 1). \tag{4.226}$$

Proof. In view of (4.151),

$$(x_{k\bar{N}+i+1}, \lambda_{k\bar{N}+i+1}, \mu_{k\bar{N}+i+1}) \in \mathcal{A}(x_{k\bar{N}+i}, (\Omega_{k\bar{N}+i+1}, w_{k\bar{N}+i+1}), \delta). \tag{4.227}$$

By (4.133) and (4.227) there exist

$$(y_t, \alpha_t, \beta_t) \in \mathcal{A}_0(x_{k\bar{N}+i}, t, \delta), \;\; t \in \Omega_{k\bar{N}+i+1} \tag{4.228}$$

such that

$$\|x_{k\bar{N}+i+1} - \sum_{t \in \Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(t) y_t\| \le \delta, \tag{4.229}$$

$$\lambda_{k\bar{N}+i+1} = \max\{\alpha_t : \ t \in \Omega_{k\bar{N}+i+1}\}, \tag{4.230}$$

$$\mu_{k\bar{N}+i+1} = \max\{\beta_t : \ t \in \Omega_{k\bar{N}+i+1}\}. \tag{4.231}$$

It follows from (4.132) and (4.228) that for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k\bar{N}+i+1}$ there exists a finite sequence $\{y_i^{(t)}\}_{i=0}^{p(t)} \subset X$ such that

$$y_0^{(t)} = x_{k\bar{N}+i}, \ y_{p(t)}^{(t)} = y_t, \tag{4.232}$$

$$\|y_j^{(t)} - P_{t_j}(y_{j-1}^{(t)})\| \le \delta \text{ for each integer } j = 1, \dots, p(t), \tag{4.233}$$

$$\alpha_t = \max\{\|y_{j+1}^{(t)} - y_j^{(t)}\| : \ j = 0, \dots, p(t) - 1\}, \tag{4.234}$$

$$\beta_t = \max\{d(y_j^{(t)}, C_{t_{j+1}}) : \ j = 0, \dots, p(t) - 1\}. \tag{4.235}$$

Let

$$t = (t_1, \dots, t_{p(t)}) \in \Omega_{k\bar{N}+i+1}$$

and

$$j \in \{1, \dots, p(t)\}.$$

It follows from (4.133), (4.219), and (A2) that

$$\begin{aligned} \|z - y_j^{(t)}\| &\le \|z - P_{t_j}(y_{j-1}^{(t)})\| + \|P_{t_j}(y_{j-1}^{(t)}) - y_j^{(t)}\| \\ &\le \|z - y_{j-1}^{(t)}\| + \delta. \end{aligned}$$

Thus we have shown that for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k\bar{N}+i+1}$ and each $j \in \{1, \dots, p(t)\}$,

$$\|z - y_j^{(t)}\| \le \|z - y_{j-1}^{(t)}\| + \delta. \tag{4.236}$$

By (4.130), (4.222), (4.232), and (4.236), for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k\bar{N}+i+1}$ and each $j \in \{0, 1, \dots, p(t)\}$,

$$\|z - y_j^{(t)}\| \le \|z - y_0^{(t)}\| + j\delta \le \|x_{k\bar{N}+i} - z\| + \bar{q}\delta. \tag{4.237}$$

By (4.232) and (4.237), for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k\bar{N}+i+1}$,

$$\|z - y_t\| = \|z - y^{(t)}_{p(t)}\| \leq \|x_{k\bar{N}+i} - z\| + \bar{q}\delta. \tag{4.238}$$

It follows from (4.124), (4.229), and (4.238) that

$$\begin{aligned}\|z - x_{k\bar{N}+i+1}\| &\leq \|z - \sum_{t \in \Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(t) y_t\| \\ &+ \|\sum_{t \in \Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(t) y_t - x_{k\bar{N}+i+1}\| \\ &\leq \sum_{t \in \Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(t)\|z - y_t\| + \delta \\ &\leq \|z - x_{k\bar{N}+i}\| + (\bar{q} + 1)\delta\end{aligned}$$

and (4.225) holds.

Assume that

$$\lambda_{k\bar{N}+i+1} > \epsilon_1. \tag{4.239}$$

In view of (4.230) and (4.239) there exists

$$s = (s_1, \dots, s_{p(s)}) \in \Omega_{k\bar{N}+i+1} \tag{4.240}$$

such that

$$\alpha_s = \lambda_{k\bar{N}+i+1} > \epsilon_1. \tag{4.241}$$

By (4.234), (4.240), and (4.241), there exists

$$j_0 \in \{1, \dots, p(s)\}$$

such that

$$\|y^{(s)}_{j_0-1} - y^{(s)}_{j_0}\| = \alpha_s > \epsilon_1. \tag{4.242}$$

In view of (4.147), (4.223), (4.224), and (4.225),

$$x_{k\bar{N}+i},\ x_{k\bar{N}+i+1} \in B(0, 3M + 1).$$

In view of (4.147), (4.219), (4.223), (4.224), (4.237), and (4.240),

$$y^{(s)}_{j_0-1},\ y^{(s)}_{j_0} \in B(0, 3M + 1). \tag{4.243}$$

We show that

$$d(y_{j_0-1}^{(s)}, C_{s_{j_0}}) \geq \epsilon_1/4. \tag{4.244}$$

Assume the contrary. Then there exists

$$\xi \in C_{s_{j_0}} \tag{4.245}$$

such that

$$\|y_{j_0-1}^{(s)} - \xi\| < \epsilon_1/4. \tag{4.246}$$

Relations (4.233), (4.245), (4.246), and (A2) imply that

$$\begin{aligned} \|y_{j_0}^{(s)} - \xi\| &\leq \|y_{j_0}^{(s)} - P_{s_{j_0}}(y_{j_0-1}^{(s)})\| + \|P_{s_{j_0}}(y_{j_0-1}^{(s)}) - \xi\| \\ &\leq \delta + \|y_{j_0-1}^{(s)} - \xi\| < \epsilon_1/4 + \delta. \end{aligned} \tag{4.247}$$

It follows from (4.147), (4.246), and (4.247) that

$$\|y_{j_0-1}^{(s)} - y_{j_0}^{(s)}\| \leq \|y_{j_0-1}^{(s)} - \xi\| + \|\xi - y_{j_0}^{(s)}\| < \epsilon_1/2 + \delta < \epsilon_1.$$

This contradicts (4.242). The contradiction we have reached proves (4.244).

By (4.219), (4.243), (4.244), and property (P1),

$$\|z - P_{s_{j_0}}(y_{j_0-1}^{(s)})\| \leq \|z - y_{j_0-1}^{(s)}\| - \gamma.$$

The relations above and (4.233) imply that

$$\begin{aligned} \|z - y_{j_0}^{(s)}\| &\leq \|z - P_{s_{j_0}}(y_{j_0-1}^{(s)})\| + \|P_{s_{j_0}}(y_{j_0-1}^{(s)}) - y_{j_0}^{(s)}\| \\ &\leq \|z - y_{j_0-1}^{(s)}\| - \gamma + \delta. \end{aligned} \tag{4.248}$$

By (4.130), (4.232), (4.236), (4.248),

$$\begin{aligned} \|z - x_{k\bar{N}+i}\| - \|z - y_s\| &= \|z - y_0^{(s)}\| - \|z - y_{p(s)}^{(s)}\| \\ &= \sum_{i=0}^{p(s)-1} [\|z - y_i^{(s)}\| - \|z - y_{i+1}^{(s)}\|] \end{aligned}$$

$$\begin{aligned} \sum\{\|z - y_i^{(s)}\| - \|z - y_{i+1}^{(s)}\| : \ i \in \{0, \ldots, p-1\} \setminus \{j_0 - 1\}\} + \|z - y_{j_0-1}^{(s)}\| - \|z - y_{j_0}^{(s)}\| \\ \geq -\delta(p(s) - 1) + \gamma - \delta = \gamma - \delta p(s) \geq \gamma - \delta \bar{q}. \end{aligned} \tag{4.249}$$

It follows from (4.124), (4.131), (4.229), (4.238), (4.240), and (4.249) that

$$\|z - x_{k\bar{N}+i+1}\| \le \|z - \sum_{t \in \Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(t) y_t\|$$

$$+\|\sum_{t \in \Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(t) y_t - x_{k\bar{N}+i+1}\|$$

$$\le \sum_{t \in \Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(t) \|z - y_t\| + \delta$$

$$= \delta + \sum \{w_{k\bar{N}+i+1}(t) \|z - y_t\| : t \in \Omega_{k\bar{N}+i+1} \setminus \{s\}\} + w_{k\bar{N}+i+1}(s) \|z - y_s\|$$

$$\le \delta + \sum \{w_{k\bar{N}+i+1}(t)(\|x_{k\bar{N}+i} - z\| + \bar{q}\delta) : t \in \Omega_{k\bar{N}+i+1} \setminus \{s\}\}$$

$$+w_{k\bar{N}+i+1}(s)(\|z - x_{k\bar{N}+i}\| - \gamma + \delta\bar{q})$$

$$\le \|z - x_{k\bar{N}+i}\| + (\bar{q}+1)\delta - w_{k\bar{N}+i+1}(s)\gamma$$

$$\le \|z - x_{k\bar{N}+i}\| + (\bar{q}+1)\delta - \Delta\gamma.$$

Thus (4.226) holds. Lemma 4.14 is proved. □

It follows from (4.137), (4.222), and Lemma 4.14 applied by induction that for all $i = 0, \dots, \bar{N} - 1$,

$$\|x_{k\bar{N}+i+1} - z\| \le \|x_{k\bar{N}+i} - z\| + \delta(\bar{q}+1), \tag{4.250}$$

$$\|x_{k\bar{N}+i+1} - z\| \le 2M + \delta(\bar{q}+1)(i+1) \le 2M + \delta(\bar{q}+1)\bar{N} \le 2M + 1, \tag{4.251}$$

$$\|x_{k\bar{N}+i} - z\| \le 2M + 1,\ i = 0, \dots, \bar{N}. \tag{4.252}$$

By (4.137), (4.220), (4.250), and Lemma 4.14,

$$\|x_{(k+1)\bar{N}} - z\| - \|x_{k\bar{N}} - z\|$$

$$= \sum_{i=0}^{\bar{N}-1} [\|x_{k\bar{N}+i+1} - z\| - \|x_{k\bar{N}+i} - z\|]$$

$$\le \delta(\bar{q}+1)(\bar{N}-1) - \Delta\gamma + \delta(\bar{q}+1) = \bar{N}\delta(\bar{q}+1) - \Delta\gamma \le -\Delta\gamma/2.$$

Thus we have shown that the following property holds:

(P14) if an integer $k \in [0, s]$ satisfies $\|x_{k\bar{N}} - z\| \le 2M$, then

$$\|x_j - z\| \le 2M + 1,\ j = k\bar{N}, \dots, (k+1)\bar{N},$$

$$\|x_{(k+1)\bar{N}} - z\| - \|x_{k\bar{N}} - z\| \le -\Delta\gamma/2. \tag{4.253}$$

In view of (4.221) and property (P14),

$$\|x_j - z\| \le 2M + 1,\ j = 0, \dots, (s+1)\bar{N} \tag{4.254}$$

and (4.253) holds for all $k = 0, \dots, s$.

By (4.253) holding for all $k = 0, \dots, s$ and (4.221),

$$2^{-1}\Delta\gamma(s+1) \le \sum_{k=0}^{s}[\|x_{k\bar{N}} - z\| - \|x_{(k+1)\bar{N}} - z\|]$$

$$= \|x_0 - z\| - \|x_{(s+1)\bar{N}} - z\| \le \|x_0 - z\| \le 2M,$$

$$s + 1 \le 4M(\Delta\gamma)^{-1}.$$

Thus we have shown that the following property holds:

(P15) If an integer $s \ge 0$ and for each integer $k \in [0, s]$ relation (4.220) holds, then

$$s \le 4M(\Delta\gamma)^{-1} - 1,$$

$$\|x_j - z\| \le 2M + 1,\ j = 0, \dots, (s+1)\bar{N},$$

$$\|x_{k\bar{N}} - z\| \le 2M,\ k = 0, \dots, s+1.$$

By (P15) and (4.146), there exists an integer $q \in [0, n_0 - 1]$ such that for each integer k satisfying $0 \le k < q$,

$$\max\{\lambda_i :\ i = k\bar{N} + 1, \dots, (k+1)\bar{N}\} > \epsilon_1,$$

$$\max\{\lambda_i :\ i = q\bar{N} + 1, \dots, (q+1)\bar{N}\} \le \epsilon_1.$$

In view of (4.219), (P15) and the choice of q,

$$\|x_{q\bar{N}} - z\| \le 2M,$$

$$\|x_j - z\| \le 2M + 1,\ j = 0, \dots, q\bar{N},$$

$$\|x_j\| \le 3M + 1,\ j = 0, \dots, q\bar{N}.$$

Assume that an integer $q \in [0, n_0 - 1]$ satisfies

$$\|x_i\| \le 3M + 1,\ i = 0, \dots, q\bar{N}, \tag{4.255}$$

$$\lambda_i \le \epsilon_1,\ i = q\bar{N} + 1, \dots, (q+1)\bar{N}. \tag{4.256}$$

Let

$$j \in \{q\bar{N}, \dots, (q+1)\bar{N} - 1\}. \tag{4.257}$$

It follows from (4.151),

$$(x_{j+1}, \lambda_{j+1}, \mu_{j+1}) \in \mathcal{A}(x_j, (\Omega_{j+1}, w_{j+1}), \delta). \tag{4.258}$$

By (4.133) and (4.258), there exist

$$(y_t^{(j)}, \alpha_t^{(j)}, \beta_t^{(j)}) \in \mathcal{A}_0(x_j, t, \delta),\ t \in \Omega_{j+1} \tag{4.259}$$

such that

$$\|x_{j+1} - \sum_{t \in \Omega_{j+1}} w_{j+1}(t) y_t^{(j)}\| \le \delta, \tag{4.260}$$

$$\lambda_{j+1} = \max\{\alpha_t^{(j)} :\ t \in \Omega_{j+1}\}, \tag{4.261}$$

$$\mu_{j+1} = \max\{\beta_t^{(j)} :\ t \in \Omega_{j+1}\}. \tag{4.262}$$

In view of (4.256), (4.257), and (4.261),

$$\alpha_t^{(j)} \le \epsilon_1,\ t \in \Omega_{j+1}. \tag{4.263}$$

It follows from (4.132) and (4.259) that for each $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{j+1}$ there exists a finite sequence $\{y_i^{(t,j)}\}_{i=0}^{p(t)} \subset X$ such that

$$y_0^{(t,j)} = x_j, \tag{4.264}$$

for each integer $i = 1, \ldots, p(t)$,

$$\|y_i^{(t,j)} - P_{t_i}(y_{i-1}^{(t,j)})\| \le \delta, \tag{4.265}$$

$$y_{p(t)}^{(t,j)} = y_t^{(j)}, \tag{4.266}$$

$$\alpha_t^{(j)} = \max\{\|y_i^{(t,j)} - y_{i-1}^{(t,j)}\| :\ i = 1, \ldots, p(t)\} \le \epsilon_1, \tag{4.267}$$

$$\beta_t = \max\{d(y_i^{(t,j)}, C_{t_{i+1}}) :\ i = 0, \ldots, p(t) - 1\}. \tag{4.268}$$

By (4.129), (4.264), (4.266), and (4.267), for each $t \in \Omega_{j+1}$ and each $i = 0, \ldots, p(t)$,

$$\|x_j - y_i^{(t,j)}\| \le i\epsilon_1 \le \bar{q}\epsilon_1,$$

$$\|x_j - y_t^{(j)}\| \le \bar{q}\epsilon_1.$$

Thus we have shown that for each $j \in \{q\bar{N}, \ldots, (q+1)\bar{N} - 1\}$ each $t \in \Omega_{j+1}$ and each $i = 0, \ldots, p(t)$,

$$\|x_j - y_i^{(t,j)}\| \le \bar{q}\epsilon_1, \ \|x_j - y_t^{(j)}\| \le \bar{q}\epsilon_1. \tag{4.269}$$

In view of (4.124), (4.147), (4.260), and (4.269), for each $j \in \{q\bar{N}, \dots, (q+1)\bar{N}-1\}$,

$$\|x_j - x_{j+1}\| \le \|x_j - \sum_{t\in\Omega_{j+1}} w_{j+1}(t)y_t^{(j)}\|$$

$$+\|\sum_{t\in\Omega_{j+1}} w_{j+1}(t)y_t^{(j)} - x_{j+1}\|$$

$$\le \sum_{t\in\Omega_{j+1}} w_{j+1}(t)\|x_j - y_t^{(j)}\| + \delta \le \epsilon_1\bar{q} + \delta \le \epsilon_1(\bar{q}+1). \tag{4.270}$$

By (4.255) and (4.270), for each $j_1, j_2 \in \{q\bar{N}, \dots, (q+1)\bar{N}\}$,

$$\|x_{j_1} - x_{j_2}\| \le \bar{N}\epsilon_1(\bar{q}+1). \tag{4.271}$$

Relations (4.145) and (4.271) imply that

$$\|x_j\| \le 3M + 3/2, \ j \in \{q\bar{N}, \dots, (q+1)\bar{N}\}. \tag{4.272}$$

Assume that

$$j \in \{q\bar{N}, \dots, (q+1)\bar{N}-1\}, \tag{4.273}$$

$$t = (t_1, \dots, t_{p(t)}) \in \Omega_{j+1}, \tag{4.274}$$

$$i \in \{1, \dots, p(t)\}. \tag{4.275}$$

We show that

$$d(y_{i-1}^{(t,j)}, C_{t_i}) \le \epsilon/2.$$

Assume the contrary. Then

$$d(y_{i-1}^{(t,j)}, C_{t_i}) > \epsilon/2. \tag{4.276}$$

It follows from (4.145), (4.269), and (4.272)–(4.275) that

$$\|y_{i-1}^{(t,j)}\| \le \|x_j\| + \bar{q}\epsilon_1 \le 3M + 2. \tag{4.277}$$

By (4.219), (4.276), and (4.277),

$$\|z - P_{t_i}(y_{i-1}^{(t,j)})\| \le \|z - y_{i-1}^{(t,j)}\| - 2\epsilon_1. \tag{4.278}$$

In view of (4.265) and (4.278),

$$\|z - y_i^{(t,j)}\| \le \|z - P_{t_i}(y_{i-1}^{(t,j)})\| + \|P_{t_i}(y_{i-1}^{(t,j)}) - y_i^{(t,j)}\|$$
$$\le \|z - y_{i-1}^{(t,j)}\| - 2\epsilon_1 + \delta.$$

The relation above and (4.267) imply that

$$2\epsilon_1 - \delta \le \|z - y_{i-1}^{(t,j)}\| - \|z - y_i^{(t,j)}\| \le \|y_{i-1}^{(t,j)} - y_i^{(t,j)}\| \le \epsilon_1,$$
$$\epsilon_1 \le \delta.$$

This contradicts (4.147). The contradiction we have reached proves that

$$d(y_{i-1}^{(t,j)}, C_{t_i}) \le \epsilon/2 \text{ for each } j \in \{q\bar{N}, \ldots, (q+1)\bar{N} - 1\},$$

each $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{j+1}$ and each $i \in \{1, \ldots, p(t)\}$. Together with (4.269) this implies that

$$d(x_j, C_{t_i}) \le \epsilon_1 \bar{q} + \epsilon/2 \tag{4.279}$$

for each $j \in \{q\bar{N}, \ldots, (q+1)\bar{N} - 1\}$, each $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{j+1}$ and each $i \in \{1, \ldots, p(t)\}$.

Let $s \in \{1, \ldots, p(t)\}$. By (4.149), there exist $j \in \{q\bar{N}, \ldots, (q+1)\bar{N} - 1\}$ and $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{j+1}$ such that

$$s \in \{t_1, \ldots, t_{p(t)}\}.$$

Together with (4.279) this implies that

$$d(x_j, C_s) \le \epsilon_1 \bar{q} + \epsilon/2.$$

Combined with (4.145) and (4.271) this implies that for each $i \in \{q\bar{N}, \ldots, (q+1)\bar{N}\}$,

$$d(x_i, C_s) \le \bar{N}(\bar{q}+1)\epsilon_1 + \epsilon_1 \bar{q} + \epsilon/2 \le \epsilon/2 + \epsilon_1(\bar{N}+1)(\bar{q}+1) \le \epsilon$$

for every $s \in \{1, \ldots, m\}$. Therefore Theorem 4.6 is proved. □

4.7 Proof of Theorem 4.8

Fix

$$z \in B(0, M) \cap C. \tag{4.280}$$

Set

$$\epsilon_0 = \epsilon(\bar{N} + 1)^{-1}(\bar{q} + 1)^{-1}. \tag{4.281}$$

By (A2), there exists $\gamma \in (0, \epsilon_0)$ such that the following property holds:

(P16) for each $i \in \{1, \ldots, m\}$, each

$$\xi \in B(0, 3M + 1) \cap C_i$$

and each $x \in B(0, 3M + 1)$ satisfying $d(x, C_i) \geq \epsilon_0/2$,

$$\|\xi - P_i(x)\| \leq \|\xi - x\| - \gamma.$$

Set

$$Q = 1 + \lfloor 2\bar{N}M(\Delta\gamma)^{-1} \rfloor. \tag{4.282}$$

Assume that

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_* \tag{4.283}$$

satisfies for each natural number j

$$\{1, \ldots, m\} \subset \cup_{i=j}^{j+\bar{N}-1}(\cup_{t \in \Omega_i}\{t_1, \ldots, t_{p(t)}\}), \tag{4.284}$$

$$x_0 \in B(0, M) \tag{4.285}$$

and $\{x_i\}_{i=1}^{\infty} \subset X$, $\{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty)$, $\{\mu_i\}_{i=1}^{\infty} \subset [0, \infty)$ satisfy for each natural number i,

$$(x_i, \lambda_i, \mu_i) \in \mathcal{A}(x_{i-1}, (\Omega_i, w_i), 0). \tag{4.286}$$

By (4.125), (4.132), (4.133), and (4.286), for each integer $i \geq 0$,

$$x_{i+1} = P_{\Omega_{i+1}, w_{i+1}}(x_i). \tag{4.287}$$

In view of (4.280) and (4.285),

$$\|x_0 - z\| \leq 2M. \tag{4.288}$$

It follows from (4.127), (4.280), (4.287), and (4.288) that

$$\|x_{i+1} - z\| \leq \|x_i - z\|, \tag{4.289}$$

$$\|z - x_i\| \leq 2M \text{ for all integers } i \geq 0. \tag{4.290}$$

Set

$$E_0 = \{i \in \{0, 1, 2, \dots\} : \ \mu_i \geq \epsilon_0/2\}. \tag{4.291}$$

Let $i \geq 0$ be an integer. By (4.133) and (4.286) there exist

$$(y_t, \alpha_t, \beta_t) \in \mathcal{A}_0(x_i, t, 0), \ t \in \Omega_{i+1} \tag{4.292}$$

such that

$$x_{i+1} = \sum_{t \in \Omega_{i+1}} w_{i+1}(t) y_t, \tag{4.293}$$

$$\lambda_{i+1} = \max\{\alpha_t : \ t \in \Omega_{i+1}\}, \tag{4.294}$$

$$\mu_{i+1} = \max\{\beta_t : \ t \in \Omega_{i+1}\}. \tag{4.295}$$

It follows from (4.132) and (4.292) that for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}$ there exists a finite sequence $\{y_j^{(t)}\}_{j=0}^{p(t)} \subset X$ such that

$$y_0^{(t)} = x_i, \ y_{p(t)}^{(t)} = y_t, \tag{4.296}$$

$$y_j^{(t)} = P_{t_j}(y_{j-1}^{(t)}) \text{ for each integer } j = 1, \dots, p(t), \tag{4.297}$$

$$\alpha_t = \max\{\|y_{j+1}^{(t)} - y_j^{(t)}\| : \ j = 0, \dots, p(t) - 1\}, \tag{4.298}$$

$$\beta_t = \max\{d(y_j^{(t)}, C_{t_{j+1}}) : \ j = 0, \dots, p(t) - 1\}. \tag{4.299}$$

Let

$$t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}.$$

By (4.280), (4.290), (4.296), (4.297), and (A2), for each integer $j = 1, \dots, p(t)$,

$$\|y_j^{(t)} - z\| = \|P_{t_j}(y_{j-1}^{(t)}) - z\| \leq \|y_{j-1}^{(t)} - z\|, \tag{4.300}$$

$$\|y_j^{(t)} - z\| \leq \|y_0^{(t)} - z\| = \|x_i - z\|, \tag{4.301}$$

$$\|y_t - z\| \leq \|x_i - z\|, \tag{4.302}$$

$$\|y_j^{(t)}\| \leq 3M, \ \|y_t\| \leq 3M. \tag{4.303}$$

Assume that

$$\beta_t \geq \epsilon_0/2. \tag{4.304}$$

By (4.299) and (4.304), there exists $j_0 \in \{1, \dots, p(t)\}$ such that

$$d(y_{j_0-1}^{(t)}, C_{t_{j_0}}) \ge \epsilon_0/2. \tag{4.305}$$

Relations (4.280), (4.285), (4.296), (4.297), (4.303), (4.305), and (P16) imply that

$$\|z - y_{j_0}^{(t)}\| = \|z - P_{t_{j_0}}(y_{j_0-1}^{(t)})\| \le \|z - y_{j_0-1}^{(t)}\| - \gamma. \tag{4.306}$$

By (4.296), (4.300), and (4.306),

$$\begin{aligned}\|z - x_i\| - \|z - y_t\| &= \|z - y_0^{(t)}\| - \|z - y_{p(t)}^{(t)}\| \\ &= \sum_{i=1}^{p(t)} (\|z - y_{i-1}^{(t)}\| - \|z - y_i^{(t)}\|) \\ &\ge \|z - y_{j_0-1}^{(t)}\| - \|z - y_{j_0}^{(t)}\| \ge \gamma.\end{aligned}$$

Thus we have shown (see (4.302)) that for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}$,

$$\|y_t - z\| \le \|x_i - z\|, \tag{4.307}$$

$$\text{if } \beta_t \ge \epsilon_0/2, \text{ then } \|z - x_i\| \ge \|z - y_t\| + \gamma. \tag{4.308}$$

It follow from (4.124), (4.293), and (4.307) that

$$\begin{aligned}\|z - x_{i+1}\| &= \|z - \sum_{t \in \Omega_{i+1}} w_{i+1}(t) y_t\| \\ &\le \sum_{t \in \Omega_{i+1}} w_{i+1}(t) \|z - x_i\| = \|z - x_i\|.\end{aligned} \tag{4.309}$$

Assume that an integer

$$i \in E_0.$$

In view of the inclusion above and (4.291),

$$\mu_{i+1} \ge \epsilon_0/2. \tag{4.310}$$

It follows from (4.295) and (4.310) that there exist

$$s = (s_1, \dots, s_{p(s)}) \in \Omega_{i+1}$$

such that

$$\mu_{i+1} = \beta_s \ge \epsilon_0/2. \tag{4.311}$$

Relations (4.308) and (4.311) imply that

$$\|z - x_i\| - \gamma \geq \|z - y_s\|. \tag{4.312}$$

By (4.124), (4.131), (4.293), and (4.312),

$$\|z - x_{i+1}\| \leq \|z - \sum_{t \in \Omega_{i+1}} w_{i+1}(t) y_t\|$$

$$\leq \sum_{t \in \Omega_{i+1}} w_{i+1}(t)\|z - y_t\|$$

$$\leq \sum \{w_{i+1}(t)\|x_i - z\| : \ t \in \Omega_{i+1} \setminus \{s\}\} + w_{i+1}(s)(\|z - x_i\| - \gamma)$$

$$\leq \|z - x_i\| - \Delta\gamma.$$

Thus we have shown that the following property holds:
(P17) for each integer $i \in E_0$,

$$\|z - x_{i+1}\| \leq \|z - x_i\| - \gamma\Delta.$$

Let n be a natural number. By (4.288) and (4.309),

$$2M \geq \|z - x_0\| \geq \|z - x_0\| - \|z - x_n\| = \sum_{i=0}^{n-1}(\|z - x_i\| - \|z - x_{i+1}\|)$$

$$\geq \sum \{\|z - x_i\| - \|z - x_{i+1}\| : \ i \in \{0, \ldots, n-1\} \cap E_0\}$$

$$\geq \Delta\gamma \mathrm{Card}(E_0 \cap \{0, \ldots, n-1\}),$$

$$\mathrm{Card}(E_0 \cap \{0, \ldots, n-1\}) \leq 2M(\Delta\gamma)^{-1}.$$

Since the relation above holds for every natural number n we conclude that

$$\mathrm{Card}(E_0) \leq 2M(\Delta\gamma)^{-1}. \tag{4.313}$$

Assume that an integer $i \geq 0$ satisfies

$$i \notin E_0. \tag{4.314}$$

Relations (4.291) and (4.314) imply that

$$\mu_{i+1} < \epsilon_0/2. \tag{4.315}$$

In view of (4.133) and (4.286), there exist

$$(y_t, \alpha_t, \beta_t) \in \mathcal{A}_0(x_i, t, 0), \; t \in \Omega_{i+1} \tag{4.316}$$

such that (4.293)–(4.295) hold. It follows from (4.132) and (4.316) that for each $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{i+1}$ there exists a finite sequence $\{y_j^{(t)}\}_{j=0}^{p(t)} \subset X$ such that (4.296)–(4.299) are true. Let

$$t = (t_1, \ldots, t_{p(t)}) \in \Omega_{i+1}. \tag{4.317}$$

By (4.137), (4.295), and (4.315),

$$\beta_t < \epsilon_0/2. \tag{4.318}$$

Relations (4.299) and (4.318) imply that for each $j = 0, \ldots, p(t) - 1$,

$$d(y_j^{(t)}, C_{t_{j+1}}) < \epsilon_0/2. \tag{4.319}$$

Let

$$j \in \{0, \ldots, p(t) - 1\}. \tag{4.320}$$

In view of (4.319) and (4.320), there exists

$$\xi \in C_{t_{j+1}} \tag{4.321}$$

such that

$$\|y_j^{(t)} - \xi\| < \epsilon_0/2. \tag{4.322}$$

It follows from (4.297), (4.320)–(4.322), and (A2) that

$$\|\xi - y_{j+1}^{(t)}\| = \|\xi - P_{t_{j+1}}(y_j^{(t)})\| \leq \|\xi - y_j^{(t)}\| < \epsilon_0/2. \tag{4.323}$$

By (4.322) and (4.323),

$$\|y_j^{(t)} - y_{j+1}^{(t)}\| < \epsilon_0. \tag{4.324}$$

In view of (4.130) and (4.324), for each $j_1, j_2 \in \{0, \ldots, p(t)\}$,

$$\|y_{j_1}^{(t)} - y_{j_2}^{(t)}\| < \epsilon_0 \bar{q}. \tag{4.325}$$

It follows from (4.296) and (4.325) that for each $j \in \{0, \ldots, p(t)\}$,

$$\|x_i - y_j^{(t)}\| < \bar{q}\epsilon_0, \; \|x_i - y_t\| < \bar{q}\epsilon_0. \tag{4.326}$$

Relations (4.319) and (4.326), for each $j \in \{0, \dots, p(t) - 1\}$,

$$d(x_i, C_{t_{j+1}}) \le \epsilon_0 \bar{q} + \epsilon_0/2.$$

Thus we have shown that for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}$,

$$d(x_i, C_{t_j}) \le \epsilon_0 \bar{q} + \epsilon_0/2, \ j = 1, \dots, p(t), \tag{4.327}$$

$$\|x_i - y_t\| \le \bar{q}\epsilon_0. \tag{4.328}$$

It follows from (4.124), (4.293), and (4.328) that

$$\|x_i - x_{i+1}\| = \|x_i - \sum_{t \in \Omega_{i+1}} w_{i+1}(t) y_t\|$$

$$\le \sum_{t \in \Omega_{i+1}} w_{i+1}(t) \|x_i - y_t\| \le \bar{q}\epsilon_0. \tag{4.329}$$

Thus we have shown that the following property holds:

(P18) if a nonnegative integer $i \notin E_0$, then for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}$, (4.327)–(4.329) hold.

Set

$$E_1 = \{i \in \{0, 1, 2, \dots\} : \ [i, i + \bar{N} - 1] \cap E_0 \ne \emptyset\}. \tag{4.330}$$

By (4.282), (4.313), and (4.330),

$$\mathrm{Card}(E_1) \le \bar{N}\mathrm{Card}(E_0) \le 2M\bar{N}(\Delta\gamma)^{-1} \le Q. \tag{4.331}$$

Assume that a nonnegative integer $j \notin E_1$. In view of (4.330),

$$\{j, \dots, j + \bar{N} - 1\} \cap E_0 \ne \emptyset. \tag{4.332}$$

It follows from (4.329), (4.332), and property (P18) that for each $i \in \{j, \dots, j + \bar{N} - 1\}$,

$$\|x_i - x_{i+1}\| \le \epsilon_0 \bar{q},$$

$$d(x_i, C_s) < \epsilon_0 \bar{q} + \epsilon_0/2 \tag{4.333}$$

for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1}$ and each $s \in \{t_1, \dots, t_{p(t)}\}$ and

$$\|x_j - x_i\| \le \bar{N}\epsilon_0 \bar{q}. \tag{4.334}$$

Let $s \in \{1, \ldots, m\}$. By (4.284), there exist $i \in \{j, \ldots, j + \bar{N} - 1\}$ and $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{i+1}$ such that $s \in \{t_1, \ldots, t_{p(t)}\}$. Together with (4.281), (4.333), and (4.324) this implies that

$$d(x_i, C_s) < \epsilon_0 \bar{q} + \epsilon_0/2$$

and

$$d(x_j, C_s) < \epsilon_0 \bar{q} + \epsilon_0/2 + \bar{N}\bar{q}\epsilon_0 < \epsilon$$

for all $s = 1, \ldots, m$. Theorem 4.8 is proved. □

Chapter 5
Dynamic String-Maximum Methods in Metric Spaces

In this chapter we study the convergence of dynamic string-maximum methods for solving common fixed point problems in a metric space. Our main goal is to obtain an approximate solution of the problem in the presence of computational errors. We show that our dynamic string-maximum algorithm generates a good approximate solution, if the sequence of computational errors is bounded from above by a constant. Moreover, for a known computational error, we find out what an approximate solution can be obtained and how many iterates one needs for this.

5.1 Preliminaries and Main Results

Let (X, d) be a metric space. For each $x \in X$ and each nonempty set $E \subset X$ put

$$d(x, E) = \inf\{d(x, y) : \ y \in E\}.$$

For each $x \in X$ and each $r > 0$ set

$$B(x, r) = \{y \in X : \ d(x, y) \le r\}.$$

Suppose that m is a natural number, $P_i : X \to X$, $i = 1, \dots, m$ are self-mappings of X and that for every $i \in \{1, \dots, m\}$,

$$\mathrm{Fix}(P_i) := \{z \in X : \ P_i(z) = z\} \ne \emptyset. \tag{5.1}$$

We suppose that

$$d(z, x) \ge d(z, P_i(x)) \tag{5.2}$$

A. Zaslavski, *Approximate Solutions of Common Fixed-Point Problems*, Springer Optimization and Its Applications 112, DOI 10.1007/978-3-319-33255-0_5

for every $i \in \{1, \dots, m\}$, every $x \in X$, and every $z \in \mathrm{Fix}(P_i)$. For every $\epsilon > 0$ and every $i \in \{1, \dots, m\}$ put

$$F_\epsilon(P_i) = \{x \in X : \ d(x, P_i(x)) \le \epsilon\}, \tag{5.3}$$

$$\tilde{F}_\epsilon(P_i) = \{y \in X : \ B(y, \epsilon) \cap F_\epsilon(P_i) \ne \emptyset\}, \tag{5.4}$$

$$F_\epsilon = \cap_{i=1}^m F_\epsilon(P_i) \tag{5.5}$$

and

$$\tilde{F}_\epsilon = \cap_{i=1}^m \tilde{F}_\epsilon(P_i) \tag{5.6}$$

A point belonging to the set F is a solution of our common fixed point problem while a point which belongs to the set $\tilde{F}_\epsilon$ is its ϵ-approximate solution.

Fix $\theta \in X$.

We apply a dynamic string method with variable strings in order to obtain a good approximative solution of the common fixed point problem. Next we describe the dynamic string method with variable strings.

By an index vector, we a mean a vector $t = (t_1, \dots, t_p)$ such that $t_i \in \{1, \dots, m\}$ for all $i = 1, \dots, p$.

For an index vector $t = (t_1, \dots, t_q)$ set

$$p(t) = q, \ P[t] = P_{t_q} \cdots P_{t_1}. \tag{5.7}$$

Denote by $\mathcal{M}$ the collection of all finite sets Ω of index vectors such that

$$\cup \{\{t_1, \dots, t_{p(t)}\} : \ t = \{t_1, \dots, t_{p(t)}\} \in \Omega\} = \{1, \dots, m\}. \tag{5.8}$$

Fix an integer

$$\bar{q} \ge m \tag{5.9}$$

and denote by $\mathcal{M}_*$ the set of all $\Omega \in \mathcal{M}$ such that

$$p(t) \le \bar{q} \ \text{ for all } t \in \Omega. \tag{5.10}$$

The dynamic string-maximum method with variable strings can now be described by the following algorithm.

Initialization: select an arbitrary $x_0 \in X$.

Iterative step: given a current iteration vector x_k pick

$$\Omega_{k+1} \in \mathcal{M}_*,$$

calculate

$$P[t](x_k), \ t \in \Omega_{k+1}$$

and choose

$$x_{k+1} \in \{P[t](x_k) : t \in \Omega_{k+1}\}$$

such that

$$d(x_k, x_{k+1}) \geq d(x_k, P[t](x_k)), \ t \in \Omega_{k+1}.$$

In order to state the main result of this section we need the following definitions. Let $\delta \geq 0$, $x \in X$ and let $t = (t_1, \dots, t_{p(t)})$ be an index vector. Define

$$\begin{aligned} A_0(x, t, \delta) = \{(y, \lambda) \in X \times R^1 : \ & \text{there is a sequence } \{y_i\}_{i=0}^{p(t)} \subset X \text{ such that} \\ & y_0 = x \text{ and for all } i = 1, \dots, p(t), \\ & d(y_i, P_{t_i}(y_{i-1})) \leq \delta, \\ & y = y_{p(t)}, \\ & \lambda = \max\{d(y_i, y_{i-1}) : i = 1, \dots, p(t)\}\}. \end{aligned} \tag{5.11}$$

Let $\delta \geq 0$, $x \in X$ and let $\Omega \in \mathcal{M}_*$. Define

$$\begin{aligned} A(x, \Omega, \delta) = \{(y, \lambda) \in X \times R^1 : \ & \text{there exist} \\ & (y_t, \lambda_t) \in A_0(x, t, \delta), \ t \in \Omega \text{ such that} \\ & (y, \lambda) \in \{(y_t, \lambda_t) : y \in \Omega\}, \\ & \lambda \geq \lambda_t, \ t \in \Omega_t\}. \end{aligned} \tag{5.12}$$

Denote by Card(A) the cardinality of a set A. Suppose that the sum over empty set is zero.

In Sect. 5.3 we prove the following result.

Theorem 5.1. *Suppose that $\bar{c} \in (0, 1)$ and that*

$$d(z, x)^2 \geq d(z, P_i(x))^2 + \bar{c} d(x, P_i(x))^2 \tag{5.13}$$

for every $i \in \{1, \dots, m\}$, every $x \in X$ and every $z \in Fix(P_i)$. Let $M > 0$,

$$\delta_0, \delta_1 \in (0, (3\bar{q})^{-1}), \ z \in B(0, M) \tag{5.14}$$

satisfy

$$B(z, \delta_0) \cap Fix(P_i) \neq \emptyset \text{ for all } i = 1, \dots, m, \tag{5.15}$$

$$\epsilon_0 = ((8\delta_1 + 16\delta_0)\bar{c}^{-1}(2M + 1)\bar{q})^{1/2}, \tag{5.16}$$

a natural number n_0 satisfy

$$n_0 \ge 8M^2\epsilon_0^{-2}\bar{c}^{-1}. \tag{5.17}$$

Assume that

$$\{\Omega_i\}_{i=1}^{\infty} \subset \mathcal{M}_*, \tag{5.18}$$

$$x_0 \in B(\theta, M) \tag{5.19}$$

and $\{x_i\}_{i=1}^{\infty} \subset X,\ \{\lambda_i\}_{i=1}^{\infty} \subset [0,\infty)$ satisfy for each natural number i,

$$(x_i, \lambda_i) \in A(x_{i-1}, \Omega_i, \delta_1). \tag{5.20}$$

Then there exists an integer $q \in [0, n_0]$ such that

$$x_i \in B(\theta, 3M),\ i = 0, \dots, q,$$
$$\lambda_{q+1} \le \epsilon_0. \tag{5.21}$$

Moreover, if an integer $q \ge 0$ satisfies $\lambda_{q+1} \le \epsilon_0$, then

$$x_q \in \tilde{F}_{\epsilon_0(\bar{q}+2)}.$$

Note that in Theorem 5.1 δ_1 is the computational error made by our computer system, we obtain a point of the set $\tilde{F}_{\epsilon_0(\bar{q}+2)}$ and in order to obtain this point we need n_0 iterations. It is not difficult to see that $\epsilon_0 = c_1(\delta_0 + \delta_1)^{1/2}$ and $n_0 = \lfloor c_2(\delta_0 + \delta_1)^{-1} \rfloor + 1$, where c_1 and c_2 are positive constants depending on M.

The next theorem is proved in Sect. 5.4.

Theorem 5.2. *Suppose that $\bar{c} \in (0, 1)$ and that*

$$d(z, x)^2 \ge d(z, P_i(x))^2 + \bar{c}d(x, P_i(x))^2 \tag{5.22}$$

for every $i \in \{1, \dots, m\}$, every $x \in X$ and every $z \in Fix(P_i)$. Let $M > 0$ be such that the following property holds:

for each $\delta > 0$ there exists $z_\delta \in B(\theta, M)$ for which

$$B(z_\delta, \delta) \cap Fixp(P_i) \ne \emptyset \text{ for all } i = 1, \dots, m.$$

Assume that $\epsilon > 0$,

$$\{\Omega_i\}_{i=1}^{\infty} \subset \mathcal{M}_*, \tag{5.23}$$

$$x_0 \in B(\theta, M) \tag{5.24}$$

and $\{x_i\}_{i=1}^{\infty} \subset X,\ \{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty)$ *satisfy for each natural number i,*

$$(x_i, \lambda_i) \in A(x_{i-1}, \Omega_i, 0). \tag{5.25}$$

Then

$$Card(\{i \in \{0, 1, \dots\} : \ x_i \notin \tilde{F}_\epsilon\}) \le \epsilon^{-2}(\bar{q} + 1)^2 \bar{c}^{-1}(4M^2 + 1).$$

The following result is proved in Sect. 5.5.

Theorem 5.3. *Suppose that for each* $\Lambda > 0$ *and each* $\lambda > 0$ *there exists* $\gamma > 0$ *such that the following assumption holds:*

(A1) for each $i \in \{1, \dots, m\}$*, each*

$$z \in B(\theta, \Lambda) \cap Fix(P_i)$$

and each $x \in B(\theta, \Lambda)$ *satisfying* $d(x, P_i(x)) \ge \lambda$*,*

$$d(z, P_i(x)) \le d(z, x) - \gamma.$$

Suppose that $\bar{M} > 0$ *and that for each* $\gamma > 0$ *there exists* $z_\gamma \in B(\theta, \bar{M})$ *for which*

$$B(z_\gamma, \gamma) \cap Fix(P_i) \ne \emptyset \text{ for all } i = 1, \dots, m.$$

Let $M > \bar{M}$*,* $\epsilon \in (0, 1)$*,*

$$\epsilon_0 = \epsilon(\bar{q} + 2)^{-1} \tag{5.26}$$

and let $\gamma_0 \in (0, 1)$ *be such that (A1) holds with*

$$\Lambda = 3M + 1,\ \lambda = \epsilon_0/2,\ \gamma = \gamma_0.$$

Let an integer

$$n_0 \ge 8M\gamma_0^{-1}, \tag{5.27}$$

$$0 < \delta < \min\{12^{-1}\bar{q}^{-1}\gamma_0, 4^{-1}\epsilon_0\}. \tag{5.28}$$

Assume that

$$\{\Omega_i\}_{i=1}^{\infty} \subset \mathcal{M}_*, \tag{5.29}$$

$$x_0 \in B(\theta, M) \tag{5.30}$$

and $\{x_i\}_{i=1}^{\infty} \subset X,\ \{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty)$ *satisfy for each natural number i,*

$$(x_i, \lambda_i) \in A(x_{i-1}, \Omega_i, \delta). \tag{5.31}$$

Then there exists an integer $q \in [0, n_0]$ *such that*

$$d(x_k, \theta) \le 3M,\ k = 0, \dots, q,$$

$$\lambda_{q+1} \le \epsilon_0.$$

Moreover, if an integer $q \ge 0$ *satisfies* $\lambda_{q+1} \le \epsilon_0$, *then*

$$x_q \in \tilde{F}_\epsilon.$$

Theorem 5.4. *Suppose that for each* $\Lambda > 0$ *and each* $\lambda > 0$ *there exists* $\gamma > 0$ *such that (A1) holds.*

Suppose that $M > 0$ *and that the following property holds:*
for each $\delta > 0$ *there exists* $z_\delta \in B(\theta, M)$ *for which*

$$B(z_\delta, \delta) \cap Fix(P_i) \neq \emptyset \text{ for all } i = 1, \dots, m.$$

Let $\epsilon > 0$. *Then there exist* $Q > 0$ *such that for each* $\{\Omega_i\}_{i=1}^\infty \subset \mathcal{M}_*$, *each* $x_0 \in B(\theta, M)$, *each* $\{x_i\}_{i=1}^\infty \subset X$ *and each* $\{\lambda_i\}_{i=1}^\infty \subset [0, \infty)$ *satisfying for each natural number i,*

$$(x_i, \lambda_i) \in A(x_{i-1}, \Omega_i, 0),$$

the inequality

$$Card(\{i \in \{0, 1, \dots\} :\ x_i \notin \tilde{F}_\epsilon\}) \le Q$$

is true.

The proof of Theorem 5.4 is given in Sect. 5.6.

Let $\delta \ge 0$, $x \in X$ and let $t = (t_1, \dots, t_{p(t)})$ be an index vector. Define

$$\begin{aligned}\tilde{A}_0(x, t, \delta) = \{(y, \lambda) \in X \times R^1 :\ & \text{there is a sequence } \{y_i\}_{i=0}^{p(t)} \subset X \text{ such that} \\ & y_0 = x \text{ and for all } i = 1, \dots, p(t), \\ & d(y_i, P_{t_i}(y_{i-1})) \le \delta, \\ & y = y_{p(t)}, \\ & \lambda = \max\{d(y_{i-1}, \mathrm{Fix}(P_{t_i})) :\ i = 1, \dots, p(t)\}\}. \end{aligned} \tag{5.32}$$

Let $\delta \ge 0$, $x \in X$ and let $\Omega \in \mathcal{M}_*$. Define

$$\tilde{A}(x, \Omega, \delta) = \{(y, \lambda) \in X \times R^1 :\ \text{there exist}$$

$$(y_t, \lambda_t) \in \tilde{A}_0(x, t, \delta),\ t \in \Omega \text{ such that}$$

$$(y, \lambda) \in \{(y_t, \lambda_t) : \; y \in \Omega\},$$
$$\lambda \geq \lambda_t, \; t \in \Omega\}. \tag{5.33}$$

Theorem 5.5. *Suppose that for each $\Lambda > 0$ and each $\lambda > 0$ there exists $\gamma > 0$ such that the following assumption holds:*

(A2) for each $i \in \{1, \ldots, m\}$, each

$$z \in B(\theta, \Lambda) \cap Fix(P_i)$$

and each $x \in B(\theta, \Lambda)$ satisfying $d(x, Fix(P_i)) \geq \lambda$,

$$d(z, P_i(x)) \leq d(z, x) - \gamma.$$

Suppose that $\epsilon \in (0, 1)$, $M > 0$ and that for each $\gamma > 0$ there exists $z_\gamma \in B(\theta, M)$ for which

$$B(z_\gamma, \gamma) \cap Fix(P_i) \neq \emptyset \text{ for all } i = 1, \ldots, m.$$

Let

$$\epsilon_0 = \epsilon(4\bar{q})^{-1} \tag{5.34}$$

and let $\gamma_0 \in (0, 1)$ be such that (A2) holds with

$$\Lambda = 3M + 1, \; \lambda = \epsilon_0/2, \; \gamma = \gamma_0.$$

Let an integer

$$n_0 \geq 8M\gamma_0^{-1}, \tag{5.35}$$

$$0 < \delta < \min\{(3\bar{q})^{-1}, 12^{-1}\epsilon_0\gamma_0\bar{q}^{-1}\}. \tag{5.36}$$

Assume that

$$\{\Omega_i\}_{i=1}^{\infty} \subset \mathcal{M}_*, \tag{5.37}$$

$$x_0 \in B(\theta, M) \tag{5.38}$$

and $\{x_i\}_{i=1}^{\infty} \subset X, \; \{\mu_i\}_{i=1}^{\infty} \subset [0, \infty)$ satisfy for each natural number i,

$$(x_i, \mu_i) \in \tilde{A}(x_{i-1}, \Omega_i, \delta). \tag{5.39}$$

Then there exists an integer $q \in [0, n_0]$ such that

$$d(x_k, \theta) \leq 3M, \; k = 0, \ldots, q,$$
$$\mu_{q+1} \leq \epsilon_0.$$

Moreover, if an integer $q \geq 0$ satisfies $\mu_{q+1} \leq \epsilon_0$, then

$$d(x_q, Fix(P_s)) \leq \epsilon,\ s = 1, \dots, m.$$

Theorem 5.5 is proved in Sect. 5.7. The proof of the next result is given in Sect. 5.8.

Theorem 5.6. *Suppose that for each $\Lambda > 0$ and each $\lambda > 0$ there exists $\gamma > 0$ such that (A2) holds.*

Suppose that $\epsilon \in (0, 1)$, $M > 0$ and that for each $\gamma > 0$ there exists $z_\gamma \in B(\theta, M)$ for which

$$B(z_\gamma, \gamma) \cap Fix(P_i) \neq \emptyset \text{ for all } i = 1, \dots, m.$$

Let a positive number

$$\epsilon_0 < \epsilon(\bar{q})^{-1}$$

be such that (A2) holds with

$$\Lambda = 3M + 1,\ \lambda = (4\bar{q})^{-1}\epsilon,\ \gamma = 2\epsilon_0,$$

let $\gamma \in (0, \epsilon_0)$ be such that (A2) holds with

$$\Lambda = 3M + 1,\ \lambda = \epsilon_0/4,$$

an integer

$$n_0 \geq 4M\gamma_0^{-1}, \tag{5.40}$$

and let

$$0 < \delta < \min\{(3\bar{q})^{-1}, 12^{-1}\gamma\bar{q}^{-1},\ 6^{-1}\epsilon_0\}. \tag{5.41}$$

Assume that

$$\{\Omega_i\}_{i=1}^{\infty} \subset \mathcal{M}_*,$$
$$x_0 \in B(\theta, M)$$

and $\{x_i\}_{i=1}^{\infty} \subset X$, $\{\mu_i\}_{i=1}^{\infty} \subset [0, \infty)$ satisfy for each natural number i,

$$(x_i, \lambda_i) \in A(x_{i-1}, \Omega_i, \delta).$$

Then there exists an integer $q \in [0, n_0]$ *such that*

$$d(x_k, \theta) \le 3M,\ k = 0, \dots, q,$$
$$\lambda_{q+1} \le \epsilon_0.$$

Moreover, if an integer $q \ge 0$ *satisfies*

$$d(x_q, \theta) \le 3M,\ \lambda_{q+1} \le \epsilon_0,$$

then

$$d(x_q, Fix(P_s)) \le \epsilon,\ s = 1, \dots, m.$$

The final result of this section is proved in Sect. 5.9.

Theorem 5.7. *Suppose that for each* $\Lambda > 0$ *and each* $\lambda > 0$ *there exists* $\gamma > 0$ *such that (A2) holds.*

Suppose that $M, \epsilon > 0$ *and that the following property holds:*
for each $\delta > 0$ *there exists* $z_\delta \in B(\theta, M)$ *for which*

$$B(z_\delta, \delta) \cap Fix(P_i) \ne \emptyset \text{ for all } i = 1, \dots, m.$$

Then there exist $Q > 0$ *such that for each* $\{\Omega_i\}_{i=1}^{\infty} \subset \mathcal{M}_*$, *each* $x_0 \in B(\theta, M)$, *each* $\{x_i\}_{i=1}^{\infty} \subset X$ *and each* $\{\mu_i\}_{i=1}^{\infty} \subset [0, \infty)$ *satisfying for each natural number i,*

$$(x_i, \mu_i) \in \tilde{A}(x_{i-1}, \Omega_i, 0),$$

the inequality

$$Card(\{i \in \{0, 1, \dots\} :\ \max\{d(x_i, Fix(P_s)) :\ s = 1, \dots, m\} > \epsilon\}) \le Q$$

holds.

5.2 Auxiliary Results

Proposition 5.8. *Let* $\gamma_1, \gamma_2 > 0$, $z, x \in X$,

$$B(z, \gamma_1) \cap Fix(P_i) \ne \emptyset \text{ for all } i = 1, \dots, m, \tag{5.42}$$

$t = (t_1, \dots, t_{p(t)})$ *be an index vector,* $\{y_i\}_{i=0}^{p(t)} \subset X$ *be such that*

$$y_0 = x \tag{5.43}$$

and for all $i = 1, \dots, p(t)$,

$$d(y_i, P_{t_i}(y_{i-1})) \le \gamma_2. \tag{5.44}$$

Then for each integer $i = 1, \dots, p(t)$,

$$d(z, y_i) \le d(z, y_{i-1}) + 2\gamma_1 + \gamma_2, \tag{5.45}$$

$$d(z, y_i) \le d(z, x) + i(2\gamma_1 + \gamma_2). \tag{5.46}$$

Proof. Let $i \in \{1, \dots, p(t)\}$. By (5.42), there exists

$$z_i \in \mathrm{Fix}(P_{t_i}) \tag{5.47}$$

such that

$$d(z, z_i) \le \gamma_1. \tag{5.48}$$

In view of (5.2), (5.44), (5.47), and (5.48),

$$\begin{aligned} d(z, y_i) &\le d(z, z_i) + d(z_i, P_{t_i}(y_{i-1})) + d(P_{t_i}(y_{i-1}), y_i) \\ &\le \gamma_1 + d(z_i, y_{i-1}) + \gamma_2 \\ &\le \gamma_1 + \gamma_2 + d(z_i, z) + d(z, y_{i-1}) \le 2\gamma_1 + \gamma_2 + d(z, y_{i-1}). \end{aligned}$$

Thus (5.45) holds for all integers $i = 1, \dots, p(t)$. By induction, together with (5.43) this implies that (5.46) holds for all $i = 1, \dots, p(t)$. Proposition 5.8 is proved. □

Proposition 5.9. *Let* $\gamma \ge 0$, $\Omega \in \mathcal{M}_*$, $\xi_0 \in X$, $\epsilon_0 \ge 0$,

$$(\xi_1, \lambda_1) \in A(\xi_0, \Omega, \gamma), \tag{5.49}$$

$$\lambda_1 \le \epsilon_0. \tag{5.50}$$

Then

$$\xi_0 \in \tilde{F}_{\epsilon_0(\bar{q}+2)}(P_s) \textit{ for all } s = 1, \dots, m.$$

Proof. In view of (5.12) and (5.49), there exist

$$(y_t, \alpha_t) \in A_0(\xi_0, t, \gamma), \ \ t \in \Omega \tag{5.51}$$

such that

$$(\xi_1, \lambda_1) \in \{(y_t, \alpha_t) : \ t \in \Omega\}, \tag{5.52}$$

$$\lambda_1 = \max\{\lambda_t : \ t \in \Omega_t\}. \tag{5.53}$$

By (5.11) and (5.51), for each $t = (t_1, \dots, t_{p(t)}) \in \Omega$, there exists a finite sequence $\{y_i^{(t)}\}_{i=0}^{p(t)} \subset X$ such that

$$y_0^{(t)} = \xi_0, \; y_{p(t)}^{(t)} = y_t, \tag{5.54}$$

$$d(y_j^{(t)}, P_{t_i}(y_{j-1}^{(t)})) \le \gamma \text{ and for all } j = 1, \dots, p(t), \tag{5.55}$$

$$\alpha_t = \max\{d(y_j^{(t)}, y_{j-1}^{(t)}) : \; j = 1, \dots, p(t)\}. \tag{5.56}$$

Relations (5.50), (5.53), and (5.56) imply that for each $t = (t_1, \dots, t_{p(t)}) \in \Omega$ and all $j = 1, \dots, p(t)$,

$$d(y_j^{(t)}, y_{j-1}^{(t)}) \le \epsilon_0. \tag{5.57}$$

It follows from (5.10), (5.54), (5.57) and the inclusion $\Omega \in \mathcal{M}_*$ that for each $t \in \Omega$ and every $j = 0, \dots, p(t)$,

$$d(\xi_0, y_j^{(t)}) \le j\epsilon_0 \le \bar{q}\epsilon_0. \tag{5.58}$$

Let $s \in \{1, \dots, m\}$. In view of (5.8) and the inclusion $\Omega \in \mathcal{M}_*$, there exist $\tau = (\tau_1, \dots, \tau_{p(\tau)}) \in \Omega_j$ and $j \in \{1, \dots, p(\tau)\}$ such that

$$s = \tau_j. \tag{5.59}$$

It follows from (5.55), (5.57), and (5.59) that

$$d(y_{j-1}^{(\tau)}, P_s(y_{j-1}^{(\tau)})) = d(y_{j-1}^{(\tau)}, P_{\tau_j}(y_{j-1}^{(\tau)}))$$

$$\le d(y_{j-1}^{(\tau)}, y_j^{(\tau)}) + d(y_j^{(\tau)}, P_{\tau_j}(y_{j-1}^{(\tau)})) \le \epsilon_0 + \gamma$$

and in view of (5.3),

$$y_{j-1}^{(\tau)} \in F_{\epsilon_0+\gamma}(P_s). \tag{5.60}$$

It follows (5.4), (5.58), (5.60) and the inequality $\gamma \le \epsilon_0$,

$$\xi_0 \in \tilde{F}_{\epsilon_0(\bar{q}+2)}(P_s)$$

for all $s \in \{1, \dots, m\}$. Proposition 5.9 is proved. □

Proposition 5.10. *Assume that $\gamma \ge 0$, $\epsilon_0 > 0$, $\xi \in X$, $\Omega \in \mathcal{M}_*$, for each $t = (t_1, \dots, t_{p(t)}) \in \Omega$, $\{y_i^{(t)}\}_{i=0}^{p(t)} \subset X$,*

$$y_0^{(t)} = \xi, \tag{5.61}$$

$$d(y_j^{(t)}, P_{t_j}(y_{j-1}^{(t)})) \le \gamma, \; j = 1, \dots, p(t), \tag{5.62}$$

$$\max\{d(y_{j-1}^{(t)}, Fix(P_{t_j})) : \ j = 1, \dots, p(t)\} \leq \epsilon_0 \tag{5.63}$$

for all $t \in \Omega$. *Then*

$$d(\xi, Fix(P_s)) \leq \bar{q}(2\epsilon_0 + \gamma) \text{ for all } s = 1, \dots, m.$$

Proof. Let

$$\delta > 0, \ t = (t_1, \dots, t_{p(t)}) \in \Omega, \ j \in \{1, \dots, p(t)\}.$$

By (5.63), there exists

$$z_j \in \mathrm{Fix}(P_{t_j}) \tag{5.64}$$

such that

$$d(y_{j-1}^{(t)}, z_j) < \epsilon_0 + \delta. \tag{5.65}$$

It follows from (5.2), (5.62), (5.64), and (5.65) that

$$\begin{aligned} d(y_j^{(t)}, z_j) &\leq d(y_j^{(t)}, P_{t_j}(y_{j-1}^{(t)})) + d(P_{t_j}(y_{j-1}^{(t)}), z_j) \\ &\leq \gamma + d(y_{j-1}^{(t)}, z_j) \leq \gamma + \epsilon_0 + \delta. \end{aligned}$$

Together with (5.65) this implies that

$$d(y_j^{(t)}, y_{j-1}^{(t)}) \leq 2\epsilon_0 + 2\delta + \gamma.$$

Since δ is an arbitrary positive number

$$d(y_j^{(t)}, y_{j-1}^{(t)}) \leq 2\epsilon_0 + \gamma.$$

Together with (5.61) this implies that for each $t = (t_1, \dots, t_{p(t)}) \in \Omega$ and each $j \in \{0, 1, \dots, p(t)\}$,

$$d(\xi, y_j^{(t)}) \leq (2\epsilon_0 + \gamma)j.$$

Combined with (5.10) and (5.63) this implies that for each $t = (t_1, \dots, t_{p(t)}) \in \Omega$ and each $j \in \{1, \dots, p(t)\}$,

$$\begin{aligned} d(\xi, \mathrm{Fix}(P_{t_j})) &\leq d(\xi, y_{j-1}^{(t)}) + d(y_{j-1}^{(t)}, \mathrm{Fix}(P_{t_j})) \\ &\leq (2\epsilon_0 + \gamma)(j-1) + \epsilon_0 \leq (2\epsilon_0 + \gamma)j \leq (2\epsilon_0 + \gamma)\bar{q}. \end{aligned} \tag{5.66}$$

Let $s \in \{1, \dots, m\}$. In view of (5.8), there exists $t = (t_1, \dots, t_{p(t)}) \in \Omega$ such that

$$s \in \{t_1, \dots, t_{p(t)}\}.$$

By (5.66),

$$d(\xi, \mathrm{Fix}(P_s)) \le \bar{q}(2\epsilon_0 + \gamma).$$

Proposition 5.10 is proved. □

5.3 Proof of Theorem 5.1

By (5.14) and (5.19),

$$d(x_0, z) \le 2M. \tag{5.67}$$

We prove the following auxiliary result.

Lemma 5.11. *Assume that a nonnegative integer k satisfies*

$$d(x_k, z) \le 2M, \tag{5.68}$$

$$\lambda_{k+1} > \epsilon_0. \tag{5.69}$$

Then

$$d(z, x_k)^2 - d(z, x_{k+1})^2 \ge 2^{-1}\epsilon_0^2 \bar{c}.$$

Proof. In view of (5.20),

$$(x_{k+1}, \lambda_{k+1}) \in A(x_k, \Omega_{k+1}, \delta_1). \tag{5.70}$$

By (5.12) and (5.70) there exist

$$(y_t, \alpha_t) \in A_0(x_k, t, \delta_1),\ \ t \in \Omega_{k+1} \tag{5.71}$$

such that

$$(x_{k+1}, \lambda_{k+1}) \in \{(y_t, \alpha_t) :\ t \in \Omega_{k+1}\}, \tag{5.72}$$

$$\lambda_{k+1} = \max\{\alpha_t :\ t \in \Omega_{k+1}\}. \tag{5.73}$$

It follows from (5.11) and (5.71) that for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}$ there exists a finite sequence $\{y_i^{(t)}\}_{i=0}^{p(t)} \subset X$ such that

$$y_0^{(t)} = x_k, \ y_{p(t)}^{(t)} = y_t, \tag{5.74}$$

$$d(y_j^{(t)}, P_{t_j}(y_{j-1}^{(t)})) \leq \delta_1 \text{ for each integer } j = 1, \dots, p(t), \tag{5.75}$$

$$\alpha_t = \max\{d(y_j^{(t)}, y_{j-1}^{(t)}) : \ j = 1, \dots, p(t)\}. \tag{5.76}$$

Proposition 5.8, (5.17), (5.71), (5.74), and (5.75) imply that for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}$ and each $j \in \{1, \dots, p(t)\}$,

$$d(y_j^{(t)}, z) \leq d(y_{j-1}^{(t)}, z) + 2\delta_0 + \delta_1, \tag{5.77}$$

$$d(y_j^{(t)}, z) \leq d(z, x_k) + j(2\delta_0 + \delta_1). \tag{5.78}$$

By (5.10), (5.18), (5.74), and (5.78), for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}$ and each $j \in \{1, \dots, p(t)\}$,

$$d(y_j^{(t)}, z) \leq d(x_k, z) + \bar{q}(2\delta_0 + \delta_1),$$
$$d(z, y_t) \leq d(z, x_k) + \bar{q}(2\delta_0 + \delta_1). \tag{5.79}$$

In view of (5.72), there exists

$$s = (s_1, \dots, s_{p(s)}) \in \Omega_{k+1}$$

such that

$$(x_{k+1}, \lambda_{k+1}) = (y_s, \alpha_s). \tag{5.80}$$

Relations (5.69) and (5.80) imply that

$$\alpha_s = \lambda_{k+1} > \epsilon_0. \tag{5.81}$$

By (5.76) and (5.81), there exists

$$j_0 \in \{1, \dots, p(s)\} \tag{5.82}$$

such that

$$\epsilon_0 < \alpha_s = d(y_{j_0}^{(s)}, y_{j_0-1}^{(s)}). \tag{5.83}$$

It follows from (5.75) and (5.82) that

$$d(y_{j_0}^{(s)}, P_{s_{j_0}}(y_{j_0-1}^{(s)})) \leq \delta_1. \tag{5.84}$$

In view of (5.15), there exists

$$z_0 \in \mathrm{Fix}(P_{s_{j_0}}) \tag{5.85}$$

such that

$$d(z, z_0) \le \delta_0. \tag{5.86}$$

Relations (5.13) and (5.85) imply that

$$\begin{aligned} &d(z_0, P_{s_{j_0}}(y^{(s)}_{j_0-1}))^2 + \bar{c}d(P_{s_{j_0}}(y^{(s)}_{j_0-1}), y^{(s)}_{j_0-1})^2 \\ &\le d(z_0, y^{(s)}_{j_0-1})^2. \end{aligned} \tag{5.87}$$

By (5.10), (5.13), (5.68), (5.74), (5.78), and (5.86),

$$\begin{aligned} d(z_0, y^{(s)}_{j_0-1})^2 &\le (d(z_0, z) + d(z, y^{(s)}_{j_0-1}))^2 \\ &\le \delta_0^2 + d(z, y^{(s)}_{j_0-1})^2 + 2\delta_0 d(z, y^{(s)}_{j_0-1}) \\ &\le d(z, y^{(s)}_{j_0-1})^2 + 2\delta_0(d(z, y^{(s)}_{j_0-1}) + \delta_0) \\ &\le d(z, y^{(s)}_{j_0-1})^2 + 2\delta_0(d(z, x_k) + j_0(2\delta_0 + \delta_1)) \\ &\le d(z, y^{(s)}_{j_0-1})^2 + 2\delta_0(2M + 1). \end{aligned} \tag{5.88}$$

In view of (5.2), (5.68), (5.74), (5.78), (5.85), and (5.86),

$$\begin{aligned} &d(z, P_{s_{j_0}}(y^{(s)}_{j_0-1})) \le d(z, z_0) + d(z_0, P_{s_{j_0}}(y^{(s)}_{j_0-1})) \\ &\le \delta_0 + d(z_0, y^{(s)}_{j_0-1}) \le \delta_0 + d(z_0, z) + d(z, y^{(s)}_{j_0-1}) \\ &\le 2\delta_0 + d(z, x_k) + (j_0 - 1)(2\delta_0 + \delta_1) \le 2M + 2\delta_0 + (j_0 - 1)(2\delta_0 + \delta_1). \end{aligned} \tag{5.89}$$

It follows (5.2), (5.10), (5.13), (5.68), (5.77), (5.85), and (5.86) that

$$\begin{aligned} &d(z, P_{s_{j_0}}(y^{(s)}_{j_0-1}))^2 \le (d(z, z_0) + d(z_0, P_{s_{j_0}}(y^{(s)}_{j_0-1})))^2 \\ &\le d(z, z_0)^2 + d(z_0, P_{s_{j_0}}(y^{(s)}_{j_0-1}))^2 + 2d(z, z_0)d(z_0, P_{s_{j_0}}(y^{(s)}_{j_0-1})) \\ &\le \delta_0^2 + d(z_0, P_{s_{j_0}}(y^{(s)}_{j_0-1}))^2 + 2\delta_0 d(z_0, y^{(s)}_{j_0-1}) \\ &\le d(z_0, P_{s_{j_0}}(y^{(s)}_{j_0-1}))^2 + 2\delta_0(d(z_0, y^{(s)}_{j_0-1}) + \delta_0) \\ &\le d(z_0, P_{s_{j_0}}(y^{(s)}_{j_0-1}))^2 + 2\delta_0(d(z, x_k) + j_0(2\delta_0 + \delta_1)) \\ &\le d(z_0, P_{s_{j_0}}(y^{(s)}_{j_0-1}))^2 + 2\delta_0(2M + 1). \end{aligned} \tag{5.90}$$

Relations (5.10), (5.13), (5.75), and (5.89) imply that

$$\begin{aligned} d(z, y_{j_0}^{(s)})^2 &\le (d(z, P_{s_{j_0}}(y_{j_0-1}^{(s)})) + d(P_{s_{j_0}}(y_{j_0-1}^{(s)}), y_{j_0}^{(s)}))^2 \\ &\le d(z, P_{s_{j_0}}(y_{j_0-1}^{(s)}))^2 + \delta_1^2 + 2\delta_1 d(z, P_{s_{j_0}}(y_{j_0-1}^{(s)})) \\ &\le d(z, P_{s_{j_0}}(y_{j_0-1}^{(s)}))^2 + 2\delta_1(d(z, P_{s_{j_0}}(y_{j_0-1}^{(s)})) + \delta_1) \\ &\le d(z, P_{s_{j_0}}(y_{j_0-1}^{(s)}))^2 + 2\delta_1(2M+1). \end{aligned} \tag{5.91}$$

By (5.87), (5.88), (5.90), and (5.91),

$$\begin{aligned} d(z, y_{j_0}^{(s)})^2 &\le d(z, P_{s_{j_0}}(y_{j_0-1}^{(s)}))^2 + 2\delta_1(2M+1) \\ &\le d(z_0, P_{s_{j_0}}(y_{j_0-1}^{(s)}))^2 + (2M+1)(2\delta_1 + 2\delta_0) \\ &\le d(z_0, y_{j_0-1}^{(s)})^2 - \bar{c} d(P_{s_{j_0}}(y_{j_0-1}^{(s)}), y_{j_0-1}^{(s)})^2 + (2M+1)(2\delta_1 + 2\delta_0) \\ &\le d(z, y_{j_0-1}^{(s)})^2 + 2\delta_0(2M+1) - \bar{c} d(P_{s_{j_0}}(y_{j_0-1}^{(s)}), y_{j_0-1}^{(s)})^2 + (2\delta_1 + 2\delta_0)(2M+1). \end{aligned} \tag{5.92}$$

In view of (5.16) and (5.83),

$$\begin{aligned} & d(P_{s_{j_0}}(y_{j_0-1}^{(s)}), y_{j_0-1}^{(s)})^2 \\ & \ge (d(y_{j_0}^{(s)}, y_{j_0-1}^{(s)}) - d(y_{j_0}^{(s)}, P_{s_{j_0}}(y_{j_0-1}^{(s)}))^2 > (\epsilon_0 - \delta_1) > (3/4)\epsilon_0^2. \end{aligned} \tag{5.93}$$

Relations (5.92) and (5.93) imply that

$$d(z, y_{j_0}^{(s)})^2 \le d(z, y_{j_0-1}^{(s)})^2 - (3/4)\epsilon_0^2 \bar{c} + (2M+1)(2\delta_1 + 4\delta_0). \tag{5.94}$$

It follows from (5.10), (5.13), (5.77), and (5.78) that for each $j = 1, \dots, p(s)$,

$$\begin{aligned} & d(y_j^{(s)}, z)^2 - d(y_{j-1}^{(s)}, z)^2 \\ & \le (d(z, y_{j-1}^{(s)}) + 2\delta_0 + \delta_1)^2 - d(y_{j-1}^{(s)}, z)^2 \\ & \le (2\delta_0 + \delta_1)^2 + 2(2\delta_0 + \delta_1) d(y_{j-1}^{(s)}, z) \\ & \le 2(2\delta_0 + \delta_1)(d(y_{j-1}^{(s)}, z) + (2\delta_0 + \delta_1)) \\ & \le 2(2\delta_0 + \delta_1)(d(x_k, z) + \bar{q}(2\delta_0 + \delta_1)) \le 2(2\delta_0 + \delta_1)(2M+1). \end{aligned} \tag{5.95}$$

By (5.10), (5.13), (5.72), (5.74), (5.94), and (5.95),

$$\begin{aligned} d(z, x_k)^2 - d(z, x_{k+1})^2 &= d(z, y_0^{(s)})^2 - d(z, y_{p(s)}^{(s)})^2 \\ &= \sum_{j=0}^{p(s)-1} (d(z, y_j^{(s)})^2 - d(z, y_{j+1}^{(s)})^2) \end{aligned}$$

$$= \sum \{d(z, y_j^{(s)})^2 - d(z, y_{j+1}^{(s)})^2 : j \in \{0, \dots, p(s-1)\} \setminus \{j_0 - 1\}\}$$
$$+ d(z, y_{j_0-1}^{(s)})^2 - d(z, y_{j_0}^{(s)})^2$$
$$\geq -2(2\delta_0 + \delta_1)(2M+1)(p(s)-1) + (3/4)\epsilon_0^2 \bar{c} - (2M+1)(2\delta_1 + 4\delta_0)$$
$$= (3/4)\epsilon_0^2 \bar{c} - (2M+1)(2\delta_1 + 4\delta_0)\bar{q} \geq 2^{-1}\epsilon_0^2 \bar{c}.$$

Lemma 5.11 is proved. □

Assume that q is a natural number such that for each nonnegative integer $k < q$,

$$\lambda_{k+1} > \epsilon_0.$$

By (5.67) and Lemma 5.11 applied by induction,

$$d(x_k, z) \leq 2M, \ k = 0, \dots, q$$

and for each nonnegative integer $k < q$,

$$d(z, x_k)^2 - d(z, x_{k+1})^2 \geq 2^{-1}\epsilon_0^2 \bar{c}. \tag{5.96}$$

It follows from (5.17), (5.67), and (5.96) that

$$4M^2 \geq d(z, x_0)^2 \geq d(z, x_0)^2 - d(z, x_q)^2$$
$$= \sum_{k=0}^{q-1} (d(z, x_k)^2 - d(z, x_{k+1})^2) \geq 2^{-1} q \epsilon_0^2 \bar{c}$$

and

$$q \leq 8M^2 \epsilon_0^{-2} \bar{c}^{-1} \leq n_0.$$

This implies that there exists a nonnegative integer $q \leq n_0$ such that for all nonnegative integers $k \leq q$,

$$d(x_k, z) \leq 2M$$

and

$$\lambda_{q+1} \leq \epsilon_0.$$

Assume that an integer q satisfies

$$\lambda_{q+1} \leq \epsilon_0.$$

In view of (5.18), (5.20), and Proposition 5.9 applied with $\gamma = \delta_1$, $\xi_0 = x_q$ and $\xi_1 = x_{q+1}$,

$$x_q \in \tilde{F}_{\epsilon_0(\bar{q}+2)}(P_s)$$

for all $s = 1, \dots, m$. Theorem 5.1 is proved. □

5.4 Proof of Theorem 5.2

Set

$$\gamma_0 = \epsilon(\bar{q} + 2)^{-1}. \tag{5.97}$$

Let $k \geq 0$ be an integer. In view of (5.25),

$$(x_{k+1}, \lambda_{k+1}) \in A(x_k, \Omega_{k+1}, 0). \tag{5.98}$$

By (5.12) and (5.98), there exist

$$(y_{k,t}, \alpha_{k,t}) \in A_0(x_k, t, 0),\ \ t \in \Omega_{k+1} \tag{5.99}$$

such that

$$(x_{k+1}, \lambda_{k+1}) \in \{(y_{k,t}, \alpha_{k,t}) :\ t \in \Omega_{k+1}\}, \tag{5.100}$$

$$\lambda_{k+1} = \max\{\alpha_{k,t} :\ t \in \Omega_{k+1}\}. \tag{5.101}$$

It follows from (5.11) and (5.99) that for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}$ there exists a finite sequence $\{y_i^{(k,t)}\}_{i=0}^{p(t)} \subset X$ such that

$$y_0^{(k,t)} = x_k,\ y_{p(t)}^{(k,t)} = y_{k,t}, \tag{5.102}$$

$$y_j^{(k,t)} = P_{t_j}(y_{j-1}^{(k,t)}) \text{ for each integer } j = 1, \dots, p(t), \tag{5.103}$$

$$\alpha_{k,t} = \max\{d(y_j^{(k,t)}, y_{j-1}^{(k,t)}) :\ j = 1, \dots, p(t)\}. \tag{5.104}$$

By (5.100), there exists

$$s^{(k)} \in \Omega_{k+1}$$

such that

$$(x_{k+1}, \lambda_{k+1}) = (y_{k,s^{(k)}}, \alpha_{k,s^{(k)}}). \tag{5.105}$$

Set

$$E = \{k \in \{0, 1, \dots\} : \ \lambda_{k+1} > \gamma_0\}. \tag{5.106}$$

Let n be a natural number and δ be an arbitrary positive number. By the assumptions of the theorem, there exists

$$z_\delta \in B(\theta, M) \tag{5.107}$$

such that

$$B(z_\delta, \delta) \cap \mathrm{Fix}(P_i) \neq \emptyset \text{ for all } i = 1, \dots, m. \tag{5.108}$$

In view of (5.24) and (5.107),

$$d(z_\delta, x_0) \le 2M. \tag{5.109}$$

Proposition 5.8, (5.10), (5.102), (5.103), and (5.108) imply that for each integer $k \ge 0$, each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}$ and each $i \in \{1, \dots, p(t)\}$,

$$d(z_\delta, y_i^{(k,t)}) \le d(z_\delta, y_{i-1}^{(k,t)}) + 2\delta, \tag{5.110}$$

$$d(z_\delta, y_i^{(k,t)}) \le d(z_\delta, x_k) + 2i\delta \le d(z_\delta, x_k) + 2\bar{q}\delta. \tag{5.111}$$

$$d(z_\delta, y_{k,t}) \le d(z_\delta, x_k) + 2\bar{q}\delta, \tag{5.112}$$

$$d(z_\delta, x_{k+1}) \le d(z_\delta, x_k) + 2\bar{q}\delta. \tag{5.113}$$

In view of (5.109) and (5.113), for all integers $k = 0, \dots, n$,

$$d(z_\delta, x_k) \le d(z, x_0) + 2k\bar{q}\delta \le 2M + 2\bar{q}n\delta. \tag{5.114}$$

By (5.107), (5.111), and (5.114), for all $k = 0, \dots, n$, each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}$ and each $i \in \{0, 1, \dots, p(t)\}$,

$$d(z_\delta, y_i^{(k,t)}) \le 2M + 2\bar{q}\delta(n + 1),$$

$$d(\theta, y_i^{(k,t)}) \le 3M + 2\bar{q}\delta(n + 1).$$

Since δ is an arbitrary positive number we conclude that for all integers $k = 0, \dots, n$, each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}$ and each $i \in \{0, 1, \dots, p(t)\}$,

$$d(\theta, y_i^{(k,t)}) \le 3M. \tag{5.115}$$

Since n is an arbitrary natural number we conclude that (5.115) holds for all integers $k \ge 0$, each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}$ and all $i \in \{0, 1, \dots, p(t)\}$.

Let n be a natural number and

$$\epsilon_0 \in (0, (2\bar{q}n)^{-1}). \tag{5.116}$$

Since the function

$$(\xi_1, \xi_2) \to d(\xi_1, \xi_2)^2, \ (\xi_1, \xi_2) \in X \times X$$

is uniformly continuous on bounded subsets of $X \times X$ there exists $\delta \in (0, 1)$ such that for each

$$(\xi_1, \xi_2), \ (\eta_1, \eta_2) \in B(\theta, 3M + 1) \times B(\theta, 3M + 1)$$

satisfying $d(\xi_i, \eta_i) \le \delta, \ i = 1, 2$ the following inequality holds:

$$|d(\xi_1, \xi_2)^2 - d(\eta_1, \eta_2)^2| \le \epsilon_0. \tag{5.117}$$

By the assumptions of the theorem, there exists

$$z \in B(\theta, M) \tag{5.118}$$

such that

$$B(z, \delta) \cap \mathrm{Fix}(P_i) \neq \emptyset \text{ for all } i = 1, \dots, m. \tag{5.119}$$

In view of (5.119), for every $i \in \{1, \dots, m\}$ there exists

$$z_i \in \mathrm{Fix}(P_i) \cap B(z, \delta). \tag{5.120}$$

Relations (5.118) and (5.120) imply that

$$z_i \in B(\theta, M + 1), \ i = 1, \dots, m. \tag{5.121}$$

Let $k \in \{0, \dots, n\}$, $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}$ and $j \in \{1, \dots, p(t)\}$. By the choice of δ, (5.115), (5.118), (5.120), and (5.121),

$$|d(z, y_{j-1}^{(k,t)})^2 - d(z_{t_j}, y_{j-1}^{(k,t)})^2| \le \epsilon_0,$$

$$|d(z, y_j^{(k,t)})^2 - d(z_{t_j}, y_j^{(k,t)})^2| \le \epsilon_0.$$

Together with (5.22), (5.103), and (5.120) this implies that

$$d(z, y_{j-1}^{(k,t)})^2 - d(z, y_j^{(k,t)})^2$$

$$\ge d(z_{t_j}, y_{j-1}^{(k,t)})^2 - d(z_{t_j}, y_j^{(k,t)})^2 - 2\epsilon_0$$

$$= d(z_{t_j}, y_{j-1}^{(k,t)})^2 - d(z_{t_j}, P_{t_j}(y_{j-1}^{(k,t)}))^2 - 2\epsilon_0$$
$$\geq \bar{c}d(y_{j-1}^{(k,t)}, P_{t_j}(y_{j-1}^{(k,t)}))^2 - 2\epsilon_0. \quad (5.122)$$

It follows from (5.10), (5.102), (5.104), and (5.122) that

$$d(z, x_k)^2 - d(z, y_{k,t})^2 = d(z, y_0^{(k,t)})^2 - d(z, y_{p(t)}^{(k,t)})^2$$
$$= \sum_{j=1}^{p(t)} (d(z, y_{j-1}^{(k,t)})^2 - d(z, y_j^{(k,t)})^2)$$
$$\geq \bar{c} \sum_{j=1}^{p(t)} d(y_{j-1}^{(k,t)}, y_j^{(k,t)})^2 - 2\epsilon_0 \bar{q} \geq \bar{c}\alpha_{k,t}^2 - 2\epsilon_0 \bar{q}. \quad (5.123)$$

In view of (5.105) and (5.123),

$$d(z, x_k)^2 - d(z, x_{k+1})^2 = d(z, x_k)^2 - d(z, y_{k,s^{(k)}})^2$$
$$\geq \bar{c}\alpha_{k,s^{(k)}}^2 - 2\epsilon_0 \bar{q} = \bar{c}\lambda_{k+1}^2 - 2\epsilon_0 \bar{q}. \quad (5.124)$$

By (5.24), (5.116), (5.118), and (5.124),

$$4M^2 \geq d(z, x_0)^2 \geq d(z, x_0)^2 - d(z, x_n)^2$$
$$= \sum_{k=0}^{n-1} (d(z, x_k)^2 - d(z, x_{k+1})^2) \geq \bar{c} \sum_{k=0}^{n-1} \lambda_{k+1}^2 - 2\epsilon_0 \bar{q} n$$

and

$$\bar{c} \sum_{k=0}^{n-1} \lambda_{k+1}^2 \leq 4M^2 + 2\epsilon_0 \bar{q} n \leq 4M^2 + 1. \quad (5.125)$$

It follows from (5.106) and (5.125) that

$$\bar{c}^{-1}(4M^2 + 1) \geq \sum_{k=0}^{n-1} \lambda_{k+1}^2$$
$$\geq \sum \{\lambda_{k+1}^2 : \ k \in \{0, \dots, n-1\} \cap E\}$$
$$\geq \gamma_0^2 \mathrm{Card}(\{k \in \{0, \dots, n-1\} \cap E\}),$$
$$\mathrm{Card}(\{k \in \{0, \dots, n-1\} \cap E\}) \leq \gamma_0^{-2} \bar{c}^{-1}(4M^2 + 1).$$

Since the relation above holds for any natural number n we deduce from (5.97) that

$$\mathrm{Card}(E) \le \gamma_0^{-2}\bar{c}^{-1}(4M^2+1) = \epsilon^{-2}(\bar{q}+1)^2\bar{c}^{-1}(4M^2+1).$$

Assume that an integer $k \ge 0$ satisfies

$$k \notin E.$$

Then

$$\lambda_{k+1} \le \gamma_0.$$

By the inequality above and Proposition 5.9 applied with $\gamma = 0$, $\epsilon_0 = \gamma_0$,

$$x_k \in \tilde{F}_{\gamma_0(\bar{q}+2)} = \tilde{F}_\epsilon$$

for any nonnegative integer k satisfying $k \notin E$. This competes the proof of Theorem 5.2. □

5.5 Proof of Theorem 5.3

By (A1) and the choice of $\gamma_0 \in (0, 1)$ the following property holds:

(P1) for each $i \in \{1, \dots, m\}$, each

$$z \in B(\theta, 3M+1) \cap \mathrm{Fix}(P_i)$$

and each $x \in B(\theta, 3M+1)$ satisfying $d(x, P_i(x)) \ge 2^{-1}\epsilon_0$,

$$d(z, P_i(x)) \le d(z, x) - \gamma_0.$$

Choose a positive number

$$\delta_0 \le (12\bar{q})^{-1}\gamma_0. \tag{5.126}$$

By the assumption of the theorem there exists

$$z \in B(\theta, \bar{M}) \tag{5.127}$$

such that

$$B(z, \delta_0) \cap \mathrm{Fix}(P_i) \ne \emptyset \text{ for all } i = 1, \dots, m. \tag{5.128}$$

In view of (5.128), for every $i \in \{1, \dots, m\}$ there exists

$$z_i \in B(z, \delta_0) \cap \mathrm{Fix}(P_i). \tag{5.129}$$

Relations (5.127) and (5.129) imply that

$$z_i \in B(\theta, \bar{M} + 1) \subset B(\theta, M + 1), \; i = 1, \dots, m. \tag{5.130}$$

Let

$$i \in \{1, \dots, m\} \text{ and } x \in B(\theta, 3M + 1) \tag{5.131}$$

satisfy

$$d(x, P_i(x)) \geq \epsilon_0/2. \tag{5.132}$$

Property (P1), (5.13), (5.129), (5.130), and (5.131) imply that

$$d(z_i, P_i(x)) \leq d(z_i, x) - \gamma_0.$$

It follows from the inequality above, (5.126) and (5.129) that

$$\begin{aligned} d(z, P_i(x)) &\leq d(z, z_i) + d(z_i, P_i(x)) \leq \delta_0 + d(z_i, x) - \gamma_0 \\ &\leq \delta_0 - \gamma_0 + d(z_i, z) + d(z, x) \\ &\leq d(z, x) + 2\delta_0 - \gamma_0 \leq d(z, x) - \gamma_0/2. \end{aligned}$$

Thus we have shown that the following property holds:

(P2) for each $i \in \{1, \dots, m\}$ and each $x \in B(\theta, 3M + 1)$ satisfying $d(x, P_i(x)) \geq 2^{-1}\epsilon_0$,

$$d(z, P_i(x)) \leq d(z, x) - \gamma_0/2.$$

Let $k \geq 0$ be an integer. In view of (5.31),

$$(x_{k+1}, \lambda_{k+1}) \in A(x_k, \Omega_{k+1}, \delta). \tag{5.133}$$

By (5.12) and (5.133) there exist

$$(y_{k,t}, \alpha_{k,t}) \in A_0(x_k, t, \delta), \; t \in \Omega_{k+1} \tag{5.134}$$

such that

$$(x_{k+1}, \lambda_{k+1}) \in \{(y_{k,t}, \alpha_{k,t}) : \; t \in \Omega_{k+1}\}, \tag{5.135}$$

$$\lambda_{k+1} = \max\{\alpha_{k,t} : \; t \in \Omega_{k+1}\}. \tag{5.136}$$

It follows from (5.11) and (5.134) that for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}$ there exists a finite sequence $\{y_i^{(k,t)}\}_{i=0}^{p(t)} \subset X$ such that

$$y_0^{(k,t)} = x_k,\ y_{p(t)}^{(k,t)} = y_{k,t}, \tag{5.137}$$

$$d(y_j^{(k,t)}, P_{t_j}(y_{j-1}^{(k,t)})) \le \delta \text{ for each integer } j = 1, \dots, p(t), \tag{5.138}$$

$$\alpha_{k,t} = \max\{d(y_j^{(k,t)}, y_{j-1}^{(k,t)}) :\ j = 1, \dots, p(t)\}. \tag{5.139}$$

Proposition 5.8, (5.128), (5.137), and (5.138) imply that for each

$$t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}$$

and each $j \in \{1, \dots, p(t)\}$,

$$d(y_j^{(k,t)}, z) \le d(y_{j-1}^{(k,t)}, z) + 2\delta_0 + \delta, \tag{5.140}$$

$$d(y_j^{(k,t)}, z) \le d(z, x_k) + j(2\delta_0 + \delta). \tag{5.141}$$

By (5.10), (5.137), and (5.141), for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}$ and each $j \in \{0, 1, \dots, p(t)\}$,

$$d(y_j^{(k,t)}, z) \le d(x_k, z) + \bar{q}(2\delta_0 + \delta), \tag{5.142}$$

$$d(z, y_{k,t}) \le d(z, x_k) + \bar{q}(2\delta_0 + \delta). \tag{5.143}$$

In view of (5.30) and (5.127),

$$d(x_0, z) \le 2M. \tag{5.144}$$

We prove the following auxiliary result.

