Chapter 8
Robust-Soft Solutions in Linear Optimization Problems with Fuzzy Parameters

Masahiro Inuiguchi

Abstract Linear optimization problems with fuzzy parameters were studied deeply and widely. Many of the approaches to fuzzy problems generate robust solutions. However, they were based on satisficing approaches so that the solutions do not maintain the optimality or suboptimality against the fluctuations in the coefficients. In this chapter, we describe a robust solution maintaining the suboptimality against the fluctuations in the coefficients. We formulate the problem as an extension of the minimax regret/maximin achievement rate problem and investigate a solution procedure based on a bisection method and a relaxation method. It is shown that the proposed solution procedure is created well so that both bisection and relaxation methods converge simultaneously.

8.1 Introduction

Due to the limit of available information, decision making problems often involve uncertainties. Traditionally two kinds of decision making problems under uncertainty have been studied: decision making problems under strict uncertainty and decision making problems under risk (see for example, [5]). In the former problems, the uncertainty is modelled by a set of possible situations where we do not know which situation is more probable than the others. In the latter problem, the uncertainty is modelled by a probability distribution. As an optimization technique under uncertainty, stochastic programming [18–20] were investigated. It treats decision

M. Inuiguchi (✉)
Graduate School of Engineering Science, Osaka University, 1-3 Machikaneyama-cho, Toyonaka, Osaka, Japan
e-mail: inuiguti@sys.es.osaka-u.ac.jp

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making problems under risk. However, recently, robust optimization [2] treating a kind of decision making problems under strict uncertainty is proposed and getting popular. In robust optimization, a solution which maintains the feasibility or the sub-optimality against the parameter fluctuation in the given range is computed. Because of this property, the solution is considered a safe decision.

As a non-traditional model of uncertainty, fuzzy set theory [4, 21] was proposed and introduced into various fields. By fuzzy set theory, we can treat the vague restriction and goals of the decision maker (DM) on constraints and objectives as well as ambiguous coefficients in optimization problems [10, 19]. As we may treat the plausibility degree of a state of nature by a fuzzy set, we can formulate intermediate problems between decision making problems under strict uncertainty and decision making problems under risk. Fuzzy mathematical programming problems [10, 19] have been formulated so as to find a solution balanced between DM's aspiration and the robustness. Those formulations are based on a satisficing approach. Namely, the solution satisfies the given constraints and goals with a certain level of parameter fluctuation and is one of the best solutions in the balance between the robustness of given constraints and the possibility of achieving goals. However, the robustness in the sense that its objective function value is kept close to the optimal value against parameter fluctuation is not always high.

In this chapter, we describe optimizing approaches to linear programming problems with fuzzy objective function coefficients. An optimizing approach implies the formulations and solution methods obtaining robust solutions in the sense that their objective function values are kept close to the optimal value against parameter fluctuation. We introduce mainly two robust optimization approaches under softness: minimax regret type and maximin achievement rate type.

This chapter is organized as follows. In next section, blind spots in fuzzy programming approaches are shown by simple numerical examples. Two optimal solution concepts are given. We describe the weakness of those optimal solution concepts. In Sect. 8.3, solution concepts based on optimization approaches are described. Robust-soft optimal solutions maintaining suboptimality against the fluctuation in coefficients are defined in two ways. Solution algorithm under given fuzzy goals is investigated in Sect. 8.4. An acceleration technique in solving the subproblem is described in Sect. 8.5. In Sect. 8.6, a simpler solution algorithm is shown when fuzzy goals are not specified. Finally, in Sect. 8.7, we give some concluding remarks.

8.2 Blind Spots in Fuzzy Programming Approaches

8.2.1 Linear Program with Fuzzy Objective Function Coefficients

We treat the following linear programming problems with fuzzy objective function coefficients:

$$\text{maximize } \mathbf{c}^\top \mathbf{x}, \quad \text{subject to } \mathbf{x} \in F, \tag{8.1}$$

where $F = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} \le \mathbf{b}\}$ is bounded. $\mathbf{A} = (a_{ij})$ is an $m \times n$ constant matrix and $\mathbf{b} = (b_1, \ldots, b_m)^\top$. $\mathbf{x} = (x_1, \ldots, x_n)^\top$ is a decision variable vector. On the other hand, objective coefficient vector $\mathbf{c} = (c_1, c_2, \ldots, c_n)^\top$ is not known precisely but imprecisely. Namely, $\mathbf{c}$ takes a value in a possible range expressed by a bounded fuzzy set Γ of $\mathbb{R}^n$ with a membership function,

$$\mu_\Gamma(\mathbf{r}) = \min_{j=1,2,\ldots,p} \mu_{\Gamma_j}(\mathbf{d}_j^\top \mathbf{r}). \tag{8.2}$$

$\mathbf{r} \in \mathbb{R}^n$, Γ_j is a fuzzy number, i.e., a normal ($\exists r_j,\ \mu_{\Gamma_j}(r_j) = 1$), convex ($\mu_{\Gamma_j}$ is quasi-concave) and bounded fuzzy set on real line ($\lim_{r_j \to \infty} \mu_{\Gamma_j}(r_j) = \lim_{r_j \to -\infty} \mu_{\Gamma_j}(r_j) = 0$) with upper semi-continuous membership function μ_{Γ_j}. $\mathbf{d}_j \in \mathbb{R}^n$ is a constraint vector. The boundedness of Γ implies $p \ge n$, in other words, $\mathrm{rank}\{\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_p\} = n$. Membership grade $\mu_\Gamma(\mathbf{r})$ can be understood as the possibility degree that $\mathbf{c} = \mathbf{r}$.

We define h-level sets and strong h-level sets by

$$[\Gamma]_h = \{\mathbf{r} \in \mathbb{R}^n \mid \mu_\Gamma(\mathbf{r}) \ge h\},\ [\Gamma_j]_h = \{\mathbf{r} \in \mathbb{R} \mid \mu_{\Gamma_j}(r) \ge h\},\ j = 1,2,\ldots,p, \tag{8.3}$$
$$(\Gamma)_h = \{\mathbf{r} \in \mathbb{R}^n \mid \mu_\Gamma(\mathbf{r}) > h\},\ (\Gamma_j)_h = \{\mathbf{r} \in \mathbb{R} \mid \mu_{\Gamma_j}(r) > h\},\ j = 1,2,\ldots,p. \tag{8.4}$$

We have

$$\begin{aligned} [\Gamma]_h &= \left\{\mathbf{r} \in \mathbb{R}^n \mid \mathbf{d}_j^\top \mathbf{r} \in [\Gamma_j]_h,\ j = 1,2,\ldots,p\right\} \\ &= \left\{\mathbf{r} \in \mathbb{R}^n \mid \inf[\Gamma_j]_h \le \mathbf{d}_j^\top \mathbf{r} \le \sup[\Gamma_j]_h,\ j = 1,2,\ldots,p\right\}, \end{aligned} \tag{8.5}$$

$$\begin{aligned} \mathrm{cl}(\Gamma)_h &= \left\{\mathbf{r} \in \mathbb{R}^n \mid \mathbf{d}_j^\top \mathbf{r} \in \mathrm{cl}(\Gamma_j)_h,\ j = 1,2,\ldots,p\right\} \\ &= \left\{\mathbf{r} \in \mathbb{R}^n \mid \inf(\Gamma_j)_h \le \mathbf{d}_j^\top \mathbf{r} \le \sup(\Gamma_j)_h,\ j = 1,2,\ldots,p\right\}, \end{aligned} \tag{8.6}$$

where $\mathrm{cl}X$ is the closure of a set $X \subseteq \mathbb{R}^n$. An h-level set $[\Gamma]_h$ is depicted in Fig. 8.1.

Given a solution $\mathbf{x} \ne \mathbf{0}$, by the extension principle, its objective function value is given as a fuzzy set $Y(\mathbf{x})$ having the following membership function $\mu_{Y(\mathbf{x})}$:

$$\mu_{Y(\mathbf{x})}(y) = \sup_{\mathbf{c}}\{\mu_\Gamma(\mathbf{c}) : \mathbf{c}^\top \mathbf{x} = y\}. \tag{8.7}$$

Note that we have $Y(\mathbf{0}) = \{0\}$.

8.2.2 Solution Comparison by Objective Function Values

To treat linear programming problems with fuzzy coefficients, necessity measure N and possibility measure Π of a fuzzy set S are defined by

$$N_Q(S) = \inf_r \max(1 - \mu_Q(r), \mu_S(r)),\ \ \Pi_Q(S) = \sup_r \min(\mu_Q(r), \mu_S(r)), \tag{8.8}$$

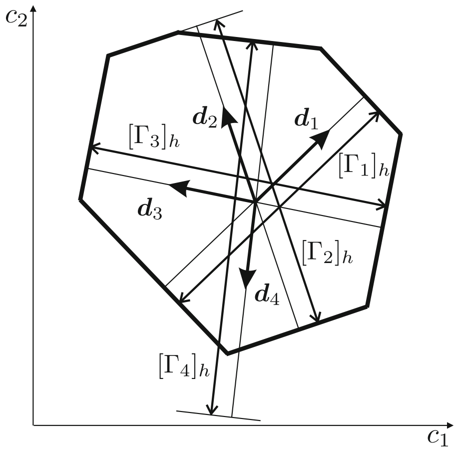

Fig. 8.1: h-Level set $[\Gamma]_h$

where μ_S and μ_Q are membership function of S and Q, respectively. $N_Q(S)$ and $\Pi_Q(S)$ evaluate to what extent the vague event expressed by fuzzy set S is necessary (certain) and possible under the possible range expressed by fuzzy set Q, respectively.