Lemma 5.12. *Assume that a nonnegative integer k satisfies*

$$d(x_k, z) \le 2M, \tag{5.145}$$

$$\lambda_{k+1} > \epsilon_0. \tag{5.146}$$

Then

$$d(z, x_k) - d(z, x_{k+1}) \ge 4^{-1}\gamma_0.$$

Proof. In view of (5.136), there exists

$$s = (s_1, \dots, s_{p(s)}) \in \Omega_{k+1}$$

such that

$$(x_{k+1}, \lambda_{k+1}) = (y_{k,s}, \alpha_{k,s}). \tag{5.147}$$

Relations (5.146) and (5.147) imply that

$$\alpha_{k,s} = \lambda_{k+1} > \epsilon_0. \tag{5.148}$$

By (5.139) and (5.148), there exists

$$j_0 \in \{1, \dots, p(s)\}$$

such that

$$\epsilon_0 < \alpha_{k,s} = d(y_{j_0}^{(k,s)}, y_{j_0-1}^{(k,s)}). \tag{5.149}$$

In view of (5.138),

$$d(y_{j_0}^{(k,s)}, P_{s_{j_0}}(y_{j_0-1}^{(k,s)})) \le \delta. \tag{5.150}$$

It follows from (5.149) and (5.150) that

$$d(y_{j_0-1}^{(k,s)}, P_{s_{j_0}}(y_{j_0-1}^{(k,s)})) \ge d(y_{j_0}^{(k,s)}, y_{j_0-1}^{(k,s)}) - d(y_{j_0}^{(k,s)}, P_{s_{j_0}}(y_{j_0-1}^{(k,s)})) > \epsilon_0 - \delta. \tag{5.151}$$

Relations (5.28) and (5.151) imply that

$$d(y_{j_0-1}^{(k,s)}, P_{s_{j_0}}(y_{j_0-1}^{(k,s)})) > \epsilon_0/2. \tag{5.152}$$

By (5.28), (5.126), (5.127), and (5.142),

$$\begin{gathered} d(z, y_{j_0-1}^{(k,s)}) \le d(z, x_k) + \bar{q}(2\delta_0 + \delta) \le 2M + 1, \\ d(y_{j_0-1}^{(k,s)}, \theta) \le 3M + 1. \end{gathered} \tag{5.153}$$

In view of (5.152), (5.153), and property (P2),

$$d(z, P_{s_{j_0}}(y_{j_0-1}^{(k,s)})) \le d(z, y_{j_0-1}^{(k,s)}) - 2^{-1}\gamma_0. \tag{5.154}$$

It follows from (5.138) and (5.154) that

$$\begin{gathered} d(z, y_{j_0}^{(k,s)}) \le d(z, P_{s_{j_0}}(y_{j_0-1}^{(k,s)})) + d(P_{s_{j_0}}(y_{j_0-1}^{(k,s)}), y_{j_0}^{(k,s)}) \\ \le d(z, y_{j_0-1}^{(k,s)}) - \gamma_0/2 + \delta, \\ d(z, y_{j_0-1}^{(k,s)}) - d(z, y_{j_0}^{(k,s)}) \ge \gamma_0/2 - \delta. \end{gathered} \tag{5.155}$$

By (5.10), (5.28), (5.126), (5.137), (5.140), and (5.147),

$$
\begin{aligned}
d(z, x_k) - d(z, x_{k+1}) &= d(x_k, z) - d(y_{k,s}, z) \\
&= d(z, y_0^{(k,s)}) - d(z, y_{p(s)}^{(k,s)}) \\
&= \sum_{j=1}^{p(s)} (d(z, y_{j-1}^{(k,s)}) - d(z, y_j^{(k,s)}) \\
&\geq -(2\delta_0 + \delta)(p(s) - 1) + d(z, y_{j_0-1}^{(k,s)}) - d(z, y_{j_0}^{(k,s)}) \\
&\geq -2(2\delta_0 + \delta)(p(s) - 1) + \gamma_0/2 - \delta \\
&\geq \gamma_0/2 - (2\delta_0 + \delta)\bar{q} \geq \gamma_0/4.
\end{aligned}
$$

Lemma 5.12 is proved. □

Assume that q is a natural number such that for each nonnegative integer $k < q$,

$$\lambda_{k+1} > \epsilon_0.$$

By (5.144) and Lemma 5.12 applied by induction,

$$d(x_k, z) \leq 2M, \ k = 0, \dots, q \tag{5.156}$$

and for each nonnegative integer $k < q$,

$$d(z, x_k) - d(z, x_{k+1}) \geq 4^{-1}\gamma_0. \tag{5.157}$$

It follows from (5.27) and (5.144) that

$$
\begin{aligned}
2M &\geq d(z, x_0) \geq d(z, x_0) - d(z, x_q) \\
&= \sum_{k=0}^{q-1} (d(z, x_k) - d(z, x_{k+1})) \geq 4^{-1} q\gamma_0
\end{aligned}
$$

and

$$q \leq 8M\gamma_0^{-1} \leq n_0.$$

This implies that there exists a nonnegative integer $q \leq n_0$ such that for all nonnegative integers $k \leq q$,

$$d(x_k, z) \leq 2M$$

and

$$\lambda_{q+1} \le \epsilon_0.$$

Assume that an integer $q \ge 0$ satisfies $\lambda_{q+1} \le \epsilon_0$. In view of (5.26), (5.28), and Proposition 5.9,

$$x_q \in \tilde{F}_{\epsilon_0(\bar{q}+2)} = \tilde{F}_\epsilon.$$

Theorem 5.3 is proved. □

5.6 Proof of Theorem 5.4

Let

$$\epsilon_0 = \epsilon(\bar{q}+2)^{-1}. \tag{5.158}$$

By the assumptions of the theorem, there exists $\gamma_0 \in (0, \epsilon_0)$ such that the following property holds:

(P3) for each $i \in \{1, \dots, m\}$, each

$$\xi \in B(\theta, M+1) \cap \mathrm{Fix}(P_i)$$

and each $x \in B(\theta, 3M+1)$ satisfying $d(x, P_i(x)) \ge \epsilon_0/2$,

$$d(\xi, P_i(x)) \le d(\xi, x) - \gamma_0.$$

Choose a positive number

$$Q = 2\gamma_0^{-1}(M+1). \tag{5.159}$$

Assume that

$$\{\Omega_i\}_{i=1}^\infty \subset \mathcal{M}_*, \tag{5.160}$$

$$x_0 \in B(\theta, M) \tag{5.161}$$

and $\{x_i\}_{i=1}^\infty \subset X,\ \{\lambda_i\}_{i=1}^\infty \subset [0,\infty)$, for each natural number i,

$$(x_i, \lambda_i) \in A(x_{i-1}, \Omega_i, 0). \tag{5.162}$$

Let $k \ge 0$ be an integer. In view of (5.162),

$$(x_{k+1}, \lambda_{k+1}) \in A(x_k, \Omega_{k+1}, 0). \tag{5.163}$$

By (5.12) and (5.163), there exist

$$(y_{k,t}, \alpha_{k,t}) \in A_0(x_k, t, 0),\ t \in \Omega_{k+1} \tag{5.164}$$

such that

$$(x_{k+1}, \lambda_{k+1}) \in \{(y_{k,t}, \alpha_{k,t}) :\ t \in \Omega_{k+1}\}, \tag{5.165}$$

$$\lambda_{k+1} = \max\{\alpha_{k,t} :\ t \in \Omega_{k+1}\}. \tag{5.166}$$

It follows from (5.11) and (5.164) that for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}$ there exists a finite sequence $\{y_i^{(k,t)}\}_{i=0}^{p(t)} \subset X$ such that

$$y_0^{(k,t)} = x_k,\ y_{p(t)}^{(k,t)} = y_{k,t}, \tag{5.167}$$

$$y_j^{(k,t)} = P_{t_j}(y_{j-1}^{(k,t)}) \text{ for each integer } j = 1, \dots, p(t), \tag{5.168}$$

$$\alpha_{k,t} = \max\{d(y_j^{(k,t)}, y_{j-1}^{(k,t)}) :\ j = 1, \dots, p(t)\}. \tag{5.169}$$

Set

$$E = \{k \in \{0, 1, \dots\} :\ \lambda_{k+1} > 2^{-1}\epsilon_0\}. \tag{5.170}$$

Let n be a natural number and δ be an arbitrary positive number which satisfy

$$\delta < (2\bar{q}(n+1))^{-1},\ \delta < (4\bar{q})^{-1}\gamma_0. \tag{5.171}$$

By the assumptions of the theorem, there exists

$$z \in B(\theta, M) \tag{5.172}$$

such that

$$B(z, \delta) \cap \mathrm{Fix}(P_i) \neq \emptyset \text{ for all } i = 1, \dots, m. \tag{5.173}$$

In view of (5.161) and (5.172),

$$d(z, x_0) \leq 2M. \tag{5.174}$$

Proposition 5.8, (5.10), (5.160), (5.162), (5.165), (5.167), and (5.173) imply that for each integer $k \geq 0$, each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}$ and each $j \in \{1, \dots, p(t)\}$,

$$d(z, y_i^{(k,t)}) \leq d(z, y_{i-1}^{(k,t)}) + 2\delta, \tag{5.175}$$

$$d(z, y_i^{(k,t)}) \leq d(z, x_k) + 2i\delta \leq d(z, x_k) + 2\bar{q}\delta, \tag{5.176}$$

$$d(z, x_{k+1}) \leq d(z, x_k) + 2\bar{q}\delta. \tag{5.177}$$

In view of (5.171), (5.174), (5.176), and (5.177), for all integers $k = 0, \dots, n$, each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}$ and each $i \in \{1, \dots, p(t)\}$,

$$d(z, x_k) \le d(z, x_0) + 2k\bar{q}\delta \le 2M + 2\bar{q}n\delta,$$

$$d(y_i^{(k,t)}, z) \le d(x_k, z) + 2\bar{q}\delta \le 2M + 2(n+1)\bar{q}\delta \le 2M + 1.$$

Together with (5.172) this implies that for each $k = 0, \dots, n$, each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}$ and each $i \in \{1, \dots, p(t)\}$,

$$y_i^{(k,t)} \in B(\theta, 3M + 1). \tag{5.178}$$

Assume that an integer $k \in \{0, \dots, n-1\}$ satisfies

$$\lambda_{k+1} > 2^{-1}\epsilon_0. \tag{5.179}$$

By (5.165), there exists

$$s \in \Omega_{k+1}$$

such that

$$(x_{k+1}, \lambda_{k+1}) = (y_{k,s}, \alpha_{k,s}). \tag{5.180}$$

In view (5.179) and (5.180),

$$2^{-1}\epsilon_0 < \lambda_{k+1} = \alpha_{k,s}. \tag{5.181}$$

Relation (5.181) implies that there exists $j_0 \in \{1, \dots, p(s)\}$ such that

$$2^{-1}\epsilon_0 < \alpha_{k,s} = d(y_{j_0}^{(k,s)}, y_{j_0-1}^{(k,s)}). \tag{5.182}$$

It follows from (5.168) and (5.182) that

$$d(y_{j_0-1}^{(k,s)}, P_{s_{j_0}}(y_{j_0-1}^{(k,s)})) > 2^{-1}\epsilon_0. \tag{5.183}$$

By (5.173), there exists

$$z_0 \in B(z, \delta) \cap \mathrm{Fix}(P_{s_{j_0}}). \tag{5.184}$$

Property (P3), (5.171), (5.172), (5.184) imply that

$$d(\theta, z_0) \le M + 1. \tag{5.185}$$

It follows from (5.168), (5.178), and (5.183)–(5.185) that

$$d(z_0, y_{j_0}^{(k,s)}) = d(z_0, P_{s_{j_0}}(y_{j_0-1}^{(k,s)})) \le d(z_0, y_{j_0-1}^{(k,s)}) - \gamma_0. \tag{5.186}$$

In view of (5.184) and (5.186),

$$d(z, y_{j_0}^{(k,s)}) \le d(z, z_0) + d(z_0, y_{j_0}^{(k,s)}) \le \delta + d(z_0, y_{j_0-1}^{(k,s)}) - \gamma_0$$

$$\le -\gamma_0 + \delta + d(z, z_0) + d(z, y_{j_0-1}^{(k,s)}) \le -\gamma_0 + 2\delta + d(z, y_{j_0-1}^{(k,s)}),$$

$$d(z, y_{j_0-1}^{(k,s)}) - d(z, y_{j_0}^{(k,s)}) \ge \gamma_0 - 2\delta. \tag{5.187}$$

By (5.10), (5.167), (5.171), (5.175), (5.180), and (5.187),

$$d(z, x_k) - d(z, x_{k+1}) = d(z, y_0^{(k,s)}) - d(z, y_{p(s)}^{(k,s)})$$

$$= \sum_{j=1}^{p(s)} (d(z, y_{j-1}^{(k,s)}) - d(z, y_j^{(k,s)}))$$

$$\ge -2\delta(p(s) - 1) + d(z, y_{j_0-1}^{(k,s)}) - d(z, y_{j_0}^{(k,s)})$$

$$\ge -2\delta(p(s) - 1) + \gamma_0 - 2\delta$$

$$\ge \gamma_0 - 2\delta\bar{q} \ge \gamma_0/2.$$

Thus the following property holds:
if $k \in \{0, \dots, n-1\}$ and $\lambda_{k+1} > 2^{-1}\epsilon_0$, then

$$d(z, x_k) - d(z, x_{k+1}) \ge 2^{-1}\gamma_0.$$

By the property above, (5.159), (5.171), (5.174), and (5.177),

$$M \ge d(z, x_0) \ge d(z, x_0) - d(z, x_n)$$

$$= \sum_{k=0}^{n-1} (d(z, x_k) - d(z, x_{k+1}))$$

$$\ge -2\bar{q}\delta n + \sum \{d(z, x_k) - d(z, x_{k+1}) :\ k \in \{0, \dots, n-1\} \cap E\}$$

$$\ge -2\bar{q}\delta n + \mathrm{Card}(\{k \in \{0, \dots, n-1\} \cap E\})\gamma_0/2,$$

$$\mathrm{Card}(\{k \in \{0, \dots, n-1\} \cap E\}) \le 2\gamma_0^{-1}(M+1) = Q.$$

Since the relation above holds for any natural number n we conclude that

$$\mathrm{Card}(\{k \in \{0, 1 \dots\} :\ \lambda_{k+1} > 2^{-1}\epsilon_0\}) = \mathrm{Card}(E) \le 2\gamma_0^{-1}(M+1) = Q.$$

Let $k \in \{0, 1, \dots\}$ satisfy $\lambda_{k+1} \leq 2^{-1}\epsilon_0$. Then in view of (5.158) and Proposition 5.9,

$$x_k \in \tilde{F}_{2^{-1}\epsilon_0(\bar{q}+2)} \subset \tilde{F}_\epsilon.$$

Theorem 5.4 is proved. □

5.7 Proof of Theorem 5.5

Let $k \geq 0$ be an integer. In view of (5.39),

$$(x_{k+1}, \mu_{k+1}) \in \tilde{A}(x_k, \Omega_{k+1}, \delta). \tag{5.188}$$

By (5.33) and (5.188) there exist

$$(y_{k,t}, \beta_{k,t}) \in \tilde{A}_0(x_k, t, \delta),\ t \in \Omega_{k+1} \tag{5.189}$$

such that

$$(x_{k+1}, \mu_{k+1}) \in \{(y_{k,t}, \beta_{k,t}) :\ t \in \Omega_{k+1}\}, \tag{5.190}$$

$$\mu_{k+1} = \max\{\beta_{k,t} :\ t \in \Omega_{k+1}\}. \tag{5.191}$$

It follows from (5.32) and (5.189) that for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}$ there exists a finite sequence $\{y_i^{(k,t)}\}_{i=0}^{p(t)} \subset X$ such that

$$y_0^{(k,t)} = x_k,\ y_{p(t)}^{(k,t)} = y_{k,t}, \tag{5.192}$$

$$d(y_j^{(k,t)}, P_{t_j}(y_{j-1}^{(k,t)})) \leq \delta \text{ for each integer } j = 1, \dots, p(t), \tag{5.193}$$

$$\beta_{k,t} = \max\{d(y_{j-1}^{(k,t)}, \mathrm{Fix}(P_{t_j})) :\ j = 1, \dots, p(t)\}. \tag{5.194}$$

By (A2) and the choice of $\gamma_0 \in (0, 1)$ the following property holds:
(P4) for each $i \in \{1, \dots, m\}$, each

$$z \in B(\theta, 3M + 1) \cap \mathrm{Fix}(P_i)$$

and each $x \in B(\theta, 3M + 1)$ satisfying $d(x, \mathrm{Fix}(P_i)) \geq 2^{-1}\epsilon_0$,

$$d(z, P_i(x)) \leq d(z, x) - \gamma_0.$$

Choose a positive number

$$\delta_0 \le (12\bar{q})^{-1}\gamma_0. \tag{5.195}$$

By the assumption of the theorem there exists

$$z \in B(\theta, M) \tag{5.196}$$

such that

$$B(z, \delta_0) \cap \mathrm{Fix}(P_i) \ne \emptyset \text{ for all } i = 1, \dots, m. \tag{5.197}$$

In view of (5.197), for every $i \in \{1, \dots, m\}$ there exists

$$z_i \in B(z, \delta_0) \cap \mathrm{Fixp}(P_i). \tag{5.198}$$

Relations (5.195), (5.196), and (5.198) imply that

$$z_i \in B(\theta, M + 1),\ \ i = 1, \dots, m. \tag{5.199}$$

Let

$$i \in \{1, \dots, m\} \text{ and } x \in B(\theta, 3M + 1) \tag{5.200}$$

satisfy

$$d(x, \mathrm{Fix}(P_i)) \ge \epsilon_0/2. \tag{5.201}$$

Property (P4) and (5.198)–(5.201) imply that

$$d(z_i, P_i(x)) \le d(z_i, x) - \gamma_0. \tag{5.202}$$

It follows from (5.195), (5.198), and (5.202) that

$$\begin{aligned} d(z, P_i(x)) &\le d(z, z_i) + d(z_i, P_i(x)) \le \delta_0 + d(z_i, x) - \gamma_0 \\ &\le \delta_0 - \gamma_0 + d(z_i, z) + d(z, x) \\ &\le d(z, x) + 2\delta_0 - \gamma_0 \le d(z, x) - \gamma_0/2. \end{aligned} \tag{5.203}$$

Thus we have shown that the following property holds:

(P5) for each $i \in \{1, \dots, m\}$ and each $x \in B(\theta, 3M+1)$ satisfying $d(x, \mathrm{Fix}(P_i)) \ge 2^{-1}\epsilon_0$,

$$d(z, P_i(x)) \le d(z, x) - \gamma_0/2.$$

Proposition 5.8, (5.192), (5.193), and (5.197) imply that for each integer $k \geq 0$, each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}$ and each $j \in \{1, \dots, p(t)\}$,

$$d(y_j^{(k,t)}, z) \leq d(y_{j-1}^{(k,t)}, z) + 2\delta_0 + \delta, \tag{5.204}$$

$$d(y_j^{(k,t)}, z) \leq d(z, x_k) + j(2\delta_0 + \delta). \tag{5.205}$$

By (5.10), (5.192), and (5.205), for each integer $k \geq 0$, each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}$ and each $j \in \{0, 1, \dots, p(t)\}$,

$$d(y_j^{(k,t)}, z) \leq d(x_k, z) + \bar{q}(2\delta_0 + \delta), \tag{5.206}$$

$$d(z, y_{k,t}) \leq d(z, x_k) + \bar{q}(2\delta_0 + \delta). \tag{5.207}$$

In view of (5.38) and (5.196),

$$d(x_0, z) \leq 2M. \tag{5.208}$$

We prove the following auxiliary result.

Lemma 5.13. *Assume that a nonnegative integer k satisfies*

$$d(x_k, z) \leq 2M, \tag{5.209}$$

$$\mu_{k+1} > \epsilon_0. \tag{5.210}$$

Then

$$d(z, x_k) - d(z, x_{k+1}) \geq 4^{-1}\gamma_0.$$

Proof. In view of (5.190), there exists

$$s = (s_1, \dots, s_{p(s)}) \in \Omega_{k+1}$$

such that

$$(x_{k+1}, \mu_{k+1}) = (y_{k,s}, \beta_{k,s}). \tag{5.211}$$

Relations (5.210) and (5.211) imply that

$$\beta_{k,s} = \mu_{k+1} > \epsilon_0. \tag{5.212}$$

By (5.194) and (5.212), there exists

$$j_0 \in \{1, \dots, p(s)\}$$

such that

$$\epsilon_0 < \beta_{k,s} = d(y^{(k,s)}_{j_0-1}, \mathrm{Fix}(P_{s_{j_0}})). \tag{5.213}$$

In view of (5.10), (5.36), (5.195), (5.196), and (5.205),

$$d(z, y^{(k,s)}_{j_0-1}) \le d(z, x_k) + \bar{q}(2\delta_0 + \delta) \le 2M + 1,$$
$$d(y^{(k,s)}_{j_0-1}, \theta) \le 3M + 1. \tag{5.214}$$

Property (P5), (5.196), (5.213), and (5.214) imply that

$$d(z, P_{s_{j_0}}(y^{(k,s)}_{j_0-1})) \le d(z, y^{(k,s)}_{j_0-1}) - \gamma_0/2. \tag{5.215}$$

It follows from (5.125) and (5.193) that

$$d(z, y^{(k,s)}_{j_0}) \le d(z, P_{s_{j_0}}(y^{(k,s)}_{j_0-1})) + d(P_{s_{j_0}}(y^{(k,s)}_{j_0-1}), y^{(k,s)}_{j_0})$$
$$\le d(z, y^{(k,s)}_{j_0-1}) - \gamma_0/2 + \delta. \tag{5.216}$$

By (5.10), (5.36), (5.192), (5.195), (5.204), (5.209), (5.211), and (5.216),

$$d(z, x_k) - d(z, x_{k+1}) = d(z, y^{(k,s)}_0) - d(z, y^{(k,s)}_{p(s)})$$
$$= \sum_{j=1}^{p(s)} (d(z, y^{(k,s)}_{j-1}) - d(z, y^{(k,s)}_j))$$
$$\ge -(2\delta_0 + \delta)(p(s) - 1) + d(z, y^{(k,s)}_{j_0-1}) - d(z, y^{(k,s)}_{j_0})$$
$$\ge \gamma_0/2 - (2\delta_0 + \delta)p(s) \ge \gamma_0/2 - (2\delta_0 + \delta)\bar{q} \ge \gamma_0/4.$$

Lemma 5.13 is proved. □

Assume that q is a natural number such that for each nonnegative integer $k < q$,

$$\mu_{k+1} > \epsilon_0. \tag{5.217}$$

By (5.208), (5.217), and Lemma 5.13 applied by induction,

$$d(x_k, z) \le 2M,\ k = 0, \ldots, q \tag{5.218}$$

and for each nonnegative integer $k < q$,

$$d(z, x_k) - d(z, x_{k+1}) \ge 4^{-1}\gamma_0. \tag{5.219}$$

It follows from (5.35), (5.208), and (5.219) that

$$2M \geq d(z, x_0) \geq d(z, x_0) - d(z, x_q)$$
$$= \sum_{k=0}^{q-1} (d(z, x_k) - d(z, x_{k+1})) \geq 4^{-1} q \gamma_0$$

and

$$q \leq 8M\gamma_0^{-1} \leq n_0.$$

This implies that there exists a nonnegative integer $q \leq n_0$ such that for all nonnegative integers $k \leq q$,

$$d(x_k, z) \leq 2M$$

and

$$\mu_{q+1} \leq \epsilon_0.$$

Assume that an integer $q \geq 0$ satisfies

$$\mu_{q+1} \leq \epsilon_0.$$

In view of (5.34), (5.36), (5.144), (5.191), (5.193), Proposition 5.10 and the relation above, for all $s = 1, \ldots, m$,

$$d(x_q, \mathrm{Fix}(P_s)) \leq \bar{q}(2\epsilon_0 + \delta) \leq \epsilon.$$

Theorem 5.5 is proved. □

5.8 Proof of Theorem 5.6

Let $k \geq 0$ be an integer. We have

$$(x_{k+1}, \lambda_{k+1}) \in A(x_k, \Omega_{k+1}, \delta). \tag{5.220}$$

By (5.12) and (5.220) there exist

$$(y_{k,t}, \alpha_{k,t}) \in A_0(x_k, t, \delta),\ \ t \in \Omega_{k+1} \tag{5.221}$$

such that

$$(x_{k+1}, \lambda_{k+1}) \in \{(y_{k,t}, \alpha_{k,t}) : t \in \Omega_{k+1}\}, \tag{5.222}$$

$$\lambda_{k+1} = \max\{\alpha_{k,t} : t \in \Omega_{k+1}\}. \tag{5.223}$$

It follows from (5.11) and (5.221) that for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}$ there exists a finite sequence $\{y_i^{(k,t)}\}_{i=0}^{p(t)} \subset X$ such that

$$y_0^{(k,t)} = x_k, \ y_{p(t)}^{(k,t)} = y_{k,t}, \tag{5.224}$$

$$d(y_j^{(k,t)}, P_{t_j}(y_{j-1}^{(k,t)})) \le \delta \text{ for each integer } j = 1, \dots, p(t), \tag{5.225}$$

$$\alpha_{k,t} = \max\{d(y_{j-1}^{(k,t)}, y_j^{(k,t)}) : j = 1, \dots, p(t)\}. \tag{5.226}$$

Choose a positive number δ_0 for which

$$\delta_0 < (12\bar{q})^{-1}\gamma, \ \delta_0 < 6^{-1}\epsilon_0. \tag{5.227}$$

By the assumption of the theorem there exists

$$z \in B(\theta, M) \tag{5.228}$$

such that

$$B(z, \delta_0) \cap \mathrm{Fix}(P_i) \neq \emptyset \text{ for all } i = 1, \dots, m. \tag{5.229}$$

In view of (5.229), for every $i \in \{1, \dots, m\}$ there exists

$$z_i \in B(z, \delta_0) \cap \mathrm{Fixp}(P_i). \tag{5.230}$$

Relations (5.227), (5.228), and (5.230) imply that

$$z_i \in B(\theta, M + 1), \ i = 1, \dots, m. \tag{5.231}$$

By (A2) and the choice of γ, (5.230) and (5.231) the following property holds:

(P6) for each $i \in \{1, \dots, m\}$ and each $x \in B(\theta, 3M + 1)$ satisfying $d(x, \mathrm{Fix}(P_i)) \ge 4^{-1}\epsilon_0$,

$$d(z_i, P_i(x)) \le d(z_i, x) - \gamma.$$

Let

$$i \in \{1, \dots, m\} \text{ and } x \in B(\theta, 3M + 1) \tag{5.232}$$

satisfy

$$d(x, \mathrm{Fix}(P_i)) \geq \epsilon_0/4. \tag{5.233}$$

By (5.230), (5.232), and (5.233),

$$d(z, P_i(x)) \leq d(z, z_i) + d(z_i, P_i(x))$$
$$\leq d(z_i, x) - \gamma + \delta_0 \leq \delta_0 - \gamma + d(z, z_i) + d(z, x)$$
$$\leq 2\delta_0 - \gamma + d(z, x).$$

Thus we have shown that the following property holds:

(P7) for each $i \in \{1, \ldots, m\}$ and each $x \in B(\theta, 3M + 1)$ satisfying $d(x, \mathrm{Fix}(P_i)) \geq 4^{-1}\epsilon_0$,

$$d(z, P_i(x)) \leq d(z, x) - \gamma + 2\delta_0.$$

Proposition 5.8, (5.224), (5.225), and (5.229) imply that for each integer $k \geq 0$, each $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{k+1}$ and each $j \in \{1, \ldots, p(t)\}$,

$$d(y_j^{(k,t)}, z) \leq d(y_{j-1}^{(k,t)}, z) + 2\delta_0 + \delta, \tag{5.234}$$

$$d(y_j^{(k,t)}, z) \leq d(z, x_k) + j(2\delta_0 + \delta). \tag{5.235}$$

By (5.10), (5.224), and (5.235), for each integer $k \geq 0$, each $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{k+1}$ and each $j \in \{0, 1, \ldots, p(t)\}$,

$$d(y_j^{(k,t)}, z) \leq d(x_k, z) + \bar{q}(2\delta_0 + \delta), \tag{5.236}$$

$$d(z, y_{k,t}) \leq d(z, x_k) + \bar{q}(2\delta_0 + \delta). \tag{5.237}$$

In view of (5.228) and the inclusion $x_0 \in B(\theta, M)$,

$$d(x_0, z) \leq 2M. \tag{5.238}$$

We prove the following auxiliary result.

Lemma 5.14. *Assume that a nonnegative integer k satisfies*

$$d(x_k, z) \leq 2M, \tag{5.239}$$

$$\lambda_{k+1} > \epsilon_0. \tag{5.240}$$

Then

$$d(z, x_k) - d(z, x_{k+1}) \geq 2^{-1}\gamma.$$

Proof. In view of (5.222), (5.223), and (5.240), there exists

$$s = (s_1, \dots, s_{p(s)}) \in \Omega_{k+1}$$

such that

$$\alpha_{k,s} = \lambda_{k+1} > \epsilon_0, \ x_{k+1} = y_{k,s}. \tag{5.241}$$

Relations (5.226) and (5.241) imply that there exists

$$j_0 \in \{1, \dots, p(s)\}$$

such that

$$\epsilon_0 < \alpha_{k,s} = d(y_{j_0-1}^{(k,s)}, y_{j_0}^{(k,s)}). \tag{5.242}$$

We show that

$$d(y_{j_0-1}^{(k,s)}, \mathrm{Fix}(P_{s_{j_0}})) \geq \epsilon_0/4.$$

Assume the contrary. Then there exists

$$\xi \in \mathrm{Fix}(P_{s_{j_0}}) \tag{5.243}$$

such that

$$d(y_{j_0-1}^{(k,s)}, \xi) < \epsilon_0/4. \tag{5.244}$$

In view of (5.21), (5.243), and (5.244),

$$d(\xi, P_{s_{j_0}}(y_{j_0-1}^{(k,s)})) \leq d(y_{j_0-1}^{(k,s)}, \xi) < \epsilon_0/4. \tag{5.245}$$

It follows from (5.41), (5.225), (5.244), and (5.245) that

$$d(y_{j_0-1}^{(k,s)}, y_{j_0}^{(k,s)}) \leq d(y_{j_0-1}^{(k,s)}, \xi) + d(\xi, y_{j_0}^{(k,s)})$$
$$< \epsilon_0/4 + d(\xi, P_{s_{j_0}}(y_{j_0-1}^{(k,s)})) + d(P_{s_{j_0}}(y_{j_0-1}^{(k,s)}), y_{j_0}^{(k,s)}) \leq \epsilon_0/4 + \epsilon_0/4 + \delta < \epsilon_0.$$

This contradicts (5.242). The contradiction we have reached proves that

$$d(y_{j_0-1}^{(k,s)}, \mathrm{Fix}(P_{s_{j_0}})) \geq \epsilon_0/4. \tag{5.246}$$

By (5.41), (5.227), (5.228), (5.236), and (5.239),

$$d(\theta, y_{j_0-1}^{(k,s)}) \leq d(\theta, z) + d(z, y_{j_0-1}^{(k,s)})$$
$$\leq M + d(z, x_k) + \bar{q}(2\delta_0 + \delta) \leq 3M + 1. \tag{5.247}$$

Property (P7), (5.246), and (5.247) imply that

$$d(z, P_{s_{j_0}}(y_{j_0-1}^{(k,s)})) \le d(z, y_{j_0-1}^{(k,s)}) - \gamma + 2\delta_0. \tag{5.248}$$

It follows from (5.225) and (5.248) that

$$\begin{aligned} d(z, y_{j_0}^{(k,s)}) &\le d(z, P_{s_{j_0}}(y_{j_0-1}^{(k,s)})) + d(P_{s_{j_0}}(y_{j_0-1}^{(k,s)}), y_{j_0}^{(k,s)}) \\ &\le d(z, y_{j_0-1}^{(k,s)}) - \gamma + 2\delta_0 + \delta. \end{aligned} \tag{5.249}$$

By (5.10), (5.41), (5.224), (5.227), (5.234), (5.241), and (5.249),

$$\begin{aligned} d(z, x_k) - d(z, x_{k+1}) &= d(z, y_0^{(k,s)}) - d(z, y_{p(s)}^{(k,s)}) \\ &= \sum_{j=1}^{p(s)} (d(z, y_{j-1}^{(k,s)}) - d(z, y_j^{(k,s)})) \\ &\ge -(2\delta_0 + \delta)(p(s) - 1) + d(z, y_{j_0-1}^{(k,s)}) - d(z, y_{j_0}^{(k,s)}) \\ &\ge \gamma - (2\delta_0 + \delta)\bar{q} \ge \gamma/2. \end{aligned}$$

Lemma 5.14 is proved. □

Assume that q is a natural number such that for each nonnegative integer $k < q$,

$$\lambda_{k+1} > \epsilon_0.$$

By (5.238) and Lemma 5.14 applied by induction,

$$d(x_k, z) \le 2M, \; k = 0, \ldots, q$$

and for each nonnegative integer $k < q$,

$$d(z, x_k) - d(z, x_{k+1}) \ge 2^{-1}\gamma. \tag{5.250}$$

It follows from (5.40), (5.238), and (5.250) that

$$\begin{aligned} 2M &\ge d(z, x_0) \ge d(z, x_0) - d(z, x_q) \\ &= \sum_{k=0}^{q-1} (d(z, x_k) - d(z, x_{k+1})) \ge 2^{-1} q\gamma \end{aligned}$$

and

$$q \le 4M\gamma^{-1} \le n_0.$$

This implies that there exists a nonnegative integer $q \le n_0$ such that for all nonnegative integers $k \le q$,

$$d(x_k, z) \le 2M$$

and

$$\lambda_{q+1} \le \epsilon_0.$$

Assume that an integer $q \ge 0$ satisfies

$$\lambda_{q+1} \le \epsilon_0, \tag{5.251}$$

$$d(x_q, \theta) \le 3M. \tag{5.252}$$

We show that for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{q+1}$ and all $j = 1, \dots, p(t)$,

$$d(y_{j-1}^{(q,t)}, \mathrm{Fix}(P_{t_j})) \le \epsilon(4\bar{q})^{-1}. \tag{5.253}$$

Assume the contrary. Then there exist $t = (t_1, \dots, t_{p(t)}) \in \Omega_{q+1}$ and $j \in \{1, \dots, p(t)\}$ such that

$$d(y_{j-1}^{(q,t)}, \mathrm{Fix}(P_{t_j})) > \epsilon(4\bar{q})^{-1}. \tag{5.254}$$

By (5.10), (5.223), (5.226), (5.229), (5.251), and (5.252),

$$d(x_q, y_{j-1}^{(q,t)}) \le \epsilon_0\bar{q},\ y_{j-1}^{(q,t)} \in B(\theta, 3M+1). \tag{5.255}$$

It follows from (A2), the choice of ϵ_0, (5.230), (5.231), (5.254), and (5.255) that

$$d(z_{t_j}, P_{t_j}(y_{j-1}^{(q,t)})) \le d(z_{t_j}, y_{j-1}^{(q,t)}) - 2\epsilon_0. \tag{5.256}$$

In view of (5.41), (5.225), (5.227), (5.230), and (5.256),

$$\begin{aligned} d(z, y_j^{(q,t)}) &\le d(z, z_{t_j}) + d(z_{t_j}, P_{t_j}(y_{j-1}^{(q,t)})) + d(P_{t_j}(y_{j-1}^{(q,t)}), y_j^{(q,t)}) \\ &\le \delta_0 + d(z_{t_j}, y_{j-1}^{(q,t)}) - 2\epsilon_0 + \delta \\ &\le -2\epsilon_0 + 2\delta_0 + \delta + d(z_{t_j}, z) + d(z, y_{j-1}^{(q,t)}) \\ &\le -2\epsilon_0 + 2\delta_0 + \delta + d(z, y_{j-1}^{(q,t)}) \end{aligned}$$

and

$$d(y_{j-1}^{(q,t)}, y_j^{(q,t)}) \ge d(y_{j-1}^{(q,t)}, z) - d(y_j^{(q,t)}, z) \ge 2\epsilon_0 - 2\delta_0 + \delta > (3/2)\epsilon_0.$$

This contradicts (5.251). The contradiction we have reached proves that (5.253) is true for all $t \in \Omega_{q+1}$ and all $j = 1, \dots, p(t)$. Together with Proposition 5.10, (5.41), (5.192), (5.193), and (5.253) this implies that for all $s = 1, \dots, m$,

$$d(x_q, \mathrm{Fix}(P_s)) \le \bar{q}(\delta + \epsilon(2\bar{q})^{-1}) \le \epsilon.$$

Theorem 5.6 is proved. □

5.9 Proof of Theorem 5.7

Let

$$\epsilon_0 = \epsilon \bar{q}^{-1}. \tag{5.257}$$

By the assumptions of the theorem and (A2), there exists $\gamma_0 \in (0, \epsilon_0)$ such that the following property holds:

(P8) for each $i \in \{1, \dots, m\}$, each

$$\xi \in B(\theta, M + 1) \cap \mathrm{Fix}(P_i)$$

and each $x \in B(\theta, 3M + 1)$ satisfying $d(x, P_i(x)) \ge \epsilon_0/2$,

$$d(\xi, P_i(x)) \le d(\xi, x) - \gamma_0.$$

Choose a positive number

$$Q \ge 2\gamma_0^{-1}(2M + 1). \tag{5.258}$$

Assume that

$$\{\Omega_i\}_{i=1}^{\infty} \subset \mathcal{M}_*, \tag{5.259}$$

$$x_0 \in B(\theta, M) \tag{5.260}$$

and $\{x_i\}_{i=1}^{\infty} \subset X$, $\{\mu_i\}_{i=1}^{\infty} \subset [0, \infty)$, for each natural number i,

$$(x_i, \mu_i) \in \tilde{A}(x_{i-1}, \Omega_i, 0). \tag{5.261}$$

Let $k \ge 0$ be an integer. In view of (5.261),

$$(x_{k+1}, \mu_{k+1}) \in \tilde{A}(x_k, \Omega_{k+1}, 0). \tag{5.262}$$

By (5.33) and (5.262), there exist

$$(y_{k,t}, \beta_{k,t}) \in \tilde{A}_0(x_k, t, 0),\ t \in \Omega_{k+1} \tag{5.263}$$

such that

$$(x_{k+1}, \mu_{k+1}) \in \{(y_{k,t}, \beta_{k,t}) :\ t \in \Omega_{k+1}\}, \tag{5.264}$$

$$\mu_{k+1} = \max\{\beta_{k,t} :\ t \in \Omega_{k+1}\}. \tag{5.265}$$

It follows from (5.32) and (5.263) that for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}$ there exists a finite sequence $\{y_i^{(k,t)}\}_{i=0}^{p(t)} \subset X$ such that

$$y_0^{(k,t)} = x_k,\ y_{p(t)}^{(k,t)} = y_{k,t}, \tag{5.266}$$

$$y_j^{(k,t)} = P_{t_j}(y_{j-1}^{(k,t)}) \text{ for each integer } j = 1, \dots, p(t), \tag{5.267}$$

$$\beta_{k,t} = \max\{d(y_{j-1}^{(k,t)}, \mathrm{Fix}(P_{t_j})) :\ j = 1, \dots, p(t)\}. \tag{5.268}$$

Set

$$E = \{k \in \{0, 1, \dots\} :\ \mu_{k+1} > 2^{-1}\epsilon_0\}. \tag{5.269}$$

Let n be a natural number and δ be an arbitrary positive number which satisfy

$$\delta < (2\bar{q}(n+1))^{-1},\ \delta < (4\bar{q})^{-1}\gamma_0. \tag{5.270}$$

By the assumptions of the theorem, there exists

$$z \in B(\theta, M) \tag{5.271}$$

such that

$$B(z, \delta) \cap \mathrm{Fix}(P_i) \neq \emptyset \text{ for all } i = 1, \dots, m. \tag{5.272}$$

In view of (5.260) and (5.272),

$$d(z, x_0) \le 2M. \tag{5.273}$$

Proposition 5.8, (5.10), (5.263), (5.264), (5.266), (5.267), and (5.272) imply that for each integer $k \ge 0$, each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}$ and each $i \in \{1, \dots, p(t)\}$,

$$d(z, y_i^{(k,t)}) \le d(z, y_{i-1}^{(k,t)}) + 2\delta, \tag{5.274}$$

$$d(z, y_i^{(k,t)}) \le d(z, x_k) + 2i\delta \le d(z, x_k) + 2\bar{q}\delta, \tag{5.275}$$

$$d(z, x_{k+1}) \le d(z, x_k) + 2\bar{q}\delta. \tag{5.276}$$

In view of (5.273), (5.275), and (5.276), for all integers $k = 0, \dots, n$, each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}$ and each $i \in \{1, \dots, p(t)\}$,

$$d(z, x_k) \le d(z, x_0) + 2k\bar{q}\delta \le 2M + 2\bar{q}n\delta, \tag{5.277}$$

$$d(y_i^{(k,t)}, z) \le d(x_k, z) + 2\bar{q}\delta \le 2M + 2(n+1)\bar{q}\delta. \tag{5.278}$$

It follows from (5.270), (5.271), and (5.278), for each $k = 0, \dots, n$, each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}$ and each $i \in \{1, \dots, p(t)\}$,

$$y_i^{(k,t)} \in B(\theta, 3M + 1). \tag{5.279}$$

Assume that an integer $k \in \{0, \dots, n-1\}$ satisfies

$$\mu_{k+1} > 2^{-1}\epsilon_0. \tag{5.280}$$

By (5.264), there exists

$$s \in \Omega_{k+1}$$

such that

$$(x_{k+1}, \mu_{k+1}) = (y_{k,s}, \beta_{k,s}). \tag{5.281}$$

In view (5.280) and (5.281),

$$\beta_{k,s} = \mu_{k+1} > 2^{-1}\epsilon_0. \tag{5.282}$$

Relations (5.268) and (5.282) imply that there exists $j_0 \in \{1, \dots, p(s)\}$ such that

$$2^{-1}\epsilon_0 < \beta_{k,s} = d(y_{j_0-1}^{(k,s)}, \mathrm{Fix}(P_{s_{j_0}})). \tag{5.283}$$

By (5.272), there exists

$$z_0 \in B(z, \delta) \cap \mathrm{Fix}(P_{s_{j_0}}). \tag{5.284}$$

Relations (5.270), (5.271), and (5.284) imply that

$$d(\theta, z_0) \le M + 1. \tag{5.285}$$

Property (P8), (5.267), (5.279), and (5.283)–(5.285) imply that

$$d(z_0, y_{j_0}^{(k,s)}) = d(z_0, P_{s_{j_0}}(y_{j_0-1}^{(k,s)})) \le d(z_0, y_{j_0-1}^{(k,s)}) - \gamma_0. \tag{5.286}$$

In view of (5.284) and (5.286),

$$d(z, y_{j_0}^{(k,s)}) \le d(z, z_0) + d(z_0, y_{j_0}^{(k,s)}) \le \delta + d(z_0, y_{j_0-1}^{(k,s)}) - \gamma_0$$
$$\le -\gamma_0 + \delta + d(z, z_0) + d(z, y_{j_0-1}^{(k,s)}) \le -\gamma_0 + 2\delta + d(z, y_{j_0-1}^{(k,s)}),$$
$$d(z, y_{j_0-1}^{(k,s)}) - d(z, y_{j_0}^{(k,s)}) \ge \gamma_0 - 2\delta. \tag{5.287}$$

By (5.10), (5.266), (5.270), (5.274), (5.281), and (5.287),

$$d(z, x_k) - d(z, x_{k+1}) = d(z, y_0^{(k,s)}) - d(z, y_{p(s)}^{(k,s)})$$
$$= \sum_{j=1}^{p(s)} (d(z, y_{j-1}^{(k,s)}) - d(z, y_j^{(k,s)}))$$
$$\ge -2\delta(p(s) - 1) + d(z, y_{j_0-1}^{(k,s)}) - d(z, y_{j_0}^{(k,s)})$$
$$\ge \gamma_0 - 2\delta\bar{q} \ge \gamma_0/2.$$

Thus the following property holds:
(P9) if $k \in \{0, \ldots, n-1\}$ and $\mu_{k+1} > 2^{-1}\epsilon_0$, then

$$d(z, x_k) - d(z, x_{k+1}) \ge 2^{-1}\gamma_0.$$

By property (P9), (5.258), (5.269), (5.270), (5.273), and (5.276),

$$M \ge d(z, x_0) \ge d(z, x_0) - d(z, x_n)$$
$$= \sum_{k=0}^{n-1} (d(z, x_k) - d(z, x_{k+1}))$$
$$\ge -2\bar{q}\delta n + \sum \{d(z, x_k) - d(z, x_{k+1}) : \ k \in \{0, \ldots, n-1\} \cap E\}$$
$$\ge -2\bar{q}\delta n + \mathrm{Card}(\{k \in \{0, \ldots, n-1\} \cap E\})\gamma_0/2,$$
$$\mathrm{Card}(\{k \in \{0, \ldots, n-1\} \cap E\}) \le 2\gamma_0^{-1}(M+1) \le Q.$$

Since the relation above holds for any natural number n we conclude that

$$\mathrm{Card}(\{k \in \{0, 1 \ldots\} : \ \mu_{k+1} > 2^{-1}\epsilon_0\}) = \mathrm{Card}(E) \le Q.$$

Let $k \in \{0, 1, \ldots\}$ satisfy

$$\mu_{k+1} \le 2^{-1}\epsilon_0. \tag{5.288}$$

Then in view of (5.263)–(5.268) and Proposition 5.10, for all $s = 1, \dots, m$,

$$d(x_k, \mathrm{Fix}(P_s)) \leq \bar{q}\epsilon_0 \leq \epsilon.$$

Theorem 5.7 is proved. □

Chapter 6
Spaces with Generalized Distances

In this chapter we study the convergence of dynamic string-maximum methods for solving common fixed point problems in a space equipped with a generalized distance. Our main goal is to obtain an approximate solution of the problem in the presence of computational errors. We show that our dynamic string-maximum algorithm generates a good approximate solution, if the sequence of computational errors is bounded from above by a constant.

6.1 Preliminaries and Main Results

Let (X, d) be a metric space. For each $x \in X$ and each nonempty set $E \subset X$ put

$$d(x, E) = \inf\{d(x, y) : \ y \in E\}.$$

For each $x \in X$ and each $r > 0$ set

$$B(x, r) = \{y \in X : \ d(x, y) \leq r\}.$$

Fix $\theta \in X$. Suppose that $D : X \times X \to [0, \infty)$, for each $(x, y) \in X \times X$, $D(x, y) = 0$ if and only if $x = y$ and that the following assumption holds:

(A1) for each $M > 0$ and each $\epsilon > 0$, there exists $\delta > 0$ such that for each $x, y \in B(\theta, M)$ which satisfy $D(x, y) \leq \delta$, the inequality $d(x, y) \leq \epsilon$ holds.

In this chapter some of the results obtained in Chap. 5 are extended for common fixed point problems on the space X equipped with the generalized distance D. Note that the class of spaces with generalized metrics includes the class of metric spaces as well as the class of spaces equipped with the Bregman distances which has many important applications. See, for example, [24, 76] and the references mentioned therein.

A. Zaslavski, *Approximate Solutions of Common Fixed-Point Problems*, Springer Optimization and Its Applications 112, DOI 10.1007/978-3-319-33255-0_6

Suppose that m is a natural number, $P_i : X \to X$, $i = 1, \dots, m$ are self-mappings of X and that for every $i \in \{1, \dots, m\}$,

$$\mathrm{Fix}(P_i) := \{z \in X : \ P_i(z) = z\} \neq \emptyset. \tag{6.1}$$

We suppose that for every $i \in \{1, \dots, m\}$,

$$D(z, x) \geq D(z, P_i(x)) \text{ for every } x \in X \text{ and every } z \in \mathrm{Fix}(P_i). \tag{6.2}$$

By an index vector, we a mean a vector $t = (t_1, \dots, t_p)$ such that $t_i \in \{1, \dots, m\}$ for all $i = 1, \dots, p$.

For an index vector $t = (t_1, \dots, t_q)$ set

$$p(t) = q, \ P[t] = P_{t_q} \cdots P_{t_1}. \tag{6.3}$$

Denote by $\mathcal{M}$ the collection of all finite sets Ω of index vectors such that

$$\cup \{\{t_1, \dots, t_{p(t)}\} : \ t = \{t_1, \dots, t_{p(t)}\} \in \Omega\} = \{1, \dots, m\}. \tag{6.4}$$

Fix an integer

$$\bar{q} \geq m \tag{6.5}$$

and denote by $\mathcal{M}_*$ the set of all $\Omega \in \mathcal{M}$ such that

$$p(t) \leq \bar{q} \ \text{ for all } t \in \Omega. \tag{6.6}$$

We suppose that

$$F := \cap_{i=1}^{m} \mathrm{Fix}(P_i) \neq \emptyset. \tag{6.7}$$

For every $\epsilon > 0$ and every $i \in \{1, \dots, m\}$ put

$$F_\epsilon(P_i) = \{x \in X : \ d(x, P_i(x)) \leq \epsilon\}, \tag{6.8}$$

$$\tilde{F}_\epsilon(P_i) = \{y \in X : \ B(y, \epsilon) \cap F_\epsilon(P_i) \neq \emptyset\}, \tag{6.9}$$

$$F_\epsilon = \cap_{i=1}^{m} F_\epsilon(P_i), \tilde{F}_\epsilon = \cap_{i=1}^{m} \tilde{F}_\epsilon(P_i). \tag{6.10}$$

A point belonging to the set F is a solution of our common fixed point problem while a point which belongs to the set $\tilde{F}_\epsilon$ is its ϵ-approximate solution.

We apply a dynamic string method with variable strings in order to obtain a good approximative solution of the common fixed point problem.

We suppose that the following assumption holds:

(A2) there exists $z_* \in F$ such that for each nonempty set $K \subset X$, the set $\{D(z_*, \xi) : \xi \in K\}$ is bounded if and only if K is bounded.

In order to state the main result of this section we need the following definitions.

Let $\delta \geq 0$, $x \in X$ and let $t = (t_1, \dots, t_{p(t)})$ be an index vector. Define

$$\begin{aligned} A_0(x, t, \delta) = \{(y, \lambda) \in X \times R^1 : \ &\text{there is a sequence } \{y_i\}_{i=0}^{p(t)} \subset X \text{ such that} \\ &y_0 = x \text{ and for all } i = 1, \dots, p(t), \\ &d(y_i, P_{t_i}(y_{i-1})) \leq \delta, \\ &y = y_{p(t)}, \\ &\lambda = \max\{D(y_i, y_{i-1}) : i = 1, \dots, p(t)\}\}. \end{aligned} \tag{6.11}$$

Let $\delta \geq 0$, $x \in X$ and let $\Omega \in \mathcal{M}_*$. Define

$$A(x, \Omega, \delta) = \{(y, \lambda) \in X \times R^1 : \ \text{there exist}$$

$$(y_t, \lambda_t) \in A_0(x, t, \delta), \ t \in \Omega \text{ such that}$$

$$(y, \lambda) \in \{(y_t, \lambda_t) : t \in \Omega\}, \tag{6.12}$$

$$\lambda \geq \lambda_t - \delta, \ t \in \Omega\}. \tag{6.13}$$

For each $x \in X$ and each nonempty set $K \subset X$ put

$$D(x, K) = \inf\{D(\xi, x) : \xi \in K\}. \tag{6.14}$$

Denote by Card(A) the cardinality of a set A. Suppose that the sum over empty set is zero.

Let $\delta \geq 0$, $x \in X$ and let $t = (t_1, \dots, t_{p(t)})$ be an index vector. Define

$$\begin{aligned} \tilde{A}_0(x, t, \delta) = \{(y, \lambda) \in X \times R^1 : \ &\text{there is a sequence } \{y_i\}_{i=0}^{p(t)} \subset X \text{ such that} \\ &y_0 = x \text{ and for all } i = 1, \dots, p(t), \\ &d(y_i, P_{t_i}(y_{i-1})) \leq \delta, \\ &y = y_{p(t)}, \\ &\lambda = \max\{D(y_{i-1}, \mathrm{Fix}(P_{t_i})) : i = 1, \dots, p(t)\}\}. \end{aligned} \tag{6.15}$$

Let $\delta \geq 0$, $x \in X$ and let $\Omega \in \mathcal{M}_*$. Define

$$\begin{aligned} \tilde{A}(x, \Omega, \delta) = \{(y, \lambda) \in X \times R^1 : \ &\text{there exist} \\ &(y_t, \lambda_t) \in \tilde{A}_0(x, t, \delta), \ t \in \Omega \text{ such that} \\ &(y, \lambda) \in \{(y_t, \lambda_t) : t \in \Omega\}, \\ &\lambda \geq \lambda_t - \delta, \ t \in \Omega\}. \end{aligned} \tag{6.16}$$

In this chapter we use the following assumptions.

(B1) There exists $\bar{c} \in (0, 1)$ such that

$$D(z, x) \geq D(z, P_i(x)) + \bar{c}D(P_i(x), x)$$

for every $i \in \{1, \ldots, m\}$, every $x \in X$ and every $z \in \mathrm{Fix}(P_i)$.

(B2) For each $\Lambda > 0$ and each $\lambda > 0$ there exists $\gamma > 0$ such that for each $i \in \{1, \ldots, m\}$, each

$$z \in B(\theta, \Lambda) \cap \mathrm{Fix}(P_i)$$

and each $x \in B(\theta, \Lambda)$ satisfying $D(P_i(x), x) \geq \lambda$,

$$D(z, P_i(x)) \leq D(z, x) - \gamma.$$

(Clearly, (B1) implies (B2).)

(B3) For each $\Lambda > 0$ and each $\lambda > 0$ there exists $\gamma > 0$ such that for each $i \in \{1, \ldots, m\}$, each

$$z \in B(\theta, \Lambda) \cap \mathrm{Fix}(P_i)$$

and each $x \in B(\theta, \Lambda)$ satisfying $D(x, \mathrm{Fix}(P_i)) \geq \lambda$,

$$D(z, P_i(x)) \leq D(z, x) - \gamma.$$

(B4) For each $\epsilon > 0$ and each $M > 0$ there exists $\delta > 0$ such that for each $i \in \{1, \ldots, m\}$, each $x \in B(\theta, M)$ and each $\xi \in \mathrm{Fix}(P_i)$ satisfying $D(\xi, x) \leq \delta$ the inequality $d(\xi, x) \leq \epsilon$ holds.

(B5) For each $\epsilon > 0$ and each $M > 0$ there exists $\delta > 0$ such that for each $\xi_1, \xi_2 \in B(\theta, M)$ satisfying $d(\xi_1, \xi_2) \leq \delta$ the inequality $D(\xi_1, \xi_2) \leq \epsilon$ holds.

In this chapter we prove the following results.

Theorem 6.1. *Suppose that (B2) holds and $M > 0$. Let $\epsilon > 0$. Then there exist $Q, M_1 > 0$ such that for each $\{\Omega_i\}_{i=1}^{\infty} \subset \mathcal{M}_*$, each $x_0 \in B(\theta, M)$, each $\{x_i\}_{i=1}^{\infty} \subset X$, and each $\{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty)$ satisfying for each natural number i,*

$$(x_i, \lambda_i) \in A(x_{i-1}, \Omega_i, 0),$$

the following relations hold:

$$x_k \in B(\theta, M_1) \text{ for all natural numbers } k,$$

$$Card(\{i \in \{0, 1, \ldots\} : \ x_i \notin \tilde{F}_\epsilon\}) \leq Q.$$

Theorem 6.2. *Suppose that (B3) and (B4) hold and $M > 0$. Let $\epsilon > 0$. Then there exist $M_1, Q > 0$ such that for each $\{\Omega_i\}_{i=1}^{\infty} \subset \mathcal{M}_*$, each $x_0 \in B(\theta, M)$, each $\{x_i\}_{i=1}^{\infty} \subset X$ and each $\{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty)$ satisfying for each natural number i,*

$$(x_i, \lambda_i) \in \tilde{A}(x_{i-1}, \Omega_i, 0)$$

the following relations hold:

$$x_k \in B(\theta, M_1) \textit{ for all natural numbers } k,$$

$$Card(\{i \in \{0, 1, \dots\} : \max\{d(x_i, Fix(P_s)) : s = 1, \dots, m\} > \epsilon\}) \le Q.$$

Theorem 6.3. *Suppose that (B3), (B4), and (B5) hold and $M, \epsilon > 0$. Then there exist $M_1, Q > 0$ such that for each $\{\Omega_i\}_{i=1}^{\infty} \subset \mathcal{M}_*$, each $x_0 \in B(\theta, M)$, each $\{x_i\}_{i=1}^{\infty} \subset X$ and each $\{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty)$ satisfying for each natural number i,*

$$(x_i, \lambda_i) \in A(x_{i-1}, \Omega_i, 0)$$

the following relations hold:

$$x_k \in B(\theta, M_1) \textit{ for all natural numbers } k,$$

$$Card(\{i \in \{0, 1, \dots\} : x_i \notin \tilde{F}_\epsilon\}) \le Q.$$

Theorem 6.4. *Suppose that the mappings P_i, $i = 1, \dots, m$ are uniformly continuous on bounded subsets of X and that (B2) holds or (B3) and (B4) hold. Let $\epsilon, M > 0$. Then there exist a natural number n_0, $\delta, M_1 > 0$ such that for each sequence of singletons,*

$$\Omega_i = \{t^{(i)}\} = \{t_1^{(i)}, \dots, t_{p(t^{(i)})}^{(i)}\} \in \mathcal{M}_*, \ i = 1, 2, \dots$$

each $x_0 \in B(\theta, M)$, each sequence $\{x_i\}_{i=1}^{\infty} \subset X$ and each sequence $\{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty)$ satisfying for each natural number i,

$$(x_i, \lambda_i) \in A_0(x_{i-1}, t^{(i)}, \delta)$$

there exists an integer $q \in [1, n_0]$ such that

$$x_i \in B(\theta, M_1) \textit{ for all integers } i = 0, \dots, q,$$

$$x_q \in \tilde{F}_\epsilon.$$

Moreover, if (B3) and (B4) hold, then

$$d(x_q, Fix(P_s)) \le \epsilon, \ s = 1, \dots, m.$$

Theorem 6.5. *Suppose that the function $D(\cdot,\cdot)$ is uniformly continuous on bounded subsets of $X\times X$ and that (B2) holds. Let $\epsilon, M > 0$. Then there exist a natural number n_0, $\delta, M_1 > 0$ such that for each sequence*

$$\{\Omega_i\}_{i=1}^{\infty} \subset \mathcal{M}_*,$$

each $x_0 \in B(\theta, M)$, each sequence $\{x_i\}_{i=1}^{\infty} \subset X$ and each sequence $\{\lambda_i\}_{i=1}^{\infty} \subset [0,\infty)$ satisfying for each natural number i,

$$(x_i, \lambda_i) \in A(x_{i-1}, \Omega_i, \delta)$$

there exists an integer $q \in [0, n_0]$ such that

$$x_i \in B(\theta, M_1) \text{ for all integers } i = 0, \dots, q,$$

$$x_q \in \tilde{F}_\epsilon.$$

In this chapter we use the following assumption.

(B6) For each $M > 0$ there exists $M_1 > 0$ such that for each $x \in B(\theta, M)$, each $i \in \{1, \dots, m\}$ and each $z \in \mathrm{Fix}(P_i)$ satisfying $D(z,x) \le D(z_*, x) + 1$ the inclusion $z \in B(\theta, M_1)$ holds.

Theorem 6.6. *Suppose that the function $D(\cdot,\cdot)$ is uniformly continuous on bounded subsets of $X\times X$ and that (B3) and (B6) hold. Let $\epsilon, M > 0$. Then there exist a natural number n_0, $\delta, M_1 > 0$ such that for each sequence*

$$\{\Omega_i\}_{i=1}^{\infty} \subset \mathcal{M}_*,$$

each $x_0 \in B(\theta, M)$, each sequence $\{x_i\}_{i=1}^{\infty} \subset X$ and each sequence $\{\lambda_i\}_{i=1}^{\infty} \subset [0,\infty)$ satisfying for each natural number i,

$$(x_i, \lambda_i) \in \tilde{A}(x_{i-1}, \Omega_i, \delta)$$

there exists an integer $q \in [0, n_0]$ such that

$$x_i \in B(\theta, M_1) \text{ for all integers } i = 0, \dots, q,$$

$$\max\{d(x_q, Fix(P_s)) : \ s = 1, \dots, m\} \le \epsilon.$$

Theorem 6.7. *Suppose that the function $D(\cdot,\cdot)$ is uniformly continuous on bounded subsets of $X\times X$ and that (B3) and (B6) hold. Let $\epsilon, M > 0$. Then there exist a natural number n_0, $\delta, M_1 > 0$ such that for each sequence*

$$\{\Omega_i\}_{i=1}^{\infty} \subset \mathcal{M}_*,$$

each $x_0 \in B(\theta, M)$*, each sequence* $\{x_i\}_{i=1}^{\infty} \subset X$ *and each sequence* $\{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty)$ *satisfying for each natural number i,*

$$(x_i, \lambda_i) \in A(x_{i-1}, \Omega_i, \delta)$$

there exists an integer $q \in [0, n_0]$ *such that*

$$x_i \in B(\theta, M_1) \textit{ for all integers } i = 0, \dots, q,$$

$$\max\{d(x_q, Fix(P_s)) : s = 1, \dots, m\} \le \epsilon.$$

It should be mentioned that examples of Bregman distances, satisfying the assumptions used in this section, can be found in Chap. 5 of [76].

6.2 Auxiliary Results

Proposition 6.8. *Assume that the function* $D(z_*, \cdot)$ *is uniformly continuous of all bounded subsets of X and that* $M_0, \epsilon > 0$*. Then there exists* $\delta_0 > 0$ *such that for each* $x \in X$ *satisfying* $D(z_*, x) \le M_0$*, each* $j \in \{1, \dots, m\}$ *and each* $y \in X$ *satisfying* $d(y, P_{t_j}(x)) \le \delta_0$ *the inequality*

$$D(z_*, y) \le D(z_*, x) + \epsilon$$

holds.

Proof. In view of (A2), there exists $M_1 > 0$ such that

$$\{x \in X : D(z_*, x) \le M_0\} \subset B(\theta, M_1). \tag{6.17}$$

Since the function $D(z_*, \cdot)$ is uniformly continuous on bounded sets there exists $\delta_0 \in (0, 1)$ such that

$$|D(z_*, \xi_1) - D(z_*, \xi_2)| \le \epsilon \tag{6.18}$$

for all $\xi_1, \xi_2 \in B(\theta, M_1 + 1)$ satisfying $d(\xi_1, \xi_2) \le \delta_0$.

Assume that

$$j \in \{1, \dots, m\},\ x \in X,\ D(z_*, x) \le M_0 \tag{6.19}$$

and that $y \in X$ satisfies

$$d(y, P_{t_j}(x)) \le \delta_0. \tag{6.20}$$

Relations (6.17) and (6.19) imply that

$$x \in B(\theta, M_1). \tag{6.21}$$

By (6.2), (6.7), (6.19) and the inclusion $z_* \in F$,

$$D(z_*, P_{t_j}(x)) \le D(z_*, x) \le M_0. \tag{6.22}$$

It follows from (6.17) and (6.22) that

$$P_{t_j}(x) \in B(\theta, M_1). \tag{6.23}$$

In view of (6.20) and (6.23),

$$y \in B(\theta, M_1 + 1). \tag{6.24}$$

By the choice of δ_0 (see (6.18)), (6.20), (6.23), and (6.24),

$$|D(z_*, y) - D(z_*, P_{t_j}(x))| \le \epsilon.$$

Together with (6.22) this implies that

$$D(z_*, y) \le D(z_*, P_{t_j}(x)) + \epsilon \le D(z_*, x) + \epsilon.$$

Proposition 6.8 is proved. □

Applying by induction Proposition 6.8 we obtain the following result.

Proposition 6.9. *Assume that the function $D(z_*, \cdot)$ is uniformly continuous of all bounded subsets of X and that $M_0, \epsilon > 0$. Then there exists $\delta_0 > 0$ such that for each $x \in X$ satisfying $D(z_*, x) \le M_0$, each index vector $t = (t_1, \dots, t_{p(t)})$ satisfying $p(t) \le \bar{q}$ and each sequence $\{y_j\}_{j=0}^{p(t)} \subset X$ which satisfies*

$$y_0 = x,$$
$$d(y_j, P_{t_j}(y_{j-1})) \le \delta_0,\ j = 1, \dots, p(t),$$

the inequalities

$$D(z_*, y_j) \le D(z_*, y_{j-1}) + \epsilon,$$
$$D(z_*, y_j) \le D(z_*, x) + \bar{q}\epsilon$$

hold for all $j \in \{1, \dots, p(t)\}$.

Proposition 6.10. *Assume that (B6) holds, the function $D(\cdot,\cdot)$ is uniformly continuous on all bounded subsets of $X \times X$ and that $M_0, \epsilon_0 > 0$. Then there exists $\delta_0 > 0$ such that for each $i \in \{1, \dots, m\}$, each $h_1, h_2 \in X$ satisfying*

$$D(z_*, h_1) \le M_0, \tag{6.25}$$

$$d(P_i(h_1), h_2) \le \delta_0 \tag{6.26}$$

and

$$D(h_2, h_1) \ge \epsilon_0 \tag{6.27}$$

the inequality

$$D(h_1, Fix(P_i)) \ge \delta_0$$

holds.

Proof. In view of (A2), there exists $M_1 > M_0$ such that

$$\{h \in X : \; D(z_*, h) \le M_0\} \subset B(\theta, M_1). \tag{6.28}$$

By (B6), there exists $M_2 > M_1 + 1$ such that the following property holds:

(i) for each $x \in B(\theta, M_1)$, each $i \in \{1, \dots, m\}$ and each $z \in \mathrm{Fix}(P_i)$ satisfying $D(z, x) \le D(z_*, x) + 1$ the inclusion $z \in B(\theta, M_2)$ holds.

In view of the uniform continuity of D, there exists

$$\delta \in (0, \min\{3^{-1}\epsilon_0, 3^{-1}\})$$

such that for each $x_1, x_2 \in B(\theta, M_2)$ satisfying $d(x_1, x_2) \le 3\delta_1$,

$$D(x_1, x_2) < \epsilon_0/2. \tag{6.29}$$

By (A1), there exists $\delta_0 \in (0, \delta_1)$ such that the following property holds:

(ii) for each $x, y \in B(\theta, M_2)$ which satisfy $D(x, y) \le \delta_0$, the inequality $d(x, y) \le \delta_1$ holds.

Assume that $i \in \{1, \dots, m\}$ and that $h_1, h_2 \in X$ satisfy (6.25)–(6.27). We show that

$$D(h_1, \mathrm{Fix}(P_i)) \ge \delta_0. \tag{6.30}$$

Assume the contrary. Then there exists

$$\xi \in \mathrm{Fix}(P_i) \tag{6.31}$$

such that

$$D(\xi, h_1) < \delta_0. \tag{6.32}$$

Relations (6.2), (6.31), and (6.32) imply that

$$D(\xi, P_i(h_1)) < \delta_0. \tag{6.33}$$

It follows from the inclusion $z_* \in F$, (6.2), (6.7), and (6.25) that

$$D(z_*, P_i(h_1)) \le D(z_*, h_1) \le M_0. \tag{6.34}$$

By (6.28) and (6.34),

$$P_i(h_1),\ h_1 \in B(\theta, M_1). \tag{6.35}$$

Property (i), (6.31), (6.32), and (6.35) imply that

$$\xi \in B(\theta, M_2). \tag{6.36}$$

It follows from property (ii), (6.32), (6.33), (6.35), and (6.36) that

$$d(\xi, h_1) \le \delta_1,\ d(\xi, P_i(h_1)) \le \delta_1.$$

This implies that

$$d(h_1, P_i(h_1)) \le 2\delta_1.$$

Together with (6.26) this implies that

$$d(h_1, h_2) \le d(h_1, P_i(h_1)) + d(P_i(h_1), h_2) \le 3\delta_1.$$

By (6.35), the relation above and the choice of δ_1 (see (6.29)),

$$D(h_2, h_1) < \epsilon_0/2.$$

This contradicts (6.27). The contradiction we have reached proves (6.30). Proposition 6.10 is proved. □

Proposition 6.11. *Assume that (B3) and (B6) hold, the function $D(\cdot,\cdot)$ is uniformly continuous on all bounded subsets of $X \times X$ and that $M_0, \epsilon_0 > 0$. Then there exists $\delta_0 > 0$ such that for each $i \in \{1, \dots, m\}$, each $h_1, h_2 \in X$ satisfying*

$$D(z_*, h_1) \le M_0,$$

$$d(P_i(h_1), h_2) \le \delta_0$$

and

$$D(h_1, h_2) \leq \delta_0$$

the inequality

$$D(h_1, Fix(P_i)) \leq \epsilon_0$$

holds.

Proof. In view of (A2), there exists $M_1 > M_0$ such that

$$\{h \in X : \; D(z_*, h) \leq M_0\} \subset B(\theta, M_1). \tag{6.37}$$

By (B6), there exists $M_2 > M_1 + 1$ such that the following property holds:

(i) for each $x \in B(\theta, M_1)$, each $i \in \{1, \dots, m\}$ and each $z \in \mathrm{Fix}(P_i)$ satisfying $D(z, x) \leq D(z_*, x) + 1$ the inclusion $z \in B(\theta, M_2)$ holds.

(B3) implies that there is $\gamma_0 > 0$ such that the following property holds:

(ii) for each $i \in \{1, \dots, m\}$, each

$$\xi \in B(\theta, M_2) \cap \mathrm{Fix}(P_i)$$

and each $x \in B(\theta, M_2)$ satisfying $D(x, \mathrm{Fix}(P_i)) \geq \epsilon_0$,

$$D(\xi, P_i(x)) \leq D(\xi, x) - \gamma_0.$$

In view of the uniform continuity of D, there exists

$$\delta_1 \in (0, \min\{\gamma_0, 1)$$

such that the following property holds:

(iii) for each $\xi_1, \xi_2, \xi_3, \xi_4 \in B(\theta, M_2)$ satisfying

$$d(\xi_1, \xi_3), \; d(\xi_2, \xi_4) \leq \delta_1,$$

we have

$$|D(\xi_1, \xi_3) - D(\xi_2, \xi_4)| \leq \gamma_0/16.$$

By (A1), there exists $\delta_0 \in (0, \delta_1)$ such that the following property holds:

(iv) for each $\xi_1, \xi_2 \in B(\theta, M_2)$ satisfying $D(\xi_1, \xi_2) \leq \delta_0$, we have

$$d(\xi_1, \xi_2) < \delta_1.$$

Assume that $i \in \{1, \ldots, m\}$ and that $h_1, h_2 \in X$ satisfy

$$D(z_*, h_1) \le M_0, \tag{6.38}$$

$$d(P_i(h_1), h_2) \le \delta_0, \tag{6.39}$$

$$D(h_1, h_2) \le \delta_0. \tag{6.40}$$

We show that

$$D(h_1, \mathrm{Fix}(P_i)) \le \epsilon_0. \tag{6.41}$$

Assume the contrary. Then

$$D(h_1, \mathrm{Fix}(P_i)) > \epsilon_0 \tag{6.42}$$

and there exists

$$\xi \in \mathrm{Fix}(P_i) \tag{6.43}$$

such that

$$D(\xi, h_1) \le D(h_1, \mathrm{Fix}(P_i)) + \delta_0. \tag{6.44}$$

Relations (6.37) and (6.38) imply that

$$h_1 \in B(\theta, M_1). \tag{6.45}$$

In view of (6.43)–(6.45) and property (i),

$$\xi \in B(\theta, M_2). \tag{6.46}$$

By (6.43)–(6.46) and property (ii),

$$D(\xi, P_i(h_1)) \le D(\xi, h_1) - \gamma_0. \tag{6.47}$$

It follows from the inclusion $z_* \in F$, (6.2), (6.7), and (6.38) that

$$D(z_*, P_i(h_1)) \le D(z_*, h_1) \le M_0. \tag{6.48}$$

In view of (6.37) and (6.48),

$$P_i(h_1) \in B(\theta, M_1). \tag{6.49}$$

By (6.39) and (6.49),

$$h_2 \in B(\theta, M_1 + 1). \tag{6.50}$$

Property (iv), (6.40), (6.45), and (6.50) imply that

$$d(h_1, h_2) \le \delta_1. \tag{6.51}$$

It follows from property (iii), (6.45), (6.46), (6.50), and (6.51) that

$$|D(\xi, h_1) - D(\xi, h_2)| \le \gamma_0/16. \tag{6.52}$$

By property (iii), (6.39), (6.46), (6.49), and (6.50),

$$|D(\xi, P_i(h_1)) - D(\xi, h_2)| \le \gamma_0/16. \tag{6.53}$$

In view of (6.47), (6.52) and (6.53),

$$\begin{aligned} D(\xi, h_2) \le \gamma_0/16 + D(\xi, P_i(h_1)) &\le D(\xi, h_1) - \gamma_0 + \gamma_0/16 \\ &\le D(\xi, h_2) - \gamma_0 + \gamma_0/8, \end{aligned}$$

a contradiction. The contradiction we have reached proves (6.41) and Proposition 6.11 itself. □

Proposition 6.12. *Assume that (B3) and (B6) hold, the function $D(\cdot, \cdot)$ is uniformly continuous on all bounded subsets of $X \times X$ and that $M_0, \epsilon_1 > 0$. Then there exists $\delta_0 > 0$ such that for each $i \in \{1, \dots, m\}$, each $h_1, h_2 \in X$ satisfying*

$$D(z_*, h_1) \le M_0, \tag{6.54}$$

$$d(P_i(h_1), h_2) \le \delta_0 \tag{6.55}$$

and

$$D(h_1, h_2) \le \delta_0 \tag{6.56}$$

the inequality

$$D(h_1, Fix(P_i)) \le \epsilon_1$$

holds.

Proof. In view of (A2), there exists $M_1 > M_0$ such that

$$\{h \in X : \ D(z_*, h) \le M_0\} \subset B(\theta, M_1). \tag{6.57}$$

By (B6), there exists $M_2 > M_1 + 1$ such that the following property holds:

(i) for each $x \in B(\theta, M_1)$, each $i \in \{1, \dots, m\}$ and each $z \in \mathrm{Fix}(P_i)$ satisfying $D(z, x) \le D(z_*, x) + 1$ the inclusion $z \in B(\theta, M_2)$ holds.

By (A1) there exists

$$\epsilon_0 \in (0, \min\{2^{-1}\epsilon_1, 2^{-1}\})$$

such that the following property holds:

(ii) for each $x_1, x_2 \in B(\theta, M_2)$ satisfying

$$D(x_1, x_2) \leq 2\epsilon_0$$

the inequality $d(x_1, x_2) \leq \epsilon_1$ holds.

By Proposition 6.11, there exists $\delta_0 > 0$ such that the following property holds:

(iii) for each $i \in \{1, \ldots, m\}$, each $h_1, h_2 \in X$ satisfying

$$D(z_*, h_1) \leq M_0,$$

$$d(P_i(h_1), h_2) \leq \delta_0$$

and

$$D(h_1, h_2) \leq \delta_0$$

the inequality

$$D(h_1, \mathrm{Fix}(P_i)) \leq \epsilon_0$$

holds.

Assume that $i \in \{1, \ldots, m\}$ and that $h_1, h_2 \in X$ satisfy (6.54)–(6.56). By property (iii) and (6.54)–(6.56),

$$D(h_1, \mathrm{Fix}(P_i)) \leq \epsilon_0. \tag{6.58}$$

In view of (6.58), there exists

$$\xi \in \mathrm{Fix}(P_i) \tag{6.59}$$

such that

$$D(\xi, h_1) \leq 2\epsilon_0. \tag{6.60}$$

Relations (6.54) and (6.57) imply that

$$h_1 \in B(\theta, M_1). \tag{6.61}$$

It follows from (6.59) to (6.61) and property (i) that

$$\xi \in B(\theta, M_2). \tag{6.62}$$

By (6.60)–(6.62) and property (ii),

$$d(h_1, \xi) \le \epsilon_1.$$

In view of (6.59), Proposition 6.12 is proved. □

6.3 Proof of Theorem 6.1

We may assume without loss of generality that

$$d(z_*, \theta) \le M_1. \tag{6.63}$$

In view of (A2), there exists $M_0 > 0$ such that

$$D(z_*, \xi) \le M_0 \text{ for all } \xi \in B(\theta, M). \tag{6.64}$$

By (A2), there exists $M_1 > M$ such that

$$\{\xi \in X : \ D(z_*, \xi) \le M_0\} \subset B(\theta, M_1). \tag{6.65}$$

Set

$$\epsilon_1 = \epsilon \bar{q}^{-1}. \tag{6.66}$$

It follows from (A1) that there exists $\epsilon_0 \in (0, \epsilon_1)$ such that the following property holds:

(i) for each $x, y \in B(\theta, M_1)$ which satisfy $D(x, y) \le \epsilon_0$, the inequality $d(x, y) \le \epsilon_1$ holds.

By (B2) there exists $\gamma_0 > 0$ such that the following property holds:

(ii) for each $i \in \{1, \dots, m\}$ and each $x \in B(\theta, M_1)$ satisfying $D(P_i(x), x) \ge \epsilon_0$ we have

$$D(z_*, P_i(x)) \le D(z_*, x) - \gamma_0.$$

Set

$$Q = M_0 \gamma_0^{-1}. \tag{6.67}$$

Assume that

$$\{\Omega_i\}_{i=1}^{\infty} \subset \mathcal{M}_*, \tag{6.68}$$

$$x_0 \in B(\theta, M) \tag{6.69}$$

and

$$\{x_i\}_{i=1}^{\infty} \subset X, \tag{6.70}$$

$$\{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty) \tag{6.71}$$

for each natural number i,

$$(x_i, \lambda_i) \in A(x_{i-1}, \Omega_i, 0). \tag{6.72}$$

By (6.64) and (6.69),

$$D(z_*, x_0) \leq M_0. \tag{6.73}$$

Let $k \geq 0$ be an integer. In view of (6.72),

$$(x_{k+1}, \lambda_{k+1}) \in A(x_k, \Omega_{k+1}, 0). \tag{6.74}$$

By (6.12), (6.13), and (6.74), there exist

$$(y_{k,t}, \alpha_{k,t}) \in A_0(x_k, t, 0),\ \ t \in \Omega_{k+1} \tag{6.75}$$

such that

$$(x_{k+1}, \lambda_{k+1}) \in \{(y_{k,t}, \alpha_{k,t}) :\ t \in \Omega_{k+1}\}, \tag{6.76}$$

$$\lambda_{k+1} = \max\{\alpha_{k,t} :\ t \in \Omega_{k+1}\}. \tag{6.77}$$

It follows from (6.11) and (6.75) that for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}$ there exists a finite sequence $\{y_i^{(k,t)}\}_{i=0}^{p(t)} \subset X$ such that

$$y_0^{(k,t)} = x_k,\ y_{p(t)}^{(k,t)} = y_{k,t}, \tag{6.78}$$

$$y_j^{(k,t)} = P_{t_j}(y_{j-1}^{(k,t)}) \text{ for each integer } j = 1, \dots, p(t), \tag{6.79}$$

$$\alpha_{k,t} = \max\{D(y_j^{(k,t)}, y_{j-1}^{(k,t)}) :\ j = 1, \dots, p(t)\}. \tag{6.80}$$

By (6.76), there exists

$$s^{(k)} \in \Omega_{k+1} \tag{6.81}$$

such that

$$(x_{k+1}, \lambda_{k+1}) = (y_{k,s^{(k)}}, \alpha_{k,s^{(k)}}). \tag{6.82}$$

Set

$$E = \{k \in \{0, 1, \dots\} : \ \lambda_{k+1} > \epsilon_0\}. \tag{6.83}$$

Let $k \geq 0$ be an integer. By (6.2), (6.78), (6.79), and (A2), for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}$ and each $j \in \{1, \dots, p(t)\}$,

$$D(z_*, y_j^{(k,t)}) \leq D(z_*, P_{t_j}(y_{j-1}^{(k,t)})) \leq D(z_*, y_{j-1}^{(k,t)}),$$
$$D(z_*, y_j^{(k,t)}) \leq D(z_*, y_0^{(k,t)}) \leq D(z_*, x_k) \tag{6.84}$$

and

$$D(z_*, y_{k,t}) \leq D(z_*, x_k). \tag{6.85}$$

In view of (6.81), (6.82), and (6.85),

$$D(z_*, x_{k+1}) = D(z_*, y_{k,s^{(k)}}) \leq D(z_*, x_k). \tag{6.86}$$

It follows from (6.65), (6.73), (6.78), (6.84), and (6.86) that for each integer $k \geq 0$, each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}$ and each $j \in \{0, 1, \dots, p(t)\}$,

$$y_j^{(k,t)} \in B(\theta, M_1), \ \ y_{k,t} \in B(\theta, M_1), \tag{6.87}$$
$$x_k \in B(\theta, M_1).$$

Assume that

$$k \in E. \tag{6.88}$$

In view of (6.83) and (6.88),

$$\lambda_{k+1} > \epsilon_0. \tag{6.89}$$

By (6.82) and (6.89),

$$\alpha_{k,s^{(k)}} > \epsilon_0. \tag{6.90}$$

It follows from (6.80) and (6.90) that there exists $j_0 \in \{1, \dots, p(s)\}$ for which

$$D(y_{j_0}^{(k,s^{(k)})}, y_{j_0-1}^{(k,s^{(k)})}) = \alpha_{k,s^{(k)}} > \epsilon_0. \tag{6.91}$$

By (6.79) and (6.91),

$$D(P_{t_{j_0}}(y_{j_0-1}^{(k,s^{(k)})}), y_{j_0-1}^{(k,s^{(k)})}) > \epsilon_0. \tag{6.92}$$

In view of the choice of γ_0, property (ii) and (6.87),

$$D(z_*, P_{t_{j_0}}(y_{j_0-1}^{(k,s^{(k)})})) \le D(z_*, y_{j_0-1}^{(k,s^{(k)})}) - \gamma_0. \tag{6.93}$$

Relations (6.79) and (6.93) imply that

$$D(z_*, y_{j_0}^{(k,s^{(k)})}) \le D(z_*, y_{j_0-1}^{(k,s^{(k)})}) - \gamma_0. \tag{6.94}$$

By (6.78), (6.81), (6.82), (6.84), and (6.94),

$$\begin{aligned} D(z_*, x_k) - D(z_*, x_{k+1}) &= D(z_*, y_0^{(k,s^{(k)})}) - D(z_*, y_{p(s^{(k)})}^{(k,s^{(k)})}) \\ &= \sum_{j=1}^{p(s^{(k)})} (D(z_*, y_{j-1}^{(k,s^{(k)})}) - D(z_*, y_j^{(k,s^{(k)})})) \\ &\ge D(z_*, y_{j_0-1}^{(k,s^{(k)})}) - D(z_*, y_{j_0}^{(k,s^{(k)})}) \ge \gamma_0. \end{aligned}$$

Thus for each $k \in E$,

$$D(z_*, x_k) - D(z_*, x_{k+1}) \ge \gamma_0. \tag{6.95}$$

Let n be a natural number. By (6.67), (6.73), (6.86), and (6.95),

$$\begin{aligned} M_0 &\ge D(z_*, x_0) - D(z_*, x_n) \\ &= \sum_{k=0}^{n-1} (D(z_*, x_k) - D(z_*, x_{k+1})) \\ &\ge \sum \{D(z_*, x_k) - D(z_*, x_{k+1}) :\ k \in \{0, \dots, n-1\} \cap E\} \\ &\ge \gamma_0 \mathrm{Card}(\{k \in \{0, \dots, n-1\} \cap E\}), \end{aligned}$$

$$\mathrm{Card}(\{k \in \{0, \dots, n-1\} \cap E\}) \le \gamma_0^{-1} M_0 = Q.$$

Since the relation above holds for any natural number n we conclude that

$$\mathrm{Card}(E) \le Q. \tag{6.96}$$

Assume that an integer $k \ge 0$ satisfies

$$k \notin E. \tag{6.97}$$

In view of (6.83) and (6.97),

$$\lambda_{k+1} \le \epsilon_0. \tag{6.98}$$

By (6.77) and (6.80), for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}$ and each $j \in \{1, \dots, p(t)\}$,

$$D(y_j^{(k,t)}, y_{j-1}^{(k,t)}) \le \epsilon_0. \tag{6.99}$$

Property (i), (6.87), and (6.99) imply that for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}$ and each $j \in \{1, \dots, p(t)\}$,

$$d(y_j^{(k,t)}, y_{j-1}^{(k,t)}) \le \epsilon_1. \tag{6.100}$$

It follows from (6.78) and (6.100) that for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}$ and each $j \in \{0, 1, \dots, p(t)\}$,

$$d(x_k, y_j^{(k,t)}) \le j\epsilon_1. \tag{6.101}$$

By (6.6), (6.8), (6.9), (6.66), (6.79), (6.100), and (6.101), for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}$ and each $j \in \{1, \dots, p(t)\}$,

$$\begin{aligned} d(y_{j-1}^{(k,t)}, P_{t_j}(y_j^{(k,t)})) &\le \epsilon_1, \\ y_{j-1}^{(k,t)} &\in F_{\epsilon_1}(P_{t_j}), \\ x_k \in \tilde{F}_{j\epsilon_1}(P_{t_j}) \subset \tilde{F}_{\bar{q}\epsilon_1}(P_{t_j}) &= \tilde{F}_{\epsilon}(P_{t_j}). \end{aligned} \tag{6.102}$$

Let $s \in \{1, \dots, m\}$. In view of (6.4), there exist $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}$ and $j \in \{, 1, \dots, p(t)\}$ such that $s = t_j$. Together with (6.102) this implies that $x_k \in \tilde{F}_{\epsilon}(P_s)$. Theorem 6.1 is proved. □

6.4 Proof of Theorem 6.2

We may assume without loss of generality that

$$d(z_*, \theta) \le M_1. \tag{6.103}$$

In view of (A2), there exists $M_0 > 0$ such that

$$D(z_*, \xi) \le M_0 \text{ for all } \xi \in B(\theta, M). \tag{6.104}$$

By (A2), there exists $M_1 > M$ such that

$$\{\xi \in X : \ D(z_*, \xi) \le M_0\} \subset B(\theta, M_1). \tag{6.105}$$

Set

$$\epsilon_1 = \epsilon \bar{q}^{-1}. \tag{6.106}$$

It follows from (B4) that there exists $\epsilon_2 > 0$ such that the following property holds:

(i) for each $i \in \{1, \dots, m\}$ and each $x \in B(\theta, M_1)$ and each $\xi \in \mathrm{Fix}(P_i)$ satisfying $D(\xi, x) \le 2\epsilon_2$ the relation

$$d(\xi, x) \le \epsilon_1/2$$

holds.

By (B3), there exists $\gamma_0 > 0$ such that the following property holds:

(ii) for each $i \in \{1, \dots, m\}$ and each $x \in B(\theta, M_1)$ satisfying

$$D(x, \mathrm{Fix}(P_i)) \ge \epsilon_2$$

the inequality

$$D(z_*, P_i(x)) \le D(z_*, x) - \gamma_0$$

holds.

Set

$$Q = M_0\gamma_0^{-1}. \tag{6.107}$$

Assume that

$$\{\Omega_i\}_{i=1}^{\infty} \subset \mathcal{M}_*, \tag{6.108}$$

$$x_0 \in B(\theta, M) \tag{6.109}$$

and $\{x_i\}_{i=1}^{\infty} \subset X$, $\{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty)$ satisfy for each natural number i,

$$(x_i, \lambda_i) \in \tilde{A}(x_{i-1}, \Omega_i, 0). \tag{6.110}$$

By (6.104) and (6.109),

$$D(z_*, x_0) \le M_0. \tag{6.111}$$

Let $k \ge 0$ be an integer. In view of (6.110),

$$(x_{k+1}, \lambda_{k+1}) \in \tilde{A}(x_k, \Omega_{k+1}, 0). \tag{6.112}$$

By (6.16) and (6.112), there exist

$$(y_{k,t}, \alpha_{k,t}) \in \tilde{A}_0(x_k, t, 0),\ t \in \Omega_{k+1} \tag{6.113}$$

such that

$$(x_{k+1}, \lambda_{k+1}) \in \{(y_{k,t}, \alpha_{k,t}) : t \in \Omega_{k+1}\}, \tag{6.114}$$

$$\lambda_{k+1} = \max\{\alpha_{k,t} : t \in \Omega_{k+1}\}. \tag{6.115}$$

It follows from (6.15) and (6.113) that for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}$ there exists a finite sequence $\{y_i^{(k,t)}\}_{i=0}^{p(t)} \subset X$ such that

$$y_0^{(k,t)} = x_k, \ y_{p(t)}^{(k,t)} = y_{k,t}, \tag{6.116}$$

$$y_j^{(k,t)} = P_{t_j}(y_{j-1}^{(k,t)}) \text{ for each integer } j = 1, \dots, p(t), \tag{6.117}$$

$$\alpha_{k,t} = \max\{D(y_{j-1}^{(k,t)}, \mathrm{Fix}(P_{t_j})) : j = 1, \dots, p(t)\}. \tag{6.118}$$

Set

$$E = \{k \in \{0, 1, \dots\} : \lambda_{k+1} > \epsilon_2\}. \tag{6.119}$$

Let $k \geq 0$ be an integer. By (6.2), (6.116), and (6.117), for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}$ and each $j \in \{1, \dots, p(t)\}$,

$$D(z_*, y_j^{(k,t)}) \leq D(z_*, P_{t_j}(y_{j-1}^{(k,t)})) \leq D(z_*, y_{j-1}^{(k,t)}), \tag{6.120}$$

$$D(z_*, y_j^{(k,t)}) \leq D(z_*, y_0^{(k,t)}) \leq D(z_*, x_k), \tag{6.121}$$

and

$$D(z_*, y_{k,t}) \leq D(z_*, x_k). \tag{6.122}$$

In view of (6.114) and (6.122),

$$D(z_*, x_{k+1}) \leq D(z_*, x_k). \tag{6.123}$$

It follows from (6.105), (6.111), (6.121), and (6.123) that for each integer $k \geq 0$, each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}$ and each $j \in \{0, 1, \dots, p(t)\}$,

$$y_j^{(k,t)} \in B(\theta, M_1), \ y_{k,t} \in B(\theta, M_1), \ x_k \in B(\theta, M_1). \tag{6.124}$$

Assume that

$$k \in E. \tag{6.125}$$

In view of (6.119) and (6.125),

$$\lambda_{k+1} > \epsilon_2. \tag{6.126}$$

In view of (6.114) there exists $s = (s_1, \dots, s_{p(s)}) \in \Omega_{k+1}$ such that

$$(x_{k+1}, \lambda_{k+1}) = (y_{k,s}, \alpha_{k,s}). \tag{6.127}$$

By (6.126) and (6.127),

$$\alpha_{k,s} > \epsilon_2. \tag{6.128}$$

It follows from (6.118) and (6.128) that there exists $j_0 \in \{1, \dots, p(s)\}$ for which

$$D(y_{j_0-1}^{(k,s)}, \mathrm{Fix}(P_{s_{j_0}})) = \alpha_{k,s} > \epsilon_2. \tag{6.129}$$

By (6.124), (6.129), and property (ii),

$$D(z_*, P_{s_{j_0}}(y_{j_0-1}^{(k,s)})) \le D(z_*, y_{j_0-1}^{(k,s)}) - \gamma_0. \tag{6.130}$$

It follows from (6.116), (6.120), and (6.127) that

$$\begin{aligned} D(z_*, x_k) - D(z_*, x_{k+1}) &= D(z_*, y_0^{(k,s)}) - D(z_*, y_{p(s)}^{(k,s)}) \\ &= \sum_{j=1}^{p(s)} (D(z_*, y_{j-1}^{(k,s)}) - D(z_*, y_j^{(k,s)})) \\ &\ge D(z_*, y_{j_0-1}^{(k,s)}) - D(z_*, y_{j_0}^{(k,s)}) \ge \gamma_0. \end{aligned}$$

Thus for each $k \in E$,

$$D(z_*, x_k) - D(z_*, x_{k+1}) \ge \gamma_0. \tag{6.131}$$

Let n be a natural number. By (6.107), (6.111), (6.123), and (6.131),

$$\begin{aligned} M_0 &\ge D(z_*, x_0) - D(z_*, x_n) \\ &= \sum_{k=0}^{n-1} (D(z_*, x_k) - D(z_*, x_{k+1})) \\ &\ge \sum \{D(z_*, x_k) - D(z_*, x_{k+1}) : \ k \in \{0, \dots, n-1\} \cap E\} \\ &\ge \gamma_0 \mathrm{Card}(\{k \in \{0, \dots, n-1\} \cap E\}), \end{aligned}$$

$$\mathrm{Card}(\{k \in \{0, \dots, n-1\} \cap E\}) \le \gamma_0^{-1} M_0 = Q.$$

Since n is an arbitrary natural number we conclude that

$$\mathrm{Card}(E) \le Q. \tag{6.132}$$

Assume that an integer $k \geq 0$ satisfies

$$k \notin E. \tag{6.133}$$

In view of (6.119) and (6.133),

$$\lambda_{k+1} \leq \epsilon_2. \tag{6.134}$$

By (6.115), (6.118), and (6.134), for each $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{k+1}$ and each $j \in \{1, \ldots, p(t)\}$,

$$D(y_{j-1}^{(k,t)}, \mathrm{Fix}(P_{t_j})) \leq \epsilon_2. \tag{6.135}$$

Let $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{k+1}$ and $j \in \{1, \ldots, p(t)\}$. In view of (6.135), there exists

$$\xi \in \mathrm{Fix}(P_{t_j}) \tag{6.136}$$

such that

$$D(\xi, y_{j-1}^{(k,t)}) \leq 2\epsilon_2. \tag{6.137}$$

It follows from (6.2), (6.117), (6.136), and (6.137) that

$$D(\xi, y_j^{(k,t)}) \leq D(\xi, P_{t_j}(y_{j-1}^{(k,t)})) \leq D(\xi, y_{j-1}^{(k,t)}) \leq 2\epsilon_2. \tag{6.138}$$

Property (i), (6.124), and (6.136)–(6.138) imply that

$$d(\xi, y_{j-1}^{(k,t)}),\ d(\xi, y_j^{(k,t)}) \leq \epsilon_1/2.$$

This implies that

$$d(y_{j-1}^{(k,t)}, y_j^{(k,t)}) \leq \epsilon_1, \tag{6.139}$$

$$d(y_{j-1}^{(k,t)}, \mathrm{Fix}(P_{t_j})) \leq \epsilon_1/2. \tag{6.140}$$

By (6.6), (6.106), (6.116), (6.139), and (6.140),

$$d(x_k, y_{j-1}^{(k,t)}) \leq (j-1)\epsilon_1,$$

$$d(x_k, \mathrm{Fix}(P_{t_j})) \leq j\epsilon_1 \leq \bar{q}\epsilon_1 \leq \epsilon. \tag{6.141}$$

Let $s \in \{1, \ldots, m\}$. In view of (6.4), there exist $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{k+1}$ and $j \in \{1, \ldots, p(t)\}$ such that $s = t_j$. By (6.141),

$$d(x_k, \mathrm{Fix}(P_s)) \leq \epsilon.$$

Theorem 6.2 is proved. □

6.5 Proof of Theorem 6.3

We may assume without loss of generality that

$$d(z_*, \theta) \le M. \tag{6.142}$$

In view of (A2), there exists $M_0 > 0$ such that

$$D(z_*, \xi) \le M_0 \text{ for all } \xi \in B(\theta, M). \tag{6.143}$$

By (A2), there exists $M_1 > M$ such that

$$\{\xi \in X : \ D(z_*, \xi) \le M_0\} \subset B(\theta, M_1). \tag{6.144}$$

Set

$$\epsilon_0 = \epsilon \bar{q}^{-1}. \tag{6.145}$$

It follows from (A1) that there exists $\epsilon_1 \in (0, \epsilon_0)$ such that the following property holds:

(i) for each $x, y \in B(\theta, M_1)$ which satisfy $D(x, y) \le \epsilon_1$, the inequality $d(x, y) \le \epsilon_0$ holds.

By (B5) there exists $\epsilon_2 \in (0, \epsilon_1)$ such that the following property holds:

(ii) for each $x, y \in B(\theta, M_1)$ satisfying $d(x, y) \le \epsilon_2$ we have

$$D(x, y) \le \epsilon_1/2.$$

It follows from (B4) that there exists $\epsilon_3 \in (0, \epsilon_2)$ such that the following property holds:

(iii) for each $i \in \{1, \ldots, m\}$, each $x \in B(\theta, M_1)$ and each $\xi \in \mathrm{Fix}(P_i)$ satisfying $D(\xi, x) \le \epsilon_3$ we have $d(x, \xi) \le \epsilon_2/2$.

In view of (B3), there exists $\gamma_0 > 0$ such that the following property holds:

(iv) for each $i \in \{1, \ldots, m\}$ and each $x \in B(\theta, M_1)$ satisfying

$$D(x, \mathrm{Fix}(P_i)) \ge \epsilon_3$$

the inequality

$$D(z_*, P_i(x)) \le D(z_*, x) - \gamma_0$$

holds.

Set

$$Q = M_0 \gamma_0^{-1}. \tag{6.146}$$

Assume that

$$\{\Omega_i\}_{i=1}^{\infty} \subset \mathcal{M}_*, \tag{6.147}$$

$$x_0 \in B(\theta, M) \tag{6.148}$$

and $\{x_i\}_{i=1}^{\infty} \subset X$, $\{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty)$, for each natural number i,

$$(x_i, \lambda_i) \in A(x_{i-1}, \Omega_i, 0). \tag{6.149}$$

By (6.143) and (6.148),

$$D(z_*, x_0) \leq M_0. \tag{6.150}$$

Let $k \geq 0$ be an integer. In view of (6.149),

$$(x_{k+1}, \lambda_{k+1}) \in A(x_k, \Omega_{k+1}, 0). \tag{6.151}$$

By (6.12), (6.13), and (6.151), there exist

$$(y_{k,t}, \alpha_{k,t}) \in A_0(x_k, t, 0), \; t \in \Omega_{k+1} \tag{6.152}$$

such that

$$(x_{k+1}, \lambda_{k+1}) \in \{(y_{k,t}, \alpha_{k,t}) : \; t \in \Omega_{k+1}\}, \tag{6.153}$$

$$\lambda_{k+1} = \max\{\alpha_{k,t} : \; t \in \Omega_{k+1}\}. \tag{6.154}$$

It follows from (6.11) and (6.152) that for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}$ there exists a finite sequence $\{y_i^{(k,t)}\}_{i=0}^{p(t)} \subset X$ such that

$$y_0^{(k,t)} = x_k, \; y_{p(t)}^{(k,t)} = y_{k,t}, \tag{6.155}$$

$$y_j^{(k,t)} = P_{t_j}(y_{j-1}^{(k,t)}) \text{ for each integer } j = 1, \dots, p(t), \tag{6.156}$$

$$\alpha_{k,t} = \max\{D(y_j^{(k,t)}, y_{j-1}^{(k,t)}) : \; j = 1, \dots, p(t)\}. \tag{6.157}$$

Set

$$E = \{k \in \{0, 1, \dots\} : \; \lambda_{k+1} > 2^{-1}\epsilon_1\}. \tag{6.158}$$

Let $k \geq 0$ be an integer. By (6.2), (6.153), (6.155), and (6.156), for each $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{k+1}$ and each $j \in \{1, \ldots, p(t)\}$,

$$D(z_*, y_j^{(k,t)}) \leq D(z_*, P_{t_j}(y_{j-1}^{(k,t)})) \leq D(z_*, y_{j-1}^{(k,t)}), \tag{6.159}$$

$$D(z_*, y_j^{(k,t)}) \leq D(z_*, y_0^{(k,t)}) \leq D(z_*, x_k), \tag{6.160}$$

$$D(z_*, y_{k,t}) \leq D(z_*, x_k) \tag{6.161}$$

and

$$D(z_*, x_{k+1}) \leq D(z_*, x_k). \tag{6.162}$$

In view of (6.144), (6.150), and (6.160)–(6.162), for each integer $k \geq 0$, each $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{k+1}$ and each $j \in \{0, 1, \ldots, p(t)\}$,

$$y_j^{(k,t)} \in B(\theta, M_1), \ y_{k,t} \in B(\theta, M_1), \ x_k \in B(\theta, M_1). \tag{6.163}$$

Assume that

$$k \in E. \tag{6.164}$$

In view of (6.158) and (6.164),

$$\lambda_{k+1} > \epsilon_1/2. \tag{6.165}$$

By (6.153), there exists $s = (s_1, \ldots, s_{p(s)}) \in \Omega_{k+1}$ such that

$$(x_{k+1}, \lambda_{k+1}) = (y_{k,s}, \alpha_{k,s}). \tag{6.166}$$

By (6.165) and (6.166),

$$\alpha_{k,s} > \epsilon_1/2. \tag{6.167}$$

It follows from (6.156), (6.157), and (6.167) that there exists $j_0 \in \{1, \ldots, p(s)\}$ for which

$$D(P_{t_{j_0}}(y_{j_0-1}^{(k,s)}), y_{j_0-1}^{(k,s)}) = D(y_{j_0}^{(k,s)}, y_{j_0-1}^{(k,s)}) = \alpha_{k,s} > \epsilon_1/2. \tag{6.168}$$

We show that

$$D(y_{j_0-1}^{(k,s)}, \mathrm{Fix}(P_{s_{j_0}})) \geq \epsilon_3.$$

Assume the contrary. Then there exists

$$\xi \in \mathrm{Fix}(P_{s_{j_0}}) \tag{6.169}$$

such that

$$D(\xi, y_{j_0-1}^{(k,s)}) < \epsilon_3. \tag{6.170}$$

By (6.2), (6.156), (6.163), (6.169), and (6.170)

$$D(\xi, y_{j_0}^{(k,s)}) = D(\xi, P_{s_{j_0}}(y_{j_0-1}^{(k,s)})) \le D(\xi, y_{j_0-1}^{(k,s)}) < \epsilon_3. \tag{6.171}$$

Property (iii), (6.170), and (6.171) imply that

$$d(\xi, y_{j_0-1}^{(k,s)}),\ d(\xi, y_{j_0}^{(k,s)}) \le \epsilon_2/2.$$

This implies that

$$d(y_{j_0-1}^{(k,s)}, y_{j_0}^{(k,s)}) \le \epsilon_2. \tag{6.172}$$

In view of (6.163) and (6.172),

$$D(y_{j_0}^{(k,s)}, y_{j_0-1}^{(k,s)}) < \epsilon_1/2.$$

This contradicts (6.168). The contradiction we have reached proves that

$$d(y_{j_0-1}^{(k,s)}, \mathrm{Fix}(P_{s_{j_0}})) \ge \epsilon_3. \tag{6.173}$$

Property (iv), (6.156), (6.163), and (6.173) that

$$D(z_*, y_{j_0}^{(k,s)}) = D(z_*, P_{s_{j_0}}(y_{j_0-1}^{(k,s)})) \le D(z_*, y_{j_0-1}^{(k,s)}) - \gamma_0.$$

The relation above and (6.155), (6.159), and (6.166) imply that

$$\begin{aligned} D(z_*, x_k) - D(z_*, x_{k+1}) &= D(z_*, y_0^{(k,s)}) - D(z_*, y_{p(s)}^{(k,s)}) \\ &= \sum_{j=1}^{p(s)} (D(z_*, y_{j-1}^{(k,s)}) - D(z_*, y_j^{(k,s)})) \\ &\ge D(z_*, y_{j_0-1}^{(k,s)}) - D(z_*, y_{j_0}^{(k,s)}) \ge \gamma_0. \end{aligned}$$

Thus for each $k \in E$,

$$D(z_*, x_k) - D(z_*, x_{k+1}) \ge \gamma_0. \tag{6.174}$$

Let n be a natural number. By (6.146), (6.150), (6.162), and (6.174),

$$M_0 \geq D(z_*, x_0) - D(z_*, x_n)$$

$$= \sum_{k=0}^{n-1}(D(z_*, x_k) - D(z_*, x_{k+1}))$$

$$\geq \sum\{D(z_*, x_k) - D(z_*, x_{k+1}) : \ k \in \{0, \dots, n-1\} \cap E\}$$

$$\geq \gamma_0 \mathrm{Card}(\{k \in \{0, \dots, n-1\} \cap E\}),$$

$$\mathrm{Card}(\{k \in \{0, \dots, n-1\} \cap E\}) \leq \gamma_0^{-1} M_0 = Q.$$

Since the relation above holds for any natural number n we conclude that

$$\mathrm{Card}(E) \leq Q.$$

Assume that an integer $k \geq 0$ satisfies

$$k \notin E. \tag{6.175}$$

In view of (6.158) and (6.175),

$$\lambda_{k+1} \leq 2^{-1}\epsilon_1. \tag{6.176}$$

Let $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}, j \in \{1, \dots, p(t)\}$. By (6.154), (6.157), and (6.176),

$$\alpha_{k,t} \leq 2^{-1}\epsilon_1, \ D(y_j^{(k,t)}, y_{j-1}^{(k,t)}) \leq 2^{-1}\epsilon_1. \tag{6.177}$$

It follows from (6.163) and (6.177) that

$$d(y_j^{(k,t)}, y_{j-1}^{(k,t)}) \leq \epsilon_0. \tag{6.178}$$

By (6.156) and (6.178),

$$y_{j-1}^{(k,t)} \in F_{\epsilon_0}(P_{t_j}). \tag{6.179}$$

In view of (6.155) and (6.178),

$$d(x_k, y_{j-1}^{(k,t)}) \leq (j-1)\epsilon_0. \tag{6.180}$$

It follows from (6.6), (6.145), (6.179), and (6.180) that

$$x_k \in \tilde{F}_{j\epsilon_0}(P_{t_j}) \subset \tilde{F}_{\bar{q}\epsilon_0}(P_{t_j}) \subset \tilde{F}_{\epsilon}(P_{t_j}).$$

Thus

$$x_k \in \tilde{F}_{\epsilon_1}(P_{t_j})$$

for all $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}$ and $j \in \{, 1, \dots, p(t)\}$. Together with (6.4) this implies that $x_k \in \tilde{F}_\epsilon$. Theorem 6.3 is proved. □

6.6 Proof of Theorem 6.4

We may assume without loss of generality that

$$\epsilon < 1,\ d(z_*, \theta) \leq M. \tag{6.181}$$

By Theorems 6.1 and 6.2, there exist a natural number n_0 and $M_1 > M + 1$ such that the following property holds:

(i) for each sequence of singletons $\Omega_i = \{t^{(i)}\} \in \mathcal{M}_*$, $i = 1, 2, \dots$, each $x_0 \in B(\theta, M)$, each $\{x_i\}_{i=1}^\infty \subset X$ and each $\{\lambda_i\}_{i=1}^\infty \subset [0, \infty)$ satisfying for each natural number i,

$$(x_i, \lambda_i) \in A_0(x_{i-1}, t^{(i)}, 0),$$

we have

$$x_k \in B(\theta, M_1 - 1) \text{ for all integers } k \geq 0$$

and there exists an integer $q \in [1, n_0]$ such that

$$x_q \in \tilde{F}_{\epsilon/2};$$

moreover, if (B3) and (B4) hold, then

$$d(x_q, \mathrm{Fix}(P_s)) \leq \epsilon/2,\ s = 1, \dots, q.$$

In view of (A2), there exists $M_2 > 0$ such that

$$D(z_*, \xi) \leq M_2 \text{ for all } \xi \in B(\theta, M_1). \tag{6.182}$$

By (A2), there exists $M_3 > M_2 + M_1 + 1$ such that

$$\{\xi \in X:\ D(z_*, \xi) \leq M_2\} \subset B(\theta, M_3 - 1). \tag{6.183}$$

Set

$$\delta_{n_0\bar{q}} = 2^{-1}\epsilon. \tag{6.184}$$

By induction we define a sequence of positive numbers $\{\delta_i\}_{i=0}^{n_0\bar{q}}$ such that for each nonnegative integer $i < n_0\bar{q}$,

$$\delta_i \le 4^{-1}\delta_{i+1} \tag{6.185}$$

and the following property holds:

(ii) for each $\xi_1, \xi_2 \in B(\theta, M_3)$ satisfying $d(\xi_1, \xi_2) \le 2\delta_i$, we have

$$d(P_j(\xi_1), P_j(\xi_2)) \le 4^{-1}\delta_{i+1},\ j = 1, \dots, m.$$

Set

$$\delta = 4^{-1}\delta_0. \tag{6.186}$$

Assume that

$$\{\Omega_i\}_{i=1}^{\infty} \subset \mathcal{M}_*, \tag{6.187}$$

for each natural number i, Ω_i is a singleton $t^{(i)} = \{t_1^{(i)}, \dots, t_{p(t^{(i)})}^{(i)}\}$,

$$x_0 \in B(\theta, M) \tag{6.188}$$

and $\{x_i\}_{i=1}^{\infty} \subset X,\ \{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty)$ satisfying for each natural number i,

$$(x_i, \lambda_i) \in A_0(x_{i-1}, \{t^{(i)}\}, \delta). \tag{6.189}$$

Let $k \ge 0$ be an integer. It follows from (6.11) and (6.189) that there exists a finite sequence $\{y_i^{(k)}\}_{i=0}^{p(t^{(k)})} \subset X$ such that

$$y_0^{(k)} = x_k,\ y_{p(t^{(k)})}^{(k)} = x_{k+1}, \tag{6.190}$$

$$d(y_j^{(k)}, P_{t_j}(y_{j-1}^{(k)})) \le \delta \text{ for each integer } j = 1, \dots, p(t^{(k)}). \tag{6.191}$$

Set

$$\tilde{x}_0 = x_0,\ y_0^{(0)} = \tilde{x}_0. \tag{6.192}$$

By induction define $\{\tilde{x}_k\}_{k=0}^{\infty} \subset X$ and $\tilde{y}_j^{(k)} \in X, j = 0, \dots, p(t^{(k)}), k = 0, 1, \dots$ such that for each integer $k \ge 0$,

$$\tilde{y}_0^{(k)} = \tilde{x}_k,\ \tilde{y}_j^{(k)} = P_{t_j}(\tilde{y}_{j-1}^{(k)}),\ j = 1, \dots, p(t^{(k)}),$$
$$\tilde{x}_{k+1} = \tilde{y}_{p(t^{(k)})}^{(k)}. \tag{6.193}$$

In view of (6.181) and (6.182),

$$D(z_*, x_0) \le M_2.$$

It follows from (6.2), (6.183), (6.192), (6.193) and the inequality above that for each integer $k \ge 0$ and each $j = 0, \dots, p(t)$,

$$D(z_*, \tilde{y}_j^{(k)}) \le D(z_*, \tilde{x}_k) \le M_2,\ \tilde{y}_j^{(k)} \in B(\theta, M_3 - 1). \tag{6.194}$$

Property (i), (6.2), (6.11), (6.187), (6.188), (6.192), and (6.193) imply that

$$\tilde{x}_k \in B(\theta, M_1 - 1) \text{ for all integers } k \ge 0$$

and that there exists an integer $q \in [1, n_0]$ such that

$$x_q \in \tilde{F}_{\epsilon/2} \tag{6.195}$$

and if (B3) and (B4) hold, then

$$d(x_q, \mathrm{Fix}(P_s)) \le \epsilon/2,\ s = 1, \dots, q. \tag{6.196}$$

We estimate

$$d(y_j^{(k)}, \tilde{y}_j^{(k)}),\ k = 0, \dots, n_0,\ j = 0, \dots, p(t^{(k)}).$$

We show that for each integer $k \in \{0, \dots, n_0 - 1\}$ and each $j = 0, \dots, p(t^{(k)})$,

$$d(\tilde{x}_k, x_k) \le \delta_{\bar{q}k},\ d(y_j^{(k)}, \tilde{y}_j^{(k)}) \le \delta_{\bar{q}k+j},$$
$$d(\tilde{x}_{n_0}, x_{n_0}) \le \delta_{\bar{q}n_0}. \tag{6.197}$$

In view of (6.192), (6.197) holds for $k = 0$ and $j = 0$.

Assume that $k \in \{0, \dots, n_0 - 1\}$ satisfies

$$d(\tilde{x}_k, x_k) \le \delta_{\bar{q}k}. \tag{6.198}$$

By (6.190), (6.193), and (6.198),

$$d(y_0^{(k)}, \tilde{y}_0^{(k)}) \le \delta_{\bar{q}k}. \tag{6.199}$$

Assume that $j \in \{0, \dots, p(t^{(k)} - 1\}$ and that

$$d(\tilde{y}_j^{(k)}, y_j^{(k)}) \le \delta_{\bar{q}k+j}. \tag{6.200}$$

In view of (6.184), (6.194), and (6.200),

$$y_j^{(k)},\ \tilde{y}_j^{(k)} \in B(\theta, M_3). \tag{6.201}$$

By inclusion, we conclude that (6.201) implies that

$$d(P_{t_{j+1}}(y_j^{(k)}), P_{t_{j+1}}(\tilde{y}_j^{(k)})) \le 4^{-1}\delta_{\bar{q}k+j+1}. \tag{6.202}$$

It follows from (6.185), (6.186), (6.191), (6.193), and (6.202) that

$$\begin{aligned} d(y_{j+1}^{(k)}, \tilde{y}_{j+1}^{(k)}) &\le d(y_{j+1}^{(k)}, P_{t_{j+1}}(y_j^{(k)})) + d(P_{t_{j+1}}(y_j^{(k)}), P_{t_{j+1}}(\tilde{y}_j^{(k)})) \\ &\le \delta + 4^{-1}\delta_{\bar{q}k+j+1} \le \delta_{\bar{q}k+j+1}. \end{aligned}$$

By induction,

$$d(\tilde{y}_j^{(k)}, y_j^{(k)}) \le \delta_{\bar{q}k+j},\ j = 0, \dots, p(t)$$

and in view of (6.6), (6.190), and (6.193),

$$d(\tilde{x}_{k+1}, x_{k+1}) = d(\tilde{y}_{p(t^{(k)})}^{(k)}, y_{p(t^{(k)})}^{(k)}) \le \delta_{\bar{q}k+p(t^{(k)})} \le \delta_{\bar{q}(k+1)}.$$

Thus we conclude that (6.197) holds for each $k \in \{0, \dots, n_0 - 1\}$ and each $j = 0, \dots, p(t^{(k)})$,

$$d(\tilde{x}_{n_0}, x_{n_0}) \le \delta_{\bar{q}n_0}. \tag{6.203}$$

It follows from (6.184), (6.194), (6.197), (6.203) and the inclusion $\tilde{x}_k \in B(\theta, M_1 - 1)$ that

$$\begin{aligned} &x_k \in B(\theta, M_1),\ k = 0, \dots, n_0, \\ &d(x_q, \tilde{x}_q) \le \delta_{n_0\bar{q}} \le \epsilon/2. \end{aligned}$$

In view of (6.195) and (6.196), this completes the proof of Theorem 6.4. □

6.7 Proof of Theorem 6.5

We may assume without loss of generality that

$$d(z_*, \theta) \le M,\ \epsilon \in (0, 1). \tag{6.204}$$

In view of (A2), there exists $M_0 > M + 1$ such that

$$D(z_*, \xi) \le M_0 \text{ for all } \xi \in B(\theta, M + 1). \tag{6.205}$$

By (A2), there exists $M_1 > M_0 + 1$ such that

$$\{\xi \in X : \; D(z_*, \xi) \le M_0 + 2\} \subset B(\theta, M_1). \tag{6.206}$$

Set

$$\epsilon_0 = \epsilon \bar{q}^{-1}. \tag{6.207}$$

It follows from (A1) that there exists $\epsilon_1 \in (0, \epsilon_0)$ such that the following property holds:

(i) for each $x, y \in B(\theta, M_1)$ which satisfy $D(x, y) \le 2\epsilon_1$, the inequality $d(x, y) \le \epsilon_0$ holds.

By (B2) there exists $\gamma_0 \in (0, 8^{-1}\epsilon_1)$ such that the following property holds:

(ii) for each $i \in \{1, \ldots, m\}$ and each $x \in B(\theta, M_1)$ satisfying $D(P_i(x), x) \ge \epsilon_1/2$ the relation

$$D(z_*, P_i(x)) \le D(z_*, x) - \gamma_0$$

holds.

Set

$$\gamma_1 = \gamma_0 (4\bar{q})^{-1}. \tag{6.208}$$

Since the function D is uniformly continuous there exists $\delta_1 \in (0, 16^{-1}\gamma_0)$ such that the following property holds:

(iii) for each $\xi_1, \xi_2, \xi_3, \xi_4 \in B(\theta, M_1)$ satisfying $d(\xi_1, \xi_3), \; d(\xi_2, \xi_4) \le \delta_1$ we have

$$|D(\xi_1, \xi_2) - D(\xi_3, \xi_4)| \le 4^{-1}\gamma_0.$$

By Proposition 6.9, there exists $\delta \in (0, \delta_1)$ such that the following property holds:

(iv) for each $x \in X$ satisfying $D(z_*, x) \le M_0 + 2$, each index vector $t = (t_1, \ldots, t_{p(t)})$ satisfying $p(t) \le \bar{q}$ and each sequence $\{y_j\}_{j=0}^{p(t)} \subset X$ which satisfies

$$y_0 = x,$$
$$d(y_j, P_{t_j}(y_{j-1})) \le \delta, \; j = 1, \ldots, p(t),$$

we have for all $j = 1, \ldots, p(t)$,

$$D(z_*, y_j) \le D(z_*, y_{j-1}) + \gamma_1,$$
$$D(z_*, y_j) \le D(z_*, x) + \bar{q}\gamma_1.$$

Choose a natural number

$$n_0 \geq 2M_0\gamma_0^{-1}. \tag{6.209}$$

Assume that

$$\{\Omega_i\}_{i=1}^{\infty} \subset \mathcal{M}_*, \tag{6.210}$$

$$x_0 \in B(\theta, M) \tag{6.211}$$

and $\{x_i\}_{i=1}^{\infty} \subset X,\ \{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty)$, for each natural number i,

$$(x_i, \lambda_i) \in A(x_{i-1}, \Omega_i, \delta). \tag{6.212}$$

Let $k \geq 0$ be an integer. In view of (6.212),

$$(x_{k+1}, \lambda_{k+1}) \in A(x_k, \Omega_{k+1}, \delta). \tag{6.213}$$

By (6.12), (6.13), and (6.213), there exist

$$(y_{k,t}, \lambda_{k,t}) \in A_0(x_k, t, \delta),\ \ t \in \Omega_{k+1} \tag{6.214}$$

such that

$$(x_{k+1}, \lambda_{k+1}) \in \{(y_{k,t}, \lambda_{k,t}) :\ t \in \Omega_{k+1}\}, \tag{6.215}$$

$$\lambda_{k+1} \geq \lambda_{k,t} - \delta \text{ for all } t \in \Omega_{k+1}. \tag{6.216}$$

It follows from (6.11) and (6.214) that for each $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{k+1}$ there exists a finite sequence $\{y_i^{(k,t)}\}_{i=0}^{p(t)} \subset X$ such that

$$y_0^{(k,t)} = x_k, \tag{6.217}$$

$$d(P_{t_j}(y_{j-1}^{(k,t)}), y_j^{(k,t)}) \leq \delta \text{ for each integer } j = 1, \ldots, p(t), \tag{6.218}$$

$$y_{p(t)}^{(k,t)} = y_{k,t}, \tag{6.219}$$

$$\lambda_{k,t} = \max\{D(y_j^{(k,t)}, y_{j-1}^{(k,t)}) :\ j = 1, \ldots, p(t)\}. \tag{6.220}$$

By (6.204) and (6.205),

$$D(z_*, x_0) \leq M_0. \tag{6.221}$$

We prove the following auxiliary result.

Lemma 6.13. *Assume that a nonnegative integer k satisfies*

$$D(z_*, x_k) \leq M_0, \tag{6.222}$$

$$\lambda_{k+1} > \epsilon_1. \tag{6.223}$$

Then

$$D(z_*, y_j^{(k,t)}) \leq M_0 + 1 \textit{ for all } t \in \Omega_{k+1} \textit{ and all } j = 1, \dots, p(t),$$

$$D(z_*, x_k) - D(z_*, x_{k+1}) \geq 2^{-1}\gamma_0.$$

Proof. By (6.6), (6.210), (6.217), (6.218), (6.222), and property (iv), for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}$ and each $j \in \{1, \dots, p(t)\}$,

$$D(z_*, y_j^{(k,t)}) \leq D(z_*, y_{j-1}^{(k,t)}) + \gamma_1, \tag{6.224}$$

$$D(z_*, y_j^{(k,t)}) \leq D(z_*, x_k) + \gamma_1 \bar{q}. \tag{6.225}$$

In view of (6.208), (6.222), and (6.225),

$$D(z_*, y_j^{(k,t)}) \leq M_0 + 1 \tag{6.226}$$

for each $t \in \Omega_{k+1}$ and each $j \in \{1, \dots, p(t)\}$.

By (6.215), there exists

$$s = (s_1, \dots, s_{p(s)}) \in \Omega_{k+1}$$

such that

$$(x_{k+1}, \lambda_{k+1}) = (y_{k,s}, \lambda_{k,s}). \tag{6.227}$$

Relations (6.223) and (6.227) imply that

$$\lambda_{k,s} = \lambda_{k+1} > \epsilon_1. \tag{6.228}$$

By (6.220) and (6.228), there exists

$$j_0 \in \{1, \dots, p(s)\}$$

such that

$$D(y_{j_0}^{(k,s)}, y_{j_0-1}^{(k,s)}) = \lambda_{k,s} > \epsilon_1. \tag{6.229}$$

In view of (6.218),

$$d(P_{s_{j_0}}(y_{j_0-1}^{(k,s)}), y_{j_0}^{(k,s)}) \leq \delta. \tag{6.230}$$

It follows from (6.208), (6.217), (6.222), and (6.225) that

$$D(z_*, y_{j_0}^{(k,s)}),\ D(z_*, y_{j_0-1}^{(k,s)}) \le D(z_*, x_k) + \gamma_1 \bar{q} \le M_0 + 1. \tag{6.231}$$

By (6.2) and (6.231),

$$D(z_*, P_{s_{j_0}}(y_{j_0-1}^{(k,s)})) \le D(z_*, y_{j_0-1}^{(k,s)}) \le M_0 + 1. \tag{6.232}$$

In view of (6.206), (6.231), and (6.232),

$$y_{j_0}^{(k,s)},\ y_{j_0-1}^{(k,s)},\ P_{s_{j_0}}(y_{j_0-1}^{(k,s)}) \in B(\theta, M_1). \tag{6.233}$$

Property (iii), (6.230), and (6.233) imply that

$$|D(P_{s_{j_0}}(y_{j_0-1}^{(k,s)}), y_{j_0-1}^{(k,s)}) - D(y_{j_0}^{(k,s)}, y_{j_0-1}^{(k,s)})| \le 4^{-1}\gamma_0. \tag{6.234}$$

It follows from (6.229) and (6.234) that

$$D(P_{s_{j_0}}(y_{j_0-1}^{(k,s)}), y_{j_0-1}^{(k,s)}) > 2^{-1}\epsilon_1. \tag{6.235}$$

Property (ii), (6.233), and (6.235) imply that

$$D(z_*, P_{s_{j_0}}(y_{j_0-1}^{(k,s)})) \le D(z_*, y_{j_0-1}^{(k,s)}) - \gamma_0. \tag{6.236}$$

By (6.204), (6.230), (6.233), and property (iii),

$$|D(z_*, y_{j_0}^{(k,s)}) - D(z_*, P_{s_{j_0}}(y_{j_0-1}^{(k,s)}))| \le \gamma_0/4. \tag{6.237}$$

In view of (6.236) and (6.237),

$$\begin{aligned} D(z_*, y_{j_0}^{(k,s)}) &\le D(z_*, P_{s_{j_0}}(y_{j_0-1}^{(k,s)})) + 4^{-1}\gamma_0 \\ &\le D(z_*, y_{j_0-1}^{(k,s)}) - \gamma_0 + 4^{-1}\gamma_0, \end{aligned}$$

$$D(z_*, y_{j_0}^{(k,s)}) \le D(z_*, y_{j_0-1}^{(k,s)}) - (3/4)\gamma_0. \tag{6.238}$$

It follows from (6.6), (6.208), (6.217), (6.219), (6.224), (6.227), and (6.238),

$$\begin{aligned} D(z_*, x_k) - D(z_*, x_{k+1}) &= D(z_*, y_0^{(k,s)}) - D(z_*, y_{p(s)}^{(k,s)}) \\ &= \sum_{j=1}^{p(s)} (D(z_*, y_{j-1}^{(k,s)}) - D(z_*, y_j^{(k,s)}) \\ &\ge -\gamma_1(p(s) - 1) + D(z_*, y_{j_0-1}^{(k,s)}) - D(z_*, y_{j_0}^{(k,s)}) \\ &\ge (3/4)\gamma_0 - \gamma_1 \bar{q} \ge 2^{-1}\gamma_0. \end{aligned}$$

Lemma 6.13 is proved. □

Assume that q is a natural number such that for each nonnegative integer $k < q$,

$$\lambda_{k+1} > \epsilon_1.$$

By (6.221) and Lemma 6.13 applied by induction,

$$D(z_*, x_k) \le M_0,\ k = 0, \dots, q$$

and for each nonnegative integer $k < q$,

$$D(z_*, x_k) - D(z_*, x_{k+1}) \ge 2^{-1}\gamma_0. \tag{6.239}$$

It follows from (6.209), (6.221), and (6.239) that

$$\begin{aligned} M_0 &\ge D(z_*, x_0) \ge D(z_*, x_0) - D(z_*, x_q) \\ &= \sum_{k=0}^{q-1}(D(z_*, x_k) - D(z_*, x_{k+1})) \ge 2^{-1}q\gamma_0 \end{aligned}$$

and

$$q \le 2M_0\gamma_0^{-1} \le n_0.$$

This implies that there exists a nonnegative integer $q \le n_0$ such that for all nonnegative integers $k \le q$,

$$D(z_*, x_k) \le M_0$$

and

$$\lambda_{q+1} \le \epsilon_1.$$

In view of (6.206),

$$x_k \in B(\theta, M_1),\ k = 0, \dots, q.$$

Assume that an integer $q \ge 0$ satisfies

$$D(z_*, x_q) \le M_0, \tag{6.240}$$

$$\lambda_{q+1} \le \epsilon_1. \tag{6.241}$$

In view of (6.216), (6.241), for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{q+1}$ and each $j \in \{1, \dots, p(t)\}$,

$$\lambda_{q,t} \le \lambda_{q+1} + \delta \le \epsilon_1 + \delta \le 2\epsilon_1, \tag{6.242}$$

$$D(y_j^{(q,t)}, y_{j-1}^{(q,t)}) \le \lambda_{q,t} \le \epsilon_1 + \delta \le 2\epsilon_1. \tag{6.243}$$

Property (iv), (6.6), (6.208), (6.212), (6.217), (6.218), and (6.240) imply that for each $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{q+1}$ and each $j \in \{1, \ldots, p(t)\}$,

$$D(z_*, y_j^{(q,t)}) \le D(z_*, x_q) + \gamma_1 \bar{q} \le M_0 + 1. \tag{6.244}$$

By (6.206) and (6.244), for each $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{q+1}$ and each $j \in \{0, 1, \ldots, p(t)\}$,

$$y_j^{(q,t)} \in B(\theta, M_1). \tag{6.245}$$

It follows from (6.243), (6.245), and property (i) that for each

$$t = (t_1, \ldots, t_{p(t)}) \in \Omega_{q+1}$$

and each $j \in \{1, \ldots, p(t)\}$,

$$d(y_{j-1}^{(q,t)}, y_j^{(q,t)}) \le \epsilon_0. \tag{6.246}$$

Let $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{q+1}$ and $j \in \{1, \ldots, p(t)\}$. In view of (6.2), (6.217), and (6.244),

$$D(z_*, P_{t_j}(y_{j-1}^{(q,t)})) \le D(z_*, y_{j-1}^{(q,t)}) \le M_0 + 1. \tag{6.247}$$

Relations (6.206) and (6.247) imply that

$$P_{t_j}(y_{j-1}^{(q,t)}) \in B(\theta, M_1). \tag{6.248}$$

By (6.218), (6.245), (6.248), and property (iii),

$$|D(y_j^{(q,t)}, y_{j-1}^{(q,t)}) - D(P_{t_j}(y_{j-1}^{(q,t)}), y_{j-1}^{(q,t)})| \le 4^{-1}\gamma_0. \tag{6.249}$$

It follows from (6.243) and (6.249) imply that

$$D(P_{t_j}(y_{j-1}^{(q,t)}), y_{j-1}^{(q,t)}) \le D(y_j^{(q,t)}), y_{j-1}^{(q,t)}) + \gamma_0/4 \le \epsilon_1 + \delta + 4^{-1}\gamma_0 \le 2\epsilon_1. \tag{6.250}$$

In view of property (i), (6.245), (6.248), and (6.250),

$$d(P_{t_j}(y_{j-1}^{(q,t)}), y_{j-1}^{(q,t)}) \le \epsilon_0. \tag{6.251}$$

Therefore (6.246) and (6.251) hold for all $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{q+1}$ and all $j \in \{1, \ldots, p(t)\}$. Together with (6.6) and (6.217) this implies that for all $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{q+1}$ and all $j_1, j_2 \in \{0, 1, \ldots, p(t)\}$,

$$d(y_{j_1}^{(q,t)}, y_{j_2}^{(q,t)}) \le \bar{q}\epsilon_0,$$

$$d(x_q, y_{j_1}^{(q,t)}),\ d(x_q, y_{j_2}^{(q,t)}) \le \bar{q}\epsilon_0. \tag{6.252}$$

By (6.207), (6.251), and (6.252), for all $t = (t_1, \dots, t_{p(t)}) \in \Omega_{q+1}$ and all $j \in \{1, \dots, p(t)\}$,

$$x_{q+1} \in \tilde{F}_{\bar{q}\epsilon_0}(P_{t_j}) = \tilde{F}_{\epsilon}(P_{t_j})$$

and in view of (6.4)

$$x_{q+1} \in \tilde{F}_{\epsilon}.$$

Theorem 6.5 is proved. □

6.8 Proof of Theorem 6.6

We may assume without loss of generality that

$$d(z_*, \theta) \le M,\ \epsilon \in (0, 1). \tag{6.253}$$

In view of (A2), there exists $M_0 > M + 1$ such that

$$D(z_*, \xi) \le M_0 \text{ for all } \xi \in B(\theta, M + 1). \tag{6.254}$$

By (A2), there exists $M_1 > M_0 + 1$ such that

$$\{\xi \in X :\ D(z_*, \xi) \le M_0 + 2\} \subset B(\theta, M_1). \tag{6.255}$$

(B6) implies that there exists $M_2 > M_1 + 1$ such that the following property holds:

(i) for each $x \in B(\theta, M_1 + 1)$, each $i \in \{1, \dots, m\}$ and each $z \in \mathrm{Fix}(P_i)$ satisfying $D(z, x) \le D(z_*, x) + 1$ the inclusion $z \in B(\theta, M_2)$ holds.

Set

$$\epsilon_0 = \epsilon(3\bar{q} + 1)^{-1}. \tag{6.256}$$

It follows from (A1) that there exists $\epsilon_1 \in (0, \epsilon_0)$ such that the following property holds:

(ii) for each $x, y \in B(\theta, M_2)$ which satisfy $D(x, y) \le 2\epsilon_1$, the inequality $d(x, y) \le \epsilon_0$ holds.

By (B3) there exists $\gamma_0 \in (0, 8^{-1}\epsilon_1)$ such that the following property holds:

(iii) for each $i \in \{1, \dots, m\}$ and each $x \in B(\theta, M_2)$ satisfying

$$D(x, \mathrm{Fix}(P_i)) \geq \epsilon_1/2$$

the relation

$$D(z_*, P_i(x)) \leq D(z_*, x) - \gamma_0$$

holds.

Set

$$\gamma_1 = \gamma_0(4\bar{q})^{-1}. \tag{6.257}$$

Since the function D is uniformly continuous there exists $\delta_1 \in (0, 16^{-1}\gamma_0)$ such that the following property holds:

(iv) for each $\xi_1, \xi_2, \xi_3, \xi_4 \in B(\theta, M_2)$ satisfying $d(\xi_1, \xi_3),\ d(\xi_2, \xi_4) \leq \delta_1$ we have

$$|D(\xi_1, \xi_2) - D(\xi_3, \xi_4)| \leq 4^{-1}\gamma_0.$$

By Proposition 6.9, there exists $\delta \in (0, \delta_1)$ such that the following property holds:

(v) for each $x \in X$ satisfying $D(z_*, x) \leq M_0 + 2$, each index vector $t = (t_1, \dots, t_{p(t)})$ satisfying $p(t) \leq \bar{q}$ and each sequence $\{y_j\}_{j=0}^{p(t)} \subset X$ which satisfies

$$y_0 = x,$$
$$d(y_j, P_{t_j}(y_{j-1})) \leq \delta_0,\ j = 1, \dots, p(t)$$

we have for all $j = 1, \dots, p(t)$,

$$D(z_*, y_j) \leq D(z_*, y_{j-1}) + \gamma_1,$$
$$D(z_*, y_j) \leq D(z_*, x) + \bar{q}\gamma_1.$$

Choose a natural number

$$n_0 \geq 2M_0\gamma_0^{-1}. \tag{6.258}$$

Assume that

$$\{\Omega_i\}_{i=1}^{\infty} \subset \mathcal{M}_*, \tag{6.259}$$
$$x_0 \in B(\theta, M) \tag{6.260}$$

and $\{x_i\}_{i=1}^{\infty} \subset X,\ \{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty)$, for each natural number i,

$$(x_i, \lambda_i) \in \tilde{A}(x_{i-1}, \Omega_i, \delta). \tag{6.261}$$

Let $k \geq 0$ be an integer. In view of (6.261),

$$(x_{k+1}, \lambda_{k+1}) \in \tilde{A}(x_k, \Omega_{k+1}, \delta). \tag{6.262}$$

By (6.16) and (6.262), there exist

$$(y_{k,t}, \lambda_{k,t}) \in \tilde{A}_0(x_k, \Omega_{k+1}, \delta),\ t \in \Omega_{k+1} \tag{6.263}$$

such that

$$(x_{k+1}, \lambda_{k+1}) \in \{(y_{k,t}, \lambda_{k,t}) :\ t \in \Omega_{k+1}\}, \tag{6.264}$$

$$\lambda_{k+1} \geq \lambda_{k,t} - \delta \text{ for all } t \in \Omega_{k+1}. \tag{6.265}$$

It follows from (6.15) and (6.263) that for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}$ there exists a finite sequence $\{y_i^{(k,t)}\}_{i=0}^{p(t)} \subset X$ such that

$$y_0^{(k,t)} = x_k, \tag{6.266}$$

$$d(P_{t_j}(y_{j-1}^{(k,t)}), y_j^{(k,t)}) \leq \delta \text{ for each integer } j = 1, \dots, p(t), \tag{6.267}$$

$$y_{p(t)}^{(k,t)} = y_{k,t}, \tag{6.268}$$

$$\lambda_{k,t} = \max\{D(y_{j-1}^{(k,t)}, \mathrm{Fix}(P_{t_j})) :\ j = 1, \dots, p(t)\}. \tag{6.269}$$

By (6.254) and (6.260),

$$D(z_*, x_0) \leq M_0. \tag{6.270}$$

We prove the following auxiliary result.

Lemma 6.14. *Assume that a nonnegative integer k satisfies*

$$D(z_*, x_k) \leq M_0, \tag{6.271}$$

$$\lambda_{k+1} > \epsilon_1. \tag{6.272}$$

Then

$$D(z_*, y_j^{(k,t)}) \leq M_0 + 1 \textit{ for all } t \in \Omega_{k+1} \textit{ and all } j = 1, \dots, p(t),$$

$$D(z_*, x_k) - D(z_*, x_{k+1}) \geq 2^{-1}\gamma_0.$$

Proof. By (6.6), (6.255), (6.257), (6.266)–(6.268), (6.271), and property (v), for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}$ and each $j \in \{1, \dots, p(t)\}$,

$$D(z_*, y_j^{(k,t)}) \le D(z_*, y_{j-1}^{(k,t)}) + \gamma_1, \tag{6.273}$$

$$D(z_*, y_j^{(k,t)}) \le D(z_*, x_k) + \gamma_1 \bar{q}, \tag{6.274}$$

$$D(z_*, y_j^{(k,t)}) \le M_0 + 1, \ y_j^{(k,t)} \in B(\theta, M_1) \tag{6.275}$$

for each $t \in \Omega_{k+1}$ and each $j \in \{0, 1, \dots, p(t)\}$.

By (6.264), there exists

$$s = (s_1, \dots, s_{p(s)}) \in \Omega_{k+1}$$

such that

$$(x_{k+1}, \lambda_{k+1}) = (y_{k,s}, \lambda_{k,s}). \tag{6.276}$$

Relations (6.276) and (6.272) imply that

$$\lambda_{k,s} = \lambda_{k+1} > \epsilon_1. \tag{6.277}$$

By (6.269) and (6.277), there exists

$$j_0 \in \{1, \dots, p(s)\}$$

such that

$$D(y_{j_0-1}^{(k,s)}, \mathrm{Fix}(P_{s_{j_0}})) = \lambda_{k,s} > \epsilon_1. \tag{6.278}$$

Property (iii), (6.275), and (6.278) imply that

$$D(z_*, P_{s_{j_0}}(y_{j_0-1}^{(k,s)})) \le D(z_*, y_{j_0-1}^{(k,s)}) - \gamma_0. \tag{6.279}$$

By (6.2) and (6.275),

$$D(z_*, P_{s_{j_0}}(y_{j_0-1}^{(k,s)})) \le D(z_*, y_{j_0-1}^{(k,s)}) \le M_0 + 1. \tag{6.280}$$

In view of (6.255), (6.275), and (6.280),

$$y_{j_0}^{(k,s)}, \ y_{j_0-1}^{(k,s)}, \ P_{s_{j_0}}(y_{j_0-1}^{(k,s)}) \in B(\theta, M_1). \tag{6.281}$$

It follows from (6.267) that

$$d(y_{j_0}^{(k,s)}, P_{s_{j_0}}(y_{j_0-1}^{(k,s)})) \le \delta. \tag{6.282}$$

Property (iv), (6.281), and (6.282) imply that

$$|D(z_*, y_{j_0}^{(k,s)}) - D(z_*, P_{s_{j_0}}(y_{j_0-1}^{(k,s)}))| \le \gamma_0/4. \tag{6.283}$$

In view of (6.283) and (6.279),

$$D(z_*, y_{j_0}^{(k,s)}) \le D(z_*, P_{s_{j_0}}(y_{j_0-1}^{(k,s)})) + 4^{-1}\gamma_0$$
$$\le D(z_*, y_{j_0-1}^{(k,s)}) - \gamma_0 + 4^{-1}\gamma_0,$$
$$D(z_*, y_{j_0}^{(k,s)}) \le D(z_*, y_{j_0-1}^{(k,s)}) - (3/4)\gamma_0. \tag{6.284}$$

It follows from (6.6), (6.257), (6.266), (6.268), (6.273), (6.276), and (6.284),

$$D(z_*, x_k) - D(z_*, x_{k+1}) = D(z_*, y_0^{(k,s)}) - D(z_*, y_{p(s)}^{(k,s)})$$
$$= \sum_{j=1}^{p(s)} (D(z_*, y_{j-1}^{(k,s)}) - D(z_*, y_j^{(k,s)}))$$
$$\ge -\gamma_1(p(s) - 1) + D(z_*, y_{j_0-1}^{(k,s)}) - D(z_*, y_{j_0}^{(k,s)})$$
$$\ge (3/4)\gamma_0 - \gamma_1\bar{q} \ge 2^{-1}\gamma_0.$$

Lemma 6.14 is proved. □

Assume that q is a natural number such that for each nonnegative integer $k < q$,

$$\lambda_{k+1} > \epsilon_1.$$

By (6.270) and Lemma 6.14 applied by induction,

$$D(z_*, x_k) \le M_0, \ k = 0, \dots, q$$

and for each nonnegative integer $k < q$,

$$D(z_*, x_k) - D(z_*, x_{k+1}) \ge 2^{-1}\gamma_0. \tag{6.285}$$

It follows from (6.258), (6.271), and (6.285) that

$$M_0 \ge D(z_*, x_0) \ge D(z_*, x_0) - D(z_*, x_q)$$
$$= \sum_{k=0}^{q-1} (D(z_*, x_k) - D(z_*, x_{k+1})) \ge 2^{-1} q\gamma_0$$

and

$$q \le 2M_0\gamma_0^{-1} \le n_0.$$

This implies that there exists a nonnegative integer $q \leq n_0$ such that for all nonnegative integers $k \leq q$,

$$D(z_*, x_k) \leq M_0 \tag{6.286}$$

and

$$\lambda_{q+1} \leq \epsilon_1. \tag{6.287}$$

By (6.255),

$$x_i \in B(\theta, M_1), \ i = 0, \dots, q. \tag{6.288}$$

In view of (6.265), (6.269), and (6.287), for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{q+1}$ and each $j \in \{1, \dots, p(t)\}$,

$$\lambda_{q,t} \leq \lambda_{q+1} + \delta \leq \epsilon_1 + \delta \leq 2\epsilon_1, \tag{6.289}$$

$$D(y_{j-1}^{(q,t)}, \mathrm{Fix}(P_{t_j})) \leq \lambda_{q,t} \leq \epsilon_1 + \delta \leq 2\epsilon_1. \tag{6.290}$$

Let $t = (t_1, \dots, t_{p(t)}) \in \Omega_{q+1}$ and $j \in \{1, \dots, p(t)\}$. Clearly, there exists

$$\xi \in \mathrm{Fix}(P_{t_j}) \tag{6.291}$$

such that

$$D(\xi, y_{j-1}^{(q,t)}) \leq D(y_{j-1}^{(q,t)}, \mathrm{Fix}(P_{t_j})) + \delta. \tag{6.292}$$

By (6.290) and (6.292),

$$D(\xi, y_{j-1}^{(q,t)}) \leq \epsilon_1 + 2\delta < 2\epsilon_1. \tag{6.293}$$

In view of (6.2) and (6.293),

$$D(\xi, P_{t_j}(y_{j-1}^{(q,t)})) \leq D(\xi, y_{j-1}^{(q,t)}) \leq \epsilon_1 + 2\delta < 2\epsilon_1. \tag{6.294}$$

It follows from (6.2), (6.6), (6.257), (6.266), (6.267), and (6.286) that

$$D(z_*, P_{t_j}(y_{j-1}^{(q,t)})) \leq D(z_*, y_{j-1}^{(q,t)}) \leq D(z_*, x_q) + \gamma_1 \bar{q} \leq M_0 + 1. \tag{6.295}$$

Relations (6.255) and (6.295) imply that

$$P_{t_j}(y_{j-1}^{(q,t)}), \ y_{j-1}^{(q,t)} \in B(\theta, M_1). \tag{6.296}$$

Property (i), (6.291), (6.292), and (6.296) imply that

$$\xi \in B(\theta, M_2). \tag{6.297}$$

By (6.293), (6.294), (6.296), and (6.297),

$$d(\xi, y_{j-1}^{(q,t)}),\ d(\xi, P_{t_j}(y_{j-1}^{(q,t)})) \le \epsilon_0. \tag{6.298}$$

This implies that

$$d(y_{j-1}^{(q,t)}, P_{t_j}(y_{j-1}^{(q,t)})) \le 2\epsilon_0. \tag{6.299}$$

In view of (6.267) and (6.299),

$$d(y_{j-1}^{(q,t)}, y_j^{(q,t)}) \le d(y_{j-1}^{(q,t)}), P_{t_j}(y_{j-1}^{(q,t)})) + d(P_{t_j}(y_{j-1}^{(q,t)}), y_j^{(q,t)}) \le 2\epsilon_0 + \delta \le 3\epsilon_0.$$

By the relation above, (6.291) and (6.298),

$$d(y_{j-1}^{(q,t)}, \mathrm{Fix}(P_{t_j})) \le \epsilon_0, \tag{6.300}$$

$$d(y_{j-1}^{(q,t)}, y_j^{(q,t)}) \le 3\epsilon_0 \tag{6.301}$$

for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{q+1}$ and each $j \in \{1, \dots, p(t)\}$. It follows from (6.6) and (6.301) that for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{q+1}$ and each $j_1, j_2 \in \{0, 1, \dots, p(t)\}$,

$$d(y_{j_1}^{(q,t)}, y_{j_2}^{(q,t)}) \le 3\epsilon_0 \bar{q}.$$

Together with (6.266) this implies that for each

$$t = (t_1, \dots, t_{p(t)}) \in \Omega_{q+1}$$

and each $j \in \{0, 1, \dots, p(t)\}$,

$$d(x_q, y_j^{(q,t)}) \le 3\bar{q}\epsilon_0.$$

Combined with (6.256) and (6.300) this implies that for each

$$t = (t_1, \dots, t_{p(t)}) \in \Omega_{q+1}$$

and each $j \in \{1, \dots, p(t)\}$,

$$d(x_q, \mathrm{Fix}(P_{t_j})) \le \epsilon_0(3\bar{q} + 1) \le \epsilon.$$

In view of (6.4),

$$d(x_q, \mathrm{Fix}(P_s)) \le \epsilon,\ s = 1, \dots, m.$$

Theorem 6.6 is proved. □

6.9 Proof of Theorem 6.7

We may assume without loss of generality that

$$d(z_*, \theta) \le M, \ \epsilon \in (0, 1). \tag{6.302}$$

In view of (A2), there exists $M_0 > M + 1$ such that

$$D(z_*, \xi) \le M_0 \text{ for all } \xi \in B(\theta, M + 1). \tag{6.303}$$

By (A2), there exists $M_1 > M_0 + 1$ such that

$$\{\xi \in X : \ D(z_*, \xi) \le M_0 + 2\} \subset B(\theta, M_1). \tag{6.304}$$

Set

$$\epsilon_0 = \epsilon(\bar{q} + 1)^{-1}. \tag{6.305}$$

It follows from (A1) that there exists $\epsilon_1 \in (0, \epsilon_0/2)$ such that the following property holds:

(i) for each $x, y \in B(\theta, M_1 + 1)$ which satisfy $D(x, y) \le 2\epsilon_1$, the inequality $d(x, y) \le \epsilon_0$ holds.

By Proposition 6.12, there exists $\epsilon_2 \in (0, 2^{-1}\epsilon_1)$ such that the following property holds:

(ii) for each $i \in \{1, \dots, m\}$, each $h_1, h_2 \in X$ satisfying

$$D(z_*, h_1) \le M_0 + 2,$$
$$d(P_i(h_1), h_2) \le 2\epsilon_2$$

and

$$D(h_1, h_2) \le 2\epsilon_2$$

the inequality

$$D(h_1, \mathrm{Fix}(P_i)) \le \epsilon_0$$

holds.

By Proposition 6.10, there exists $\epsilon_3 \in (0, \epsilon_2)$ such that the following property holds:

(iii) for each $i \in \{1, \dots, m\}$, each $h_1, h_2 \in X$ satisfying

$$D(z_*, h_1) \le M_0 + 2,$$
$$d(P_i(h_1), h_2) \le \epsilon_3$$

and

$$D(h_2, h_1) \geq \epsilon_2$$

the inequality

$$D(h_1, \mathrm{Fix}(P_i)) \geq \epsilon_3$$

holds.

By (B3) there exists $\gamma_0 \in (0, 8^{-1}\epsilon_3)$ such that the following property holds:

(iv) for each $i \in \{1, \dots, m\}$ and each $x \in B(\theta, M_1+1)$ satisfying $D(x, \mathrm{Fix}(P_i)) \geq \epsilon_3$ the relation

$$D(z_*, P_i(x)) \leq D(z_*, x) - \gamma_0$$

holds.

Set

$$\gamma_1 = \gamma_0(4\bar{q})^{-1}. \tag{6.306}$$

Since the function D is uniformly continuous there exists $\delta_1 \in (0, 16^{-1}\gamma_0)$ such that the following property holds:

(v) for each $\xi_1, \xi_2, \xi_3, \xi_4 \in B(\theta, M_1 + 1)$ satisfying $d(\xi_1, \xi_3),\ d(\xi_2, \xi_4) \leq \delta_1$ we have

$$|D(\xi_1, \xi_2) - D(\xi_3, \xi_4)| \leq 4^{-1}\gamma_0.$$

By Proposition 6.9, there exists $\delta \in (0, \delta_1)$ such that the following property holds:

(vi) for each $x \in X$ satisfying $D(z_*, x) \leq M_0 + 2$, each index vector $t = (t_1, \dots, t_{p(t)})$ satisfying $p(t) \leq \bar{q}$ and each sequence $\{y_j\}_{j=0}^{p(t)} \subset X$ which satisfies

$$y_0 = x,$$
$$d(y_j, P_{t_j}(y_{j-1})) \leq \delta,\ j = 1, \dots, p(t)$$

we have for all $j = 1, \dots, p(t)$,

$$D(z_*, y_j) \leq D(z_*, y_{j-1}) + \gamma_1,$$
$$D(z_*, y_j) \leq D(z_*, x) + \bar{q}\gamma_1.$$

Choose a natural number

$$n_0 \geq 2M_0\gamma_0^{-1}. \tag{6.307}$$

Assume that

$$\{\Omega_i\}_{i=1}^{\infty} \subset \mathcal{M}_*, \tag{6.308}$$

$$x_0 \in B(\theta, M) \tag{6.309}$$

and

$$\{x_i\}_{i=1}^{\infty} \subset X,\ \{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty), \tag{6.310}$$

for each natural number i,

$$(x_i, \lambda_i) \in A(x_{i-1}, \Omega_i, \delta). \tag{6.311}$$

Let $k \geq 0$ be an integer. In view of (6.310) and (6.311),

$$(x_{k+1}, \lambda_{k+1}) \in A(x_k, \Omega_{k+1}, \delta). \tag{6.312}$$

By (6.12), (6.13), and (6.312), there exist

$$(y_{k,t}, \lambda_{k,t}) \in A_0(x_k, \Omega_{k+1}, \delta),\ t \in \Omega_{k+1} \tag{6.313}$$

such that

$$(x_{k+1}, \lambda_{k+1}) \in \{(y_{k,t}, \lambda_{k,t}) :\ t \in \Omega_{k+1}\}, \tag{6.314}$$

$$\lambda_{k+1} \geq \lambda_{k,t} - \delta \text{ for all } t \in \Omega_{k+1}. \tag{6.315}$$

It follows from (6.11) and (6.313) that for each $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{k+1}$ there exists a finite sequence $\{y_i^{(k,t)}\}_{i=0}^{p(t)} \subset X$ such that

$$y_0^{(k,t)} = x_k, \tag{6.316}$$

$$d(P_{t_j}(y_{j-1}^{(k,t)}), y_j^{(k,t)}) \leq \delta \text{ for each integer } j = 1, \ldots, p(t), \tag{6.317}$$

$$y_{p(t)}^{(k,t)} = y_{k,t}, \tag{6.318}$$

$$\lambda_{k,t} = \max\{D(y_j^{(k,t)}, y_{j-1}^{(k,t)}) :\ j = 1, \ldots, p(t)\}. \tag{6.319}$$

By (6.302) and (6.303),

$$D(z_*, x_0) \leq M_0. \tag{6.320}$$

We prove the following auxiliary result.

Lemma 6.15. *Assume that a nonnegative integer k satisfies*

$$D(z_*, x_k) \le M_0, \tag{6.321}$$

$$\lambda_{k+1} > \epsilon_2. \tag{6.322}$$

Then

$$D(z_*, x_k) - D(z_*, x_{k+1}) \ge 2^{-1}\gamma_0.$$

Proof. By (6.6), (6.316), (6.317), (6.320), and property (vi), for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}$ and each $j \in \{1, \dots, p(t)\}$,

$$D(z_*, y_j^{(k,t)}) \le D(z_*, y_{j-1}^{(k,t)}) + \gamma_1, \tag{6.323}$$

$$D(z_*, y_j^{(k,t)}) \le D(z_*, x_k) + \gamma_1 \bar{q}. \tag{6.324}$$

In view of (6.304), (6.306), (6.316), (6.321), and (6.324),

$$D(z_*, y_j^{(k,t)}) \le M_0 + 1,\ \ y_j^{(k,t)} \in B(\theta, M_1) \tag{6.325}$$

for each $t \in \Omega_{k+1}$ and each $j \in \{0, 1, \dots, p(t)\}$.

By (6.314), there exists

$$s = (s_1, \dots, s_{p(s)}) \in \Omega_{k+1}$$

such that

$$(x_{k+1}, \lambda_{k+1}) = (y_{k,s}, \lambda_{k,s}). \tag{6.326}$$

Relations (6.322) and (6.326) imply that

$$\lambda_{k,s} = \lambda_{k+1} > \epsilon_2. \tag{6.327}$$

By (6.319) and (6.327), there exists

$$j_0 \in \{1, \dots, p(s)\}$$

such that

$$D(y_{j_0}^{(k,s)}, y_{j_0-1}^{(k,s)}) = \lambda_{k,s} > \epsilon_2. \tag{6.328}$$

In view of (6.317), (6.320), and property (iii),

$$D(y_{j_0-1}^{(k,s)}, \mathrm{Fix}(P_{s_{j_0}})) \ge \epsilon_3. \tag{6.329}$$

Property (iv), (6.325), and (6.329) imply that

$$D(z_*, P_{s_{j_0}}(y_{j_0-1}^{(k,s)})) \le D(z_*, y_{j_0-1}^{(k,s)}) - \gamma_0. \tag{6.330}$$

By (6.2) and (6.325),

$$D(z_*, P_{s_{j_0}}(y_{j_0-1}^{(k,s)})) \le D(z_*, y_{j_0-1}^{(k,s)}) \le M_0 + 1. \tag{6.331}$$

In view of (6.304), (6.325), and (6.331),

$$y_{j_0}^{(k,s)},\ y_{j_0-1}^{(k,s)},\ P_{s_{j_0}}(y_{j_0-1}^{(k,s)}) \in B(\theta, M_1). \tag{6.332}$$

It follows from (6.317) that

$$d(y_{j_0}^{(k,s)}, P_{s_{j_0}}(y_{j_0-1}^{(k,s)})) \le \delta. \tag{6.333}$$

Property (v), (6.302), (6.332), and (6.333) imply that

$$|D(z_*, y_{j_0}^{(k,s)}) - D(z_*, P_{s_{j_0}}(y_{j_0-1}^{(k,s)}))| \le \gamma_0/4. \tag{6.334}$$

In view of (6.330) and (6.334),

$$\begin{aligned} D(z_*, y_{j_0}^{(k,s)}) &\le D(z_*, P_{s_{j_0}}(y_{j_0-1}^{(k,s)})) + 4^{-1}\gamma_0 \\ &\le D(z_*, y_{j_0-1}^{(k,s)}) - \gamma_0 + 4^{-1}\gamma_0, \end{aligned}$$

$$D(z_*, y_{j_0}^{(k,s)}) \le D(z_*, y_{j_0-1}^{(k,s)}) - (3/4)\gamma_0. \tag{6.335}$$

It follows from (6.6), (6.306), (6.316), (6.318), (6.323), (6.326), and (6.335),

$$\begin{aligned} D(z_*, x_k) - D(z_*, x_{k+1}) &= D(z_*, y_0^{(k,s)}) - D(z_*, y_{p(s)}^{(k,s)}) \\ &= \sum_{j=1}^{p(s)} (D(z_*, y_{j-1}^{(k,s)}) - D(z_*, y_j^{(k,s)})) \\ &\ge -\gamma_1(p(s) - 1) + D(z_*, y_{j_0-1}^{(k,s)}) - D(z_*, y_{j_0}^{(k,s)}) \\ &\ge (3/4)\gamma_0 - \gamma_1 \bar{q} \ge 2^{-1}\gamma_0. \end{aligned}$$

Lemma 6.15 is proved. □

Assume that q is a natural number such that for each nonnegative integer $k < q$,

$$\lambda_{k+1} > \epsilon_2.$$

By (6.320) and Lemma 6.15 applied by induction,

$$D(z_*, x_k) \le M_0,\ k = 0, \dots, q$$

and for each nonnegative integer $k < q$,

$$D(z_*, x_k) - D(z_*, x_{k+1}) \ge 2^{-1}\gamma_0. \tag{6.336}$$

It follows from (6.307), (6.320), and (6.336) that

$$M_0 \ge D(z_*, x_0) \ge D(z_*, x_0) - D(z_*, x_q)$$
$$= \sum_{k=0}^{q-1}(D(z_*, x_k) - D(z_*, x_{k+1})) \ge 2^{-1}q\gamma_0$$

and

$$q \le 2M_0\gamma_0^{-1} \le n_0.$$

This implies that there exists a nonnegative integer $q \le n_0$ such that for all nonnegative integers $k \le q$,

$$D(z_*, x_k) \le M_0 \tag{6.337}$$

and

$$\lambda_{q+1} \le \epsilon_2. \tag{6.338}$$

In view of (6.337),

$$x_i \in B(\theta, M_1),\ i = 0, \dots, q.$$

Let $t = (t_1, \dots, t_{p(t)}) \in \Omega_{q+1}$ and $j \in \{1, \dots, p(t)\}$. By (6.315), (6.319), and (6.338),

$$\lambda_{q,t} \le \lambda_{q+1} + \delta \le \epsilon_2 + \delta \le 2\epsilon_2, \tag{6.339}$$

$$D(y_j^{(q,t)}, y_{j-1}^{(q,t)}) \le \lambda_{q,t} \le 2\epsilon_2. \tag{6.340}$$

Property (vi), (6.6), (6.306), (6.316), (6.317), and (6.337) imply that

$$D(z_*, y_{j-1}^{(q,t)}),\ D(z_*, y_j^{(q,t)}) \le M_0 + 1. \tag{6.341}$$

In view of (6.304) and (6.341),

$$y_j^{(q,t)},\ y_{j-1}^{(q,t)} \in B(\theta, M_1). \tag{6.342}$$

It follows from property (i), (6.340), and (6.342) that

$$d(y_j^{(q,t)}, y_{j-1}^{(q,t)}) \le \epsilon_0. \tag{6.343}$$

By (6.317), (6.340), and (6.341),

$$d(y_{j-1}^{(q,t)}, \mathrm{Fix}(P_{t_j})) \le \epsilon_0 \tag{6.344}$$

for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{q+1}$ and each $j \in \{1, \dots, p(t)\}$. Relations (6.6), (6.316), and (6.343) imply that for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{q+1}$ and each $j \in \{0, 1, \dots, p(t)\}$,

$$d(x_q, y_j^{(q,t)}) \le \epsilon_0 \bar{q}. \tag{6.345}$$

By (6.305), (6.344), and (6.345), for all $t = (t_1, \dots, t_{p(t)}) \in \Omega_{q+1}$ and all $j \in \{1, \dots, p(t)\}$,

$$d(x_q, \mathrm{Fix}(P_{t_j})) \le \epsilon_0(\bar{q} + 1) = \epsilon.$$

Together with (6.4) this implies that

$$d(x_q, \mathrm{Fix}(P_s)) \le \epsilon,\ s = 1, \dots, m.$$

Theorem 6.7 is proved. □

Chapter 7
Abstract Version of CARP Algorithm

In this chapter we study the convergence of an abstract version of the algorithm which is called in the literature as component-averaged row projections or CARP. This algorithm was introduced for solving a convex feasibility problem in a finite-dimensional space, when a given collection of sets is divided into blocks in such a manner that all sets belonging to every block are subsets of a vector subspace associated with the block. All the blocks are processed in parallel and the algorithm operates in vector subspaces of the whole vector space. This method becomes efficient, in particular, when the dimensions of the subspaces are essentially smaller than the dimension of the whole space. In this chapter we study CARP for problems in a normed space, which is not necessarily finite-dimensional. Our main goal is to obtain an approximate solution of the problem in the presence of computational errors. We show that our dynamic string-averaging algorithm generates a good approximate solution, if the sequence of computational errors is bounded from above by a constant. Moreover, for a known computational error, we find out what an approximate solution can be obtained and how many iterates one needs for this.

7.1 Preliminaries and Main Results

In [53] Gordon and Gordon studied a convex feasibility problem in a finite-dimensional space and introduced an algorithm which is called in the literature as component-averaged row projections or CARP. According to CARP, a given collection of sets is divided into blocks in such a manner that all sets belonging to every block are subsets of a vector subspace associated with the block. Here we study CARP for problems in a normed space, which is not necessarily finite-dimensional, under the presence of computational errors.

A. Zaslavski, *Approximate Solutions of Common Fixed-Point Problems*, Springer Optimization and Its Applications 112, DOI 10.1007/978-3-319-33255-0_7

Let $(Z, \|\cdot\|)$ be a normed space. For each $x \in Z$ and each $r > 0$ set

$$B_Z(x,r) = \{y \in Z : \|x - y\| \le r\}.$$

For each $x \in Z$ and each nonempty set $D \subset Z$ put

$$d_Z(x,D) = \inf\{\|x - y\| : y \in D\}.$$

Let $(Y_i, \|\cdot\|)$, $i = 1, \dots, p$ be normed spaces. The vector space

$$Y_1 \times \cdots \times Y_p = \prod_{i=1}^{p} Y_i$$

is equipped with the norm

$$\|y\| = \|(y_1, \dots, y_p)\| = (\sum_{i=1}^{p} \|y_i\|^2)^{1/2}, \ y = (y_1, \dots, y_p) \in \prod_{i=1}^{p} Y_i.$$

Suppose that $(X_i, \|\cdot\|)$, $i = 1, \dots, l$ are normed spaces and

$$X = \prod_{i=1}^{l} X_i.$$

Let m be a natural number,

$$C_i \subset X, \ i = 1, \dots, m$$

be nonempty closed subsets of X and let there exist a finite set $\mathcal{E}$ of index vectors $\tau = (\tau_1, \dots, \tau_p)$ such that

$$\tau_i \in \{1, \dots, m\} \text{ for each } i \in \{1, \dots, p\}, \tag{7.1}$$

$$\tau_{i_1} < \tau_{i_2} \text{ for each pair } i_1, i_2 \in \{\tau_1, \dots, \tau_p\} \text{ such that } i_1 < i_2, \tag{7.2}$$

$$\cup\{\{\tau_1, \dots, \tau_p\} : (\tau_1, \dots, \tau_p) \in \mathcal{E}\} = \{1, \dots, m\}. \tag{7.3}$$

For each $\tau = (\tau_1, \dots, \tau_q) \in \mathcal{E}$ set

$$p(\tau) = q. \tag{7.4}$$

Let $j \in \{1, \dots, l\}$. Denote by $\widehat{X}_j$ the set of all $x = (x_1, \dots, x_l) \in X$ such that $x_i = 0$ for all $i \in \{1, \dots, l\} \setminus \{j\}$. Clearly, $\widehat{X}_j$ is a vector subspace of X and it is equipped with the norm induced by the norm of X. Evidently, X_j and $\widehat{X}_j$ are isometric in a natural way.

Let $\tau = (\tau_1, \dots, \tau_{p(\tau)}) \in \mathcal{E}$. We suppose that there exists an index vector $\hat{\tau} = (\hat{\tau}_1, \dots, \hat{\tau}_{p(\hat{\tau})})$ such that

$$\hat{\tau}_i \in \{1, \dots, l\},\ i \in \{1, \dots, p(\hat{\tau})\}, \tag{7.5}$$

$$\hat{\tau}_{i_1} < \hat{\tau}_{i_2} \text{ for each pair } i_1, i_2 \in \{1, \dots, p(\hat{\tau})\} \text{ such that } i_1 < i_2 \tag{7.6}$$

and that for each $s \in \{\tau_1, \dots, \tau_{p(\tau)}\}$ there exists a closed set

$$C_{\tau,s} \subset \sum\{\widehat{X}_i :\ i \in \{\hat{\tau}_1, \dots, \hat{\tau}_{p(\hat{\tau})}\}\} \tag{7.7}$$

such that

$$C_s = C_{\tau,s} + \sum\{\widehat{X}_i :\ i \in \{1, \dots, l\} \setminus \{\hat{\tau}_1, \dots, \hat{\tau}_{p(\hat{\tau})}\}\}. \tag{7.8}$$

For each $\tau = (\tau_1, \dots, \tau_p) \in \mathcal{E}$ set

$$\widehat{X}_\tau = \sum\{\widehat{X}_i :\ i \in \{\hat{\tau}_1, \dots, \hat{\tau}_{p(\hat{\tau})}\}\}. \tag{7.9}$$

We suppose that

$$\cup\{\{\hat{\tau}_1, \dots, \hat{\tau}_{p(\hat{\tau})}\} :\ \tau \in \mathcal{E}\} = \{1, \dots, l\} \tag{7.10}$$

and that for each $\tau = (\tau_1, \dots, \tau_{p(\tau)}) \in \mathcal{E}$ and each $s \in \{\tau_1, \dots, \tau_{p(\tau)})$ there exists a mapping

$$P_{\tau,s} : \widehat{X}_\tau \to \widehat{X}_\tau \tag{7.11}$$

such that

$$P_{\tau,s}(z) = z \text{ for all } z \in C_{\tau,s}, \tag{7.12}$$

$$\|z - x\| \geq \|z - P_{\tau,s}(x)\| \tag{7.13}$$

for all $z \in C_{\tau,s}$ and all $x \in \widehat{X}_\tau$.

We consider index vectors $t = (t_1, \dots, t_p)$ such that $t_i \in \{1, \dots, m\}$. For each index vector $t = (t_1, \dots, t_q)$ set

$$p(t) = q. \tag{7.14}$$

Let $\bar{c} \in (0, 1)$. In this chapter we use the following assumptions.

(A1) For each $\tau = (\tau_1, \dots, \tau_p) \in \mathcal{E}$ and each $s \in \{\tau_1, \dots, \tau_p\}$,

$$P_{\tau,s}(\widehat{X}_\tau) = C_{\tau,s}, \tag{7.15}$$

$$\|z - x\|^2 \geq \|z - P_{\tau,s}(x)\|^2 + \bar{c}\|x - P_{\tau,s}(x)\|^2 \tag{7.16}$$

for all $z \in C_{\tau,s}$ and all $x \in \widehat{X}_\tau$.

(A2) For each $\Lambda > 0$ and each $\lambda > 0$ there exists $\gamma > 0$ such that for each $\tau = (\tau_1, \dots, \tau_{p(\tau)}) \in \mathcal{E}$, each $s \in \{\tau_1, \dots, \tau_{p(\tau)}\}$, each $z \in C_{\tau,s}$ satisfying $\|z\| \le \Lambda$ and each $x \in \widehat{X}_\tau$ satisfying

$$\|x\| \le \Lambda,\ d_{\widehat{X}_\tau}(x, C_{\tau,s}) \ge \lambda$$

the inequality

$$\|z - P_{\tau,s}(x)\| \le \|z - x\| - \gamma$$

holds.

Let $\tau = (\tau_1, \dots, \tau_p) \in \mathcal{E}$. Consider a mapping $\pi_\tau : X \to \widehat{X}_\tau$ such that for each $x = (x_1, \dots, x_l) \in X$,

$$\pi_\tau(x) = (\pi_{\tau,1}(x), \dots, \pi_{\tau,l}(x)),$$

where for each $i \in \{1, \dots, l\}$,

$$\pi_{\tau,i}(x) = x_i \text{ if } i \in \{\hat{\tau}_1, \dots, \hat{\tau}_{p(\hat{\tau})}\} \tag{7.17}$$

and

$$\pi_{\tau,i}(x) = 0 \text{ if } i \notin \{\hat{\tau}_1, \dots, \hat{\tau}_{p(\hat{\tau})}\}. \tag{7.18}$$

We suppose that

$$C := \cap_{s=1}^m C_s \neq \emptyset.$$

Denote by Card(A) the cardinality of a set A. Suppose that the sum over empty set is zero.

For each $i \in \{1, \dots, l\}$ set

$$m_i = \text{Card}(\{\tau = (\tau_1, \dots, \tau_{p(\tau)}) \in \mathcal{E} :\ i \in \{\hat{\tau}_1, \dots, \hat{\tau}_{p(\hat{\tau})}\}\}). \tag{7.19}$$

We consider linear operators

$$B_1 : X \to X,\ B_2 : X \to X$$

such that for each $i \in \{1, \dots, l\}$ and each $x \in \widehat{X}_i$,

$$B_1(x) = m_i^{-1} x,\ B_2(x) = m_i^{1/2} x. \tag{7.20}$$

Fix an integer

$$\bar{q} \ge m \text{ and } \Delta \in (0, m^{-1}]. \tag{7.21}$$

Let $\tau = (\tau_1, \ldots, \tau_{p(\tau)}) \in \mathcal{E}$ and let an index vector $t = (t_1, \ldots, t_{p(t)})$ be such that

$$t_i \in \{\tau_1, \ldots, \tau_{p(\tau)}\},\ i = 1, \ldots, p(t).$$

Set

$$P[t] = P_{\tau,t_{p(t)}} \cdots P_{\tau,t_1}. \tag{7.22}$$

By (7.12), (7.13), and (7.22), for each $x \in \cap\{C_{\tau,s} :\ s \in \{\tau_1, \ldots, \tau_{p(\tau)}\}\}$,

$$P[t](x) = x \tag{7.23}$$

and for each $z \in \cap\{C_{\tau,s} :\ s \in \{\tau_1, \ldots, \tau_{p(\tau)}\}\}$, each $x \in \widehat{X}_\tau$,

$$\|P[t](x) - P[t](z)\| = \|P[t](x) - z\| \le \|x - z\|. \tag{7.24}$$

Denote by $\mathcal{M}_\tau$ the collection of all pairs (Ω, w), where Ω is a finite set of index vectors $t = (t_1, \ldots, t_{p(t)})$ such that

$$t_i \in \{\tau_1, \ldots, \tau_{p(\tau)}\},\ i = 1, \ldots, p(t), \tag{7.25}$$

$$\cup\{\{t_1, \ldots, t_{p(t)}\} :\ t = (t_1, \ldots, t_{p(t)}) \in \Omega\} = \{\tau_1, \ldots, \tau_{p(\tau)}\}, \tag{7.26}$$

$$p(t) \le \bar{q},\ t \in \Omega, \tag{7.27}$$

$$w : \Omega \to (0, \infty) \text{ satisfies } \sum_{t \in \Omega} w(t) = 1, \tag{7.28}$$

$$w(t) \ge \Delta,\ t \in \Omega. \tag{7.29}$$

Let $(\Omega, w) \in \mathcal{M}_\tau$. Define

$$P_{\Omega,w}(x) = \sum_{t \in \Omega} w(t) P[t](x),\ x \in \widehat{X}_\tau. \tag{7.30}$$

By (7.23), (7.24), (7.28), (7.30) and the convexity of the norm, for each $z \in \cap\{C_{\tau,s} :$ $s \in \{\tau_1, \ldots, \tau_{p(\tau)}\}\}$ and each $x \in \widehat{X}_\tau$,

$$P_{\Omega,w}(z) = z, \tag{7.31}$$

$$\|P_{\Omega,w}(z) - P_{\Omega,w}(x)\| = \|z - P_{\Omega,w}(x)\| \le \|z - x\|. \tag{7.32}$$

The dynamic string-averaging method with variable strings and variable weights can now be described by the following algorithm.

Initialization: select an arbitrary $x_0 \in X$.

Iterative step: given a current iteration vector x_k pick

$$(\Omega_{\tau,k+1}, w_{\tau,k+1}) \in \mathcal{M}_\tau,\ \tau \in \mathcal{E}$$

and calculate the next iteration vector x_{k+1} by

$$x_{k+1} = B_1(\sum_{\tau \in \mathcal{E}} (P_{\Omega_{\tau,k+1}, w_{\tau,k+1}}(\pi_\tau(x_k))).$$

In order to state the main results of this chapter we need the following definitions.

Let $\delta \geq 0$, $x \in \widehat{X}_\tau$, $\tau = (\tau_1, \dots, \tau_{p(\tau)}) \in \mathcal{E}$ and $t = (t_1, \dots, t_{p(t)})$ be an index vector such that

$$t_i \in \{\tau_1, \dots, \tau_{p(\tau)}\},\ i = 1, \dots, p(t).$$

Define

$$\begin{gathered} A_{\tau,0}(x, t, \delta) = \{(y, \lambda) \in \widehat{X}_\tau \times R^1 : \\ \text{there is a sequence } \{y_i\}_{i=0}^{p(t)} \subset X_\tau \text{ such that} \\ y_0 = \pi_\tau(x) = x \text{ and for all } i = 1, \dots, p(t), \\ \|y_i - P_{\tau,t_i}(y_{i-1})\| \leq \delta, \\ y = y_{p(t)}, \\ \lambda = \max\{\|y_i - y_{i-1}\| :\ i = 1, \dots, p(t)\}\}. \end{gathered} \tag{7.33}$$

Let $\tau = (\tau_1, \dots, \tau_{p(\tau)}) \in \mathcal{E}$, $x \in \widehat{X}_\tau$, $\delta \geq 0$ and let $(\Omega, w) \in \mathcal{M}_\tau$. Define

$$\begin{gathered} A_\tau(x, (\Omega, w), \delta) = \{(y, \lambda) \in \widehat{X}_\tau \times R^1 : \text{ there exist} \\ (y_t, \lambda_t) \in A_{\tau,0}(x, t, \delta),\ t \in \Omega \text{ such that} \\ \|y - \sum_{t \in \Omega} w(t) y_t\| \leq \delta,\ \lambda = \max\{\lambda_t :\ t \in \Omega\}\}. \end{gathered} \tag{7.34}$$

Let $x \in X$, $\delta \geq 0$, $(\Omega_\tau, w_\tau) \in \mathcal{M}_\tau$, $\tau \in \mathcal{E}$. Define

$$\begin{gathered} A(x, \{(\Omega_\tau, w_\tau)\}_{\tau \in \mathcal{E}}, \delta) = \{(y, \lambda) \in X \times R^1 : \text{ there exist} \\ (y_\tau, \lambda_\tau) \in A_\tau(\pi_\tau(x), (\Omega_\tau, w_\tau), \delta),\ \tau \in \mathcal{E} \text{ such that} \\ \|y - B_1(\sum_{\tau \in \mathcal{E}} y_\tau)\| \leq \delta,\ \lambda = \max\{\lambda_\tau :\ \tau \in \mathcal{E}\}\}. \end{gathered} \tag{7.35}$$

Let $\delta \geq 0$, $\tau = (\tau_1, \dots, \tau_{p(\tau)}) \in \mathcal{E}$, $t = (t_1, \dots, t_{p(t)})$ be an index vector such that

$$t_i \in \{\tau_1, \dots, \tau_{p(\tau)}\},\ i = 1, \dots, p(t)$$

and $x \in \widehat{X}_\tau$.

Define

$$\tilde{A}_{\tau,0}(x,t,\delta) = \{(y,\lambda) \in \widehat{X}_\tau \times R^1 :$$
$$\text{there is a sequence } \{y_i\}_{i=0}^{p(t)} \subset X_\tau \text{ such that}$$
$$y_0 = \pi_\tau(x) = x \text{ and for all } i = 1,\dots,p(t),$$
$$\|y_i - P_{\tau,t_i}(y_{i-1})\| \le \delta,$$
$$y = y_{p(t)},$$
$$\lambda = \max\{d_{\widehat{X}_\tau}(y_{i-1}, C_{\tau,t_i}) : \ i = 1,\dots,p(t)\}\}. \tag{7.36}$$

Let $\tau = (\tau_1,\dots,\tau_{p(\tau)}) \in \mathcal{E}$, $x \in \widehat{X}_\tau$, $\delta \ge 0$ and let $(\Omega, w) \in \mathcal{M}_\tau$. Define

$$\tilde{A}_\tau(x,(\Omega,w),\delta) = \{(y,\lambda) \in \widehat{X}_\tau \times R^1 : \ \text{there exist}$$
$$(y_t,\lambda_t) \in \tilde{A}_{\tau,0}(x,t,\delta), \ t \in \Omega \text{ such that}$$
$$\|y - \sum_{t\in\Omega} w(t)y_t\| \le \delta, \ \lambda = \max\{\lambda_t : \ t \in \Omega\}\}. \tag{7.37}$$

Let $x \in X$, $\delta \ge 0$, $(\Omega_\tau, w_\tau) \in \mathcal{M}_\tau$, $\tau \in \mathcal{E}$. Define

$$\tilde{A}(x,\{(\Omega_\tau,w_\tau)\}_{\tau\in\mathcal{E}},\delta) = \{(y,\lambda) \in X \times R^1 : \ \text{there exist}$$
$$(y_\tau,\lambda_\tau) \in \tilde{A}_\tau(\pi_\tau(x),(\Omega_\tau,w_\tau),\delta), \ \tau \in \mathcal{E} \text{ such that}$$
$$\|y - B_1(\sum_{\tau\in\mathcal{E}} y_\tau)\| \le \delta, \ \lambda = \max\{\lambda_\tau : \ \tau \in \mathcal{E}\}\}. \tag{7.38}$$

In this chapter we prove the following results.

Theorem 7.1. *Suppose that (A1) holds. Let $M > 0$ satisfy*

$$B_X(0,M) \cap C \ne \emptyset \tag{7.39}$$

and let $\epsilon > 0$. Assume that

$$(\Omega_{i,\tau}, w_{i,\tau}) \in \mathcal{M}_\tau, \ \tau \in \mathcal{E}, \tag{7.40}$$
$$x_0 \in B_X(0,M), \tag{7.41}$$

$\{x_i\}_{i=1}^\infty \subset X$, $\{\lambda_i\}_{i=1}^\infty \subset [0,\infty)$ *and that for each natural number i,*

$$(x_i,\lambda_i) \in A(x_{i-1},\{(\Omega_{i,\tau},w_{i,\tau})\}_{\tau\in\mathcal{E}},0). \tag{7.42}$$

Then

$$Card(\{i \in \{0,1,\dots\} : \max\{d_X(x_i, C_s) : s = 1,\dots,m\} > \epsilon\})$$
$$\leq 4\bar{c}^{-1}\Delta^{-1}\epsilon^{-2}\bar{q}^{-2}M^2 \max\{m_i : i = 1,\dots,l\}.$$

Theorem 7.2. *Suppose that (A2) holds. Let* $M > 0$ *satisfy*

$$B_X(0,M) \cap C \neq \emptyset$$

and let $\epsilon > 0$. *Then there exists a constant* $Q > 0$ *such that for each sequence*

$$(\Omega_{i,\tau}, w_{i,\tau}) \in \mathcal{M}_\tau,\ \tau \in \mathcal{E},\ i = 0,1,\dots,$$

each

$$x_0 \in B_X(0,M),$$

each sequence $\{x_i\}_{i=1}^\infty \subset X$ *and each sequence* $\{\lambda_i\}_{i=1}^\infty \subset [0,\infty)$ *satisfying for each natural number i,*

$$(x_i, \lambda_i) \in \tilde{A}(x_{i-1}, \{(\Omega_{i,\tau}, w_{i,\tau})\}_{\tau\in\mathcal{E}}, 0)$$

the inequality

$$Card(\{i \in \{0,1,\dots\} : \max\{d_X(x_i, C_s) : s = 1,\dots,m\} > \epsilon\}) \leq Q$$

holds.

Let

$$M_1 = \max\{m_i : i = 1,\dots,l\},\ M_2 = \min\{m_i : i = 1,\dots,l\}. \tag{7.43}$$

Theorem 7.3. *Suppose that (A1) holds,* $M > 0$ *satisfy*

$$B_X(0,M) \cap C \neq \emptyset, \tag{7.44}$$

a positive number δ *satisfies*

$$\delta \leq (2(\bar{q}+1))^{-1},\ \delta < M_1^{-1}, \tag{7.45}$$

$$\epsilon_0 = \max\{(16\Delta^{-1}\bar{c}^{-1}\delta M_1^{1/2}(2MM_1^{1/2}+1))^{1/2},$$
$$(Card(\mathcal{E})32\bar{c}^{-1}\Delta^{-1}\delta(2M(M_1M_2^{-1})^{1/2}+1)\bar{q}+1)^{1/2}\}, \tag{7.46}$$

$$\epsilon_1 = \epsilon_0(\bar{q}+1) \tag{7.47}$$

and that a natural number n_0 *satisfies*

$$n_0 \geq 64M^2 M_1 \Delta^{-1} \bar{c}^{-1} \epsilon_0^{-2}. \tag{7.48}$$

Assume that

$$\{(\Omega_{i,\tau}, w_{i,\tau})\}_{i=1}^{\infty} \subset \mathcal{M}_\tau,\ \tau \in \mathcal{E}, \tag{7.49}$$

$$x_0 \in B_X(0, M), \tag{7.50}$$

$\{x_i\}_{i=1}^{\infty} \subset X,\ \{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty)$ *and that for each natural number i,*

$$(x_i, \lambda_i) \in A(x_{i-1}, \{(\Omega_{i,\tau}, w_{i,\tau})\}_{\tau \in \mathcal{E}}, \delta). \tag{7.51}$$

Then there exists a nonnegative integer $q \leq n_0$ *such that*

$$\|x_i\| \leq M + 2M(M_1 M_2^{-1})^{1/2} \text{ for all nonnegative integers } i \leq q$$

and

$$\lambda_{q+1} \leq \epsilon_0.$$

Moreover, if an integer $q \geq 0$ *satisfies* $\lambda_{q+1} \leq \epsilon_0$*, then*

$$d(x_q, C_s) \leq \epsilon_1,\ s = 1, \ldots, m.$$

Note that in Theorem 7.3 δ is the computational error made by our computer system, we obtain a point $x \in X$ satisfying $d(x, C_s) \leq \epsilon_1$, $s = 1, \ldots, m$ and in order to obtain this point we need n_0 iterations. It is not difficult to see that $\epsilon_1 = c_1(\delta)^{1/2}$ and $n_0 = \lfloor c_2(\delta^{-1} \rfloor$, where c_1 and c_2 are positive constants depending on M.