There are various ways to compare two fuzzy numbers Z_1 and $Z_2 \subseteq \mathbb{R}$. The following two indices are often used in the literature:

$$\mathrm{POS}(Z_1 \geq Z_2) = \sup_{r_1,r_2}\{\min(\mu_{Z_1}(r_1), \mu_{Z_2}(r_2)) : r_1 \geq r_2\}, \tag{8.9}$$

$$\mathrm{NES}(Z_1 \geq Z_2) = 1 - \sup_{r_1,r_2}\{\min(\mu_{Z_1}(r_1), \mu_{Z_2}(r_2)) : r_1 < r_2\}. \tag{8.10}$$

where μ_{Z_1} and μ_{Z_2} are membership functions of Z_1 and Z_2, and Z_1 and Z_2 are considered possible ranges of ambiguous numbers ζ_1 and ζ_2. Namely, we have a fuzzy set $Z_1 \times Z_2 \subseteq \mathbb{R}^2$ showing the possible ranges of (ζ_1, ζ_2) defined by a membership function $\mu_{Z_1 \times Z_2}(r_1, r_2) = \min(\mu_{Z_1}(r_1), \mu_{Z_2}(r_2))$. As an event we consider "ζ_1 is not smaller than ζ_2" which can be represented by a set (a binary relation) $\geq = \{(r_1, r_2) \in \mathbb{R}^2 \mid r_1 \geq r_2\}$. Then we have $\mathrm{POS}(Z_1 \geq Z_2) = \Pi_{Z_1 \times Z_2}(\geq)$ and $\mathrm{NES}(Z_1 \geq Z_2) = N_{Z_1 \times Z_2}(\geq)$. Namely, possibility degree $\mathrm{POS}(Z_1 \geq Z_2)$ shows to what extent Z_1 is possibly larger than or equal to Z_2. Similarly, Necessity degree $\mathrm{NES}(Z_1 \geq Z_2)$ shows to what extent Z_1 is necessarily larger than or equal to Z_2.

When Z_1 and Z_2 are closed intervals $[z_1^{\mathrm{L}}, z_1^{\mathrm{R}}]$ and $[z_2^{\mathrm{L}}, z_2^{\mathrm{R}}]$, respectively, we have

$$\mathrm{POS}(Z_1 \geq Z_2) = 1 \Leftrightarrow z_1^{\mathrm{R}} \geq z_2^{\mathrm{L}}, \qquad \mathrm{NES}(Z_1 \geq Z_2) = 0 \Leftrightarrow z_1^{\mathrm{L}} < z_2^{\mathrm{R}}. \tag{8.11}$$

Those equivalences remarkably show their meanings and difference.

A comparison index between two fuzzy numbers is often applied to the comparison of fuzzy objective function values discarding their interaction in literature. Next example demonstrates the inadequacy caused by the desertion of the interaction.

Example 1. Let $n = 2$ and $\Gamma = [1,2] \times [-2,-1]$. Namely, we consider a case when each objective function coefficient is given by a closed interval. Consider two feasible solutions $\mathbf{x}^1 = (2,1)^\top$ and $\mathbf{x}^2 = (3,1)^\top$. We have $Y(\mathbf{x}^1) = [0,3]$ and $Y(\mathbf{x}^2) = [1,5]$.

Let us apply the first equation of (8.11) discarding the interaction between $Z_1 = Y(\mathbf{x}^1)$ and $Z_2 = Y(\mathbf{x}^2)$. We obtain POS($Z_1 \geq Z_2$)= 1 which implies that the objective function value of $\mathbf{x}^1$ can be larger than or equal to that of $\mathbf{x}^2$. On the other hand, we have

$$\mathbf{c}^\top\mathbf{x}^1 = 2c_1 + c_2 < 3c_1 + c_2 = \mathbf{c}^\top\mathbf{x}^2,\ \ \forall c_1 \in [1,2],\ \forall c_2 \in [-2,-1]. \tag{8.12}$$

This insists that the objective function value of $\mathbf{x}^1$ can never be larger than or equal to that of $\mathbf{x}^2$. Because the realized values of c_1 and c_2 are common independent on the selection of a feasible solution of Problem (8.1), the latter result is correct. Therefore, the direct application of index POS($Z_1 \geq Z_2$) is not adequate for the problem setting.

Similarly, from the second equation of (8.11), we obtain NES($Z_2 \geq Z_1$)= 0. This implies that there exists $(c_1,c_2)^\top \in \Gamma$ such that the objective function value of $\mathbf{x}^2$ is less than that of $\mathbf{x}^1$. However, this is neither true. As is shown in (8.12), for all $(c_1,c_2)^\top \in \Gamma$, the objective function value of $\mathbf{x}^2$ is larger than that of $\mathbf{x}^1$.

Now we emphasize the reason why indices defined by (8.9) and (8.10) do not work in the case of Example 1. Let ζ_1 and ζ_2 be variables taking values in Z_1 and Z_2, respectively. In the indices defined by (8.9) and (8.10), it is implicitly assumed that the possible range of ζ_2 does not depend on the realization of ζ_1 and also possible range of ζ_1 does not depend on the realization of ζ_2.

In Example 1, we set $Z_1 = Y(\mathbf{x}^1)$ and $Z_2 = Y(\mathbf{x}^2)$. Namely, they are possible ranges of $\zeta_1 = \mathbf{c}^\top\mathbf{x}^1$ and $\zeta_2 = \mathbf{c}^\top\mathbf{x}^2$, respectively. Both ζ_1 and ζ_2 depend on the realization of variable vector $\mathbf{c}$ taking a vector value in $\Gamma = [1,2] \times [-2,-1]$. Because of this fact, the implicit assumption in (8.9) and (8.10) does not hold. For example, when $\zeta_1 = \mathbf{c}^\top\mathbf{x}^1 = 0$, the possible realizations of $\mathbf{c} \in \Gamma$ are in

$$\{(c_1,c_2)^\top \in \mathbb{R}^2 : 2c_1 + c_2 = 0, 1 \leq c_1 \leq 2, -2 \leq c_2 \leq -1\} = \{(1,-2)\}. \tag{8.13}$$

Namely, from the information $\zeta_1 = 0$, in this case, we know the realization of $\mathbf{c}$ uniquely as $(1,-2)^\top$. It implies that ζ_2 is also uniquely known as $\zeta_2 = (1,-2)^\top\mathbf{x}^1 = 1$. Generally, when $\zeta_1 = q$, the possible range of ζ_2 is given by

$$\{3c_1 + c_2 : 2c_1 + c_2 = q,\ 1 \leq c_1 \leq 2,\ -2 \leq c_2 \leq -1\}. \tag{8.14}$$

This varies depending on ζ_1's realization q. Therefore, ζ_2 interacts with ζ_1. Similarly, ζ_1 interacts with ζ_2.

Since the implicit assumption of (8.9) and (8.10) does not hold, indices defined by (8.9) and (8.10) cannot be applied directly to the comparison between fuzzy objective function values. For the comparison between fuzzy objective function values, the following modified indices [7, 9] are adequate:

$$\mathrm{POS}(\mathbf{c}^\top \mathbf{x}^1 \geq \mathbf{c}^\top \mathbf{x}^2) = \sup_{\mathbf{r}} \{\mu_\Gamma(\mathbf{r}) : \mathbf{r}^\top \mathbf{x}^1 \geq \mathbf{r}^\top \mathbf{x}^2\}, \tag{8.15}$$

$$\mathrm{NES}(\mathbf{c}^\top \mathbf{x}^1 \geq \mathbf{c}^\top \mathbf{x}^2) = 1 - \sup_{\mathbf{r}} \{\mu_\Gamma(\mathbf{r}) : \mathbf{r}^\top \mathbf{x}^1 < \mathbf{r}^\top \mathbf{x}^2\}. \tag{8.16}$$

In literature, such desertion often appears when a fuzzy objective function is treated by a comparison of fuzzy numbers. Moreover, we note that under other interpretations of fuzzy coefficients, this discussion about the inadequacy is not valid. For example, when a fuzzy objective function is regarded as a collection of objective functions, e.g., a collection of utility functions of many decision makers, the above discussion cannot be applied.