Theorem 7.4. *Suppose that (A2) holds,* $M > 0$ *satisfy*

$$B_X(0, M) \cap C \neq \emptyset,$$

$\epsilon \in (0, 1)$. *Then there exist* $\epsilon_0 > 0$*,* $\delta > 0$ *and a natural number* n_0 *such that for each*

$$\{(\Omega_{i,\tau}, w_{i,\tau})\}_{i=1}^{\infty} \subset \mathcal{M}_\tau,\ \tau \in \mathcal{E},$$

each

$$x_0 \in B_X(0, M),$$

each $\{x_i\}_{i=1}^{\infty} \subset X$ *and each* $\{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty)$ *satisfying for each natural number i,*

$$(x_i, \lambda_i) \in A(x_{i-1}, \{(\Omega_{i,\tau}, w_{i,\tau})\}_{\tau \in \mathcal{E}}, \delta)$$

there exists a nonnegative integer $q \le n_0$ *such that for all nonnegative integers* $i \le q$,

$$\|x_i\| \le M + 2M(M_1 M_2^{-1})^{1/2}$$

and

$$\lambda_{q+1} \le \epsilon_0.$$

Moreover, if an integer $q \ge 0$ *satisfies*

$$\|x_q\| \le M + 2M(M_1 M_2^{-1})^{1/2}$$

$\lambda_{q+1} \le \epsilon_0$, *then*

$$d(x_q, C_s) \le \epsilon,\ s = 1, \dots, m.$$

7.2 Auxiliary Results

Lemma 7.5. *Let* $z, x \in X$. *Then*

$$\sum_{\tau \in \mathcal{E}} \|\pi_\tau(z) - \pi_\tau(x)\|^2 = \|B_2(z - x)\|^2.$$

Proof. Let

$$z = (z_1, \dots, z_l\},\ x = (x_1, \dots, x_l)$$

and

$$\tau = (\tau_1, \dots, \tau_{p(\tau)}) \in \mathcal{E}.$$

By the definition of π_τ (see (7.17), (7.18)),

$$\|\pi_\tau(z) - \pi_\tau(x)\|^2 = \sum\{\|z_i - x_i\|^2 :\ i \in \{\hat{\tau}_1, \dots, \hat{\tau}_{p(\hat{\tau})}\}\}. \tag{7.52}$$

In view of (7.19), (7.20), and (7.52),

$$\sum_{\tau \in \mathcal{E}} \|\pi_\tau(z) - \pi_\tau(x)\|^2 = \sum_{i=1}^{l} \|z_i - x_i\|^2 m_i = \|B_2(z - x)\|^2.$$

Lemma 7.5 is proved. □

Lemma 7.6. *Let* $z \in X$, $x_\tau \in \widehat{X}_\tau$, $\tau \in \mathcal{E}$,

$$x = B_1(\sum_{\tau \in \mathcal{E}} x_\tau). \tag{7.53}$$

Then

$$\|B_2(z - x)\|^2 \le \sum_{\tau \in \mathcal{E}} \|\pi_\tau(z) - x_\tau\|^2.$$

Proof. Let

$$z = (z_1, \dots, z_l\}, \ x_\tau = (x_{\tau,1}, \dots, x_{\tau,l}), \ \tau \in \mathcal{E}$$

and

$$x = (x_1, \dots, x_l).$$

In view of (7.10) and (7.53), for each $i \in \{1, \dots, l\}$,

$$x_i = m_i^{-1}(\sum \{x_{\tau,i} : \ \tau \in \mathcal{E}, \ i \in \{\hat{\tau}_1, \dots, \hat{\tau}_{p(\hat{\tau})}\}\}). \tag{7.54}$$

By (7.19), (7.54) and the convexity of the function $\| \cdot \|^2$, for each $i \in \{1, \dots, l\}$,

$$\|z_i - x_i\|^2 = \|z_i - m_i^{-1}(\sum \{x_{\tau,i} : \ \tau \in \mathcal{E}, \ i \in \{\hat{\tau}_1, \dots, \hat{\tau}_{p(\hat{\tau})}\}\})\|^2$$

$$= \| \sum \{m_i^{-1}(z_i - x_{\tau,i}) : \ \tau \in \mathcal{E}, \ i \in \{\hat{\tau}_1, \dots, \hat{\tau}_{p(\hat{\tau})}\}\}\|^2$$

$$\le \sum \{m_i^{-1}\|z_i - x_{\tau,i}\|^2 : \ \tau \in \mathcal{E}, \ i \in \{\hat{\tau}_1, \dots, \hat{\tau}_{p(\hat{\tau})}\}\},$$

$$m_i\|z_i - x_i\|^2 \le \sum \{\|z_i - x_{\tau,i}\|^2 : \ \tau \in \mathcal{E}, \ i \in \{\hat{\tau}_1, \dots, \hat{\tau}_{p(\hat{\tau})}\}\}. \tag{7.55}$$

It follows from (7.20) and (7.55) that

$$\|B_2(z - x)\|^2 = \sum_{i=1}^{l} m_i\|z_i - x_i\|^2$$

$$\le \sum_{i=1}^{l} \sum \{\|z_i - x_{\tau,i}\|^2 : \ \tau \in \mathcal{E}, \ i \in \{\hat{\tau}_1, \dots, \hat{\tau}_{p(\hat{\tau})}\}\} = \sum_{\tau \in \mathcal{E}} \|\pi_\tau(z) - x_\tau\|^2.$$

Lemma 7.6 is proved. □

7.3 Proof of Theorem 7.1

By (7.39), there exists

$$z_* = (z_{*,1}, \ldots, z_{*,l}) \in B(0, M) \cap C. \tag{7.56}$$

Set

$$\gamma_0 = \epsilon(\bar{q})^{-1}. \tag{7.57}$$

Let $i \geq 0$ be an integer. By (7.42),

$$(x_{i+1}, \lambda_{i+1}) \in A(x_i, \{(\Omega_{i+1,\tau}, w_{i+1,\tau})\}_{\tau \in \mathcal{E}}, 0). \tag{7.58}$$

In view of (7.35) and (7.58), there exist

$$(y_{i,\tau}, \lambda_{i,\tau}) \in A_\tau(\pi_\tau(x_i), (\Omega_{i+1,\tau}, w_{i+1,\tau}), 0), \ \ \tau \in \mathcal{E} \tag{7.59}$$

such that

$$x_{i+1} = B_1(\sum_{\tau \in \mathcal{E}} y_{i,\tau}), \tag{7.60}$$

$$\lambda_{i+1} = \max\{\lambda_{i,\tau} : \ \tau \in \mathcal{E}\}. \tag{7.61}$$

By (7.34) and (7.59), for each $\tau \in \mathcal{E}$, there exist

$$(y_t^{(i,\tau)}, \lambda_t^{(i,\tau)}) \in A_{\tau,0}(\pi_\tau(x_i), t, 0), \ \ t \in \Omega_{i+1,\tau} \tag{7.62}$$

such that

$$y_{i,\tau} = \sum_{t \in \Omega_{i+1,\tau}} w_{i+1,\tau}(t) y_t^{(i,\tau)}, \tag{7.63}$$

$$\lambda_{i,\tau} = \max\{\lambda_t^{(i,\tau)} : \ t \in \Omega_{i+1,\tau}\}. \tag{7.64}$$

By (7.33) and (7.62), for each $\tau \in \mathcal{E}$ and each $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{i+1,\tau}$ there exists a sequence

$$\{y_{t,j}^{(i,\tau)}\}_{j=0}^{p(t)} \subset \widehat{X}_\tau$$

such that

$$y_{t,0}^{(i,\tau)} = \pi_\tau(x_i), \tag{7.65}$$

for all $j = 1, \dots, p(t)$,

$$y_{t,j}^{(i,\tau)} = P_{\tau,t_j}(y_{t,j-1}^{(i,\tau)}), \tag{7.66}$$

$$y_t^{(i,\tau)} = y_{t,p(t)}^{(i,\tau)}, \tag{7.67}$$

$$\lambda_t^{(i,\tau)} = \max\{\|y_{t,j}^{(i,\tau)} - y_{t,j-1}^{(i,\tau)}\| : j = 1, \dots, p(t)\}. \tag{7.68}$$

Let

$$\tau = (\tau_1, \dots, \tau_{p(\tau)}) \in \mathcal{E},\ t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1,\tau}.$$

By (7.56), (7.66), and (A1), for each integer j satisfying $0 \le j < p(t)$,

$$\|\pi_\tau(z_*) - y_{t,j}^{(i,\tau)}\|^2 - \|\pi_\tau(z_*) - y_{t,j+1}^{(i,\tau)}\|^2$$
$$= \|\pi_\tau(z_*) - y_{t,j}^{(i,\tau)}\|^2 - \|\pi_\tau(z_*) - P_{\tau,t_{j+1}}(y_{t,j}^{(i,\tau)})\|^2 \ge \bar{c}\|y_{t,j}^{(i,\tau)} - y_{t,j+1}^{(i,\tau)}\|^2. \tag{7.69}$$

In view of (7.65) and (7.67)–(7.69),

$$\|\pi_\tau(z_*) - \pi_\tau(x_i)\|^2 - \|\pi_\tau(z_*) - y_t^{(i,\tau)}\|^2$$
$$= \|\pi_\tau(z_*) - y_{t,0}^{(i,\tau)}\|^2 - \|\pi_\tau(z_*) - y_{t,p(t)}^{(i,\tau)}\|^2$$
$$= \sum_{j=0}^{p(t)-1} (\|\pi_\tau(z_*) - y_{t,j}^{(i,\tau)}\|^2 - \|\pi_\tau(z_*) - y_{t,j+1}^{(i,\tau)}\|^2)$$
$$\ge \sum_{j=0}^{p(t)-1} \bar{c}\|y_{t,j}^{(i,\tau)} - y_{t,j+1}^{(i,\tau)}\|^2 \ge \bar{c}(\lambda_t^{(i,\tau)})^2. \tag{7.70}$$

It follows from (7.28), (7.29), (7.63), (7.64), and (7.70) that

$$\|\pi_\tau(z_*) - y_{i,\tau}\|^2 = \|\pi_\tau(z_*) - \sum_{t \in \Omega_{i+1,\tau}} w_{i+1,\tau}(t) y_t^{(i,\tau)}\|^2$$
$$\le \sum_{t \in \Omega_{i+1,\tau}} w_{i+1,\tau}(t)\|\pi_\tau(z_*) - y_t^{(i,\tau)}\|^2$$
$$\le \sum_{t \in \Omega_{i+1,\tau}} w_{i+1,\tau}(t)(\|\pi_\tau(z_*) - \pi_\tau(x_\tau)\|^2 - \bar{c}(\lambda_t^{(i,\tau)})^2)$$
$$\le \|\pi_\tau(z_*) - \pi_\tau(x_i)\|^2 - \bar{c}\Delta \sum_{t \in \Omega_{i+1,\tau}} (\lambda_t^{(i,\tau)})^2$$
$$\le \|\pi_\tau(z_*) - \pi_\tau(x_i)\|^2 - \bar{c}\Delta(\lambda_{i,\tau})^2. \tag{7.71}$$

By (7.61) and (7.71),

$$\sum_{\tau\in\mathcal{E}}(\|\pi_\tau(z_*) - y_{i,\tau}\|^2) \le \sum_{\tau\in\mathcal{E}}(\|\pi_\tau(z_*) - \pi_\tau(x_i)\|^2) - \bar{c}\Delta\lambda_{i+1}^2. \tag{7.72}$$

Lemma 7.5 implies that

$$\sum_{\tau\in\mathcal{E}}(\|\pi_\tau(z_*) - \pi_\tau(x_i)\|^2) = \|B_2(z_* - x_i)\|^2. \tag{7.73}$$

In view of (7.60) and Lemma 7.6,

$$\begin{aligned}\|B_2(z_* - x_{i+1})\|^2 &= \|B_2(z_* - B_1(\sum_{\tau\in\mathcal{E}} y_{i,\tau}))\|^2 \\ &\le \sum_{\tau\in\mathcal{E}} \|\pi_\tau(z_* - y_{i,\tau})\|^2. \end{aligned}\tag{7.74}$$

It follows from (7.72) to (7.74) that

$$\|B_2(z_* - x_{i+1})\|^2 \le \|B_2(z_* - x_i)\|^2 - \bar{c}\Delta\lambda_{i+1}^2. \tag{7.75}$$

By (7.19), (7.20), (7.41), (7.56), and (7.75), for each natural number n,

$$\begin{aligned} 4M^2 \max\{m_i :\ i = 1, \dots, l\} &\ge \|B_2(z_* - x_0)\|^2 \\ &\ge \|B_2(z_* - x_0)\|^2 - \|B_2(z_* - x_n)\|^2 \\ = \sum_{i=0}^{n-1}(\|B_2(z_* - x_i)\|^2 - \|B_2(z_* - x_{i+1})\|^2) &\ge \bar{c}\Delta \sum_{i=0}^{n-1} \lambda_{i+1}^2 \\ &\ge \bar{c}\Delta\gamma_0^2 \mathrm{Card}(\{i \in \{0, \dots .n-1\} :\ \lambda_{i+1} \ge \gamma_0\}). \end{aligned}$$

Since the relation above holds for any natural number n we conclude in view of (7.57) that

$$\begin{aligned} &\mathrm{Card}(\{i \in \{0, 1, \dots, \} :\ \lambda_{i+1} \ge \gamma_0\}) \\ &\le 4\bar{c}^{-1}\Delta^{-1}\gamma_0^{-2}M^2 \max\{m_i :\ i = 1, \dots, l\} \\ &= 4\bar{c}^{-1}\Delta^{-1}\epsilon^{-2}\bar{q}^2 M^2 \max\{m_i :\ i = 1, \dots, l\}. \end{aligned}$$

Assume that an integer $i \ge 0$ satisfies

$$\lambda_{i+1} \le \gamma_0. \tag{7.76}$$

By (7.61), (7.64), (7.66), (7.76), and (A1), for each $\tau = (\tau_1, \dots, \tau_{p(\tau)}) \in \mathcal{E}$, each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1,\tau}$ and each $j = 1, \dots, p(t)$,

$$\|y_{t,j}^{(i,\tau)} - y_{t,j-1}^{(i,\tau)}\| \leq \gamma_0 \tag{7.77}$$

$$d_{\widehat{X}_\tau}(y_{t,j-1}^{(i,\tau)}, C_{\tau,t_j})| \leq \gamma_0. \tag{7.78}$$

Let

$$\tau = (\tau_1, \dots, \tau_{p(\tau)}) \in \mathcal{E}.$$

In view of (7.27), (7.65), (7.77), and (7.78), for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1,\tau}$ and each $j = 0, 1, \dots, p(t)$,

$$\|\pi_\tau(x_i) - y_{t,j}^{(i,\tau)}\| \leq \gamma_0 j \leq \gamma_0 \bar{q}$$

and for each $j = 1, \dots, p(t)$,

$$\begin{aligned} d_{\widehat{X}_\tau}(\pi_\tau(x_i), C_{\tau,t_j}) &\leq \|\pi_\tau(x_i) - y_{t,j-1}^{(i,\tau)}\| + d_{\widehat{X}_\tau}(y_{t,j-1}^{(i,\tau)}, C_{\tau,t_j}) \\ &\leq \gamma_0(j-1) + \gamma_0 \leq \gamma_0 \bar{q} = \epsilon. \end{aligned} \tag{7.79}$$

By (7.79), for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1,\tau}$ and each $j = 1, \dots, p(t)$,

$$d_X(x_i, C_{t_j}) \leq \epsilon.$$

Together with (7.10) and (7.26) this implies that

$$d_X(x_i, C_s) \leq \epsilon, \ \ s = 1, \dots, m.$$

Theorem 7.1 is proved. □

7.4 Proof of Theorem 7.2

There exist

$$z_* \in B_X(0, M) \cap C. \tag{7.80}$$

Choose

$$\gamma_0 = (2\bar{q})^{-1}\epsilon. \tag{7.81}$$

By (A2), there exists $\delta > 0$ such that the following property holds:

(i) for each $\tau = (\tau_1, \dots, \tau_{p(\tau)}) \in \mathcal{E}$, each $s \in \{\tau_1, \dots, \tau_{p(\tau)}\}$ and each $x \in \widehat{X}_\tau$ satisfying

$$\|x\| \le 2MM_1 + M,\ d_{\widehat{X}_\tau}(x, C_{\tau,s}) \ge \gamma_0$$

we have

$$\|\pi_\tau(z_*) - P_{\tau,s}(x)\| \le \|\pi_\tau(z_*) - x\| - \delta.$$

Set

$$Q = 4M^2 M_1 (\Delta\delta)^{-2}. \tag{7.82}$$

Assume that

$$(\Omega_{i,\tau}, w_{i,\tau}) \in \mathcal{M}_\tau,\ \tau \in \mathcal{E},\ i = 0, 1, \dots, \tag{7.83}$$

$$x_0 \in B_X(0, M), \tag{7.84}$$

$\{x_i\}_{i=1}^\infty \subset X$ and $\{\lambda_i\}_{i=1}^\infty \subset [0, \infty)$ satisfy for each natural number i,

$$(x_i, \lambda_i) \in \tilde{A}(x_{i-1}, \{(\Omega_{i,\tau}, w_{i,\tau})\}_{\tau \in \mathcal{E}}, 0). \tag{7.85}$$

Let $i \ge 0$ be an integer. In view of (7.85),

$$(x_{i+1}, \lambda_{i+1}) \in \tilde{A}(x_i, \{(\Omega_{i+1,\tau}, w_{i+1,\tau})\}_{\tau \in \mathcal{E}}, 0). \tag{7.86}$$

By (7.38) and (7.86), there exist

$$(y_{i,\tau}, \lambda_{i,\tau}) \in \tilde{A}_\tau(\pi_\tau(x_i), (\Omega_{i+1,\tau}, w_{i+1,\tau}), 0),\ \ \tau \in \mathcal{E} \tag{7.87}$$

such that

$$x_{i+1} = B_1(\sum_{\tau \in \mathcal{E}} y_{i,\tau}), \tag{7.88}$$

$$\lambda_{i+1} = \max\{\lambda_{i,\tau} :\ \tau \in \mathcal{E}\}. \tag{7.89}$$

By (7.37) and (7.87), for each $\tau \in \mathcal{E}$, there exist

$$(y_t^{(i,\tau)}, \lambda_t^{(i,\tau)}) \in \tilde{A}_{\tau,0}(\pi_\tau(x_i), t, 0),\ \ t \in \Omega_{i+1,\tau} \tag{7.90}$$

such that

$$y_{i,\tau} = \sum_{t \in \Omega_{i+1,\tau}} w_{i+1,\tau}(t) y_t^{(i,\tau)}, \tag{7.91}$$

$$\lambda_{i,\tau} = \max\{\lambda_t^{(i,\tau)} :\ t \in \Omega_{i+1,\tau}\}. \tag{7.92}$$

By (7.36) and (7.90), for each $\tau \in \mathcal{E}$ and each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1,\tau}$ there exists a sequence

$$\{y_{t,j}^{(i,\tau)}\}_{j=0}^{p(t)} \subset \widehat{X}_\tau$$

such that

$$y_{t,0}^{(i,\tau)} = \pi_\tau(x_i), \tag{7.93}$$

for all $j = 1, \dots, p(t)$,

$$y_{t,j}^{(i,\tau)} = P_{\tau,t_j}(y_{t,j-1}^{(i,\tau)}), \tag{7.94}$$

$$y_t^{(i,\tau)} = y_{t,p(t)}^{(i,\tau)}, \tag{7.95}$$

$$\lambda_t^{(i,\tau)} = \max\{d_{\widehat{X}_\tau}(y_{t,j-1}^{(i,\tau)}, C_{\tau+t_j}) : j = 1, \dots, p(t)\}. \tag{7.96}$$

In view of (7.13), (7.80), (7.93), and (7.94), for each $\tau \in \mathcal{E}$, each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1,\tau}$ and each $j \in \{1, \dots, p(t)\}$,

$$\|\pi_\tau(z_*) - y_{t,j}^{(i,\tau)}\| = \|\pi_\tau(z_*) - P_{\tau,t_j}(y_{t,j-1}^{(i,\tau)})\| \le \|\pi_\tau(z_*) - y_{t,j-1}^{(i,\tau)}\|, \tag{7.97}$$

$$\|\pi_\tau(z_*) - y_{t,j}^{(i,\tau)}\| \le \|\pi_\tau(z_*) - \pi_\tau(x_i)\|. \tag{7.98}$$

Let

$$\tau = (\tau_1, \dots, \tau_{p(\tau)}) \in \mathcal{E},\ t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1,\tau}.$$

By (7.95) and (7.98),

$$\|\pi_\tau(z_*) - y_t^{(i,\tau)}\| \le \|\pi_\tau(z_*) - \pi_\tau(x_i)\|. \tag{7.99}$$

It follows from (7.28), (7.91), (7.99) and the convexity of the norm that

$$\|\pi_\tau(z_*) - y_{i,\tau}\| = \|\pi_\tau(z_*) - \sum_{t \in \Omega_{i+1,\tau}} w_{i+1,\tau}(t) y_t^{(i,\tau)}\|$$

$$\le \sum_{t \in \Omega_{i+1,\tau}} w_{i+1,\tau}(t) \|\pi_\tau(z_*) - y_t^{(i,\tau)}\| \le \|\pi_\tau(z_*) - \pi_\tau(x_i)\|. \tag{7.100}$$

Lemmas 7.5 and 7.6, (7.88), and (7.100) imply that

$$\|B_2(z_* - x_{i+1})\|^2 = \|B_2(z_* - B_1(\sum_{\tau \in \mathcal{E}} y_{i,\tau}))\|^2$$

$$\le \sum_{\tau \in \mathcal{E}} \|\pi_\tau(z_*) - y_{i,\tau}\|^2 \le \sum_{\tau \in \mathcal{E}} \|\pi_\tau(z_*) - \pi_\tau(x_i)\|^2 = \|B_2(z_* - x_i)\|^2,$$

$$\|B_2(z_* - x_{i+1})\| \le \|B_2(z_* - x_i)\|. \tag{7.101}$$

It follows from (7.20), (7.43), (7.80), (7.84), and (7.101) that for all integers $i \geq 0$,

$$\|B_2(z_* - x_i)\| \leq \|B_2(z_* - x_0)\| \leq 2M \max\{m_j^{1/2} : j = 1, \dots, l\},$$
$$\|z_* - x_i\| \leq 2M \max\{m_j^{1/2} : j = 1, \dots, l\}(\min\{m_j^{1/2} : j = 1, \dots, l\})^{-1}$$
$$= 2MM_1M_2^{-1}. \tag{7.102}$$

In view of (7.80), (7.93), and (7.98), for each integer $i \geq 0$, each $\tau \in \mathcal{E}$, each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1,\tau}$ and each $j \in \{0, 1, \dots, p(t)\}$,

$$\|\pi_\tau(z_*) - y_{t,j}^{(i,\tau)}\| \leq 2M,$$
$$\|y_{t,j}^{(i,\tau)}\| \leq M + 2MM_1M_2^{-1}. \tag{7.103}$$

Assume that an integer $i \geq 0$ satisfies

$$\lambda_{i+1} \geq \gamma_0. \tag{7.104}$$

By (7.89) and (7.104), there exists

$$\tau_* = (\tau_{*,1}, \dots, \tau_{*,p(\tau_*)}) \in \mathcal{E}$$

such that

$$\lambda_{i,\tau_*} = \lambda_{i+1} \geq \gamma_0. \tag{7.105}$$

In view of (7.92) and (7.105), there exists $s = (s_1, \dots, s_{p(s)}) \in \Omega_{i+1,\tau_*}$ such that

$$\lambda_s^{(i,\tau_*)} = \lambda_{i,\tau_*} \geq \gamma_0. \tag{7.106}$$

Relations (7.96) and (7.106) imply that there exists $j_0 \in \{1, \dots, p(s)\}$ such that

$$d_{\widehat{X}_{\tau_*}}(y_{s,j_0-1}^{(i,\tau_*)}, C_{\tau_*,s_{j_0}}) = \lambda_s^{(i,\tau_*)} \geq \gamma_0. \tag{7.107}$$

By (7.103), (7.107) and property (i),

$$\|\pi_{\tau_*}(z_*) - P_{\tau_*,s_{j_0}}(y_{s,j_0-1}^{(i,\tau_*)})\| \leq \|\pi_{\tau_*}(z_*) - y_{s,j_0-1}^{(i,\tau_*)}\| - \delta.$$

Together with (7.94) this implies that

$$\|\pi_{\tau_*}(z_*) - y_{s,j_0}^{(i,\tau_*)}\| \leq \|\pi_{\tau_*}(z_*) - y_{s,j_0-1}^{(i,\tau_*)}\| - \delta. \tag{7.108}$$

In view of (7.93), (7.95), (7.97), and (7.108),

$$\|\pi_{\tau_*}(z_*) - \pi_{\tau_*}(x_i)\| - \|\pi_{\tau_*}(z_*) - y_s^{(i,\tau_*)}\|$$
$$= \|\pi_{\tau_*}(z_*) - y_{s,0}^{(i,\tau_*)}\| - \|\pi_{\tau_*}(z_*) - y_{s,p(s)}^{(i,\tau_*)}\|$$
$$= \sum_{j=1}^{p(s)} (\|\pi_{\tau_*}(z_*) - y_{s,j-1}^{(i,\tau_*)}\| - \|\pi_{\tau_*}(z_*) - y_{s,j}^{(i,\tau_*)}\|)$$
$$\geq \|\pi_{\tau_*}(z_*) - y_{s,j_0-1}^{(i,\tau_*)}\| - \|\pi_{\tau_*}(z_*) - y_{s,j_0}^{(i,\tau_*)}\| \geq \delta. \tag{7.109}$$

It follows from (7.28), (7.91), (7.99), (7.109) and the convexity of the norm that

$$\|\pi_{\tau_*}(z_*) - y_{i,\tau_*}\| = \|\pi_{\tau_*}(z_*) - \sum_{t \in \Omega_{i+1,\tau_*}} w_{i+1,\tau_*}(t) y_t^{(i,\tau_*)}\|$$
$$\leq \sum_{t \in \Omega_{i+1,\tau_*}} w_{i+1,\tau_*}(t) \|\pi_{\tau_*}(z_*) - y_t^{(i,\tau_*)}\|$$
$$\leq \|\pi_{\tau_*}(z_*) - \pi_{\tau_*}(x_i)\| \sum \{w_{i+1,\tau_*}(t) : \ t \in \Omega_{i+1,\tau_*} \setminus \{s\}\}$$
$$+ w_{i+1,\tau_*}(s)(\|\pi_{\tau_*}(z_*) - \pi_{\tau_*}(x_i)\| - \delta)$$
$$\leq \|\pi_{\tau_*}(z_*) - \pi_{\tau_*}(x_i)\| - \Delta\delta. \tag{7.110}$$

Lemmas 7.5 and 7.6, (7.88), (7.106), and (7.110) implies that

$$\|B_2(z_* - x_{i+1})\|^2 = \|B_2(z_* - B_1(\sum_{\tau \in \mathcal{E}} y_{i,\tau}))\|^2$$
$$\leq \sum_{\tau \in \mathcal{E}} \|\pi_\tau(z_*) - y_{i,\tau}\|^2$$
$$\leq \sum \{\|\pi_\tau(z_*) - y_{i,\tau}\|^2 : \ \tau \in \mathcal{E} \setminus \{\tau_*\}\} + \|\pi_{\tau_*}(z_*) - y_{i,\tau_*}\|^2$$
$$\leq \sum \{\|\pi_\tau(z_*) - \pi_\tau(x_i)\|^2 : \ \tau \in \mathcal{E} \setminus \{\tau_*\}\}$$
$$+ (\|\pi_{\tau_*}(z_*) - \pi_{\tau_*}(x_i)\| - \Delta\delta)^2$$
$$= \sum_{\tau \in \mathcal{E}} \|\pi_\tau(z_*) - \pi_\tau(x_i)\|^2 - (\Delta\delta)^2$$
$$= \|B_2(z_* - x_i)\|^2 - (\Delta\delta)^2,$$
$$\|B_2(z_* - x_{i+1})\|^2 \leq \|B_2(z_* - x_i)\|^2 - (\Delta\delta)^2 \tag{7.111}$$

for each integer $i \geq 0$ satisfying $\lambda_{i+1} \geq \gamma_0$. By (7.14), (7.21), (7.80), (7.84), and (7.101), for each natural number n,

$$4M^2 \max\{m_i : i = 1, \dots, l\} \geq \|B_2(z_* - x_0)\|^2$$
$$\geq \|B_2(z_* - x_0)\|^2 - \|B_2(z_* - x_n)\|^2$$
$$= \sum_{i=0}^{n-1} (\|B_2(z_* - x_i)\|^2 - \|B_2(z_* - x_{i+1})\|^2)$$
$$\geq (\Delta\delta)^2 \mathrm{Card}(\{i \in \{0, \dots .n-1\} : \lambda_{i+1} \geq \gamma_0\}).$$

Since the relation above holds for any natural number n we conclude that

$$\mathrm{Card}(\{i \in \{0, \dots, \} : \lambda_{i+1} \geq \gamma_0\})$$
$$\leq (\Delta\delta)^{-2} 4M^2 \max\{m_i : i = 1, \dots, l\} = Q.$$

Assume that an integer $i \geq 0$ satisfies

$$\lambda_{i+1} < \gamma_0. \tag{7.112}$$

By (7.112), for each $\tau = (\tau_1, \dots, \tau_{p(\tau)}) \in \mathcal{E}$, each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1,\tau}$ and each $j = 1, \dots, p(t)$,

$$\lambda_{i,\tau} < \gamma_0, \ \lambda_t^{(i,\tau)} < \gamma_0,$$
$$d_{\widehat{X}_\tau}(y_{t,j-1}^{(i,\tau)}, C_{\tau,t_j}) < \gamma_0. \tag{7.113}$$

Let

$$\tau = (\tau_1, \dots, \tau_{p(\tau)}) \in \mathcal{E}, \ t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1,\tau}, \ j \in \{1, \dots, p(t)\}.$$

In view of (7.113), there exists

$$\xi \in C_{\tau,t_j} \tag{7.114}$$

such that

$$\|y_{t,j-1}^{(i,\tau)} - \xi\| < \gamma_0. \tag{7.115}$$

By (7.13), (7.94), (7.114), and (7.115),

$$\|y_{t,j}^{(i,\tau)} - \xi\| = \|P_{\tau,t_j}(y_{t,j-1}^{(i,\tau)}) - \xi\| \leq \|y_{t,j-1}^{(i,\tau)} - \xi\| < \gamma_0.$$

Together with (7.115) this implies that

$$\|y_{t,j-1}^{(i,\tau)} - y_{t,j}^{(i,\tau)}\| < 2\gamma_0.$$

Combined with (7.93) this implies that

$$\|\pi_\tau(x_i) - y^{(i,\tau)}_{t,j-1}\| < 2\gamma_0(j-1). \tag{7.116}$$

It follows from (7.27), (7.113), and (7.116) that

$$d_{\widehat{X}_\tau}(\pi_\tau(x_i), C_{\tau,t_j}) < 2\gamma_0 j \le 2\gamma_0\bar{q}. \tag{7.117}$$

By (7.8), (7.81), and (7.117),

$$d_X(x_i, C_{t_j}) < 2\gamma_0\bar{q} = \epsilon$$

for each $j = 1, \dots, p(t)$, each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1,\tau}$ and each $\tau \in \mathcal{E}$. Together with (7.3) and (7.26) this implies that

$$d_X(x_i, C_s) \le \epsilon, \ \ s = 1, \dots, m.$$

Theorem 7.2 is proved. □

7.5 Proof of Theorem 7.3

By (7.44), there exists

$$z_* \in B_X(0, M) \cap C. \tag{7.118}$$

In view of (7.50) and (7.118),

$$\|x_0 - z_*\| \le 2M. \tag{7.119}$$

It follows from (7.10), (7.43), and (7.119) that

$$\|B_2(x_0 - z_*)\| \le M_1^{1/2}\|x_0 - z_*\| \le 2MM^{1/2}. \tag{7.120}$$

Let $i \ge 0$ be an integer. In view of (7.51),

$$(x_{i+1}, \lambda_{i+1}) \in A(x_i, \{(\Omega_{i+1,\tau}, w_{i+1,\tau})\}_{\tau\in\mathcal{E}}, \delta). \tag{7.121}$$

By (7.35) and (7.121), there exist

$$(y_{i,\tau}, \lambda_{i,\tau}) \in A_\tau(\pi_\tau(x_i), (\Omega_{i+1,\tau}, w_{i+1,\tau}), \delta), \ \ \tau \in \mathcal{E} \tag{7.122}$$

such that

$$\|x_{i+1} - B_1(\sum_{\tau\in\mathcal{E}} y_{i,\tau})\| \le \delta, \tag{7.123}$$

$$\lambda_{i+1} = \max\{\lambda_{i,\tau} : \ \tau \in \mathcal{E}\}. \tag{7.124}$$

By (7.34) and (7.122), for each $\tau \in \mathcal{E}$ there exist

$$(y_t^{(i,\tau)}, \lambda_t^{(i,\tau)}) \in A_{\tau,0}(\pi_\tau(x_i), t, \delta),\ t \in \Omega_{i+1,\tau} \tag{7.125}$$

such that

$$\|y_{i,\tau} - \sum_{t \in \Omega_{i+1,\tau}} w_{i+1,\tau}(t) y_t^{(i,\tau)}\| \le \delta, \tag{7.126}$$

$$\lambda_{i,\tau} = \max\{\lambda_t^{(i,\tau)} :\ t \in \Omega_{i+1,\tau}\}. \tag{7.127}$$

In view of (7.125), for each $\tau \in \mathcal{E}$ and each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1,\tau}$ there exist

$$\{y_{t,j}^{(i,\tau)}\}_{j=0}^{p(t)} \subset \widehat{X}_\tau$$

such that

$$y_{t,0}^{(i,\tau)} = \pi_\tau(x_i) \tag{7.128}$$

and for all $j = 1, \dots, p(t)$,

$$\|y_{t,j}^{(i,\tau)} - P_{\tau,t_j}(y_{t,j-1}^{(i,\tau)})\| \le \delta, \tag{7.129}$$

$$y_t^{(i,\tau)} = y_{t,p(t)}^{(i,\tau)}, \tag{7.130}$$

$$\lambda_t^{(i,\tau)} = \max\{\|y_{t,j}^{(i,\tau)} - y_{t,j-1}^{(i,\tau)}\| :\ j = 1, \dots, p(t)\}. \tag{7.131}$$

By (7.13), (7.118), and (7.129), for each $\tau \in \mathcal{E}$, each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1,\tau}$ and each $j \in \{1, \dots, p(t)\}$,

$$\begin{aligned} \|\pi_\tau(z_*) - y_{t,j}^{(i,\tau)}\| &\le \|\pi_\tau(z_*) - P_{\tau,t_j}(y_{t,j-1}^{(i,\tau)})\| + \|P_{\tau,t_j}(y_{t,j-1}^{(i,\tau)}) - y_{t,j}^{(i,\tau)}\| \\ &\le \|\pi_\tau(z_*) - y_{t,j-1}^{(i,\tau)}\| + \delta. \end{aligned} \tag{7.132}$$

We prove the following auxiliary result.

Lemma 7.7. *Assume that an integer $k \ge 0$ satisfies*

$$\|B_2(x_k - z_*)\| \le 2MM_1^{1/2}, \tag{7.133}$$

$$\lambda_{k+1} > \epsilon_0. \tag{7.134}$$

Then

$$\|B_2(x_{k+1} - z_*)\|^2 \le \|B_2(x_k - z_*)\|^2 - 16^{-1}\Delta\bar{c}\epsilon_0^2.$$

Proof. By (7.20), (7.43), and (7.133),

$$\|x_k - z_*\| \le \|B_2(x_k - z_*)\| M_2^{-1/2} \le 2M(M_1 M_2^{-1})^{1/2}. \tag{7.135}$$

In view of (7.27), (7.128), and (7.132), for each $\tau \in \mathcal{E}$, each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1,\tau}$ and each $j \in \{0, 1, \dots, p(t)\}$,

$$\begin{aligned}\|\pi_\tau(z_*) - y_{t,j}^{(k,\tau)}\| &\le \|\pi_\tau(z_*) - \pi_\tau(x_k)\| + j\delta \\ &\le \|\pi_\tau(z_*) - \pi_\tau(x_k)\| + \bar{q}\delta. \end{aligned} \tag{7.136}$$

It follows from (7.45), (7.132), (7.135), and (7.136) that for each $\tau \in \mathcal{E}$, each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1,\tau}$ and each $j \in \{1, \dots, p(t)\}$,

$$\begin{aligned} &\|\pi_\tau(z_*) - y_{t,j}^{(k,\tau)}\|^2 - \|\pi_\tau(z_*) - y_{t,j-1}^{(k,\tau)}\|^2 \\ &\le (\|\pi_\tau(z_*) - y_{t,j}^{(k,\tau)}\| - \|\pi_\tau(z_*) - y_{t,j-1}^{(k,\tau)}\|)(\|\pi_\tau(z_*) - y_{t,j}^{(k,\tau)}\| + \|\pi_\tau(z_*) - y_{t,j-1}^{(k,\tau)}\|) \\ &\le \delta(\|\pi_\tau(z_*) - y_{t,j}^{(k,\tau)}\| + \|\pi_\tau(z_*) - y_{t,j-1}^{(k,\tau)}\|) \\ &\le \delta(2\|\pi_\tau(z_*) - \pi_\tau(x_k)\| + 2) \le 2\delta(2M(M_1 M_2^{-1})^{1/2} + 1). \end{aligned} \tag{7.137}$$

Relations (7.124) and (7.134) imply that there exists $\xi \in \mathcal{E}$ such that

$$\epsilon_0 < \lambda_{k+1} = \lambda_{k,\xi}. \tag{7.138}$$

By (7.127) and (7.138), there exists $s = (s_1, \dots, s_{p(s)}) \in \Omega_{k+1,\xi}$ such that

$$\epsilon_0 < \lambda_{k,\xi} = \lambda_s^{(k,\xi)}. \tag{7.139}$$

In view of (7.131) and (7.139), there exists $j_0 \in \{1, \dots, p(s)\}$ such that

$$\|y_{s,j_0-1}^{(k,\xi)} - y_{s,j_0}^{(k,\xi)}\| = \lambda_s^{(k,\xi)} > \epsilon_0. \tag{7.140}$$

It follows from (7.129) and (7.140) that

$$\begin{aligned} &\|y_{s,j_0-1}^{(k,\xi)} - P_{\xi,s_{j_0}}(y_{s,j_0-1}^{(k,\xi)})\| \\ &\ge \|y_{s,j_0-1}^{(k,\xi)} - y_{s,j_0}^{(k,\xi)}\| - \|P_{\xi,s_{j_0}}(y_{s,j_0-1}^{(k,\xi)}) - y_{s,j_0}^{(k,\xi)}\| > \epsilon_0 - \delta. \end{aligned} \tag{7.141}$$

Assumption (A1), (7.46), (7.118), and (7.141) imply that

$$\begin{aligned} &\|\pi_\xi(z_*) - P_{\xi,s_{j_0}}(y_{s,j_0-1}^{(k,\xi)})\|^2 \\ &\le \|\pi_\xi(z_*) - y_{s,j_0-1}^{(k,\xi)}\|^2 - \bar{c}\|y_{s,j_0-1}^{(k,\xi)} - P_{\xi,s_{j_0}}(y_{s,j_0-1}^{(k,\xi)})\|^2 \\ &\le \|\pi_\xi(z_*) - y_{s,j_0-1}^{(k,\xi)}\|^2 - \bar{c}(\epsilon_0 - \delta)^2. \end{aligned} \tag{7.142}$$

By (7.13), (7.45), (7.118), (7.129), (7.135), and (7.136),

$$\begin{aligned}
&\|\pi_\xi(z_*) - y_{s,j_0}^{(k,\xi)}\|^2 - \|\pi_\xi(z_*) - P_{\xi,s_{j_0}}(y_{s,j_0-1}^{(k,\xi)})\|^2 \\
&= (\|\pi_\xi(z_*) - y_{s,j_0}^{(k,\xi)}\| - \|\pi_\xi(z_*) - P_{\xi,s_{j_0}}(y_{s,j_0-1}^{(k,\xi)})\|)(\|\pi_\xi(z_*) \\
&\quad - y_{s,j_0}^{(k,\xi)}\| + \|\pi_\xi(z_*) - P_{\xi,s_{j_0}}(y_{s,j_0-1}^{(k,\xi)})\|) \\
&\le 2\|y_{s,j_0}^{(k,\xi)} - P_{\xi,s_{j_0}}(y_{s,j_0-1}^{(k,\xi)})\|(\|\pi_\xi(z_*) - \pi_\xi(x_k)\| + 1) \\
&\le 2\delta(2M(M_1M_2^{-1})^{1/2} + 1). \qquad (7.143)
\end{aligned}$$

In view of (7.142) and (7.143),

$$\begin{aligned}
&\|\pi_\xi(z_*) - y_{s,j_0}^{(k,\xi)}\|^2 \le \|\pi_\xi(z_*) - P_{\xi,s_{j_0}}(y_{s,j_0-1}^{(k,\xi)})\|^2 + 2\delta(2M(M_1M_2^{-1})^{1/2} + 1) \\
&\le \|\pi_\xi(z_*) - y_{s,j_0-1}^{(k,\xi)}\|^2 - \bar{c}(\epsilon_0 - \delta)^2 + 2\delta(2M(M_1M_2^{-1})^{1/2} + 1). \qquad (7.144)
\end{aligned}$$

By (7.17), (7.46), (7.128), (7.130), (7.137), and (7.144),

$$\begin{aligned}
&\|\pi_\xi(z_*) - \pi_\xi(x_k)\|^2 - \|\pi_\xi(z_*) - y_s^{(k,\xi)}\|^2 \\
&= \|\pi_\xi(z_*) - y_{s,0}^{(k,\xi)}\|^2 - \|\pi_\xi(z_*) - y_{s,p(s)}^{(k,\xi)}\|^2 \\
&= \sum_{j=1}^{p(s)} (\|\pi_\xi(z_*) - y_{s,j-1}^{(k,\xi)}\|^2 - \|\pi_\xi(z_*) - y_{s,j}^{(k,\xi)}\|^2) \\
&\ge \sum \{\|\pi_\xi(z_*) - y_{s,j-1}^{(k,\xi)}\|^2 - \|\pi_\xi(z_*) - y_{s,j}^{(k,\xi)}\|^2 : \ j \in \{1, \dots, p(s)\} \setminus \{j_0\}\} \\
&\quad + \|\pi_\xi(z_*) - y_{s,j_0-1}^{(k,\xi)}\|^2 - \|\pi_\xi(z_*) - y_{s,j_0}^{(k,\xi)}\|^2 \\
&\ge -2\delta(2M(M_1M_2^{-1})^{1/2} + 1)(p(s) - 1) - 2\delta(2M(M_1M_2^{-1})^{1/2} + 1) + 4^{-1}\bar{c}\epsilon_0^2 \\
&\ge 4^{-1}\bar{c}\epsilon_0^2 - 2\delta(2M(M_1M_2^{-1})^{1/2} + 1)\bar{q} \ge 8^{-1}\bar{c}\epsilon_0^2. \qquad (7.145)
\end{aligned}$$

Let $\tau \in \mathcal{E}$ and $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1,\tau}$. In view of (7.130) and (7.136),

$$\begin{aligned}
&\|\pi_\tau(z_*) - y_t^{(k,\tau)}\| = \|\pi_\tau(z_*) - y_{t,p(t)}^{(k,\tau)}\| \\
&\le \|\pi_\tau(z_*) - \pi_\tau(x_k)\| + \bar{q}\delta. \qquad (7.146)
\end{aligned}$$

It follows from (7.27), (7.128), (7.130), and (7.137) that

$$\begin{aligned}
&\|\pi_\tau(z_*) - \pi_\tau(x_k)\|^2 - \|\pi_\tau(z_*) - y_t^{(k,\tau)}\|^2 \\
&= \|\pi_\tau(z_*) - y_{t,0}^{(k,\tau)}\|^2 - \|\pi_\tau(z_*) - y_{t,p(t)}^{(k,\tau)}\|^2
\end{aligned}$$

$$= \sum_{j=1}^{p(t)} (\|\pi_\tau(z_*) - y_{t,j-1}^{(k,\tau)}\|^2 - \|\pi_\tau(z_*) - y_{t,j}^{(k,\tau)}\|^2)$$

$$\geq -2\delta p(s)(2M(M_1 M_2^{-1})^{1/2} + 1) \geq -2\delta \bar{q}(2M(M_1 M_2^{-1})^{1/2} + 1). \tag{7.147}$$

By (7.28), (7.146) and convexity of the norm,

$$\|\pi_\tau(z_*) - \sum_{t \in \Omega_{k+1,\tau}} w_{k+1,\tau}(t) y_t^{(k,\tau)}\| \leq \sum_{t \in \Omega_{k+1,\tau}} w_{k+1,\tau}(t) \|\pi_\tau(z_*) - y_t^{(k,\tau)}\|$$

$$\leq \|\pi_\tau(z_*) - \pi_\tau(x_k)\| + \bar{q}\delta. \tag{7.148}$$

In view of (7.28), (7.147) and convexity of the function $\|\cdot\|^2$,

$$\|\pi_\tau(z_*) - \sum_{t \in \Omega_{k+1,\tau}} w_{k+1,\tau}(t) y_t^{(k,\tau)}\|^2 \leq \sum_{t \in \Omega_{k+1,\tau}} w_{k+1,\tau}(t) \|\pi_\tau(z_*) - y_t^{(k,\tau)}\|^2$$

$$\leq \|\pi_\tau(z_*) - \pi_\tau(x_k)\|^2 + 2\bar{q}\delta(2M(M_1 M_2^{-1})^{1/2}) + 1). \tag{7.149}$$

Relations (7.126) and (7.148) imply that

$$\|\pi_\tau(z_*) - y_{k,\tau}\| \leq \|\pi_\tau(z_*) - \sum_{t \in \Omega_{k+1,\tau}} w_{k+1,\tau}(t) y_t^{(k,\tau)}\|$$

$$+ \|\sum_{t \in \Omega_{k+1,\tau}} w_{k+1,\tau}(t) y_t^{(k,\tau)} - y_{k,\tau}\|$$

$$\leq \|\pi_\tau(z_*) - \pi_\tau(x_k)\| + (\bar{q} + 1)\delta. \tag{7.150}$$

By (7.45), (7.126), (7.135), (7.148), and (7.150),

$$\|\pi_\tau(z_*) - y_{k,\tau}\|^2 - \|\pi_\tau(z_*) - \sum_{t \in \Omega_{k+1,\tau}} w_{k+1,\tau}(t) y_t^{(k,\tau)}\|^2$$

$$= (\|\pi_\tau(z_*) - y_{k,\tau}\| - \|\pi_\tau(z_*) - \sum_{t \in \Omega_{k+1,\tau}} w_{k+1,\tau}(t) y_t^{(k,\tau)}\|)$$

$$\times (\|\pi_\tau(z_*) - y_{k,\tau}\| + \|\pi_\tau(z_*) - \sum_{t \in \Omega_{k+1,\tau}} w_{k+1,\tau}(t) y_t^{(k,\tau)}\|)$$

$$\leq \delta(2\|\pi_\tau(z_*) - \pi_\tau(x_k)\| + 2(\bar{q} + 1)\delta)$$

$$\leq \delta(4M(M_1 M_2^{-1})^{1/2} + 1). \tag{7.151}$$

It follows from (7.28), (7.29), (7.145), (7.147) and convexity of the function $\|\cdot\|^2$ that

$$\|\pi_\xi(z_*) - \sum_{t\in\Omega_{k+1,\xi}} w_{k+1,\xi}(t)y_t^{(k,\xi)}\|^2$$
$$\le \sum_{t\in\Omega_{k+1,\xi}} w_{k+1,\xi}(t)\|\pi_\xi(z_*) - y_t^{(k,\xi)}\|^2$$
$$= \sum\{w_{k+1,\xi}(t)\|\pi_\xi(z_*) - y_t^{(k,\xi)}\|^2 :\ t \in \Omega_{k+1,\xi} \setminus \{s\}\}$$
$$+w_{k+1,\xi}(s)\|\pi_\xi(z_*) - y_s^{(k,\xi)}\|^2$$
$$\le (\|\pi_\xi(z_*) - \pi_\xi(x_k)\|^2$$
$$+2\delta\bar{q}(2M(M_1M_2^{-1})^{1/2} + 1)) \sum\{w_{k+1,\xi}(t) :\ t \in \Omega_{k+1,\xi} \setminus \{s\}\}$$
$$+w_{k+1,\xi}(s)(\|\pi_\xi(z_*) - \pi_\xi(x_k)\|^2 - 4^{-1}\bar{c}\epsilon_0^2 + 2\delta(2M(M_1M_2^{-1})^{1/2} + 1)\bar{q})$$
$$\le \|\pi_\xi(z_*) - \pi_\xi(x_k)\|^2 + 2\delta\bar{q}(2M(M_1M_2^{-1})^{1/2} + 1) - 4^{-1}\Delta\bar{c}\epsilon_0^2. \tag{7.152}$$

Relations (7.151) and (7.152) imply that

$$\|\pi_\xi(z_*) - y_{k,\xi}\|^2$$
$$\le \|\pi_\xi(z_*) - \sum_{t\in\Omega_{k+1,\xi}} w_{k+1,\xi}(t)y_t^{(k,\xi)}\|^2 + 2\delta(2M(M_1M_2^{-1})^{1/2} + 1)$$
$$\le \|\pi_\xi(z_*) - \pi_\xi(x_k)\|^2$$
$$+2\delta(\bar{q} + 1)(2M(M_1M_2^{-1})^{1/2} + 1) - 4^{-1}\Delta\bar{c}\epsilon_0^2. \tag{7.153}$$

In view of (7.149) and (7.151), for each $\tau \in \mathcal{E}$,

$$\|\pi_\tau(z_*) - y_{k,\tau}\|^2$$
$$\le \|\pi_\tau(z_*) - \sum_{t\in\Omega_{k+1,\tau}} w_{k+1,\tau}(t)y_t^{(k,\tau)}\|^2$$
$$+2\delta(2M(M_1M_2^{-1})^{1/2} + 1)$$
$$\le \|\pi_\xi(z_*) - \pi_\xi(x_k)\|^2 + 2\delta(\bar{q} + 1)(2M(M_1M_2^{-1})^{1/2} + 1). \tag{7.154}$$

By (7.153) and (7.154),

$$\sum_{\tau\in\mathcal{E}} \|\pi_\tau(z_*) - y_{k,\tau}\|^2 = \sum_{\tau\in\mathcal{E}\setminus\{\xi\}} \|\pi_\tau(z_*) - y_{k,\tau}\|^2 + \|\pi_\xi(z_*) - y_{k,\xi}\|^2$$

$$\leq \sum\{\|\pi_\tau(z_*) - \pi_\tau(x_k)\|^2 + 2\delta(\bar{q}+1)(2M(M_1M_2^{-1})^{1/2}+1) : \tau \in \mathcal{E} \setminus \{\xi\}\}$$
$$+\|\pi_\xi(z_*) - \pi_\xi(x_k)\|^2 + 2\delta(\bar{q}+1)(2M(M_1M_2^{-1})^{1/2}+1) - 4^{-1}\Delta\bar{c}\epsilon_0^2$$
$$= \sum_{\tau\in\mathcal{E}} \|\pi_\tau(z_*) - \pi_\tau(x_k)\|^2 + \mathrm{Card}(\mathcal{E})2\delta(\bar{q}+1)(2M(M_1M_2^{-1})^{1/2}+1) - 4^{-1}\Delta\bar{c}\epsilon_0^2. \tag{7.155}$$

It follows from (7.46) and (7.155) that

$$\sum_{\tau\in\mathcal{E}} \|\pi_\tau(z_*) - y_{k,\tau}\|^2 \leq \sum_{\tau\in\mathcal{E}} \|\pi_\tau(z_*) - \pi_\tau(x_k)\|^2 - 8^{-1}\Delta\bar{c}\epsilon_0^2. \tag{7.156}$$

Lemmas 7.5 and 7.6 and (7.156) imply that

$$\|B_2(z_* - x_k)\|^2 - 8^{-1}\Delta\bar{c}\epsilon_0^2 \geq \|B_2(z_* - B_1(\sum_{\tau\in\mathcal{E}} y_{k,\tau}))\|^2, \tag{7.157}$$

$$\|B_2(z_* - B_1(\sum_{\tau\in\mathcal{E}} y_{k,\tau}))\| \leq \|B_2(z_* - x_k)\|. \tag{7.158}$$

In view of (7.10), (7.43), (7.45), (7.123), (7.133), and (7.158),

$$\|B_2(z_* - x_{k+1})\|^2 - \|B_2(z_* - B_1(\sum_{\tau\in\mathcal{E}} y_{k,\tau}))\|^2$$
$$= (\|B_2(z_* - x_{k+1})\| - \|B_2(z_* - B_1(\sum_{\tau\in\mathcal{E}} y_{k,\tau}))\|)$$
$$\times(\|B_2(z_* - x_{k+1})\| + \|B_2(z_* - B_1(\sum_{\tau\in\mathcal{E}} y_{k,\tau}))\|)$$
$$\leq \delta M_1^{1/2}(2\|B_2(z_* - B_1(\sum_{\tau\in\mathcal{E}} y_{k,\tau}))\| + \delta M_1^{1/2})$$
$$\leq \delta M_1^{1/2}(2\|B_2(z_* - x_k)\| + \delta M_1^{1/2}) \leq \delta M_1^{1/2}(2MM_1^{1/2}+1). \tag{7.159}$$

By (7.47), (7.157), and (7.159),

$$\|B_2(z_* - x_{k+1})\|^2 \leq \|B_2(z_* - B_1(\sum_{\tau\in\mathcal{E}} y_{k,\tau}))\|^2 + \delta M_1^{1/2}(2MM_1^{1/2}+1)$$
$$\leq \|B_2(z_* - x_k)\|^2 - 8^{-1}\Delta\bar{c}\epsilon_0^2 + \delta M_1^{1/2}(2MM_1^{1/2}+1)$$
$$\leq \|B_2(z_* - x_k)\|^2 - 16^{-1}\Delta\bar{c}\epsilon_0^2.$$

Lemma 7.7 is proved. □

Assume that q is a natural number such that for each integer $k \in [0, q-1]$,

$$\lambda_{k+1} > \epsilon_0. \tag{7.160}$$

By (7.120), (7.160) and Lemma 7.7 applied by induction,

$$\|B_2(x_k - z_*)\| \le 2MM_1^{1/2},\ k = 0, \dots, q$$

and for all $k = 0, \dots, q-1$,

$$\|B_2(z_* - x_{k+1})\|^2 \le \|B_2(z_* - x_k)\|^2 - 16^{-1}\Delta\bar{c}\epsilon_0^2. \tag{7.161}$$

In view of (7.48), (7.120), and (7.161),

$$4M^2M_1 \ge \|B_2(z_* - x_0)\|^2 \ge \|B_2(z_* - x_0)\|^2 - \|B_2(z_* - x_q)\|^2$$

$$= \sum_{k=0}^{q-1}(\|B_2(z_* - x_k)\|^2 - \|B_2(z_* - x_{k+1})\|^2) \ge 16^{-1}\bar{q}\Delta\bar{c}\epsilon_0^2,$$

$$q \le 64M^2M_1\Delta^{-1}\bar{c}^{-1}\epsilon_0^{-2} \le n_0.$$

Therefore there exists an integer $q \in [0, n_0]$ such that for each integer k satisfying $0 \le k < q$,

$$\lambda_{k+1} > \epsilon_0,$$

$$\lambda_{q+1} \le \epsilon_0$$

and for all integers $k = 0, \dots, q$,

$$\|B_2(x_k - z_*)\| \le 2MM_1^{1/2}.$$

By (7.20), (7.43), and (7.118), for all $k = 0, \dots, q$,

$$\|x_k - z_*\| \le 2M(M_1M_2^{-1})^{1/2},\ \|x_k\| \le 2M(M_1M_2^{-1})^{1/2} + M.$$

Assume that an integer $q \ge 0$ satisfies

$$\lambda_{q+1} \le \epsilon_0. \tag{7.162}$$

By (7.124), (7.127), (7.131), and (7.162), for each $\tau \in \mathcal{E}$, each

$$t = (t_1, \dots, t_{p(t)}) \in \Omega_{q+1,\tau}$$

and each $j = 1, \dots, p(t)$,

$$\lambda_{q,\tau} \le \epsilon_0,$$
$$\lambda_t^{(q,\tau)} \le \epsilon_0,$$
$$\|y_{t,j}^{(q,\tau)} - y_{t,j-1}^{(q,\tau)}\| \le \epsilon_0. \tag{7.163}$$

Let

$$\tau \in \mathcal{E},\ t = (t_1, \dots, t_{p(t)}) \in \Omega_{q+1,\tau}.$$

Assumption (A1), (7.163), and (7.129) imply that for each $j = 1, \dots, p(t)$,

$$d_{\widehat{X}_\tau}(y_{t,j-1}^{(q,\tau)}, C_{\tau,t_j}) \le \|y_{t,j-1}^{(q,\tau)} - P_{\tau,t_j}(y_{t,j-1}^{(q,\tau)})\|$$
$$\le \|y_{t,j-1}^{(q,\tau)} - y_{t,j}^{(q,\tau)}\| + \|y_{t,j}^{(q,\tau)} - P_{\tau,t_j}(y_{t,j-1}^{(q,\tau)})\| \le \epsilon_0 + \delta. \tag{7.164}$$

In view of (7.128) and (7.163), for each $j = 0, \dots, p(t)$,

$$\|\pi_\tau(x_q) - y_{t,j}^{(q,\tau)}\| \le j\epsilon_0. \tag{7.165}$$

It follows from (7.27), (7.46), (7.47), (7.164), and (7.165) that for each $j = 1, \dots, p(t)$,

$$d_{\widehat{X}_\tau}(\pi_\tau(x_q), C_{\tau,t_j}) \le \|\pi_\tau(x_q) - y_{t,j-1}^{(q,\tau)}\| + d_{\widehat{X}_\tau}(y_{t,j-1}^{(q,\tau)}, C_{\tau,t_j})$$
$$\le (j-1)\epsilon_0 + \epsilon_0 + \delta \le (\bar{q}+1)\epsilon_0 = \epsilon_1$$

and

$$d(x_q, C_{t_j}) \le \epsilon_1$$

for each $\tau \in \mathcal{E}$, each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{q+1,\tau}$ and each $j = 1, \dots, p(t)$. Together with (7.3) and (7.26) this implies that $d(x_q, C_s) \le \epsilon_1$ for all $s = 1, \dots, m$. Theorem 7.3 is proved. □

7.6 Proof of Theorem 7.4

We may assume that $\epsilon < 1$. There exists

$$z_* \in B_X(0, M) \cap C. \tag{7.166}$$

Let

$$\epsilon_1 = \epsilon \bar{q}^{-1}. \tag{7.167}$$

By (7.166) and (A2) there exists

$$\epsilon_0 \in (0, \epsilon_1)$$

such that the following property holds:

(i) for each $\tau = (\tau_1, \dots, \tau_{p(\tau)}) \in \mathcal{E}$, each $s \in \{\tau_1, \dots, \tau_{p(\tau)}\}$ and each $x \in \widehat{X}_\tau$ satisfying

$$\|x\| \le M + 1 + 2M(M_1 M_2^{-1})^{1/2}, \ d_{\widehat{X}_\tau}(x, C_{\tau,s}) \ge \epsilon_1$$

the inequality

$$\|\pi_\tau(z_*) - P_{\tau,s}(x)\| \le \|\pi_\tau(z_*) - x\| - 2\epsilon_0$$

holds.

By (7.166) and (A2) there exists a positive number

$$\gamma < \min\{\epsilon_0/3, 1\}$$

such that the following property holds:

(ii) for each $\tau = (\tau_1, \dots, \tau_{p(\tau)}) \in \mathcal{E}$, each $s \in \{\tau_1, \dots, \tau_{p(\tau)}\}$ and each $x \in \widehat{X}_\tau$ satisfying

$$\|x\| \le M + 1 + 2M(M_1 M_2^{-1})^{1/2}, \ d_{\widehat{X}_\tau}(x, C_{\tau,s}) \ge \epsilon_0/3$$

the inequality

$$\|\pi_\tau(z_*) - P_{\tau,s}(x)\| \le \|\pi_\tau(z_*) - x\| - \gamma$$

holds.

Choose a positive number δ such that

$$16\delta(\bar{q} + 1)(2M(M_1 M_2^{-1})^{1/2} + 1)\mathrm{Card}(\mathcal{E}) \le \Delta^2\gamma^2,$$
$$\delta M_1 < 1, \ \delta M_1^{1/2}(4MM_1^{1/2} + 1) \le 16^{-1}\Delta^2\gamma^2. \tag{7.168}$$

Choose a natural number

$$n_0 \ge 64M^2 M_1 \Delta^{-2}\gamma^{-2}. \tag{7.169}$$

Assume that

$$\{(\Omega_{i,\tau}, w_{i,\tau})\}_{i=1}^{\infty} \subset \mathcal{M}_\tau,\ \ \tau \in \mathcal{E}, \tag{7.170}$$

$$x_0 \in B_X(0, M), \tag{7.171}$$

$\{x_i\}_{i=1}^{\infty} \subset X$, $\{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty)$ and that for each natural number i,

$$(x_i, \lambda_i) \in A(x_{i-1}, \{(\Omega_{i,\tau}, w_{i,\tau})\}_{\tau \in \mathcal{E}}, \delta). \tag{7.172}$$

By (7.20), (7.43), (7.166), and (7.171)

$$\|x_0 - z_*\| \le 2M,$$

$$\|B_2(x_0 - z_*)\| \le M_1^{1/2}\|x_0 - z\| \le 2MM_1^{1/2}. \tag{7.173}$$

Let $i \ge 0$ be an integer. In view of (7.172),

$$(x_{i+1}, \lambda_{i+1}) \in A(x_i, \{(\Omega_{i+1,\tau}, w_{i+1,\tau})\}_{\tau \in \mathcal{E}}, \delta). \tag{7.174}$$

By (7.35) and (7.174), there exist

$$(y_{i,\tau}, \lambda_{i,\tau}) \in A_\tau(\pi_\tau(x_i), (\Omega_{i+1,\tau}, w_{i+1,\tau}), \delta),\ \ \tau \in \mathcal{E} \tag{7.175}$$

such that

$$\|x_{i+1} - B_1(\sum_{\tau \in \mathcal{E}} y_{i,\tau})\| \le \delta, \tag{7.176}$$

$$\lambda_{i+1} = \max\{\lambda_{i,\tau} :\ \tau \in \mathcal{E}\}. \tag{7.177}$$

By (7.34) and (7.175), for each $\tau \in \mathcal{E}$ there exist

$$(y_t^{(i,\tau)}, \lambda_t^{(i,\tau)}) \in A_{\tau,0}(\pi_\tau(x_i), t, \delta),\ \ t \in \Omega_{i+1,\tau} \tag{7.178}$$

such that

$$\|y_{i,\tau} - \sum_{t \in \Omega_{i+1,\tau}} w_{i+1,\tau}(t) y_t^{(i,\tau)}\| \le \delta, \tag{7.179}$$

$$\lambda_{i,\tau} = \max\{\lambda_t^{(i,\tau)} :\ t \in \Omega_{i+1,\tau}\}. \tag{7.180}$$

In view of (7.33) and (7.178), for each $\tau \in \mathcal{E}$ and each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1,\tau}$ there exist

$$\{y_{t,j}^{(i,\tau)}\}_{j=0}^{p(t)} \subset \widehat{X}_\tau$$

such that

$$y_{t,0}^{(i,\tau)} = \pi_\tau(x_i), \tag{7.181}$$

and for all $j = 1, \dots, p(t)$,

$$\|y_{t,j}^{(i,\tau)} - P_{\tau,t_j}(y_{t,j-1}^{(i,\tau)})\| \le \delta, \tag{7.182}$$

$$y_t^{(i,\tau)} = y_{t,p(t)}^{(i,\tau)}, \tag{7.183}$$

$$\lambda_t^{(i,\tau)} = \max\{\|y_{t,j}^{(i,\tau)} - y_{t,j-1}^{(i,\tau)}\| : j = 1, \dots, p(t)\}. \tag{7.184}$$

By (7.13), (7.166), and (7.182), for each $\tau \in \mathcal{E}$, each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{i+1,\tau}$ and each $j \in \{1, \dots, p(t)\}$,

$$\begin{aligned}\|\pi_\tau(z_*) - y_{t,j}^{(i,\tau)}\| &\le \|\pi_\tau(z_*) - P_{\tau,t_j}(y_{t,j-1}^{(i,\tau)})\| + \|P_{\tau,t_j}(y_{t,j-1}^{(i,\tau)}) - y_{t,j}^{(i,\tau)}\| \\ &\le \|\pi_\tau(z_*) - y_{t,j-1}^{(i,\tau)}\| + \delta. \end{aligned}\tag{7.185}$$

We prove the following auxiliary result.

Lemma 7.8. *Assume that an integer $k \ge 0$ satisfies*

$$\|B_2(x_k - z_*)\| \le 2MM_1^{1/2}, \tag{7.186}$$

$$\lambda_{k+1} > \epsilon_0. \tag{7.187}$$

Then

$$\|B_2(x_{k+1} - z_*)\|^2 \le \|B_2(x_k - z)\|^2 - 16^{-1}\Delta^2\gamma^2.$$

Proof. By (7.20), (7.43), and (7.186),

$$\|x_k - z_*\| \le \|B_2(x_k - z_*)\|M_2^{-1/2} \le 2M(M_1M_2^{-1})^{1/2}. \tag{7.188}$$

In view of (7.27), (7.181), and (7.185), for each $\tau \in \mathcal{E}$, each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1,\tau}$ and each $j \in \{0, 1, \dots, p(t)\}$,

$$\begin{aligned}\|\pi_\tau(z_*) - y_{t,j}^{(k,\tau)}\| &\le \|\pi_\tau(z_*) - \pi_\tau(x_k)\| + j\delta \\ &\le \|\pi_\tau(z_*) - \pi_\tau(x_k)\| + \bar{q}\delta, \end{aligned}\tag{7.189}$$

$$\|\pi_\tau(z_*) - y_{t,j}^{(k,\tau)}\| \le \|z_* - x_k\| + \bar{q}\delta \tag{7.190}$$

and in view of (7.166), (7.168), and (7.188),

$$\|y_{t,j}^{(k,\tau)}\| \le 2M(M_1M_2^{-1})^{1/2} + 1 + \|z_*\| \le M + 1 + 2M(M_1M_2^{-1})^{1/2}. \tag{7.191}$$

Relations (7.177) and (7.187) imply that there exists $\xi \in \mathcal{E}$ such that

$$\epsilon_0 < \lambda_{k+1} = \lambda_{k,\xi}. \tag{7.192}$$

By (7.180) and (7.192), there exists $s = (s_1, \dots, s_{p(s)}) \in \Omega_{k+1,\xi}$ such that

$$\epsilon_0 < \lambda_{k,\xi} = \lambda_s^{(k,\xi)}. \tag{7.193}$$

In view of (7.184) and (7.193), there is $j_0 \in \{1, \dots, p(s)\}$ such that

$$\|y_{s,j_0-1}^{(k,\xi)} - y_{s,j_0}^{(k,\xi)}\| = \lambda_s^{(k,\xi)} > \epsilon_0. \tag{7.194}$$

We show that

$$d_{\widehat{X}_\xi}(y_{s,j_0-1}^{(k,\xi)}, C_{\xi,s_{j_0}}) \geq \epsilon_0/3. \tag{7.195}$$

Assume the contrary. Then there exists

$$h \in C_{\xi,s_{j_0}} \tag{7.196}$$

such that

$$\|y_{s,j_0-1}^{(k,\xi)} - h\| < \epsilon_0/3. \tag{7.197}$$

By (7.13), (7.196), and (7.197),

$$\|h - P_{\xi,s_{j_0}}(y_{s,j_0-1}^{(k,\xi)})\| \leq \|h - y_{s,j_0-1}^{(k,\xi)}\| < \epsilon_0/3. \tag{7.198}$$

It follows from (7.168), (7.182), (7.197), and (7.198) that

$$\begin{aligned}\|y_{s,j_0-1}^{(k,\xi)} - y_{s,j_0}^{(k,\xi)}\| &\leq \|y_{s,j_0-1}^{(k,\xi)} - h\| + \|h - P_{\xi,s_{j_0}}(y_{s,j_0-1}^{(k,\xi)})\| \\ &+ \|P_{\xi,s_{j_0}}(y_{s,j_0-1}^{(k,\xi)}) - y_{s,j_0}^{(k,\xi)}\| < \epsilon_0/3 + \epsilon_0/3 + \delta < \epsilon_0.\end{aligned}$$

This contradicts (7.194). The contradiction we have reached proves (7.195).

By (7.191), (7.195) and property (ii),

$$\|\pi_\xi(z_*) - P_{\xi,s_{j_0}}(y_{s,j_0-1}^{(k,\xi)})\| \leq \|\pi_\xi(z_*) - y_{s,j_0-1}^{(k,\xi)}\| - \gamma. \tag{7.199}$$

In view of (7.182) and (7.199),

$$\begin{aligned}&\|\pi_\xi(z_*) - y_{s,j_0}^{(k,\xi)}\| \\ &\leq \|\pi_\xi(z_*) - P_{\xi,s_{j_0}}(y_{s,j_0-1}^{(k,\xi)})\| + \|P_{\xi,s_{j_0}}(y_{s,j_0-1}^{(k,\xi)}) - y_{s,j_0}^{(k,\xi)}\| \\ &\leq \|\pi_\xi(z_*) - y_{s,j_0-1}^{(k,\xi)}\| - \gamma + \delta.\end{aligned} \tag{7.200}$$

It follows from (7.27), (7.181), (7.183), (7.185), and (7.200) that

$$\begin{aligned}
&\|\pi_\xi(z_*) - \pi_\xi(x_k)\| - \|\pi_\xi(z_*) - y_s^{(k,\xi)}\| \\
&= \|\pi_\xi(z_*) - \pi_\xi(y_{s,0}^{(k,\xi)})\| - \|\pi_\xi(z_*) - y_{s,p(s)}^{(k,\xi)}\| \\
&= \sum_{j=1}^{p(s)} (\|\pi_\xi(z_*) - y_{s,j-1}^{(k,\xi)}\| - \|\pi_\xi(z_*) - y_{s,j}^{(k,\xi)}\|) \\
&= \sum \{\|\pi_\xi(z_*) - y_{s,j-1}^{(k,\xi)}\| - \|\pi_\xi(z_*) - y_{s,j}^{(k,\xi)}\| : \ j \in \{1, \dots, p(s)\} \setminus \{j_0\}\} \\
&\quad + \|\pi_\xi(z_*) - y_{s,j_0-1}^{(k,\xi)}\| - \|\pi_\xi(z_*) - y_{s,j_0}^{(k,\xi)}\| \\
&\geq -\delta(p(s) - 1) + \gamma - \delta \geq \gamma - \delta\bar{q}.
\end{aligned} \tag{7.201}$$

Let

$$\tau \in \mathcal{E}, \ t = (t_1, \dots, t_{p(t)} \in \Omega_{k+1,\tau}.$$

In view of (7.183) and (7.189),

$$\begin{aligned}
\|\pi_\tau(z_*) - y_t^{(k,\tau)}\| &= \|\pi_\tau(z_*) - y_{t,p(t)}^{(k,\tau)}\| \\
&\leq \|\pi_\tau(z_*) - \pi_\tau(x_k)\| + \bar{q}\delta.
\end{aligned} \tag{7.202}$$

By (7.28), (7.202) and convexity of the norm,

$$\begin{aligned}
&\|\pi_\tau(z_*) - \sum_{t \in \Omega_{k+1,\tau}} w_{k+1,\tau}(t) y_t^{(k,\tau)}\| \\
&\leq \sum_{t \in \Omega_{k+1,\tau}} w_{k+1,\tau}(t) \|\pi_\tau(z_*) - y_t^{(k,\tau)}\| \\
&\leq \|\pi_\tau(z_*) - \pi_\tau(x_k)\| + \bar{q}\delta.
\end{aligned} \tag{7.203}$$

It follows from (7.28), (7.29), (7.201), and (7.202) that

$$\begin{aligned}
&\|\pi_\xi(z_*) - \sum_{t \in \Omega_{k+1,\xi}} w_{k+1,\xi}(t) y_t^{(k,\xi)}\| \leq \sum_{t \in \Omega_{k+1,\xi}} w_{k+1,\xi}(t) \|\pi_\xi(z_*) - y_t^{(k,\xi)}\| \\
&= \sum \{w_{k+1,\xi}(t) \|\pi_\xi(z_*) - y_t^{(k,\xi)}\| : \ t \in \Omega_{k+1,\xi} \setminus \{s\}\} \\
&\quad + w_{k+1,\xi}(s) \|\pi_\xi(z_*) - y_s^{(k,\xi)}\| \\
&\leq (\|\pi_\xi(z_*) - \pi_\xi(x_k)\| + \bar{q}\delta) \sum \{w_{k+1,\xi}(t) : \ t \in \Omega_{k+1,\xi} \setminus \{s\}\} \\
&\quad + w_{k+1,\xi}(s)(\|\pi_\xi(z_*) - \pi_\xi(x_k)\| - \gamma + \delta\bar{q}) \\
&\leq \|\pi_\xi(z_*) - \pi_\xi(x_k)\| + \delta\bar{q} - \Delta\gamma.
\end{aligned} \tag{7.204}$$

Relations (7.179) and (7.203) imply that for each $\tau \in \mathcal{E}$,

$$\|\pi_\tau(z_*) - y_{k,\tau}\| \le \|\pi_\tau(z_*) - \sum_{t \in \Omega_{k+1,\tau}} w_{k+1,\tau}(t) y_t^{(k,\tau)}\|$$
$$+\|\sum_{t \in \Omega_{k+1,\tau}} w_{k+1,\tau}(t) y_t^{(k,\tau)} - y_{k,\tau}\|$$
$$\le \|\pi_\tau(z_*) - \pi_\tau(x_k)\| + \delta(\bar{q}+1). \tag{7.205}$$

In view of (7.168), (7.179), and (7.204),

$$\|\pi_\xi(z_*) - y_{k,\xi}\| \le \|\pi_\xi(z_*) - \sum_{t \in \Omega_{k+1,\xi}} w_{k+1,\xi}(t) y_t^{(k,\xi)}\|$$
$$+\|\sum_{t \in \Omega_{k+1,\xi}} w_{k+1,\xi}(t) y_t^{(k,\xi)} - y_{k,\xi}\|$$
$$\le \|\pi_\xi(z_*) - \pi_\xi(x_k)\| + \delta(\bar{q}+1) - \Delta\gamma$$
$$\le \|\pi_\xi(z_*) - \pi_{xi}(x_k)\| - 2^{-1}\Delta\gamma. \tag{7.206}$$

By (7.205), (7.168), and (7.188), for each $\tau \in \mathcal{E}$,

$$\|\pi_\tau(z_*) - y_{k,\tau}\|^2 \le \|\pi_\tau(z_*) - \pi_\tau(x_k)\|^2$$
$$+2\delta(\bar{q}+1)(\|\pi_\tau(z_*) - \pi_\tau(x_k)\|^2 + \delta(\bar{q}+1))$$
$$\le \|\pi_\tau(z_*) - \pi_\tau(x_k)\|^2 + 2\delta(\bar{q}+1)(2M(M_1 M_2^{-1})^{1/2} + 1). \tag{7.207}$$

Relation (7.206) implies that

$$\|\pi_\xi(z_*) - y_{k,\xi}\|^2 \le (\|\pi_\xi(z_*) - \pi_\xi(x_k)\| - 2^{-1}\Delta\gamma)^2$$
$$\le (\|\pi_\xi(z_*) - \pi_\xi(x_k)\| + 2^{-1}\Delta\gamma)(\|\pi_\xi(z_*) - \pi_\xi(x_k)\| - 2^{-1}\Delta\gamma)$$
$$= \|\pi_\xi(z_*) - \pi_\xi(x_k)\|^2 - 4^{-1}\Delta^2\gamma^2. \tag{7.208}$$

It follows from (7.208), (7.207), and (7.168) that

$$\sum_{\tau \in \mathcal{E}} \|\pi_\tau(z_*) - y_{k,\tau}\|^2 \le \sum_{\tau \in \mathcal{E}} \|\pi_\tau(z_*) - \pi_\tau(x_k)\|^2$$
$$+2\delta(\bar{q}+1)(2M(M_1 M_2^{-1})^{1/2} + 1)\mathrm{Card}(\mathcal{E}) - 4^{-1}\Delta^2\gamma^2$$
$$\le \sum_{\tau \in \Omega_{k+1}} \|\pi_\tau(z_*) - \pi_\tau(x_k)\|^2 - 8^{-1}\Delta^2\gamma^2. \tag{7.209}$$

By Lemmas 7.5 and 7.6 and (7.209),

$$\|B_2(z_* - B_1(\sum_{\tau\in\mathcal{E}} y_{k,\tau}))\|^2 \le \sum_{\tau\in\mathcal{E}} \|\pi_\tau(z_*) - y_{k,\tau}\|^2$$
$$\le \sum_{\tau\in\mathcal{E}} \|\pi_\tau(z_*) - \pi_\tau(x_k)\|^2 - 8^{-1}\Delta^2\gamma^2$$
$$= \|B_2(z_* - x_k)\|^2 - 8^{-1}\Delta^2\gamma^2. \tag{7.210}$$

In view of (7.210),

$$\|B_2(z_* - B_1(\sum_{\tau\in\mathcal{E}} y_{k,\tau}))\| \le \|B_2(z_* - x_k)\|. \tag{7.211}$$

It follows from (7.20), (7.43), (7.168), (7.176), (7.186), and (7.211) that

$$\|B_2(z_* - x_{k+1})\|^2 - \|B_2(z_* - B_1(\sum_{\tau\in\mathcal{E}} y_{k,\tau}))\|^2$$
$$= (\|B_2(z_* - x_{k+1})\| - \|B_2(z_* - B_1(\sum_{\tau\in\mathcal{E}} y_{k,\tau}))\|)$$
$$\times(\|B_2(z_* - x_{k+1})\| + \|B_2(z_* - B_1(\sum_{\tau\in\mathcal{E}} y_{k,\tau}))\|)$$
$$\le \delta M_1^{1/2}(2\|B_2(z_* - B_1(\sum_{\tau\in\mathcal{E}} y_{k,\tau}))\| + \delta M_1^{1/2})$$
$$\le \delta M_1^{1/2}(2\|B_2\|(z_* - x_k)\| + \delta M_1^{1/2}) \le \delta M_1^{1/2}(4MM_1^{1/2} + 1). \tag{7.212}$$

By (7.168), (7.210), and (7.212),

$$\|B_2(z_* - x_{k+1})\|^2 \le \|B_2(z_* - B_1(\sum_{\tau\in\mathcal{E}} y_{k,\tau}))\|^2 + \delta M_1^{1/2}(4MM_1^{1/2} + 1)$$
$$\le \|B_2(z_* - x_k)\|^2 - 8^{-1}\Delta^2\gamma^2 + \delta M_1^{1/2}(4MM_1^{1/2} + 1)$$
$$\le \|B_2(z_* - x_k)\|^2 - 16^{-1}\Delta^2\gamma^2.$$

Lemma 7.8 is proved. □

Assume that q is a natural number such that for each integer $k \in [0, q-1]$,

$$\lambda_{k+1} > \epsilon_0. \tag{7.213}$$

By (7.173), (7.213) and Lemma 7.8 applied by induction,

$$\|B_2(x_k - z_*)\| \le 2MM_1^{1/2},\ k = 0, \dots, q \tag{7.214}$$

and for all $k = 0, \dots, q-1$,

$$\|B_2(z_* - x_{k+1})\|^2 \le \|B_2(z_* - x_k)\|^2 - 16^{-1}\Delta^2\gamma^{\cdot} \tag{7.215}$$

In view of (7.169), (7.214), and (7.215),

$$4M^2M_1 \ge \|B_2(z_* - x_0)\|^2 \ge \|B_2(z_* - x_0)\|^2 - \|B_2(z_* - x_q)\|^2$$

$$= \sum_{k=0}^{q-1}(\|B_2(z_* - x_k)\|^2 - \|B_2(z_* - x_{k+1})\|^2) \ge 16^{-1}q\Delta^2\gamma^2,$$

$$q \le 64M^2M_1\Delta^{-2}\gamma^{-2} \le n_0.$$

Therefore there exists an integer $q \in [0, n_0]$ such that for each integer k satisfying $0 \le k < q$,

$$\lambda_{k+1} > \epsilon_0,$$

$$\lambda_{q+1} \le \epsilon_0$$

and for all integers $k = 0, \dots, q$,

$$\|B_2(x_k - z_*)\| \le 2MM_1^{1/2},$$

$$\|x_k - z_*\| \le 2M(M_1M_2^{-1})^{1/2}, \ \|x_k\| \le 2M(M_1M_2^{-1})^{1/2} + M.$$

Assume that an integer $q \ge 0$ satisfies

$$\|x_q\| \le 2M(M_1M_2^{-1})^{1/2} + M, \tag{7.216}$$

$$\lambda_{q+1} \le \epsilon_0. \tag{7.217}$$

By (7.177), (7.180), (7.184), and (7.217), for each $\tau \in \mathcal{E}$, each

$$t = (t_1, \dots, t_{p(t)}) \in \Omega_{q+1,\tau}$$

and each $j = 1, \dots, p(t)$,

$$\lambda_{q,\tau} \le \epsilon_0,$$

$$\lambda_t^{(q,\tau)} \le \epsilon_0,$$

$$\|y_{t,j}^{(q,\tau)} - y_{t,j-1}^{(q,\tau)}\| \le \epsilon_0. \tag{7.218}$$

In view of (7.27), (7.167), (7.181), (7.216), and (7.218), for each $\tau \in \mathcal{E}$, each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{q+1,\tau}$ and each $j = 0, 1, \dots, p(t)$,

$$\|\pi_\tau(x_q) - y_{t,j}^{(q,\tau)}\| \le \epsilon_0 j \le \epsilon_0 \bar{q}, \tag{7.219}$$

$$\|y_{t,j}^{(q,\tau)}\| \le 2M(M_1M_2^{-1})^{1/2} + M + 1. \tag{7.220}$$

Assume that

$$\tau \in \mathcal{E},\ t = (t_1, \dots, t_{p(t)}) \in \Omega_{q+1,\tau},\ j \in \{1, \dots, p(t)\}.$$

We show that

$$d_{\widehat{X}_\tau}(y_{t,j-1}^{(q,\tau)}, C_{\tau,t_j}) \le \epsilon_1. \tag{7.221}$$

Assume the contrary. Then

$$d_{\widehat{X}_\tau}(y_{t,j-1}^{(q,\tau)}, C_{\tau,t_j}) > \epsilon_1$$

and together with (7.220) and property (i) this implies that

$$\|\pi_\tau(z_*) - P_{\tau,t_j}(y_{t,j-1}^{(q,\tau)})\| \le \|\pi_\tau(z_*) - y_{t,j-1}^{(q,\tau)}\| - 2\epsilon_0. \tag{7.222}$$

By (7.182) and (7.222),

$$\begin{aligned}
&\|\pi_\tau(z_*) - y_{t,j}^{(q,\tau)}\| \\
&\le \|\pi_\tau(z_*) - P_{\tau,t_j}(y_{t,j-1}^{(q,\tau)})\| + \|P_{\tau,t_j}(y_{t,j-1}^{(q,\tau)}) - y_{t,j}^{(q,\tau)}\| \\
&\le \|\pi_\tau(z_*) - y_{t,j-1}^{(q,\tau)}\| - 2\epsilon_0 + \delta
\end{aligned}$$

and in view of (7.218),

$$\begin{aligned}
2\epsilon_0 - \delta &\le \|\pi_\tau(z_*) - y_{t,j-1}^{(q,\tau)}\| - \|\pi_\tau(z_*) - y_{t,j}^{(q,\tau)}\| \\
&\le \|y_{t,j}^{(q,\tau)} - y_{t,j-1}^{(q,\tau)}\| \le \epsilon_0.
\end{aligned}$$

This contradicts (7.168). The contradiction we have reached proves (7.221).

By (7.27), (7.167), (7.219), and (7.221),

$$\begin{aligned}
d_{\widehat{X}_\tau}(\pi_\tau(x_q), C_{\tau,t_j}) &\le \|\pi_\tau(x_q) - y_{t,j-1}^{(q,\tau)}\| + d_{\widehat{X}_\tau}(y_{t,j-1}^{(q,\tau)}, C_{\tau,t_j}) \\
&\le \epsilon_0(j-1) + \epsilon_1 \le \epsilon_1 \bar{q}
\end{aligned}$$

and

$$d(x_q, C_{t_j}) \le \epsilon$$

for each $\tau \in \mathcal{E}$ each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{q+1,\tau}$ and each $j = 1, \dots, p(t)$. Together with (7.10) and (7.26) this implies that $d(x_q, C_s) \le \epsilon$ for all $s = 1, \dots, m$. Theorem 7.4 is proved. □

Chapter 8
Proximal Point Algorithm

In a Hilbert space, we study the convergence of an iterative proximal point method to a common zero of a finite family of maximal monotone operators under the presence of computational errors. Most results known in the literature establish the convergence of proximal point methods, when computational errors are summable. In this chapter, the convergence of the method is established for nonsummable computational errors. We show that the proximal point method generates a good approximate solution, if the sequence of computational errors is bounded from above by a constant. Moreover, for a known computational error, we find out what an approximate solution can be obtained and how many iterates one needs for this.

8.1 Preliminaries and Main Results

Let $(X, \langle \cdot, \cdot \rangle)$ be a Hilbert space equipped with an inner product $\langle \cdot, \cdot \rangle$ which induces the norm $\| \cdot \|$.

A multifunction $T : X \to 2^X$ is called a monotone operator if and only if

$$\langle z - z', w - w' \rangle \geq 0 \quad \forall z, z', w, w' \in X$$
$$\text{such that } w \in T(z) \text{ and } w' \in T(z'). \tag{8.1}$$

It is called maximal monotone if, in addition, the graph

$$\{(z, w) \in X \times X : \; w \in T(z)\}$$

is not properly contained in the graph of any other monotone operator $T' : X \to 2^X$. A fundamental problem consists in determining an element z such that $0 \in T(z)$. For example, if T is the subdifferential ∂f of a lower semicontinuous convex function

A. Zaslavski, *Approximate Solutions of Common Fixed-Point Problems*, Springer Optimization and Its Applications 112, DOI 10.1007/978-3-319-33255-0_8

$f : X \to (-\infty, \infty]$, which is not identically infinity, then T is maximal monotone (see [71, 73]), and the relation $0 \in T(z)$ means that z is a minimizer of f.

Let $T : X \to 2^X$ be a maximal monotone operator. The proximal point algorithm generates, for any given sequence of positive real numbers and any starting point in the space, a sequence of points and the goal is to show the convergence of this sequence. Note that in a general infinite-dimensional Hilbert space this convergence is usually weak. The proximal algorithm for solving the inclusion $0 \in T(z)$ is based on the fact established by Minty [70], who showed that, for each $z \in X$ and each $c > 0$, there is a unique $u \in X$ such that

$$z \in (I + cT)(u),$$

where $I : X \to X$ is the identity operator ($Ix = x$ for all $x \in X$).

The operator

$$P_{c,T} := (I + cT)^{-1} \tag{8.2}$$

is therefore single-valued from all of X onto X (where c is any positive number). It is also nonexpansive:

$$\|P_{c,T}(z) - P_{c,T}(z')\| \le \|z - z'\| \text{ for all } z, z' \in X \tag{8.3}$$

and

$$P_{c,T}(z) = z \text{ if and only if } 0 \in T(z). \tag{8.4}$$

Following the terminology of Moreau [73] $P_{c,T}$ is called the proximal mapping associated with cT.

The proximal point algorithm generates, for any given sequence $\{c_k\}_{k=0}^{\infty}$ of positive real numbers and any starting point $z^0 \in X$, a sequence $\{z^k\}_{k=0}^{\infty} \subset X$, where

$$z^{k+1} := P_{c_k,T}(z^k),\ k = 0, 1, \ldots$$

It is not difficult to see that the

$$\text{graph}(T) := \{(x, w) \in X \times X :\ w \in T(x)\}$$

is closed in the norm topology of $X \times X$.

Set

$$F(T) = \{z \in X :\ 0 \in T(z)\}. \tag{8.5}$$

Usually algorithms considering in the literature generate sequences which converge weakly to an element of $F(T)$. In this chapter, for a given $\epsilon > 0$, we are interested to find a point x for which there is $y \in T(x)$ such that $\|y\| \le \epsilon$. This point x is considered as an ϵ-approximate solution.

For every point $x \in X$ and every nonempty set $A \subset X$ define

$$d(x, A) := \inf\{\|x - y\| : \ y \in A\}.$$

For every point $x \in X$ and every positive number r put

$$B(x, r) = \{y \in X : \ \|x - y\| \le r\}.$$

We denote by $\mathrm{Card}(A)$ the cardinality of the set A.

We apply the proximal point algorithm in order to obtain a good approximation of a point which is a common zero of a finite family of maximal monotone operators and a common fixed point of a finite family of quasi-nonexpansive operators.

Let $\mathcal{L}_1$ be a finite set of maximal monotone operators $T : X \to 2^X$ and $\mathcal{L}_2$ be a finite set of mappings $T : X \to X$. We suppose that the set $\mathcal{L}_1 \cup \mathcal{L}_2$ is nonempty. (Note that one of the sets $\mathcal{L}_1$ or $\mathcal{L}_2$ may be empty.)

Let $\bar{c} \in (0, 1]$ and let $\bar{c} = 1$, if $\mathcal{L}_2 = \emptyset$.

We suppose that

$$F(T) \neq \emptyset \text{ for any } T \in \mathcal{L}_1 \tag{8.6}$$

and that for every mapping $T \in \mathcal{L}_2$,

$$\mathrm{Fix}(T) := \{z \in X : \ T(z) = z\} \neq \emptyset, \tag{8.7}$$

$$\|z - x\|^2 \ge \|z - T(x)\|^2 + \bar{c}\|x - T(x)\|^2$$
$$\text{for all } x \in X \text{ and all } z \in \mathrm{Fix}(T). \tag{8.8}$$

Let $\bar{\lambda} > 0$ and let $\bar{\lambda} = \infty$ and $\bar{\lambda}^{-1} = 0$, if $\mathcal{L}_1 = \emptyset$. Let a natural number

$$l \ge \mathrm{Card}(\mathcal{L}_1 \cup \mathcal{L}_2). \tag{8.9}$$

Denote by $\mathcal{R}$ the set of all mappings

$$S : \{0, 1, 2, \dots\} \to \mathcal{L}_2 \cup \{P_{c,T} : \ T \in \mathcal{L}_1, \ c \in [\bar{\lambda}, \infty)\}$$

such that the following properties hold:

(P1) for every nonnegative integer p and every mapping $T \in \mathcal{L}_2$ there exists an integer $i \in \{p, \dots, p + l - 1\}$ satisfying $S(i) = T$;

(P2) for every nonnegative integer p and every monotone operator $T \in \mathcal{L}_1$ there exist an integer $i \in \{p, \dots, p+l-1\}$ and a number $c \ge \bar{\lambda}$ satisfying that $S(i) = P_{c,T}$.

Suppose that

$$F := (\cap_{T \in \mathcal{L}_1} F(T)) \cap (\cap_{Q \in \mathcal{L}_2} \mathrm{Fix}(Q)) \neq \emptyset. \tag{8.10}$$

Let $\epsilon > 0$. For every monotone operator $T \in \mathcal{L}_1$ define

$$F_\epsilon(T) = \{x \in X : \ T(x) \cap B(0, \epsilon) \neq \emptyset\} \tag{8.11}$$

and for every mapping $T \in \mathcal{L}_2$ set

$$\text{Fix}_\epsilon(T) = \{x \in X : \ \|T(x) - x\| \leq \epsilon\}. \tag{8.12}$$

Define

$$F_\epsilon = (\cap_{T \in \mathcal{L}_1} F_\epsilon(T)) \cap (\cap_{Q \in \mathcal{L}_2} \text{Fix}_\epsilon(Q)), \tag{8.13}$$

$$\tilde{F}_\epsilon = (\cap_{T \in \mathcal{L}_1} \{x \in X : \ d(x, F_\epsilon(T)) \leq \epsilon\})$$

$$\cap(\cap_{Q \in \mathcal{L}_2} \{x \in X : \ d(x, \text{Fix}_\epsilon(Q)) \leq \epsilon\}). \tag{8.14}$$

We are interested to find solutions of the inclusion $x \in F$. In order to meet this goal we apply algorithms generated by mappings $S \in \mathcal{R}$. More precisely, we associate with every mapping $S \in \mathcal{R}$ the algorithm which generates, for every starting point $x_0 \in X$, a sequence of points $\{x_k\}_{k=0}^\infty \subset X$ such that

$$x_{k+1} := [S(k)](x_k), \ k = 0, 1, \ldots .$$

According to the results known in the literature, this sequence should converge weakly to a point of the set F. In this chapter, we study the behavior of the sequences generated by mappings $S \in \mathcal{R}$ taking into account computational errors which are always present in practice. Namely, in practice the algorithm associate with a mapping $S \in \mathcal{R}$ generates a sequence of points $\{x_k\}_{k=0}^\infty$ such that for every nonnegative integer k the inequality

$$\|x_{k+1} - [S(k)](x_k)\| \leq \delta$$

holds with a positive constant δ which depends only on our computer system. Surely, in this situation one cannot expect that the sequence $\{x_k\}_{k=0}^\infty$ converges to the set F. The goal of this chapter is to understand what subset of X attracts all sequences $\{x_k\}_{k=0}^\infty$ generated by algorithms associated with mappings $S \in \mathcal{R}$. The main result of this chapter (Theorem 8.1 stated below) shows that this subset of X is the set $\tilde{F}_\epsilon$ with some $\epsilon > 0$ depending on δ (see (8.16)).

Our goal is also, for a given $\epsilon > 0$, to find a point $x \in \tilde{F}_\epsilon$. This point x is considered as an ϵ-approximate solution of our inclusion associated with the family of operators $\mathcal{L}_1 \cup \mathcal{L}_2$. We will prove the following result (Theorem 8.1), which shows that an ϵ-approximate solution can be obtained after $l(n_0 - 1)$ iterations of the algorithm associated with $S \in \mathcal{R}$ and under the presence of computational errors bounded from above by a constant δ, where δ and n_0 are constants depending on ϵ (see (8.16) and (8.17)).

Theorem 8.1. *Let* $M > 0$, $\epsilon \in (0, 1]$ *be such that*

$$B(0, M) \cap F \neq \emptyset, \tag{8.15}$$

a positive number δ *satisfy*

$$\delta \leq 2^{-1}(2M + 2)^{-1}(l + 1)^{-1}\bar{c}32^{-1}\epsilon^2(\max\{\bar{\lambda}^{-1}, (1 + 2l)\})^{-2} \tag{8.16}$$

and let a natural number n_0 *satisfy*

$$n_0 > 128M^2\bar{c}^{-1}\epsilon^{-2}(\max\{\bar{\lambda}^{-1},\ (1 + 2l)\})^2. \tag{8.17}$$

Assume that

$$S \in \mathcal{R},\ \{x_k\}_{k=0}^{\infty} \subset X,\ \|x_0\| \leq M, \tag{8.18}$$

$$\|x_{k+1} - [S(k)](x_k)\| \leq \delta,\ k = 0, 1, \ldots. \tag{8.19}$$

Then there exists an integer $q \in [0, n_0 - 1]$ *such that*

$$||x_k|| \leq 3M + 1 \textit{ for all integers } k = 0, \ldots, (q + 1)l$$

and that for each integer $k \in [ql, \ldots, (q + 1)l - 1]$,

$$\|x_k - x_{k+1}\| \leq 2^{-1}\epsilon(\max\{\bar{\lambda}^{-1},\ (1 + 2l)\})^{-1}. \tag{8.20}$$

Moreover, if an integer $q \geq 0$ *is such that for each integer* $k \in [ql, \ldots, (q + 1)l - 1]$ *(8.20) holds and that* $\|x_k\| \leq 3M + 1$, *then for each pair* $i, j \in [ql, \ldots, (q + 1)l]$,

$$\|x_i - x_j\| \leq 4^{-1}\epsilon$$

and for each integer $i \in [ql, (q + 1)l]$,

$$x_i \in \tilde{F}_\epsilon.$$

Theorem 8.1 is proved in Sect. 8.3. Note that in Theorem 8.1 δ is the computational error made by our computer system, we obtain a point of the set $\tilde{F}_\epsilon$ and in order to obtain this point we need $n_0 l$ iterations. It is not difficult to see that $\epsilon = c_1\delta^{1/2}$ and $n_0 = \lfloor c_2\delta^{-1} \rfloor$, where c_1 and c_2 are positive constants depending on M.

The next result is proved in Sect. 8.4.

Theorem 8.2. *Let* $M > 0$, $\epsilon > 0$,

$$B(0, M) \cap F \neq \emptyset.$$

Assume that

$$S \in \mathcal{R},\ \{x_k\}_{k=0}^{\infty} \subset X,\ \|x_0\| \le M,$$
$$x_{k+1} = [S(k)](x_k),\ k = 0, 1, \dots .$$

Then

$$Card(\{i \in \{0, 1, \dots\} :\ x_i \notin \tilde{F}_\epsilon\}) \le 4M^2\bar{c}^{-1}l\epsilon^{-2}(\min\{l^{-1}, \bar{\lambda}\})^{-2}.$$

Theorem 8.3. *Suppose that the space X is finite-dimensional, every mapping $T \in \mathcal{L}_2$ is continuous and that $M, \epsilon > 0$. Then there exists a natural number n_0 such that for each $S \in \mathcal{R}$ and each $\{x_k\}_{k=0}^{\infty} \subset X$ satisfying*

$$\|x_0\| \le M,$$
$$x_{k+1} = [S(k)](x_k) \textit{ for all integers } k \ge 0,$$

the inequality $d(x_k, F) < \epsilon$ holds for all integers $k \ge n_0$.

Theorem 8.3 is proved in Sect. 8.5.

Theorem 8.4. *Suppose that the space X is finite-dimensional, for every mapping $T \in \mathcal{L}_2$,*

$$\|T(y_1) - T(y_2)\| \le \|y_1 - y_2\| \textit{ for all } y_, y_2 \in X \tag{8.21}$$

and that $M, \epsilon > 0$. Then there exist a natural number n_0 and a positive number δ such that for every mapping $S \in \mathcal{R}$ and every sequence $\{x_k\}_{k=0}^{\infty} \subset X$ which satisfies

$$\|x_0\| \le M,$$
$$\|x_{k+1} - [S(k)](x_k)\| \le \delta,\ k = 0, \dots, n_0 - 1,$$

the inequality $d(x_{n_0}, F) < \epsilon$ is valid.

Theorem 8.4 easily follows from the following result, which is proved in Sect. 8.6.

Theorem 8.5. *Suppose that the space X is finite-dimensional, for all mappings $T \in \mathcal{L}_2$ inequality (8.21) is valid, $M, \epsilon_0 > 0$, let a natural number n_0 be as guaranteed by Theorem 8.3 with $\epsilon = \epsilon_0/2$ and let $\delta = \epsilon_0(2n_0)^{-1}$. Then for every mapping $S \in \mathcal{R}$ and every sequence $\{x_k\}_{k=0}^{\infty} \subset X$ which satisfies*

$$\|x_0\| \le M,$$
$$\|x_{k+1} - [S(k)](x_k)\| \le \delta,\ k =, 0 \dots, n_0 - 1$$

the inequality $d(x_{n_0}, F) < \epsilon_0$ is valid.

Theorem 8.4 easily implies the following result.

Theorem 8.6. *Suppose that the space X is finite-dimensional and that for all mappings $T \in \mathcal{L}_2$ inequality (8.21) is valid. Let $M, \epsilon > 0$ and let a natural number n_0 and $\delta > 0$ be as guaranteed by Theorem 8.4. Assume that $S \in \mathcal{R}$ and that a sequence $\{x_k\}_{k=0}^{\infty} \subset B(0, M)$ satisfies*

$$\|x_{k+1} - [S(k)](x_k)\| \leq \delta, \ k = 0, 1, \ldots.$$

Then $d(x_k, F) \leq \epsilon$ for all integers $k \geq n_0$.

Theorem 8.6 easily implies the following result.

Theorem 8.7. *Suppose that the space X is finite-dimensional and that all mappings $T \in \mathcal{L}_2$ are continuous and satisfy inequality (8.21). Let $M > 0$, $\{\delta_k\}_{k=0}^{\infty}$ be a sequence of positive numbers such that $\lim_{k\to\infty} \delta_k = 0$ and let $\epsilon > 0$. Then there exists a natural number n_ϵ such that for every mapping $S \in \mathcal{R}$ and every sequence $\{x_k\}_{k=0}^{\infty} \subset B(0, M)$ which satisfies*

$$\|x_{k+1} - [S(k)](x_k)\| \leq \delta_k$$

for all integers $k \geq 0$ the inequality $d(x_k, F) \leq \epsilon$ is valid for all integers $k \geq n_\epsilon$.

Theorem 8.8. *Suppose that the space X is finite-dimensional, every mapping $T \in \mathcal{L}_2$ satisfies inequality (8.21) and that the set F is bounded. Let $M, \epsilon > 0$. Then there exist a positive number δ_* and a natural number n_0 such that for every mapping $S \in \mathcal{R}$ and every sequence $\{x_k\}_{k=0}^{\infty} \subset X$ which satisfy*

$$\|x_0\| \leq M,$$

$$\|x_{k+1} - [S(k)](x_k)\| \leq \delta_*, \ k = 0, 1, \ldots,$$

the inequality $d(x_k, F) \leq \epsilon$ is valid for all integers $k \geq n_0$.

Theorem 8.8 is proved in Sect. 8.7.

Theorem 8.9. *Suppose that the space X is finite-dimensional, every mapping $T \in \mathcal{L}_2$ satisfies inequality (8.21) and that the set F be bounded and let $M > 0$. Then there exists a positive number δ such that the following assertion holds.*

Assume that $\{\delta_k\}_{k=0}^{\infty} \subset (0, \delta]$ satisfies

$$\lim_{k\to\infty} \delta_k = 0$$

and that $\epsilon > 0$. Then there exists a natural number n_ϵ such that for every mapping $S \in \mathcal{R}$ and every sequence $\{x_k\}_{k=0}^{\infty} \subset X$ which satisfy

$$\|x_0\| \leq M,$$

$$\|x_{k+1} - [S(k)](x_k)\| \leq \delta_k$$

for all integers $k \geq 0$ *the inequality* $d(x_k, F) \leq \epsilon$ *is valid for all integers* $k \geq n_\epsilon$.

Theorem 8.9 is proved in Sect. 8.8.

In the following results we study the asymptotic behavior of algorithms associated with $S \in \mathcal{R}$ when X is a general Hilbert space.

The next result follows easily from Theorem 8.1.

Theorem 8.10. *Let* $M > 0$, $\epsilon \in (0, 1]$ *be such that*

$$B(0, M) \cap F \neq \emptyset,$$

a positive number δ *satisfy (8.16) and let a natural* n_0 *satisfy (8.17). Assume that*

$$S \in \mathcal{R},\ \{x_k\}_{k=0}^{\infty} \subset B(0, M),$$
$$\|x_{k+1} - [S(k)](x_k)\| \leq \delta,\ k = 0, 1, \ldots.$$

Then for each integer $p \geq 0$ *there is an integer* $q \in [0, n_0 - 1]$ *such that for each integer* $i \in [p + ql, p + (q + 1)l]$,

$$x_i \in \tilde{F}_\epsilon.$$

Theorem 8.10 implies the following result.

Theorem 8.11. *Let* $M > 0$, $\{\delta_k\}_{k=0}^{\infty}$ *be a sequence of positive numbers such that* $\lim_{k\to\infty} \delta_k = 0$ *and let* $\epsilon > 0$. *Then there exists an integer* $n_\epsilon \geq 1$ *such that for every mapping* $S \in \mathcal{R}$ *and every sequence* $\{x_k\}_{k=0}^{\infty} \subset B(0, M)$ *which satisfy*

$$\|x_{k+1} - [S(k)](x_k)\| \leq \delta_k,\ k = 0, 1, \ldots$$

and each integer $p \geq n_\epsilon$ *there exists an integer* $q \in [0, n_\epsilon - 1]$ *such that for every integer* $i \in [p + ql, p + (q + 1)l]$,

$$x_i \in \tilde{F}_\epsilon.$$

The next result follows easily from Theorem 8.1 applied by induction.

Proposition 8.12. *Let* $\epsilon \in (0, 1]$, *the set* $\tilde{F}_\epsilon$ *be bounded,*

$$\sup\{\|z\| :\ z \in \tilde{F}_\epsilon\} + 1 < M,$$

a positive number δ *satisfy (8.16) and let a natural* n_0 *satisfy (8.17). Assume that*

$$S \in \mathcal{R},\ \{x_k\}_{k=0}^{\infty} \subset X,\ \|x_0\| \leq M,$$
$$\|x_{k+1} - [S(k)](x_k)\| \leq \delta,\ k = 0, 1, \ldots.$$

Then there is a strictly increasing sequence of integers $\{j_p\}_{p=0}^{\infty}$ *such that*

$$0 \le j_0 \le l(n_0 - 1),$$
$$l \le j_{p+1} - j_p \le n_0 l \text{ for all integers } p \ge 0,$$

for any integer $p \ge 0$ *and for all integers* $i \in [j_p, j_p + l]$,

$$x_i \in \tilde{F}_\epsilon$$

and that

$$\|x_k\| \le 3M + 1 \text{ for all integers } k \ge 0.$$

Theorem 8.1 and Proposition 8.12 imply the following result.

Theorem 8.13. *Let* $\epsilon \in (0, 1]$, *the set* $\tilde{F}_\epsilon$ *be bounded,*

$$\sup\{\|z\| : z \in \tilde{F}_\epsilon\} + 1 < M_0, \tag{8.22}$$

$M = 3M_0 + 1$, *a positive number* δ *satisfy (8.16) and let a natural* n_0 *satisfy (8.17). Assume that*

$$S \in \mathcal{R}, \ \{x_k\}_{k=0}^{\infty} \subset X, \ \|x_0\| \le M_0,$$
$$\|x_{k+1} - [S(k)](x_k)\| \le \delta, \ k = 0, 1, \ldots.$$

Then

$$\|x_k\| \le M \text{ for all integers } k \ge 0$$

and for every nonnegative integer p there exists an integer $q \in [0, n_0 - 1]$ *such that* $x_i \in \tilde{F}_\epsilon$ *for all integers* $i = p + ql, \ldots, p + (q + 1)l$.

Theorems 8.1 and 8.13 imply the following result.

Theorem 8.14. *Let* $\epsilon \in (0, 1]$, *the set* $\tilde{F}_\epsilon$ *be bounded, inequality (8.22) hold,* $M = 3M_0 + 1$, *a positive number* δ *satisfy (8.16) and let* $\{\delta_k\}_{k=0}^{\infty} \subset (0, \delta]$ *be a sequence of positive numbers such that* $\lim_{k\to\infty} \delta_k = 0$. *Then for each* $\gamma \in (0, \epsilon]$ *there is natural number* n_γ *such that for each* $S \in \mathcal{R}$, *each sequence* $\{x_k\}_{k=0}^{\infty} \subset X$ *satisfying*

$$\|x_0\| \le M_0,$$
$$\|x_{k+1} - [S(k)](x_k)\| \le \delta_k$$

for all integers $k \ge 0$ *and for each integer* $p \ge n_\gamma$ *there is an integer* $q \in [0, n_\gamma - 1]$ *such that* $x_i \in \tilde{F}_\gamma$ *for each integer* $i \in [p + ql, p + (q + 1)l]$.

The following theorem is proved in Sect. 8.9.

Theorem 8.15. *Let* $\epsilon_0 \in (0, 1]$*, the set* $\tilde{F}_{\epsilon_0}$ *be bounded, for all* $T \in \mathcal{L}_2$ *(8.21) hold,*

$$\sup\{\|y\| : y \in F_{\epsilon_0}\} + 1 < M_0,$$

$\epsilon \in (0, \epsilon_0]$ *and* $\gamma \in (0, 1)$*. Then there exist an integer* $\hat{n} \geq 1$ *and a positive number* δ *such that for every mapping* $S \in \mathcal{R}$ *and every sequence* $\{x_k\}_{k=0}^{\infty} \subset X$ *which satisfy*

$$\|x_0\| \leq M_0,$$
$$\|x_{k+1} - [S(k)](x_k)\| \leq \delta, \ k = 0, 1, \ldots,$$

for every nonnegative integer p and every integer $n \geq \hat{n}$*,*

$$n^{-1} Card(\{i \in \{p, \ldots, p + n - 1\} : x_i \notin \tilde{F}_{\epsilon}\}) < \gamma.$$

The results of this chapter are generalizations of the results of [94] which were obtained under an additional assumption that all operators $T \in \mathcal{L}_2$ are nonexpansive.

8.2 Auxiliary Results

It is easy to see that the following lemma holds.

Lemma 8.16. *Let* $z, x_0, x_1 \in X$*. Then*

$$2^{-1}\|z - x_0\|^2 - 2^{-1}\|z - x_1\|^2 - 2^{-1}\|x_0 - x_1\|^2 = \langle x_0 - x_1, x_1 - z\rangle.$$

Lemma 8.17. *Assume that* $S \in \mathcal{R}$*,*

$$z \in F, \tag{8.23}$$

the integers p, q *satisfy* $0 \leq p < q$*,*

$$\{\epsilon_k\}_{k=p}^{q-1} \subset (0, \infty), \ \{x_k\}_{k=p}^{q} \subset X$$

and that for all integers $k \in \{p, \ldots, q - 1\}$*,*

$$\|x_{k+1} - [S(k)](x_k)\| \leq \epsilon_k. \tag{8.24}$$

Then, for every integer $k \in \{p + 1 \ldots, q\}$ *the following inequality holds:*

$$||z - x_k|| \leq \|z - x_p\| + \sum_{i=p}^{k-1} \epsilon_i.$$

Proof. Let an integer $k \in \{p, \dots, q-1\}$. By (8.3)–(8.5), (8.8), (8.10), (8.23), and (8.24),

$$\|z - x_{k+1}\| \le \|z - [S(k)](x_k)\| + \|S(k)(x_k) - x_{k+1}\| \le \|z - x_k\| + \epsilon_k.$$

This implies the validity of Lemma 8.17. □

Lemma 8.18. *Assume that for every mapping* $T \in \mathcal{L}_2$

$$\|T(x) - T(y)\| \le \|x - y\| \text{ for all } x, y \in X, \tag{8.25}$$

$S \in \mathcal{R}$*, the integers* p, q *satisfy* $0 \le p < q$*,*

$$\{\epsilon_k\}_{k=p}^{q-1} \subset (0, \infty),\ \{x_k\}_{k=p}^{q} \subset X,\ \{y_k\}_{k=p}^{q} \subset X,\ y_p = x_p$$

and that for all integers $k \in \{p, \dots, q-1\}$*,*

$$y_{k+1} = [S(k)](y_k),\ \|x_{k+1} - [S(k)](x_k)\| \le \epsilon_k. \tag{8.26}$$

Then, for every integer $k \in \{p+1 \dots, q\}$ *the following inequality holds:*

$$\|y_k - x_k\| \le \sum_{i=p}^{k-1} \epsilon_i. \tag{8.27}$$

Proof. We prove the lemma by induction. In view of (8.26) and the equality $x_p = y_p$ inequality (8.27) holds for $k = p + 1$.

Assume that an integer j satisfies $p + 1 \le j \le q$, (8.27) holds for all $k = p + 1, \dots, j$ and that $j < q$.

By (8.3), (8.27), (8.25), and (8.26) with $k = j$,

$$\begin{aligned} \|y_{j+1} - x_{j+1}\| &\le \|[S(j)](y_j) - x_{j+1}\| \\ &\le \|[S(j)](y_j) - [S(j)](x_j)\| + \|[S_j](x_j) - x_{j+1}\| \\ &\le \|y_j - x_j\| + \epsilon_j \le \sum_{i=p}^{j-1} \epsilon_i + \epsilon_j = \sum_{i=p}^{j} \epsilon_i \end{aligned}$$

and (8.27) holds for all $k = p + 1, \dots, j + 1$. Therefore we showed by induction that (8.27) holds for all $k = p + 1, \dots, q$. This completes the proof of Lemma 8.18. □

Lemma 8.19. *Let*

$$A \in \mathcal{L}_2 \cup \{P_{c,T} :\ T \in \mathcal{L}_1,\ c \in [\bar{\lambda}, \infty)\},\ x \in X,$$
$$z \in F. \tag{8.28}$$

Then

$$\|z-x\|^2 - \|z-A(x)\|^2 - \bar{c}\|x-A(x)\|^2 \geq 0. \tag{8.29}$$

Proof. There are two cases:

(i) $T \in \mathcal{L}_2$;
(ii) there exist a mapping $T \in \mathcal{L}_1$,a number $c \in [\bar{\lambda}, \infty)$ such that $A = P_{c,T}$.

If (i) holds, then (8.29) follows from (8.8) and (8.28). Assume that (ii) holds. Then by Lemma 8.16,

$$2^{-1}\|z-x\|^2 - 2^{-1}\|z-A(x)\|^2 - 2^{-1}\|x-A(x)\|^2 = \langle x-A(x), A(x)-z\rangle. \tag{8.30}$$

By (ii) and (8.2),

$$A(x) = P_{c,T}(x) \text{ and } x \in (I+cT)(A(x)),$$
$$x - A(x) \in cT(A(x)). \tag{8.31}$$

By (8.1), (8.5), (8.28), (8.30), and (8.31), Eq. (8.29) holds. Lemma 8.19 is proved. □

Lemma 8.20. *Let the space X be finite-dimensional, every mapping $T \in \mathcal{L}_2$ be continuous and $M, \epsilon > 0$. Then there exists a positive number δ such that for every mapping $T \in \mathcal{L}_2$ and every point $x \in B(0, M)$ which satisfies $d(x, Fix_\delta(T)) \leq \delta$, the inequality $d(x, Fix(T)) \leq \epsilon$ is valid.*

Proof. Since the set $\mathcal{L}_2$ is finite it is sufficient to show that for every mapping $T \in \mathcal{L}_2$ there exists a positive number δ such that

$$\text{if } x \in B(0, M) \text{ satisfies } d(x, \mathrm{Fix}_\delta(T)) \leq \delta, \text{ then } d(x, \mathrm{Fix}(T)) \leq \epsilon.$$

Assume the contrary. Then there exist

$$T \in \mathcal{L}_2, \tag{8.32}$$

$$x_k \in B(0, M) \tag{8.33}$$

satisfying

$$d(x_k, \mathrm{Fix}_{1/k}(T)) \leq k^{-1}, k = 0, 1, \ldots \tag{8.34}$$

such that

$$d(x_k, \mathrm{Fix}(T)) > \epsilon,\ k = 0, 1, \ldots. \tag{8.35}$$

By (8.33), extracting a subsequence and re-indexing, we can assume without loss of generality that these exists

$$x := \lim_{k\to\infty} x_k. \tag{8.36}$$

By (8.33), (8.34), (8.36) and continuity of the mapping T,

$$x \in B(0, M) \cap \mathrm{Fix}(T)$$

and for all sufficiently large natural numbers k, $\|x - x_k\| < \epsilon/2$. This contradicts (8.35). The contradiction we have reached proves Lemma 8.20. □

Lemma 8.21. *Let the space X be finite-dimensional and $M, \epsilon > 0$. Then there exists a positive number δ such that for every mapping $T \in \mathcal{L}_1$ and every point $x \in B(0, M) \cap F_\delta(T)$, the inequality $d(x, F(T)) \le \epsilon$ is valid.*

Proof. Since the set $\mathcal{L}_1$ is finite it is sufficient to show that for every mapping $T \in \mathcal{L}_1$ there exists a positive number δ such that

$$\text{if } x \in B(0, M) \cap F_\delta(T), \text{ then } d(x, F(T)) \le \epsilon.$$

Assume the contrary. Then there exist an operator

$$T \in \mathcal{L}_1 \tag{8.37}$$

and

$$x_k \in B(0, M) \cap F_{1/k}(T), \ k = 0, 1, \dots \tag{8.38}$$

such that

$$d(x_k, F(T)) > \epsilon, \ k = 0, 1, \dots. \tag{8.39}$$

By (8.38), for every integer $k \ge 1$, there exists a point

$$y_k \in T(x_k) \cap B(0, k^{-1}). \tag{8.40}$$

In view of (8.36), extracting a subsequence and re-indexing, if necessary we may assume without loss of generality that these exists a point

$$x := \lim_{k\to\infty} x_k. \tag{8.41}$$

Since the graph of the operator T is closed, (8.40) and (8.41) imply the inclusion $x \in F(T)$ and that for all sufficiently large natural numbers k, we have $\|x_k - x\| < \epsilon/2$. This contradicts (8.39). The contradiction we have reached proves Lemma 8.21. □

Lemma 8.22. *Let the space X be finite-dimensional, E_i, $i = 1, \dots, k$ are closed subset of the space X,*

$$E = \cap_{i=1}^{k} E_i \neq \emptyset$$

and let $M, \epsilon > 0$. Then there exists a positive number δ such that for every point $x \in B(0, M)$ satisfying $d(x, E_i) \leq \delta$, $i = 1, \dots, k$ the inequality $d(x, E) \leq \epsilon$ is valid.

Proof. Assume the contrary. Then for every natural number p there exists a point $x_p \in B(0, M)$ which satisfy

$$d(x_p, E_i) \leq p^{-1}, \ i = 1, \dots, k, \dots, d(x_p, E) > \epsilon. \tag{8.42}$$

Extracting a subsequence and re-indexing, if necessary, we may assume without loss of generality that there exists a point

$$x = \lim_{p \to \infty} x_p.$$

It is not difficult to see that $x \in B(0, M)$, $x \in \cap_{i=1}^{k} E_i = E$ and that for all sufficiently large natural numbers p,

$$\|x_p - x\| < \epsilon/2.$$

This contradicts (8.42). The contradiction we have reached completes the proof of Lemma 8.22. □

8.3 Proof of Theorem 8.1

Fix

$$\epsilon_0 = 2^{-1}\epsilon(\max\{\bar{\lambda}^{-1}, (1 + 2l)\})^{-1} \tag{8.43}$$

and a point

$$z \in B(0, M) \cap F. \tag{8.44}$$

Assume that $\tilde{q} \in [0, n_0 - 1]$ is an integer such that for each integer $p \in [0, \tilde{q}]$,

$$\max\{\|x_i - x_{i+1}\| : \ i \in \{pl, \dots, (p + 1)l - 1\}\} > \epsilon_0. \tag{8.45}$$

By (8.18) and (8.44),

$$\|x_0 - z\| \leq 2M. \tag{8.46}$$

Assume that an integer $p \in [0, \tilde{q}]$ and that

$$\|x_{pl} - z\| \leq 2M. \tag{8.47}$$

By (8.16), (8.18), (8.19), (8.44), (8.47), and Lemma 8.17, for each integer $i \in \{1, \dots, l\}$,

$$\|z - x_{pl+i}\| \leq \|z - x_{pl+i-1}\| + \delta, \tag{8.48}$$

$$\|z - x_{pl+i}\| \leq \|z - x_{pl}\| + \delta i \leq 2M + l\delta \leq 2M + 1. \tag{8.49}$$

In view of (8.45) there exists an integer $m \in \{pl, \dots, (p+1)l - 1\}$ such that

$$\|x_m - x_{m+1}\| > \epsilon_0. \tag{8.50}$$

Set

$$u = [S(m)](x_m). \tag{8.51}$$

By (8.19) and (8.51) we have

$$\|x_{m+1} - u\| \leq \delta. \tag{8.52}$$

By (8.18), (8.44), (8.51), Lemma 8.19, the definition of $\mathcal{R}$, properties (P1) and (P2),

$$\bar{c}\|u - x_m\|^2 \leq \|z - x_m\|^2 - \|z - u\|^2. \tag{8.53}$$

In view of (8.16), (8.43), (8.50) and (8.52),

$$\|x_m - u\| \geq \|x_m - x_{m+1}\| - \|x_{m+1} - u\| > \epsilon_0 - \delta > \epsilon_0/2. \tag{8.54}$$

It follows from (8.53) and (8.54) that

$$\|z - u\|^2 \leq \|z - x_m\|^2 - \bar{c}\|u - x_m\|^2 \leq \|z - x_m\|^2 - \bar{c}(\epsilon_0/2)^2. \tag{8.55}$$

In view of (8.3), (8.4), (8.8), (8.16), (8.44), (8.47), (8.49), and (8.51),

$$\|z - u\| \leq \|z - x_m\| \leq 2M + l\delta \leq 2M + 1. \tag{8.56}$$

By (8.16), (8.52), (8.55), and (8.56),

$$\begin{aligned}
\|z - x_{m+1}\|^2 &= \|z - u + u - x_{m+1}\|^2 \\
&\leq \|z - u\|^2 + \|u - x_{m+1}\|^2 + 2\|z - u\|\|u - x_{m+1}\| \\
&\leq \|z - x_m\|^2 - \bar{c}(\epsilon_0^2)/4 + \delta^2 + 2\delta(2M + 1) \\
&\leq \|z - x_m\|^2 - \bar{c}(\epsilon_0^2)/4 + 2\delta(2M + 2).
\end{aligned} \tag{8.57}$$

In view of (8.16) and (8.47)–(8.49), for all integers $k \in \{pl, \dots, (p+1)l - 1\}$,

$$\|z - x_k\|^2 - \|z - x_{k+1}\|^2 \ge \|z - x_k\|^2 - (\|z - x_k\| + \delta)^2$$
$$\ge -\delta^2 - 2\delta\|z - x_k\| \ge -\delta^2 - 2\delta(2M + 1) \ge -2\delta(2M + 2). \tag{8.58}$$

Relations (8.16), (8.43), (8.57), and (8.58) imply that

$$\|z - x_{pl}\|^2 - \|z - x_{(p+1)l}\|^2 = \sum_{k=pl}^{(p+1)l-1} [\|z - x_k\|^2 - \|z - x_{k+1}\|^2]$$
$$\ge -2\delta l(2M + 2) + \bar{c}(\epsilon_0)^2/4 - 2\delta(2M + 2)$$
$$= \bar{c}(\epsilon_0^2)/4 - 2\delta(2M + 2)(l + 1) \ge \bar{c}(\epsilon_0^2)/8. \tag{8.59}$$

It follows from (8.47) and (8.59) that

$$\|z - x_{(p+1)l}\| \le \|z - x_{pl}\| \le 2M.$$

Thus we show that the following property holds:
(P3) If an integer $p \in [0, \tilde{q}]$ satisfies

$$\|x_{pl} - z\| \le 2M, \tag{8.60}$$

then

$$\|x_k - z\| \le 2M + 1 \text{ for all integers } k = pl, \dots, (p+1)l \tag{8.61}$$

and

$$\|z - x_{pl}\|^2 - \|z - x_{(p+1)l}\|^2 \ge \bar{c}\epsilon_0^2/8. \tag{8.62}$$

It follows from (8.46), (8.60)–(8.62), and property (P3) that (8.62) is valid for all integers $p \in [0, \tilde{q}]$. By (8.17), (8.43), (8.46), and (8.62), we have

$$(\bar{c}\epsilon_0^2/8)(\tilde{q} + 1) \le \sum_{p=0}^{\tilde{q}} [\|z - x_{pl}\|^2 - \|z - x_{(p+1)l}\|^2] \le \|z - x_0\|^2 \le 4M^2$$

and

$$\tilde{q} + 1 \le 32M^2\bar{c}^{-1}\epsilon_0^{-2} < n_0.$$

We assumed that $\tilde{q} \in [0, n_0 - 1]$ is an integer such that for each integer $p \in [0, \tilde{q}]$ (8.45) holds and showed that $\tilde{q} + 1 < n_0$.

This implies that there exists an integer $q \in [0, n_0 - 1]$ such that for every integer p satisfying $0 \le p < q$,

$$\max\{\|x_i - x_{i+1}\| : i \in \{pl, \dots, (p+1)l - 1\}\} > \epsilon_0$$
$$= 2^{-1}\epsilon(\max\{\bar{\lambda}^{-1}, (1 + 2l)\})^{-1}, \tag{8.63}$$
$$\max\{\|x_i - x_{i+1}|| : i \in \{ql, \dots, (q+1)l - 1\}\} \le \epsilon_0. \tag{8.64}$$

In view of (8.46), (8.63), and property (P3) with ($\tilde{q} = q - 1$),

$$\|x_{ql} - z\| \le 2M, \tag{8.65}$$
$$\|x_k - z\| \le 2M + 1 \text{ for all } k = 0, \dots, ql. \tag{8.66}$$

Relations (8.43), (8.64), and (8.65) imply that for every integer $i \in \{ql, \dots, (q+1)l\}$, we have

$$\|x_i - z\| \le \|x_i - x_{ql}\| + \|x_{ql} - z\| \le 2M + l\epsilon_0 \le 2M + 1.$$

Combined with (8.44) this implies that

$$\|x_i\| \le 3M + 1, \; i = ql, \dots, (q+1)l.$$

Together with (8.66) and (8.44) this implies that

$$\|x_i\| \le 3M + 1, \; i = 0, \dots, (q+1)l. \tag{8.67}$$

Assume that $q \ge 0$ is an integer such that for each integer $i \in \{ql, \dots, (q+1)l - 1\}$,

$$\|x_i - x_{i+1}\| \le 2^{-1}\epsilon(\max\{\bar{\lambda}^{-1}, (1 + 2l)\})^{-1} = \epsilon_0, \tag{8.68}$$
$$\|x_i\| \le 3M + 1, \; i = ql, \dots, (q+1)l - 1. \tag{8.69}$$

In view of (8.43) and (8.68), for every pair of integers $i, j \in \{ql, \dots, (q+1)l\}$, we have

$$\|x_i - x_j\| \le l\epsilon_0 \le \epsilon/4. \tag{8.70}$$

Let $j \in \{ql, \dots, (q+1)l\}$. Assume that $T \in \mathcal{L}_2$. Property (P1) implies that there exists an integer $i_T \in \{ql, \dots, (q+1)l - 1\}$ such that

$$S(i_T) = T. \tag{8.71}$$

It follows from (8.16), (8.19), (8.43), (8.68), and (8.71) that

$$\|x_{i_T} - T(x_{i_T})\| = \|x_{i_T} - S(i_T)(x_{i_T})\| \le \delta + \|x_{i_T} - x_{i_T+1}\| \le \delta + \epsilon_0 \le \epsilon. \tag{8.72}$$

Relations (8.12), (8.70), and (8.72) imply that

$$d(x_j, \mathrm{Fix}_\epsilon(T)) \le \epsilon \text{ for all } T \in \mathcal{L}_2. \tag{8.73}$$

Assume that $T \in \mathcal{L}_1$. Property (P2) and (8.19) imply that there exist

$$k_T \in \{ql, \dots, (q+1)l - 1\},\ c \ge \bar{\lambda} \tag{8.74}$$

such that

$$S(k_T) = P_{c,T},\ \|x_{k_T+1} - P_{c,T}(x_{k_T})\| \le \delta. \tag{8.75}$$

Set

$$y = P_{c,T}(x_{k_T}). \tag{8.76}$$

In view of (8.2) and (8.76), we have

$$x_{k_T} \in (I + cT)(y),\ x_{k_T} - y \in cT(y),\ c^{-1}(x_{k_T} - y) \in T(y). \tag{8.77}$$

It follows from (8.16), (8.43), (8.68), and (8.74)–(8.76) that

$$\begin{aligned}\|c^{-1}(x_{k_T} - y)\| &\le \bar{\lambda}^{-1}(\|x_{k_T} - x_{k_T+1}\| + \|x_{k_T+1} - y\|)\\ &\le \bar{\lambda}^{-1}(\epsilon_0 + \delta) \le 2\bar{\lambda}^{-1}\epsilon_0 \le \epsilon.\end{aligned}$$

Combined with (8.77) this implies the inclusion $y \in F_\epsilon(T)$. Together with (8.16), (8.70), and (8.74)–(8.76) this implies that

$$d(x_j, F_\epsilon(T)) \le \|x_j - y\| \le \|x_j - x_{k_T+1}\| + \|x_{k_T+1} - y\| \le \epsilon/4 + \delta \le \epsilon,$$
$$d(x_j, F_\epsilon(T)) \le \epsilon$$

for all $T \in \mathcal{L}_1$. This completes the proof of Theorem 8.1. □

8.4 Proof of Theorem 8.2

Put

$$\gamma = \min\{\epsilon l^{-1},\ \epsilon\bar{\lambda}\}. \tag{8.78}$$

There exists a point

$$z \in B(0, M) \cap F. \tag{8.79}$$

By (8.79) and the assumptions of the theorem, we have

$$\|z - x_0\| \le 2M. \tag{8.80}$$

It follows from Lemma 8.19, (8.79), and (8.80) that for all integers $k \ge 0$,

$$\|z - x_k\|^2 - \|z - x_{k+1}\|^2 - \bar{c}\|x_k - x_{k+1}\|^2 \ge 0, \tag{8.81}$$

$$\|z - x_{k+1}\| \le \|z - x_k\|, \tag{8.82}$$

$$\|z - x_k\| \le M. \tag{8.83}$$

Define

$$E = \{i \in \{0, 1, \dots, \} : \ \|x_{i+1} - x_i\| \ge \gamma\}. \tag{8.84}$$

Let n be a natural number. In view of (8.80) and (8.81), we have

$$\begin{aligned}
4M^2 &\ge \|x_0 - z\|^2 \ge \|x_0 - z\|^2 - \|x_n - z\|^2 \\
&= \sum_{i=0}^{n-1}(\|x_i - z\|^2 - \|x_{i+1} - z\|^2) \ge \sum_{i=0}^{n-1} \|x_i - x_{i+1}\|^2 \\
&\ge \bar{c}\gamma^2 \mathrm{Card}(\{i \in \{0, \dots, n-1\} : \ \|x_{i+1} - x_i\| \ge \gamma\}),
\end{aligned}$$

$$\mathrm{Card}(\{i \in \{0, \dots, n-1\} : \ \|x_{i+1} - x_i\| \ge \gamma\}) \le 4M^2\gamma^{-2}\bar{c}^{-1}.$$

Since the relation above holds for every natural number n we conclude that

$$\mathrm{Card}(E) \le 4M^2\gamma^{-2}\bar{c}^{-1}. \tag{8.85}$$

Define

$$E_0 = \{i \in \{0, 1, \dots\} : \ [i, i + l - 1] \cap E \ne \emptyset\}. \tag{8.86}$$

By (8.85) and (8.86), we have

$$\mathrm{Card}(E_0) \le 4M^2\gamma^{-2}l\bar{c}^{-1}. \tag{8.87}$$

Let

$$j \in \{0, 1, \dots\} \setminus E_0. \tag{8.88}$$

In view of (8.86) and (8.88),

$$\{j, \dots, j + l - 1\} \cap E = \emptyset. \tag{8.89}$$

Relations (8.84) and (8.89) imply that

$$\|x_{i+1} - x_i\| \leq \gamma,\ \ i = j, \dots, j + l - 1. \tag{8.90}$$

By (8.78) and (8.90), for every integer $i \in \{l, \dots, j + l\}$, we have

$$\|x_j - x_i\| \leq l\gamma \leq \epsilon. \tag{8.91}$$

Let $T \in \mathcal{L}_2$. It follows from property (P1) that there exists an integer $i \in \{j, \dots, j + l - 1\}$ such that

$$S(i) = T. \tag{8.92}$$

In view of (8.90) and (8.92),

$$\|x_i - T(x_i)\| \leq \gamma. \tag{8.93}$$

By (8.78), (8.91), and (8.93), we have

$$d(x_j, \mathrm{Fix}_\epsilon(T)) \leq \epsilon \text{ for all } T \in \mathcal{L}_2.$$

Let $T \in \mathcal{L}_1$. Property (P2) implies that there exist

$$i \in \{j, \dots, j + l - 1\},\ \ c \geq \bar{\lambda} \tag{8.94}$$

such that

$$S(i) = P_{c,T},\ \ x_{i+1} = P_{c,T}(x_i). \tag{8.95}$$

It follows from (8.2) and (8.95) that

$$\begin{aligned} x_i &\in (I + cT)(x_{i+1}), \\ x_i - x_{i+1} &\in cT(x_{i+1}), \\ c^{-1}(x_i - x_{i+1}) &\in T(x_{i+1}). \end{aligned} \tag{8.96}$$

By (8.78), (8.90), and (8.94), we have

$$\|c^{-1}(x_i - x_{i+1})\| \leq \bar{\lambda}^{-1}\gamma \leq \epsilon. \tag{8.97}$$

Relations (8.91), (8.94), (8.96), and (8.97) imply that

$$\begin{aligned} x_{i+1} &\in F_\epsilon(T), \\ d(x_j, F_\epsilon(T)) &\leq \|x_j - x_{i+1}\| \leq \epsilon \end{aligned}$$

for all $T \in \mathcal{L}_1$. Hence

$$x_j \in \tilde{F}_\epsilon.$$

Theorem 8.2 is proved. □

8.5 Proof of Theorem 8.3

Fix a point

$$z \in F. \tag{8.98}$$

We may assume without loss of generality that

$$M > \|z\|. \tag{8.99}$$

Lemma 8.22 implies that there exists a number $\delta \in (0, \epsilon/2)$ such that the following property holds:

(P4) For every point $x \in B(0, 4M + 4)$ which satisfies

$$d(x, F(T)) \le \delta \text{ for all } T \in \mathcal{L}_1,$$
$$d(x, \mathrm{Fix}(T)) \le \delta \text{ for all } T \in \mathcal{L}_2$$

we have $d(x, F) \le \epsilon/2$.

Lemmas 8.20 and 8.21 imply that there exists a positive number

$$\gamma < \min\{\delta, M, 1\}/2 \tag{8.100}$$

such that the following properties hold:

(P5) For every mapping $T \in \mathcal{L}_2$ and every point $x \in B(0, 4M + 4)$ which satisfies

$$d(x, \mathrm{Fix}_\gamma(T)) \le \gamma$$

we have $d(x, \mathrm{Fix}(T)) \le \delta/2$.

(P6) For every operator $T \in \mathcal{L}_1$ and every point $x \in B(0, 4M + 4) \cap F_\gamma(T)$ we have $d(x, F(T)) \le \delta/2$.

By Theorem 8.1 there is an integer $n_0 \ge 1$ such that the following property holds:

(P7) for every mapping $S \in \mathcal{R}$ and every sequence $\{x_k\}_{k=0}^{\infty} \subset X$ which satisfies

$$\|x_0\| \le M, \ x_{k+1} = [S(k)(x_k)] \text{ for all integers } k \ge 0 \tag{8.101}$$

there exists an integer $p \in [0, n_0 - 1]$ such that

$$d(x_p, \mathrm{Fix}_\gamma(T)) \le \gamma \text{ for all } T \in \mathcal{L}_2, \tag{8.102}$$

$$d(x_p, F_\gamma(T)) < \gamma \text{ for all } T \in \mathcal{L}_1. \tag{8.103}$$

Assume that $S \in \mathcal{R}$, $\{x_k\}_{k=0}^\infty \subset X$ and (8.101) holds. In view of (8.98), (8.99), (8.101), and Lemma 8.19, the sequence $\{\|x_k - z\|\}_{k=0}^\infty$ is decreasing,

$$\|x_k\| \le 3M \text{ for all integers } k \ge 0. \tag{8.104}$$

Property (P7) and (8.101) imply there exists an integer $p \in [0, n_0 - 1]$ such that (8.102) and (8.103) are valid.

It follows from (8.102), (8.104), and property (P5) that for every mapping $T \in \mathcal{L}_2$ we have

$$d(x_p, \mathrm{Fix}(T)) \le \delta. \tag{8.105}$$

Let $T \in \mathcal{L}_1$ be a monotone operator. In view of (8.103) there exists a point

$$\xi \in F_\gamma(T) \tag{8.106}$$

such that

$$\|x_p - \xi\| < \gamma. \tag{8.107}$$

Relations (8.100), (8.104), and (8.107) imply that

$$\|\xi\| < \|x_p\| + \gamma < 3M + 1. \tag{8.108}$$

Property (P6) (with $x = \xi$), (8.106), and (8.108) imply that

$$d(\xi, F(T)) \le \delta/2. \tag{8.109}$$

In view of (8.100), (8.107), and (8.109),

$$d(x_p, F(T)) \le \|x_p - \xi\| + d(\xi, F(T)) < \gamma + \delta/2 < \delta$$

and

$$d(x_p, F(T)) < \delta \text{ for all } T \in \mathcal{L}_1. \tag{8.110}$$

It follows from property (P4), (8.104), (8.105), and (8.110) that the inequality $d(x_p, F) \le \epsilon/2$ holds. Combined with (8.4) and (8.6) this implies that

$$d(x_i, F) < \epsilon \text{ for all integers } i \ge p.$$

Theorem 8.3 is proved. □

8.6 Proof of Theorem 8.5

Assume that

$$S \in \mathcal{R},\ \{x_k\}_{k=0}^{\infty} \subset X, \tag{8.111}$$

$$\|x_0\| \le M,\ \|x_{k+1} - [S(k)](x_k)\| \le \delta,\ k = 0, 1, \dots, n_0 - 1. \tag{8.112}$$

Set

$$y_0 = x_0,\ y_{k+1} = [S(k)](y_k),\ k = 0, 1, \dots \tag{8.113}$$

It follows from (8.111) to (8.113), the choice of n_0 and Theorem 8.3 that

$$d(y_{n_0}, F) \le \epsilon_0/2. \tag{8.114}$$

Lemma 8.18, (8.4), and (8.111)–(8.113) imply that

$$\|y_{n_0} - x_{n_0}\| \le n_0\delta = \epsilon_0/2.$$

Combined with (8.114) this implies that

$$d(x_{n_0}, F) \le \|x_{n_0} - y_{n_0}\| + d(y_{n_0}, F) \le \epsilon_0.$$

Theorem 8.5 is proved. □

8.7 Proof of Theorem 8.8

We may assume without loss of generality that

$$M > 1 + \sup\{\|z\| :\ z \in F\},\ \epsilon < 1. \tag{8.115}$$

Theorem 8.4 implies that there exist a positive number δ and an integer $n_0 \ge 1$ such that the following property holds:

(P8) For every mapping $S \in \mathcal{R}$ and every sequence $\{x_k\}_{k=0}^{\infty} \subset X$ which satisfies

$$\|x_0\| \le M,\ \|x_{k+1} - [S(k)](x_k)\| \le \delta,\ k = 0, 1, \dots, n_0 - 1$$

we have $d(x_{n_0}, F) \le \epsilon/4$.

Set

$$\delta_* = \min\{\delta,\ (\epsilon/4)n_0^{-1}\}. \tag{8.116}$$

Assume that

$$S \in \mathcal{R},\ \{x_k\}_{k=0}^{\infty} \subset X,\ \|x_0\| \le M, \tag{8.117}$$

$$\|x_{k+1} - [S(k)](x_k)\| \le \delta_*,\ k = 0, 1, \dots. \tag{8.118}$$

Property (P8) and (8.115)–(8.118) imply that

$$d(x_{n_0}, F) \le \epsilon/4,\ \|x_{n_0}\| \le M. \tag{8.119}$$

By induction applying (P8) we obtain that

$$d(x_{jn_0}, F) \le \epsilon/4,\ \|x_{jn_0}\| \le M \tag{8.120}$$

for any natural number j.

Let j be a natural number. Set

$$y_{jn_0} = x_{jn_0},\ y_{k+1} = [S(k)](y_k),\ k = jn_0, \dots, 2jn_0 - 1. \tag{8.121}$$

It follows from (8.21), (8.116)–(8.118), (8.121), and Lemma 8.18 that for all $k = jn_0 + 1, \dots, (j+1)n_0$ we have

$$\|y_k - x_k\| \le n_0\delta_* \le \epsilon/4. \tag{8.122}$$

Since the set F is closed and bounded in view of (8.120) there exists a point $z \in F$ such that

$$\|x_{jn_0} - z\| = d(x_{jn_0}, F) \le \epsilon/4.$$

It follows from the equation above, Lemma 8.19 and (8.121) that for all integers $k = jn_0 + 1, \dots, (j+1)n_0$ we have

$$\|y_k - z\| \le \|y_{jn_0} - z\| = \|x_{jn_0} - z\| \le \epsilon/4.$$

In view of the equation above, (8.122) and the inclusion $z \in F$ for all integers $k = jn_0 + 1, \dots, (j+1)n_0$,

$$d(x_k, F) \le \|x_k - z\| \le \|x_k - y_k\| + \|y_k - z\| \le \epsilon/4 + \epsilon/4.$$

Since j is any natural number we conclude that $d(x_k, F) \le \epsilon/2$ for all integers $k \ge n_0$.

Theorem 8.8 is proved. □

8.8 Proof of Theorem 8.9

We may assume without loss of generality that

$$M > 2 + \sup\{\|z\| : z \in F\}. \tag{8.123}$$

Theorem 8.8 implies that there exist a positive number δ and an integer $n_0 \geq 1$ such that the following property holds:

(P9) For every mapping $S \in \mathcal{R}$ and every sequence $\{x_k\}_{k=0}^{\infty} \subset X$ which satisfies

$$\|x_0\| \leq M, \ \|x_{k+1} - [S(k)](x_k)\| \leq \delta, \ k = 0, 1, \ldots,$$

the inequality $d(x_k, F) \leq 1$ is valid for all integers $k \geq n_0$.

Assume that

$$\{\delta_k\}_{k=0}^{\infty} \subset (0, \delta], \ \lim_{k \to \infty} \delta_k = 0, \ \epsilon > 0. \tag{8.124}$$

We may assume without loss of generality that $\epsilon < 1$.

Theorem 8.8 implies that there exist $\delta_0 \in (0, \delta)$ and a natural number n_* such that the following property holds:

(P10) for every mapping $S \in \mathcal{R}$ and every sequence $\{x_k\}_{k=0}^{\infty} \subset X$ which satisfies

$$\|x_0\| \leq M, \ \|x_{k+1} - [S(k)](x_k)\| \leq \delta_*, \ k = 0, 1, \ldots,$$

the inequality $d(x_k, F) \leq \epsilon$ is valid for all integers $k \geq n_*$.

Evidently, there exists an integer $p \geq 1$ such that

$$\delta_k < \delta_* \text{ for all integers } k \geq p. \tag{8.125}$$

Set

$$n_\epsilon = n_0 + p + n_*. \tag{8.126}$$

Assume that

$$S \in \mathcal{R}, \ \{x_k\}_{k=0}^{\infty} \subset X, \ \|x_0\| \leq M, \tag{8.127}$$
$$\|x_{k+1} - [S(k)](x_k)\| \leq \delta_k, \ k = 0, 1, \ldots.$$

It follows from (8.124), (P10), and (8.127) that

$$d(x_k, F) \leq 1 \text{ for all integers } k \geq n_0 \tag{8.128}$$

and by (8.123),

$$\|x_k\|| \leq M \text{ for all integers } k \geq n_0. \tag{8.129}$$

In view of (8.125) and (8.127) for all integers $k \geq n_0 + p$, we have

$$\|x_{k+1} - [S(k)](x_k)\| \leq \delta_*. \tag{8.130}$$

Relations (8.126), (8.129), (8.130), and property (P10) applied to the sequence $\{x_k\}_{k=n_0+p}^{\infty}$ imply that the inequality $d(x_k, F) \leq \epsilon$ holds for all integers $k \geq n_0 + p + n_* = n_\epsilon$. Theorem 8.9 is proved. □

8.9 Proof of Theorem 8.15

Theorem 8.13 implies that there exists a positive number δ_0 such that the following property holds:

(P11) For every mapping $S \in \mathcal{R}$ and every sequence $\{x_k\}_{k=0}^{\infty} \subset X$ which satisfies

$$\|x_0\| \leq M_0, \ \|x_{k+1} - [S(k)](x_k)\| \leq \delta_0, \ k = 0, 1, \dots,$$

the inequality $\|x_k\| \leq 3M_0 + 1$ is valid for all nonnegative integers k.

Fix a positive number ϵ_1 such that

$$\epsilon_1(\bar{\lambda}^{-1} + 2l + 1) \leq \epsilon/2 \tag{8.131}$$

and a point

$$z \in F. \tag{8.132}$$

Choose an integer $\hat{n} \geq 2n_0$ such that

$$l(1 + (4M_0 + 2)^2 \bar{c}^{-1} \epsilon_1^{-2}) < (\gamma/2)\hat{n}. \tag{8.133}$$

Fix a positive number $\delta < \delta_0$ such that

$$2\hat{n}\delta \leq \epsilon/2. \tag{8.134}$$

Assume that

$$S \in \mathcal{R}, \ \{x_k\}_{k=0}^{\infty} \subset X, \ \|x_0\| \leq M_0, \tag{8.135}$$

$$\|x_{k+1} - [S(k)](x_k)\| \leq \delta, \ k = 0, 1, \dots. \tag{8.136}$$

It follows from (8.135), property (P11), (8.136), and the inequality $\delta < \delta_0$ that

$$\|x_k\| \leq 3M_0 + 1 \text{ for all integers } k \geq 0. \tag{8.137}$$

We claim that for every nonnegative integer p,

$$\hat{n}^{-1}\mathrm{Card}(\{i \in \{p, \dots, p + \hat{n} - 1\} :\ x_i \notin \tilde{F}_\epsilon\}) < \gamma/2. \tag{8.138}$$

Let $p \geq 0$ be an integer. Define

$$y_p = x_p,\ y_{i+1} = [S(i)](y_i) \text{ for all integers } i \geq p. \tag{8.139}$$

It follows from (8.135), Lemma 8.19, (8.132), (8.137), (8.139), and the inclusion $F \subset B(0, M_0)$ that

$$\bar{c}\|y_i - y_{i+1}\|^2 \leq \|z - y_i\|^2 - \|z - y_{i+1}\|^2 \text{ for all integers } i \geq p,$$

$$\bar{c} \sum_{i=p}^{p+\hat{n}-1} \|y_i - y_{i+1}\|^2 \leq \|z - y_p\|^2 \leq \|z - x_p\|^2 \leq (4M_0 + 2)^2. \tag{8.140}$$

In view of (8.140), we have

$$\mathrm{Card}(\{i \in \{p, \dots, p+\hat{n}-1\} :\ \|y_i - y_{i+1}\| \geq \epsilon_1\}) \leq (4M_0 + 2)^2 \bar{c}^{-1} \epsilon_1^{-2}. \tag{8.141}$$

Define

$$E_1 = \{i \in \{p, \dots, p + \hat{n} - 1\} :\ \|y_i - y_{i+1}\| < \epsilon_1\}, \tag{8.142}$$

$$E_2 = \{p, \dots, p + \hat{n} - 1\} \setminus E_1, \tag{8.143}$$

$$E = \{j \in \{p, \dots, p + \hat{n} - 1\} :\ \{j, \dots, j + l - 1\} \subset E_1\}. \tag{8.144}$$

In view of (8.141)–(8.144), we have

$$\begin{aligned} \mathrm{Card}(\{p, \dots, p + \hat{n} - 1\} \setminus E) &\leq l + l\mathrm{Card}(E_2) \\ &\leq (1 + (4M_0 + 2)^2 \bar{c}^{-1} \epsilon_1^{-2})l. \end{aligned} \tag{8.145}$$

Assume that

$$q \in E. \tag{8.146}$$

It follows from (8.142), (8.144), and (8.146) that

$$\{q, \dots, q + l - 1\} \subset E_1$$

and for every integer $i \in \{q, \dots, q + l - 1\}$ we have

$$\|y_i - y_{i+1}\| < \epsilon_1. \tag{8.147}$$

In view of (8.147) for every pair of integers $i, j \in \{q, \dots, q+l\}$,

$$\|y_i - y_j\| < l\epsilon_1. \tag{8.148}$$

Let $k \in \{q, \dots, q+l-1\}$. Assume that

$$T \in \mathcal{L}_2. \tag{8.149}$$

Property (P1) and (8.135) imply that there exists an integer

$$i_T \in \{q, \dots, q+l-1\} \tag{8.150}$$

such that

$$S(i_T) = T. \tag{8.151}$$

It follows from (8.135), (8.139), (8.147), (8.150), and (8.151) that

$$\epsilon_1 > \|y_{i_T+1} - y_{i_T}\| = \|y_{i_T} - T(y_{i_T})\|. \tag{8.152}$$

In view of (8.131), (8.148), (8.150), and (8.152) we have

$$d(y_k, \mathrm{Fix}_\epsilon(T)) \le \|y_k - y_{i_T}\| < l\epsilon_1 \le \epsilon/4,$$
$$d(y_k, \mathrm{Fix}_\epsilon(T)) \le \epsilon/4 \tag{8.153}$$

for all mappings $T \in \mathcal{L}_2$.

Assume that

$$T \in \mathcal{L}_1. \tag{8.154}$$

In view of (8.135), (8.154), property (P2), and (8.2), there exist

$$j_T \in \{q, \dots, q+l-1\},\ c \ge \bar{\lambda} \tag{8.155}$$

such that

$$S(j_T) = P_{c,T}. \tag{8.156}$$

Relations (8.2), (8.139), and (8.156) imply that

$$y_{j_{T+1}} = P_{c,T}(y_{j_T}),\ y_{j_T} \in (I + cT)(y_{j_T+1}),$$
$$y_{j_T} - y_{j_T+1} \in cT(y_{j_T+1}),$$
$$c^{-1}(y_{j_T} - y_{j_T+1}) \in T(y_{j_T+1}). \tag{8.157}$$

It follows from (8.147), (8.155), and (8.157) that $y_{j_{T+1}} \in F_{\bar{\lambda}^{-1}\epsilon_1}(T)$. Combined with (8.148) and (8.155) this implies that

$$d(y_k, F_{\bar{\lambda}^{-1}\epsilon_1}(T)) < l\epsilon_1 \text{ for all } T \in \mathcal{L}_1. \tag{8.158}$$

Note that (8.153) and (8.158) hold for all $k \in \{q, \ldots, q+l-1\}$.

It follows from Lemma 8.18, (8.21), (8.135), (8.136), (8.139), (8.144), and (8.146) that for all integers $k = q, \ldots, q+l-1$,

$$\|y_k - x_k\| \le \hat{n}\delta. \tag{8.159}$$

In view of (8.131), (8.134), (8.158), and (8.159), for every mapping $T \in \mathcal{L}_1$, we have

$$d(x_q, F_{\bar{\lambda}^{-1}\epsilon_1}(T)) \le \|x_q - y_q\| + d(y_q, F_{\bar{\lambda}^{-1}\epsilon_1}(T)) < \hat{n}\delta + l\epsilon_1,$$
$$d(x_q, F_\epsilon(T)) < \epsilon/2 + \hat{n}\delta < \epsilon$$

and

$$d(x_q, F_\epsilon(T)) < \epsilon \text{ for all } T \in \mathcal{L}_1. \tag{8.160}$$

By (8.134), (8.153), and (8.159), for every mapping $T \in \mathcal{L}_2$ we have

$$d(x_q, \mathrm{Fix}_\epsilon(T)) \le \|x_q - y_q\| + d(y_q, \mathrm{Fix}_\epsilon(T)) \le \hat{n}\delta + \epsilon/4 < \epsilon. \tag{8.161}$$

Hence in view of (8.160) and (8.161),

$$x_q \in \tilde{F}_\epsilon \text{ for each } q \in E. \tag{8.162}$$

It follows from (8.145) and (8.162) that

$$\mathrm{Card}(\{i \in \{p, \ldots, p+\hat{n}-1\} : \ x_i \notin \tilde{F}_\epsilon\})$$
$$\le \mathrm{Card}(\{i \in \{p, \ldots, p+\hat{n}-1\} \setminus E) \le l(1 + (4M_0+2)^2 \bar{c}^{-1}\epsilon_1^{-2})$$

and by (8.133), we have

$$\hat{n}^{-1}\mathrm{Card}(\{i \in \{p, \ldots, p+\hat{n}-1\} : \ x_i \notin \tilde{F}_\epsilon\})$$
$$\le \hat{n}^{-1} l(1 + (4M_0+2)^2 \bar{c}^{-1}\epsilon_1^{-2}) < \gamma/2.$$

We have shown that for every nonnegative integer p,

$$\hat{n}^{-1}\mathrm{Card}(\{i \in \{p, \ldots, p+\hat{n}-1\} : \ x_i \notin \tilde{F}_\epsilon\}) < \gamma/2. \tag{8.163}$$

Assume that an integer $p \geq 0$ and an integer $n \geq \hat{n}$. There exist integers $k \geq 1$ and $q \in [0, \hat{n} - 1]$ such that

$$n = k\hat{n} + q. \tag{8.164}$$

In view of (8.163) and (8.164),

$$\begin{aligned}
& n^{-1}\mathrm{Card}(\{i \in \{p, \dots, p + n - 1\} : \ x_i \notin \tilde{F}_\epsilon\}) \\
& \leq (\hat{n}k)^{-1}[\mathrm{Card}(\{i \in \{p, \dots, p + \hat{n}k - 1\} : \ x_i \notin \tilde{F}_\epsilon\}) \\
& + \mathrm{Card}(\{i \in \{p + \hat{n}k, \dots, p + (\hat{n} + 1)k - 1\} : \ x_i \notin \tilde{F}_\epsilon\})] \\
& \leq k^{-1} \sum_{j=0}^{k-1} \hat{n}^{-1}\mathrm{Card}(\{i \in \{p + \hat{n}j, \dots, p + \hat{n}(j + 1) - 1\} : \ x_i \notin \tilde{F}_\epsilon\}) \\
& + \hat{n}^{-1}\mathrm{Card}(\{i \in \{p + \hat{n}k, \dots, p + (\hat{n} + 1)k - 1\} : \ x_i \notin \tilde{F}_\epsilon\}) \\
& < \gamma/2 + \gamma/2 = \gamma.
\end{aligned}$$

Theorem 8.15 is proved. □

Chapter 9
Dynamic String-Averaging Proximal Point Algorithm

In a Hilbert space, we study the convergence of a dynamic string-averaging proximal point method to a common zero of a finite family of maximal monotone operators under the presence of computational errors. We show that the algorithm generates a good approximate solution, if the sequence of computational errors is bounded from above by a constant. Moreover, for a known computational error, we find out what an approximate solution can be obtained and how many iterates one needs for this.

9.1 Preliminaries and Main Results

Let $(X, \langle \cdot, \cdot \rangle)$ be a Hilbert space equipped with an inner product $\langle \cdot, \cdot \rangle$ which induces the complete norm $\| \cdot \|$.

For each $x \in X$ and each nonempty set $A \subset X$ put

$$d(x, A) = \inf\{\|x - y\| : \ y \in A\}.$$

For each $x \in X$ and each $r > 0$ set

$$B(x, r) := \{y \in X : \ \|x - y\| \le r\}.$$

Denote by $\mathrm{Card}(A)$ the cardinality of a set A. The sum over an empty set is assumed to be zero.

Recall (see Sect. 8.1) that a multifunction $T : X \to 2^X$ is called a monotone operator if

$$\langle z - z', w - w' \rangle \ \ge 0 \quad \forall z, z', w, w' \in X$$
$$\text{such that } w \in T(z) \text{ and } w' \in T(z').$$

A. Zaslavski, *Approximate Solutions of Common Fixed-Point Problems*, Springer Optimization and Its Applications 112, DOI 10.1007/978-3-319-33255-0_9

It is called maximal monotone if, in addition, the graph

$$\{(z, w) \in X \times X : \ w \in T(z)\}$$

is not properly contained in the graph of any other monotone operator $T' : X \to 2^X$.

Let $T : X \to 2^X$ be a maximal monotone operator. Then (see Sect. 8.1) for each $z \in X$ and each $c > 0$, there is a unique $u \in X$ such that

$$z \in (I + cT)(u),$$

where $I : X \to X$ is the identity operator ($Ix = x$ for all $x \in X$).

The operator

$$P_{c,T} := (I + cT)^{-1} \tag{9.1}$$

is therefore single-valued from all of X onto X (where c is any positive number). It is also nonexpansive:

$$\|P_{c,T}(z) - P_{c,T}(z')\| \le \|z - z'\| \text{ for all } z, z' \in X \tag{9.2}$$

and

$$P_{c,T}(z) = z \text{ if and only if } 0 \in T(z) \tag{9.3}$$

(see Sect. 8.1).

Set

$$F(T) = \{z \in X : \ 0 \in T(z)\}. \tag{9.4}$$

Let $\mathcal{L}_1$ be a finite set of maximal monotone operators $T : X \to 2^X$ and $\mathcal{L}_2$ be a finite set of mappings $T : X \to X$. We suppose that the set $\mathcal{L}_1 \cup \mathcal{L}_2$ is nonempty. (Note that one of the sets $\mathcal{L}_1$ or $\mathcal{L}_2$ may be empty.)

Let $\bar{c} \in (0, 1]$ and let $\bar{c} = 1$, if $\mathcal{L}_2 = \emptyset$.

We suppose that

$$F(T) \neq \emptyset \text{ for any } T \in \mathcal{L}_1 \tag{9.5}$$

and that for each $T \in \mathcal{L}_2$,

$$\mathrm{Fix}(T) := \{z \in X : \ T(z) = z\} \neq \emptyset, \tag{9.6}$$

$$\|z - x\|^2 \ge \|z - T(x)\|^2 + \bar{c}\|x - T(x)\|^2 \tag{9.7}$$

$$\text{for all } x \in X \text{ and all } z \in \mathrm{Fix}(T).$$

Suppose that

$$F := (\cap_{T\in\mathcal{L}_1}F(T)) \cap (\cap_{Q\in\mathcal{L}_2}\mathrm{Fix}(Q)) \neq \emptyset. \tag{9.8}$$

Let $\epsilon > 0$. For any $T \in \mathcal{L}_1$ set

$$F_\epsilon(T) = \{x \in X :\ T(x) \cap B(0,\epsilon) \neq \emptyset\} \tag{9.9}$$

and for any $T \in \mathcal{L}_2$ put

$$\mathrm{Fix}_\epsilon(T) = \{x \in X :\ \|T(x) - x\| \le \epsilon\}. \tag{9.10}$$

Set

$$\tilde{F}_\epsilon = (\cap_{T\in\mathcal{L}_1}\{x \in X :\ d(x, F_\epsilon(T)) \le \epsilon\})$$
$$\cap(\cap_{Q\in\mathcal{L}_2}\{x \in X :\ d(x, \mathrm{Fix}_\epsilon(Q)) \le \epsilon\}). \tag{9.11}$$

Let $\bar{\lambda} > 0$ and let $\bar{\lambda} = \infty$ and $\bar{\lambda}^{-1} = 0$, if $\mathcal{L}_1 = \emptyset$. Set

$$\mathcal{L} = \mathcal{L}_2 \cup \{P_{c,T} :\ T \in \mathcal{L}_1,\ c \in [\bar{\lambda}, \infty)\}. \tag{9.12}$$

Next we describe the dynamic string-averaging method with variable strings and weights.

By a mapping vector, we a mean a vector $T = (T_1, \dots, T_p)$ such that $T_i \in \mathcal{L}$ for all $i = 1, \dots, p$.

For a mapping vector $T = (T_1, \dots, T_q)$ set

$$p(T) = q,\ P[T] = T_q \cdots T_1. \tag{9.13}$$

It is easy to see that for each mapping vector $T = (T_1, \dots, T_p)$,

$$P[T](x) = x \text{ for all } x \in F, \tag{9.14}$$

$$\|P[T](x) - P[T](y)\| = \|(x) - P[T](y)\| \le \|x - y\| \tag{9.15}$$

for every $x \in F$ and every $y \in X$.

Denote by $\mathcal{M}$ the collection of all pairs (Ω, w), where Ω is a finite set of mapping vectors and

$$w : \Omega \to (0, \infty) \text{ be such that } \sum_{T\in\Omega} w(T) = 1. \tag{9.16}$$

Let $(\Omega, w) \in \mathcal{M}$. Define

$$P_{\Omega,w}(x) = \sum_{T\in\Omega} w(T)P[T](x),\ x \in X. \tag{9.17}$$

It is not difficult to see that

$$P_{\Omega,w}(x) = x \text{ for all } x \in F, \tag{9.18}$$

$$\|P_{\Omega,w}(x) - P_{\Omega,w}(y)\| = \|x - P_{\Omega,w}(y)\| \le \|x - y\| \tag{9.19}$$

for all $x \in F$ and all $y \in X$.

The dynamic string-averaging method with variable strings and variable weights can now be described by the following algorithm.

Initialization: select an arbitrary $x_0 \in X$.

Iterative step: given a current iteration vector x_k pick a pair

$$(\Omega_{k+1}, w_{k+1}) \in \mathcal{M}$$

and calculate the next iteration vector x_{k+1} by

$$x_{k+1} = P_{\Omega_{k+1},w_{k+1}}(x_k).$$

Fix a number

$$\Delta \in (0, \mathrm{Card}(\mathcal{L}_1 \cup \mathcal{L}_2)^{-1}) \tag{9.20}$$

and natural numbers $\bar{N}$ and $\bar{q}$ satisfying

$$\bar{q} \ge \mathrm{Card}(\mathcal{L}_1 \cup \mathcal{L}_2). \tag{9.21}$$

Denote by $\mathcal{M}_*$ the set of all $(\Omega, w) \in \mathcal{M}$ such that

$$p(T) \le \bar{q} \ \text{ for all } T \in \Omega, \tag{9.22}$$

$$w(T) \ge \Delta \text{ for all } T \in \Omega. \tag{9.23}$$

Denote by $\mathcal{R}$ the set of all sequences

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$$

such that the following properties hold:

(P1) for each integer $j \ge 1$ and each $S \in \mathcal{L}_2$ there exist $k \in \{j, \dots, j + \bar{N} - 1\}$, $T = (T_1, \dots, T_{p(T)}) \in \Omega_k$ such that

$$S \in \{T_1, \dots, T_{p(T)}\};$$

(P2) for each integer $j \ge 1$ and each $S \in \mathcal{L}_1$ there exist $k \in \{j, \dots, j + \bar{N} - 1\}$, $T = (T_1, \dots, T_{p(T)}) \in \Omega_k$ and $c \ge \bar{\lambda}$ such that

$$P_{c,S} \in \{T_1, \dots, T_{p(T)}\}.$$

In order to state our main results we need the following definitions.

Let $\delta \geq 0$, $x \in X$ and let $T = (T_1, \ldots, T_{p(t)})$ be a mapping vector. Define

$$\begin{aligned} A_0(x, T, \delta) = \{(y, \lambda) \in X \times R^1 : \ &\text{there is a sequence } \{y_i\}_{i=0}^{p(t)} \subset X \text{ such that} \\ &y_0 = x \text{ and for all } i = 1, \ldots, p(t), \\ &\|y_i - T_i(y_{i-1})\| \leq \delta, \\ &y = y_{p(T)}, \\ &\lambda = \max\{\|y_i - y_{i-1}\| : \ i = 1, \ldots, p(T)\}\}. \end{aligned} \tag{9.24}$$

Let $\delta \geq 0$, $x \in X$ and let $(\Omega, w) \in \mathcal{M}$. Define

$$\begin{aligned} A(x, (\Omega, w), \delta) = \{(y, \lambda) \in X \times R^1 : \ &\text{there exist} \\ &(y_T, \lambda_T) \in A_0(x, T, \delta), \ T \in \Omega \text{ such that} \\ \|y - \sum_{T \in \Omega} w(T) y_T\| \leq \delta, \ &\lambda = \max\{\lambda_T : \ T \in \Omega\}\}. \end{aligned} \tag{9.25}$$

In this chapter we prove the following two results.

Theorem 9.1. *Let $M > 0$ satisfy*

$$B(0, M) \cap F \neq \emptyset, \tag{9.26}$$

$\delta > 0$ satisfy

$$\delta \leq (2\bar{q}\bar{N})^{-1}, \tag{9.27}$$

a natural number n_0 satisfy

$$n_0 \geq M^2 \delta^{-1} (\bar{q} + 1)^{-1} (2M + 4)^{-1} (4\bar{N})^{-1}, \tag{9.28}$$

$$\epsilon_0 = (64 \Delta^{-1} \delta (\bar{q} + 1)(2M + 4) 4\bar{N})^{1/2} \bar{c}^{-1/2}, \tag{9.29}$$

and let

$$\epsilon_1 = (\bar{q} + 1)(\bar{N} + 4)\epsilon_0 \max\{\bar{\lambda}^{-1}, 1\}. \tag{9.30}$$

Assume that

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \in \mathcal{R}, \tag{9.31}$$

$$x_0 \in B(0, M) \text{ and } \{x_i\}_{i=1}^{\infty} \subset X, \ \{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty) \tag{9.32}$$

satisfy for each natural number i,

$$(x_i, \lambda_i) \in A(x_{i-1}, (\Omega_i, w_i), \delta). \tag{9.33}$$

Then there exists an integer $q \in [0, n_0]$ *such that*

$$\|x_i\| \leq 3M + 1,\ i = 0, \dots, q\bar{N}, \tag{9.34}$$

$$\lambda_i \leq \epsilon_0,\ i = q\bar{N} + 1, \dots, (q + 1)\bar{N}. \tag{9.35}$$

Moreover, if an integer $q \geq 0$ *satisfies (9.35), then for each* $i = q\bar{N}, \dots, (q + 1)\bar{N}$,

$$x_i \in \tilde{F}_{\epsilon_1}$$

and

$$\|x_i - x_j\| \leq (\bar{q} + 1)\bar{N}\epsilon_0$$

for each $i, j \in \{q\bar{N}, \dots, (q + 1)\bar{N}\}$.

Note that in Theorem 9.1 δ is the computational error made by our computer system, we obtain a point of the set $\tilde{F}_{\epsilon_1}$ and in order to obtain this point we need $n_0\bar{N}$ iterations. It is not difficult to see that $\epsilon_1 = c_1\delta^{1/2}$ and $n_0 = \lfloor c_2(\delta^{-1}\rfloor$, where c_1 and c_2 are positive constants depending on M.

Theorem 9.2. *Let* $M, \epsilon > 0$ *satisfy*

$$B(0, M) \cap F \neq \emptyset. \tag{9.36}$$

Assume that

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \in \mathcal{R}, \tag{9.37}$$

$$x_0 \in B(0, M), \{x_i\}_{i=1}^{\infty} \subset X,\ \{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty) \tag{9.38}$$

satisfy for each natural number i,

$$(x_i, \lambda_i) \in A(x_{i-1}, (\Omega_i, w_i), 0). \tag{9.39}$$

Then

$$Card(\{i \in \{0, 1, \dots\} : \ x_i \notin \tilde{F}_\epsilon\}) \leq 4\bar{N}M^2\bar{c}^{-1}\Delta^{-1}\epsilon^{-2}(\bar{N} + 1)^2\bar{q}^2.$$

9.2 Proof of Theorem 9.1

By (9.26) there exists

$$z \in B(0, M) \cap F. \tag{9.40}$$

Let $k \geq 0$ be an integer. By (9.33),

$$(x_{k+1}, \lambda_{k+1}) \in A(x_k, (\Omega_{k+1}, w_{k+1}), \delta). \tag{9.41}$$

By (9.25) and (9.41) there exist

$$(y_{k,T}, \lambda_{k,T}) \in A_0(x_k, T, \delta), \ T \in \Omega_{k+1} \tag{9.42}$$

such that

$$\|x_{k+1} - \sum_{T \in \Omega_{k+1}} w_{k+1}(T) y_{k,T}\| \leq \delta, \tag{9.43}$$

$$\lambda_{k+1} = \max\{\lambda_{k,T} : \ T \in \Omega_{k+1}\}. \tag{9.44}$$

Let

$$T = (T_1, \dots, T_{p(T)}) \in \Omega_{k+1}.$$

It follows from (9.24) and (9.42) that there exists a finite sequence

$$\{y_i^{(k,T)}\}_{i=0}^{p(T)} \subset X$$

such that

$$y_0^{(k,T)} = x_k, \ y_{p(T)}^{(k,T)} = y_{k,T}, \tag{9.45}$$

$$\|y_i^{(k,T)} - T_i(y_{i-1}^{(k,T)})\| \leq \delta \text{ for each integer } i = 1, \dots, p(T), \tag{9.46}$$

$$\lambda_{k,T} = \max\{\|y_i^{(k,T)} - y_{i-1}^{(k,T)}\| : \ i = 1, \dots, p(T)\}. \tag{9.47}$$

Let

$$T = (T_1, \dots, T_{p(T)}) \in \Omega_{k+1},$$
$$i \in \{1, \dots, p(T)\}. \tag{9.48}$$

By (9.12), (9.37), (9.40), (9.48) and Lemma 8.19,

$$\|z - y_{i-1}^{(k,T)}\|^2 \geq \|z - T_i(y_{i-1}^{(k,T)})\|^2 + \bar{c}\|y_{i-1}^{(k,T)} - T_i(y_{i-1}^{(k,T)})\|^2. \tag{9.49}$$

Relations (9.46) and (9.49) imply that

$$\|z - y_i^{(k,T)}\| \le \|z - T_i(y_{i-1}^{(k,T)})\| + \|T_i(y_{i-1}^{(k,T)}) - y_i^{(k,T)}\| \\ \le \|z - y_{i-1}^{(k,T)}\| + \delta. \tag{9.50}$$

In view of (9.22), (9.45), and (9.50), for all $i = 1, \dots, p(T)$,

$$\|z - y_i^{(k,T)}\| \le \|z - x_k\| + i\delta \le \|z - x_k\| + \bar{q}\delta. \tag{9.51}$$

It follows from (9.45) and (9.51) that

$$\|z - y_{k,T}\| \le \|z - x_k\| + \bar{q}\delta. \tag{9.52}$$

By (9.16), (9.43), (9.52) and the convexity of the norm $\|\cdot\|$,

$$\|z - x_{k+1}\| \le \|z - \sum_{T \in \Omega_{k+1}} w_{k+1}(T) y_{k,T}\| + \|\sum_{T \in \Omega_{k+1}} w_{k+1}(T) y_{k,T} - x_{k+1}\| \\ \le \sum_{T \in \Omega_{k+1}} w_{k+1}(T)\|z - y_{k,T}\| + \delta \le \|z - x_k\| + (\bar{q} + 1)\delta. \tag{9.53}$$

In view (9.32) and (9.40),

$$\|x_0 - z\| \le 2M. \tag{9.54}$$

Assume that a nonnegative integer s satisfies for each integer $k \in [0, s]$,

$$\max\{\lambda_i :\ i = k\bar{N} + 1, \dots, (k+1)\bar{N}\} > \epsilon_0. \tag{9.55}$$

We prove the following auxiliary result.

Lemma 9.3. *Assume that an integer $k \in [0, s]$ satisfies*

$$\|x_{k\bar{N}} - z\| \le 2M \tag{9.56}$$

and that

$$i \in [0, \bar{N} - 1]. \tag{9.57}$$

Then

$$\|x_{k\bar{N}+i+1} - z\|^2 \le \|x_{k\bar{N}+i} - z\|^2 + \delta(\bar{q} + 1)(4M + 2) \tag{9.58}$$

and if $\lambda_{k\bar{N}+i+1} > \epsilon_0$, then

$$\|x_{k\bar{N}+i+1} - z\|^2 - \|x_{k\bar{N}+i} - z\|^2 \le -8^{-1}\Delta\epsilon_0^2\bar{c}. \tag{9.59}$$

Proof. In view of (9.27), (9.53), (9.56), and (9.57),

$$\|x_{k\bar{N}+i+1} - z\|, \ \|x_{k\bar{N}+i} - z\| \le \|x_{k\bar{N}} - z\| + (i+1)(\bar{q}+1)\delta$$
$$\le 2M + \bar{N}(\bar{q}+1)\delta \le 2M + 1. \tag{9.60}$$

By (9.53) and (9.60),

$$\|x_{k\bar{N}+i+1} - z\|^2 - \|x_{k\bar{N}+i} - z\|^2$$
$$(\|x_{k\bar{N}+i+1} - z\| - \|x_{k\bar{N}+i} - z\|)(\|x_{k\bar{N}+i+1} - z\| + \|x_{k\bar{N}+i} - z\|)$$
$$\le \delta(\bar{q}+1)(4M+2)$$

and (9.58) holds.

Assume that

$$\lambda_{k\bar{N}+i+1} > \epsilon_0. \tag{9.61}$$

In view of (9.53),

$$\|x_{k\bar{N}+i} - z\| \le \|x_{k\bar{N}} - z\| + i(\bar{q}+1)\delta. \tag{9.62}$$

Relations (9.52) and (9.62) imply that for each $T \in \Omega_{k\bar{N}+i+1}$,

$$\|z - y_{k\bar{N}+i,T}\| \le \|z - x_{k\bar{N}+i}\| + \bar{q}\delta. \tag{9.63}$$

It follows from (9.27), (9.56), (9.57), (9.62), and (9.63) that for each $T \in \Omega_{k\bar{N}+i+1}$,

$$\|z - y_{k\bar{N}+i,T}\|^2 - \|z - x_{k\bar{N}+i}\|^2$$
$$= (\|z - y_{k\bar{N}+i,T}\| - \|z - x_{k\bar{N}+i}\|)(\|z - y_{k\bar{N}+i,T}\| + \|z - x_{k\bar{N}+i}\|)$$
$$\le \bar{q}\delta(2\|z - x_{k\bar{N}+i}\| + \bar{q}\delta)$$
$$\le \bar{q}\delta(2\|z - x_{k\bar{N}}\| + 2i(\bar{q}+1)\delta + \bar{q}\delta)$$
$$\le \bar{q}\delta(4M + (\bar{q}+1)\delta(2i+1))$$
$$\le \bar{q}\delta(4M + 2(\bar{q}+1)\delta\bar{N}) \le \bar{q}\delta(4M+2). \tag{9.64}$$

In view of (9.44) and (9.61) there exists

$$S = (S_1, \dots, S_{p(S)}) \in \Omega_{k\bar{N}+i+1} \tag{9.65}$$

such that

$$\epsilon_0 < \lambda_{k\bar{N}+i+1} = \lambda_{k\bar{N}+i,S}. \tag{9.66}$$

By (9.47) and (9.66), there exists

$$j_0 \in \{1, \dots, p(S)\}$$

such that

$$\epsilon_0 < \lambda_{k\bar{N}+i,S} = \|y_{j_0}^{(k\bar{N}+i,S)} - y_{j_0-1}^{(k\bar{N}+i,S)}\|. \tag{9.67}$$

By (9.27), (9.45), (9.51), (9.53), (9.56), and (9.57), for each $j \in \{1, \dots, p(S)\}$,

$$\|z - y_j^{(k\bar{N}+i,S)}\|,\ \|z - y_{j-1}^{(k\bar{N}+i,S)}\| \le \|z - x_{k\bar{N}+i}\| + \bar{q}\delta, \tag{9.68}$$

$$\|z - y_j^{(k\bar{N}+i,S)}\| \le \|z - y_{j-1}^{(k\bar{N}+i,S)}\| + \delta, \tag{9.69}$$

$$\begin{aligned}
&\|z - y_j^{(k\bar{N}+i,S)}\|^2 - \|z - y_{j-1}^{(k\bar{N}+i,S)}\|^2 \\
&(\|z - y_j^{(k\bar{N}+i,S)}\| - \|z - y_{j-1}^{(k\bar{N}+i,S)}\|)(\|z - y_j^{(k\bar{N}+i,S)}\| + \|z - y_{j-1}^{(k\bar{N}+i,S)}\|) \\
&\le \delta(2\|z - x_{k\bar{N}+i}\| + 2\bar{q}\delta) \\
&\le \delta(2\|z - x_{k\bar{N}}\| + 2(\bar{q}+1)\delta i + 2\bar{q}\delta) \\
&\le \delta(4M + 2(\bar{q}+1)\delta(i+1)) \le \delta(4M+2).
\end{aligned} \tag{9.70}$$

In view of (9.49),

$$\begin{aligned}
&\|z - y_{j_0-1}^{(k\bar{N}+i,S)}\|^2 \\
&\ge \|z - S_{j_0}(y_{j_0-1}^{(k\bar{N}+i,S)})\|^2 + \bar{c}\|y_{j_0-1}^{(k\bar{N}+i,S)} - S_{j_0}(y_{j_0-1}^{(k\bar{N}+i,S)})\|^2.
\end{aligned} \tag{9.71}$$

It follows from (9.29), (9.46), (9.67) that

$$\begin{aligned}
&\|y_{j_0-1}^{(k\bar{N}+i,S)} - S_{j_0}(y_{j_0-1}^{(k\bar{N}+i,S)})\| \\
&\ge \|y_{j_0-1}^{(k\bar{N}+i,S)} - y_{j_0}^{(k\bar{N}+i,S)}\| - \|y_{j_0}^{(k\bar{N}+i,S)} - S_{j_0}(y_{j_0-1}^{(k\bar{N}+i,S)})\| \\
&> \epsilon_0 - \delta > \epsilon_0/2.
\end{aligned} \tag{9.72}$$

Relations (9.71) and (9.72) imply that

$$\|z - y_{j_0-1}^{(k\bar{N}+i,S)}\|^2 - \bar{c}\epsilon_0^2/4 \ge \|z - S_{j_0}(y_{j_0-1}^{(k\bar{N}+i,S)})\|^2. \tag{9.73}$$

In view of (9.73),

$$\|z - S_{j_0}(y_{j_0-1}^{(k\bar{N}+i,S)})\| \le \|z - y_{j_0-1}^{(k\bar{N}+i,S)}\|. \tag{9.74}$$

By (9.46), (9.53), (9.56), (9.68), and (9.74),

$$\begin{aligned}
&\|z - y_{j_0}^{(k\bar{N}+i,S)}\|^2 - \|z - S_{j_0}(y_{j_0-1}^{(k\bar{N}+i,S)})\|^2 \\
&= (\|z - y_{j_0}^{(k\bar{N}+i,S)}\| - \|z - S_{j_0}(y_{j_0-1}^{(k\bar{N}+i,S)})\|) \\
&\times(\|z - y_{j_0}^{(k\bar{N}+i,S)}\| + \|z - S_{j_0}(y_{j_0-1}^{(k\bar{N}+i,S)})\|) \\
&\le \|y_{j_0}^{(k\bar{N}+i,S)} - S_{j_0}(y_{j_0-1}^{(k\bar{N}+i,S)})\|(2\|z - x_{k\bar{N}+i}\| + 2\bar{q}\delta) \\
&\le \delta(2\|z - x_{k\bar{N}}\| + 2i(\bar{q}+1)\delta + 2\bar{q}\delta) \\
&\le \delta(4M + 2\bar{N}(\bar{q}+1)\delta) \le \delta(4M+2).
\end{aligned} \tag{9.75}$$

In view of (9.73) and (9.75),

$$\begin{aligned}
\|z - y_{j_0}^{(k\bar{N}+i,S)}\|^2 &\le \|z - S_{j_0}(y_{j_0-1}^{(k\bar{N}+i,S)})\|^2 + \delta(4M+2) \\
&\le \|z - y_{j_0-1}^{(k\bar{N}+i,S)}\|^2 - \bar{c}\epsilon_0^2/4 + \delta(4M+2).
\end{aligned} \tag{9.76}$$

It follows from (9.22), (9.45), (9.70), and (9.76) that

$$\begin{aligned}
&\|z - x_{k\bar{N}+i}\|^2 - \|z - y_{k\bar{N}+i,S}\|^2 \\
&= \|z - y_0^{(k\bar{N}+i,S)}\|^2 - \|z - y_{p(S)}^{(k\bar{N}+i,S)}\|^2 \\
&= \sum_{j=1}^{p(S)} (\|z - y_{j-1}^{(k\bar{N}+i,S)}\|^2 - \|z - y_j^{(k\bar{N}+i,S)}\|^2) \\
&= \sum \{\|z - y_{j-1}^{(k\bar{N}+i,S)}\|^2 - \|z - y_j^{(k\bar{N}+i,S)}\|^2 : \ j \in \{1, \dots, p(S)\} \setminus \{j_0\}\} \\
&+ \|z - y_{j_0-1}^{(k\bar{N}+i,S)}\|^2 - \|z - y_{j_0}^{(k\bar{N}+i,S)}\|^2 \\
&\ge -\delta(4M+2)(p(S)-1) + \bar{c}\epsilon_0^2/4 - \delta(4M+2) \\
&\ge \bar{c}\epsilon_0^2/4 - \delta(4M+2)\bar{q}.
\end{aligned} \tag{9.77}$$

By (9.16), (9.23), (9.64), (9.65), (9.77) and the convexity of the function $\|\cdot\|^2$,

$$\begin{aligned}
&\|z - \sum_{T \in \Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(T) y_{k\bar{N}+i,T}\|^2 \\
&\le \sum_{T \in \Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(T)\|z - y_{k\bar{N}+i,T}\|^2 \\
&= \sum \{w_{k\bar{N}+i+1}(T)\|z - y_{k\bar{N}+i,T}\|^2 : \ T \in \Omega_{k\bar{N}+i+1} \setminus \{S\}\}
\end{aligned}$$

$$+w_{k\bar{N}+i+1}(S)\|z-y_{k\bar{N}+i,S}\|^2$$
$$\leq \sum\{w_{k\bar{N}+i+1}(T)(\|z-x_{k\bar{N}+i}\|^2$$
$$+\bar{q}\delta(4M+2)):\ T\in\Omega_{k\bar{N}+i+1}\setminus\{S\}\}$$
$$+w_{k\bar{N}+i+1}(S)(\|z-x_{k\bar{N}+i}\|^2-\bar{c}\epsilon_0^2/4+\delta(4M+2)\bar{q})$$
$$\leq \|z-x_{k\bar{N}+i}\|^2+\bar{q}\delta(4M+2)-\Delta\bar{c}\epsilon_0^2/4. \tag{9.78}$$

By (9.29), (9.53), (9.56), (9.57), and (9.78),

$$\|z-\sum_{T\in\Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(T)y_{k\bar{N}+i,T}\| \leq \|z-x_{k\bar{N}+i}\|$$
$$\leq \|z-x_{k\bar{N}}\|+i(\bar{q}+1)\delta \leq 2M+\delta(\bar{q}+1)(\bar{N}-1). \tag{9.79}$$

In view of (9.79),

$$\|z-x_{k\bar{N}+i+1}\|$$
$$\leq \|z-\sum_{T\in\Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(T)y_{k\bar{N}+i,T}\|$$
$$+\|\sum_{T\in\Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(T)y_{k\bar{N}+i,T}-x_{k\bar{N}+i+1}\|$$
$$\leq 2M+\delta(\bar{q}+1)(\bar{N}-1)+\delta. \tag{9.80}$$

It follows from (9.27), (9.43), (9.79), and (9.80) that

$$\|z-x_{k\bar{N}+i+1}\|^2-\|z-\sum_{T\in\Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(T)y_{k\bar{N}+i,T}\|^2$$
$$=(\|z-x_{k\bar{N}+i+1}\|-\|z-\sum_{T\in\Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(T)y_{k\bar{N}+i,T}\|)$$
$$\times(\|z-x_{k\bar{N}+i+1}\|+\|z-\sum_{T\in\Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(T)y_{k\bar{N}+i,T}\|)$$
$$\leq \|x_{k\bar{N}+i+1}-\sum_{T\in\Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(T)y_{k\bar{N}+i,T}\|(4M+2)\leq\delta(4M+2). \tag{9.81}$$

By (9.29), (9.78), and (9.81),

$$\|z-x_{k\bar{N}+i+1}\|^2 \leq \|z-\sum_{T\in\Omega_{k\bar{N}+i+1}} w_{k\bar{N}+i+1}(T)y_{k\bar{N}+i,T}\|+\delta(4M+2)$$

$$\leq \|z - x_{k\bar{N}+i}\|^2 + (\bar{q}+1)\delta(4M+2) - \Delta\bar{c}\epsilon_0^2/4$$
$$\leq \|z - x_{k\bar{N}+i}\|^2 - \Delta\bar{c}\epsilon_0^2/8.$$

Lemma 9.3 is proved. □

Lemma 9.4. *Assume that an integer* $k \in [0, s]$ *satisfies*

$$\|x_{k\bar{N}} - z\| \leq 2M.$$

Then

$$\|x_{k\bar{N}} - z\|^2 - \|x_{(k+1)\bar{N}} - z\|^2 \geq \Delta\bar{c}\epsilon_0^2/16.$$

Proof. By Lemma 9.3, for all $i \in [0, \bar{N}-1]$,

$$\|x_{k\bar{N}+i+1} - z\|^2 \leq \|x_{k\bar{N}+i} - z\|^2 + \delta(\bar{q}+1)(4M+2). \tag{9.82}$$

In view of (9.55), there exists

$$j_0 \in \{0, \ldots, \bar{N}-1\} \tag{9.83}$$

such that

$$\lambda_{k\bar{N}+j_0+1} > \epsilon_0. \tag{9.84}$$

Lemma 9.3, (9.29), (9.82), (9.83), and (9.84) imply that

$$\|x_{k\bar{N}} - z\|^2 - \|x_{(k+1)\bar{N}} - z\|^2$$
$$= \sum_{i=0}^{\bar{N}-1}(\|x_{k\bar{N}+i} - z\|^2 - \|x_{k\bar{N}+i+1} - z\|^2)$$
$$\geq -(\bar{N}-1)\delta(\bar{q}+1)(4M+2) + 8^{-1}\Delta\epsilon_0^2\bar{c} \geq 16^{-1}\Delta\epsilon_0^2\bar{c}.$$

Lemma 9.4 is proved. □

By (9.54) and Lemma 9.4 applied by induction, for all integers $k = 0, \ldots, s$,

$$\|x_{k\bar{N}} - z\|^2 - \|x_{(k+1)\bar{N}} - z\|^2 \geq 16^{-1}\Delta\epsilon_0^2\bar{c}, \tag{9.85}$$
$$\|x_{(k+1)\bar{N}} - z\| \leq \|x_{k\bar{N}} - z\| \leq 2M. \tag{9.86}$$

It follows from (9.27), (9.53), and (9.86) that for all integers $k = 0, \ldots, s$ and all $i = 0, \ldots, \bar{N}$,

$$\|x_{k\bar{N}+i} - z\| \leq \|x_{k\bar{N}} - z\| + i\delta(\bar{q}+1) \leq \|x_{k\bar{N}} - z\| + \bar{N}\delta(\bar{q}+1) \leq 2M+1.$$

Together with (9.40) this implies that for all $i = 0, \dots, (s+1)\bar{N}$,

$$\|x_i - z\| \le 2M + 1, \ \|x_i\| \le 3M + 1. \tag{9.87}$$

Relations (9.54) and (9.85) imply that

$$4M^2 \ge \|x_0 - z\|^2 \ge \|x_0 - z\|^2 - \|x_{(s+1)\bar{N}} - z\|^2$$

$$= \sum_{k=0}^{s} (\|x_{k\bar{N}} - z\|^2 - \|x_{(k+1)\bar{N}} - z\|^2) \ge 16^{-1}(s+1)\Delta\epsilon_0^2\bar{c},$$

$$s + 1 \le 64M^2\Delta^{-1}\bar{c}^{-1}\epsilon_0^{-2}.$$

Thus we have shown that the following property holds:

(P3) if an integer $s \ge 0$ and for each integer $k \in [0, s]$, (9.55) holds, then (see (9.87))

$$s \le 64M^2\Delta^{-1}\bar{c}^{-1}\epsilon_0^{-2} - 1,$$

$$\|x_k\| \le 3M + 1, \ k = 0, \dots, (s+1)\bar{N}.$$

Property (P3), (9.28), (9.29), and (9.55) imply that there exists an integer $q \in \{0, \dots, n_0\}$ such that for each integer k satisfying $0 \le k < q$,

$$\max\{\lambda_i : \ i = k\bar{N} + 1, \dots, (k+1)\bar{N}\} > \epsilon_0,$$

$$\max\{\lambda_i : \ i = q\bar{N} + 1, \dots, (q+1)\bar{N}\} \le \epsilon_0.$$

By property (P3), (9.40), (9.54), (9.55), the choice of q and the inequalities above,

$$\|x_k\| \le 3M + 1, \ k = 0, \dots, q\bar{N}.$$

Assume that $q \ge 0$ is an integer and that

$$\lambda_i \le \epsilon_0, \ i = q\bar{N} + 1, \dots, (q+1)\bar{N}. \tag{9.88}$$

In view of (9.44), (9.47), and (9.88), for each $j \in \{q\bar{N}, \dots, (q+1)\bar{N} - 1\}$, each $T = (T_1, \dots, T_{p(T)}) \in \Omega_{j+1}$ and each $i \in \{1, \dots, p(T)\}$,

$$\lambda_{j+1} \le \epsilon_0, \ \lambda_{j,T} \le \epsilon_0, \tag{9.89}$$

$$\|y_{i-1}^{(j,T)} - y_i^{(j,T)}\| \le \epsilon_0. \tag{9.90}$$

By (9.22), (9.45), and (9.90), for each $j \in \{q\bar{N}, \dots, (q+1)\bar{N} - 1\}$, each $T = (T_1, \dots, T_{p(T)}) \in \Omega_{j+1}$ and each $i \in \{1, \dots, p(T)\}$,

$$\|x_j - y_i^{(j,T)}\| \le \epsilon_0 i \le \epsilon_0 \bar{q}, \tag{9.91}$$

$$\|x_j - y_{j,T}\| \le \epsilon_0 \bar{q}. \tag{9.92}$$

It follows from (9.29), (9.46), and (9.91) that for each $j \in \{q\bar{N}, \dots, (q+1)\bar{N} - 1\}$, each $T = (T_1, \dots, T_{p(T)}) \in \Omega_{j+1}$ and each $i \in \{1, \dots, p(T)\}$ that

$$\|x_j - T_i(y_{i-1}^{(j,T)})\| \le \|x_j - y_i^{(j,T)}\| + \|y_i^{(j,T)} - T_i(y_{i-1}^{(j,T)})\| \le \epsilon_0 \bar{q} + \delta \le \epsilon_0(\bar{q}+1). \tag{9.93}$$

By (9.16), (9.29), (9.43), (9.92) and the convexity of the norm, for each $j \in \{q\bar{N}, \dots, (q+1)\bar{N} - 1\}$,

$$\|x_{j+1} - x_j\| \le \|x_{j+1} - \sum_{T \in \Omega_{j+1}} w_{j+1}(T) y_{j,T}\| + \|\sum_{T \in \Omega_{j+1}} w_{j+1}(T) y_{j,T} - x_j\|$$

$$\le \delta + \sum_{T \in \Omega_{j+1}} w_{j+1}(T) \|y_{j,T} - x_j\| \le \delta + \epsilon_0 \bar{q} \le \epsilon_0(\bar{q}+1). \tag{9.94}$$

In view of (9.91) and (9.93), for each $j \in \{q\bar{N}, \dots, (q+1)\bar{N} - 1\}$, each $T = (T_1, \dots, T_{p(T)}) \in \Omega_{j+1}$ and each $i \in \{1, \dots, p(T)\}$,

$$\|y_{i-1}^{(j,T)} - T_i(y_{i-1}^{(j,T)})\| \le \|y_{i-1}^{(j,T)} - x_j\| + \|x_j - T_i(y_{i-1}^{(j,T)})\| \le \epsilon_0(2\bar{q}+1). \tag{9.95}$$

Relation (9.94) implies that for all $j_1, j_2 \in \{q\bar{N}, \dots, (q+1)\bar{N}\}$,

$$\|x_{j_1} - x_{j_2}\| \le \epsilon_0 \bar{N}(\bar{q}+1). \tag{9.96}$$

Let

$$Q \in \mathcal{L}_2. \tag{9.97}$$

Property (P1), (9.31) and (9.97) imply that there exist $j \in \{q\bar{N}, \dots, (q+1)\bar{N} - 1\}$, $T = (T_1, \dots, T_{p(T)}) \in \Omega_{j+1}$ and $s \in \{1, \dots, p(T)\}$ such that

$$Q = T_s. \tag{9.98}$$

In view of (9.95) and (9.98),

$$y_{s-1}^{(j,T)} \in \mathrm{Fix}_{\epsilon_0(2\bar{q}+1)}(Q). \tag{9.99}$$

By (9.91) and (9.99),

$$d(x_j, \mathrm{Fix}_{\epsilon_0(2\bar{q}+1)}(Q)) \le \|x_j - y_{s-1}^{(j,T)}\| \le \epsilon_0 \bar{q}.$$

Together with (9.30) and (9.96) this implies that for all $i \in \{q\bar{N}, \dots, (q+1)\bar{N}\}$,

$$d(x_i, \mathrm{Fix}_{\epsilon_0(2\bar{q}+1)}(Q)) \le \epsilon_0(\bar{q}+1)(\bar{N}+1) \le \epsilon_1 \tag{9.100}$$

for all $Q \in \mathcal{L}_2$.