Using $\mathrm{POS}(\mathbf{c}^\top \mathbf{x}^1 \geq \mathbf{c}^\top \mathbf{x}^2)$ and $\mathrm{NES}(\mathbf{c}^\top \mathbf{x}^1 \geq \mathbf{c}^\top \mathbf{x}^2)$ of (8.15) and (8.16), we can define fuzzy sets of necessary and possible non-inferior solutions by the following membership functions:

$$\mu_{NnS}(\mathbf{x}^2) = \begin{cases} 1 - \sup_{\mathbf{x}^1 \in F} \mathrm{POS}(\mathbf{c}^\top \mathbf{x}^1 \geq \mathbf{c}^\top \mathbf{x}^2), & \text{if } \mathbf{x} \in F, \\ 0, & \text{if } \mathbf{x} \notin F, \end{cases} \tag{8.17}$$

$$\mu_{\Pi nS}(\mathbf{x}^2) = \begin{cases} 1 - \sup_{\mathbf{x}^1 \in F} \mathrm{NES}(\mathbf{c}^\top \mathbf{x}^1 \geq \mathbf{c}^\top \mathbf{x}^2), & \text{if } \mathbf{x} \in F, \\ 0, & \text{if } \mathbf{x} \notin F. \end{cases} \tag{8.18}$$

We note that necessary non-inferior solution set NnS is defined by $\mathrm{POS}(\mathbf{c}^\top \mathbf{x}^1 \geq \mathbf{c}^\top \mathbf{x}^2)$ while possible non-inferior solution set NnS is defined by $\mathrm{NES}(\mathbf{c}^\top \mathbf{x}^1 \geq \mathbf{c}^\top \mathbf{x}^2)$.

8.2.3 Necessity and Possibility Measure Optimization

When fuzzy goals G_1 and G_2 showing vaguely the required and desirable levels of objective function value are given, for example, Problem (8.1) can be treated as the following biobjective programming problem (see [10]):

$$\text{maximize } \left(N_{Y(\mathbf{x})}(G_1), \Pi_{Y(\mathbf{x})}(G_2)\right), \quad \text{subject to } \mathbf{x} \in F. \tag{8.19}$$

We assume that membership functions of G_1 and G_2 are non-decreasing.

When Γ is crisp and membership functions of G_1 and G_2 are increasing, Problem (8.19) is reduced to

$$\text{maximize } \left(\min_{\mathbf{c} \in \Gamma} \mathbf{c}^\top \mathbf{x}, \max_{\mathbf{c} \in \Gamma} \mathbf{c}^\top \mathbf{x}\right), \quad \text{subject to } \mathbf{x} \in F. \tag{8.20}$$

Completely optimal solutions to Problems (8.19) and (8.20), which maximize the both objective functions at the same time, have been regarded as the best solutions. However, as exemplified in the next example, such a complete optimal solution is not always the best solution.

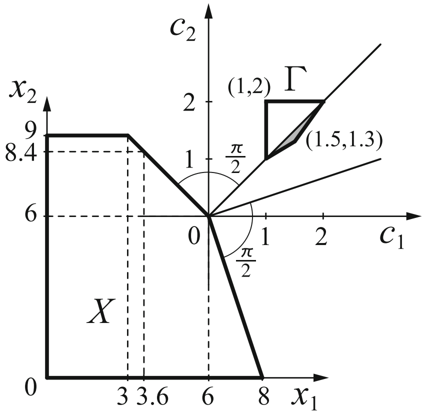

Fig. 8.2: The problem of Example 2

Example 2. Let us consider the following linear programming problem with uncertain objective coefficients:

$$\begin{aligned}&\text{maximize } c_1x_1 + c_2x_2,\\ &\text{subject to } x_1 + x_2 \le 12,\ \ 3x_1 + x_2 \le 24,\\ &\qquad\qquad x_2 \le 9,\ -x_1 \le 0,\ -x_2 \le 0,\end{aligned} \tag{8.21}$$

where $\mathbf{c} = (c_1, c_2)^\top$ is restricted by

$$\begin{aligned}\Gamma = \{(c_1, c_2,)^\top \mid &-4 \le 7c_1 - 5c_2 \le 4,\\ &2 \le -3c_1 + 5c_2 \le 9,\ 0 \le c_2 \le 2,\ 1 \le c_1 \le 3\}.\end{aligned} \tag{8.22}$$

For every $\mathbf{r} = (r_1, r_2) \in \Gamma$, we have $(1,1)^\top \le \mathbf{r} \le (2,2)^\top$, $(1,1)^\top \in \Gamma$ and $(2,2)^\top \in \Gamma$. The biobjective programming problem becomes

$$\text{maximize}(x_1 + x_2,\ 2x_1 + 2x_2),\ \ \text{subject to } \mathbf{x} = (x_1, x_2)^\top \in F. \tag{8.23}$$

This problem has a completely optimal solution $\mathbf{x}^* = (6,6)^\top$.

The solution is illustrated in Fig. 8.2. The shaded area of Fig. 8.2 is the set of $\mathbf{c} \in \Gamma$ which makes $(6,6)^\top$ optimal. This shaded area is small relatively to Γ. From the viewpoint of optimality, $(6,6)^\top$ is not very robust because it easily fails to be optimal. However, this solution is robust in the sense that the objective function value is never less than $6+6=12$ as far as $\mathbf{c}$ fluctuates in $[1,2] \times [1,2]$.

8.3 Optimization Approaches

8.3.1 Possible and Necessary Optimal Solutions

Let $P(\mathbf{x}) = \{\mathbf{r} \mid \mathbf{r}^\top \mathbf{x} = \max_{\mathbf{y}\in F} \mathbf{r}^\top \mathbf{y}\}$. Possibly optimal solution set ΠS and necessarily optimal solution set NS are proposed for Problem (8.1) (see [11]). They are defined as fuzzy sets with the following membership functions, respectively:

$$\mu_{\Pi S}(\mathbf{x}) = \begin{cases} \sup\limits_{\mathbf{r}\in P(\mathbf{x})} \mu_\gamma(\mathbf{r}), & \text{if } \mathbf{x} \in F, \\ 0, & \text{if } \mathbf{x} \notin F, \end{cases} \tag{8.24}$$

$$\mu_{NS}(\mathbf{x}) = \begin{cases} \sup\limits_{\mathbf{r}\notin P(\mathbf{x})} 1 - \mu_\Gamma(\mathbf{r}), & \text{if } \mathbf{x} \in F, \\ 0, & \text{if } \mathbf{x} \notin F. \end{cases} \tag{8.25}$$

We have $\mu_{NS}(\mathbf{x}) > 0 \Rightarrow \mu_{\Pi S}(\mathbf{x}) = 1$. $\mu_{\Pi S}(\mathbf{x})$ and $\mu_{NS}(\mathbf{x})$ are called the possible optimality degree and necessary optimality degree of solution $\mathbf{x}$, respectively.

When a feasible solution is given, we will be interested in the degrees of $\mu_{\Pi S}$ and μ_{NS}. This topic is studied in [11] when $p = n$ and $\mathbf{d}_j = \mathbf{e}_j$, where $\mathbf{e}_j$ is a unit vector whose j-th component is one. The method is easily extended to the general case.

A solution such that $\mu_{NS}(\mathbf{x}) > 0$ (resp. $\mu_{\Pi S}(\mathbf{x}) > 0$) is called a necessarily (resp. possibly) optimal solution. In Example 2, the solutions on the line segment from $(6,6)^\top$ to $(3,9)^\top$ are possibly optimal solutions and there is no necessarily optimal solution. Generally, a necessarily optimal solution does not always exist but usually there are a lot of possibly optimal solutions. If a necessarily optimal solution exists, the solution is the most rational solution. Since F and Γ are bounded, any possibly optimal solution can be expressed as a convex combination of possibly optimal basic solutions. The number of possibly optimal solutions are finite because of the boundedness of F. An enumeration method of all possible optimal basic solutions together with possible optimality degrees $\mu_{\Pi S}$ is proposed in [8].

We note that the possibly and necessary optimal solutions sets equal to possibly and necessary non-inferior solution sets, i.e., we have $\Pi S = \Pi nS$ and $NS = NnS$.

8.3.2 Robust-Soft Optimal Solutions

Since in many cases no solution with positive necessary optimality degree exists, let us weaken the concept of the necessary optimality. To this end, we introduce the concept of soft optimality. If the objective function value of a feasible solution is slightly smaller than the optimal value, the solution can be regarded as a suboptimal solution. From this point of view, we define a suboptimal solution set $T(\mathbf{c})$ to a linear programming problem with an objective function vector $\mathbf{c}$ as a fuzzy set with a membership function,

$$\mu_{T(\mathbf{c})}(\mathbf{x}) = \begin{cases} \mu_{Dif}\left(\max_{\mathbf{y}\in F} \mathbf{c}^\top \mathbf{y} - \mathbf{c}^\top \mathbf{x}\right), & \text{if } \mathbf{x} \in F, \\ 0, & \text{if } \mathbf{x} \notin F, \end{cases} \tag{8.26}$$

or

$$\mu_{T(\mathbf{c})}(\mathbf{x}) = \begin{cases} \mu_{Rat}\left(\dfrac{\mathbf{c}^\top \mathbf{x}}{\max_{\mathbf{y}\in F} \mathbf{c}^\top \mathbf{y}}\right), & \text{if } \mathbf{x} \in F, \\ 0, & \text{if } \mathbf{x} \notin F, \end{cases} \tag{8.27}$$

where μ_{Dif} is assumed to be upper semi-continuous and non-increasing. μ_{Rat} is assumed to be upper semi-continuous, and non-decreasing if $\max_{\mathbf{y}\in F} \mathbf{c}^\top \mathbf{x} > 0$ and non-increasing if $\max_{\mathbf{y}\in F} \mathbf{c}^\top \mathbf{x} < 0$. While (8.26) is useful whenever the decision maker takes care of the difference from the optimal value, (8.27) is useful when $\max_{\mathbf{y}\in F} \mathbf{c}^\top \mathbf{x} \neq 0$ and the decision maker takes care of the achievement rate based on the optimal value.

Using T, the necessarily soft optimal solution set NT is defined by

$$\mu_{NT}(\mathbf{x}) = \inf_{\mathbf{c}} \max\left(1 - \mu_{\Gamma}(\mathbf{c}), \mu_{T(\mathbf{c})}(\mathbf{x})\right). \tag{8.28}$$

It should be noted that NT with $T(\mathbf{c})$ defined by (8.27) is useful only when there exists $\mathbf{y} \in F$ such that $\min_{\mathbf{c}\in \mathrm{cl}(\Gamma)_h} \mathbf{c}^\top \mathbf{y} > 0$ for all $h \in [0,1)$ or when $\max_{\mathbf{y}\in F} \mathbf{c}^\top \mathbf{y} < 0$ for all $\mathbf{c} \in (\Gamma)_0$. When we define a fuzzy set $V(\mathbf{x})$ of $\mathbf{c} \in \mathbb{R}^n$ for $\mathbf{x} \in \mathbb{R}^n$ by $\mu_{V(\mathbf{x})}(\mathbf{c}) = \mu_{T(\mathbf{c})}(\mathbf{x})$, we obtain $\mu_{NT}(\mathbf{x}) = N_{\Gamma}(V(\mathbf{x}))$, i.e., the necessarily soft optimal solution set NT is defined by using a necessity measure.

When $\mu_{Dif}(r)$ takes 1 for $r \leq 0$ and 0 for $r > 0$, fuzzy set NT is reduced to NS. Similarly, in the case of $\max_{\mathbf{y}\in F} \mathbf{c}^\top \mathbf{x} > 0$, when $\mu_{Ra}(r)$ takes 1 for $r \geq 1$ and 0 for $r < 1$, fuzzy set NT is reduced to NS. In the case of $\max_{\mathbf{y}\in F} \mathbf{c}^\top \mathbf{x} < 0$, when $\mu_{Ra}(r)$ takes 1 for $r \leq 1$ and 0 for $r > 1$, fuzzy set NT is reduced to NS.

8.4 Solution Algorithms Under Given Fuzzy Goals

When μ_{Dif} or μ_{Rat} is given by the decision maker, the best solution among NT is a solution with highest necessary soft optimality degree $\mu_{NT}(\mathbf{x})$. This problem is formulated as

$$\text{maximize} \ \ \mu_{NT}(\mathbf{x}). \tag{8.29}$$

This formulation was already proposed in [13]. While the objective function values of a solution $\mathbf{x}$ are independent of F in Problem (8.19), the objective function value depends on F in Problem (8.29). In this sense, both objective function values of Problem (8.19) are *independent* from other feasible solutions but the objective function value of Problem (8.29) is *depends* on others. Let $\hat{\mathbf{x}}$ and $\hat{h}$ be an optimal solution and the optimal value of Problem (8.29). Then we have

$$\forall \mathbf{c} \in (\Gamma)_{1-\hat{h}}, \ \mu_{T(\mathbf{c})}(\hat{\mathbf{x}}) \geq \hat{h}. \tag{8.30}$$

This implies that the suboptimality degree is guaranteed as at least $\hat{h}$ as far as $\mathbf{c}$ takes a value in $(\Gamma)_{1-\hat{h}}$. In this sense, Problem (8.29) produces a solution which is robust in suboptimality.

When $T(\mathbf{c})$ is defined by (8.26) and $\mathbf{d}_j = \mathbf{e}_j$, $j = 1, 2, \ldots, p = n$, the equivalent problem and a solution algorithm based on bisection and relaxation methods are shown in [13]. In this chapter, we describe the result when $\mathbf{d}_j$, $j = 1, 2, \ldots, p$ are general.

For the sake of simplicity, we consider the following three cases:

Case (I): $T(\mathbf{c})$ is defined by (8.26),
Case (II): $T(\mathbf{c})$ is defined by (8.27) under the assumption that there exists $\mathbf{y} \in F$ such that $\min_{\mathbf{c} \in (\Gamma)_h} \mathbf{c}^\top \mathbf{y} > 0$ for all $h \in (0, 1]$,
Case (III): $T(\mathbf{c})$ is defined by (8.27) under the assumption that $\max_{\mathbf{y} \in F} \mathbf{c}^\top \mathbf{y} < 0$ for all $\mathbf{c} \in (\Gamma)_0$

The procedure is the same among those three cases but subproblems are different. We note that in cases (II) and (III) we implicitly assume that $(\Gamma)_0$ is bounded.

We investigate (8.29) when $T(\mathbf{c})$ is defined by (8.26) and (8.27). Introducing an auxiliary variable h, from the upper semi-continuity of μ_{Rat}, Problem (8.29) is reduced to

$$\begin{cases} \text{maximize } h, \text{ subject to } \mathbf{x} \in F, \ \mu_{Dif}\left(\max_{\substack{\mathbf{c} \in \mathrm{cl}(\Gamma)_{1-h} \\ \mathbf{y} \in F}} (\mathbf{c}^\top \mathbf{y} - \mathbf{c}^\top \mathbf{x}) \right) \geq h, \\ \hfill \text{in case (I)}, \\ \text{maximize } h, \text{ subject to } \mathbf{x} \in F, \ \mu_{Rat}\left(\min_{\substack{\mathbf{c} \in \mathrm{cl}(\Gamma)_{1-h} \\ \mathbf{y} \in F,\ \mathbf{c}^\top \mathbf{y} > 0}} \dfrac{\mathbf{c}^\top \mathbf{x}}{\mathbf{c}^\top \mathbf{y}} \right) \geq h, \text{ in case (II)}, \\ \text{maximize } h, \text{ subject to } \mathbf{x} \in F, \ \mu_{Rat}\left(\max_{\substack{\mathbf{c} \in \mathrm{cl}(\Gamma)_{1-h} \\ \mathbf{y} \in F}} \dfrac{\mathbf{c}^\top \mathbf{x}}{\mathbf{c}^\top \mathbf{y}} \right) \geq h, \text{ in case (III)}. \end{cases} \tag{8.31}$$

As h increases, $\mathrm{cl}(\Gamma)_{1-h}$ enlarges and thus, (1) the maximum value of $(\mathbf{c}^\top \mathbf{y} - \mathbf{c}^\top \mathbf{x})$ under $\mathbf{c} \in \mathrm{cl}(\Gamma)_{1-h}$ and $\mathbf{y} \in F$ increases, (2) the minimum value of $\mathbf{c}^\top \mathbf{x}/\mathbf{c}^\top \mathbf{y}$ under $\mathbf{c} \in \mathrm{cl}(\Gamma)_{1-h}$, $\mathbf{y} \in F$ and $\mathbf{c}^\top \mathbf{y} > 0$ decreases and (3) the maximum value of $\mathbf{c}^\top \mathbf{x}/\mathbf{c}^\top \mathbf{y}$ under $\mathbf{c} \in \mathrm{cl}(\Gamma)_{1-h}$ and $\mathbf{y} \in F$ increases. Therefore, Problem (8.31) can be solved by a bisection method with respect to h with checking

$$\begin{cases} \mu_{Dif}\left(\min_{\mathbf{x}\in F}\max_{\substack{\mathbf{c}\in \mathrm{cl}(\Gamma)_{1-h}\\ \mathbf{y}\in F}}(\mathbf{c}^\top\mathbf{y}-\mathbf{c}^\top\mathbf{x})\right)\geq h, \text{ in case (I)},\\ \mu_{Rat}\left(\max_{\mathbf{x}\in F}\min_{\substack{\mathbf{c}\in \mathrm{cl}(\Gamma)_{1-h}\\ \mathbf{y}\in F,\ \mathbf{c}^\top\mathbf{y}>0}}\frac{\mathbf{c}^\top\mathbf{x}}{\mathbf{c}^\top\mathbf{y}}\right)\geq h, \text{ in case (II)},\\ \mu_{Rat}\left(\min_{\mathbf{x}\in F}\max_{\substack{\mathbf{c}\in \mathrm{cl}(\Gamma)_{1-h}\\ \mathbf{y}\in F}}\frac{\mathbf{c}^\top\mathbf{x}}{\mathbf{c}^\top\mathbf{y}}\right)\geq h, \text{ in case (III)}, \end{cases} \tag{8.32}$$

If (8.32) is satisfied, we know that the optimal value of Problem (8.31) is not less than h, and examine (8.32) again with an increased h. Otherwise, we know that the optimal value of Problem (8.31) is less than h and examine (8.32) again with a decreased h. Repeating this procedure, the possible range of the optimal value of Problem (8.31) reduced and we stop the procedure when the range becomes small enough.