Let

$$Q \in \mathcal{L}_1. \tag{9.101}$$

Property (P2), (9.31) and (9.101) imply that there exist $j \in \{q\bar{N}, \dots, (q+1)\bar{N}-1\}$, $T = (T_1, \dots, T_{p(T)}) \in \Omega_{j+1}$,

$$s \in \{1, \dots, p(T)\},\ c \ge \bar{\lambda} \tag{9.102}$$

such that

$$P_{Q,c} = T_s. \tag{9.103}$$

By (9.95) and (9.101)–(9.103),

$$\|y_{s-1}^{(j,T)} - P_{Q,c}(y_{s-1}^{(j,T)})\| \le \epsilon_0(2\bar{q}+1). \tag{9.104}$$

Set

$$\xi = P_{Q,c}(y_{s-1}^{(j,T)}). \tag{9.105}$$

In view of (9.1) and (9.105),

$$\begin{aligned} y_{s-1}^{(j,T)} &\in (I + cQ)(\xi), \\ y_{s-1}^{(j,T)} - \xi &\in cQ(\xi), \\ c^{-1}(y_{s-1}^{(j,T)} - \xi) &\in Q(\xi). \end{aligned} \tag{9.106}$$

It follows from (9.29), (9.30), (9.46), (9.90), (9.102), (9.103), and (9.105) that

$$\begin{aligned} &\|c^{-1}(y_{s-1}^{(j,T)} - \xi)\| \le \bar{\lambda}^{-1}(\|y_{s-1}^{(j,T)} - y_s^{(j,T)}\| + \|y_s^{(j,T)} - \xi)\|) \\ &= \bar{\lambda}^{-1}(\epsilon_0 + \|y_s^{(j,T)} - T_s(y_{s-1}^{(j,T)})\|) \le \bar{\lambda}^{-1}(\epsilon_0 + \delta) \le 2\bar{\lambda}^{-1}\epsilon_0 \le \epsilon_1. \end{aligned} \tag{9.107}$$

By (9.106) and (9.107),

$$\xi \in F_{\epsilon_1}(Q). \tag{9.108}$$

In view of (9.104), (9.105) and (9.108),

$$d(y_{s-1}^{(j,T)}, F_{\epsilon_1}(Q)) \le \epsilon_0(2\bar{q}+1). \tag{9.109}$$

It follows from (9.91) and (9.109) that

$$d(x_j, F_{\epsilon_1}(Q)) \le \|x_j - y_{s-1}^{(j,T)}\| + d(y_{s-1}^{(j,T)}, F_{\epsilon_1}(Q))$$
$$\le \epsilon_0\bar{q} + \epsilon_0(2\bar{q} + 1) = \epsilon_0(3\bar{q} + 1).$$

Together with (9.30) and (9.96) this implies that for all $i \in \{q\bar{N}, \dots, (q + 1)\bar{N}\}$,

$$d(x_i, F_{\epsilon_1}(Q)) \le \epsilon_0(\bar{q} + 1)(\bar{N} + 4) \le \epsilon_1$$

for all $Q \in \mathcal{L}_1$. This implies that

$$x_i \in \tilde{F}_{\epsilon_1} \text{ for all } i \in \{q\bar{N}, \dots, (q + 1)\bar{N}\}.$$

Theorem 9.1 is proved. □

9.3 Proof of Theorem 9.2

Set

$$\gamma_0 = \epsilon(\bar{N} + 1)^{-1}\bar{q}^{-1} \min\{1, \bar{\lambda}\}. \tag{9.110}$$

By (9.36) there exists

$$z \in B(0, M) \cap F. \tag{9.111}$$

Let $k \ge 0$ be an integer. By (9.39),

$$(x_{k+1}, \lambda_{k+1}) \in A(x_k, (\Omega_{k+1}, w_{k+1}), 0). \tag{9.112}$$

By (9.25) and (9.112) there exists

$$(y_{k,T}, \lambda_{k,T}) \in A_0(x_k, T, 0),\ T \in \Omega_{k+1} \tag{9.113}$$

such that

$$x_{k+1} = \sum_{T \in \Omega_{k+1}} w_{k+1}(T) y_{k,T}, \tag{9.114}$$

$$\lambda_{k+1} = \max\{\lambda_{k,T} :\ T \in \Omega_{k+1}\}. \tag{9.115}$$

Let

$$T = (T_1, \dots, T_{p(T)}) \in \Omega_{k+1}.$$

It follows from (9.24) and (9.113) that there exists $\{y_i^{(k,T)}\}_{i=0}^{p(T)} \subset X$ such that

$$y_0^{(k,T)} = x_k, \ y_{p(T)}^{(k,T)} = y_{k,T}, \tag{9.116}$$

$$y_i^{(k,T)} = T_i(y_{i-1}^{(k,T)}) \text{ for each integer } i = 1, \dots, p(T), \tag{9.117}$$

$$\lambda_{k,T} = \max\{\|y_i^{(k,T)} - y_{i-1}^{(k,T)}\| : \ i = 0, \dots, p(T)\}. \tag{9.118}$$

Let

$$\begin{aligned} T = (T_1, \dots, T_{p(T)}) \in \Omega_{k+1}, \\ i \in \{1, \dots, p(T)\}. \end{aligned} \tag{9.119}$$

By (9.12), (9.37), (9.111), (9.119), (9.117) and Lemma 8.19,

$$\begin{aligned} \|z - y_{i-1}^{(k,T)}\|^2 &\geq \|z - T_i(y_{i-1}^{(k,T)})\|^2 + \bar{c}\|y_{i-1}^{(k,T)} - T_i(y_{i-1}^{(k,T)})\|^2 \\ &= \|z - y_i^{(k,T)}\|^2 + \bar{c}\|y_{i-1}^{(k,T)} - y_i^{(k,T)})\|^2. \end{aligned} \tag{9.120}$$

It follows from (9.116), (9.118), and (9.120) that

$$\begin{aligned} \|z - x_k\|^2 - \|z - y_{k,T}\|^2 &= \|z - y_0^{(k,T)}\|^2 - \|z - y_{p(T)}^{(k,T)}\|^2 \\ &= \sum_{j=1}^{p(T)} (\|z - y_{j-1}^{(k,T)}\|^2 - \|z - y_j^{(k,T)}\|^2) \\ &\geq \bar{c} \sum_{j=1}^{p(T)} \|y_{j-1}^{(k,T)} - y_j^{(k,T)}\|^2 \geq \bar{c}\lambda_{k,T}^2. \end{aligned} \tag{9.121}$$

By (9.16), (9.23), (9.114), (9.121), (9.145) and the convexity of the function $\|\cdot\|^2$,

$$\begin{aligned} \|z - x_{k+1}\|^2 &= \|z - \sum_{T \in \Omega_{k+1}} w_{k+1}(T) y_{k,T}\|^2 \\ &\leq \sum_{T \in \Omega_{k+1}} w_{k+1}(T) \|z - y_{k,T}\|^2 \\ &\leq \sum_{T \in \Omega_{k+1}} w_{k+1}(T)(\|z - x_k\|^2 - \bar{c}\lambda_{k,T}^2) \\ &\leq \|z - x_k\|^2 - \bar{c}\Delta \sum_{T \in \Omega_{k+1}} \lambda_{k,T}^2 \leq \|z - x_k\|^2 - \bar{c}\Delta\lambda_{k+1}^2. \end{aligned} \tag{9.122}$$

In view of (9.38), (9.111), and (9.122), for each natural number n,

$$\begin{aligned}4M^2 &\geq \|z - x_0\|^2 \geq \|z - x_0\|^2 - \|z - x_n\|^2 \\ &= \sum_{i=0}^{n-1} [\|z - x_i\|^2 - \|z - x_{i+1}\|^2] \\ &\geq \sum_{i=0}^{n-1} \bar{c}\Delta\lambda_{i+1}^2 \geq \bar{c}\Delta\gamma_0^2 \text{Card}(\{i \in \{0, \dots, n-1\} : \lambda_{i+1} \geq \gamma_0\}).\end{aligned}$$

Since the relation above holds for any natural number n we conclude that

$$\text{Card}(\{i \in \{0, 1, \dots\} : \lambda_{i+1} \geq \gamma_0\}) \leq 4M^2\bar{c}^{-1}\Delta^{-1}\gamma_0^{-2}. \tag{9.123}$$

Set

$$\begin{aligned}E = \{k \in \{0, 1, \dots\} : &\text{ there is an integer } i \in [k, k + \bar{N} - 1] \\ &\text{such that } \lambda_{i+1} \geq \gamma_0\}.\end{aligned} \tag{9.124}$$

By (9.123) and (9.124),

$$\begin{aligned}\text{Card}(E) &\leq 4M^2\bar{N}\bar{c}^{-1}\Delta^{-1}\gamma_0^{-2} \\ &\leq 4M^2\bar{N}\bar{c}^{-1}\Delta^{-1}\epsilon^{-2}\bar{q}^2(\bar{N} + 1)^2(\min\{1, \bar{\lambda}\})^{-2}.\end{aligned} \tag{9.125}$$

Assume that an integer $q \geq 0$ satisfies

$$q \notin E. \tag{9.126}$$

In view of (9.124) and (9.126),

$$\lambda_{k+1} < \gamma_0 \text{ for all } k \in \{q, \dots, q + \bar{N} - 1\}. \tag{9.127}$$

It follows from (9.22), (9.115)–(9.118), (9.127), and the convexity of the norm that for each $k \in \{q, \dots, q + \bar{N} - 1\}$, each $T = (T_1, \dots, T_{p(T)}) \in \Omega_{k+1}$ and each $j \in \{1, \dots, p(T)\}$,

$$\gamma_0 > \|y_j^{(k,T)} - y_{j-1}^{(k,T)}\| = \|y_{j-1}^{(k,T)} - T_j(y_{j-1}^{(k,T)})\|, \tag{9.128}$$

$$\|x_k - y_j^{(k,T)}\|, \ \|x_k - y_{j-1}^{(k,T)}\| \leq \gamma_0 j \leq \bar{q}\gamma_0, \tag{9.129}$$

$$\|x_k - x_{k+1}\| \leq \bar{q}\gamma_0. \tag{9.130}$$

Relation (9.130) implies that for each $k_1, k_2 \in \{q, \dots, q + \bar{N}\}$,

$$\|x_{k_1} - x_{k_2}\| \leq \gamma_0\bar{N}\bar{q}. \tag{9.131}$$

Let

$$Q \in \mathcal{L}_2. \tag{9.132}$$

Property (P1) and (9.132) imply that there exist

$$k \in \{q, \dots, q+\bar{N}-1\},\ T = (T_1, \dots, T_{p(T)}) \in \Omega_{k+1},\ s \in \{1, \dots, p(T)\} \tag{9.133}$$

such that

$$Q = T_s. \tag{9.134}$$

In view of (9.128), (9.133), and (9.134),

$$y_{s-1}^{(k,T)} \in \mathrm{Fix}_{\gamma_0}(Q). \tag{9.135}$$

By (9.129), (9.133), and (9.135),

$$d(x_k, \mathrm{Fix}_{\gamma_0}(Q)) \le \|x_k - y_{s-1}^{(k,T)}\| \le \gamma_0 \bar{q}. \tag{9.136}$$

It follows from (9.110), (9.131), (9.133), and (9.136) that

$$d(x_q, \mathrm{Fix}_{\gamma_0}(Q)) \le \|x_q - x_k\| + d(x_k, \mathrm{Fix}_{\gamma_0}(Q)) \le \bar{N}\bar{q}\gamma_0 + \bar{q}\gamma_0$$

and

$$d(x_q, \mathrm{Fix}_{\epsilon}(Q)) \le \epsilon \text{ for all } Q \in \mathcal{L}_2. \tag{9.137}$$

Let

$$Q \in \mathcal{L}_1. \tag{9.138}$$

Property (P2), (9.37), and (9.138) imply that there exist

$$k \in \{q, \dots, q+\bar{N}-1\},\ T = (T_1, \dots, T_{p(T)}) \in \Omega_{k+1},\ s \in \{1, \dots, p(T)\},\ c \ge \bar{\lambda} \tag{9.139}$$

such that

$$P_{Q,c} = T_s. \tag{9.140}$$

By (9.117), (9.128), (9.139), and (9.140),

$$\gamma_0 > \|y_{s-1}^{(k,T)} - T_s(y_{s-1}^{(k,T)})\| = \|y_{s-1}^{(k,T)} - P_{Q,c}(y_{s-1}^{(k,T)})\|. \tag{9.141}$$

By (9.1), (9.117), (9.139), and (9.140),

$$y_{s-1}^{(k,T)} \in (I + cQ)(y_s^{(k,T)}). \tag{9.142}$$

In view of (9.142),

$$\begin{aligned} y_{s-1}^{(k,T)} - y_s^{(k,T)} &\in cQ(y_s^{(k,T)}), \\ c^{-1}(y_{s-1}^{(j,T)} - y_s^{(k,T)}) &\in Q(y_s^{(k,T)}). \end{aligned} \tag{9.143}$$

It follows from (9.110), (9.128), (9.139), and (9.143) that

$$\begin{aligned} \|c^{-1}(y_{s-1}^{(k,T)} - y_s^{(k,T)})\| &\leq \bar{\lambda}^{-1}\gamma_0, \\ y_s^{(k,T)} &\in F_{\bar{\lambda}^{-1}\gamma_0}(Q) \subset F_\epsilon(Q). \end{aligned} \tag{9.144}$$

In view of (9.129), (9.139), and (9.144),

$$d(x_k, F_\epsilon(Q)) \leq \|x_k - y_s^{(k,T)}\| \leq \bar{q}\gamma_0. \tag{9.145}$$

By (9.111), (9.131), and (9.145),

$$\begin{aligned} d(x_q, F_\epsilon(Q)) &\leq \|x_q - x_k\| + d(x_k, F_\epsilon(Q)) \\ &\leq \gamma_0\bar{q}\bar{N} + \gamma_0\bar{q}, \\ d(x_q, F_\epsilon(Q)) &\leq \epsilon \text{ for all } Q \in \mathcal{L}_1. \end{aligned}$$

Together with (9.137) this implies that

$$x_q \in \tilde{F}_\epsilon.$$

Theorem 9.2 is proved. □

Chapter 10
Convex Feasibility Problems

We use subgradient projection algorithms for solving convex feasibility problems. We show that almost all iterates, generated by a subgradient projection algorithm in a Hilbert space, are approximate solutions. Moreover, we obtain an estimate of the number of iterates which are not approximate solutions. In a finite-dimensional case, we study the behavior of the subgradient projection algorithm in the presence of computational errors. Provided computational errors are bounded, we prove that our subgradient projection algorithm generates a good approximate solution after a certain number of iterates.

10.1 Iterative Methods in Infinite-Dimensional Spaces

Let $(X, \langle \cdot, \cdot \rangle)$ be a Hilbert space with an inner product $\langle \cdot, \cdot \rangle$, which induces a complete norm $\| \cdot \|$. For each $x \in X$ and each nonempty set $A \subset X$ put

$$d(x, A) := \inf\{||x - y|| : \ y \in A\}.$$

For each $x \in X$ and each $r > 0$ set

$$B(x, r) = \{y \in X : \ ||x - y|| \leq r\}.$$

It is well known that the following proposition holds (see Fact 1.5 and Lemma 2.4 of [7]).

Proposition 10.1. *Let C be a nonempty, closed and convex subset of X. Then, for each $x \in X$, there is a unique point $P_C(x) \in C$ satisfying*

$$\|x - P_C(x)\| = d(x, C).$$

A. Zaslavski, *Approximate Solutions of Common Fixed-Point Problems*, Springer Optimization and Its Applications 112, DOI 10.1007/978-3-319-33255-0_10

Moreover, $\|P_C(x) - P_C(y)\| \le \|x - y\|$ *for all* $x, y \in X$ *and, for each* $x \in X$ *and each* $z \in C$,

$$\langle z - P_C(x), x - P_C(x)\rangle \le 0,$$
$$\|z - P_C(x)\|^2 + \|x - P_C(x)\|^2 \le \|z - x\|^2. \tag{10.1}$$

Let $f : X \to R^1$ be a continuous and convex function such that

$$\{x \in X :\ f(x) \le 0\} \ne \emptyset. \tag{10.2}$$

Let $y_0 \in X$. Then the set

$$\partial f(y_0) := \{l \in X :\ f(y) - f(y_0) \ge \langle l, y - y_0\rangle \text{ for all } y \in X\} \tag{10.3}$$

is the subdifferential of f at the point y_0 [72, 77]. For any $l \in \partial f(y_0)$, in view of (10.3),

$$\{x \in X :\ f(x) \le 0\} \subset \{x \in X :\ f(y_0) + \langle l, x - y_0\rangle \le 0\}. \tag{10.4}$$

It is well known that the following lemma holds (see Lemma 7.3 of [7]).

Lemma 10.2. *Let* $y_0 \in X, f(y_0) > 0$, $l \in \partial f(y_0)$ *and let*

$$D := \{x \in X :\ f(y_0) + \langle l, x - y_0\rangle \le 0\}.$$

Then $l \ne 0$ *and* $P_D(y_0) = y_0 - f(y_0)\|l\|^{-2}l$.

Denote by $\mathcal{N}$ the set of all nonnegative integers. Let m be a natural number, $\mathbb{I} = \{1, \dots, m\}$ and $f_i : X \to R^1$, $i \in \mathbb{I}$, be convex and continuous functions. For each $i \in \mathbb{I}$ set

$$C_i := \{x \in X :\ f_i(x) \le 0\},$$
$$C := \cap_{i \in \mathbb{I}} C_i = \cap_{i \in \mathbb{I}}\{x \in X :\ f_i(x) \le 0\}.$$

Suppose that

$$C \ne \emptyset.$$

A point $x \in C$ is called a solution of our feasibility problem. For a given $\epsilon > 0$, a point $x \in X$ is called an ϵ-approximate solution of the feasibility problem if $f_i(x) \le \epsilon$ for all $i \in \mathbb{I}$. We apply the subgradient projection method in order to obtain a good approximative solution of the feasibility problem.

Consider a natural number $\bar{p} \ge m$. Denote by $\mathbb{S}$ the set of all mappings $S : \mathcal{N} \to \mathbb{I}$ such that the following property holds:

(P1) For each integer $N \in \mathcal{N}$ and each $i \in \mathbb{I}$, there is $n \in \{N, \dots, N + \bar{p} - 1\}$ such that $S(n) = i$.

We want to find approximate solutions of the inclusion $x \in C$. In order to meet this goal we apply algorithms generated by $S \in \mathbb{S}$.

For each $x \in X$, each number $\epsilon \geq 0$ and each $i \in \mathbb{I}$ set

$$A_i(x, \epsilon) := \{x\} \text{ if } f_i(x) \leq \epsilon \tag{10.5}$$

and, in view of Lemma 10.2,

$$A_i(x, \epsilon) := x - f_i(x)\{\|l\|^{-2} l : \ l \in \partial f_i(x)\} \text{ if } f_i(x) > \epsilon. \tag{10.6}$$

We associate with any $S \in \mathbb{S}$ the algorithm which generates, for any starting point $x_0 \in X$, a sequence $\{x_n\}_{n=0}^{\infty} \subset X$ such that, for each integer $n \geq 0$,

$$x_{n+1} \in A_{S(n)}(x_n, 0).$$

Note that by Lemma 10.2 the sequence $\{x_n\}_{n=0}^{\infty}$ is well defined, and that for each integer $n \geq 0$, if $f_{S(n)}(x_n) > 0$, then $x_{n+1} = P_{D_n}(x_n)$, where

$$D_n = \{x \in X : \ f(x_n) + \langle l_n, x - x_n \rangle \leq 0\} \text{ and } l_n \in \partial f_{S(n)}(x_n).$$

We will prove the following result (Theorem 10.3) which shows that, for the subgradient projection method considered in the chapter, almost all iterates are good approximate solutions. Denote by Card(A) the cardinality of the set A.

Theorem 10.3. *Let*

$$b > 0, \ \epsilon \in (0, 1], \ \Lambda > 0, \ \gamma \in [0, \epsilon], \tag{10.7}$$

$$c \in B(0, b) \cap C, \tag{10.8}$$

$$|f_i(u) - f_i(v)| \leq \Lambda \|u - v\|, \ u, v \in B(0, 3b + 1), \ i \in \mathbb{I}, \tag{10.9}$$

let a positive number ϵ_0 satisfy

$$\epsilon_0 \leq \epsilon \Lambda^{-1} \tag{10.10}$$

and let a natural number n_0 satisfy

$$4\bar{p}\epsilon_0^{-2} b^2 \leq n_0. \tag{10.11}$$

Assume that

$$S \in \mathbb{S}, \ x_0 \in B(0, b), \tag{10.12}$$

and that for each integer $n \geq 0$,

$$x_{n+1} \in A_{S(n)}(x_n, \gamma). \tag{10.13}$$

Then

$$\|x_n\| \leq 3b \text{ for all integers } n \geq 0 \tag{10.14}$$

and

$$Card(\{N \in \mathcal{N} : \max\{\|x_{n+1}-x_n\| : n = N, \dots, N+\bar{p}-1\} > \epsilon_0\}) \leq n_0. \tag{10.15}$$

Moreover, if an integer $N \geq 0$ *satisfies*

$$\|x_{n+1} - x_n\| \leq \epsilon_0, \ n = N, \dots, N + \bar{p} - 1,$$

then, for all integers $n, m \in \{N, \dots, N + \bar{p}\}$, $\|x_n - x_m\| \leq \bar{p}\epsilon_0$ *and for all integers* $n = N, \dots, N + \bar{p}$ *and each* $i \in \mathbb{I}$, $f_i(x_n) \leq \epsilon(\bar{p} + 1)$.

Theorem 10.3 was obtained in [96].

10.2 Proof of Theorem 10.3

By (10.5), (10.6), and (10.13), there exists a sequence $\{l_n\}_{n=0}^{\infty} \subset X$ such that

$$x_{n+1} = x_n \text{ if } f_{S(n)}(x_n) \leq \gamma \tag{10.16}$$

and

$$\text{if } f_{S(n)}(x_n) > \gamma, \text{ then } l_n \in \partial f_{S(n)}(x_n) \text{ and}$$
$$x_{n+1} = x_n - f_{S(n)}(x_n)\|l_n\|^{-2} l_n. \tag{10.17}$$

By (10.16), (10.17), (10.8), (10.12), (10.4), Lemma 10.2, and Proposition 10.1 for all integers $n \geq 0$,

$$\|c - x_{n+1}\| \leq \|c - x_n\| \leq 2b, \tag{10.18}$$

$$\|x_n\| \leq 3b \text{ for all integers } n \geq 0. \tag{10.19}$$

Assume that an integer $N \geq 0$ and that

$$\|x_{n+1} - x_n\| \leq \epsilon_0 \text{ for } n = N, \dots, N + \bar{p} - 1. \tag{10.20}$$

This implies that for all $n, m \in \{N, \dots, N + \bar{p}\}$,

$$\|x_n - x_m\| \le \bar{p}\epsilon_0. \tag{10.21}$$

Let $i \in \mathbb{I}$. By (P1), there is $m \in \{N, \dots, N + \bar{p} - 1\}$ such that

$$S(m) = i. \tag{10.22}$$

We show that

$$f_i(x_m) = f_{S(m)}(x_m) \le \epsilon. \tag{10.23}$$

Assume the contrary. Then

$$f_i(x_m) > \epsilon. \tag{10.24}$$

By (10.20), (10.24), (10.7), (10.16), (10.17), and (10.22),

$$\epsilon_0 \ge \|x_{m+1} - x_m\| = \|f_{S(m)}(x_m)\|l_m\|^{-2} l_m\| > \epsilon\|l_m\|^{-1}. \tag{10.25}$$

By (10.24), (10.22), (10.16), (10.17), and (10.7), $l_m \in \partial f_{S(m)}(x_m)$. Combined with (10.19) and (10.9) this implies that

$$\|l_m\| \le \Lambda. \tag{10.26}$$

In view of (10.25) and (10.26), $\epsilon_0 > \epsilon\Lambda^{-1}$. This inequality contradicts (10.10). This contradiction proves (10.23).

Let $n \in \{N, \dots, N + \bar{p}\}$. It follows from (10.22), (10.23), (10.19), (10.9), (10.21), and (10.10) that

$$\begin{aligned} f_i(x_n) &\le f_{S(m)}(x_m) + |f_{S(m)}(x_n) - f_{S(m)}(x_m)| \\ &\le \epsilon + \Lambda\|x_n - x_m\| \le \epsilon + \Lambda\bar{p}\epsilon_0 \le \epsilon(\bar{p} + 1) \end{aligned}$$

and

$$f_i(x_n) \le \epsilon(\bar{p} + 1) \text{ for } n = N, \dots, N + \bar{p}, \tag{10.27}$$

for all integers $i \in \mathbb{I}$.

Thus we have shown that the following property holds:

(P2) if an integer $N \ge 0$ and (10.20) holds, then (10.27) is valid for all $i \in \mathbb{I}$.

Set

$$E_1 = \{n \in \mathcal{N} : \|x_n - x_{n+1}\| \le \epsilon_0\}, \tag{10.28}$$

$$E_2 = \mathcal{N} \setminus E_1, \tag{10.29}$$

$$E_3 = \{n \in \mathcal{N} : \{n, \dots, n + \bar{p} - 1\} \cap E_2 \ne \emptyset\}. \tag{10.30}$$

By (10.18), (10.29), (10.28), (10.16), (10.17), (10.8), Lemma 10.2, and Proposition 10.1 (see (10.1)), for any natural number n,

$$4b^2 \geq \|c - x_0\|^2 \geq \|c - x_0\|^2 - \|c - x_n\|^2$$
$$= \sum_{m=0}^{n-1}[\|c - x_m\|^2 - \|c - x_{m+1}\|^2] \geq \sum_{m \in E_2 \cap [0,n-1]} [\|c - x_m\|^2 - \|c - x_{m+1}\|^2]$$
$$\geq \sum_{m \in E_2 \cap [0,n-1]} \|x_m - x_{m+1}\|^2 \geq \epsilon_0^2 \operatorname{Card}(E_2 \cap [0, n-1])$$

and

$$\operatorname{Card}(E_2 \cap [0, n-1]) \leq 4\epsilon_0^{-2} b^2.$$

Since the inequality above holds for any natural number n, we conclude that

$$\operatorname{Card}(E_2) \leq 4\epsilon_0^{-2} b^2. \tag{10.31}$$

By (10.31), (10.30), and (10.11),

$$\operatorname{Card}(E_3) \leq \operatorname{Card}(E_2)\bar{p} \leq 4\epsilon_0^{-2} b^2 \bar{p} \leq n_0.$$

This completes the proof of Theorem 10.3. □

10.3 Iterative Methods in Finite-Dimensional Spaces

We use all the notation and the definitions introduced in Sect. 10.1 and suppose that all the assumptions made in Sect. 10.1 hold. In this section, we suppose that the space X is finite-dimensional. The results presented in the section were obtained in [96].

We prove the following result, which describes the asymptotic behavior of the subgradient projection method without computational errors.

Theorem 10.4. *Let $b > 0$, $\epsilon \in (0, 1]$ and*

$$c \in B(0, b) \cap C. \tag{10.32}$$

Then there exist a natural number n_0 and $\gamma_0 \in (0, \epsilon]$ such that the following assertion holds.

Assume that

$$\gamma \in [0, \gamma_0],\ S \in \mathbb{S},\ x_0 \in B(0, b) \tag{10.33}$$

and that, for each integer $n \geq 0$,

$$x_{n+1} \in A_{S(n)}(x_n, \gamma). \tag{10.34}$$

Then $\|x_n\| \leq 3b$ *for all integers* $n \geq 0$,

$$f_i(x_n) \leq \epsilon \text{ for all } i \in \mathbb{I} \text{ and all integers } n \geq n_0$$

and $d(x_n, C) \leq \epsilon$ *for all integers* $n \geq n_0$.

Theorem 10.4 is proved in Sect. 10.5.

For each $x \in X$, each $\delta \geq 0$, each $\tilde{\delta} \geq 0$ and each $i \in \mathbb{I}$ set

$$A_i(x, \tilde{\delta}, \delta) := \{x\} \text{ if } f_i(x) \leq \tilde{\delta}, \tag{10.35}$$

and, if $f_i(x) > \tilde{\delta}$, then set

$$A_i(x, \tilde{\delta}, \delta) := \{x - f_i(x)\|l\|^{-2} l : \ l \in \partial f_i(x) + B(0, \delta), \ l \neq 0\} + B(0, \delta). \tag{10.36}$$

The following theorem is one of our main results of this chapter. It describes the behavior of iterates under the presence of computational errors which occur in the calculations of subgradients as well in the calculations of iterates themselves.

Theorem 10.5. *Let* $b > 0$, $\epsilon \in (0, 1]$, *(10.32) hold and let*

$$c_i \in B(0, b) \text{ and } f_i(c_i) < 0, \ i \in \mathbb{I}. \tag{10.37}$$

Then, there exist a natural number n_0 *and* $\delta > 0$ *such that the following assertion holds.*

Assume that

$$\tilde{\delta}_n \in [0, \delta], \ n \in \mathcal{N}, \ S \in \mathbb{S}, \ x_0 \in B(0, b), \tag{10.38}$$

and that, for each integer $n \geq 0$,

$$x_{n+1} \in A_{S(n)}(x_n, \tilde{\delta}_n, \delta). \tag{10.39}$$

Then $\|x_n\| \leq 3b + 1, \ n = 0, \ldots, n_0$, $d(x_{n_0}, C) \leq \epsilon$ *and* $f_i(x_{n_0}) \leq \epsilon$ *for all* $i \in \mathbb{I}$.

This result is proved in Sect. 10.6. Theorem 10.5 easily implies the following result.

Theorem 10.6. *Let* $b > 0$, $\epsilon \in (0, 1]$, *(10.32), and (10.37) hold and let a natural number* n_0 *and* $\delta > 0$ *be given, as guaranteed by Theorem 10.5.*

Assume that (10.38) holds, for each integer $n \geq 0$, *(10.39) holds and that a sequence* $\{x_n\}_{n=0}^{\infty} \subset B(0, b)$. *Then, for all integers* $n \geq n_0$,

$$d(x_n, C) \leq \epsilon \text{ and } f_i(x_n) \leq \epsilon \text{ for all } i \in \mathbb{I}.$$

Theorem 10.6 easily implies the following result.

Theorem 10.7. *Let $b > 0$, (10.32) and (10.37) hold, $\{\delta_n\}_{n=0}^{\infty}$ be a sequence of positive numbers such that $\lim_{n\to\infty} \delta_n = 0$ and let $\epsilon \in (0, 1]$. Then there exists a natural number n_ϵ such that the following assertion holds.*

Assume that $\tilde{\delta}_n \in [0, \delta_n]$ for all $n \in \mathcal{N}$, $S \in \mathbb{S}$, $\{x_n\}_{n=0}^{\infty} \subset B(0, b)$ and that, for each integer $n \geq 0$,

$$x_{n+1} \in A_{S(n)}(x_n, \tilde{\delta}_n, \delta_n).$$

Then, for all integers $n \geq n_\epsilon$, $d(x_n, C) \leq \epsilon$ and $f_i(x_n) \leq \epsilon$ for all $i \in \mathbb{I}$.

In the last two theorems we consider the case when the set C is bounded.

Theorem 10.8. *Suppose that the set C is bounded, (10.32) and (10.37) hold with $b > 0$ and $b_0, \epsilon > 0$. Then there exist a natural number n_0 and $\delta > 0$ such that the following assertion holds.*

Assume that

$$\tilde{\delta}_n \in [0, \delta], \; n \in \mathcal{N}, \; S \in \mathbb{S}, \tag{10.40}$$

$$x_0 \in B(0, b_0) \tag{10.41}$$

and that, for each integer $n \geq 0$,

$$x_{n+1} \in A_{S(n)}(x_n, \tilde{\delta}_n, \delta). \tag{10.42}$$

Then, for all integers $n \geq n_0$, $d(x_n, C) \leq \epsilon$ and $f_i(x_n) \leq \epsilon$ for all $i \in \mathbb{I}$.

Proof. We may assume without any loss of generality that

$$b_0 > \sup\{\|z\| : \; z \in C\} + 4, \; b_0 > \|c_i\|, \; i \in I, \; b > 3b_0 + 1 \text{ and } \epsilon < 1. \tag{10.43}$$

By Theorem 10.5, there exist a natural number n_1 and $\delta_1 > 0$ such that the following property holds:

(P3) for each $\tilde{\delta}_n \in [0, \delta_1]$, $n \in \mathcal{N}$, each $S \in \mathbb{S}$, each $\{x_n\}_{n=0}^{\infty} \subset X$ such that $\|x_0\| \leq b_0$ and that, for each integer $n \geq 0$,

$$x_{n+1} \in A_{S(n)}(x_n, \tilde{\delta}_n, \delta_1),$$

we have $\|x_n\| \leq 3b_0 + 1$, $n = 0, \ldots, n_1$ and $d(x_{n_1}, C) \leq \epsilon$.

By Theorem 10.6, there exist a natural number n_0 and $\delta \in (0, \delta_1)$ such that the following property holds:

(P4) if (10.38) holds and if for each integer $n \geq 0$, (10.39) holds and if $\{x_n\}_{n=0}^{\infty} \subset B(0, b)$, then for all integers $n \geq n_0$,

$$d(x_n, C) \leq \epsilon \text{ and } f_i(x_n) \leq \epsilon \text{ for all } i \in \mathbb{I}. \tag{10.44}$$

Assume that $\tilde{\delta}_n \in [0, \delta]$, $n \in \mathcal{N}$, $S \in \mathbb{S}$, $\{x_n\}_{n=0}^{\infty} \subset X$, (10.41) holds and (10.42) holds for each integer $n \geq 0$. By (P3), (10.41)–(10.43) and the inequality $\delta < \delta_1$,

$$\|x_{nn_1}\| \leq b_0,\ n \in \mathcal{N} \text{ and } \|x_n\| \leq 3b_0 + 1,\ n \in \mathcal{N}. \tag{10.45}$$

By (10.45), (10.43), (10.41), (10.42), and (P4), (10.44) holds for all integers $n \geq n_0$. This completes the proof of Theorem 10.8. □

Theorems 10.7 and 10.8 easily imply the following result.

Theorem 10.9. *Let (10.32) and (10.37) hold with $b > 0$ and the set C be bounded. Then there exists $\delta > 0$ such that the following assertion holds.*

Assume that a sequence $\{\delta_n\}_{n=0}^{\infty} \subset [0, \delta]$ satisfy $\lim_{n\to\infty} \delta_n = 0$ and let $\epsilon > 0$. Then there exists a natural number n_ϵ such that, for each $\tilde{\delta}_n \in [0, \delta_n]$, $n \in \mathcal{N}$, each $S \in \mathbb{S}$ and each $\{x_n\}_{n=0}^{\infty} \subset X$ which satisfies $\|x_0\| \leq b$ and

$$x_{n+1} \in A_{S(n)}(x_n, \tilde{\delta}_n, \delta_n) \text{ for each integer } n \geq 0,$$

the following relations hold:

$$d(x_n, C) \leq \epsilon \text{ and } f_i(x_n) \leq \epsilon \text{ for all } i \in \mathbb{I} \text{ and all integers } n \geq n_\epsilon.$$

10.4 Auxiliary Results

We use the notation and the definitions introduced in Sect. 10.3 and suppose that all the assumptions made there hold.

Lemma 10.10. *Let $M > 0$, $\gamma_1 > 0$. Then there exists $\gamma_2 > 0$ such that, for each $x \in B(0, M)$ satisfying $f_i(x) \leq \gamma_2$, $i \in \mathbb{I}$, the inequality $d(x, C) \leq \gamma_1$ holds.*

Proof. Assume the contrary. Then, for any natural number n, there is $x_n \in B(0, M)$ such that

$$f_i(x_n) \leq 1/n,\ i \in \mathbb{I} \text{ and } d(x_n, C) > \gamma_1. \tag{10.46}$$

Extracting a subsequence and re-indexing, if necessary, we may assume without any loss of generality that there is $x = \lim_{n\to\infty} x_n$. It is easy to see that $x \in B(0, M)$, $f_i(x) \leq 0$ for all $i \in \mathbb{I}$ and $x \in C$. Clearly,

$$d(x_n, C) \leq \|x_n - x\| < \gamma_1/2$$

for all sufficiently large natural numbers n. This contradicts (10.46). The contradiction we have reached completes the proof of Lemma 10.10. □

10.5 Proof of Theorem 10.4

Since the functions $f_i,\ i \in \mathbb{I}$ are convex [50], there exists $\Lambda > 0$ such that

$$|f_i(u) - f_i(v)| \le \Lambda\|u - v\| \text{ for all } u, v \in B(0, 3b + 1),\ i \in \mathbb{I}. \tag{10.47}$$

Choose a positive number $\gamma_1 < \epsilon$ such that

$$\Lambda\gamma_1 < \epsilon. \tag{10.48}$$

By Lemma 10.10, there exists $\gamma_2 \in (0, \epsilon)$ such that the following property holds:

(P5) for each $y \in B(0, 3b + 1)$ satisfying $f_i(y) \le \gamma_2$, $i \in \mathbb{I}$ we have $d(y, C) < \gamma_1$.

Choose a positive number γ_0 such that

$$\gamma_0 < \gamma_1 \text{ and } \gamma_0(\bar{p} + 1) < \gamma_2. \tag{10.49}$$

By (10.47) and Theorem 10.3 (with $\epsilon = \gamma_0$), there exists a natural number n_0 such that the following property holds:

(P6) Let $\gamma \in [0, \gamma_0]$, $S \in \mathbb{S}$, $x_0 \in B(0, b)$ and let for each integer $n \ge 0$,

$$x_{n+1} \in A_{S(n)}(x_n, \gamma).$$

Then

$$\|x_n\| \le 3b \text{ for all integers } n \ge 0, \tag{10.50}$$

and there is an integer $q \in [0, n_0]$ such that

$$f_i(x_q) \le \gamma_0(\bar{p} + 1),\ i \in \mathbb{I}. \tag{10.51}$$

Assume that (10.33) holds and that (10.34) holds for each integer $n \ge 0$. Together with (P6) this implies that (10.50) holds and that there is an integer $q \in [0, n_0]$ such that (10.51) holds. By (10.49) and (10.51), $f_i(x_q) \le \gamma_2$ for all $i \in \mathbb{I}$. Together with (P5) and (10.51), this implies that $d(x_q, C) < \gamma_1$ and that there is $\tilde{z} \in X$ such that

$$\tilde{z} \in C \text{ and } \|x_q - \tilde{z}\| < \gamma_1. \tag{10.52}$$

By (10.52), (10.33), (10.34), (10.5), (10.6), (10.4). Lemma 10.2, and Proposition 10.1 (see (10.1)),

$$\|x_n - \tilde{z}\| < \gamma_1 < \epsilon \text{ for all integers } n \ge q. \tag{10.53}$$

In view of (10.53) and (10.50), $\|\tilde{z}\| \le 3b+1$. Together with (10.52), (10.47), (10.53), (10.50), and (10.48), this implies that for all integers $n \ge n_0$ and all $i \in \mathbb{I}$,

$$f_i(x_n) \le f_i(\tilde{z}) + |f_i(x_n) - f_i(\tilde{z})| \le \Lambda\|x_n - \tilde{z}\| < \Lambda\gamma_1 < \epsilon.$$

This completes the proof of Theorem 10.4. □

10.6 Proof of Theorem 10.5

Put

$$r = \min\{-f_i(c_i) : i \in \mathbb{I}\}. \tag{10.54}$$

By (10.54) and (10.37),

$$r > 0. \tag{10.55}$$

Since the functions f_i, $i \in \mathbb{I}$ are convex [50], there exists $\Lambda > 0$ such that

$$|f_i(u) - f_i(v)| \le \Lambda \|u - v\| \text{ for all } u, v \in B(0, 3b + 2),\ i \in \mathbb{I}, \tag{10.56}$$

$$|f_i(u)| \le \Lambda \text{ for all } u \in B(0, 3b + 2),\ i \in \mathbb{I}. \tag{10.57}$$

By Theorem 10.4, there exist a natural number n_0 and $\bar{\gamma}_0 \in (0, \epsilon]$ such that the following property holds:

(P7) If $\gamma \in [0, \bar{\gamma}_0]$, $S \in \mathbb{S}$, $x_0 \in B(0, b)$ and, if for each integer $n \ge 0$,

$$x_{n+1} \in A_{S(n)}(x_n, \gamma),$$

then

$$\|x_n\| \le 3b \text{ for all integers } n \ge 0, \tag{10.58}$$

$$f_i(x_n) \le \epsilon/4 \text{ for all } i \in \mathbb{I} \text{ and all integers } n \ge n_0, \tag{10.59}$$

$$d(x_n, C) \le \epsilon/4 \text{ for all integers } n \ge n_0. \tag{10.60}$$

By (10.56), for each $u \in B(0, 3b + 1)$, all $i \in \mathbb{I}$ and each $g \in \partial f_i(u)$,

$$\|g\| \le \Lambda. \tag{10.61}$$

Let

$$u \in B(0, 3b + 1),\ i \in \mathbb{I},\ f_i(u) > 0,\ g \in \partial f_i(u). \tag{10.62}$$

By (10.62), (10.54), (10.55), and (10.37),

$$-r \ge f_i(c_i) > f_i(c_i) - f_i(u) \ge \langle g, c_i - u\rangle \ge -\|g\|(4b + 1)$$

and

$$\|g\| > r(4b + 1)^{-1}. \tag{10.63}$$

We have shown that the following property holds:

(P8) if $u \in B(0, 3b + 1)$, $i \in \mathbb{I}$, $f_i(u) > 0$ and if $g \in X$ satisfies

$$d(g, \partial f_i(u)) \le r(4b + 1)^{-1}4^{-1},$$

then $\|g\| > r(4b + 1)^{-1}2^{-1}$.

For each $\gamma \ge 0$ denote by $\mathcal{M}_\gamma$ the set of all sequences $\{x_n\}_{n=0}^{\infty} \subset X$ for which $\|x_0\| \le b$, and there exist $\tilde{\gamma}_n \in [0, \gamma]$, $n \in \mathcal{N}$, $S \in \mathbb{S}$ such that, for each integer $n \ge 0$,

$$x_{n+1} \in A_{S(n)}(x_n, \tilde{\gamma}_n, \gamma).$$

By induction we show that for all $m = 0, \dots, n_0$ the following assertion holds.

(A) For each $\gamma > 0$ there exists $\delta > 0$ such that, for each $\{x_n\}_{n=0}^{\infty} \in \mathcal{M}_\delta$, there is $\{y_n\}_{n=0}^{\infty} \in \mathcal{M}_0$ such that $\|y_n - x_n\| \le \gamma$, $n = 0, \dots, m$.

Clearly, for $m = 0$ this assertion holds. Assume that assertion (A) holds for $m = q$ where $q \in [0, n_0 - 1]$ is an integer. We show that (A) holds for $m = q + 1$. Since (A) holds for $m = q$, it follows from (P7) and (10.58) that there is $\gamma_0 > 0$ such that

$$\gamma_0 < 2^{-1} \text{ and } \gamma_0 < 4^{-1}r(4b + 1)^{-1} \tag{10.64}$$

and that, for each $\{y_n\}_{n=0}^{\infty} \in \mathcal{M}_{\gamma_0}$,

$$\|y_n\| \le 3b + 1/2,\ n = 0, \dots, q. \tag{10.65}$$

Assume that assertion (A) does not hold for $m = q + 1$. Then there exists $\gamma > 0$ such that for each natural number j there is

$$\{x_n^{(j)}\}_{n=0}^{\infty} \in \mathcal{M}_{\gamma_0/j} \tag{10.66}$$

such that

$$\max\{\|y_n - x_n^{(j)}\| :\ n = 0, \dots, q + 1\} > \gamma \text{ for each } \{y_n\}_{n=0}^{\infty} \in \mathcal{M}_0. \tag{10.67}$$

By (10.66) and the choice of γ_0 (see (10.65)), for all natural numbers j,

$$\|x_n^{(j)}\| \le 3b + 1/2,\ n = 0, \dots, q. \tag{10.68}$$

By the definition of $\mathcal{M}_\gamma$, $\gamma \ge 0$, (10.36), (10.37), and (10.66), for each integer $j \ge 1$ there is

$$\tilde{\gamma}_{j,n} \in [0, \gamma_0/j],\ n \in \mathcal{N},\ S_j \in \mathbb{S},\ \{g_n^{(j)}\}_{n=0}^{\infty} \subset X \tag{10.69}$$

such that, for each integer $n \in \{0, \dots, q+1\}$ satisfying $f_{S_j(n)}(x_n^{(j)}) > \tilde{\gamma}_{j,n}$,

$$g_n^{(j)} \neq 0 \text{ and } d(g_n^{(j)}, \partial f_{S_j(n)}(x_n^{(j)})) \leq \gamma_0/j, \tag{10.70}$$

$$\|x_{n+1}^{(j)} - (x_n^{(j)} - f_{S_j(n)}(x_n^{(j)})\|g_n^{(j)}\|^{-2} g_n^{(j)})\| \leq \gamma_0/j \tag{10.71}$$

and that, for each integer $n \in \{0, \dots, q+1\}$ satisfying $f_{S_j(n)}(x_n^{(j)}) \leq \tilde{\gamma}_{j,n}$,

$$g_n^{(j)} = 0, \; x_{n+1}^{(j)} = x_n^{(j)}. \tag{10.72}$$

Extracting a subsequence and re-indexing, if necessary, we may assume that

$$S_j(n) = S_1(n) \text{ for all natural numbers } j \text{ and all } n \in \mathcal{N}. \tag{10.73}$$

Put

$$S(n) = S_1(n), \; n \in \mathcal{N}. \tag{10.74}$$

By (10.74), (10.73), (10.70), (10.68), (10.64), (P8), and (10.61), for all $j = 1, 2, \dots$ and all $n = 0, \dots, q$,

$$\text{if } f_{S(n)}(x_n^{(j)}) > \tilde{\gamma}_{j,n}, \text{ then } 2^{-1}(4b+1)^{-1}r \leq \|g_n^{(j)}\| \leq \Lambda + 1. \tag{10.75}$$

Assume that j is a natural number. We estimate $\|x_{q+1}^{(j)}\|$. If $f_{S(q)}(x_q^{(j)}) \leq \tilde{\gamma}_{j,q}$, then, in view of (10.74), (10.72), and (10.68),

$$\|x_{q+1}^{(j)}\| = \|x_q^{(j)}\| \leq 3b + 2^{-1}. \tag{10.76}$$

If $f_{S(q)}(x_q^{(j)}) > \tilde{\gamma}_{j,q}$, then by (10.70), (10.71), (10.74), (10.64), (10.68), (10.57), and (10.75),

$$\begin{aligned} \|x_{q+1}^{(j)}\| &\leq \gamma_0 j^{-1} + \|x_q^{(j)} - f_{S(q)}(x_q^{(j)})\|g_q^{(j)}\|^{-2} g_q^{(j)}\| \\ &\leq 1 + 3b + 2^{-1} + \Lambda\|g_q^{(j)}\|^{-1} \leq 3/2 + 3b + 2\Lambda(4b+1)r^{-1}. \end{aligned}$$

Thus for all $j = 1, 2, \dots,$

$$\|x_{q+1}^{(j)}\| \leq 3/2 + 3b + 2\Lambda(4b+1)r^{-1}. \tag{10.77}$$

By (10.77) and (10.68), extracting a subsequence and re-indexing, if necessary, we may assume without any loss of generality that for any $n \in \{0, \dots, q+1\}$ there is

$$y_n = \lim_{j \to \infty} x_n^{(j)} \tag{10.78}$$

and that for any $n \in \{0, \dots, q+1\}$ one of the following cases holds:

$$f_{S(n)}(x_n^{(j)}) \le \tilde{\gamma}_{j,n} \text{ for all natural numbers } j, \tag{10.79}$$

$$f_{S(n)}(x_n^{(j)}) > \tilde{\gamma}_{j,n} \text{ for all natural numbers } j. \tag{10.80}$$

For all $n = 0, 1, 2, \dots, q+1$ and all $j = 1, 2, \dots$ choose $\tilde{g}_n^{(j)} \in X$ as follows:

$$\tilde{g}_n^{(j)} = 0 \text{ if } f_{S(n)}(x_n^{(j)}) \le \tilde{\gamma}_{j,n}, \tag{10.81}$$

$$\text{if } f_{S(n)}(x_n^{(j)}) > \tilde{\gamma}_{j,n}, \text{ then } \tilde{g}_n^{(j)} \in \partial f_{S(n)}(x_n^{(j)}),\ \|\tilde{g}_n^{(j)} - g_n^{(j)}\| \le 2\gamma_0 j^{-1}. \tag{10.82}$$

In view (10.69)–(10.74), $\tilde{g}_n^{(j)}$, $n = 0, 1, 2, \dots, q+1$, $j = 1, 2, \dots$ are well-defined. Set

$$E = \{n \in \{0, \dots, q\} : f_{S(n)}(x_n^{(j)}) > \tilde{\gamma}_{j,n} \text{ for all } j = 1, 2, \dots\}. \tag{10.83}$$

By (10.75), (10.83), (10.79), and (10.80), extracting a subsequence and re-indexing, if necessary, we may assume without any loss of generality that, for each $n \in E$ there is

$$g_n = \lim_{j \to \infty} g_n^{(j)}. \tag{10.84}$$

For each $n \in \{0, \dots, q+1\} \setminus E$ set

$$g_n = 0. \tag{10.85}$$

Let $n \in \{0, \dots, q\}$. There are two cases:

$$f_{S(n)}(y_n) > 0; \tag{10.86}$$

$$f_{S(n)}(y_n) \le 0. \tag{10.87}$$

Consider the case (10.86). By (10.78), we may assume without any loss of generality that

$$f_{S(n)}(x_n^{(j)}) > 2^{-1} f_{S(n)}(y_n) > 0 \text{ for all natural numbers } j. \tag{10.88}$$

Then, in view of (10.69)–(10.72) and (10.88) for all sufficiently large natural numbers j,

$$\|x_{n+1}^{(j)} - (x_n^{(j)} - f_{S(n)}(x_n^{(j)})\|g_n^{(j)}\|^{-2} g_n^{(j)})\| \le \gamma_0/j. \tag{10.89}$$

By (10.78)–(10.80), (10.88), (10.82), and (10.84) for each $u \in X$,

$$f_{S(n)}(u) - f_{S(n)}(y_n) = \lim_{j\to\infty}(f_{S(n)}(u) - f_{S(n)}(x_n^{(j)}))$$
$$\geq \lim_{j\to\infty}\langle \tilde{g}_n^{(j)}, u - x_n^{(j)}\rangle = \langle g_n, u - y_n\rangle$$

and

$$g_n \in \partial f_{S(n)}(y_n). \tag{10.90}$$

By (10.78), (10.89), (10.84), (10.75), (10.88), (10.79), and (10.80),

$$y_{n+1} = \lim_{j\to\infty} x_{n+1}^{(j)}$$
$$= \lim_{j\to\infty}[x_n^{(j)} - f_{S(n)}(x_n^{(j)})\|g_n^{(j)}\|^{-2}g_n^{(j)}]$$
$$= y_n - f_{S(n)}(y_n)\|g_n\|^{-2}g_n. \tag{10.91}$$

Consider the case (10.87). If $n \notin E$, then by (10.78)–(10.80), (10.83), and (10.69)–(10.72), $x_{n+1}^{(j)} = x_n^{(j)}$ for all natural numbers j and

$$y_{n+1} = \lim_{j\to\infty} x_{n+1}^{(j)} = \lim_{j\to\infty} x_n^{(j)} = y_n. \tag{10.92}$$

Assume that

$$n \in E. \tag{10.93}$$

By (10.83) and (10.93), for each natural numbers j,

$$f_{S(n)}(x_n^{(j)}) > \tilde{\gamma}_{j,n}. \tag{10.94}$$

By (10.87), (10.94), (10.78), and (10.69),

$$f_{S(n)}(y_n) = 0,\ \lim_{j\to\infty} f_{S(n)}(x_n^{(j)}) = 0. \tag{10.95}$$

By (10.78), (10.94), (10.69)–(10.72), (10.74), (10.95), and (10.75),

$$y_{n+1} = \lim_{j\to\infty} x_{n+1}^{(j)} = \lim_{j\to\infty}[x_n^{(j)} - f_{S(n)}(x_n^{(j)})\|g_n^{(j)}\|^{-2}g_n^{(j)}]$$
$$= \lim_{j\to\infty} x_n^{(j)} - [\lim_{j\to\infty} f_{S(n)}(x_n^{(j)})\|g_n^{(j)}\|^{-2}g_n^{(j)})] = \lim_{j\to\infty} x_n^{(j)} = y_n.$$

Thus in both cases (10.87) implies that

$$y_{n+1} = y_n. \tag{10.96}$$

Thus (10.86) implies (10.90), (10.91), and (10.87) implies (10.96). Clearly, there are $y_n \in X$ for all integers $n \geq q+1$ such that $\{y_n\}_{n=0}^{\infty} \in \mathcal{M}_0$. By (10.78), for all sufficiently large natural numbers j, we have $\|x_n^{(j)} - y_n\| < \gamma/2$, $n = 0, 1, \ldots, q+1$. This contradicts (10.67). The contradiction we have reached proves that assertion (A) holds with $m = q+1$. Thus by induction we have shown that assertion (A) holds with $m = n_0$.

Fix a positive number γ_1 such that

$$\gamma_1 < \epsilon/4,\ \gamma_1 < 1/2,\ \gamma_1 < (\epsilon/2)\Lambda^{-1}. \tag{10.97}$$

By (A) with $m = n_0$, there is $\delta > 0$ such that, for each $\{x_n\}_{n=0}^{\infty} \in \mathcal{M}_\delta$ there is $\{y_n\}_{n=0}^{\infty} \subset \mathcal{M}_0$ for which

$$\|y_n - x_n\| \leq \gamma_1,\ n = 0, \ldots, n_0. \tag{10.98}$$

Let

$$\{x_n\}_{n=0}^{\infty} \in \mathcal{M}_\delta. \tag{10.99}$$

By (10.99) and the choice of δ there is

$$\{y_n\}_{n=0}^{\infty} \in \mathcal{M}_0 \tag{10.100}$$

such that (10.98) holds. By (10.100), (P7), and the definition of $\mathcal{M}_0$,

$$\|y_n\| \leq 3b \text{ for all integers } n \geq 0, \tag{10.101}$$

$$f_i(y_{n_0}) \leq \epsilon/4,\ i \in \mathbb{I}, \tag{10.102}$$

$$d(y_{n_0}, C) \leq \epsilon/4. \tag{10.103}$$

By (10.98), (10.103), and (10.97),

$$d(x_{n_0}, C) \leq \|x_{n_0} - y_{n_0}\| + d(y_{n_0}, C) < \epsilon/2. \tag{10.104}$$

By (10.101), (10.98), and (10.97),

$$\|x_n\| \leq 3b + 1/2,\ n = 0, \ldots, n_0. \tag{10.105}$$

By (10.102), (10.101), (10.105), (10.56), (10.98), and (10.97), for any $i \in \mathbb{I}$,

$$\begin{aligned} f_i(x_{n_0}) &\leq f_i(y_{n_0}) + |f_i(x_{n_0}) - f_i(y_{n_0})| \leq \epsilon/4 + \Lambda\|x_{n_0} - y_{n_0}\| \\ &\leq \epsilon/4 + \gamma_1\Lambda < \epsilon/2 + \epsilon/4. \end{aligned}$$

Theorem 10.5 is proved. □

10.7 Dynamic String-Averaging Methods in Infinite-Dimensional Spaces

Let $(X, \langle \cdot, \cdot \rangle)$ be a Hilbert space with an inner product $\langle \cdot, \cdot \rangle$, which induces a complete norm $\| \cdot \|$. For each $x \in X$ and each nonempty set $A \subset X$ put

$$d(x, A) := \inf\{\| |x - y\| : \ y \in A\}.$$

For each $x \in X$ and each $r > 0$ set

$$B(x, r) = \{y \in X : \ \|x - y\| \leq r\}.$$

We use the notation, definitions, and assumptions introduced in Sect. 10.1.

Let $f : X \to R^1$ be a continuous and convex function such that

$$\{x \in X : \ f(x) \leq 0\} \neq \emptyset. \tag{10.106}$$

Let $y_0 \in X$. Recall that the set

$$\partial f(y_0) := \{l \in X : \ f(y) - f(y_0) \geq \langle l, y - y_0 \rangle \text{ for all } y \in X\} \tag{10.107}$$

is the subdifferential of f at the point y_0 [72, 77]. For any $g \in \partial f(y_0)$, in view of (10.107),

$$\{x \in X : \ f(x) \leq 0\} \subset \{x \in X : \ f(y_0) + \langle g, x - y_0 \rangle \leq 0\}. \tag{10.108}$$

Let m be a natural number and $f_i : X \to R^1$, $i = 1, \ldots, m$, be convex and continuous functions. For each $i \in \{1, \ldots, m\}$ set

$$C_i := \{x \in X : \ f_i(x) \leq 0\}, \tag{10.109}$$

$$C := \cap_{i=1}^m C_i = \cap_{i=1}^m \{x \in X : \ f_i(x) \leq 0\}. \tag{10.110}$$

Suppose that

$$C \neq \emptyset.$$

A point $x \in C$ is called a solution of our feasibility problem. For a given $\epsilon > 0$, a point $x \in X$ is called an ϵ-approximate solution of the feasibility problem if $f_i(x) \leq \epsilon$ for all $i = 1, \ldots .m$. We apply the dynamic string-averaging subgradient projection method in order to obtain a good approximative solution of the feasibility problem.

Denote by $\mathcal{N}$ the set of all nonnegative integers.

By an index vector, we mean a vector $t = (t_1, \ldots, t_p)$ such that $t_i \in \{1, \ldots, m\}$ for all $i = 1, \ldots, p$.

For an index vector $t = (t_1, \dots, t_q)$ set

$$p(t) = q. \tag{10.111}$$

Denote by $\mathcal{M}$ the collection of all pairs (Ω, w), where Ω is a finite set of index vectors and

$$w : \Omega \to (0, \infty) \text{ be such that } \sum_{t \in \Omega} w(t) = 1. \tag{10.112}$$

Fix a number

$$\Delta \in (0, m^{-1}] \tag{10.113}$$

and an integer

$$\bar{q} \geq m. \tag{10.114}$$

Denote by $\mathcal{M}_*$ the set of all $(\Omega, w) \in \mathcal{M}$ such that

$$p(t) \leq \bar{q} \ \text{ for all } t \in \Omega, \tag{10.115}$$

$$w(t) \geq \Delta \text{ for all } t \in \Omega. \tag{10.116}$$

Fix a natural number $\bar{N}$.

For each $x \in X$, each number $\epsilon \geq 0$ and each $i \in \{1, \dots, m\}$ set

$$A_i(x, \epsilon) := \{x\} \text{ if } f_i(x) \leq \epsilon \tag{10.117}$$

and, in view of Lemma 10.2,

$$A_i(x, \epsilon) = x - f_i(x)\{\|g\|^{-2}g : \ g \in \partial f_i(x)\} \text{ if } f_i(x) > \epsilon. \tag{10.118}$$

Let $\epsilon \geq 0$, $x \in X$ and let $t = (t_1, \dots, t_{p(t)})$ be an index vector. Define

$$A_0(t, x, \epsilon) = \{(y, \lambda) \in X \times R^1 : \ \text{there is a sequence } \{y_i\}_{i=0}^{p(t)} \subset X \text{ such that}$$

$$y_0 = x, \tag{10.119}$$

for each $i = 1, \dots, p(t)$,

$$y_i \in A_{t_i}(y_{i-1}, \epsilon), \tag{10.120}$$

$$y = y_{p(t)}, \tag{10.121}$$

$$\lambda = \max\{\|y_i - y_{i-1}\| : \ i = 1, \dots, p(t)\}\}. \tag{10.122}$$

Let $\epsilon \geq 0$, $x \in X$ and let $(\Omega, w) \in \mathcal{M}$. Define

$$A(x, (\Omega, w), \epsilon) = \{(y, \lambda) \in X \times R^1 : \text{ there exist}$$
$$(y_t, \lambda_t) \in A_0(t, x, \epsilon),\ t \in \Omega \text{ such that}$$
$$y = \sum_{t \in \Omega} w(t) y_t, \tag{10.123}$$
$$\lambda = \max\{\lambda_t :\ t \in \Omega\}\}. \tag{10.124}$$

Denote by Card(A) the cardinality of a set A. Suppose that the sum over empty set is zero.

Theorem 10.11. *Let*

$$M_0 > 0,\ \epsilon \in (0, 1),\ M_1 > 0,\ \gamma \in [0, \epsilon], \tag{10.125}$$
$$B(0, M_0) \cap C \neq \emptyset, \tag{10.126}$$
$$|f_i(u) - f_i(v)| \leq M_1 \|u - v\|,\ u, v \in B(0, 3M_0 + 1),\ i \in \{1, \dots, m\}, \tag{10.127}$$
$$\epsilon_0 \in (0, \epsilon M_1^{-1}] \tag{10.128}$$

and let a natural number n_0 satisfy

$$n_0 \geq 4M_0^2 \epsilon_0^{-2} \Delta^{-1} \bar{N}. \tag{10.129}$$

Assume that

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*, \tag{10.130}$$

satisfies for each natural number j,

$$\{1, \dots, m\} \subset \cup_{i=j}^{j+\bar{N}-1} (\cup_{t \in \Omega_i} \{t_1, \dots, t_{p(t)}\}), \tag{10.131}$$
$$x_0 \in B(0, M_0), \tag{10.132}$$
$$\{x_i\}_{i=1}^{\infty} \subset X,\ \{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty) \tag{10.133}$$

satisfy for each natural number i,

$$(x_i, \lambda_i) \in A(x_{i-1}, (\Omega_i, w_i), \gamma). \tag{10.134}$$

Then

$$\|x_i\| \leq 3M_0 \textit{ for all integers } i \geq 0 \tag{10.135}$$

and

$$Card(\{n \in \mathcal{N} :\ \max\{\lambda_i :\ i = n+1, \dots, n+\bar{N}\} > \epsilon_0\}) \le n_0. \tag{10.136}$$

Moreover, if an integer $n \ge 0$ *satisfies*

$$\lambda_i \le \epsilon_0,\ i = n+1, \dots, n+\bar{N}, \tag{10.137}$$

then, for all integers $i, j \in \{n, \dots, n+\bar{N}\}$, $\|x_i - x_j\| \le \bar{N}\bar{q}\epsilon_0$ *and for all integers* $j \in \{n, \dots, n+\bar{N}\}$ *and each* $s \in \{1, \dots, m\}$,

$$f_s(x_j) \le \epsilon(\bar{q}(\bar{N}+1)+1).$$

10.8 Proof of Theorem 10.11

Let n be a natural number. In view of (10.134),

$$(x_n, \lambda_n) \in A(x_{n-1}, (\Omega_n, w_n), \gamma). \tag{10.138}$$

By (10.124) and (10.124), for any $t \in \Omega_n$ there exists

$$(y_{n,t}, \lambda_{n,t}) \in A_0(t, x_{n-1}, \gamma) \tag{10.139}$$

such that

$$x_n = \sum_{t \in \Omega_n} w_n(t) y_{n,t}, \tag{10.140}$$

$$\lambda_n = \max\{\lambda_{n,t} :\ t \in \Omega_n\}. \tag{10.141}$$

Let

$$t = (t_1, \dots, t_{p(t)}) \in \Omega_n. \tag{10.142}$$

By (10.119), (10.139), and (10.142), there is a sequence $\{y_{n,t,i}\}_{i=0}^{p(t)} \subset X$ such that

$$y_{n,t,0} = x_{n-1}, \tag{10.143}$$

$$y_{n,t,i} \in A_{t_i}(y_{n,t,i-1}, \gamma),\ i = 1, \dots, p(t), \tag{10.144}$$

$$y_{n,t} = y_{n,t,p(t)}, \tag{10.145}$$

$$\lambda_{n,t} = \max\{\|y_{n,t,i} - y_{n,t,i-1}\| :\ i = 1, \dots, p(t)\}. \tag{10.146}$$

In view of (10.126), there exists

$$z \in B(0, M_0) \cap C. \tag{10.147}$$

Let n be a natural number,

$$t = (t_1, \dots, t_{p(t)}) \in \Omega_n, \ i = 1, \dots, p(t). \tag{10.148}$$

By (10.117), (10.118), and (10.144), the following properties hold:
(P1) if $f_{t_i}(y_{n,t,i-1}) \le \gamma$, then $y_{n,t,i} = y_{n,t,i-1}$;
(P2) if $f_{t_i}(y_{n,t,i-1}) > \gamma$, then there is

$$g_{n,t,i} \in \partial f_{t_i}(y_{n,t,i-1}) \tag{10.149}$$

such that

$$y_{n,t,i} = y_{n,t,i-1} - f_{t_i}(y_{n,t,i-1}) \|g_{n,t,i}\|^{-2} g_{n,t,i}. \tag{10.150}$$

If (P1) holds, the we set $g_{n,t,i} = 0$. Set

$$D_{n,t,i} = \{x \in X : \ f_{t_i}(y_{n,t,i-1}) + \langle g_{n,t,i}, x - y_{n,t,i-1} \rangle \le 0\}. \tag{10.151}$$

Clearly, if $f_{t_i}(y_{n,t,i-1}) \le \gamma$, then

$$\|z - y_{n,t,i}\| = \|z - y_{n,t,i-1}\|. \tag{10.152}$$

Assume that

$$f_{t_i}(y_{n,t,i-1}) > \gamma.$$

Property (P2), Lemma 10.2 and (10.149)–(10.151) imply that

$$g_{n,t,i} \neq 0,$$
$$y_{n,t,i} = P_{D_{n,t,i}}(y_{n,t,i-1}). \tag{10.153}$$

It follows from (10.108), (10.110), (10.147), (10.149), and (10.151) that

$$z \in C \subset \{x \in X : \ f_{t_i}(x) \le 0\} \subset D_{n,t,i}. \tag{10.154}$$

Proposition 10.1, (10.151), and (10.153) imply that

$$\|z - y_{n,t,i}\|^2 + \|y_{n,t,i} - y_{n,t,i-1}\|^2 \le \|z - y_{n,t,i-1}\|^2. \tag{10.155}$$

Thus we have shown that the following property holds:

(P3) if $f_{t_i}(y_{n,t,i-1}) > \gamma$, then (10.155) holds.
In view of (10.152) and (10.153),

$$\|z - y_{n,t,i}\| \le \|z - y_{n,t,i-1}\| \text{ for all } t = 1, \dots, p(t). \tag{10.156}$$

By (10.143), (10.147), and (10.156), for all $i = 1, \dots, p(t)$,

$$\|y_{n,t,i}\| \le \|y_{n,t,i} - z\| + \|z\|$$
$$\le \|y_{n,t,0} - z\| + M_0 \le \|x_{n-1} - z\| + M_0. \tag{10.157}$$

It follows from (10.143), (10.145), and (10.156) that

$$\|z - x_{n-1}\| = \|z - y_{n,t,0}\| \ge \|z - y_{n,t,p(t)}\| = \|z - y_{n,t}\| \tag{10.158}$$

for all $t \in \Omega_n$. By (10.112), (10.140), (10.158) and the convexity of the norm,

$$\|z - x_n\| = \|z - \sum_{t \in \Omega_n} w_n(t) y_{n,t}\|$$
$$\le \sum_{t \in \Omega_n} w_n(t) \|z - y_{n,t}\| \le \|z - x_{n-1}\|.$$

Thus

$$\|z - x_n\| \le \|z - x_{n-1}\| \text{ for all integers } n \ge 1. \tag{10.159}$$

In view of (10.132) and (10.147),

$$\|z - x_0\| \le 2M_0. \tag{10.160}$$

It follows from (10.159) and (10.160) that

$$\|z - x_n\| \le 2M_0 \text{ for all integers } n \ge 0. \tag{10.161}$$

By (10.157) and (10.161), for all natural numbers n, all $t \in \Omega_n$ and all $i \in \{1, \dots, p(t)\}$,

$$\|y_{n,t,i}\| \le M_0 + \|x_{n-1} - z\| \le 3M_0. \tag{10.162}$$

Relations (10.132) and (10.161) that

$$\|x_n\| \le 3M_0 \text{ for all } n \in \mathcal{N}. \tag{10.163}$$

Assume that $n \ge 0$ is an integer such that

$$\lambda_k \le \epsilon_0, \; k = n + 1, \dots, n + \bar{N}. \tag{10.164}$$

In view of (10.141), (10.146), and (10.164), for each $k \in \{n+1, \dots, n+\bar{N}\}$, each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}$ and each $i \in \{1, \dots, p(t)\}$,

$$\|y_{k,t,i} - y_{k,t,i-1}\| \le \epsilon_0. \tag{10.165}$$

By (10.115), (10.143), and (10.165), for each $k \in \{n+1, \dots, n+\bar{N}\}$, each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}$ and each $i \in \{1, \dots, p(t)\}$,

$$\|x_{k-1} - y_{k,t,i}\| \le \bar{q}\epsilon_0. \tag{10.166}$$

Relations (10.112), (10.140), (10.145), and (10.166) imply that

$$\|x_{k-1} - x_k\| \le \bar{q}\epsilon_0. \tag{10.167}$$

It follows from (10.164) and (10.167) that for all $k, m \in \{n, \dots, n+\bar{N}\}$

$$\|x_k - x_m\| \le \bar{N}\bar{q}\epsilon_0. \tag{10.168}$$

Let

$$s \in \{1, \dots, m\}.$$

By (10.131), there exist

$$k \in \{n+1, \dots, n+\bar{N}\},\ t = (t_1, \dots, t_{p(t)}) \in \Omega_k \tag{10.169}$$

such that

$$s \in \{t_1, \dots, t_{p(t)}\}. \tag{10.170}$$

In view of (10.170), there is $i \in \{1, \dots, p(t)\}$ such that

$$s = t_i.$$

We show that

$$f_s(y_{k,t,i-1}) \le \epsilon. \tag{10.171}$$

Assume the contrary. Then by (10.125),

$$f_{t_i}(y_{k,t,i-1}) > \epsilon \ge \gamma. \tag{10.172}$$

Property (P2), (10.150), (10.165), (10.169), and (10.172) imply that

$$\begin{aligned} \epsilon_0 &\ge \|y_{k,t,i} - y_{k,t,i-1}\| \\ &= \|f_{t_i}(y_{k,t,i-1})\|g_{k,t,i}\|^{-2}g_{k,t,i}\| > \|g_{k,t,i}\|^{-1}\epsilon. \end{aligned} \tag{10.173}$$

It follows from (10.143), (10.162), (10.163), and (10.169) that

$$\|y_{k,t,i-1}\| \le 3M_0. \tag{10.174}$$

By (10.127), (10.149), and (10.174),

$$\|g_{k,t,i}\| \le M_1. \tag{10.175}$$

In view of (10.173) and (10.175),

$$\epsilon_0 > \epsilon M_1^{-1}.$$

This contradicts (10.128). The contradiction we have reached proves (10.171). Relations (10.143), (10.162), (10.163), (10.166), and (10.169) imply that

$$\|y_{k,t,i-1}\| \le 3M_0, \ \|x_{k-1}\| \le 3M_0, \tag{10.176}$$

$$\|x_{k-1} - y_{k,t,i-1}\| \le \bar{q}\epsilon_0. \tag{10.177}$$

By (10.127), (10.128), (10.170), (10.171), (10.176), and (10.177),

$$|f_s(x_{k-1}) - f_s(y_{k,t,i-1})| \le M_1\bar{q}\epsilon_0,$$

$$f_s(x_{k-1}) \le f_s(y_{k,t,i-1}) + M_1\bar{q}\epsilon_0 \le \epsilon(\bar{q}+1). \tag{10.178}$$

Let $j \in \{n, \dots, n+\bar{N}\}$. In view of (10.168) and (10.169),

$$\|x_j - x_{k-1}\| \le \bar{N}\bar{q}\epsilon_0. \tag{10.179}$$

It follows from (10.127), (10.128), (10.163), (10.178), and (10.179) that

$$f_s(x_j) \le f_s(x_{k-1}) + M_1\bar{N}\bar{q}\epsilon_0$$

$$\le \epsilon(\bar{q}+1) + \bar{N}\bar{q}\epsilon \le \epsilon(\bar{q}(\bar{N}+1)+1) \tag{10.180}$$

for all $s \in \{1, \dots, m\}$.

Set

$$E_1 = \{n \in \mathcal{N} : \ \lambda_{n+1} \le \epsilon_0\}, \tag{10.181}$$

$$E_2 = \mathcal{N} \setminus E_1, \tag{10.182}$$

$$E_3 = \{n \in \mathcal{N} : \ \{n, \dots, n+\bar{N}-1\} \cap E_2\} \ne \emptyset. \tag{10.183}$$

Let

$$n \in E_2. \tag{10.184}$$

In view of (10.181), (10.182), and (10.184),

$$\lambda_{n+1} > \epsilon_0. \tag{10.185}$$

By (10.112), (10.140) and the convexity of the function $\|\cdot\|^2$,

$$\begin{aligned}
&\|z - x_n\|^2 - \|z - x_{n+1}\|^2 \\
&= \|z - x_n\|^2 - \|z - \sum_{t \in \Omega_{n+1}} w_{n+1}(t) y_{n+1,t}\|^2 \\
&\geq \|z - x_n\|^2 - \sum_{t \in \Omega_{n+1}} w_{n+1}(t) \|z - y_{n+1,t}\|^2.
\end{aligned} \tag{10.186}$$

In view of (10.141) and (10.185), there exists

$$\hat{t} \in \Omega_{n+1} \tag{10.187}$$

such that

$$\epsilon_0 < \lambda_{n+1} = \lambda_{n+1,\hat{t}}. \tag{10.188}$$

It follows from (10.112), (10.143), (10.145), (10.156), (10.186), and (10.187) that

$$\begin{aligned}
&\|z - x_n\|^2 - \|z - x_{n+1}\|^2 \\
&\geq \sum_{t \in \Omega_{n+1}} w_{n+1}(t)[\|z - x_n\|^2 - \|z - y_{n+1,t}\|^2] \\
&= \sum_{t \in \Omega_{n+1}} w_{n+1}(t)[\|z - y_{n+1,t,0}\|^2 - \|z - y_{n+1,t,p(t)}\|^2] \\
&\geq w_{n+1}(\hat{t})[\|z - y_{n+1,\hat{t},0}\|^2 - \|z - y_{n+1,\hat{t},p(\hat{t})}\|^2].
\end{aligned} \tag{10.189}$$

Property (P3), (10.116), (10.152), (10.155), (10.156), and (10.189) imply that

$$\begin{aligned}
&\|z - x_n\|^2 - \|z - x_{n+1}\|^2 \\
&\geq \Delta[\|z - y_{n+1,\hat{t},0}\|^2 - \|z - y_{n+1,\hat{t},p(\hat{t})}\|^2] \\
&\geq \Delta \sum_{i=1}^{p(\hat{t})} [\|z - y_{n+1,\hat{t},i-1}\|^2 - \|z - y_{n+1,\hat{t},i}\|^2] \\
&\geq \Delta \sum_{i=1}^{p(\hat{t})} \|y_{n+1,\hat{t},i-1} - y_{n+1,\hat{t},i}\|^2.
\end{aligned} \tag{10.190}$$

By (10.146), (10.188), and (10.190),

$$\|z - x_n\|^2 - \|z - x_{n+1}\|^2 \geq \Delta\epsilon_0^2 \text{ for all } t \in E_2. \tag{10.191}$$

In view of (10.159), (10.160), and (10.191), for any natural number n,

$$4M^2 \geq \|z - x_0\|^2 \geq \|z - x_0\|^2 - \|z - x_{n+1}\|^2$$

$$\sum_{i=0}^{n}[\|z - x_i\|^2 - \|z - x_{i+1}\|^2]$$

$$\geq \sum\{\|z - x_i\|^2 - \|z - x_{i+1}\|^2 : i \in [0, n] \cap E_2\}$$

$$\geq \mathrm{Card}([0, n] \cap E_2)\Delta\epsilon_0^2,$$

$$\mathrm{Card}([0, n] \cap E_2) \leq 4M_0^2(\Delta\epsilon_0^2)^{-1}.$$

Since the inequality above holds for any natural number n,

$$\mathrm{Card}(E_2) \leq 4M_0^2(\Delta\epsilon_0^2)^{-1}.$$

In view of the relation above, (10.129) and (10.183),

$$\mathrm{Card}(E_3) \leq \bar{N}\mathrm{Card}(E_2) \leq 4M_0^2(\Delta\epsilon_0^2)^{-1}\bar{N} \leq n_0.$$

This completes the proof of Theorem 10.11. □

10.9 Dynamic String-Averaging Methods in Finite-Dimensional Spaces

We use the notation, definitions, and assumptions introduced in Sects. 10.1 and 10.7.

Suppose that the space X is finite-dimensional. We prove the following result.

Theorem 10.12. *Let $M_0 > 0$, $\epsilon \in (0, 1)$,*

$$B(0, M_0) \cap C \neq \emptyset, \tag{10.192}$$

Then there exist a natural number n_0 and $\gamma_0 \in (0, \epsilon)$ such that the following assertion holds.

Assume that

$$\gamma \in [0, \gamma_0], \tag{10.193}$$

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*, \tag{10.194}$$

satisfies for each natural number j,

$$\{1, \ldots, m\} \subset \cup_{i=j}^{j+\bar{N}-1} (\cup_{t \in \Omega_i} \{t_1, \ldots, t_{p(t)}\}), \tag{10.195}$$

$$x_0 \in B(0, M_0), \tag{10.196}$$

$$\{x_i\}_{i=1}^{\infty} \subset X, \ \{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty) \tag{10.197}$$

satisfy for each natural number i,

$$(x_i, \lambda_i) \in A(x_{i-1}, (\Omega_i, w_i), \gamma). \tag{10.198}$$

Then

$$\|x_i\| \le 3M_0 \text{ for all integers } i \ge 0,$$
$$f_i(x_n) \le \epsilon \text{ for all } i \in \{1, \ldots, m\} \text{ and all integers } n \ge n_0,$$
$$d(x_n, C) \le \epsilon \text{ for all integers } n \ge n_0.$$

10.10 Proof of Theorem 10.12

Since the functions f_i, $i \in \{1, \ldots, m\}$ are convex [50], there exists $M_1 > 0$ such that

$$|f_i(u) - f_i(v)| \le M_1 \|u - v\| \text{ for all } u, v \in B(0, 3M_0 + 1), \ i \in \{1, \ldots, m\}. \tag{10.199}$$

Choose a positive number

$$\gamma_1 < \min\{\epsilon, \ M_1^{-1}\epsilon\}. \tag{10.200}$$

By Lemma 10.10, there exists $\gamma_2 \in (0, \epsilon)$ such that the following property holds:

(P4) for each $y \in B(0, 3M_0 + 1)$ satisfying $f_i(y) \le \gamma_2$, $i \in \{1, \ldots, m\}$, the inequality $d(y, C) \le \gamma_1/2$ holds.

Choose a positive number γ_0 such that

$$\gamma_0 < \gamma_1 \text{ and } (\bar{N} + 1)\gamma_0(\bar{q} + 1) < \gamma_2. \tag{10.201}$$

By (10.199) and Theorem 10.11 (with $\epsilon = \gamma_0$), there exists a natural number n_0 such that the following property holds:

(P5) let $\gamma \in [0, \gamma_0]$,

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*,$$

for each natural number j, (10.195) hold,

$$x_0 \in B(0, M_0),$$
$$\{x_i\}_{i=1}^{\infty} \subset X, \ \{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty),$$

for each natural number i (10.198) hold. Then

$$\|x_n\| \le 3M_0 \text{ for all integers } n \ge 0 \tag{10.202}$$

and there exists an integer $q \in [0, n_0]$ such that

$$f_i(x_q) \le \gamma_0(\bar{q}(\bar{N}+1)+1),\ i \in \{1,\dots,m\}. \tag{10.203}$$

Let

$$\gamma \in [0, \gamma_0],\ \{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*, \tag{10.204}$$

for each natural number j, (10.195) holds, (10.196) is true,

$$\{x_i\}_{i=1}^{\infty} \subset X,\ \{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty)$$

and for each natural number i (10.198) holds. Property (P5), (10.196), (10.198), and (10.204) imply that (10.202) holds and there exists an integer $q \in [0, n_0]$ such that (10.203) holds. It follows from (10.201) to (10.203) and (P4) that

$$d(x_q, C) \le 2^{-1}\gamma_1. \tag{10.205}$$

In view of (10.205), there exists $\tilde{z} \in X$ such that

$$\tilde{z} \in C,\ \|x_q - \tilde{z}\| < \gamma_1. \tag{10.206}$$

Proposition 10.1, Lemma 10.2, (10.119), (10.124), (10.198), (10.200), and (10.206) imply that

$$\|x_n - \tilde{z}\| \le \|x_q - \tilde{z}\| < \gamma_1 < \epsilon. \tag{10.207}$$

By (10.200), (10.202), and (10.206),

$$\|\tilde{z}\| \le 3M_0 + 1. \tag{10.208}$$

In view of (10.124), (10.199), (10.200), (10.202), (10.206), and (10.208), for all integers $n \ge n_0$ and all $i \in \{1,\dots,m\}$,

$$\begin{aligned} f_i(x_n) &\le f_i(\tilde{z}) + |f_i(x_n) - f_i(\tilde{z})| \\ &\le M_1\|x_n - \tilde{z}\| < M_1\gamma_1 < \epsilon. \end{aligned}$$

Theorem 10.12 is proved. □

10.11 Problems in Finite-Dimensional Spaces with Computational Errors

We use all the notation, definitions, and assumptions introduced in Sects. 10.7 and 10.9. In particular, we assume that the space X is finite-dimensional.

For each $x \in X$, each $\epsilon \geq 0$, each $\bar{\epsilon} \geq 0$ and each $i \in \{1, \ldots, m\}$ set

$$A_i(x, \bar{\epsilon}, \epsilon) := \{x\} \text{ if } f_i(x) \leq \bar{\epsilon} \tag{10.209}$$

and if $f_i(x) > \bar{\epsilon}$, then set

$$A_i(x, \bar{\epsilon}, \epsilon) = \{x - f_i(x)\|g\|^{-2}g : \; g \in \partial f_i(x) + B(0, \epsilon), \; g \neq 0\} + B(0, \epsilon). \tag{10.210}$$

Let $x \in X$ and let $t = (t_1, \ldots, t_{p(t)})$ be an index vector, $\epsilon \geq 0$, $\bar{\epsilon} \geq 0$. Define

$$A_0(t, x, \bar{\epsilon}, \epsilon) = \{(y, \lambda) \in X \times R^1 : \text{ there is a sequence } \{y_i\}_{i=0}^{p(t)} \subset X \text{ such that}$$

$$y_0 = x, \tag{10.211}$$

for each $i = 1, \ldots, p(t)$,

$$y_i \in A_{t_i}(y_{i-1}, \bar{\epsilon}, \epsilon), \tag{10.212}$$

$$y = y_{p(t)}, \tag{10.213}$$

$$\lambda = \max\{\|y_i - y_{i-1}\| : \; i = 1, \ldots, p(t)\}\}. \tag{10.214}$$

Let $x \in X$, $(\Omega, w) \in \mathcal{M}$, $\epsilon \geq 0$, $\bar{\epsilon} \geq 0$. Define

$$A(x, (\Omega, w), \bar{\epsilon}, \epsilon) = \{(y, \lambda) \in X \times R^1 : \text{ there exist}$$

$$(y_t, \lambda_t) \in A_0(t, x, \bar{\epsilon}, \epsilon), \; t \in \Omega \text{ such that}$$

$$\|y - \sum_{t \in \Omega} w(t) y_t\| \leq \epsilon, \; \lambda = \max\{\lambda_t : \; t \in \Omega\}\}. \tag{10.215}$$

We prove the following result.

Theorem 10.13. *Let $M_0 > 0$, $\epsilon \in (0, 1)$,*

$$B(0, M_0) \cap C \neq \emptyset, \tag{10.216}$$

$$\{z \in X : \; f_i(z) < 0\} \cap B(0, M_0) \neq \emptyset \text{ for all } i = 1, \ldots, m. \tag{10.217}$$

Then there exist a natural number n_0 and $\delta > 0$ such that the following assertion holds.

Assume that

$$\tilde{\delta}_n \in [0, \delta],\ n \in \mathcal{N} \setminus \{0\}, \tag{10.218}$$

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*, \tag{10.219}$$

satisfies for each natural number j,

$$\{1, \dots, m\} \subset \cup_{i=j}^{j+\bar{N}-1} (\cup_{t \in \Omega_i} \{t_1, \dots, t_{p(t)}\}), \tag{10.220}$$

$$x_0 \in B(0, M_0), \tag{10.221}$$

$$\{x_i\}_{i=1}^{\infty} \subset X,\ \{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty) \tag{10.222}$$

and that for each natural number i,

$$(x_i, \lambda_i) \in A(x_{i-1}, (\Omega_i, w_i), \tilde{\delta}_i, \delta). \tag{10.223}$$

Then

$$\|x_n\| \le 3M_0 + 1 \textit{ for all integers } n = 0, \dots, n_0,$$

$$d(x_{n_0}, C) \le \epsilon,$$

$$f_i(x_{n_0}) \le \epsilon,\ i \in \{1, \dots, m\}.$$

10.12 Proof of Theorem 10.13

In view of (10.217), for each $i \in \{1, \dots, m\}$, there exists

$$z_i \in B(0, M_0) \tag{10.224}$$

such that

$$f_i(z_i) < 0. \tag{10.225}$$

Set

$$r = \min\{-f_i(z_i) :\ i \in \{1, \dots, m\}\}. \tag{10.226}$$

By (10.225) and (10.226),

$$r > 0. \tag{10.227}$$

Since the functions $f_i,\ i = 1,\dots,m$ are convex [50], there exists $\Lambda > 0$ such that

$$|f_i(u) - f_i(v)| \le \Lambda\|u - v\| \text{ for all } u, v \in B(0, 3M_0 + 2),\ i = 1,\dots,m, \tag{10.228}$$

$$|f_i(u)| \le \Lambda \text{ for all } u \in B(0, 3M_0 + 2),\ i = 1,\dots,m. \tag{10.229}$$

By Theorem 10.12, there exist a natural number n_0 and $\bar{\gamma}_0 \in (0, \epsilon)$ such that the following property holds:

(P6) For each $\gamma \in [0, \tilde{\gamma}]$, each

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$$

satisfying (10.220) for all integers $j \ge 1$, each pair of sequences

$$\{x_i\}_{i=1}^{\infty} \subset X,\ \{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty)$$

which satisfy

$$x_0 \in B(0, M_0), \tag{10.230}$$

$$(x_i, \lambda_i) \in A(x_{i-1}, (\Omega_i, w_i), \gamma) \text{ for all natural numbers } i \tag{10.231}$$

we have

$$\|x_n\| \le 3M_0 \text{ for all integers } n \ge 0, \tag{10.232}$$

$$f_i(x_n) \le \epsilon/4 \text{ for all } i \in \{1,\dots,m\} \text{ and all integers } n \ge n_0, \tag{10.233}$$

$$d(x_n, C) \le \epsilon/4 \text{ for all integers } n \ge n_0. \tag{10.234}$$

By the choice of Λ (see (10.228)), for each $u \in B(0, 3M_0 + 1)$, all $i \in \{1,\dots,m\}$,

$$\partial f_i(u) \subset B(0, \Lambda). \tag{10.235}$$

We show that the following property holds:

(P7) for each $M \ge M_0$, each $i \in \{1,\dots,m\}$, each $u \in B(0, 3M + 1)$ satisfying $f_i(u) > 0$ and each $g \in X$ which satisfies

$$d(g, \partial f_i(u)) \le r(4M + 1)^{-1}4^{-1},$$

we have

$$\|g\| > r(M + 1)^{-1}2^{-1}.$$

Let

$$M \ge M_0,\ u \in B(0, 3M + 1),\ i \in \{1,\dots,m\},\ f_i(u) > 0 \tag{10.236}$$

and let $g \in X$ satisfy (10.235). Let

$$\xi \in \partial f_i(u). \tag{10.237}$$

By (10.224), (10.226), (10.236), and (10.237),

$$-r \geq f_i(z_i) > f_i(z_i) - f_i(u) \geq \langle \xi, z_i - u \rangle$$
$$\geq -\|\xi\|\|z_i - u\| \geq -\|\xi\|(4M + 1),$$
$$\|\xi\| \geq r(4M + 1)^{-1}$$

and

$$\partial f_i(u) \subset \{\xi \in X : \ \|\xi\| \geq r(4M + 1)^{-1}\}.$$

Together with (10.235) this implies that

$$\|g\| > 2^{-1} r(4M + 1)^{-1}.$$

Thus (P7) holds.

Property (P7) implies that the following property holds:

(P8) let $M \geq M_0$, $i \in \{1, \dots, m\}$, $u \in B(0, 3M + 1)$ satisfy $f_i(u) > 0$, $g \in X$ satisfy

$$d(g, \partial f_i(u)) \leq r(4M + 1)^{-1} 4^{-1}$$

and

$$u' \in u - f_i(u)\|g\|^{-2} g + B(0, 1).$$

Then

$$\|u'\| \leq 3M + 2 + 2(4M + 1)r^{-1} f_i(u).$$

For each $\gamma \geq 0$ denote by $\mathcal{K}_\gamma$ the set of all sequences $\{x_n\}_{n=1}^\infty \subset X$ such that

$$\|x_0\| \leq M_0$$

and there exist $\{\lambda_i\}_{i=1}^\infty \subset [0, \infty)$,

$$\tilde{\gamma}_n \in [0, \gamma], \ n \in \mathcal{N} \setminus \{0\}, \tag{10.238}$$

$$\{(\Omega_i, w_i)\}_{i=1}^\infty \subset \mathcal{M}_*, \tag{10.239}$$

satisfying (10.220) such that

$$(x_i, \lambda_i) \in A(x_{i-1}, (\Omega_i, w_i), \tilde{\gamma}_i, \gamma) \text{ for all integers } i \geq 1. \tag{10.240}$$

By induction we show that for all $m = 0, \dots, n_0$ the following assertion holds.