In order to check the validity of (8.32), we should solve the min-max or max-min problem included in (8.32). Let us look into a solution method for these min-max and max-min problems.

Let $\mathbf{c}^j : (0,1] \to (\Gamma)_0$, $j = 1,2,\ldots,k$ be vector functions such that $\mathbf{c}^j(h) \in \mathrm{cl}(\Gamma)_h$, for all $h \in [0,1)$. These vector functions are generated through the algorithm proposed later in this chapter. Each of these function values is usually obtained at a vertex of $\mathrm{cl}(\Gamma)_h$ by solving a linear programming problem defined by an index set $Q_j = \{q_{j1}, q_{j2},\ldots,q_{jn}\} \subseteq P = \{1,2,\ldots,p\}$ and a 0-1 vector $B_j = (\beta_{j1},\beta_{j2},\ldots,\beta_{jn})^\top \subseteq \{0,1\}^n$. We assume that $q_{j1} < q_{j2} < \cdots < q_{jn}$. Namely, given $h \in (0,1]$, the function value is obtained as $\mathbf{c}$-value of an optimal solution $(\mathbf{c}^\top,\delta_1,\delta_2,\ldots,\delta_n)^\top$ to the following linear programming problem:

$$\begin{aligned} &\text{minimize } \sum_{i=1}^{n}\delta_i,\\ &\text{subject to } \inf(\Gamma_{q_{ji}})_h+\delta_i=\mathbf{d}_{q_{ji}}^\top\mathbf{c}\leq \sup(\Gamma_{q_{ji}})_h,\ \text{for } i\in N \text{ such that } \beta_{ji}=0,\\ &\qquad\qquad \inf(\Gamma_{q_{ji}})_h\leq \mathbf{d}_{q_{ji}}^\top\mathbf{c}=\sup(\Gamma_{q_{ji}})_h-\delta_i,\ \text{for } i\in N \text{ such that } \beta_{ji}=1,\\ &\qquad\qquad \inf(\Gamma_q)_h\leq \mathbf{d}_q^\top\mathbf{c}\leq \sup(\Gamma_q)_h,\ q\in P\backslash Q_j\\ &\qquad\qquad \delta_i\geq 0,\ i=1,2,\ldots,n, \end{aligned} \tag{8.33}$$

where $N = \{1,2,\ldots,n\}$.

When $\mathbf{y}^j \in F$, $j = 1,2,\ldots,k$ are given under fixed $h \in (0,1]$, a relaxation problem of the min-max/max-min problem in (8.32) is obtained as the following linear programming problem:

$$\begin{cases} \text{minimize } r, \text{ subject to } \mathbf{x} \in F, \ \mathbf{c}^j(1-h)^\top \mathbf{y}^j - \mathbf{c}^j(1-h)^\top \mathbf{x} \le r, \ j \in K, \\ \hfill \text{in case (I)}, \\ \text{maximize } r, \text{ subject to } \mathbf{x} \in F, \ \dfrac{\mathbf{c}^j(1-h)^\top \mathbf{x}}{\mathbf{c}^j(1-h)^\top \mathbf{y}^j} \ge r, \ j \in K, \quad \text{in case (II)}, \\ \text{minimize } r, \text{ subject to } \mathbf{x} \in F, \ \dfrac{\mathbf{c}^j(1-h)^\top \mathbf{x}}{\mathbf{c}^j(1-h)^\top \mathbf{y}^j} \le r, \ j \in K, \ \text{in case (III)}, \end{cases} \tag{8.34}$$

where $K = \{1,2,\ldots,k\}$ and the given solution $\mathbf{y}^j \in F$ satisfies $\mathbf{c}^j(1-h)^\top \mathbf{y}^j > 0$ when $T(\mathbf{c})$ is defined by (8.27) and $\max_{\mathbf{y}\in F} \mathbf{c}^\top \mathbf{y} > 0$ for all $\mathbf{c} \in (\Gamma)_0$. Note that the number of possible Q_j's is at most ${}_p\mathrm{C}_n = n!/(p!(n-p)!)$ and that the number of possible B_j's is 2^n. The value of $\mathbf{c}^j(h)$ is determined by solving Problem (8.33) for a given $h \in (0,1]$. Then the number of all possible vector functions $\mathbf{c}^j$ is $2^n n!/(p!(n-p)!)$. Because $\mathrm{cl}(\Gamma)_{1-h}$ is a polytope for each $h \in (0,1]$, any element $\mathbf{c} \in \mathrm{cl}(\Gamma)_{1-h}$ can be represented by a convex combinations of the vertices of $\mathrm{cl}(\Gamma)_{1-h}$. Let $V(\mathrm{cl}(\Gamma)_{1-h})$ be the set of vertices of $\mathrm{cl}(\Gamma)_{1-h}$. Then we have $V(\mathrm{cl}(\Gamma)_{1-h}) = \{\mathbf{c}^j(1-h),\ j = 1,2,\ldots,2^n n!/(p!(n-p)!)\}$. Hence, when $k = 2^n n!/(p!(n-p)!)$, Problem (8.34) is equivalent to the min-max/max-min problem in (8.32).

Let $\mathbf{x}^0$ and r^0 be an optimal solution and the optimal value of Problem (8.34), respectively. Since Problem (8.34) is a relaxed problem, we should examine whether $\mathbf{x}^0$ is an optimal solution to the min-max/max-min problem in (8.32) or not under the fixed $h \in (0,1]$. This can be done by checking whether the optimal value of the following problem is not less than r^0:

$$\begin{cases} \text{maximize } \mathbf{c}^\top \mathbf{y} - \mathbf{c}^\top \mathbf{x}^0, \ \text{subject to } \mathbf{c} \in \mathrm{cl}(\Gamma)_{1-h}, \ \mathbf{y} \in F, \ \text{in case (I)}, \\ \text{minimize } \dfrac{\mathbf{c}^\top \mathbf{x}^0}{\mathbf{c}^\top \mathbf{y}}, \ \text{subject to } \mathbf{c} \in \mathrm{cl}(\Gamma)_{1-h}, \ \mathbf{y} \in F, \ \mathbf{c}^\top \mathbf{y} > 0, \quad \text{in case (II)}, \\ \text{maximize } \dfrac{\mathbf{c}^\top \mathbf{x}^0}{\mathbf{c}^\top \mathbf{y}}, \ \text{subject to } \mathbf{c} \in \mathrm{cl}(\Gamma)_{1-h}, \ \mathbf{y} \in F, \quad \text{in case (III)}, \end{cases} \tag{8.35}$$

where we note that constraint $\mathbf{c} \in \mathrm{cl}(\Gamma)_{1-h}$ is represent by a system of linear inequalities because we have

$$\mathbf{c} \in \mathrm{cl}(\Gamma)_{1-h} \text{ if and only if } \inf(\Gamma_q)_{1-h} \le \mathbf{d}_q^\top \mathbf{c} \le \sup(\Gamma_q)_{1-h}, \ q \in P. \tag{8.36}$$

If the optimal value of Problem (8.35) is not greater/less than r^0, $\mathbf{x}^0$ is an optimal solution to the min-max/max-min problem in (8.32). Otherwise, we add $\mathbf{c}^{k+1} : (0,1] \to (\Gamma)_0$ which satisfies

$$\begin{cases} \mathbf{c}^{k+1}(1-h)^\top \mathbf{y}^j - \mathbf{c}^{k+1}(1-h)^\top \mathbf{x} > r^0, \ \exists j \in K, \text{ in case (I)}, \\ \dfrac{\mathbf{c}^{k+1}(1-h)^\top \mathbf{x}}{\mathbf{c}^{k+1}(1-h)^\top \mathbf{y}^j} < r^0, \ \exists j \in K, \text{ in case (II)}, \\ \dfrac{\mathbf{c}^{k+1}(1-h)^\top \mathbf{x}}{\mathbf{c}^{k+1}(1-h)^\top \mathbf{y}^j} > r^0, \ \exists j \in K, \text{ in case (III)}, \end{cases} \tag{8.37}$$

for the fixed $h \in (0,1]$ and update $k = k+1$. Such function $\mathbf{c}^{k+1}$ can be determined by using an optimal solution to Problem (8.35). Namely, there exists an optimal solution $\mathbf{c}$ to Problem (8.35) which has at least n independent $\mathbf{d}_q$ such that $\mathbf{d}_q^\top \mathbf{c} = \inf(\Gamma_q)_{1-h}$ or $\mathbf{d}_q^\top \mathbf{c} = \sup(\Gamma_q)_{1-h}$. This implies that we can find Q_{k+1} and B_{k+1} corresponding to $\mathbf{c}^{k+1}$ whose function value is obtained by solving (8.33) with substitution $j = k+1$.

From the above discussion, we obtain the following solution algorithm based on the bisection method and the relaxation procedure with an admissible error $\varepsilon > 0$.

Algorithm 1

Step 1. Select Q_1 and B_1 arbitrarily in order to define $\mathbf{c}^1 : (0,1] \to (\Gamma)_0$. Let $\mathbf{y}^1$ be an optimal solution to the following linear programming problem:

$$\text{maximize } \mathbf{c}^1(0.5)^\top \mathbf{y}, \quad \text{subject to } \mathbf{y} \in F. \tag{8.38}$$

Step 2. Set $h^{\mathrm{L}} = 0$, $h^{\mathrm{U}} = 1$, $k = 1$ and $\mathbf{x}^0 = \mathbf{y}^1$.
Step 3. Set $h = \frac{1}{2}(h^{\mathrm{L}} + h^{\mathrm{U}})$ and let $\mathbf{y}^{k+1}$ and r^k be an optimal solution and the optimal value of Problem (8.35), respectively.
Step 4. If $\mu_{Dif}(r^k) \geq h$ or $\mu_{Rat}(r^k) \geq h$ then update $h^{\mathrm{L}} = h$ and return to Step 3.
Step 5. If $h^{\mathrm{U}} - h^{\mathrm{L}} \leq \varepsilon$ then terminate the algorithm. If $h^{\mathrm{U}} \leq \varepsilon$ then there is no feasible solution $\mathbf{x}$ such that $\mu_{NT}(\mathbf{x}) > \varepsilon$ and otherwise the optimal solution is obtained as $\mathbf{x}^0$.
Step 6. Construct $\mathbf{c}^{k+1}$ from a pair (Q_{k+1}, B_{k+1}) corresponding to an optimal solution of the latest problem solved at Step 3. If there is no $j \in \{1,2,\ldots,k\}$ such that $\mathbf{c}^j = \mathbf{c}^{k+1}$ and $\mathbf{c}^j(1-h)^\top \mathbf{y}^j = \mathbf{c}^{k+1}(1-h)^\top \mathbf{y}^{k+1}$ then update $k = k+1$.
Step 7. Set $h = \frac{1}{2}(h^{\mathrm{L}} + h^{\mathrm{U}})$. In cases (I) and (III), obtain an optimal solution $(\mathbf{x}^*, r^*)$ to Problem (8.34) and go to Step 8. In case (II), update $\mathbf{y}^j$ as an optimal solution to $\text{maximize}_{\mathbf{y} \in F}\ \mathbf{c}^j(1-h)^\top \mathbf{y}$.
Step 8. If $\mu_{Dif}(r^*) < h$ or $\mu_{Rat}(r^*) < h$, then set $h^{\mathrm{U}} = h$ and return to Step 7. Otherwise, set $\mathbf{x}^0 = \mathbf{x}^*$ and return to Step 3.

In this algorithm, we use μ_{Dif} in case (I), and μ_{Rat} in cases (II) and (III). In case (II), we update $\mathbf{y}^j$ at Step 7 so that we have $\mathbf{c}^j(1-h)^\top \mathbf{y}^j > 0$, $j = 1,2,\ldots,k$. The existence of such $\mathbf{y}^j \in F$ is guaranteed by the assumption described in case (II). Furthermore, we do not solve the max-min problem in (8.32) at each fixed h but solve it simultaneously with optimizing h so as to obtain a solution of Problem (8.31).

To prove the convergence of Algorithm 1, we use the following proposition.

Proposition 1. *If h^{L} is not updated at Step 4 in iteration $k \geq 2$, for the pair $(\mathbf{c}^{k+1}, \mathbf{y}^{k+1})$ obtained by solving Problem (8.35) at Step 3, there is no $l \leq k$ such that $\mathbf{c}^l = \mathbf{c}^{k+1}$ and $\mathbf{c}^j(1-h)^{\top}\mathbf{y}^j = \mathbf{c}^{k+1}(1-h)^{\top}\mathbf{y}^{k+1}$.*

Proof. Assume there exists $l \leq k$ such that $\mathbf{c}^l = \mathbf{c}^{k+1}$ and $\mathbf{c}^j(1-h)^{\top}\mathbf{y}^j = \mathbf{c}^{k+1}(1-h)^{\top}\mathbf{y}^{k+1}$ for the pair $(\mathbf{c}^{k+1}, \mathbf{y}^{k+1})$ corresponding to Problem (8.35) solved at Step 3 when h^{L} is not updated at Step 4 in iteration $k \geq 2$. Since h^{L} is not updated at Step 4, r^k satisfies $\mu_{Dif}(r^k) < h$ in case (I), and $\mu_{Rat}(r^k) < h$ in cases (II) and (III). On the other hand, because $k \geq 2$, it has returned to Step 3 from Step 8 and thus r^* of the optimal solution $(\mathbf{x}^*, r^*)$ obtained at the last visit of Step 7 satisfies $\mu_{Dif}(r^*) \geq h$ in case (I), and $\mu_{Rat}(r^*) \geq h$ in cases (II) and (III). Therefore, we have

$$\mu_{Dif}(r^k) < \mu_{Dif}(r^*) \text{ in case (I) and } \mu_{Rat}(r^k) < \mu_{Rat}(r^*) \text{ in cases (II) and (III).} \quad (*)$$

If we fix $\mathbf{y}$ at a feasible solution in Problem (8.35), the problem becomes a linear programming problem or a linear fractional programming problem with a decision variable vector $\mathbf{c}$. From the theories of linear and linear fractional programming [1, 3], the optimal solution $\mathbf{c}$ exists at an extreme point of $\mathrm{cl}(\Gamma)_{1-h}$. Together with this fact, the assumption of the existence of $l \leq k$ such that $\mathbf{c}^l = \mathbf{c}^{k+1}$ and $\mathbf{c}^j(1-h)^{\top}\mathbf{y}^j = \mathbf{c}^{k+1}(1-h)^{\top}\mathbf{y}^{k+1}$ implies

$$r^k = \mathbf{c}^{k+1}(1-h)^{\top}\mathbf{y}^{k+1} - \mathbf{c}^{k+1}(1-h)^{\top}\mathbf{x}^0 = \mathbf{c}^l(1-h)^{\top}\mathbf{y}^l - \mathbf{c}^l(1-h)^{\top}\mathbf{x}^0.$$

Because r^* is the optimal value of Problem (8.34), we have $r^k \leq r^*$ in cases (I) and (III), and $r^k \geq r^*$ in case (II). This implies

$$\mu_{Dif}(r^k) \geq \mu_{Dif}(r^*) \text{ in case (I) and } \mu_{Rat}(r^k) \geq \mu_{Rat}(r^*) \text{ in cases (II) and (III).}$$

This contradicts $(*)$. □

From Proposition 1, in every iteration, h^{L} is updated or a new pair $(\mathbf{c}^{k+1}, \mathbf{y}^{k+1})$ is added. The update of h^{L} reduces the difference $h^{\mathrm{U}} - h^{\mathrm{L}}$ to half or to less than half. The number of pairs $(\mathbf{c}^j, \mathbf{y}^j)$ is finite because the number of pairs (Q_j, B_j) is finite and F is bounded. Hence Algorithm 8.4 terminates in a finite number of iterations.

8.5 Solving the Subproblem

In Algorithm 1, all problems other than Problem (8.35) are linear programming problems and solved easily. However, Problem (8.35) is neither a linear programming problem nor a concave/convex programming problem but a convex maximization/concave minimization problem. To solve this problem, several approaches such as two-phase method, outer approximation method, cutting hyperplane method and so on were proposed (see [12, 14, 15, 17]) when Γ is a crisp set. In two-phase method, all possibly optimal extreme points $\mathbf{z}^l \in F$, $l = 1,2,\ldots,q$ such that $\mu_{\Pi S}(\mathbf{z}^l) > 0$ are enumerated before the execution of Algorithm 1 and at Step 3 of Algorithm 1 we solve q linear programming problems/linear fractional programming problems (8.35) with fixing $\mathbf{y} = \mathbf{z}^l$, $l = 1,2,\ldots,q$. In case 1, we may apply a post optimality technique of linear programming for the change of objective coefficient vector when we solve q linear programming problems (8.35) with fixing $\mathbf{y} = \mathbf{z}^l$, $l = 1,2,\ldots,q$. Therefore, those q problems are solved sequentially without reinitialization of simplex tableau.

On the other hand, in cases (II) and (III), we cannot apply this technique directly. In case (III), because we have $\mathbf{c}^\top\mathbf{x}^0 < 0$ and $\mathbf{c}^\top\mathbf{y} < 0$ for any $\mathbf{y} \in F$, we obtain

$$\min_{\substack{\mathbf{c}\in \mathrm{cl}(\Gamma)_{1-h} \\ \mathbf{y}\in F}} \frac{\mathbf{c}^\top\mathbf{y}}{\mathbf{c}^\top\mathbf{x}^0} = \frac{1}{\displaystyle\max_{\substack{\mathbf{c}\in \mathrm{cl}(\Gamma)_{1-h} \\ \mathbf{y}\in F,}} \frac{\mathbf{c}^\top\mathbf{x}^0}{\mathbf{c}^\top\mathbf{y}}}. \tag{8.39}$$

Applying the linear fractional programming technique, the minimization problem in the left-hand side of (8.39) is reduced to

$$\text{minimize } \hat{\mathbf{c}}^\top\mathbf{y}, \ \text{ subject to } \hat{\mathbf{c}}^\top\mathbf{x}^0 = -1, \ \mathbf{y} \in F, \ \frac{\hat{\mathbf{c}}}{t} \in \mathrm{cl}(\Gamma)_{1-h}, t \geq 0. \tag{8.40}$$

To this reduced problem, we can apply the post optimality technique and thus q linear fractional programming problems are solved efficiently. We note an optimal solution to Problem (8.35) is obtained as $(\hat{\mathbf{c}}/t, \mathbf{y})$ from the obtained optimal solution $(\hat{\mathbf{c}}, \mathbf{y}, t)$ to Problem (8.40).