(A) For each $\gamma > 0$ there exists $\delta > 0$ such that, for each $\{x_n\}_{n=0}^{\infty} \in \mathcal{K}_\delta$, there is $\{y_n\}_{n=0}^{\infty} \in \mathcal{K}_0$ such that $\|y_n - x_n\| \le \gamma,\ n = 0, \dots, m.$

Clearly, for $m = 0$ this assertion holds. Assume that assertion (A) holds for $m = q$ where $q \in [0, n_0 - 1]$ is an integer. We show that (A) holds for $m = q + 1$. Set

$$M_1 = (M_0 + 1)(1 + r^{-1}) \tag{10.241}$$

and for each integer $i \ge 1$ set

$$\begin{aligned} M_{i+1} = 18(M_i + 1)(1 + r^{-1})(1 + \sup\{|f_s(h)| :\\ h \in B(0, 3M_i + 1),\ s = 1, \dots, m\}). \end{aligned} \tag{10.242}$$

Since (A) holds for $m = q$, it follows from (P8) that there is $\gamma_0 > 0$ such that

$$\gamma_0 < 2^{-1} \text{ and } \gamma_0 < 4^{-1} r(4M_q + 1)^{-1} \tag{10.243}$$

and that, for each $\{y_n\}_{n=0}^{\infty} \in \mathcal{K}_{\gamma_0}$,

$$\|y_n\| \le 3M_0 + 1/2,\ n = 0, \dots, q. \tag{10.244}$$

Assume that assertion (A) does not hold for $m = q + 1$. Then there exists $\gamma > 0$ such that for each natural number j there is

$$\{x_n^{(j)}\}_{n=0}^{\infty} \in \mathcal{K}_{\gamma_0/j} \tag{10.245}$$

such that

$$\max\{\|y_n - x_n^{(j)}\| :\ n = 0, \dots, q + 1\} > \gamma \text{ for each } \{y_n\}_{n=0}^{\infty} \in \mathcal{K}_0. \tag{10.246}$$

By (10.245) and the choice of γ_0 (see (10.244)), for all natural numbers j,

$$\|x_n^{(j)}\| \le 3M_0 + 1/2,\ n = 0, \dots, q. \tag{10.247}$$

By the definition of $\mathcal{K}_\gamma$, $\gamma \ge 0$, for each integer $j \ge 1$ there exist

$$\{\lambda_i^{(j)}\}_{i=1}^{\infty} \subset [0, \infty), \tag{10.248}$$

$$\tilde{\gamma}_{j,n} \in [0, \gamma_0/j],\ n \in \mathcal{N} \setminus \{0\}, \tag{10.249}$$

$$\{(\Omega_i^{(j)}, w_i^{(j)})\}_{i=1}^{\infty} \subset \mathcal{M}_* \tag{10.250}$$

such that for each natural number s,

$$\{1,\dots,m\} \subset \cup_{i=s}^{s+\bar{N}-1}(\cup_{t\in\Omega_i^{(j)}}\{t_1,\dots,t_{p(t)}\}), \tag{10.251}$$

$$(x_i^{(j)},\lambda_i^{(j)}) \in A(x_{i-1}^{(j)},(\Omega_i^{(j)},w_i^{(j)}),\tilde{\gamma}_{j,i},\gamma_0/j) \tag{10.252}$$

for all integers $i \geq 1$.

In view of (10.115) and (10.150), extracting a subsequence and re-indexing if necessary, we may assume that

$$\Omega_i^{(j)} = \Omega_i^{(1)} \text{ for all pairs of natural numbers } i,j. \tag{10.253}$$

Set

$$\Omega_i = \Omega_i^{(1)} \text{ for all natural numbers } i. \tag{10.254}$$

Let j be a natural number. By (10.215) and (10.252), for each natural number s there exist

$$(y_t^{(j,s)},\lambda_t^{(j,s)}) \in A_0(t,x_{s-1}^{(j)},\tilde{\gamma}_{j,s},\gamma_0/j),\ t\in\Omega_s \tag{10.255}$$

such that

$$\lambda_s^{(j)} = \max\{\lambda_t^{(j,s)}:\ t\in\Omega_s\}, \tag{10.256}$$

$$\|x_s^{(j)} - \sum_{t\in\Omega_s} w_s^{(j)}(t)y_t^{(j,s)}\| \leq \gamma_0/j. \tag{10.257}$$

By (10.211)–(10.214) and (10.255), for each natural number s and each

$$t \in \Omega_s, \tag{10.258}$$

there exists finite a sequence

$$\{y_{t,i}^{(j,s)}\}_{i=0}^{p(t)} \subset X \tag{10.259}$$

such that

$$y_{t,0}^{(j,s)} = x_{s-1}^{(j)}, \tag{10.260}$$

for each $i = 1,\dots,p(t)$,

$$y_{t,i}^{(j,s)} \in A_{t_i}(y_{t,i-1}^{(j,s)},\tilde{\gamma}_{j,s},\gamma_0/j), \tag{10.261}$$

$$y_t^{(j,s)} = y_{t,p(t)}^{(j,s)}, \tag{10.262}$$

$$\lambda_t^{(j,s)} = \max\{\|y_{t,i}^{(j,s)} - y_{t,i-1}^{(j,s)}\|:\ i=1,\dots,p(t)\}. \tag{10.263}$$

By (10.209), (10.210), and (10.261), for each natural number s, each $t = (t_1, \dots, t_{p(t)}) \in \Omega_s$ and each $i \in \{1, \dots, p(t)\}$, if

$$f_{t_i}(y_{t,i-1}^{(j,s)}) \le \tilde{\gamma}_{j,s}, \tag{10.264}$$

then

$$y_{t,i}^{(j,s)} = y_{t,i-1}^{(j,s)}; \tag{10.265}$$

if

$$f_{t_i}(y_{t,i-1}^{(j,s)}) > \tilde{\gamma}_{j,s},$$

then there exists

$$g_{t,i}^{(j,s)} \in \partial f_{t_i}(y_{t,i-1}^{(j,s)}) + B(0, \gamma_0/j) \setminus \{0\} \tag{10.266}$$

such that

$$y_{t,i}^{(j,s)} \in y_{t,i-1}^{(j,s)} - f_{t_i}(y_{t,i-1}^{(j,s)}) \|g_{t,i}^{(j,s)}\|^{-2} g_{t,i}^{(j,s)} + B(0, \gamma_0/j). \tag{10.267}$$

For each natural number s, each $t = (t_1, \dots, t_{p(t)} \in \Omega_s$ and each $i \in \{1, \dots, p(t)\}$ satisfying (10.264) set

$$g_{t,i}^{(j,s)} = 0. \tag{10.268}$$

Let s be a natural number such that

$$s \le q + 1 \text{ and } t = (t_1, \dots, t_{p(t)}) \in \Omega_s. \tag{10.269}$$

By induction we show that for all $i = 1, \dots, p(t)$,

$$\|y_{t,i}^{(j,s)}\| \le 3M_i + 1. \tag{10.270}$$

In view of (10.247) and (10.260), (10.270) holds for $i = 0$.

Assume that an integer $i \in \{1, \dots, p(t)\}$ satisfies

$$\|y_{t,i-1}^{(j,s)}\| \le 3M_{i-1} + 1. \tag{10.271}$$

If (10.264) is true, then by (10.242), (10.265), and (10.271),

$$\|y_{t,i}^{(j,s)}\| = \|y_{t,i-1}^{(j,s)}\| \le 3M_{i-1} + 1 \le 3M_i + 1. \tag{10.272}$$

Assume that

$$f_{t_i}(y_{t,i-1}^{(j,s)}) > \tilde{\gamma}_{j,s}. \tag{10.273}$$

By (10.273), there exists $g^{(j,s)}_{t,i}$ satisfying (10.266) such that $y^{(j,s)}_{t,i}$ satisfies (10.267). It follows from (10.241), (10.242), (10.243), (10.267), (10.271), (10.273), and (P8) that

$$\|y^{(j,s)}_{t,i}\| \le 3M_{i-1} + 2 + 2(4M_{i-1} + 1)r^{-1} \sup\{|f_{t_i}(\eta)| : \eta \in B(0, 3M_{i-1} + 1)\} \le M_i.$$

Thus (10.270) holds for all $i = 0, \dots, p(t)$. Hence

$$\|y^{(j,s)}_{t,i}\| \le 3M_i + 1 \le 3M_q + 1 \tag{10.274}$$

for all natural numbers j, each natural number $s \le q+1$, each $t = (t_1, \dots, t_{p(t)}) \in \Omega_s$ and all $i = 0, \dots, p(t)$.

Since the functions f_i, $i = 1, \dots, m$ are Lipschitz on bounded subsets of R^n it follows from (10.214), (10.266), and (10.268) and property (P8) that there exists a constant $\tilde{M} > 0$ such that

$$\|g^{(j,s)}_{t,i}\| \le \tilde{M} \tag{10.275}$$

for all natural numbers j, each natural number $s \le q+1$, each $t = (t_1, \dots, t_{p(t)}) \in \Omega_s$ and all $i = 1, \dots, p(t)$.

By (10.247), (10.274), and (10.275), extracting a subsequence and re-indexing, if necessary, we may assume without any loss of generality that for any $s \in \{0, \dots, q\}$ there is

$$x_s = \lim_{j\to\infty} x^{(j)}_s \tag{10.276}$$

and that for every natural number $s \le q + 1$ and every $t = (t_1, \dots, t_{p(t)}) \in \Omega_s$ there exist

$$y^{(s)}_{t,i} = \lim_{j\to\infty} y^{(j,s)}_{t,i} \text{ for all } i = 0, \dots, p(t), \tag{10.277}$$

$$g^{(s)}_{t,i} = \lim_{j\to\infty} g^{(j,s)}_{t,i}, \; i = 1, \dots, p(t), \tag{10.278}$$

for all integers $s \ge 1$ and all $t \in \Omega_s$ there exists

$$w_s(t) = \lim_{j\to\infty} w^{(j)}_s(t). \tag{10.279}$$

For each natural number $s \le q + 1$ and each $t \in \Omega_s$ set

$$y^{(s)}_t = y^{(s)}_{t,p(t)}. \tag{10.280}$$

In view of (10.112), (10.116), and (10.279), for each integer $s \geq 1$,

$$\sum_{t \in \Omega_s} w_s(t) = 1, \tag{10.281}$$

$$w_s(t) \geq \Delta, \ t \in \Omega_s, \tag{10.282}$$

$$\{(\Omega_s, w_s) : \ s = 1, 2, \ldots\} \subset \mathcal{M}_*. \tag{10.283}$$

By (10.245) and (10.276),

$$\|x_0\| \leq M_0. \tag{10.284}$$

Assume that

$$s \in \{0, \ldots, q\}. \tag{10.285}$$

It follows from (10.257), (10.276), (10.278), (10.279), and (10.280) that

$$\begin{aligned} & x_s - \sum_{t \in \Omega_s} w_s(t) y_t^{(s)} \\ & = \lim_{j \to \infty} x_s^{(j)} - \sum_{t \in \Omega_s} \lim_{j \to \infty} w_s^{(j)}(t) \lim_{j \to \infty} y_{t,p(t)}^{(j,s)} \\ & = \lim_{j \to \infty} [x_s^{(j)} - \sum_{t \in \Omega_s} w_s^{(j)}(t) y_t^{(j,s)}] = 0 \end{aligned} \tag{10.286}$$

for all $s \in \{0, \ldots, q\}$.

Let

$$s \in \{0, \ldots, q+1\} \tag{10.287}$$

and

$$t = (t_1, \ldots, t_{p(t)}) \in \Omega_s. \tag{10.288}$$

In view of (10.260) and (10.276), if

$$s \geq 1, \tag{10.289}$$

then

$$x_{s-1} = \lim_{j \to \infty} x_{s-1}^{(j)} = \lim_{j \to \infty} y_{t,0}^{(j,s)} = y_{t,0}^{(s)}. \tag{10.290}$$

Let

$$i \in \{1, \dots, p(t)\}. \tag{10.291}$$

There are two cases:

$$f_{t_i}(y_{t,i-1}^{(s)}) > 0; \tag{10.292}$$

$$f_{t_i}(y_{t_{i-1}}^{(s)}) \le 0. \tag{10.293}$$

Assume that (10.292) holds. By (10.249), (10.266), (10.267), (10.277), and (10.292), for all sufficiently large natural numbers j,

$$f_{t_i}(y_{t,i-1}^{(j,s)}) > 2^{-1} f_{t_i}(y_{t,i-1}^{(s)}) > \gamma_0/j \ge \tilde{\gamma}_{j,s} \tag{10.294}$$

and

$$\|y_{t,i}^{(j,s)} - y_{t,i-1}^{(j,s)} + f_{t_i}(y_{t,i-1}^{(j,s)})\|g_{t,i}^{(j,s)}\|^{-2} g_{t,i}^{(j,s)}\| \le \gamma_0/j. \tag{10.295}$$

By (10.266), (10.277), (10.278), and (10.294), for each $u \in X$,

$$f_{t_i}(u) - f_{t_i}(y_{t,i-1}^{(s)}) = \lim_{j\to\infty} (f_{t_i}(u) - f_{t_i}(y_{t,i-1}^{(j,s)}))$$

$$\ge \lim_{j\to\infty} \langle g_{t,i}^{(j,s)}, u - y_{t,i-1}^{(j,s)} \rangle \ge \langle g_{t,i}^{(s)}, u - y_{t,i-1}^{(s)} \rangle,$$

$$g_{t,i}^{(s)} \in \partial f_{t_i}(y_{t,i-1}^{(s)}). \tag{10.296}$$

It follows from (10.266), (10.274), (10.277), (10.278), (10.293)–(10.295), and (P7) that

$$y_{t,i}^{(s)} = \lim_{j\to\infty} y_{t,i}^{(j,s)}$$

$$= \lim_{j\to\infty} [y_{t,i-1}^{(j,s)} + f_{t_i}(y_{t,i-1}^{(j,s)})\|g_{t,i}^{(j,s)}\|^{-2} g_{t,i}^{(j,s)}]$$

$$= y_{t,i-1}^{(s)} + \|g_{t,i}^{(s)}\|^{-2} g_{t,i}^{(s)} f_{t_i}(y_{t,i-1}^{(s)}). \tag{10.297}$$

Assume that (10.293) holds. There are two cases:

$$f_{t_i}(y_{t,i-1}^{(j,s)}) < \tilde{\gamma}_{j,i-1} \text{ for infinitely many integers } j \ge 1, \tag{10.298}$$

$$f_{t_i}(y_{t,i-1}^{(j,s)}) < \tilde{\gamma}_{j,i-1} \text{ only for finite numbers of integers } j \ge 1. \tag{10.299}$$

If (10.298) holds, then by (10.264), (10.265), and (10.277),

$$y_{t,i}^{(s)} = \lim_{j\to\infty} y_{t,i}^{(j,s)} = \lim_{j\to\infty} y_{t,i-1}^{(j,s)} = y_{t,i-1}^{(s)}.$$

Assume that (10.299) holds. Then there exists $j_0 \in \mathcal{N} \setminus \{0\}$ such that

$$f_{t_i}(y_{t,i-1}^{(j,s)}) \geq \tilde{\gamma}_{j,i-1} \text{ for all integers } j \geq j_0. \tag{10.300}$$

By (10.277), (10.293), and (10.300),

$$f_{t_i}(y_{t,i-1}^{(s)}) = 0. \tag{10.301}$$

It follows from (10.267), (10.277), (10.300), and (10.301) that

$$y_{t,i}^{(s)} = \lim_{j\to\infty} y_{t,i}^{(j,s)}$$

$$= \lim_{j\to\infty} [y_{t,i-1}^{(j,s)} + \|g_{t,i}^{(j,s)}\|^{-2} g_{t,i}^{(j,s)} f_{t_i}(y_{t,i-1}^{(j,s)})] = y_{t,i-1}^{(j,s)}.$$

Thus

$$y_{t,i}^{(s)} = y_{t,i-1}^{(s)} \tag{10.302}$$

in both cases.

Set

$$x_{q+1} = \sum_{t\in\Omega_{q+1}} w_{q+1}(t) y_t^{(q+1)}. \tag{10.303}$$

In view of (10.257), (10.262), (10.277), (10.279), (10.280), and (10.303),

$$x_{q+1} = \lim_{j\to\infty} x_{q+1}^{(j)}.$$

Clearly, there exist $x_s \in X$ for all integers $s > q+1$ such that $\{x_i\}_{i=0}^{\infty} \in \mathcal{K}_0$. For all sufficiently large natural numbers j,

$$\|x_n^{(j)} - x_n\| < \gamma/2, \ n = 0, \ldots, q+1.$$

This contradicts (10.246). The contradiction we have reached proves that (A) holds for $m = q+1$. By induction we showed that (A) holds for $m = n_0$.

Fix a positive number γ_1 such that

$$\gamma_1 \leq \epsilon/4, \ \gamma_1 \leq \Lambda^{-1}\epsilon/4.$$

By (A) with $m = n_0$ there is $\delta > 0$ such that for each $\{x_n\}_{n=0}^{\infty} \in \mathcal{K}_\delta$, there is $\{y_n\}_{n=0}^{\infty} \in \mathcal{K}_0$ such that

$$\|y_n - x_n\| \leq \gamma_1, \ n = 0, \ldots, n_0. \tag{10.304}$$

Let $\{x_n\}_{n=0}^{\infty} \in \mathcal{K}_\delta$. By the choice of δ, there is

$$\{y_n\}_{n=0}^{\infty} \in \mathcal{K}_0 \tag{10.305}$$

such that (10.304) hold. Property (P6) and (10.305) imply that

$$\|y_n\| \le 3M_0 \text{ for all integers } n \ge 0, \tag{10.306}$$

$$f_i(y_n) \le \epsilon/4 \text{ for all } i \in \{1, \dots, m\} \text{ and all integers } n \ge n_0, \tag{10.307}$$

$$d(x_n, C) \le \epsilon/4 \text{ for all integers } n \ge n_0. \tag{10.308}$$

In view of (10.304)–(10.308),

$$d(x_{n_0}, C) \le \|x_{n_0} - y_{n_0}\| + d(y_{n_0}, C) \le \epsilon/2,$$

$$\|x_n\| \le 3M_0 + 2^{-1},\ n = 0, \dots, n_0. \tag{10.309}$$

By (10.228), (10.304), (10.306), (10.307), (10.309) and the inequalities

$$\gamma_1 \le \epsilon/4,\ \gamma_1 \le \Lambda^{-1}\epsilon/4,$$

for all $i \in \{1, \dots, m\}$,

$$f_i(x_{n_0}) \le f_i(y_{n_0}) + |f_i(x_{n_0}) - f_i(y_{n_0})| \le \epsilon/4 + \Lambda\|x_{n_0} - y_{n_0}\| < \epsilon.$$

Theorem 10.13 is proved. □

10.13 Extensions

Theorem 10.13 implies the following result.

Theorem 10.14. *Let* $M_0 > 0,\ \epsilon \in (0, 1)$,

$$B(0, M_0) \cap C \neq \emptyset,$$

$$\{z \in X :\ f_i(z) < 0\} \cap B(0, M_0) \neq \emptyset \text{ for all } i = 1, \dots, m.$$

Let a natural number n_0 *and* $\delta > 0$ *be as guaranteed by Theorem 10.13. Assume that (10.218)–(10.223) hold and* $\{x_n\}_{i=1}^{\infty} \subset B(0, M_0)$. *Then for all integers* $n \ge n_0$,

$$d(x_n, C) \le \epsilon,$$

$$f_i(x_n) \le \epsilon,\ i \in \{1, \dots, m\}.$$

Theorem 10.14 easily implies the following result.

Theorem 10.15. *Let* $M_0 > 0$,

$$B(0, M_0) \cap C \neq \emptyset,$$

$$\{z \in X : f_i(z) < 0\} \cap B(0, M_0) \neq \emptyset \textit{ for all } i = 1, \dots, m,$$

$\{\delta_n\}_{n=0}^{\infty} \subset (0, \infty)$, $\lim_{n\to\infty} \delta_n = 0$, $\epsilon \in (0, 1)$. *Then there exist a natural number* n_ϵ *such that the following assertion holds.*

Assume that

$$\tilde{\delta}_n \in [0, \delta_n],\ n \in \mathcal{N} \setminus \{0\},$$

(10.219)–(10.222) hold,

$$\{x_i\}_{i=0}^{\infty} \subset B(0, M_0)$$

for each natural number i,

$$(x_i, \lambda_i) \in A(x_{i-1}, (\Omega_i, w_i), \tilde{\delta}_i, \delta_i).$$

Then for all integers $n \geq n_\epsilon$,

$$d(x_n, C) \leq \epsilon,$$

$$f_i(x_n) \leq \epsilon,\ i \in \{1, \dots, m\}.$$

Theorem 10.16. *Suppose that the set C is bounded,* $M_0 > 0$,

$$B(0, M_0) \cap C \neq \emptyset,$$

$$\{z \in X : f_i(z) < 0\} \cap B(0, M_0) \neq \emptyset \textit{ for all } i = 1, \dots, m,$$

$M_1 > 0$, $\epsilon \in (0, 1)$. *Then there exist a natural number* n_0 *and* $\delta > 0$ *such that the following assertion holds.*

Assume that

$$\tilde{\delta}_n \in [0, \delta],\ n \in \mathcal{N} \setminus \{0\}, \tag{10.310}$$

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*, \tag{10.311}$$

satisfies (10.220) for each natural number j,

$$x_0 \in B(0, M_1), \tag{10.312}$$

$$\{x_i\}_{i=1}^{\infty} \subset X,\ \{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty) \tag{10.313}$$

satisfy for each natural number i,

$$(x_i, \lambda_i) \in A(x_{i-1}, (\Omega_i, w_i), \tilde{\delta}_i, \delta). \tag{10.314}$$

Then for all integers $n \geq n_0$,

$$d(x_n, C) \leq \epsilon,$$
$$f_i(x_{n_0}) \leq \epsilon,\ i \in \{1, \dots, m\}.$$

Proof. For each $i \in \{1, \dots, m\}$, there exists $z_i \in X$ such that

$$z_i \in B(0, M_0),\ f_i(z_i) < 0.$$

We may assume without loss of generality that

$$M_1 > \sup\{\|z\| :\ z \in C\} + 4, \tag{10.315}$$
$$M_0 > 3M_1 + 1. \tag{10.316}$$

By Theorem 10.13, there exist a natural number n_1 and $\gamma_1 > 0$ such that the following assertion holds.

(i) for each

$$\tilde{\delta}_n \in [0, \gamma_1],\ n \in \mathcal{N} \setminus \{0\},$$

each

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*,$$

satisfying (10.220) for each natural number j, each

$$x_0 \in B(0, M_1),$$

each pair of sequences

$$\{x_i\}_{i=1}^{\infty} \subset X,\ \{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty)$$

satisfying for each natural number i,

$$(x_i, \lambda_i) \in A(x_{i-1}, (\Omega_i, w_i), \tilde{\delta}_i, \gamma_1)$$

we have

$$\|x_i\| \leq 3M_1 + 1 \text{ for all integers } i = 0, \dots, n_1,$$
$$d(x_{n_1}, C) \leq \epsilon.$$

By Theorem 10.14, there exist a natural number n_0 and $\delta \in (0, \gamma_1)$ such that the following property hold:

(ii) for each

$$\tilde{\delta}_n \in [0, \delta], \ n \in \mathcal{N} \setminus \{0\},$$

each

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$$

satisfying (10.220) for each natural number j and (10.221)–(10.223), and each

$$\{x_n\}_{n=0}^{\infty} \subset B(0, M_0),$$

for all integers $n \geq n_0$ we have

$$d(x_n, C) \leq \epsilon,$$
$$f_i(x_n) \leq \epsilon, \ i \in \{1, \dots, m\}.$$

Assume that (10.310) and (10.311) hold, for all integers $j \geq 1$ (10.220) is true and that (10.312), (10.313) hold. By (i) and (10.315), (10.314) is true for all integers $i \geq 1$,

$$\|x_{nn_1}\| \leq M_1, \ n \in \mathcal{N}, \ \|x_n\| \leq 3M_1 + 1, \ n \in \mathcal{N}. \tag{10.317}$$

In view of property (ii), (10.316), and (10.317), for all integers $n \geq n_0$,

$$d(x_n, C) \leq \epsilon,$$
$$f_i(x_n) \leq \epsilon, \ i \in \{1, \dots, m\}.$$

This completes the proof of Theorem 10.16. □

Theorem 10.16 implies the following result.

Theorem 10.17. *Let $M_0 > 0$, the set C be bounded,*

$$\{z \in X : \ f_i(z) < 0\} \neq \emptyset \text{ for all } i = 1, \dots, m.$$

Then there exists $\delta > 0$ such that the following assertion holds.

Assume that

$$\{\delta_n\}_{n=0}^{\infty} \subset (0, \delta), \ \lim_{n \to \infty} \delta_n = 0, \ \epsilon > 0.$$

Then there exists a natural number n_ϵ such that for each $\tilde{\delta}_n \in [0, \delta_n]$, $n \in \mathcal{N} \setminus \{0\}$, each

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$$

satisfying (10.220) for each natural number j, each

$$x_0 \in B(0, M_0),$$

each $\{x_i\}_{i=1}^{\infty} \subset X$*, each* $\{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty)$ *satisfying for each natural number i,*

$$(x_i, \lambda_i) \in A(x_{i-1}, (\Omega_i, w_i), \tilde{\delta}_i, \delta_i),$$

the inequalities

$$d(x_n, C) \le \epsilon,$$
$$f_i(x_n) \le \epsilon, \; i \in \{1, \dots, m\}$$

hold for all integers $n \ge n_\epsilon$.

Chapter 11
Iterative Subgradient Projection Algorithm

In this chapter we study convergence of iterative subgradient projection algorithms for solving convex feasibility problems in a general Hilbert space. Our goal is to obtain an approximate solution of the problem in the presence of computational errors. We show that our subgradient projection algorithm generates a good approximate solution, if the sequence of computational errors is bounded from above by a constant. Moreover, for a known computational error, we find out what an approximate solution can be obtained and how many iterates one needs for this.

11.1 Preliminaries

Let $(X, \langle \cdot, \cdot \rangle)$ be a Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ which induces a complete norm $\| \cdot \|$.

For every point $x \in X$ and every nonempty set $A \subset X$ define

$$d(x, A) = \inf\{\|x - y\| : \ y \in A\}.$$

For every point $x \in X$ and every positive number r put

$$B(x, r) = \{y \in X : \ \|x - y\| \le r\}.$$

In view of Proposition 10.1, for every nonempty closed convex subset C of the space X and every point $x \in X$ there exists a unique point $P_C(x) \in C$ which satisfies

$$\|x - P_C(x)\| = \inf\{\|x - y\| : \ y \in C\}, \tag{11.1}$$

$$\|P_C(x) - P_C(y)\| \le \|x - y\| \text{ for all } x, y \in X \tag{11.2}$$

A. Zaslavski, *Approximate Solutions of Common Fixed-Point Problems*, Springer Optimization and Its Applications 112, DOI 10.1007/978-3-319-33255-0_11

and for every point $x \in X$ and every point $z \in C$,

$$\langle z - P_C(x), x - P_C(x) \rangle \le 0,$$
$$\|z - P_C(x)\|^2 + \|x - P_C(x)\|^2 \le \|z - x\|^2. \tag{11.3}$$

Denote by $\mathcal{N}$ a set of all nonnegative integers. We recall the following useful facts on convex functions given already in Chap. 10.

Let $f : X \to R^1$ be a continuous convex function such that

$$\{x \in X : f(x) \le 0\} \ne \emptyset. \tag{11.4}$$

Let $y_0 \in X$. Then

$$\partial f(y_0) = \{l \in X : f(y) - f(y_0) \ge \langle l, y - y_0 \rangle \text{ for all } y \in X\} \tag{11.5}$$

is the subdifferential of f at the point y_0 [72, 77].

For every $l \in \partial f(y_0)$ it follows from (11.5) that

$$\{x \in X : f(x) \le 0\} \subset \{x \in X : f(y_0) + \langle l, x - y_0 \rangle \le 0\}. \tag{11.6}$$

It is well known that the following lemma holds.

Lemma 11.1. *Let $y_0 \in X, f(y_0) > 0$, $l \in \partial f(y_0)$ and let*

$$D = \{x \in X : f(y_0) + \langle l, x - y_0 \rangle \le 0\}.$$

The $l \ne 0$ and

$$P_D(y_0) = y_0 - f(y_0)\|l\|^{-2} l.$$

Let us now describe the convex feasibility problem studied in the chapter and iterative subgradient projection algorithms which will be used for its solving.

Let m be a natural number and $f_i : X \to R^1$, $i = 1, \dots, m$ be convex continuous functions.

For every integer $i = 1, \dots, m$ put

$$C_i = \{x \in X : f_i(x) \le 0\},$$
$$C = \cap_{i=1}^m C_i = \cap_{i=1}^m \{x \in X : f_i(x) \le 0\}.$$

We suppose that

$$C \ne \emptyset.$$

A point $x \in C$ is called a solution of our feasibility problem. For a given positive number ϵ a point $x \in X$ is called an ϵ-approximate solution of the feasibility problem if

$$f_i(x) \le \epsilon \text{ for all } i = 1, \dots, m.$$

In this chapter we apply the subgradient projection method in order to obtain a good approximative solution of the feasibility problem.

Let a natural number $\bar{m} \geq m$. Denote by $\mathcal{R}$ the set of all mappings $S : \mathcal{N} \to \{1, \dots m\}$ such that the following property holds:

(P1) For every nonnegative integer p and every $j \in \{1, \dots, m\}$ there exists an integer $i \in \{p, \dots, p + \bar{m} - 1\}$ for which $S(i) = j$.

We are interested to find approximate solutions of the inclusion $x \in C$. In order to meet this goal we apply algorithms generated by $S \in \mathcal{R}$. More precisely, we associate with any $S \in \mathcal{R}$ the algorithm which generates, for every starting point $x_0 \in X$, a sequence $\{x_k\}_{k=0}^{\infty} \subset X$ such that for each integer $k \geq 0$,

$$x_{k+1} = x_k \text{ if } f_{S(k)}(x_k) \leq 0$$

and

$$x_{k+1} = x_k - f_{S(k)}(x_k)\|l_k\|^{-2} l_k$$

if $f_{S(k)}(x_k) > 0$, where $l_k \in \partial f_{S(k)}(x_k)$.

Note that by Lemma 11.1 the sequence $\{x_k\}_{k=0}^{\infty}$ is well defined and that for each integer $k \geq 0$, if $f_{S(k)}(x_k) > 0$, then

$$x_{k+1} = P_{D_k}(x_k),$$

where

$$D_k = \{x \in X : f(x_k) + \langle l_k, x - x_k \rangle \leq 0\}$$

and $l_k \in \partial f_{S(k)}(x_k)$.

The algorithms of this type are well known in the literature, where their convergence was studied as $k \to \infty$ (see [7] and the references mentioned there). In this chapter, we study the behavior of the sequences generated by the algorithm, associated with $S \in \mathcal{R}$, taking into account computational errors which are always present in practice. Namely, in practice the algorithm, associated with $S \in \mathcal{R}$, generates, for any starting point $x_0 \in X$, sequences $\{x_k\}_{k=0}^{\infty} \subset X$ and $\{l_k\}_{k=0}^{\infty} \subset X$ such that for each integer $k \geq 0$,

$$x_{k+1} = x_k \text{ if } f_{S(k)}(x_k) \leq \delta$$

and

$$\|x_{k+1} - x_k + f_{S(k)}(x_k)\|l_k\|^{-2} l_k\| \leq \delta,$$

if $f_{S(k)}(x_k) > \delta$, where $l_k \in X \setminus \{0\}$ satisfies $d(l_k, \partial f_{S(k)}(x_k)) \leq \delta$, with a constant $\delta > 0$ which depends only on our computer system. Surely, in this situation one cannot expect that the sequence $\{x_k\}_{k=0}^{\infty}$ converges to a point of C. The goal of this chapter is to understand what subset of X attracts all sequences $\{x_k\}_{k=0}^{\infty}$ generated by the algorithm. The results of this chapter were obtained in [97].

11.2 The First Main Result

Suppose that m is a natural number and that $f_i : X \to R^1$, $i = 1, \dots, m$ are convex continuous functions.

For every integer $i = 1, \dots, m$ put

$$C_i = \{x \in X : \ f_i(x) \le 0\}$$

and set

$$C = \cap_{i=1}^{m} C_i = \cap_{i=1}^{m} \{x \in X : \ f_i(x) \le 0\}.$$

Suppose that

$$C \ne \emptyset.$$

Let a natural number $\bar{m} \ge m$. Recall that $\mathcal{R}$ is the set of all mappings $S : \mathcal{N} \to \{1, \dots m\}$ which possess property (P1).

We study the behavior of iterates generated by $S \in \mathcal{R}$ taking into account computational errors bounded from above by a positive constant δ which depends only on our computer system.

Suppose that $M > 0$, $M_0 > 0$, and $M_1 > 2$ be such that

$$B(0, M) \cap \{x \in X : \ f_i(x) \le 0, \ i = 1, \dots, m\} \ne \emptyset, \tag{11.7}$$

$$f_i(B(0, 3M + 3)) \subset [-M_0, M_0], \ i = 1, \dots, m, \tag{11.8}$$

$$|f_i(u) - f_i(v)| \le (M_1 - 2)\|u - v\|$$
$$\text{for all } u, v \in B(0, 3M + 4) \text{ and all } i = 1, \dots, m. \tag{11.9}$$

Let $\Delta \in (0, 1]$ and let $\delta \in (0, 1]$ satisfy

$$8(4M + 4)\bar{m}^3\delta(1 + 16M_0\Delta^{-2}(4M + 3)^2) \le 1, \tag{11.10}$$

$$\Delta \ge 2M_1(8\delta(4M + 4)\bar{m}(1 + 16M_0\Delta^{-2}(4M + 3)^2))^{1/2}. \tag{11.11}$$

Let

$$\epsilon_0 = (8\delta(4M + 4)\bar{m}(1 + 16M_0\Delta^{-2}(4M + 3)^2))^{1/2} \tag{11.12}$$

and a natural number

$$n_0 > 32M^2\epsilon_0^{-2}. \tag{11.13}$$

The following theorem is our first main result which will be proved in Sect. 11.5.

Theorem 11.2. *Assume that*

$$S \in \mathcal{R},\ \{x_k\}_{k=0}^{\infty} \subset X,\ \|x_0\| \le M, \{l_k\}_{k=0}^{\infty} \subset X \tag{11.14}$$

be such that for each integer $k \ge 0$*,*

$$x_{k+1} = x_k,\ l_k = 0,\ \textit{if}\, f_{S(k)}(x_k) \le \Delta, \tag{11.15}$$

$$\textit{if}\, f_{S(k)}(x_k) > \Delta,\ \textit{then}$$

$$l_k \in X \setminus \{0\},\ d(l_k, \partial f_{S(k)}(x_k)) < \delta \tag{11.16}$$

and

$$\|x_{k+1} - x_k + f_{S(k)}(x_k)\|l_k\|^{-2} l_k\| \le \delta. \tag{11.17}$$

Then there exists an integer $q \in [0, n_0 - 1]$ *such that* $\|x_k\| \le 3M + 1$ *for all integers* $k = 0, \ldots, (q+1)\bar{m}$ *and for each integer* $k \in \{q\bar{m}, \ldots, (q+1)\bar{m} - 1\}$*,*

$$\|x_k - x_{k+1}\| \le \epsilon_0. \tag{11.18}$$

Moreover, if an integer $q \ge 0$ *be such that for all integers* $k \in \{q\bar{m}, \ldots, (q+1)\bar{m} - 1\}$ *(11.18) holds and* $\|x_k\| \le 3M + 1$*, then for each* $i, j \in \{q\bar{m}, \ldots, (q+1)\bar{m}\}$*,*

$$\|x_i - x_j\| \le \bar{m}\epsilon_0$$

and for each $k \in \{q\bar{m}, \ldots, (q+1)\bar{m}\}$ *and each* $i \in \{1, \ldots, m\}$*,*

$$f_i(x_k) \le \Delta + M_1\bar{m}\epsilon_0 \le \Delta(\bar{m}+1).$$

Theorem 11.2 is useful in answering the following questions.

Assume that the upper bound for computational errors δ, which depends on our computer system, is known, and we need to know what approximate solution can be obtained and how many iterates should be done. According to Theorem 11.2, we obtain a $\Delta(\bar{m}+1)$-approximate solution after $\bar{m}(n_0 - 1)$ iterations of the algorithm associated with $S \in \mathcal{R}$, where constants Δ and n_0 are defined by (11.10)–(11.13).

Assume now that a constant Δ is given, and we are interested to know what should be an upper bound δ for computational errors, produced by a computer system, and how many iterates should be done in order to obtain $\Delta(\bar{m}+1)$-approximate solution. These constants (δ and n_0) again can be found by (11.10)–(11.13).

Theorem 11.2 is also useful if the upper bound δ for computational errors, produced by a computer system, is known, it is given an upper bound for a number of iterates and we are interested to find Δ.

Note that Theorem 11.2 provides the estimations for the constants δ and n_0, which follow from (11.10) to (11.13). Namely, $\delta = c_1\Delta^4$ and $n_0 = c_2\Delta^{-2}$, where c_1 and c_2 are positive constants depending on M.

Theorem 11.7 also answers an important question how we can find an iteration number i for which x_i is a $\Delta(\bar{m}+1)$-approximate solution. By Theorem 11.2 we need just to find the smallest integer $q \in [0, \dots, n_0 - 1]$ such that for each integer $k \in [q\bar{m}, \dots, (q+1)\bar{m} - 1]$ (11.18) holds and that $||x_k|| \leq 3M + 1$. Then x_i is a $\Delta(\bar{m}+1)$-approximate solution for all integers $i \in [q\bar{m}, (q+1)\bar{m}]$.

In the proof of Theorem 11.2 we use the following auxiliary results which is proved in Sect. 11.4.

Lemma 11.3. *Let*

$$\delta_0 = \delta(1 + 16M_0\Delta^{-2}(4M+3)^2), \tag{11.19}$$

an integer $j \in \{1, \dots, m\}$,

$$x \in B(0, 3M+3),\ f_j(x) > \Delta, \tag{11.20}$$

$$z \in B(0, M) \cap C \tag{11.21}$$

and

$$\xi \in \partial f_j(x),\ \ l \in B(\xi, \delta). \tag{11.22}$$

Then $l \neq 0$, $\xi \neq 0$,

$$y := x - f_j(x)\|\xi\|^{-2}\xi \tag{11.23}$$

satisfy

$$\|y - z\| \leq \|z - x\|,\ \ \|y - z\|^2 \leq \|z - x\|^2 - \|x - y\|^2 \tag{11.24}$$

and for each

$$u \in B(x - f_j(x)\|l\|^{-2}l, \delta) \tag{11.25}$$

the following inequalities hold:

$$\|u - y\| \leq \delta_0, \tag{11.26}$$

$$\|u - z\| \leq \delta_0 + \|x - z\|. \tag{11.27}$$

11.3 The Second Main Result

Assume that m is a natural number and that $f_i : X \to R^1$, $i = 1, \dots, m$ are convex continuous functions.

For every integer $i = 1, \dots, m$ put

$$C_i = \{x \in X : f_i(x) \le 0\}.$$

Set

$$C = \cap_{i=1}^m C_i = \cap_{i=1}^m \{x \in X : f_i(x) \le 0\}.$$

Suppose that

$$C \ne \emptyset.$$

In this section we present our second main result (Theorem 11.4) which describes the behavior of iterates generated by the subgradient projection method under an assumption that for each $i \in \{1, \dots, m\}$, $\{x \in X : f_i(x) < 0\} \ne \emptyset$.

Let a natural number $\bar{m} \ge m$. Recall that $\mathcal{R}$ is the set of all mappings $S : \mathcal{N} \to \{1, \dots m\}$ which possess property (P1).

Suppose that $M > 0$, $M_0 > 0$, $M_1 > 2$ be such that

$$B(0, M) \cap \{x \in X : f_i(x) \le 0, \ i = 1, \dots, m\} \ne \emptyset, \tag{11.28}$$

$$B(0, M) \cap \{x \in X : f_i(x) < 0\} \ne \emptyset \tag{11.29}$$

for all $i \in \{1, \dots, m\}$,

$$f_i(B(0, 3M + 3)) \subset [-M_0, M_0], \ i = 1, \dots, m, \tag{11.30}$$

$$|f_i(u) - f_i(v)| \le (M_1 - 2)||u - v|| \text{ for all } u, v \in B(0, 3M + 4), \ i = 1, \dots, m. \tag{11.31}$$

By (11.29) there exists $\Delta \in (0, 1]$ such that for each $i = 1, \dots, m$,

$$B(0, M) \cap \{x \in X : f_i(x) \le -\Delta\} \ne \emptyset. \tag{11.32}$$

Let $\delta \in (0, 1]$ satisfy

$$\delta \le 2^{-1}\Delta(4M + 3)^{-1}, \tag{11.33}$$

$$8(4M + 4)\bar{m}^3\delta(1 + 16M_0\Delta^{-2}(4M + 3)^2) \le 1, \tag{11.34}$$

let

$$\epsilon_0 = (8\delta(4M+4)\bar{m}(1+16M_0\Delta^{-2}(4M+3)^2))^{1/2} \tag{11.35}$$

and let a natural number

$$n_0 > 32M^2\epsilon_0^{-2}. \tag{11.36}$$

The following theorem is the second main result of this chapter.

Theorem 11.4. *Assume that*

$$S \in \mathcal{R},\ \epsilon \in [0, 2M_1\epsilon_0],\ \{x_k\}_{k=0}^{\infty} \subset X,\ \|x_0\| \le M, \{l_k\}_{k=0}^{\infty} \subset X \tag{11.37}$$

be such that for each integer $k \ge 0$,

$$x_{k+1} = x_k,\ l_k = 0,\ \text{if}\, f_{S(k)}(x_k) \le \epsilon, \tag{11.38}$$

if $f_{S(k)}(x_k) > \epsilon$, *then*

$$l_k \in X \setminus \{0\},\ d(l_k, \partial f_{S(k)}(x_k)) < \delta \tag{11.39}$$

and

$$\|x_{k+1} - x_k + f_{S(k)}(x_k)\|l_k\|^{-2} l_k\| \le \delta. \tag{11.40}$$

Then there exists an integer $q \in [0, n_0 - 1]$ *such that* $\|x_k\| \le 3M+1$ *for all integers* $k = 0, \dots, (q+1)\bar{m}$ *and that for each integer* $k \in \{q\bar{m}, \dots, (q+1)\bar{m} - 1\}$,

$$\|x_k - x_{k+1}\| \le \epsilon_0. \tag{11.41}$$

Moreover, if an integer $q \ge 0$ *be such that for all* $k \in \{q\bar{m}, \dots, (q+1)\bar{m}-1\}$ *(11.41) holds and* $\|x_k\| \le 3M+1$, *then for each* $i, j \in \{q\bar{m}, \dots, (q+1)\bar{m}\}$,

$$\|x_i - x_j\| \le \bar{m}\epsilon_0$$

and for each $k \in \{q\bar{m}, \dots, (q+1)\bar{m}\}$ *and each* $i \in \{1, \dots, m\}$,

$$f_i(x_k) \le M_1(\bar{m}+2)\epsilon_0.$$

Note that in Theorem 11.4 the upper bound for computational errors produced by our computer system is δ and we obtain a $M_1(\bar{m}+2)\epsilon_0$-approximate solution after $\bar{m}(n_0-1)$ iterations of the algorithm associated with $S \in \mathcal{R}$, where constants ϵ_0 and n_0 are defined by (11.35) and (11.36). Namely, $\epsilon_0 = c_1\delta^{1/2}$ and $n_0 = c_2\delta^{-1}$, where c_1 and c_2 are positive constants. These estimates are better than estimates provided by Theorem 11.2.

Theorem 11.4 is proved in Sect. 11.5. In its proof we use the following auxiliary results proved in Sect. 11.4.

Lemma 11.5. *Let*

$$\delta_0 = \delta(1 + 16M_0\Delta^{-2}(4M + 3)^2), \tag{11.42}$$

an integer $j \in \{1, \dots, m\}$,

$$x \in B(0, 3M + 3),\ f_j(x) > 0, \tag{11.43}$$

$$z \in C, \tag{11.44}$$

$$\xi \in \partial f_j(x),\ l \in B(\xi, \delta). \tag{11.45}$$

Then $l \neq 0$, $\xi \neq 0$,

$$y := x - f_j(x)\|\xi\|^{-2}\xi \tag{11.46}$$

satisfy

$$\|y - z\| \leq \|z - x\|,\ \|y - z\|^2 \leq \|z - x\|^2 - \|x - y\|^2 \tag{11.47}$$

and for each

$$u \in B(x - f_j(x)\|l\|^{-2}l, \delta) \tag{11.48}$$

the following inequalities hold:

$$\|u - y\| \leq \delta_0, \tag{11.49}$$

$$\|u - z\| \leq \delta_0 + \|x - z\|. \tag{11.50}$$

11.4 Proofs of Lemmas 11.3 and 11.5

We prove Lemmas 11.3 and 11.5 simultaneously. Define

$$D = \{v \in X :\ f_j(x) + \langle \xi, v - x\rangle \leq 0\}. \tag{11.51}$$

In view of (11.20)–(11.22) (in the case of Lemma 11.3) and in view of (11.43)–(11.45) (in the case of Lemma 11.5), $\xi \neq 0$.

Lemma 11.1, (11.20)–(11.23), and (11.51) (in the case of Lemma 11.3) and Lemma 11.1, (11.43)–(11.46) and (11.51) (in the case of Lemma 11.5) imply that

$$P_D(x) = y. \tag{11.52}$$

By (11.21), (11.22), (in the case of Lemma 11.3), (11.44), (11.45) (in the case of Lemma 11.5), (11.3), (11.6), (11.51), and (11.52),

$$\|z - y\|^2 = \|z - P_D(x)\|^2 \leq \|z - x\|^2 - \|x - y\|^2. \tag{11.53}$$

It is clear that (11.53) implies (11.24) and (11.47).

It follows from (11.20), (11.22), (11.8), (11.9) (in the case of Lemma 11.3) and it follows from (11.43), (11.45), (11.30), and (11.31) (in the case of Lemma 11.5) that

$$f_j(x) \leq M_0, \tag{11.54}$$

$$\|\xi\| \leq M_1 - 2. \tag{11.55}$$

By (11.55), (11.22) (in the case of Lemma 11.3) and (11.45) (in the case of Lemma 11.5),

$$\|l\| \leq M_1 - 1. \tag{11.56}$$

In the case of Lemma 11.3 put

$$z_j = z. \tag{11.57}$$

In the case of Lemma 11.5 it follows from (11.32) that there exists a point

$$z_j \in B(0, M) \text{ such that } f_j(z_j) \leq -\Delta. \tag{11.58}$$

By (11.57), (11.21), (11.20), (11.22) (in the case of Lemma 11.3) and by (11.58), (11.43), (11.45) (in the case of Lemma 11.5),

$$\begin{aligned} -\Delta \geq f_j(z_j) - f_j(x) \geq \langle \xi, z_j - x \rangle \geq -\|\xi\| \|z_j - x\| \\ \geq -\|\xi\|(4M + 3) \end{aligned}$$

and

$$\|\xi\| \geq \Delta(4M + 3)^{-1}. \tag{11.59}$$

Relations (11.22), (11.11) (in the case of Lemma 11.3), and (11.45), (11.34) (in the case of Lemma 11.5) and (11.59) imply that

$$\|l\| \geq \|\xi\| - \delta \geq \Delta(4M + 3)^{-1} - \delta \geq 2^{-1}\Delta(4M + 3)^{-1}. \tag{11.60}$$

In view of (11.60),

$$l \neq 0.$$

Let

$$u \in B(x - f_j(x)\|l\|^{-2}l, \delta). \tag{11.61}$$

(see (11.15), (11.48)).

It follows from (11.61), (11.23), (11.54), (11.20) (in the case of Lemma 11.3) and from (11.61), (11.46), (11.54), (11.43) (in the case of Lemma 11.5) that

$$\begin{aligned} \|u - y\| &\le \delta + \|x - f_j(x)\|l\|^{-2}l - y\| \\ &\le \delta + \|f_j(x)\|\xi\|^{-2}\xi - f_j(x)\|l\|^{-2}l\| \\ &\le \delta + M_0\|\|\xi\|^{-2}\xi - \|l\|^{-2}l\|. \end{aligned} \tag{11.62}$$

In view of (11.22), (11.59), (11.11), (11.60) (in the case of Lemma 11.3) and in view of (11.45), (11.59), (11.33), and (11.60) (in the case of Lemma 11.5),

$$\begin{aligned} \|\|\xi\|^{-2}\xi - \|l\|^{-2}l\| &\le \|l\|^{-2}\|l - \xi\| + \|\xi\|\|\|\xi\|^{-2} - \|l\|^{-2}| \\ &\le \|l\|^{-2}\|l - \xi\| + \|l\|^{-2}\|\xi\|^{-1}|\|l\|^2 - \|\xi\|^2| \\ &\le \|l\|^{-2}[\delta + \|\xi\|^{-1}\delta(\|l\| + \|\xi\|)] \\ &\le \|l\|^{-2}\delta[1 + \|\xi\|^{-1}(2\|\xi\| + \delta)] \le 4\delta\|l\|^{-2} \\ &\le 16\delta(4M + 3)^2\Delta^{-2}. \end{aligned} \tag{11.63}$$

Relations (11.63), (11.62), and (11.19) (in the case of Lemma 11.3) and (11.63), (11.62), (11.42) (in the case of Lemma 11.5) and (11.28) imply that

$$\|u - y\| \le \delta + \delta M_0 16\Delta^{-2}(4M + 3)^2 = \delta_0. \tag{11.64}$$

Hence (11.26) and (11.49) hold. It follows from (11.64) and (11.53) that

$$\|u - z\| \le \|u - y\| + \|y - z\| \le \delta_0 + \|z - x\|.$$

This completes the proof of Lemmas 11.3 and 11.5. □

11.5 Proofs of Theorems 11.2 and 11.4

We prove Theorems 11.2 and 11.4 simultaneously. In view of (11.7) (in the case of Theorem 11.2) and in view of (11.18) (in the case of Theorem 11.4) there exists a point

$$z \in B(0, M) \text{ such that } f_i(z) \le 0, \ i = 1, \dots, m. \tag{11.65}$$

In the case of Theorem 11.2 put

$$\epsilon = \Delta. \tag{11.66}$$

Set

$$\delta_0 = \delta(1 + 16M_0\Delta^{-2}(4M + 3)^2). \tag{11.67}$$

Assume that $\tilde{q} \in [0, n_0 - 1]$ is an integer such that for every integer $p \in [0, \tilde{q}]$ we have

$$\max\{\|x_i - x_{i+1}\| : i \in \{p\bar{m}, \dots, (p + 1)\bar{m} - 1\}\} > \epsilon_0. \tag{11.68}$$

In view of (11.65), (11.14) (in the case of Theorem 11.2) and in view of (11.37) (in the case of Theorem 11.4)

$$\|x_0 - z\| \leq 2M. \tag{11.69}$$

Assume that an integer $p \in [0, \tilde{q}]$ and that

$$\|x_{p\bar{m}} - z\| \leq 2M. \tag{11.70}$$

Now assume that an integer $k \in \{p\bar{m}, \dots, (p + 1)\bar{m} - 1]$ satisfies

$$\|x_k - z\| \leq 2M + \delta_0(k - p\bar{m}). \tag{11.71}$$

(Note that in view of (11.70) inequality (11.71) is valid with $k = p\bar{m}$.) There are two cases:

$$f_{S(k)}(x_k) \leq \epsilon; \tag{11.72}$$

$$f_{S(k)}(x_k) > \epsilon. \tag{11.73}$$

If inequality (11.72) is true, then $x_{k+1} = x_k$ (see (11.5), (11.66), (11.38)). Assume that inequality (11.73) is valid. Then it follows from (11.73), (11.66), (11.17), (11.16) (in the case of Theorem 11.2) and it follows from (11.39), (11.40) (in the case of Theorem 11.4) that there exist

$$\xi_k \in \partial f_{S(k)}(x_k),\ l_k \in B(\xi_k, \delta) \setminus \{0\} \tag{11.74}$$

such that

$$\|x_{k+1} - x_k + f_{S(k)}(x_k)\|l_k\|^{-2}l_k\| \leq \delta. \tag{11.75}$$

In the case of Theorem 11.2 by (11.67), (11.71), (11.19), (11.10) and in the case of Theorem 11.4 by (11.71), (11.67), (11.34) we have

$$\|x_k - z\| \leq 2M + \delta_0 \bar{m} \leq 2M + 1. \tag{11.76}$$

Relations (11.76) and (11.65) imply that

$$\|x_k\| \leq 3M + 1. \tag{11.77}$$

In the case of Theorem 11.2 in view of (11.19), (11.67), (11.77), (11.73), (11.65), (11.74), (11.66), (11.76), Lemma 11.3 applied with $x = x_k$, $\xi = \xi_k$, $l = l_k$, $u = x_{k+1}$ and in the case of Theorem 11.4 in view of (11.42), (11.67), (11.77), (11.65), (11.73), (11.74)–(11.76), Lemma 11.5 applied with $x = x_k$, $\xi = \xi_k$, $l = l_k$,

$$\|x_{k+1} - z\| \leq \delta_0 + \|x_k - z\|. \tag{11.78}$$

Together with (11.71) this implies that

$$\|x_{k+1} - z\| \leq 2M + \delta_0(k + 1 - p\bar{m}).$$

Hence the inequality above and (11.78) are true in both cases. Therefore by induction we have shown that (11.71) is valid for all integers $k = p\bar{m}, \dots, (p+1)\bar{m}$ and that (11.78) is true for all integers $k = p\bar{m}, \dots, (p+1)\bar{m} - 1$. Combined with (11.67), (11.10) (in the case of Theorem 11.2) and with (11.34) (in the case of Theorem 11.4) this implies that

$$\|x_k - z\| \leq 2M + 1, \; k = p\bar{m}, \dots, (p+1)\bar{m}. \tag{11.79}$$

By (11.79), (11.78) which holds for all $k = p\bar{m}, \dots, (p+1)\bar{m} - 1$, (11.67), (11.10) (in the case of Theorem 11.2) and by (11.34) (in the case of Theorem 11.4), for all integers $k = p\bar{m}, \dots, (p+1)\bar{m} - 1$, we have

$$\begin{aligned} \|z - x_k\|^2 - \|z - x_{k+1}\|^2 &\geq \|z - x_k\|^2 - (\|z - x_k\| + \delta_0)^2 \\ &\geq -2\delta_0\|z - x_k\| - \delta_0^2 \geq -2\delta_0(2M + 2). \end{aligned} \tag{11.80}$$

In view of (11.68) there exists an integer

$$j \in \{p\bar{m}, \dots, (p+1)\bar{m} - 1\} \tag{11.81}$$

for which

$$\|x_{j+1} - x_j\| > \epsilon_0. \tag{11.82}$$

By (11.82), (11.66), (11.15)–(11.17) (in the case of Theorem 11.2) and by (11.38)–(11.40) (in the case of Theorem 11.4),

$$f_{S(j)}(x_j) > \epsilon. \tag{11.83}$$

By (11.83), (11.62), (11.16), (11.17) (in the case of Theorem 11.2) and by (11.83), (11.39) (in the case of Theorem 11.4), that there exists a point

$$\xi_j \in \partial f_{S(j)}(x_j) \tag{11.84}$$

such that

$$\|\xi_j - l_j\| \le \delta. \tag{11.85}$$

Relations (11.84) and (11.83) imply that $\xi_j \neq 0$. Set

$$y = x_j - f_{S(j)}(x_j)\|\xi_j\|^{-2}\xi_j. \tag{11.86}$$

In the case of Theorem 11.2 relations (11.67), (11.79), (11.65), (11.83), (11.66), (11.84), (11.81), (11.85), (11.86), (11.17), and Lemma 11.3 applied with $x = x_j$, $\xi = \xi_j$, $l = l_j$, $u = x_{j+1}$ and in the case of Theorem 11.4 relations (11.67), (11.79), (11.65), (11.83), (11.84), (11.81), (11.85), (11.86), (11.40) and Lemma 11.5 applied with $x = x_j$, $\xi = \xi_j$, $l = l_j$, $u = x_{j+1}$ imply that

$$\|x_{j+1} - y\| \le \delta_0, \tag{11.87}$$

$$\|z - y\|^2 \le \|z - x_j\|^2 - \|x_j - y\|^2. \tag{11.88}$$

By (11.82), (11.87), (11.67), (11.12), (11.10) (in the case of Theorem 11.2), (11.34) and by (11.35) (in the case of Theorem 11.4), we have

$$\|x_j - y\| \ge \|x_j - x_{j+1}\| - \|y - x_{j+1}\| > \epsilon_0 - \delta_0 \ge \epsilon_0/2. \tag{11.89}$$

Relations (11.88) and (11.89) imply that

$$\|z - y\|^2 \le \|z - x_j\|^2 - \epsilon_0^2/4. \tag{11.90}$$

In view of (11.90), (11.79), and (11.81),

$$\|z - y\| \le 2M + 1. \tag{11.91}$$

By (11.90), (11.87), (11.91), (11.67), (11.10) (in the case of Theorem 11.2) and by (11.34) (in the case of Theorem 11.4), we have

$$\|z - x_{j+1}\|^2 = \|z - y + y - x_{j+1}\|^2$$
$$\leq \|z - y\|^2 + \|y - x_{j+1}\|^2 + 2\|z - y\| \|y - x_{j+1}\|$$
$$\leq \|z - x_j\|^2 - \epsilon_0^2/4 + \delta_0^2 + 2\delta_0(2M + 1)$$
$$\leq \|z - x_j\|^2 - \epsilon_0^2/4 + \delta_0(4M + 4). \tag{11.92}$$

It follows from (11.80), (11.92), (11.67), (11.81), (11.12) (in the case of Theorem 11.2) and from (11.35) (in the case of Theorem 11.4) that

$$\|z - x_{p\bar{m}}\|^2 - \|z - x_{(p+1)\bar{m}}\|^2 = \sum_{k=p\bar{m}}^{(p+1)\bar{m}-1} [\|z - x_k\|^2 - \|z - x_{k+1}\|^2]$$
$$\geq -\delta_0(4M + 4)\bar{m} + \epsilon_0^2/4 \geq \epsilon_0^2/8. \tag{11.93}$$

We have shown that the following property holds:

(P2) If an integer $p \in [0, \tilde{q}]$ satisfies (11.70), then (11.79) and (11.93) are valid and (11.78) is true for all integers $k = p\bar{m}, \ldots, (p + 1)\bar{m} - 1$.

It follows from (11.93), property (P2), (11.69), (11.13), and (11.36) that

$$(\tilde{q} + 1)\epsilon_0^2/8 \leq \sum_{p=0}^{\tilde{q}} [\|z - x_{p\bar{m}}\|^2 - \|z - x_{(p+1)\bar{m}}\|^2] \leq \|z - x_0\|^2 \leq 4M^2,$$

$$\tilde{q} + 1 \leq 32M^2\epsilon_0^{-2} < n_0.$$

Thus we assumed that an integer $\tilde{q} \in [0, n_0 - 1]$ and that (11.68) is true for every integer $p \in [0, \tilde{q}]$ and showed that $\tilde{q} + 1 < n_0$. This implies that there exists an integer $q \in [0, n_0 - 1]$ such that for each integer p satisfying $0 \leq p < q$,

$$\max\{\|x_i - x_{i+1}\| : i = p\bar{m}, \ldots, (p + 1)\bar{m} - 1\} > \epsilon_0, \tag{11.94}$$
$$\max\{\|x_i - x_{i+1}\| : i = q\bar{m}, \ldots, (q + 1)\bar{m} - 1\} \leq \epsilon_0. \tag{11.95}$$

It follows from (11.69), property (P2) (with $\tilde{q} = q - 1$), (11.93) and (11.79) that

$$\|x_{q\bar{m}} - z\| \leq 2M, \tag{11.96}$$
$$\|x_k - z\| \leq 2M + 1 \text{ for all } k = 0, \ldots, q\bar{m}. \tag{11.97}$$

Relations (11.96), (11.95), (11.10), (11.12), (11.34), and (11.35) imply that for every integer $i = q\bar{m}, \ldots, (q + 1)\bar{m}$,

$$\|x_i - z\| \leq \|x_i - x_{q\bar{m}}\| + \|x_{q\bar{m}} - z\| \leq \bar{m}\epsilon_0 + 2M \leq 2M + 1. \tag{11.98}$$

In view of (11.97), (11.98), and (11.65), we have

$$\|x_k\| \le 3M + 1,\ k = 0, \dots, (q+1)\bar{m}. \tag{11.99}$$

Assume that an integer $q \ge 0$ and that for each integer $k \in [q\bar{m}, \dots, (q+1)\bar{m} - 1]$,

$$\|x_k - x_{k+1}\| \le \epsilon_0, \tag{11.100}$$

$$\|x_k\| \le 3M + 1. \tag{11.101}$$

It follows from (11.101), (11.100), (11.12), (11.34), (11.10), and (11.35) that

$$\|x_{(q+1)\bar{m}}\| \le 3M + 2. \tag{11.102}$$

By (11.100) for each pair of integers $i, j \in \{q\bar{m}, \dots, (q+1)\bar{m}\}$,

$$\|x_i - x_j\| \le \bar{m}\epsilon_0. \tag{11.103}$$

First complete the proof of Theorem 11.2.

Let $p \in \{1, \dots, m\}$. Property (P1) implies that there exists an integer $j \in \{q\bar{m}, \dots, (q+1)\bar{m} - 1\}$ for which

$$S(j) = p. \tag{11.104}$$

We claim that $f_p(x_j) \le \Delta$. Assume the contrary. Then

$$f_p(x_j) = f_{S(j)}(x_j) > \Delta \tag{11.105}$$

and in view of (11.100), (11.105), (11.15)–(11.17), and (11.12),

$$\begin{aligned} \epsilon_0 \ge \|x_{j+1} - x_j\| &\ge \|f_{S(j)}(x_j)\|l_j\|^{-2} l_j\| \\ -\|x_{j+1} - x_j + f_{S(j)}(x_j)\|l_j\|^{-2} l_j\| &> \Delta\|l_j\|^{-1} - \delta, \\ 2\epsilon_0\|l_j\| &> \Delta. \end{aligned} \tag{11.106}$$

It follows from (11.105), (11.15), and (11.16) that there exists a point

$$\xi_j \in \partial f_{S(j)}(x_j) \tag{11.107}$$

such that

$$\|l_j - \xi_j\| < \delta. \tag{11.108}$$

Relations (11.107), (11.101), (11.9), (11.10), and (11.108) imply that

$$\|\xi_j\| \leq M_1 - 2, \ \|l_j\| \leq M_1 - 1. \tag{11.109}$$

In view of (11.106) and (11.109), we have $2\epsilon_0 M_1 > \Delta$. This contradicts (11.11) and (11.12). The contradiction we have reached proves that

$$f_p(x_j) \leq \Delta. \tag{11.110}$$

Let $i \in \{q\bar{m}, \ldots, (q+1)\bar{m}\}$. Relations (11.101) and (11.102) imply that

$$\|x_i\| \leq 3M + 2. \tag{11.111}$$

It follows from (11.104), (11.110), (11.111), (11.101), (11.9), (11.103), (11.11), and (11.12) that

$$f_p(x_i) = f_{S(j)}(x_i) \leq f_{S(j)}(x_j) + |f_{S(j)}(x_i) - f_{S(j)}(x_j)|$$
$$\leq \Delta + (M_1 - 2)\|x_i - x_j\| \leq \Delta + M_1 \bar{m}\epsilon_0,$$
$$f_p(x_i) \leq \Delta + M_1 \bar{m}\epsilon_0 \leq \Delta(\bar{m} + 1),$$

for all $p \in \{1, \ldots, m\}$ all $i \in \{q\bar{m}, \ldots, (q+1)\bar{m}\}$. This completes the proof of Theorem 11.2.

Now we complete the proof of Theorem 11.4. Let $p \in \{1, \ldots, m\}$. Property (P1) implies that there exists $j \in \{q\bar{m}, \ldots, (q+1)\bar{m} - 1\}$ such that (11.104) is trues. We claim that $f_p(x_j) \leq 2M_1\epsilon_0$. Assume the contrary. Then

$$f_p(x_j) = f_{S(j)}(x_j) > 2M_1\epsilon_0 \tag{11.112}$$

and in view of (11.100), (11.112), (11.37)–(11.40), (11.35), and (11.33), we have

$$\epsilon_0 \geq \|x_{j+1} - x_j\| \geq \|f_{S(j)}(x_j)\|l_j\|^{-2} l_j\|$$
$$-\|x_{j+1} - x_j + f_{S(j)}(x_j)\|l_j\|^{-2} l_j\| > 2M_1\epsilon_0 \|l_j\|^{-1} - \delta,$$
$$2\epsilon_0 > 2M_1\epsilon_0\|l_j\|^{-1}. \tag{11.113}$$

It follows from (11.112) and (11.37)–(11.40) that there exists a point $\xi_j \in X$ which satisfies (11.107) and (11.108). By (11.31), (11.107), (11.108) and (11.101), (11.109) is true. In view of (11.113) and (11.109), we have

$$2\epsilon_0 > 2M_1\epsilon_0(M_1 - 1)^{-1},$$

a contradiction. The contradiction we have reached proves that

$$f_p(x_j) \leq 2M_1\epsilon_0. \tag{11.114}$$

Let $i \in \{q\bar{m}, \dots, (q+1)\bar{m}\}$. Relations (11.101) and (11.102) imply that (11.111) is valid. It follows from (11.104), (11.114), (11.111), (11.101), (11.31), and (11.103) that

$$f_p(x_i) = f_{S(j)}(x_i) \le f_{S(j)}(x_j) + |f_{S(j)}(x_i) - f_{S(j)}(x_j)|$$
$$\le 2M_1\epsilon_0 + (M_1 - 2)\|x_i - x_j\| \le 2M_1\epsilon_0 + (M_1 - 2)\bar{m}\epsilon_0 \le M_1\epsilon_0(\bar{m} + 2),$$
$$f_p(x_i) \le M_1(\bar{m} + 2)\epsilon_0$$

for all integers $p \in \{1, \dots, m\}$ and all integers $i \in \{q\bar{m}, \dots, (q+1)\bar{m}\}$. This completes the proof of Theorem 11.4. □

11.6 The Third Main Result

Assume that m is a natural number and that $f_i : X \to R^1$, $i = 1, \dots, m$ are convex continuous functions.

For every integer $i = 1, \dots, m$ define

$$C_i = \{x \in X : f_i(x) \le 0\}.$$

Set

$$C = \cap_{i=1}^m C_i = \cap_{i=1}^m \{x \in X : f_i(x) \le 0\}.$$

Suppose that

$$C \ne \emptyset.$$

In this section we present the third main result of the chapter (Theorem 11.7) which describes the behavior of iterates generated by the subgradient projection method under assumptions that C is bounded and that $\{x \in X : f_i(x) < 0, \ i = 1, \dots, m\} \ne \emptyset$.

Let a natural number $\bar{m} \ge m$. Recall that $\mathcal{R}$ is the set of all mappings $S : \mathcal{N} \to \{1, \dots m\}$ which possess property (P1).

Suppose that

$$\{x \in X : f_i(x) < 0, \ i = 1, \dots, m\} \ne \emptyset.$$

Thus there exists $\Delta \in (0, 1]$ such that

$$\{x \in X : f_i(x) \le -\Delta, \ i = 1, \dots, m\} \ne \emptyset. \tag{11.115}$$

We assume that the set C is bounded.

Proposition 11.6. *Let* $\tilde{M} > 0$ *and*

$$C \subset B(0, \tilde{M}). \tag{11.116}$$

Then

$$\{x \in X : f_i(x) \le 1, \ i = 1, \dots, m\} \subset B(0, \tilde{M}(\Delta + 2)\Delta^{-1}). \tag{11.117}$$

Proof. Assume that a point $x \in X$ satisfies

$$f_i(x) \le 1, \ i = 1, \dots, m. \tag{11.118}$$

In view of (11.115), there exists a point $z \in X$ such that

$$f_i(z) \le -\Delta, \ i = 1, \dots, m. \tag{11.119}$$

For every integer $i \in \{1, \dots, m\}$, the convexity of f_i, (11.118) and (11.119) imply that

$$f_i((\Delta + 1)^{-1}z + \Delta(\Delta + 1)^{-1}x) \le (\Delta + 1)^{-1}f_i(z) + \Delta(\Delta + 1)^{-1}f_i(x)$$
$$\le -(\Delta + 1)^{-1}\Delta + \Delta(\Delta + 1)^{-1} \le 0$$

and

$$(\Delta + 1)^{-1}z + \Delta(\Delta + 1)^{-1}x \in C.$$

Combined with (11.116) this implies that

$$\|(\Delta + 1)^{-1}z + \Delta(\Delta + 1)^{-1}x\| \le \tilde{M}. \tag{11.120}$$

In view of (11.116), (11.119), and (11.120), we have

$$\|\Delta(\Delta + 1)^{-1}x\| \le \|(\Delta + 1)^{-1}z + \Delta(\Delta + 1)^{-1}x\| + \| - (\Delta + 1)^{-1}z\|$$
$$\le \tilde{M} + (\Delta + 1)^{-1}\tilde{M} \le \tilde{M}(\Delta + 2)(\Delta + 1)^{-1},$$
$$\|x\| \le \tilde{M}(\Delta + 2)\Delta^{-1}.$$

This completes the proof of Proposition 11.6. □

Let $M > 0$, $M_0 > 0$, $M_1 > 2$ be such that

$$\{x \in X : f_i(x) \le 1, \ i = 1, \dots, m\} \subset B(0, M), \tag{11.121}$$
$$f_i(B(0, 3M + 3)) \subset [-M_0, M_0], \ i = 1, \dots, m, \tag{11.122}$$

$$|f_i(u) - f_i(v)| \leq (M_1 - 2)\|u - v\| \text{ for all } u, v \in B(0, 3M + 4),\ i = 1, \dots, m. \tag{11.123}$$

Let $\delta \in (0, 1]$ satisfy (11.34) and

$$M_1(\bar{m} + 2)(8(4M + 4)\bar{m}\delta(1 + 16M_0\Delta^{-2}(4M + 3)^2))^{1/2} \leq 1, \tag{11.124}$$

ϵ_0 be defined by (11.35) and let a natural umber n_0 satisfy (11.36).

Note that all the assumptions made for Theorem 11.4 hold.

The next theorem is the third main result of the chapter.

Theorem 11.7. *Assume that*

$$S \in \mathcal{R},\ \epsilon \in [0, 2M_1\epsilon_0],\ \{x_k\}_{k=0}^{\infty} \subset X,\ \|x_0\| \leq M, \{l_k\}_{k=0}^{\infty} \subset X \tag{11.125}$$

be such that for each integer $k \geq 0$,

$$x_{k+1} = x_k,\ l_k = 0,\ \textit{if}\ f_{S(k)}(x_k) \leq \epsilon \tag{11.126}$$

and if $f_{S(k)}(x_k) > \epsilon$, then

$$l_k \in X \setminus \{0\},\ d(l_k, \partial f_{S(k)}(x_k)) < \delta, \tag{11.127}$$

$$\|x_{k+1} - x_k + f_{S(k)}(x_k)\|l_k\|^{-2}l_k\| \leq \delta. \tag{11.128}$$

Then there exists an integer $q_0 \in [0, n_0 - 1]$ such that for each integer $i \geq q_0\bar{m}$,

$$d(x_i, C) \leq (\bar{m} + 2)\epsilon_0(1 + 2M_1M\Delta^{-1}).$$

In this section we say that $x \in X$ is a γ-approximate solution of our feasibility problem with $\gamma > 0$ if $d(x, C) \leq \gamma$.

Note that in Theorem 11.7 the upper bound for computational errors produced by our computer system is δ and we obtain a $(1 + 2M_1M\Delta^{-1})(\bar{m} + 2)\epsilon_0$-approximate solution after $\bar{m}(n_0 - 1)$ iterations of the algorithm associated with $S \in \mathcal{R}$, where constants ϵ_0 and n_0 are defined by (11.35) and (11.36). Namely, $\epsilon_0 = c_1\delta^{1/2}$ and $n_0 = c_2\delta^{-1}$, where c_1 and c_2 are positive constants. The convergence in Theorem 11.7 is stronger than in Theorems 11.2 and 11.4 because x_i is $(1 + 2M_1M\Delta^{-1})(\bar{m} + 2)\epsilon_0$-approximate solution for all integers $i \geq (n_0 - 1)\bar{m}$.

11.7 Auxiliary Results for Theorem 11.7

In this section we use the notation and the assumptions made in Sect. 11.6.

Lemma 11.8. *Let $x \in X$, $\lambda \in (0, 1]$ and*

$$f_i(x) \leq \lambda,\ i = 1, \dots, m. \tag{11.129}$$

Then

$$d(x, C) \leq 2M\lambda(\lambda + \Delta)^{-1}.$$

Proof. Fix a point $z \in X$ for which

$$f_i(z) \leq -\Delta, \ i = 1, \dots, m. \tag{11.130}$$

Since the functions f_i, $i = 1, \dots, m$ are convex relations (11.129) and (11.130) imply that for all integers $i = 1, \dots, m$,

$$\begin{aligned} & f_i(\lambda(\lambda + \Delta)^{-1}z + \Delta(\lambda + \Delta)^{-1}x) \\ & \leq \lambda(\lambda + \Delta)^{-1}f_i(z) + \Delta(\lambda + \Delta)^{-1}f_i(x) \\ & \leq -\lambda(\lambda + \Delta)^{-1}\Delta + \Delta(\lambda + \Delta)^{-1}\lambda = 0. \end{aligned} \tag{11.131}$$

Hence

$$\lambda(\lambda + \Delta)^{-1}z + \Delta(\lambda + \Delta)^{-1}x \in C. \tag{11.132}$$

It follows from (11.121), (11.129), and (11.130) that

$$\|x\|, \ \|z\| \leq M. \tag{11.133}$$

In view of (11.132) and (11.133), we have

$$\begin{aligned} d(x, C) & \leq \|\lambda(\lambda + \Delta)^{-1}z + \Delta(\lambda + \Delta)^{-1}x - x\| \\ & = \lambda(\lambda + \Delta)^{-1}\|z - x\| = 2M\lambda(\lambda + \Delta)^{-1}. \end{aligned}$$

Lemma 11.8 is proved. □

Lemma 11.9. *Let $x \in X$ satisfy*

$$d(x, C) \leq 1. \tag{11.134}$$

Then for $i = 1, \dots, m$, $f_i(x) \leq M_1 d(x, C)$.

Proof. Let $\gamma \in (0, 1)$. There exists a point

$$z \in C \tag{11.135}$$

satisfying

$$\|x - z\| \leq d(x, C) + \gamma < 2. \tag{11.136}$$

Relations (11.121), (11.135), and (11.136) imply that

$$\|z\| \le M, \tag{11.137}$$

$$\|x\| \le M + 2. \tag{11.138}$$

It follows from (11.137), (11.138), (11.123), and (11.136) that for every integer $i = 1, \dots, m$,

$$|f_i(x) - f_i(z)| \le \|x - z\|M_1 \le M_1 d(x, C) + M_1\gamma. \tag{11.139}$$

Since γ is any number belonging to $(0, 1)$, it follows from (11.135) and (11.139) that for all $i = 1, \dots, m, f_i(x) \le M_1 d(x, C)$. Lemma 11.9 is proved. □

11.8 Proof of Theorem 11.7

We have already mentioned that all the assumptions made for Theorem 11.4 hold.

Define

$$\delta_0 = \delta(1 + 16M_0\Delta^{-2}(4M + 3)). \tag{11.140}$$

Note that in view of (11.124) and (11.35), we have

$$M_1\epsilon_0(\bar{m} + 2) \le 1. \tag{11.141}$$

Since $C \subset B(0, M)$ we conclude that in the proof of Theorem 11.4 the following properties were established (for the sequence $\{x_{i+l}\}_{i=0}^{\infty}$):

(P3) Let $z \in C$ and an integer $l \ge 0$ be such that $\|x_l\| \le M$. Then there exists an integer $q \in [0, n_0 - 1]$ such that:

$$\|x_i\| \le 3M + 1,\ i = l, \dots, (q + 1)\bar{m} + l \tag{11.142}$$

(see (11.99));

for each integer p satisfying $0 \le p < q$,

$$\max\{\|x_i - x_{i+1}\| :\ i \in \{p\bar{m} + l, \dots, (p + 1)\bar{m} + l - 1\}\} > \epsilon_0, \tag{11.143}$$

$$\max\{\|x_i - x_{i+1}\| :\ i \in \{q\bar{m} + l, \dots, (q + 1)\bar{m} + l - 1\}\} \le \epsilon_0 \tag{11.144}$$

(see (11.94) and (11.95));

for each integer p satisfying $0 \le p < q$,

$$\|z - x_{p\bar{m}+l}\|^2 - \|z - x_{(p+1)\bar{m}+l}\|^2 \ge \epsilon_0^2/8 \tag{11.145}$$

(see (P2) and (11.93));

$$\|x_{i+1} - z\| \le \delta_0 + \|x_i - z\| \tag{11.146}$$

for all integers i satisfying $l \le i < q\bar{m} + l$ (see (P2) and (11.79))

and

(P4) Assume that an integer $q \ge 0$, an integer $l \ge 0$, $\|x_l\| \le M$,

$$\|x_i\| \le 3M + 1,\ \|x_{i+1} - x_i\| \le \epsilon_0,\ i = l + q\bar{m}. \dots, (q+1)\bar{m} - 1 + l.$$

Then

$$f_p(x_i) \le M_1\epsilon_0(\bar{m} + 2),\ i = l + q\bar{m}, \dots, (q+1)\bar{m} + l, \tag{11.147}$$
$$p = 1, \dots, m.$$

Assume that an integer $l \ge 0$ be such that $\|x_l\| \le M$ and let an integer $q \in [0, n_0 - 1]$ be as guaranteed by property (P3) which in its turn implies in view of (11.145) that for every integer p satisfying $0 \le p < q$,

$$d(x_{(p+1)\bar{m}+l}, C) \le d(x_{p\bar{m}+l}, C).$$

This implies that

$$d(x_{p\bar{m}+l}, C) \le d(x_l, C) \text{ for all } p = 0, \dots, q. \tag{11.148}$$

By (11.146) and (11.148),

$$d(x_i, C) \le d(x_l, C) + \delta_0\bar{m} \text{ for all } i = l, \dots, l + q\bar{m}. \tag{11.149}$$

In view of the choice of q, property (P3), (11.142), (11.144) and property (P4), (11.147) is valid. By (11.141), (11.147), and Lemma 11.8, for all integers $i = l + q\bar{m}, \dots, l + (q+1)\bar{m}$, we have

$$d(x_i, C) \le 2MM_1\epsilon_0(\bar{m} + 2)(M_1\epsilon_0(\bar{m} + 2) + \Delta)^{-1} \le 2M_1M\epsilon_0(\bar{m} + 2)\Delta^{-1}. \tag{11.150}$$

It follows from (11.121) and (11.147) that

$$x_i \in B(0, M),\ i = l + q\bar{m}, \dots, (q+1)\bar{m}. \tag{11.151}$$

Thus we have shown that the following property holds:

(P5) Let an integer $l \ge 0$ be such that $\|x_l\| \le M$ and let an integer $q \in [0, n_0 - 1]$ be as guaranteed by property (P3). Then (11.147), (11.149), (11.150), and (11.151) hold.

In view of (11.125), there exists an integer $q_0 \in [0, n_0 - 1]$ such that property (P3) holds with $l = 0$ and $q = q_0$. It follows from the choice of q_0, property (P5), (11.125), (11.147), and (11.150) that

$$f_p(x_i) \le M_1\epsilon_0(\bar{m}+2),\ i = q_0\bar{m}, \dots, (q_0+1)\bar{m},\ p = 1, \dots, m, \tag{11.152}$$

$$d(x_i, C) \le 2M_1M\epsilon_0(\bar{m}+2)\Delta^{-1} \text{ for all integers } i = q_0\bar{m}, \dots, (q_0+1)\bar{m}. \tag{11.153}$$

We claim that

$$d(x_i, C) \le 2M_1M\epsilon_0(\bar{m}+2)\Delta^{-1} + \delta_0\bar{m} \text{ for all integers } i \ge q_0\bar{m}.$$

Assume the contrary. Then there exists an integer

$$p > q_0\bar{m} \tag{11.154}$$

which satisfies

$$d(x_p, C) > 2M_1M\epsilon_0(\bar{m}+2)\Delta^{-1} + \delta_0\bar{m} \tag{11.155}$$

In view of (11.153)–(11.155), we have

$$p > q_0(\bar{m}+1). \tag{11.156}$$

We may assume without loss of generality that

$$d(x_i, C) \le 2M_1M\epsilon_0(\bar{m}+2)\Delta^{-1} + \delta_0\bar{m} \tag{11.157}$$
$$\text{for all integers } i \text{ satisfying } (q_0+1)\bar{m} \le i < p.$$

It follows from (11.152) and (11.156) that there exists an integer l_0 satisfying

$$(q_0+1)\bar{m} \le l_0 \le p, \tag{11.158}$$

$$f_k(x_{l_0}) \le M_1\epsilon_0(\bar{m}+2),\ k = 1, \dots, m, \tag{11.159}$$

$$\max\{f_k(x_i) :\ k = 1, \dots, m\} > M_1\epsilon_0(\bar{m}+2) \tag{11.160}$$
$$\text{for all integers } i \text{ satisfying } l_0 < i \le p.$$

In view of (11.141), (11.159) and Lemma 11.8, we have

$$d(x_{l_0}, C) \le 2MM_1\epsilon_0(\bar{m}+2)\Delta^{-1}. \tag{11.161}$$

Relations (11.155), (11.158), and (11.161) imply that

$$l_0 < p. \tag{11.162}$$

By view of (11.121), (11.141), and (11.159),

$$\|x_{l_0}\| \le M. \tag{11.163}$$

Let an integer $q \in [0, n_0 - 1]$ be as guaranteed by (P3) with $l = l_0$. It follows from (11.163), the choice of q, (P5) with $l = l_0$, (11.147), (11.149), (11.150), and (11.161) that

$$d(x_i, C) \le 2MM_1\epsilon_0(\bar{m} + 2)\Delta^{-1} + \delta_0\bar{m},\ i = l_0, \ldots, l_0 + (q + 1)\bar{m}, \tag{11.164}$$

$$f_t(x_i) \le M_1\epsilon_0(\bar{m} + 2),\ t = 1, \ldots, m,\ i = l_0 + q\bar{m}, \ldots, l_0 + (q + 1)\bar{m}. \tag{11.165}$$

By (11.155), (11.162), and (11.164),

$$p > l_0 + (q + 1)\bar{m}. \tag{11.166}$$

Equations (11.165) and (11.166) contradict (11.158)–(11.160).

The contradiction we have reached proves that for all integers $i \ge q_0\bar{m}$, in view of (11.35), (11.140),

$$d(x_i, C) \le 2MM_1\epsilon_0(\bar{m} + 2)\Delta^{-1} + \delta_0\bar{m}$$

$$\le 2MM_1\epsilon_0(\bar{m} + 2)\Delta^{-1} + \epsilon_0\bar{m} \le (1 + 2MM_1\Delta^{-1})\epsilon_0(\bar{m} + 2).$$

Theorem 11.7 is proved. □

Chapter 12
Dynamic String-Averaging Subgradient Projection Algorithm

In this chapter we study convergence of dynamic string-averaging subgradient projection algorithms for solving convex feasibility problems in a general Hilbert space. Our goal is to obtain an approximate solution of the problem in the presence of computational errors. We show that our subgradient projection algorithm generates a good approximate solution, if the sequence of computational errors is bounded from above by a constant. Moreover, for a known computational error, we find out what an approximate solution can be obtained and how many iterates one needs for this.

12.1 Preliminaries and the First Main Result

Let $(X, \langle \cdot, \cdot \rangle)$ be a Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ which induces a complete norm $\| \cdot \|$.

For each $x \in X$ and each nonempty set $A \subset X$ put

$$d(x, A) = \inf\{\|x - y\| : \ y \in A\}.$$

For each $x \in X$ and each $r > 0$ set

$$B(x, r) = \{y \in X : \ \|x - y\| \leq r\}.$$

By Proposition 10.1, for each nonempty closed convex subset C of X and each $x \in X$ there is a unique point $P_C(x) \in C$ satisfying

$$\|x - P_C(x)\| = \inf\{\|x - y\| : \ y \in C\}, \tag{12.1}$$

$$\|P_C(x) - P_C(y)\| \leq \|x - y\| \text{ for all } x, y \in X \tag{12.2}$$

A. Zaslavski, *Approximate Solutions of Common Fixed-Point Problems*, Springer Optimization and Its Applications 112, DOI 10.1007/978-3-319-33255-0_12

and for each $x \in X$ and each $z \in C$,

$$\langle z - P_C(x), x - P_C(x) \rangle \le 0,$$
$$\|z - P_C(x)\|^2 + \|x - P_C(x)\|^2 \le \|z - x\|^2. \tag{12.3}$$

Denote by $\mathcal{N}$ a set of all nonnegative integers. We recall the following useful facts on convex functions given already in Chap. 10.

Let $f : X \to R^1$ be a continuous convex function such that

$$\{x \in X : f(x) \le 0\} \neq \emptyset. \tag{12.4}$$

Let $y_0 \in X$. Then

$$\partial f(y_0) = \{l \in X : f(y) - f(y_0) \ge \langle l, y - y_0 \rangle \text{ for all } y \in X\} \tag{12.5}$$

is the subdifferential of f at the point y_0 [72, 77].

For any $l \in \partial f(y_0)$ it follows from (12.5) that

$$\{x \in X : f(x) \le 0\} \subset \{x \in X : f(y_0) + \langle l, x - y_0 \rangle \le 0\}. \tag{12.6}$$

By Lemma 11.1, the following property holds:

(P1) For each $y_0 \in X$ satisfying $f(y_0) > 0$, each $l \in \partial f(y_0)$ and

$$D = \{x \in X : f(y_0) + \langle l, x - y_0 \rangle \le 0\}$$

we have $l \neq 0$ and

$$P_D(y_0) = y_0 - f(y_0)\|l\|^{-2} l.$$

Let us now describe the convex feasibility problem studied in the chapter and dynamic string-averaging subgradient projection algorithms which will be used for its solving.

Let m be a natural number and $f_i : X \to R^1$, $i = 1, \dots, m$ be convex continuous functions.

For each $i = 1, \dots, m$ set

$$C_i = \{x \in X : f_i(x) \le 0\},$$
$$C = \cap_{i=1}^m C_i = \cap_{i=1}^m \{x \in X : f_i(x) \le 0\}.$$

Suppose that

$$C \neq \emptyset.$$

A point $x \in C$ is called a solution of our feasibility problem. For a given $\epsilon > 0$ a point $x \in X$ is called an ϵ-approximate solution of the feasibility problem if

$$f_i(x) \le \epsilon \text{ for all } i = 1, \dots, m.$$

In this chapter we apply a dynamic string-averaging subgradient projection method with variable strings and weights in order to obtain a good approximative solution of the feasibility problem.

Next we describe the dynamic string-averaging subgradient method with variable strings and weights.

By an index vector, we a mean a vector $t = (t_1, \dots, t_p)$ such that $t_i \in \{1, \dots, m\}$ for all $i = 1, \dots, p$.

For an index vector $t = (t_1, \dots, t_q)$ set

$$p(t) = q. \tag{12.7}$$

Denote by $\mathcal{M}$ the collection of all pairs (Ω, w), where Ω is a finite set of index vectors and

$$w : \Omega \to (0, \infty) \text{ be such that } \sum_{t \in \Omega} w(t) = 1. \tag{12.8}$$

Fix a number

$$\bar{\Delta} \in (0, m^{-1}] \tag{12.9}$$

and an integer

$$\bar{q} \ge m. \tag{12.10}$$

Denote by $\mathcal{M}_*$ the set of all $(\Omega, w) \in \mathcal{M}$ such that

$$p(t) \le \bar{q} \text{ for all } t \in \Omega, \tag{12.11}$$

$$w(t) \ge \bar{\Delta} \text{ for all } t \in \Omega. \tag{12.12}$$

For each $x \in X$, each $\epsilon \ge 0$, each $\bar{\epsilon} \ge 0$, and each $i \in \{1, \dots, m\}$ set

$$A_i(x, \bar{\epsilon}, \epsilon) := \{x\} \text{ if } f_i(x) \le \bar{\epsilon} \tag{12.13}$$

and if $f_i(x) > \bar{\epsilon}$, then set

$$A_i(x, \bar{\epsilon}, \epsilon) = \{x - f_i(x)\|l\|^{-2} l : \ l \in \partial f_i(x) + B(0, \epsilon), \ l \ne 0\} + B(0, \epsilon). \tag{12.14}$$

Let $x \in X$ and let $t = (t_1, \dots, t_{p(t)})$ be an index vector, $\epsilon \ge 0$, $\bar{\epsilon} \ge 0$. Define

$$A_0(t, x, \bar{\epsilon}, \epsilon) = \{(y, \lambda) \in X \times R^1 : \text{ there is a sequence } \{y_i\}_{i=0}^{p(t)} \subset X \text{ such that}$$
$$y_0 = x, \tag{12.15}$$

for each $i = 1, \dots, p(t)$,

$$y_i \in A_{t_i}(y_{i-1}, \bar{\epsilon}, \epsilon), \tag{12.16}$$
$$y = y_{p(t)}, \tag{12.17}$$
$$\lambda = \max\{\|y_i - y_{i-1}\| : i = 1, \dots, p(t)\}\}. \tag{12.18}$$

Let $x \in X$, $(\Omega, w) \in \mathcal{M}$, $\epsilon \ge 0$, $\bar{\epsilon} \ge 0$. Define

$$A(x, (\Omega, w), \bar{\epsilon}, \epsilon) = \{(y, \lambda) \in X \times R^1 : \text{ there exist}$$
$$(y_t, \lambda_t) \in A_0(t, x, \bar{\epsilon}, \epsilon),\ t \in \Omega \tag{12.19}$$

such that

$$\|y - \sum_{t \in \Omega} w(t) y_t\| \le \epsilon, \tag{12.20}$$
$$\lambda = \max\{\lambda_t : t \in \Omega\}\}. \tag{12.21}$$

Fix a natural number $\bar{N}$.
Suppose that $M > 0$, $M_0 > 0$ and $M_1 > 2$ be such that

$$B(0, M) \cap \{x \in X : f_i(x) \le 0,\ i = 1, \dots, m\} \ne \emptyset, \tag{12.22}$$
$$f_i(B(0, 3M + 3)) \subset [-M_0, M_0],\ i = 1, \dots, m, \tag{12.23}$$
$$|f_i(u) - f_i(v)| \le (M_1 - 2)\|u - v\|$$
$$\text{for all } u, v \in B(0, 3M + 4) \text{ and all } i = 1, \dots, m. \tag{12.24}$$

Let $\Delta \in (0, 1]$ and let $\delta \in (0, 1]$ satisfy

$$8(2(1 + \bar{q})(1 + \bar{N}))^3 \bar{\Delta}^{-3}(4M + 4)\delta(1 + 16M_0\Delta^{-2}(4M + 3)^2) \le 1, \tag{12.25}$$
$$\Delta \ge 2M_1(8\delta(4M + 4)\bar{\Delta}^{-1}(2(1 + \bar{q})(1 + \bar{N}))(1 + 16M_0\Delta^{-2}(4M + 3)^2))^{1/2}\bar{\Delta}^{-1/2}. \tag{12.26}$$

Let

$$\epsilon_0 = (8\delta\bar{\Delta}^{-1}(4M + 4)2(1 + \bar{q})(1 + \bar{N})(1 + 16M_0\Delta^{-2}(4M + 3)^2))^{1/2} \tag{12.27}$$

and a natural number

$$n_0 > 32M^2\epsilon_0^{-2}\bar{\Delta}^{-1}. \tag{12.28}$$

The following theorem is our first main result which will be proved in Sect. 12.2.

Theorem 12.1. *Assume that*

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_* \tag{12.29}$$

satisfies for each natural number j,

$$\{1,\dots,m\} \subset \cup_{i=j}^{j+\bar{N}-1}(\cup_{t\in\Omega_i}\{t_1,\dots,t_{p(t)}\}), \tag{12.30}$$

$$x_0 \in B(0,M), \tag{12.31}$$

$$\{x_i\}_{i=1}^{\infty} \subset X,\ \{\lambda_i\}_{i=1}^{\infty} \subset [0,\infty) \tag{12.32}$$

satisfy for each natural number i,

$$(x_i,\lambda_i) \in A(x_{i-1},(\Omega_i,w_i),\Delta,\delta). \tag{12.33}$$

Then there exists an integer $q \in [0, n_0 - 1]$ *such that*

$$\|x_i\| \le 3M+1 \text{ for all integers } i = 0,\dots,q\bar{N}, \tag{12.34}$$

$$\lambda_i \le \epsilon_0,\ i = q\bar{N}+1,\dots,(q+1)\bar{N}. \tag{12.35}$$

Moreover, if an integer $q \ge 0$ *satisfies (12.35) and* $\|x_{q\bar{N}}\| \le 3M+1$*, then*

$$\|x_{k_1} - x_{k_2}\| \le \delta\bar{N}$$

for all $k_1, k_2 \in \{q\bar{N},\dots,(q+1)\bar{N}\}$ *and for all* $i \in \{q\bar{N},\dots,(q+1)\bar{N}\}$ *and all* $s \in \{1,\dots,m\}$

$$f_s(x_i) \le \Delta + M_1\bar{N}\delta.$$

Note that in Theorem 12.1 δ is the computational error made by our computer system, we obtain a point x satisfying

$$f_s(x) \le \Delta + M_1\bar{N}\delta$$

for all $s = 1,\dots,m$ and in order to obtain this point we need $(n_0-1)\bar{N}$ iterations. It is not difficult to see that $\Delta + M_1\bar{N}\delta = c_1\delta^{1/4}$ and $n_0 = \lfloor c_2\delta^{-1/2}\rfloor$, where c_1 and c_2 are positive constants which do not depend on δ.