In case (II), we cannot obtain a similar result to (8.39). This is because there is no guarantee that we have $\mathbf{c}^\top\mathbf{x}^0 > 0$ for all $\mathbf{c} \in \mathrm{cl}(\Gamma)_{1-h}$ at Step 3. However, because of the assumption described in case (II), we have a solution $\mathbf{y} \in F$ such that $\mathbf{c}^\top\mathbf{y} > 0$ for all $\mathbf{c} \in \mathrm{cl}(\Gamma)_{1-h}$. In order to ensure $\mathbf{c}^\top\mathbf{x}^0 > 0$ for all $\mathbf{c} \in \mathrm{cl}(\Gamma)_{1-h}$, we can add $\mathbf{c}^{k+1} \in \mathrm{cl}(\Gamma)_{1-h}$ such that ${\mathbf{c}^{k+1}}^\top\mathbf{x}^0 < 0$, iteratively at Step 6 with the replacement of Step 3 by the following step:

Step 3'. Set $h = \frac{1}{2}(h^{\mathrm{L}} + h^{\mathrm{U}})$ and solve the following linear programming problem:

$$\text{minimize } \ {\mathbf{x}^0}^\top\mathbf{c}, \ \text{ subject to } \ \mathbf{c} \in \mathrm{cl}(C)_{1-h}. \tag{8.41}$$

If the optimal value is negative, let $\bar{\mathbf{c}}$ be the obtained optimal solution and $\mathbf{y}^{k+1}$ an optimal solution to

$$\text{maximize } \ \bar{\mathbf{c}}^\top\mathbf{y}, \ \text{ subject to } \ \mathbf{y} \in F, \tag{8.42}$$

and then go to Step 6. Otherwise, let $\mathbf{y}^{k+1}$ and r^k be an optimal solution and the optimal value of Problem (8.35), respectively.

Because F is bounded, we obtain some $\mathbf{x}^0$ such that $\mathbf{c}^\top\mathbf{x}^0 > 0$ for all $\mathbf{c} \in \mathrm{cl}(\Gamma)_{1-h}$ in a finite iterations. If $\mathbf{c}^\top\mathbf{x}^0 > 0$ for all $\mathbf{c} \in \mathrm{cl}(\Gamma)_{1-h}$ is ensured, we have

$$\max_{\substack{\mathbf{c}\in \mathrm{cl}(\Gamma)_{1-h}\\ \mathbf{y}\in F, \mathbf{c}^\top \mathbf{y}>0}} \frac{\mathbf{c}^\top \mathbf{y}}{\mathbf{c}^\top \mathbf{x}^0} = \frac{1}{\displaystyle\min_{\substack{\mathbf{c}\in \mathrm{cl}(\Gamma)_{1-h}\\ \mathbf{y}\in F,\ \mathbf{c}^\top \mathbf{y}>0}} \frac{\mathbf{c}^\top \mathbf{x}^0}{\mathbf{c}^\top \mathbf{y}}}, \tag{8.43}$$

Applying a linear fractional programming technique, the maximization problem in the left-hand side problem of (8.43) is reduced to a bilinear programming problem,

$$\begin{aligned} &\text{maximize } \hat{\mathbf{c}}^\top \mathbf{y},\\ &\text{subject to } \hat{\mathbf{c}}^\top \mathbf{x}^0 = 1,\ \mathbf{y} \in F,\ \frac{\hat{\mathbf{c}}}{t} \in \mathrm{cl}(\Gamma)_{1-h}. \end{aligned} \tag{8.44}$$

Let $(\hat{\mathbf{c}}, \mathbf{y}, t)$ be an optimal solution to Problem (8.44). Then solution $(\hat{\mathbf{c}}/t, \mathbf{y})$ is an optimal solution to Problem (8.35). The post optimality technique of linear programming problem can be applied to the reduced problem (8.44).

The transformations of the fractional programming problems in cases (II) and (III) are useful when the outer approximation method is used for solving Problem (8.35). By the numerical experiments reported in [14], the outer approximation method solves Problem (8.35) efficiently. An outer approximation algorithm is shown as follows.

Algorithm 2

Step 1. Initialize $p = 0$ and obtain a polytope Y_0 such that $F \subseteq Y_0$.
Step 2. Enumerate all elements of $\Pi B(Y_p)$.
Step 3. Calculate $f(\mathbf{y})$ for all $\mathbf{y} \in \Pi B(Y_p)$.

In case (I): let $\mathbf{y}^p$ be a solution which maximizes $f(\mathbf{y})$ subject to $\mathbf{y} \in \Pi B(Y_p)$. Moreover, let $\bar{\mathbf{c}}^p$ be a $\mathbf{c} \in \Gamma$ such that $f(\mathbf{y}^p) = \mathbf{c}^{\mathrm{T}}(\mathbf{y}^p - \mathbf{x}^0)$.
In case (II): In case (II) let $\mathbf{y}^p$ be a solution which minimizes $f(\mathbf{y})$ subject to $\mathbf{y} \in \Pi B(Y_p)$. Moreover, let $\bar{\mathbf{c}}^p$ be a $\mathbf{c} \in \Gamma$ such that $f(\mathbf{y}^p) = \mathbf{c}^{\mathrm{T}}\mathbf{y}^p/\mathbf{c}^\top \mathbf{x}^0$.
In case (III): let $\mathbf{y}^p$ be a solution which maximizes $f(\mathbf{y})$ subject to $\mathbf{y} \in \Pi B(Y_p)$. Moreover, let $\bar{\mathbf{c}}^p$ be a $\mathbf{c} \in \Gamma$ such that $f(\mathbf{y}^p) = \mathbf{c}^{\mathrm{T}}\mathbf{y}^p/\mathbf{c}^\top \mathbf{x}^0$.

Step 4. In cases (I) and (III) if $f(\mathbf{y}^p) \le r^0$, terminate the algorithm with setting $r^k = r^0$. In case (II) if $f(\mathbf{y}^p) \ge r^0$, terminate the algorithm with setting $r^k = r^0$.
Step 5. If $\mathbf{y}^p \in F$, terminate the algorithm with setting $\mathbf{c}^k = \bar{\mathbf{c}}^p$, $\mathbf{z}^k = \mathbf{y}^p$ and $r^k = f(\mathbf{y}^p)$.
Step 6. Solve a linear programming problem,

$$\underset{\mathbf{y}\in F}{\text{maximize}}\ \bar{\mathbf{c}}^{p\mathrm{T}}\mathbf{y}, \tag{8.45}$$

and let $\mathbf{w}^p$ be an optimal solution. Let Z be a set defined by constraints whose corresponding slack variables are nonbasic at the optimal solution $\mathbf{w}^p$.
Step 7. Update $Y_{p+1} = Y_p \cap Z$ and $p = p+1$. Return to Step 2.

In the algorithm above, $\Pi B(Y_p)$ is the set of all possibly extreme points with positive possible optimality degrees of Problem (8.1) where F is replaced with Y_p. $f(\mathbf{y})$ is defined by

$$f(\mathbf{y}) = \begin{cases} \max\limits_{\mathbf{c}\in\Gamma}(\mathbf{c}^\top\mathbf{y} - \mathbf{c}^\top\mathbf{x}^0), & \text{in case (I)}, \\ \min\limits_{\mathbf{c}\in\Gamma} \dfrac{\mathbf{c}^\top\mathbf{x}^0}{\mathbf{c}^\top\mathbf{y}}, & \text{in case (II)}, \\ \max\limits_{\mathbf{c}\in\Gamma} \dfrac{\mathbf{c}^\top\mathbf{x}^0}{\mathbf{c}^\top\mathbf{y}}, & \text{in case (III)}. \end{cases} \tag{8.46}$$

Extreme points of $\Pi B(Y_{p+1})$ can be obtained easily from extreme points of $\Pi B(Y_p)$ (see [14]).

In [14], the outer approximation method and other possible solution methods for Problem (8.35) are described in case (I). Moreover the results of numerical experiments in comparison among possible solutions methods for Problem (8.35) are explained in [14].

8.6 Solution Algorithms Under Unknown Goals

The determination of μ_{Dif} or μ_{Rat} can be difficult in some situations. Instead of giving μ_{Dif} or μ_{Rat}, the decision maker may tell to what extent the fluctuation of coefficients should be taken care of. In this situation, the decision maker specifies $h^0 \in (0,1]$ so that we consider all $\mathbf{c} \in (\Gamma)_{1-h^0}$. Under such a situation, we consider

$$\begin{cases} \text{minimize } q_{Dif}, \ \text{ subject to } \mu_{NT}(\mathbf{x}) \geq h^0, \ \text{ in case (I)}, \\ \text{minimize } |q_{Rat} - 1|, \ \text{ subject to } \mu_{NT}(\mathbf{x}) \geq h^0, \ \text{ in cases (II) and (III)}, \end{cases} \tag{8.47}$$

where we define

$$\mu_{Dif}(r) = \begin{cases} 1, \text{ if } r \leq q_{Dif}, \\ 0, \text{ if } r > q_{Dif}, \end{cases} \qquad \mu_{Rat}(r) = \begin{cases} 1, \text{ if } r \leq |q_{Rat} - 1|, \\ 0, \text{ otherwise}. \end{cases} \tag{8.48}$$

Those problems produce a robust solution. Let $\hat{\mathbf{x}}$ and $\hat{q}$ be an optimal solution and the optimal value of Problem (8.47). Then we have

$$\begin{cases} \forall \mathbf{c} \in (\Gamma)_{1-h^0}, \ \forall \mathbf{y} \in F, \ \mathbf{c}^\top\mathbf{y} - \mathbf{c}^\top\hat{\mathbf{x}} \leq \hat{q}, \ \text{ in case (I)}, \\ \forall \mathbf{c} \in (\Gamma)_{1-h^0}, \ \forall \mathbf{y} \in F, \ \left| \dfrac{\mathbf{c}^\top\hat{\mathbf{x}}}{\mathbf{c}^\top\mathbf{y}} - 1 \right| \leq \hat{q}, \ \text{ in cases (II) and (III)}. \end{cases} \tag{8.49}$$

In this section, we show a simpler solution procedure to Problem (8.47). Problem (8.47) is reduced to

$$\begin{cases} \text{minimize } \max\limits_{\substack{\mathbf{c}\in \mathrm{cl}(\Gamma)_{1-h^0} \\ \mathbf{y}\in F}} (\mathbf{c}^\top\mathbf{y} - \mathbf{c}^\top\mathbf{x}), \ \text{ subject to } \mathbf{x} \in F, \ \text{ in case (I)}, \\ \text{maximize } \min\limits_{\substack{\mathbf{c}\in \mathrm{cl}(\Gamma)_{1-h^0} \\ \mathbf{y}\in F, \ \mathbf{c}^\top\mathbf{y}>0}} \dfrac{\mathbf{c}^\top\mathbf{x}}{\mathbf{c}^\top\mathbf{y}}, \ \text{ subject to } \mathbf{x} \in F, \ \text{ in case (II)}, \\ \text{minimize } \max\limits_{\substack{\mathbf{c}\in \mathrm{cl}(\Gamma)_{1-h^0} \\ \mathbf{y}\in F, \ \mathbf{c}^\top\mathbf{y}>0}} \ , \ \text{ subject to } \mathbf{x} \in F, \ \text{ in case (III)}. \end{cases} \tag{8.50}$$

In each case, this problem is the same as the min-max or max-min problem appeared in the argument of the membership function in (8.32). Thus, the discussion in Section 4 is valid also for Problem (8.50). Because h^0 is fixed, Problem (8.50) is easier than Problem (8.31). Based on the relaxation procedure, we have the following algorithm.

Algorithm 3

Step 1. Select $\mathbf{c}^1 : (0,1] \to (\Gamma)_0$. Let $\mathbf{y}^1$ be an optimal solution to a linear programming problem,

$$\text{maximize } {\mathbf{c}^1}^\top \mathbf{y}, \quad \text{subject to } \mathbf{y} \in F. \tag{8.51}$$

Set $k = 1$, $\mathbf{x}^0 = \mathbf{y}^1$, and $r^0 = 0$ in case (I) and $r^0 = 1$ in cases (II) and (III).

Step 2. Let $(\mathbf{c}^{k+1}, \mathbf{y}^{k+1})$ and r^k be an optimal solution and the optimal value of Problem (8.35) with $h = h^0$, respectively.

Step 3. If $r^k \le r^0 + \varepsilon$ in cases (I) and (III) and if $r^k \ge r^0 - \varepsilon$ in case (II) then terminate the algorithm. The optimal solution is obtained as $\mathbf{x}^0$.

Step 4. Update $k = k+1$. Update $(\mathbf{x}^0, r^0)$ with an optimal solution to the following problem:

$$\begin{cases} \text{minimize } r, \text{ subject to } \mathbf{x} \in F, \ {\mathbf{c}^j}^\top \mathbf{y}^j - {\mathbf{c}^j}^\top \mathbf{x} \le r, \ j \in K, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in case (I)}, \\ \text{maximize } r, \text{ subject to } \mathbf{x} \in F, \ \dfrac{{\mathbf{c}^j}^\top \mathbf{x}}{{\mathbf{c}^j}^\top \mathbf{y}^j} \ge r, \ j \in K, \quad \text{in case (II)}, \\ \text{minimize } r, \text{ subject to } \mathbf{x} \in F, \ \dfrac{{\mathbf{c}^j}^\top \mathbf{x}}{{\mathbf{c}^j}^\top \mathbf{y}^j} \le r, \ j \in K, \quad \text{in case (III)}. \end{cases} \tag{8.52}$$

Return to Step 2.

Example 3. Let us apply the approach under unknown goal to the problem of Example 2 with $h^0 = 0.5$. We define $T(\mathbf{c})$ by (8.27). In the problem, we can confirm that $\min_{\mathbf{c} \in (\Gamma)_0} \mathbf{c}^\top \mathbf{y} > 0$, for any $\mathbf{y} \in F$. Namely, we consider case (II). Since Γ is crisp in this problem, there is no difference by the choice of $h^0 \in (0,1]$. Setting $\varepsilon = 0.00001$, we applied Algorithm 3. The computation process is shown in Table 8.1. The obtained solution is

$$(x_1, x_2)^\top = (3.6, 8.4)^\top, \tag{8.53}$$

and its location is shown in Fig. 8.2. As shown in Fig. 8.2, reflecting the shape of Γ, i.e., the fact that Γ has a small right lower part, the obtained solution is located near an extreme point $(x_1, x_2) = (3, 9)$ rather than $(x_1, x_2) = (6, 6)$.

In order to see the correspondences between $\mathbf{c}^i \in \mathrm{cl}(\Gamma)_h$ and pair (Q_i, B_i) which are used in Algorithm 1, we note that $\mathbf{c}^1 = (1.5, 1.3)^\top$ is a solution to Problem (8.33) with $Q_1 = \{1, 2\}$ and $B_1 = (1, 0)$ and $\mathbf{c}^2 = (1, 2)$ is a solution to Problem (8.33) with $Q_2 = \{3, 4\}$ and $B_2 = (1, 0)$.

Table 8.1: Computation process

Step 1	We select $\mathbf{c}^1(h_0) = (1.5, 1.3)^\top$. We obtain $\mathbf{y}^1 = (6,6)^\top$. Set $k = 1$, $\mathbf{x}^0 = \mathbf{y}^1$ and $r^0 = 1$.
Step 2	Solve Problem (8.35) with $h = h^0$. We obtain $(\mathbf{c}^2, \mathbf{y}^2) = ((1,2)^\top, (3,9)^\top)$ and $r^1 = 0.857143$.
Step 3	$r^1 = 0.857143 < r^0 - \varepsilon = 1 - 0.00001$. Continue.
Step 4	We update $k = 2$. We obtain $\mathbf{x}^0 = (3.6, 8.4)^\top$ and $r^0 = 0.971429$. Return to Step 2.
Step 2	Solve Problem (8.35) with $h = h^0$. We obtain $(\mathbf{c}^3, \mathbf{y}^3) = ((1.5, 1.3)^\top, (6,6)^\top)$ and $r^1 = 0.971429$.
Step 3	$r^1 = 0.971429 \geq r^0 - \varepsilon = 0.971429 - 0.00001$. Terminate the algorithm. The optimal solution is $\mathbf{x}^0 = (3.6, 8.4)^\top$

8.7 Concluding Remarks

In this chapter, we described robust optimization approaches to linear programming problems with fuzzy parameters. We explained the necessary care of the interaction between objective function values of two solutions when they are compared. The insufficiency of satisficing approaches is exemplified by a simple example. Two optimal solution sets are defined and their properties are briefly described. To overcome the weak points of two optimality concepts under uncertainty, robust-soft optimality concept is introduced. The necessarily soft optimal solution set is defined. Two suboptimal solution sets are considered and then solution approaches to two necessarily soft optimal solution sets are investigated. Both cases use the same main solution procedure although the subproblems are different. The solution procedure is based on a bisection method and a relaxation method and combined successfully so that both methods converge simultaneously. Nevertheless, the solution procedure is generally much more difficult than that by the satisficing approach. When fuzzy goals are unknown, we do not need to use the bisection method and the solution procedure becomes simpler. However, the reduced problem is still a non-convex optimization problem. We hope that global optimization techniques [6, 16] as well as computer technologies would be developed so that problems in optimization approaches would be solved in a practically acceptable computation time.

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