12.2 Proof of Theorem 12.1

By (12.22), there exists

$$z \in B(0, M) \text{ such that } f_i(z) \leq 0, \ i = 1, \dots, m. \tag{12.36}$$

Set

$$\delta_0 = \delta(1 + 16M_0\Delta^{-2}(4M + 3)^2). \tag{12.37}$$

Let n be a natural number. In view of (12.33),

$$(x_n, \lambda_n) \in A(x_{n-1}, (\Omega_n, w_n), \Delta, \delta). \tag{12.38}$$

By (12.19)–(12.21) and (12.38), there exist

$$(y_{n,t}, \lambda_{n,t}) \in A_0(t, x_{n-1}, \Delta, \delta), \ t \in \Omega_n \tag{12.39}$$

such that

$$\|x_n - \sum_{t \in \Omega_n} w_n(t)y_{n,t}\| \leq \delta, \tag{12.40}$$

$$\lambda_n = \max\{\lambda_{n,t} : \ t \in \Omega_n\}. \tag{12.41}$$

It follows from (12.15)–(12.18) and (12.39) that for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_n$ there exists a sequence $\{y_{n,t,i}\}_{i=0}^{p(t)} \subset X$ such that

$$y_{n,t,0} = x_{n-1}, \tag{12.42}$$

$$y_{n,t,i} \in A_{t_i}(y_{n,t,i-1}, \Delta, \delta), \ i = 1, \dots, p(t), \tag{12.43}$$

$$y_{n,t} = y_{n,t,p(t)}, \tag{12.44}$$

$$\lambda_{n,t} = \max\{\|y_{n,t,i} - y_{n,t,i-1}\| : \ i = 1, \dots, p(t)\}. \tag{12.45}$$

Assume that a nonnegative integer $\tilde{q} \leq n_0 - 1$ satisfies for each integer $p \in [0, \tilde{q}]$,

$$\max\{\lambda_i : \ i = p\bar{N} + 1, \dots, (p + 1)\bar{N}\} > \epsilon_0. \tag{12.46}$$

Relations (12.31) and (12.36) imply that

$$\|x_0 - z\| \leq 2M. \tag{12.47}$$

Assume that an integer $p \in [0, \tilde{q}]$ satisfies

$$\|x_{p\bar{N}} - z\| \leq 2M. \tag{12.48}$$

Assume that an integer k satisfies

$$k \in \{p\bar{N}, \dots, (p+1)\bar{N}-1\}, \tag{12.49}$$

$$\|x_k - z\| \le 2M + \delta_0(k - p\bar{N})(\bar{q}+1). \tag{12.50}$$

(Note that in view of (12.48), inequality (12.50) holds for $k = p\bar{N}$.)

Let

$$t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}. \tag{12.51}$$

By (12.43) and (12.51), for each $j \in \{1, \dots, p(t)\}$, we have

$$y_{k+1,t,j} \in A_{t_j}(y_{k+1,t,j-1}, \Delta, \delta), \tag{12.52}$$

$$y_{k+1,t,0} = x_k. \tag{12.53}$$

In view of (12.50) and (12.53),

$$\|y_{k+1,t,0} - z\| \le 2M + \delta_0(k - p\bar{N})(\bar{q}+1). \tag{12.54}$$

Assume that an integer

$$j \in \{1, \dots, p(t)\}, \tag{12.55}$$

$$\|y_{k+1,t,j-1} - z\| \le 2M + \delta_0(k - p\bar{N})(\bar{q}+1) + \delta_0(j-1). \tag{12.56}$$

(Note that in view of (12.54), (12.56) is true for $j = 1$.) By (12.13), (12.14), and (12.52), if

$$f_{t_j}(y_{k+1,t,j-1}) \le \Delta,$$

then

$$y_{k+1,t,j} = y_{k+1,t,j-1}.$$

Assume that

$$f_{t_j}(y_{k+1,t,j-1}) > \Delta. \tag{12.57}$$

It follows from (12.11), (12.25), (12.36), (12.37), (12.49), (12.55), and (12.56) that

$$\begin{aligned}\|y_{k+1,t,j-1}\| &\le 3M + \delta_0(k - p\bar{N})(1+\bar{q}) + \delta_0(j-1) \\ &\le 3M + \delta_0(1+\bar{N})(1+\bar{q}) < 3M + 1.\end{aligned} \tag{12.58}$$

Lemma 11.3, (12.14), (12.22)–(12.28), (12.30), (12.36), (12.37), (12.52), (12.57), and (12.58) imply that

$$\|y_{k+1,t,j} - z\| \le \delta_0 + \|y_{k+1,t,j-1} - z\|. \tag{12.59}$$

Clearly, (12.59) hold in both cases. By (12.56) and (12.59),

$$\|y_{k+1,t,j} - z\| \le 2M + \delta_0(k - p\bar{N})(\bar{q} + 1) + \delta_0 j. \tag{12.60}$$

Thus we have shown by induction that (12.60) holds for all $j = 0, \ldots, p(t)$ and that (12.59) holds for all $j = 1, \ldots, p(t)$. By (12.25), (12.36), (12.37), (12.49) and (12.60),

$$\|y_{k+1,t,j} - z\| \le 2M + \delta_0(\bar{N} + 1)(\bar{q} + 1) \le 2M + 1, \tag{12.61}$$

$$\|y_{k+1,t,j}\| \le 3M + 1. \tag{12.62}$$

In view of (12.44) and (12.61),

$$\|y_{k+1,t} - z\| \le 2M + 1. \tag{12.63}$$

Thus we have shown that the following property holds:

(P2) for each $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{k+1}$ and each $j = 0, \ldots, p(t)$, (12.60)–(12.62) hold and for each $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{k+1}$ and each $j = 1, \ldots, p(t)$, (12.59) and (12.63) hold.

Let

$$t = (t_1, \ldots, t_{p(t)}) \in \Omega_{k+1},\ j \in \{1, \ldots, p(t)\}.$$

Property (P2), (12.25), (12.37), (12.59), and (12.61) imply that

$$\begin{aligned}
\|y_{k+1,t,j} - z\|^2 &\le (\|y_{k+1,t,j-1} - z\| + \delta_0)^2 \\
&\le \|y_{k+1,t,j-1} - z\|^2 + \delta_0^2 + 2\delta_0\|y_{k+1,t,j-1} - z\| \\
&\le \|y_{k+1,t,j-1} - z\|^2 + 2\delta_0(2M + 1 + \delta_0) \\
&\le \|y_{k+1,t,j-1} - z\|^2 + 2\delta_0(2M + 2).
\end{aligned}$$

Thus we have shown that the following property holds:

(P3) for each $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{k+1}$ and each $j \in \{1, \ldots, p(t)\}$,

$$\|y_{k+1,t,j} - z\|^2 - \|y_{k+1,t,j-1} - z\|^2 \le 2\delta_0(2M + 2).$$

Property (P3), (12.11), (12.42), and (12.44) imply that for all

$$t = (t_1, \ldots, t_{p(t)}) \in \Omega_{k+1}$$

we have

$$\begin{aligned}
&\|y_{k+1,t} - z\|^2 - \|x_k - z\|^2 \\
&= \|y_{k+1,t,p(t)} - z\|^2 - \|y_{k+1,t,0} - z\|^2 \\
&= \sum_{i=1}^{p(t)} [\|y_{k+1,t,i} - z\|^2 - \|y_{k+1,t,i-1} - z\|^2] \\
&\le 2\delta_0(2M+2)p(t) \le 2\delta_0\bar{q}(2M+2). \qquad (12.64)
\end{aligned}$$

Property (P2), (12.11), (12.44), and (12.60) holding for all $t \in \Omega_{k+1}$ and all $j = 0, \dots, p(t)$, imply that

$$\|y_{k+1,t} - z\| = \|y_{k+1,t,p(t)} - z\| \le 2M + \delta_0((1+\bar{q})(k - p\bar{N}) + \bar{q}). \qquad (12.65)$$

By (12.8), (12.37), (12.40), (12.65) and the convexity of the norm,

$$\begin{aligned}
\|z - x_{k+1}\| &\le \|z - \sum_{t \in \Omega_{k+1}} w_{k+1}(t) y_{k+1,t}\| + \|\sum_{t \in \Omega_{k+1}} w_{k+1}(t) y_{k+1,t} - x_{k+1}\| \\
&\le \sum_{t \in \Omega_{k+1}} w_{k+1}(t)\|z - y_{k+1,t}\| + \delta \\
&\le 2M + \delta_0((1+\bar{q})(k - p\bar{N}) + \bar{q}) + \delta \le 2\bar{M} + \delta_0(1+\bar{q})(k - p\bar{N}) + \delta_0(\bar{q}+1) \\
&\le 2M + \delta_0(1+\bar{q})(k - p\bar{N} + 1),
\end{aligned}$$

$$\|x_{k+1} - z\| \le 2M + \delta_0(1+\bar{q})(k - p\bar{N} + 1). \qquad (12.66)$$

In view of (12.37), (12.49), (12.50), (12.55), and (12.66),

$$\|x_k - z\|,\ \|x_{k+1} - z\| \le 2M + 1. \qquad (12.67)$$

By (12.8), (12.64), (12.67) and the convexity of the function $\|\cdot\|^2$,

$$\begin{aligned}
\|\sum_{t \in \Omega_{k+1}} w_{k+1}(t) y_{k+1,t} - z\|^2 &\le \sum_{t \in \Omega_{k+1}} w_{k+1}(t)\|y_{k+1,t} - z\|^2 \\
&\le \|x_k - z\|^2 + 2\delta_0\bar{q}(2M+2). \qquad (12.68)
\end{aligned}$$

Property (P2), (12.8), and (12.63) imply that

$$\|\sum_{t \in \Omega_{k+1}} w_{k+1}(t) y_{k+1,t} - z\| \le 2M + 1.$$

By the relation above, (12.37), (12.40), and (12.68),

$$\|x_{k+1} - z\|^2 \le (\|x_{k+1} - \sum_{t \in \Omega_{k+1}} w_{k+1}(t) y_{k+1,t}\| + \| \sum_{t \in \Omega_{k+1}} w_{k+1}(t) y_{k+1,t} - z\|)^2$$
$$\le (\delta + \| \sum_{t \in \Omega_{k+1}} w_{k+1}(t) y_{k+1,t} - z\|)^2$$
$$\le \delta^2 + 2\delta(2M+1) + \|x_k - z\|^2 + 2\delta_0 \bar{q}(2M+2)$$
$$\le \|x_k - z\|^2 + 2\delta_0(\bar{q}(2M+2) + 2M + 3). \tag{12.69}$$

Assume that

$$\lambda_{k+1} > \epsilon_0. \tag{12.70}$$

In view of (12.41) and (12.70), there exists

$$s = (s_1, \dots, s_{p(s)}) \in \Omega_{k+1} \tag{12.71}$$

such that

$$\epsilon_0 < \lambda_{k+1} = \lambda_{k+1,s}. \tag{12.72}$$

It follows from (12.45) and (12.72) that there exists

$$j_0 \in \{1, \dots, p(s)\} \tag{12.73}$$

such that

$$\epsilon_0 < \lambda_{k+1,s} = \|y_{k+1,s,j_0} - y_{k+1,s,j_0-1}\|. \tag{12.74}$$

By (12.13), (12.43), and (12.74),

$$f_{s_{j_0}}(y_{k+1,s,j_0-1}) > \Delta. \tag{12.75}$$

It follows from (12.13), (12.14), (12.43), and (12.75) that there exist

$$\xi \in \partial f_{s_{j_0}}(y_{k+1,s,j_0-1}),\ l \in B(\xi, \delta) \setminus \{0\} \tag{12.76}$$

such that

$$y_{k+1,s,j_0} \in B(y_{k+1,s,j_0-1} - f_{s_{j_0}}(y_{k+1,s,j_0-1})\|l\|^{-2} l, \delta). \tag{12.77}$$

Relations (12.22), (12.75), and (12.76) imply that

$$\xi \neq 0. \tag{12.78}$$

Set

$$y = y_{k+1,s,j_0-1} - f_{s_{j_0}}(y_{k+1,s,j_0-1})\|\xi\|^{-2}\xi. \tag{12.79}$$

Property (P2), (12.62), and Lemma 11.3 applied with $x = y_{k+1,s,j_0-1}$, $j = s_{j_0}$, $u = y_{k+1,s,j_0}$ imply that

$$\|y_{k+1,s,j_0} - y\| \leq \delta_0, \tag{12.80}$$

$$\|y - z\|^2 \leq \|z - y_{k+1,s,j_0-1}\|^2 - \|y_{k+1,s,j_0-1} - y\|^2. \tag{12.81}$$

In view of (12.27), (12.37), (12.74), and (12.80),

$$\begin{aligned}\|y_{k+1,s,j_0-1} - y\| &\geq \|y_{k+1,s,j_0-1} - y_{k+1,s,j_0}\| - \|y - y_{k+1,s,j_0}\| \\ &> \epsilon_0 - \delta_0 > 2^{-1}\epsilon_0.\end{aligned} \tag{12.82}$$

By (12.61), (12.80), (12.82), and property (P2),

$$\begin{aligned}\|y_{k+1,s,j_0} - z\|^2 &\leq (\|y_{k+1,s,j_0} - y\| + \|y - z\|)^2 \\ &\leq \|y - z\|^2 + \delta_0^2 + 2\delta_0\|y - z\| \\ \leq \|y - z\|^2 + 2\delta_0(\|y - z\| + \delta_0) &\leq \|y - z\|^2 + 2\delta_0(2M + 1 + \delta_0) \\ &\leq \|y - z\|^2 + 2\delta_0(2M + 2).\end{aligned} \tag{12.83}$$

Relations (12.81), (12.82), and (12.83) imply that

$$\begin{aligned}\|y_{k+1,s,j_0} - z\|^2 &\leq \|z - y_{k+1,s,j_0-1}\|^2 - \|y_{k+1,s,j_0-1} - y\|^2 + 2\delta_0(2M + 2) \\ &\leq \|z - y_{k+1,s,j_0-1}\|^2 - 4^{-1}\epsilon_0^2 + 2\delta_0(2M + 2).\end{aligned} \tag{12.84}$$

It follows from (12.42), (12.44), (12.71), (12.73), (12.84), and (P3) that

$$\begin{aligned}&\|y_{k+1,s} - z\|^2 - \|x_k - z\|^2 \\ &= \|y_{k+1,s,p(s)} - z\|^2 - \|y_{k+1,s,0} - z\|^2 \\ &= \sum_{i=1}^{p(s)}[\|y_{k+1,s,i} - z\|^2 - \|y_{k+1,s,i-1} - z\|^2] \\ &= \sum\{\|y_{k+1,s,i} - z\|^2 - \|y_{k+1,s,i-1} - z\|^2 :\ i \in \{1, \dots, p(s)\} \setminus \{j_0\}\} \\ &\quad + \|y_{k+1,s,j_0} - z\|^2 - \|y_{k+1,s,j_0-1} - z\|^2 \\ &\leq 2\delta_0(2M + 2)(p(s) - 1) - 4^{-1}\epsilon_0^2 + 2\delta_0(2M + 2) \leq -4^{-1}\epsilon_0^2 + 2\delta_0\bar{q}(2M + 2).\end{aligned} \tag{12.85}$$

By (12.8), (12.12), (12.71), (12.85), properties (P2) and (P3), and the convexity of the function $\|\cdot\|^2$,

$$\|z - \sum_{t\in\Omega_{k+1}} w_{k+1}(t)y_{k+1,t}\|^2 \le \sum_{t\in\Omega_{k+1}} w_{k+1}(t)\|z - y_{k+1,t}\|^2$$

$$\le \sum\{w_{k+1}(t)\|z - y_{k+1,t}\|^2 : \ t \in \Omega_{k+1} \setminus \{s\}\} + w_{k+1}(s)\|z - y_{k+1,s}\|^2$$

$$\le (\|x_k - z\|^2 + 2\delta_0\bar{q}(2M + 2)) \sum\{w_{k+1}(t) : \ t \in \Omega_{k+1} \setminus \{s\}\}$$

$$+w_{k+1}(s)(\|z - x_k\|^2 - 2^{-1}\epsilon_0^2\bar{q}\delta_0(2M + 2))$$

$$\le \|z - x_k\|^2 + 2\bar{q}\delta_0(2M + 2) - 4^{-1}\epsilon_0^2\bar{\Delta}. \tag{12.86}$$

In view of (12.26), (12.27), (12.37), and (12.67),

$$\|z - \sum_{t\in\Omega_{k+1}} w_{k+1}(t)y_{k+1,t}\| \le \|x_k - z\| \le 2M + 1. \tag{12.87}$$

It follows from (12.37), (12.40), (12.86), and (12.87) that

$$\|z - x_{k+1}\|^2 \le (\|z - \sum_{t\in\Omega_{k+1}} w_{k+1}(t)y_{k+1,t}\| + \|\sum_{t\in\Omega_{k+1}} w_{k+1}(t)y_{k+1,t} - x_{k+1}\|)^2$$

$$\le \|z - \sum_{t\in\Omega_{k+1}} w_{k+1}(t)y_{k+1,t}\|^2 + \delta^2 + 2\delta\|z - \sum_{t\in\Omega_{k+1}} w_{k+1}(t)y_{k+1,t}\|$$

$$\le \|x_k - z\|^2 + 2\delta_0\bar{q}(2M + 2) - 4^{-1}\bar{\Delta}\epsilon_0^2 + 2\delta(2M + 1) + \delta^2$$

$$\le \|x_k - z\|^2 + 2\delta_0(\bar{q} + 1)(2M + 2) - 4^{-1}\bar{\Delta}\epsilon_0^2. \tag{12.88}$$

Therefore we have shown that the following property holds:
(P4) if an integer $k \in \{p\bar{N}, \dots, (p + 1)\bar{N} - 1\}$ satisfies

$$\|x_k - z\| \le 2M + \delta_0(\bar{q} + 1)(k - p\bar{N}),$$

then

$$\|x_{k+1} - z\| \le 2M + \delta_0(\bar{q} + 1)(k + 1 - p\bar{N})$$

(see (12.66)),

$$\|x_{k+1} - z\|^2 \le \|x_k - z\|^2 + 2\delta_0(\bar{q}(2M + 2) + 2M + 3)$$

(see (12.69)) and if in addition $\lambda_{k+1} > \epsilon_0$, then

$$\|x_{k+1} - z\|^2 \le \|x_k - z\|^2 + 2\delta_0(\bar{q} + 1)(2M + 2) - 4^{-1}\bar{\Delta}\epsilon_0^2$$

(see (12.88)).

Property (P4) and (12.48) imply that

$$\|x_k - z\| \le 2M + \delta_0(\bar{q} + 1)(k - p\bar{N}),\ k = p\bar{N}, \dots, (p + 1)\bar{N}, \tag{12.89}$$

$$\|x_{k+1} - z\|^2 \le \|x_k - z\|^2 + 2\delta_0(\bar{q}(2M + 2) + 2M + 3),\ k = p\bar{N}, \dots, (p + 1)\bar{N} - 1. \tag{12.90}$$

In view of (12.46), there exists

$$k_0 \in \{p\bar{N}, \dots, (p + 1)\bar{N} - 1\}$$

such that

$$\lambda_{k_0+1} > \epsilon_0. \tag{12.91}$$

Property (P4), (12.89), and (12.91) imply that

$$\|x_{k_0+1} - z\|^2 \le \|x_{k_0} - z\|^2 + 2\delta_0(\bar{q} + 1)(2M + 2) - 4^{-1}\bar{\Delta}\epsilon_0^2. \tag{12.92}$$

It follows from (12.27), (12.37), (12.90), and (12.92) that

$$\begin{aligned}
&\|z - x_{p\bar{N}}\|^2 - \|z - x_{(p+1)\bar{N}}\|^2 \\
&= \sum_{k=p\bar{N}}^{(p+1)\bar{N}-1} (\|z - x_k\|^2 - \|z - x_{k+1}\|^2) \\
&\ge -(\bar{N} - 1)2\delta_0(\bar{q}(2M + 2) + 2M + 3) + \|z - x_{k_0}\|^2 - \|z - x_{k_0+1}\|^2 \\
&\ge 4^{-1}\bar{\Delta}\epsilon_0^2 - 2\delta_0(\bar{q} + 1)(2M + 3)\bar{N} \ge 8^{-1}\bar{\Delta}\epsilon_0^2,
\end{aligned}$$

$$\|z - x_{p\bar{N}}\|^2 - \|z - x_{(p+1)\bar{N}}\|^2 \ge 8^{-1}\bar{\Delta}\epsilon_0^2. \tag{12.93}$$

Thus we have shown that the following property holds:

(P5) if an integer $p \in [0, \tilde{q}]$ satisfies $\|x_{p\bar{N}} - z\| \le 2M$, then (12.93) holds and for all $k \in \{p\bar{N}, \dots, (p + 1)\bar{N}\}$,

$$\|x_k - z\| \le 2M + 1$$

(see (12.25), (12.37), and (12.89)).

Property (P5) and (12.47) imply that

$$\|x_{p\bar{N}} - z\| \le 2M \text{ for all } p = 0, \dots, \tilde{q} + 1, \tag{12.94}$$

for all $p = 0, \dots, \tilde{q}$, (12.93) holds and

$$\|x_i - z\| \le 2M + 1,\ i = 0, \dots, (\tilde{q} + 1)\bar{N}.$$

By (12.28), (12.47), and (12.93),

$$4M^2 \ge \|x_0 - z\|^2 \ge \|x_0 - z\|^2 - \|x_{(\tilde{q}+1)\bar{N}} - z\|^2$$

$$= \sum_{p=0}^{\tilde{q}} (\|x_{p\bar{N}} - z\|^2 - \|x_{(p+1)\bar{N}} - z\|^2) \ge 8^{-1}(\tilde{q} + 1)\bar{\Delta}\epsilon_0^2,$$

$$\tilde{q} + 1 \le 32M^2\bar{\Delta}^{-1}\epsilon_0^{-2} < n_0.$$

Thus we have shown that the following property holds:

(P6) if an integer $\tilde{q} \in [0, n_0 - 1]$ and if for each integer $p \in [0, \tilde{q}]$ (12.46) holds, then

$$\|x_i\| \le 3M + 1,\ i = 0, \dots, (\tilde{q} + 1)\bar{N},$$

$$\tilde{q} + 1 < n_0.$$

In view of property (P6), there exists a nonnegative integer $q < n_0$ such that for each nonnegative integer $p < q$,

$$\max\{\lambda_i :\ i = p\bar{N} + 1, \dots, (p + 1)\bar{N}\} > \epsilon_0,$$

$$\max\{\lambda_i :\ i = q\bar{N} + 1, \dots, (q + 1)\bar{N}\} \le \epsilon_0,$$

$$\|x_i\| \le 3M + 1,\ i = 0, \dots, q\bar{N}.$$

Assume that a nonnegative integer q satisfies

$$\|x_{q\bar{N}}\| \le 3M + 1, \tag{12.95}$$

$$\lambda_i \le \epsilon_0,\ i = q\bar{N} + 1, \dots, (q + 1)\bar{N}. \tag{12.96}$$

By (12.39), (12.41), (12.45), and (12.96), for each $k = q\bar{N}, \dots, (q + 1)\bar{N} - 1$, each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}$ and each $j = 1, \dots, p(t)$,

$$\|y_{k+1,t,j} - y_{k+1,t,j-1}\| \le \epsilon_0. \tag{12.97}$$

Assume that

$$k \in \{q\bar{N}, \dots, (q+1)\bar{N} - 1\} \tag{12.98}$$

and

$$\|x_k\| \le 3M + 1 + (k - q\bar{N})\epsilon_0(\bar{q} + 1). \tag{12.99}$$

(Note that by (12.95), (12.99) is true for $k = q\bar{N}$.)

Let $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}$. By (12.42) and (12.99),

$$\|y_{k+1,t,0}\| \le 3M + 1 + (k - q\bar{N})\epsilon_0(\bar{q} + 1). \tag{12.100}$$

It follows from (12.44), (12.97), and (12.100),

$$\|y_{k+1,t,j}\| \le 3M + 1 + \epsilon_0\bar{q} + \epsilon_0(\bar{q} + 1)(k - q\bar{N}), \tag{12.101}$$

$$j = 0, \dots, p(t),$$

$$\|y_{k+1,t}\| \le 3M + 1 + \epsilon_0(\bar{q} + 1)(k - q\bar{N}) + \epsilon_0\bar{q}. \tag{12.102}$$

In view of (12.8), (12.40), and (12.102),

$$\begin{aligned}\|x_{k+1}\| &\le 3M + 1 + \epsilon_0(\bar{q} + 1)(k - q\bar{N}) + \epsilon_0\bar{q} + \delta \\ &\le 3M + 1 + \epsilon_0(\bar{q} + 1)(k + 1 - q\bar{N}).\end{aligned}$$

Thus by induction we shown that (12.99) holds for all $k = q\bar{N}, \dots, (q+1)\bar{N}$ and for all $k = q\bar{N}, \dots, (q+1)\bar{N} - 1$, all $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}$ and all $j = 0, \dots, p(t)$, in view of (12.101),

$$\|y_{k+1,t,j}\| \le 3M + 1 + \epsilon_0(\bar{q} + 1)(k + 1 - q\bar{N}). \tag{12.103}$$

By (12.25), (12.27), and (12.99)

$$\|x_k\| \le 3M + 2,\ k = q\bar{N}, \dots, (q+1)\bar{N}. \tag{12.104}$$

Relations (12.25), (12.27), and (12.103) imply that

$$\|y_{k+1,t,j}\| \le 3M + 2,\ k = q\bar{N}, \dots, (q+1)\bar{N} - 1,\ t \in \Omega_{k+1},\ j = 0, \dots, p(t). \tag{12.105}$$

Let

$$k \in \{q\bar{N}, \dots, (q+1)\bar{N} - 1\},\ t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1},\ j \in \{1, \dots, p(t)\}.$$

We show that

$$f_{t_j}(y_{k+1,t,j-1}) \le \Delta. \tag{12.106}$$

Assume the contrary. Then

$$f_{t_j}(y_{k+1,t,j-1}) > \Delta. \tag{12.107}$$

By (12.13), (12.14), (12.43), and (12.107),

$$\begin{aligned} y_{k+1,t,j} \in \{ & y_{k+1,t,j-1} - f_{t_j}(y_{k+1,t,j-1})\|l\|^{-2}l : \\ & l \in \partial f_{t_j}(y_{k+1,t,j-1}) + B(0,\delta),\ l \ne 0\} + B(0,\delta). \end{aligned} \tag{12.108}$$

In view of (12.108), there exists

$$l \in (\partial f_{t_j}(y_{k+1,t,j-1}) + B(0,\delta)) \setminus \{0\} \tag{12.109}$$

such that

$$\|y_{k+1,t,j} - (y_{k+1,t,j-1} - f_{t_j}(y_{k+1,t,j-1})\|l\|^{-2}l)\| \le \delta. \tag{12.110}$$

By (12.24), (12.105), and (12.109),

$$\|l\| \le M_1. \tag{12.111}$$

It follows from (12.27), (12.97), (12.107), and (12.111),

$$\epsilon_0 \ge \|y_{k+1,t,j} - y_{k+1,t,j-1}\|$$

$$\ge f_{t_j}(y_{k+1,t,j-1})\|l\|^{-1} - \delta > \Delta M_1^{-1} - \delta,$$

$$\Delta < 2M_1\epsilon_0.$$

This contradicts (12.26). The contradiction we have reached proves that (12.106) holds. In view of (12.13), (12.43), and (12.106),

$$y_{k+1,t,j} = y_{k+1,t,j-1}.$$

This relation, (12.106), and (12.42) imply that for all $j = 0, \dots, p(t)$,

$$y_{k+1,t,j} = x_k, \tag{12.112}$$

$$f_{t_j}(x_k) \le \Delta,\ j = 1, \dots, p(t). \tag{12.113}$$

It follows from (12.12), (12.40), (12.44), and (12.112) that

$$\|x_{k+1} - x_k\| \le \delta. \tag{12.114}$$

By (12.114), for all $k_1, k_2 \in \{q\bar{N}, \dots, (q+1)\bar{N}\}$,

$$\|x_{k_1} - x_{k_2}\| \le \delta\bar{N}. \tag{12.115}$$

Let $s \in \{1, \dots, m\}$ and $i \in \{q\bar{N}, \dots, (q+1)\bar{N}\}$. By (12.30), there exist

$$k \in \{q\bar{N}, \dots, (q+1)\bar{N} - 1\},\ t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1} \tag{12.116}$$

such that

$$s \in \{t_1, \dots, t_{p(t)}\}. \tag{12.117}$$

By (12.113), (12.116), and (12.117),

$$f_s(x_k) \le \Delta.$$

By the relation above, the inclusion $i \in \{q\bar{N}, \dots, (q+1)\bar{N}\}$, (12.24), (12.104), (12.115), and (12.116),

$$f_s(x_i) \le f_s(x_k) + |f_s(x_i) - f_s(x_k)| \le M_1\|x_k - x_i\| + \Delta \le \Delta + M_1\bar{N}\delta.$$

Theorem 12.1 is proved. □

12.3 The Second Main Result

Let m be a natural number and $f_i : X \to R^1$, $i = 1, \dots, m$ be convex continuous functions.

For each $i = 1, \dots, m$ set

$$C_i = \{x \in X : f_i(x) \le 0\},$$
$$C = \cap_{i=1}^m C_i = \cap_{i=1}^m \{x \in X : f_i(x) \le 0\}.$$

Suppose that

$$C \ne \emptyset.$$

We use the notation and definitions introduced in Sect. 12.1.

Suppose that $M > 0$, $M_0 > 0$ and $M_1 > 2$ be such that

$$B(0, M) \cap \{x \in X : f_i(x) \le 0,\ i = 1, \dots, m\} \ne \emptyset, \tag{12.118}$$

$$B(0, M) \cap \{x \in X : f_i(x) < 0\} \ne \emptyset, \tag{12.119}$$

for all $i \in \{1, \dots, m\}$,

$$f_i(B(0, 3M + 3)) \subset [-M_0, M_0],\ i = 1, \dots, m, \tag{12.120}$$

$$|f_i(u) - f_i(v)| \le (M_1 - 2)\|u - v\|$$
$$\text{for all } u, v \in B(0, 3M + 4) \text{ and all } i = 1, \dots, m. \tag{12.121}$$

In view of (12.119), there exists $\Delta \in (0, 1]$ such that for each $i = 1, \dots, m$,

$$B(0, M) \cap \{x \in X : f_i(x) \le -\Delta\} \ne \emptyset. \tag{12.122}$$

Let $\delta \in (0, 1]$ satisfy

$$\delta \le 2^{-1}\Delta(4M + 3)^{-1}, \tag{12.123}$$

$$8(2(1 + \bar{q})(1 + \bar{N}))^3 \bar{\Delta}^{-1}(4M + 4)\delta(1 + 16M_0\Delta^{-2}(4M + 3)^2) \le 1, \tag{12.124}$$

let

$$\epsilon_0 = (8\delta\bar{\Delta}^{-1}(4M + 4)2(1 + \bar{q})(1 + \bar{N})(1 + 16M_0\Delta^{-2}(4M + 3)^2))^{1/2} \le 1 \tag{12.125}$$

and a natural number

$$n_0 > 32M^2\epsilon_0^{-2}\bar{\Delta}^{-1}. \tag{12.126}$$

The following theorem is our second main result which will be proved in Sect. 12.4.

Theorem 12.2. *Assume that*

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_* \tag{12.127}$$

satisfies for each natural number j,

$$\{1, \dots, m\} \subset \cup_{i=j}^{j+\bar{N}-1}(\cup_{t\in\Omega_i}\{t_1, \dots, t_{p(t)}\}), \tag{12.128}$$

$$x_0 \in B(0, M), \tag{12.129}$$

$$\epsilon \in [0, 2M_1\epsilon_0], \tag{12.130}$$

$$\{x_i\}_{i=1}^{\infty} \subset X,\ \{\lambda_i\}_{i=1}^{\infty} \subset [0, \infty)$$

satisfy for each natural number i,

$$(x_i, \lambda_i) \in A(x_{i-1}, (\Omega_i, w_i), \epsilon, \delta). \tag{12.131}$$

Then there exists an integer $q \in [0, n_0 - 1]$ such that

$$\|x_i\| \leq 3M + 1 \text{ for all integers } i = 0, \dots, q\bar{N}, \tag{12.132}$$

$$\lambda_i \leq \epsilon_0, \ i = q\bar{N} + 1, \dots, (q+1)\bar{N}. \tag{12.133}$$

Moreover, if an integer $q \geq 0$ satisfies (12.133) and $\|x_{q\bar{N}}\| \leq 3M + 1$, then

$$\|x_{k_1} - x_{k_2}\| \leq \epsilon_0 \bar{N}(\bar{q} + 1)$$

for all $k_1, k_2 \in \{q\bar{N}, \dots, (q+1)\bar{N}\}$ and for all $i \in \{q\bar{N}, \dots, (q+1)\bar{N}\}$ all $s \in \{1, \dots, m\}$

$$f_s(x_i) \leq M_1 \epsilon_0 (\bar{N} + 1)(\bar{q} + 2).$$

Note that in Theorem 12.2 δ is the computational error made by our computer system, we obtain a point x satisfying

$$f_s(x) \leq M_1 \epsilon_0 (\bar{N} + 1)(\bar{q} + 2)$$

for all $s = 1, \dots, m$ and in order to obtain this point we need $(n_0 - 1)\bar{N}$ iterations. It is not difficult to see that

$$M_1 \epsilon_0 (\bar{N} + 1)(\bar{q} + 2) = c_1 \delta^{1/2}$$

and $n_0 = \lfloor c_2 \delta^{-1} \rfloor$, where c_1 and c_2 are positive constants which do not depend on δ.

12.4 Proof of Theorem 12.2

By (12.118), there exists

$$z \in B(0, M) \text{ such that } f_i(z) \leq 0, \ i = 1, \dots, m. \tag{12.134}$$

Set

$$\delta_0 = \delta(1 + 16 M_0 \Delta^{-2} (4M + 3)^2). \tag{12.135}$$

Let n be a natural number. In view of (12.131),

$$(x_n, \lambda_n) \in A(x_{n-1}, (\Omega_n, w_n), \epsilon, \delta). \tag{12.136}$$

By (12.19)–(12.21) and (12.136), there exist

$$(y_{n,t}, \lambda_{n,t}) \in A_0(t, x_{n-1}, \epsilon, \delta),\ t \in \Omega_n \tag{12.137}$$

such that

$$\|x_n - \sum_{t \in \Omega_n} w_n(t) y_{n,t}\| \le \delta, \tag{12.138}$$

$$\lambda_n = \max\{\lambda_{n,t} :\ t \in \Omega_n\}. \tag{12.139}$$

It follows from (12.15)–(12.18) and (12.137) that for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_n$ there exists a sequence $\{y_{n,t,i}\}_{i=0}^{p(t)} \subset X$ such that

$$y_{n,t,0} = x_{n-1}, \tag{12.140}$$

$$y_{n,t,i} \in A_{t_i}(y_{n,t,i-1}, \epsilon, \delta),\ i = 1, \dots, p(t), \tag{12.141}$$

$$y_{n,t} = y_{n,t,p(t)}, \tag{12.142}$$

$$\lambda_{n,t} = \max\{\|y_{n,t,i} - y_{n,t,i-1}\| :\ i = 1, \dots, p(t)\}. \tag{12.143}$$

Assume that a nonnegative integer $\tilde{q} \le n_0 - 1$ satisfies for each integer $p \in [0, \tilde{q}]$,

$$\max\{\lambda_i :\ i = p\bar{N} + 1, \dots, (p+1)\bar{N}\} > \epsilon_0. \tag{12.144}$$

Relations (12.129) and (12.134) imply that

$$\|x_0 - z\| \le 2M. \tag{12.145}$$

Assume that an integer $p \in [0, \tilde{q}]$ satisfies

$$\|x_{p\bar{N}} - z\| \le 2M. \tag{12.146}$$

Assume that an integer k satisfies

$$k \in \{p\bar{N}, \dots, (p+1)\bar{N} - 1\}, \tag{12.147}$$

$$\|x_k - z\| \le 2M + \delta_0(k - p\bar{N})(\bar{q} + 1). \tag{12.148}$$

(Note that in view of (12.146), inequality (12.148) holds for $k = p\bar{N}$.)

Let

$$t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}. \tag{12.149}$$

By (12.141) and (12.149), for each $j \in \{1, \dots, p(t)\}$, we have

$$y_{k+1,t,j} \in A_{t_j}(y_{k+1,t,j-1}, \epsilon, \delta), \tag{12.150}$$

$$y_{k+1,t,0} = x_k. \tag{12.151}$$

In view of (12.151) and (12.148),

$$\|y_{k+1,t,0} - z\| \le 2M + \delta_0(k - p\bar{N})(\bar{q} + 1). \tag{12.152}$$

Assume that an integer

$$j \in \{1, \dots, p(t)\} \tag{12.153}$$

satisfies

$$\|y_{k+1,t,j-1} - z\| \le 2M + \delta_0(k - p\bar{N})(\bar{q} + 1) + \delta_0(j - 1). \tag{12.154}$$

(Note that in view of (12.152), (12.154) is true for $j = 1$.) By (12.13) and (12.150), if

$$f_{t_j}(y_{k+1,t,j-1}) \le \epsilon,$$

then

$$y_{k+1,t,j} = y_{k+1,t,j-1}.$$

Assume that

$$f_{t_j}(y_{k+1,t,j-1}) > \epsilon. \tag{12.155}$$

It follows from (12.11), (12.125), (12.134), (12.135), (12.147), (12.153), and (12.154) that

$$\|y_{k+1,t,j-1}\| \le 3M + \delta_0(1 + \bar{N})(1 + \bar{q}) < 3M + 1. \tag{12.156}$$

Lemma 11.5, (12.14), (12.134), (12.150), (12.155), and (12.156) imply that

$$\|y_{k+1,t,j} - z\| \le \delta_0 + \|y_{k+1,t,j-1} - z\|. \tag{12.157}$$

Clearly, (12.157) holds in both cases. By (12.157),

$$\|y_{k+1,t,j} - z\| \le 2M + \delta_0(k - p\bar{N})(\bar{q} + 1) + \delta_0 j. \tag{12.158}$$

Thus we have shown by induction that (12.158) holds for all $j = 0, \dots, p(t)$ and that (12.157) holds for all $j = 1, \dots, p(t)$. By (12.11), (12.125), (12.134), (12.135), (12.147), and (12.158), for all $j = 0, \dots, p(t)$,

$$\|y_{k+1,t,j} - z\| \le 2M + \delta_0(\bar{N} + 1)(\bar{q} + 1) \le 2M + 1, \tag{12.159}$$

$$\|y_{k+1,t,j}\| \le 3M + 1. \tag{12.160}$$

In view of (12.142) and (12.159),

$$\|y_{k+1,t} - z\| \le 2M + 1. \tag{12.161}$$

Thus we have shown that the following property holds:

(P7) for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}$ and each $j = 0, \dots, p(t)$, (12.158)–(12.160) hold and for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}$ and each $j = 1, \dots, p(t)$, (12.157) and (12.161) are valid.

Let

$$t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1},\ j \in \{1, \dots, p(t)\}.$$

Property (P7) and (12.157) imply that

$$\|y_{k+1,t,j} - z\|^2 \le (\|y_{k+1,t,j-1} - z\| + \delta_0)^2$$

$$\le \|y_{k+1,t,j-1} - z\|^2 + \delta_0^2 + 2\delta_0\|y_{k+1,t,j-1} - z\|$$

$$\le \|y_{k+1,t,j} - z\|^2 + 2\delta_0(2M + 1 + \delta_0)$$

$$\le \|y_{k+1,t,j-1} - z\|^2 + 2\delta_0(2M + 2).$$

Thus we have shown that the following property holds:

(P8) for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}$ and each $j \in \{1, \dots, p(t)\}$,

$$\|y_{k+1,t,j} - z\|^2 - \|y_{k+1,t,j-1} - z\|^2 \le 2\delta_0(2M + 2). \tag{12.162}$$

Property (P8), (12.140), and (12.142) imply that for each

$$t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1},$$

$$\|y_{k+1,t} - z\|^2 - \|x_k - z\|^2$$

$$= \|y_{k+1,t,p(t)} - z\|^2 - \|y_{k+1,t,0} - z\|^2$$

$$= \sum_{i=1}^{p(t)} [\|y_{k+1,t,i} - z\|^2 - \|y_{k+1,t,i-1} - z\|^2]$$

$$\le 2\delta_0(2M + 2)p(t) \le 2\delta_0\bar{q}(2M + 2). \tag{12.163}$$

Property (P7), (12.11), (12.142), and (12.158) holding for all $t \in \Omega_{k+1}$ and all $j = 0, \dots, p(t)$, imply that

$$\|y_{k+1,t} - z\| = \|y_{k+1,t,p(t)} - z\| \le 2M + \delta_0((1+\bar{q})(k - p\bar{N}) + \bar{q}). \tag{12.164}$$

By (12.8), (12.138), (12.164) and the convexity of the norm,

$$\|z - x_{k+1}\| \le \|z - \sum_{t \in \Omega_{k+1}} w_{k+1}(t) y_{k+1,t}\| + \|\sum_{t \in \Omega_{k+1}} w_{k+1}(t) y_{k+1,t} - x_{k+1}\|$$

$$\le \sum_{t \in \Omega_{k+1}} w_{k+1}(t)\|z - y_{k+1,t}\| + \delta$$

$$\le 2M + \delta_0((1+\bar{q})(k - p\bar{N}) + \bar{q}) + \delta \le 2\bar{M} + \delta_0(1+\bar{q})(k - p\bar{N} + 1),$$

$$\|x_{k+1} - z\| \le 2M + \delta_0(1+\bar{q})(k - p\bar{N} + 1). \tag{12.165}$$

In view of (12.125), (12.135), (12.147), (12.148), and (12.165),

$$\|x_k - z\|, \ \|x_{k+1} - z\| \le 2M + 1. \tag{12.166}$$

By (12.8), (12.163) and the convexity of the function $\|\cdot\|^2$,

$$\|\sum_{t \in \Omega_{k+1}} w_{k+1}(t) y_{k+1,t} - z\|^2 \le \sum_{t \in \Omega_{k+1}} w_{k+1}(t)\|y_{k+1,t} - z\|^2$$

$$\le \|x_k - z\|^2 + 2\delta_0 \bar{q}(2M+2). \tag{12.167}$$

By (12.125), (12.135), (12.167),

$$\|\sum_{t \in \Omega_{k+1}} w_{k+1}(t) y_{k+1,t} - z\| \le 2M + 2. \tag{12.168}$$

By the relation above, (12.138), (12.167), and (12.168),

$$\|x_{k+1} - z\|^2 \le (\|x_{k+1} - \sum_{t \in \Omega_{k+1}} w_{k+1}(t) y_{k+1,t}\| + \|\sum_{t \in \Omega_{k+1}} w_{k+1}(t) y_{k+1,t} - z\|)^2$$

$$\le (\delta + \|\sum_{t \in \Omega_{k+1}} w_{k+1}(t) y_{k+1,t} - z\|)^2$$

$$\le \delta^2 + 2\delta(2M+2) + \|x_k - z\|^2 + 2\delta_0 \bar{q}(2M+2)$$

$$\le \|x_k - z\|^2 + 2\delta_0(\bar{q}(2M+2) + 2M + 3). \tag{12.169}$$

It follows from (12.140), (12.142), and (P7) that for each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}$,

$$\|y_{k+1,t} - z\| - \|x_k - z\| = \|y_{k+1,t,p(t)} - z\| - \|y_{k+1,t,0} - z\|$$
$$= \sum_{i=1}^{p(t)} [\|y_{k+1,t,i} - z\| - \|y_{k+1,t,i-1} - z\|] \le \delta_0 p(t) \le \delta_0 \bar{q}. \tag{12.170}$$

In view of (12.138), (12.168), and (12.170),

$$\|x_{k+1} - z\| \le \|x_{k+1} - \sum_{t \in \Omega_{k+1}} w_{k+1}(t) y_{k+1,t}\| + \|\sum_{t \in \Omega_{k+1}} w_{k+1}(t) y_{k+1,t} - z\|$$
$$\le \delta + \sum_{t \in \Omega_{k+1}} w_{k+1}(t) \|y_{k+1,t} - z\|$$
$$\le \|x_k - z\| + \delta_0 \bar{q} + \delta \le \|x_k - z\| + \delta_0(\bar{q} + 1). \tag{12.171}$$

Assume that

$$\lambda_{k+1} > \epsilon_0. \tag{12.172}$$

In view of (12.139) and (12.172), there exists

$$s = (s_1, \dots, s_{p(s)}) \in \Omega_{k+1} \tag{12.173}$$

such that

$$\epsilon_0 < \lambda_{k+1} = \lambda_{k+1,s}. \tag{12.174}$$

It follows from (12.143) and (12.174) that there exists

$$j_0 \in \{1, \dots, p(s)\} \tag{12.175}$$

such that

$$\epsilon_0 < \lambda_{k+1,s} = \|y_{k+1,s,j_0} - y_{k+1,s,j_0-1}\|. \tag{12.176}$$

By (12.13), (12.14), (12.141), and (12.176),

$$f_{s_{j_0}}(y_{k+1,s,j_0-1}) > \epsilon. \tag{12.177}$$

It follows from (12.14), (12.141), and (12.177) that there exist

$$\xi \in \partial f_{s_{j_0}}(y_{k+1,s,j_0-1}), \tag{12.178}$$
$$l \in B(\xi, \delta) \setminus \{0\} \tag{12.179}$$

such that

$$y_{k+1,s,j_0} \in B(y_{k+1,s,j_0-1} - f_{s_{j_0}}(y_{k+1,s,j_0-1})\|l\|^{-2}l, \delta). \tag{12.180}$$

Relations (12.118), (12.177), and (12.178) imply that

$$\xi \neq 0.$$

Set

$$y = y_{k+1,s,j_0-1} - f_{s_{j_0}}(y_{k+1,s,j_0-1})\|\xi\|^{-2}\xi. \tag{12.181}$$

Property (P7), (12.134), (12.160), (12.177)–(12.181), and Lemma 11.5 applied with $x = y_{k+1,s,j_0-1}$, $j = s_{j_0}$, $u = y_{k+1,s,j_0}$ imply that

$$\|y_{k+1,s,j_0} - y\| \leq \delta_0, \tag{12.182}$$

$$\|y - z\|^2 \leq \|z - y_{k+1,s,j_0-1}\|^2 - \|y_{k+1,s,j_0-1} - y\|^2. \tag{12.183}$$

In view of (12.125), (12.135), (12.176), and (12.182),

$$\begin{aligned} \|y_{k+1,s,j_0-1} - y\| &\geq \|y_{k+1,s,j_0-1} - y_{k+1,s,j_0}\| - \|y - y_{k+1,s,j_0}\| \\ &> \epsilon_0 - \delta_0 > 2^{-1}\epsilon_0. \end{aligned} \tag{12.184}$$

By (12.135), (12.159), (12.182), (12.183), and property (P7),

$$\begin{aligned} \|y_{k+1,s,j_0} - z\|^2 &\leq (\|y_{k+1,s,j_0} - y\| + \|y - z\|)^2 \\ &\leq \|y - z\|^2 + \delta_0^2 + 2\delta_0\|y - z\| \\ \leq \|y - z\|^2 + 2\delta_0(\|y - z\| + \delta_0) &\leq \|y - z\|^2 + 2\delta_0(2M + 2). \end{aligned} \tag{12.185}$$

Relations (12.183)–(12.185) imply that

$$\begin{aligned} \|y_{k+1,s,j_0} - z\|^2 &\leq \|z - y_{k+1,s,j_0-1}\|^2 - \|y_{k+1,s,j_0-1} - y\|^2 + 2\delta_0(2M + 2) \\ &\leq \|z - y_{k+1,s,j_0-1}\|^2 - 4^{-1}\epsilon_0^2 + 2\delta_0(2M + 2). \end{aligned} \tag{12.186}$$

It follows from (12.140), (12.142), (12.175), (12.186), and (P8) that

$$\begin{aligned} &\|y_{k+1,s} - z\|^2 - \|x_k - z\|^2 \\ &= \|y_{k+1,s,p(s)} - z\|^2 - \|y_{k+1,s,0} - z\|^2 \\ &= \sum_{i=1}^{p(s)} [y_{k+1,s,i} - z\|^2 - \|y_{k+1,s,i-1} - z\|^2] \end{aligned}$$

$$= \sum\{\|y_{k+1,s,i} - z\|^2 - \|y_{k+1,s,i-1} - z\|^2 : i \in \{1, \dots, p(s)\} \setminus \{j_0\}\}$$
$$+\|y_{k+1,s,j_0} - z\|^2 - \|y_{k+1,s,j_0-1} - z\|^2$$
$$\le 2\delta_0(2M + 2)(p(s) - 1) - 4^{-1}\epsilon_0^2 + 2\delta_0(2M + 2) \le -4^{-1}\epsilon_0^2 + 2\delta_0\bar{q}(2M + 2). \tag{12.187}$$

By (12.8), (12.12), (12.163), (12.173), (12.187), and the convexity of the function $\|\cdot\|^2$,

$$\|z - \sum_{t\in\Omega_{k+1}} w_{k+1}(t)y_{k+1,t}\|^2 \le \sum_{t\in\Omega_{k+1}} w_{k+1}(t)\|z - y_{k+1,t}\|^2$$
$$\sum\{w_{k+1}(t)\|z - y_{k+1,t}\|^2 : t \in \Omega_{k+1} \setminus \{s\}\} + w_{k+1}(s)\|z - y_{k+1,s}\|^2$$
$$\le (\|x_k - z\|^2 + 2\delta_0\bar{q}(2M + 2)) \sum\{w_{k+1}(t) : t \in \Omega_{k+1} \setminus \{s\}\}$$
$$+w_{k+1}(s)(\|z - x_k\|^2 - 4^{-1}\epsilon_0^2 + 2\bar{q}\delta_0(2M + 2))$$
$$\le \|z - x_k\|^2 + 2\bar{q}\delta_0(2M + 2) - 4^{-1}\epsilon_0^2\bar{\Delta}. \tag{12.188}$$

In view of (12.125), (12.135), (12.148), and (12.188),

$$\|z - \sum_{t\in\Omega_{k+1}} w_{k+1}(t)y_{k+1,t}\| \le \|x_k - z\| \le 2M + 1. \tag{12.189}$$

It follows from (12.138), (12.181), and (12.189) that

$$\|z - x_{k+1}\|^2 \le (\|z - \sum_{t\in\Omega_{k+1}} w_{k+1}(t)y_{k+1,t}\| + \|\sum_{t\in\Omega_{k+1}} w_{k+1}(t)y_{k+1,t} - x_{k+1}\|)^2$$
$$\le \|z - \sum_{t\in\Omega_{k+1}} w_{k+1}(t)y_{k+1,t}\|^2 + \delta^2 + 2\delta\|z - \sum_{t\in\Omega_{k+1}} w_{k+1}(t)y_{k+1,t}\|$$
$$\le \|x_k - z\|^2 + 2\delta_0\bar{q}(2M + 2) - 4^{-1}\bar{\Delta}\epsilon_0^2 + 2\delta(2M + 2)$$
$$\le \|x_k - z\|^2 + 2\delta_0(\bar{q} + 1)(2M + 2) - 4^{-1}\bar{\Delta}\epsilon_0^2.$$

Therefore we have shown that the following property holds:
(P9) if an integer $k \in \{p\bar{N}, \dots, (p + 1)\bar{N} - 1\}$ satisfies

$$\|x_k - z\| \le 2M + \delta_0(\bar{q} + 1)(k - p\bar{N}),$$

then

$$\|x_{k+1} - z\| \le 2M + \delta_0(\bar{q} + 1)(k + 1 - p\bar{N})$$

(see (12.165)), in view of (12.171),

$$\|x_{k+1} - z\| \le \|x_k - z\| + \delta_0(\bar{q} + 1),$$

by (12.169),

$$\|x_{k+1} - z\|^2 \le \|x_k - z\|^2 + 2\delta_0(\bar{q}(2M + 2) + 2M + 3)$$

and if in addition $\lambda_{k+1} > \epsilon_0$, then

$$\|x_{k+1} - z\|^2 \le \|x_k - z\|^2 + 2\delta_0(\bar{q} + 1)(2M + 2) - 4^{-1}\bar{\Delta}\epsilon_0^2.$$

Property (P9) and (12.146) imply that

$$\|x_k - z\| \le 2M + \delta_0(\bar{q}+1)(k - p\bar{N}),\ k = p\bar{N}, \dots, (p + 1)\bar{N}, \tag{12.190}$$

$$\|x_{k+1} - z\|^2 \le \|x_k - z\|^2 + 2\delta_0(\bar{q}(2M + 2) + 2M + 3),\ k = p\bar{N}, \dots, (p + 1)\bar{N} - 1. \tag{12.191}$$

In view of (12.144), there exists

$$k_0 \in \{p\bar{N}, \dots, (p + 1)\bar{N} - 1\}$$

such that

$$\lambda_{k_0+1} > \epsilon_0. \tag{12.192}$$

Property (P9) and (12.192) imply that

$$\|x_{k_0+1} - z\|^2 \le \|x_{k_0} - z\|^2 + 2\delta_0(\bar{q} + 1)(2M + 2) - 4^{-1}\bar{\Delta}\epsilon_0^2. \tag{12.193}$$

It follows from (12.125), (12.135), (12.191), and (12.193) that

$$\begin{aligned} &\|z - x_{p\bar{N}}\|^2 - \|z - x_{(p+1)\bar{N}}\|^2 \\ &= \sum_{k=p\bar{N}}^{(p+1)\bar{N}-1} (\|z - x_k\|^2 - \|z - x_{k+1}\|^2) \\ &\ge -(\bar{N} - 1)2\delta_0(\bar{q}(2M + 2) + 2M + 3) + \|z - x_{k_0}\|^2 - \|z - x_{k_0+1}\|^2 \\ &\ge 4^{-1}\bar{\Delta}\epsilon_0^2 - 2\delta_0(\bar{q} + 1)(2M + 3)\bar{N} \ge 8^{-1}\bar{\Delta}\epsilon_0^2. \end{aligned} \tag{12.194}$$

Thus we have shown that the following property holds:

(P10) if an integer $p \in [0, \tilde{q}]$ satisfies $\|x_{p\bar{N}} - z\| \le 2M$, then (12.194) holds and (in view of (12.125), (12.135) and (12.190)) for all $k \in \{p\bar{N}, \dots, (p+1)\bar{N}\}$,

$$\|x_k - z\| \le 2M + 1$$

and for all $k \in \{p\bar{N}, \dots, (p+1)\bar{N} - 1\}$,

$$\|x_{k+1} - z\| \le \|x_k - z\| + \delta_0(\bar{q} + 1)$$

(see (P9)).

Property (P10) and (12.145) imply that

$$\|x_{p\bar{N}} - z\| \le 2M \text{ for all } p = 0, \dots, \tilde{q} + 1, \tag{12.195}$$

for all $p = 0, \dots, \tilde{q}$, (12.194) holds and

$$\|x_i - z\| \le 2M + 1, \ i = 0, \dots, (\tilde{q} + 1)\bar{N}. \tag{12.196}$$

By (12.126), (12.145) and (12.194),

$$4M^2 \ge \|x_0 - z\|^2 \ge \|x_0 - z\|^2 - \|x_{(\tilde{q}+1)\bar{N}} - z\|^2$$

$$= \sum_{p=0}^{\tilde{q}} (\|x_{p\bar{N}} - z\|^2 - \|x_{(p+1)\bar{N}} - z\|^2) \ge 8^{-1}(\tilde{q} + 1)\bar{\Delta}\epsilon_0^2,$$

$$\tilde{q} + 1 \le 32M^2\bar{\Delta}^{-1}\epsilon_0^{-2} < n_0.$$

Thus we have shown that the following property holds:

(P11) if an integer $\tilde{q} \in [0, n_0 - 1]$ and if for each integer $p \in [0, \tilde{q}]$ (12.144) holds, then, in view of (12.196), (12.198), and (P10),

$$\|x_k\| \le 3M + 1, \ k = 0, \dots, (\tilde{q} + 1)\bar{N},$$

$$\|x_{(\tilde{q}+1)\bar{N}} - z\| \le 2M,$$

$$\|x_{k+1} - z\| \le \|x_k - z\| + \delta_0(\bar{q} + 1), \ k = 0, \dots, (\tilde{q} + 1)\bar{N} - 1,$$

$$\tilde{q} + 1 < n_0.$$

In view of property (P11), there exists a nonnegative integer $q < n_0$ such that for each nonnegative integer $p < q$,

$$\max\{\lambda_i : \ i = p\bar{N} + 1, \dots, (p + 1)\bar{N}\} > \epsilon_0,$$

$$\max\{\lambda_i : \ i = q\bar{N} + 1, \dots, (q + 1)\bar{N}\} \le \epsilon_0,$$

$$\|x_i\| \le 3M + 1, \ i = 0, \dots, q\bar{N}.$$

Assume that a nonnegative integer q satisfies

$$\|x_{q\bar{N}}\| \le 3M + 1, \tag{12.197}$$

$$\lambda_i \le \epsilon_0,\ i = q\bar{N} + 1, \dots, (q+1)\bar{N}. \tag{12.198}$$

By (12.139), (12.141), and (12.198), for each $k = q\bar{N}, \dots, (q+1)\bar{N} - 1$, each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}$ and each $j = 1, \dots, p(t)$,

$$\|y_{k+1,t,j} - y_{k+1,t,j-1}\| \le \epsilon_0. \tag{12.199}$$

Assume that

$$k \in \{q\bar{N}, \dots, (q+1)\bar{N} - 1\} \tag{12.200}$$

and

$$\|x_k\| \le 3M + 1 + (k - q\bar{N})\epsilon_0(\bar{q} + 1). \tag{12.201}$$

(Note that by (12.197), (12.201) is true for $k = q\bar{N}$.)

Let $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}$. By (12.140) and (12.201),

$$\|y_{k+1,t,0}\| \le 3M + 1 + (k - q\bar{N})\epsilon_0(\bar{q} + 1). \tag{12.202}$$

It follows from (12.142), (12.199), and (12.202) that for all $j = 0, \dots, p(t)$,

$$\|y_{k+1,t,j}\| \le 3M + 1 + \epsilon_0\bar{q} + \epsilon_0(\bar{q} + 1)(k - q\bar{N}),$$

$$\|y_{k+1,t}\| \le 3M + 1 + \epsilon_0(\bar{q} + 1)(k - q\bar{N}) + \epsilon_0\bar{q}.$$

In view of the relation above, (12.125), (12.135), and (12.138),

$$\begin{aligned}\|x_{k+1}\| &\le 3M + 1 + \epsilon_0(\bar{q} + 1)(k - q\bar{N}) + \epsilon_0\bar{q} + \delta \\ &\le 3M + 1 + \epsilon_0(\bar{q} + 1)(k + 1 - q\bar{N}).\end{aligned}$$

Thus by induction we show that (12.201) holds for all $k = q\bar{N}, \dots, (q+1)\bar{N}$ and that for all $k = q\bar{N}, \dots, (q+1)\bar{N}-1$, all $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}$ and all $j = 0, \dots, p(t)$,

$$\|y_{k+1,t,j}\| \le 3M + 1 + \epsilon_0(\bar{q} + 1)(k + 1 - q\bar{N}).$$

By (12.124), (12.125), and (12.201),

$$\|x_k\| \le 3M + 2,\ k = q\bar{N}, \dots, (q+1)\bar{N}, \tag{12.203}$$

$$\|y_{k+1,t,j}\| \le 3M + 2,\ k = q\bar{N}, \dots, (q+1)\bar{N} - 1,\ t \in \Omega_{k+1},\ j = 0, \dots, p(t). \tag{12.204}$$

Let

$$k \in \{q\bar{N}, \dots, (q+1)\bar{N}-1\},\ t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1},\ j \in \{1, \dots, p(t)\}. \tag{12.205}$$

We show that

$$f_{t_j}(y_{k+1,t,j-1}) \le 2M_1\epsilon_0. \tag{12.206}$$

Assume the contrary. Then

$$f_{t_j}(y_{k+1,t,j-1}) > 2M_1\epsilon_0. \tag{12.207}$$

By (12.14), (12.130), (12.141), (12.205) and (12.207), there exists

$$l \in (\partial f_{t_j}(y_{k+1,t,j-1}) + B(0,\delta)) \setminus \{0\} \tag{12.208}$$

such that

$$\|y_{k+1,t,j} - (y_{k+1,t,j-1} - f_{t_j}(y_{k+1,t,j-1})\|l\|^{-2}l)\| \le \delta. \tag{12.209}$$

By (12.121), (12.204), (12.205) and (12.208),

$$\|l\| \le M_1. \tag{12.210}$$

It follows from (12.199), (12.205), (12.207), (12.209), and (12.210) that

$$\epsilon_0 \ge \|y_{k+1,t,j} - y_{k+1,t,j-1}\|$$
$$\ge f_{t_j}(y_{k+1,t,j-1})\|l\|^{-1} - \delta > 2M_1\epsilon_0 M_1^{-1} - \delta > 2\epsilon_0 - \delta.$$

This contradicts (12.125). The contradiction we have reached proves that (12.206) holds. Thus the following property holds:

(P12) for each $k \in \{q\bar{N}, \dots, (q+1)\bar{N}-1\}$, each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}$ and each $j \in \{1, \dots, p(t)$, (12.206) holds.

By (12.140), (12.142), and (12.199), for each $k \in \{q\bar{N}, \dots, (q+1)\bar{N}-1\}$, each $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}$ and each $j \in \{0, 1, \dots, p(t)$,

$$\|x_k - y_{k+1,t,j}\| \le \epsilon_0\bar{q}, \tag{12.211}$$

$$\|x_k - y_{k+1,t}\| \le \epsilon_0\bar{q}. \tag{12.212}$$

In view of (12.125), (12.138), and (12.212),

$$\|x_k - x_{k+1}\| \le \epsilon_0 q + \delta \le \epsilon_0(\bar{q}+1). \tag{12.213}$$

It follows from (12.213) that for each $k_1, k_2 \in \{q\bar{N}, \dots, (q+1)\bar{N}\}$

$$\|x_{k_1} - x_{k_2}\| \le \epsilon_0(\bar{q}+1)\bar{N}. \tag{12.214}$$

Let

$$s \in \{1, \dots, m\},\ i \in \{q\bar{N}, \dots, (q+1)\bar{N}\}. \tag{12.215}$$

By (12.128), there exist

$$k \in \{q\bar{N}, \dots, (q+1)\bar{N} - 1\},\ t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1},\ j \in \{1, \dots, p(t)\} \tag{12.216}$$

such that

$$s = t_j. \tag{12.217}$$

By (12.121), (12.203), (12.204), (12.211), (12.216), (12.217), and (P12),

$$\begin{aligned} f_s(x_k) &\le f_{t_j}(y_{k+1,t,j-1}) + |f_{t_j}(x_k) - f_{t_j}(y_{k+1,t,j-1})| \\ &\le 2M_1\epsilon_0 + M_1\|x_k - y_{k+1,t,j-1}\| \le 2M_1\epsilon_0 + M_1\epsilon_0\bar{q}. \end{aligned} \tag{12.218}$$

In view of (12.201), (12.203), (12.214)–(12.216), and (12.218),

$$\begin{aligned} f_s(x_i) &\le f_s(x_k) + |f_s(x_i) - f_s(x_k)| \le 2M_1\epsilon_0 + M_1\epsilon_0\bar{q} + M_1\|x_k - x_i\| \\ &\le M_1\epsilon_0(\bar{q}+2) + M_1\epsilon_0(\bar{q}+1)\bar{N} \le M_1\epsilon_0(\bar{q}+2)(\bar{N}+1). \end{aligned}$$

Theorem 12.2 is proved. □

12.5 The Third Main Result

Let m be a natural number and $f_i : X \to R^1$, $i = 1, \dots, m$ be convex continuous functions.

For each $i = 1, \dots, m$ set

$$C_i = \{x \in X :\ f_i(x) \le 0\},$$
$$C = \cap_{i=1}^m C_i = \cap_{i=1}^m \{x \in X :\ f_i(x) \le 0\}.$$

Suppose that

$$C \ne \emptyset.$$

In this section we state the third main result of this chapter under the assumptions that the set C is bounded and that

$$\{x \in X : f_i(x) < 0, \ i = 1, \dots, m\} \neq \emptyset.$$

This result will be proved in the next section.

We use the notation and definitions introduced in Sects. 12.1 and 12.3.

Suppose that

$$\{x \in X : f_i(x) < 0, \ i = 1, \dots, m\} \neq \emptyset.$$

Therefore there exists $\Delta \in (0, 1]$ such that

$$\{x \in X : f_i(x) \leq -\Delta, \ i = 1, \dots, m\} \neq \emptyset. \tag{12.219}$$

Suppose that the set C is bounded.

Let $M > 0$, $M_0 > 0$, $M_1 > 2$ be such that

$$\{x \in X : f_i(x) \leq 1, \ i = 1, \dots, m\} \subset B(0, M), \tag{12.220}$$

$$f_i(B(0, 3M + 3)) \subset [-M_0, M_0], \ i = 1, \dots, m, \tag{12.221}$$

$$|f_i(u) - f_i(v)| \leq (M_1 - 2)\|u - v\| \text{ for all } u, v \in B(0, 3M + 4), \ i = 1, \dots, m. \tag{12.222}$$

Let $\delta \in (0, 1]$ satisfy

$$8(2(1 + \bar{q})(1 + \bar{N}))^3 \bar{\Delta}^{-1}(4M + 4)\delta(1 + 16M_0\Delta^{-2}(4M + 3)^2) \leq 1, \tag{12.223}$$

$$M_1(\bar{N} + 1)(\bar{q} + 2)(8(2(1 + \bar{q})(1 + \bar{N}))(4M + 4)\delta(1 + 16M_0\Delta^{-2}(4M + 3)^2))^{1/2} \leq 1, \tag{12.224}$$

$$\epsilon_0 = (8\delta\bar{\Delta}^{-1}(4M + 4)2(1 + \bar{q})(1 + \bar{N})(1 + 16M_0\Delta^{-2}(4M + 3)^2))^{1/2} \tag{12.225}$$

and a natural number

$$n_0 > 32M^2\epsilon_0^{-2}\bar{\Delta}^{-1}. \tag{12.226}$$

Note that all the assumptions made for Theorem 12.2 hold.

The next theorem is the third main result of the chapter.

Theorem 12.3. *Assume that*

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$$

satisfies for each natural number j,

$$\{1,\dots,m\} \subset \cup_{i=j}^{j+\bar{N}-1}(\cup_{t\in\Omega_i}\{t_1,\dots,t_{p(t)}\}), \tag{12.227}$$

$$x_0 \in B(0,M), \tag{12.228}$$

$$\epsilon \in [0, 2M_1\epsilon_0], \tag{12.229}$$

$$\{x_i\}_{i=1}^{\infty} \subset X,\ \{\lambda_i\}_{i=1}^{\infty} \subset [0,\infty)$$

satisfy for each natural number i,

$$(x_i,\lambda_i) \in A(x_{i-1},(\Omega_i,w_i),\epsilon,\delta). \tag{12.230}$$

Then there exists an integer $q_0 \in [0, n_0-1]$ such that for every integer $i \ge q_0\bar{N}$,

$$d(x_i,C) \le 2\epsilon_0 M_1 M(\bar{q}+2)(\bar{N}+1)\Delta^{-1} + (\bar{q}+1)\bar{N}\delta(1+16M_0\Delta^{-2}(4M+3)^2).$$

Note that in Theorem 12.3 δ is the computational error made by our computer system, $\epsilon_0 = c_1\delta^{1/2}$ and $n_0 = \lfloor c_2\delta^{-1}\rfloor$, where c_1 and c_2 are positive constants which do not depend on δ.

12.6 Proof of Theorem 12.3

Set

$$\delta_0 = \delta(1+16M_0\Delta^{-2}(4M+3)^2). \tag{12.231}$$

Note that by (12.224) and (12.225),

$$M_1\epsilon_0(\bar{q}+2)(\bar{N}+1) \le 1. \tag{12.232}$$

Since $C \subset B(0,M)$ we conclude that in the proof of Theorem 12.2 (see (P10) and (P11)) the following properties were established (for the sequence $\{x_{i+l}\}_{i=0}^{\infty}$):

(P12) Let $z \in C$ and an integer $l \ge 0$ be such that $\|x_l\| \le M$. Then there exists an integer $q \in [0, n_0-1]$ such that:

$$\|x_i\| \le 3M+1,\ i = l,\dots,q\bar{N}+l;$$

for each nonnegative integer $p < q$,

$$\max\{\lambda_i :\ i = p\bar{N}+l+1,\dots,(p+1)\bar{N}+l\} > \epsilon_0,$$

$$\max\{\lambda_i :\ i = q\bar{N}+l+1,\dots,(q+1)\bar{N}+l\} \le \epsilon_0,$$

$$\|x_i\| \le 3M+2,\ i = q\bar{N}+l,\dots,(q+1)\bar{N}+l,$$

for each integer p satisfying $0 \le p < q$,

$$\|z - x_{p\bar{N}+l}\|^2 - \|z - x_{(p+1)\bar{N}+l}\|^2 \ge \bar{\Delta}\epsilon_0^2/8; \tag{12.233}$$

for every $i \in \{l, \dots, l + q\bar{N} - 1\}$,

$$\|x_{i+1} - z\| \le \|x_i - z\| + \delta_0(q + 1) \tag{12.234}$$

and

(P13) Assume that an integer $q \ge 0$, an integer $l \ge 0$, $\|x_l\| \le M$,

$$\|x_{q\bar{N}+l}\| \le 3M + 1,$$
$$\lambda_i \le \epsilon_0,\ i = l + q\bar{N} + 1, \dots, (q + 1)\bar{N} + l.$$

Then

$$f_s(x_i) \le M_1\epsilon_0(\bar{q} + 2)(\bar{N} + 1),\ i \in \{l + q\bar{N}, \dots, (q + 1)\bar{N} + l\},\ s \in \{1, \dots, m\}. \tag{12.235}$$

Assume that an integer $l \ge 0$ be such that $\|x_l\| \le M$ and let an integer $q \in [0, n_0 - 1]$ be as guaranteed by property (P12) which in its turn implies in view of (12.233) that for each integer p satisfying $0 \le p < q$,

$$d(x_{(p+1)\bar{N}+l}, C) \le d(x_{p\bar{N}+l}, C),$$
$$d(x_{p\bar{N}+l}, C) \le d(x_l, C) \text{ for all } p = 0, \dots, q$$

and in view of (12.134),

$$d(x_i, C) \le d(x_l, C) + \delta_0\bar{N}(\bar{q} + 1) \text{ for all } i = l, \dots, l + q\bar{N}. \tag{12.236}$$

By the choice of q and (P13), for every $i \in \{l + q\bar{N}, \dots, (q + 1)\bar{N} + l\}$ and every $s \in \{1, \dots, m\}$, (12.235) is true. Together with (12.232), (12.235), and Lemma 11.8 this implies that for all integers $i \in \{l + q\bar{N}, \dots, l + (q + 1)\bar{N}\}$,

$$\begin{aligned} d(x_i, C) &\le 2M(M_1\epsilon_0(\bar{q} + 2)(\bar{N} + 1))(M_1\epsilon_0(\bar{q} + 2)(\bar{N} + 1) + \Delta)^{-1} \\ &\le 2M_1M\epsilon_0(\bar{q} + 2)(\bar{N} + 1)\Delta^{-1}. \end{aligned} \tag{12.237}$$

By (12.210), (12.232), and (12.235),

$$x_i \in B(0, M),\ i = l + q\bar{N}, \dots, (q + 1)\bar{N} + l. \tag{12.238}$$

Thus we have shown that the following property holds:

(P14) Let an integer $l \geq 0$ be such that $\|x_l\| \leq M$ and let an integer $q \in [0, n_0 - 1]$ be as guaranteed by (P12). Then (12.235), (12.236) are valid and for all $i = q\bar{N} + l, \dots, (q+1)\bar{N} + l$, (12.237), and (12.238) hold.

By (12.228), there is an integer $q_0 \in [0, n_0 - 1]$ such that (P12) holds with $l = 0$ and $q = q_0$. By the choice of q_0, (P12), (P14), (12.235), and (12.237),

$$f_s(x_i) \leq M_1\epsilon_0(\bar{q} + 2)(\bar{N} + 1),\ i = q_0\bar{N}, \dots, (q_0 + 1)\bar{N},\ s = 1, \dots, m, \tag{12.239}$$

$$d(x_i, C) \leq 2M_1M\epsilon_0(\bar{q} + 2)(\bar{N} + 1)\Delta^{-1} \text{ for all integers } i = q_0\bar{N}, \dots, (q_0 + 1)\bar{N}. \tag{12.240}$$

We show that

$$d(x_i, C) \leq 2M_1M\epsilon_0(\bar{q} + 2)\Delta^{-1}(\bar{N} + 1) + \delta_0\bar{N}(\bar{q} + 1) \text{ for all integers } i \geq q_0\bar{N}. \tag{12.241}$$

Assume the contrary. Then there is an integer

$$p > q_0\bar{N} \tag{12.242}$$

such that

$$d(x_p, C) > 2M_1M\epsilon_0(\bar{q} + 2)(\bar{N} + 1)\Delta^{-1} + \delta_0\bar{N}(\bar{q} + 1). \tag{12.243}$$

By (12.240), (12.242), and (12.243),

$$p > (q_0 + 1)\bar{N}. \tag{12.244}$$

We may assume without loss of generality that

$$d(x_i, C) \leq 2M_1M\epsilon_0(\bar{q} + 2)(\bar{N} + 1)\Delta^{-1} + \delta_0(\bar{q} + 1)\bar{N} \tag{12.245}$$
$$\text{for all integers } i \text{ satisfying } (q_0 + 1)\bar{N} \leq i < p.$$

By (12.239) and (12.244), there exists an integer l_0 such that

$$(q_0 + 1)\bar{N} \leq l_0 \leq p, \tag{12.246}$$

$$f_s(x_{l_0}) \leq M_1\epsilon_0(\bar{q} + 2)(\bar{N} + 1),\ s = 1, \dots, m, \tag{12.247}$$

$$\max\{f_s(x_i) :\ s = 1, \dots, m\} > M_1\epsilon_0(\bar{q} + 2)(\bar{N} + 1) \tag{12.248}$$
$$\text{for all integers } i \text{ satisfying } l_0 < i \leq p.$$

By (12.247) and Lemma 11.8,

$$d(x_{l_0}, C) \leq 2MM_1\epsilon_0(\bar{q} + 2)(\bar{N} + 1)\Delta^{-1}. \tag{12.249}$$

By (12.246), (12.243), and (12.249),

$$l_0 < p.$$

In view of (12.220), (12.232), and (12.247),

$$\|x_{l_0}\| \leq M. \tag{12.250}$$

Let an integer $q \in [0, n_0 - 1]$ be as guaranteed by (P12) with $l = l_0$. By (12.235), (12.236), (12.249), and (P14), for all $i = l_0, \ldots, l_0 + (q+1)\bar{N}$,

$$d(x_i, C) \leq 2MM_1\epsilon_0(\bar{q} + 2)(\bar{N} + 1)\Delta^{-1} + \delta_0\bar{N}(\bar{q} + 1), \tag{12.251}$$

$$f_s(x_i) \leq M_1\epsilon_0(\bar{q} + 2)(\bar{N} + 1),\ s = 1, \ldots, m,\ i = l_0 + q\bar{N}, \ldots, l_0 + (q+1)\bar{N}. \tag{12.252}$$

By (12.243) and (12.251),

$$p > l_0 + (q+1)\bar{N}.$$

The equation above and (12.252) contradict (12.248).

The contradiction we have reached proves that (12.241). Theorem 12.3 is proved. □

References

1. Aleyner, A., & Reich, S. (2008). Block-iterative algorithms for solving convex feasibility problems in Hilbert and Banach spaces. *Journal of Mathematical Analysis and Applications, 343*, 427–435.
2. Allevi, E., Gnudi, A., & Konnov, I. V. (2006). The proximal point method for nonmonotone variational inequalities. *Mathematical Methods of Operations Research, 63*, 553–565.
3. Alsulami, S. M., & Takahashi, W. (2015). Iterative methods for the split feasibility problem in Banach spaces. *Journal of Nonlinear and Convex Analysis, 16*, 585–596.
4. Bacak, M. (2012). Proximal point algorithm in metric spaces. *Israel Journal Mathematics, 160*, 1–13.
5. Bauschke, H. H. (1995). A norm convergence result on random products of relaxed projections in Hilbert space. *Transactions of the American Mathematical Society, 347*, 1365–1373.
6. Bauschke, H. H., & Borwein, J. M. (1993). On the convergence of von Neumann's alternating projection algorithm for two sets. *Set-Valued Analysis, 1*, 185–212.
7. Bauschke, H. H., & Borwein, J. M. (1996). On projection algorithms for solving convex feasibility problems. *SIAM Review, 38*, 367–426.
8. Bauschke, H. H., & Borwein, J. M., & Combettes, P. L. (2003). Bregman monotone optimization algorithms. *SIAM Journal on Control and Optimization, 42*, 596–636.
9. Bauschke, H. H., Borwein, J. M., Wang, X., & Yao, L. (2012). Construction of pathological maximally monotone operators on non-reflexive Banach spaces. *Set-Valued and Variational Analysis, 20*, 387–415.
10. Bauschke, H. H., Deutsch, F., Hundal, H., & Park, S.-H. (2003). Accelerating the convergence of the method of alternating projections. *Transactions of the American Mathematical Society, 355*, 3433–3461.
11. Bauschke, H. H., Goebel, R., Lucet, Y., & Wang, X. (2008). The proximal average: Basic theory. *SIAM Journal on Optimization, 19*, 766–785.
12. Bauschke, H. H., & Koch, V. R. (2015). Projection methods: Swiss army knives for solving feasibility and best approximation problems with halfspaces. *Contemporary Mathematics, 636*, 1–40.
13. Bauschke, H. H., Matoušková, E., & Reich, S. (2004). Projection and proximal point methods: Convergence results and counterexamples. *Nonlinear Analysis, 56*, 715–738.
14. Bauschke, H., Moffat, S., & Wang, X. (2012). Firmly nonexpansive mappings and maximally monotone operators: Correspondence and duality. *Set-Valued and Variational Analysis, 20*, 131–153.

A. Zaslavski, *Approximate Solutions of Common Fixed-Point Problems*, Springer Optimization and Its Applications 112, DOI 10.1007/978-3-319-33255-0

15. Bregman, L. M. (1967). A relaxation method of finding a common point of convex sets and its application to the solution of problems in convex programming. *Zhurnal Vychislitel'noi Matematiki i Matematicheskoi Fiziki, 7*, 620–631.
16. Bruck, R. E. (2010). On the random product of orthogonal projections in Hilbert space II. *Contemporary Mathematics, 513*, 65–98.
17. Burachik, R. S., & Dutta, J. (2010). Inexact proximal point methods for variational inequality problems. *SIAM Journal on Optimization, 20*, 2653–2678.
18. Burachik, R. S., & Iusem, A. N. (1998). A generalized proximal point algorithm for the variational inequality problem in a Hilbert space. *SIAM Journal on Optimization, 8*, 197–216.
19. Burachik, R. S., Iusem, A. N., & Svaiter, B. F. (1997). Enlargement of monotone operators with applications to variational inequalities. *Set-Valued Analysis, 5*, 159–180.
20. Burachik, R. S., Lopes, J. O., & Da Silva, G. J. P. (2009). An inexact interior point proximal method for the variational inequality. *Computational and Applied Mathematics, 28*, 15–36.
21. Burachik, R. S., & Scheimberg, S. (2001). A proximal point method for the variational inequality problem in Banach spaces. *SIAM Journal on Control and Optimization, 39*, 1633–1649.
22. Butnariu, D., Censor, Y., & Reich, S. (Ed.). (2001). *Inherently parallel algorithms in feasibility and optimization and their applications*. Amsterdam: Elsevier Science Publishers.
23. Butnariu, D., Davidi, R., Herman, G. T., & Kazantsev, I. G. (2007). Stable convergence behavior under summable perturbations of a class of projection methods for convex feasibility and optimization problems. *IEEE Journal of Selected Topics in Signal Processing, 1*, 540–547.
24. Butnariu, D., & Iusem, A. N. (2000). *Totally convex functions for fixed points computation and infinite dimensional optimization*. Dordrecht: Kluwer.
25. Butnariu, D., & Kassay, G. (2008). A proximal-projection method for finding zeros of set-valued operators. *SIAM Journal on Control and Optimization, 47*, 2096–2136.
26. Butnariu, D., Reich, S., & Zaslavski, A. J. (2008). Stable convergence theorems for infinite products and powers of nonexpansive mappings. *Numerical Functional Analysis and Optimization, 29*, 304–323.
27. Cegielski, A. (2012). *Iterative methods for fixed point problems in Hilbert spaces. Lecture notes in mathematics* (Vol. 2057). Berlin/Heidelberg: Springer.
28. Ceng, L. C., Hadjisavvas, N., & Wong, N. C. (2010). Strong convergence theorem by a hybrid extragradient-like approximation method for variational inequalities and fixed point problems. *Journal of Global Optimization, 46*, 635–646.
29. Ceng, L. C., Mordukhovich, B. S., & Yao, J. C. (2010). Hybrid approximate proximal method with auxiliary variational inequality for vector optimization. *Journal of Optimization Theory and Applications, 146*, 267–303.
30. Censor, Y. (1981). Row-action methods for huge and sparse systems and their applications. *SIAM Review, 23*, 444–466.
31. Censor, Y., & Cegielski, A. (2015). Projection methods: An annotated bibliography of books and reviews. *Optimization, 64*, 2343–2358.
32. Censor, Y., Chen, W., Combettes, P. L., Davidi, R., & Herman, G. T. (2012). On the effectiveness of projection methods for convex feasibility problems with linear inequality constraints. *Computational Optimization and Applications, 51*, 1065–1088.
33. Censor, Y., Davidi, R., & Herman, G. T. (2010). Perturbation resilience and superiorization of iterative algorithms. *Inverse Problems, 26*, 1–12.
34. Censor, Y., Elfving, T., & Herman, G. T. (2001). Averaging strings of sequential iterations for convex feasibility problems. In D. Butnariu, Y. Censor, & S. Reich, (Eds.), *Inherently parallel algorithms in feasibility and optimization and their applications* (pp. 101–113). Amsterdam: North-Holland.
35. Censor, Y., Elfving, T., Herman, G. T., & Nikazad, T. (2008). Diagonally-relaxed orthogonal projection methods. *SIAM Journal on Scientific Computing, 30*, 473–504.

36. Censor, Y., Gibali, A., & Reich, S. (2011). The subgradient extragradient method for solving variational inequalities in Hilbert space. *Journal of Optimization Theory and Applications, 148*, 318–335.
37. Censor, Y., & Lent, A. (1982). Cyclic subgradient projections. *Mathematical Program, 24*, 233–235.
38. Censor, Y., & Segal, A. (2009). On the string averaging method for sparse common fixed point problems. *International Transactions in Operational Research, 16*, 481–494.
39. Censor, Y., & Segal, A. (2010). On string-averaging for sparse problems and on the split common fixed point problem. *Contemporary Mathematics, 513*, 125–142.
40. Censor, Y., & Tom, E. (2003). Convergence of string-averaging projection schemes for inconsistent convex feasibility problems. *Optimization Methods and Software, 18*, 543–554.
41. Censor, Y., & Zaslavski, A. J. (2013). Convergence and perturbation resilience of dynamic string-averaging projection methods. *Computational Optimization and Applications, 54*, 65–76.
42. Censor, Y., & Zenios, S. A. (1992). The proximal minimization algorithm with D-functions. *Journal of Optimization Theory and Applications, 73*, 451–464.
43. Censor, Y., & Zenios, S. (1997). *Parallel optimization: Theory, algorithms and applications*. New York: Oxford University Press.
44. Chuong, T. D., Mordukhovich, B. S., & Yao, J. C. (2011). Hybrid approximate proximal algorithms for efficient solutions in for vector optimization. *Journal of Nonlinear Convex Analysis, 12*, 861–864.
45. Cimmino, G. (1938). Calcolo approssimato per le soluzioni dei sistemi di equazioni lineari. *La Ricerca Scientifica (Roma), 1*, 326–333.
46. Combettes, P. L. (1996). The convex feasibility problem in image recovery. *Advances in Imaging and Electron Physics, 95*, 155–270.
47. Combettes, P. L. (1997). Hilbertian convex feasibility problems: Convergence of projection methods. *Applied Mathematics and Optimization, 35*, 311–330.
48. Crombez, G. (2002). Finding common fixed points of strict paracontractions by averaging strings of sequential iterations. *Journal of Nonlinear and Convex Analysis, 3*, 345–351.
49. De Pierro, A. R., & Iusem, A. N. (1988). A finitely convergent row-action method for the convex feasibility problem. *Applied Mathematics and Optimization, 17*, 225–235.
50. Ekeland, I., & Temam, R. (1976). *Convex analysis and variational problems*. Amsterdam/Oxford: North-Holland.
51. Facchinei, F., & Pang, J. S. (2003). *Finite-dimensional variational inequalities and complementarity problems* (Vols. I and II). New York: Springer.
52. Flam, S. D., & Zowe, J. (1990). Relaxed outer projections, weighted averages and convex feasibility. *BIT, 30*, 289–300.
53. Gordon, D., & Gordon, R. (2005). Component-averaged row projections: A robust block-parallel scheme for sparse linear systems. *SIAM Journal on Scientific Computing, 27*, 1092–117.
54. Gubin, L. G., Polyak, B. T., & Raik, E. V. (1967). The method of projections for finding the common point of convex sets. *USSR Computational Mathematics and Mathematical Physics, 7*, 1–24.
55. Gwinner, J., & Raciti, F. (2009). On monotone variational inequalities with random data. *Journal of Mathematical Inequalities, 3*, 443–453.
56. Hager, W. W., & Zhang, H. (2007). Asymptotic convergence analysis of a new class of proximal point methods. *SIAM Journal on Control and Optimization, 46*, 1683–1704.
57. Huebner, E., & Tichatschke, R. (2008). Relaxed proximal point algorithms for variational inequalities with multi-valued operators. *Optimization Methods and Software, 23*, 847–877.
58. Iusem, A., & Nasri, M. (2007). Inexact proximal point methods for equilibrium problems in Banach spaces. *Numerical Functional Analysis and Optimization, 28*, 1279–1308.
59. Iusem, A., & Resmerita, E. (2010). A proximal point method in nonreflexive Banach spaces. *Set-Valued and Variational Analysis, 18*, 109–120.

60. Kaplan, A., & Tichatschke, R. (2007). Bregman-like functions and proximal methods for variational problems with nonlinear constraints. *Optimization, 56*, 253–265.
61. Kassay, G. (1985). The proximal points algorithm for reflexive Banach spaces. *Studia Universitatis Babes-Bolyai Mathematica, 30*, 9–17.
62. Konnov, I. V. (1997). On systems of variational inequalities. *Russian Mathematics, 41*, 79–88.
63. Konnov, I. V. (2001). *Combined relaxation methods for variational inequalities*. Berlin/Heidelberg: Springer.
64. Konnov, I. V. (2006). Partial proximal point method for nonmonotone equilibrium problems. *Optimization Methods and Software, 21*, 373–384.
65. Konnov, I. V. (2008). Nonlinear extended variational inequalities without differentiability: Applications and solution methods. *Nonlinear Analysis, 69*, 1–13.
66. Konnov, I. V. (2009). A descent method with inexact linear search for mixed variational inequalities. *Russian Mathematics (Iz. VUZ), 53*, 29–35.
67. Lopez, G., Martin, V., & Xu, H. K. (2010). Halpern's iteration for nonexpansive mappings. *Contemporary Mathematics, 513*, 211–230.
68. Marino, G., & Xu, H. K. (2004). Convergence of generalized proximal point algorithms. *Communications in Pure and Applied Analysis, 3*, 791–808.
69. Martinet, B. (1978). Pertubation des methodes d'optimisation: Application. *RAIRO Analyse Numérique, 12*, 153–171.
70. Minty, G. J. (1962). Monotone (nonlinear) operators in Hilbert space. *Duke Mathematical Journal, 29*, 341–346.
71. Minty, G. J. (1964). On the monotonicity of the gradient of a convex function. *Pacific Journal of Mathematics, 14*, 243–247.
72. Mordukhovich, B. S. (2006). *Variational analysis and generalized differentiation, I: Basic theory*. Berlin: Springer.
73. Moreau, J. J. (1965). Proximite et dualite dans un espace Hilbertien. *Bulletin de la Société Mathématique de France, 93*, 273–299.
74. ODHara, J. G., Pillay, P., & Xu, H. K. (2006). Iterative approaches to convex feasibility problems in Banach spaces. *Nonlinear Analysis, 64*, 2022–2042.
75. Reich, S. (1983). A limit theorem for projections. *Linear and Multilinear Algebra, 13*, 281–290.
76. Reich, S., & Zaslavski, A. J. (2014). *Genericity in nonlinear analysis*. New York: Springer.
77. Rockafellar, R. T. (1970). *Convex analysis*. Princeton: Princeton University Press.
78. Rockafellar, R. T. (1976). Augmented Lagrangians and applications of the proximal point algorithm in convex programming. *Mathematics of Operations Research, 1*, 97–116.
79. Rockafellar, R. T. (1976). Monotone operators and the proximal point algorithm. *SIAM Journal on Control and Optimization, 14*, 877–898.
80. Sahu, D. R., Wong, N. C., & Yao, J. C. (2011). A generalized hybrid steepest-descent method for variational inequalities in Banach spaces. *Fixed Point Theory and Applications, 2011*, 28.
81. Sahu, D. R., Wong, N. C., & Yao, J. C. (2012). A unified hybrid iterative method for solving variational inequalities involving generalized pseudocontractive mappings. *SIAM Journal on Control and Optimization, 50*, 2335–2354.
82. Solodov, M. V., & Svaiter, B. F. (2000). Error bounds for proximal point subproblems and associated inexact proximal point algorithms. *Mathematical Programming, 88*, 371–389.
83. Solodov, M. V., & Svaiter, B. F. (2001). A unified framework for some inexact proximal point algorithms. *Numerical Functional Analysis and Optimization, 22*, 1013–1035.
84. Takahashi, W. (2014). The split feasibility problem in Banach spaces. *Journal of Nonlinear and Convex Analysis, 15*, 1349–1355.
85. Takahashi, W. (2015). The split feasibility problem and the shrinking projection method in Banach spaces. *Journal of Nonlinear and Convex Analysis, 16*, 1449–1459.
86. Takahashi, W., & Iiduka, H. (2008). Weak convergence of a projection algorithm for variational inequalities in a Banach space. *Journal of Mathematical Analysis and Applications, 339*, 668–679.

87. Verma, R. U. (2010). New approach to the eta-proximal point algorithm and nonlinear variational inclusion problems. *Applied Mathematics and Computation, 217*, 3155–3165.
88. Xu, H. K. (2006). A regularization method for the proximal point algorithm. *Journal of Global Optimization, 36*, 115–125.
89. Xu, H. K. (2010). Iterative methods for the split feasibility problem in infinite-dimensional Hilbert spaces. *Inverse Problems, 26*, 1–17.
90. Xu, H. K., & Kim, T. H. (2003). Convergence of hybrid steepest descent methods for variational inequalities. *Journal of Optimization Theory and Applications, 119*, 184–201.
91. Zaslavski, A. J. (2010). Convergence of a proximal method in the presence of computational errors in Hilbert spaces. *SIAM Journal on Optimization, 20*, 2413–2421.
92. Zaslavski, A. J. (2011). Maximal monotone operators and the proximal point algorithm in the presence of computational errors. *Journal of Optimization theory and Applications, 150*, 20–32.
93. Zaslavski, A. J. (2012). Convergence of projection algorithms to approximate solutions of convex feasibility problems. *Communications on Applied Nonlinear Analysis, 19*, 99–105.
94. Zaslavski, A. J. (2012). Proximal point algorithm for finding a common zero of a finite family of maximal monotone operators in the presence of computational errors. *Nonlinear Analysis, 75*, 6071–6087.
95. Zaslavski, A. J. (2012). Solving for (approximate) convex feasibility under finite precision. *Nonlinear Studies, 19*, 653–660.
96. Zaslavski, A. J. (2013). Subgradient projection algorithms and approximate solutions of convex feasibility problems. *Journal of Optimization Theory and Applications, 157*, 803–819.
97. Zaslavski, A. J. (2013). Subgradient projection algorithms for convex feasibility problems in the presence of computational errors. *Journal of Approximation Theory, 175*, 19–42.
98. Zaslavski, A. J. (2014). Dynamic string-averaging projection methods for convex feasibility problems in the presence of computational errors. *Journal of Nonlinear and Convex Analysis, 15*, 1–14.
99. Zaslavski, A. J. (2014). Approximate solutions of common fixed point problems. *Communications on Applied Nonlinear Analysis, 22*, 80–89.

Index

A. Zaslavski, *Approximate Solutions of Common Fixed-Point Problems*, Springer Optimization and Its Applications 112, DOI 10.1007/978-3-319-33255-0