Chapter 5
The State of Robust Optimization

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Abstract This survey presents a broad overview of the developments in robust optimization over the past 5 years, i.e., between 2011 and 2015. We highlight the advancement of knowledge both with respect to the theory of robust optimization and application areas. From a theoretical standpoint, we describe novel findings in static and multi-stage decision making, the connection with stochastic optimization, distributional robustness and robust nonlinear optimization. In terms of application areas, we consider inventory and logistics, finance, revenue management and health care. We conclude with guidelines for researchers interested in immunizing their problem against uncertainty.

5.1 Introduction

A classical assumption in mathematical programming is that the input data is perfectly known; however, in practice this is a rather rare situation and researchers have attempted to take data uncertainty into account since the seminal work of Charnes and Cooper [35] on chance-constrained programming. Unfortunately, many settings in today's fast-changing environments do not lend themselves to a probabilistic description of uncertainty. Robust optimization was first proposed in the early 1970s in order to provide a decision-making framework when probabilistic models are either unavailable or intractable, and has been the focus of significant research attention from the 1990s onwards.

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Robust optimization assumes that the uncertain data belongs to a convex and bounded set, called uncertainty set, and aims to find a solution that would remain feasible for all possible instances of the data parameters while achieving the best possible worst-case performance, as measured by the objective for the worst-case realization of the parameters. The specific choice of the set naturally plays an important role in terms of tractability and insightfulness of the optimal solution. Key to the tractability of robust optimization is the ability to optimize this worst-case criterion efficiently in presence of two conflicting imperatives: (1) a high level of robustness (protection against uncertainty) (2) the attainment of high-quality objective values (close to the objective of the nominal problem).

Soyster [107] took the first step toward the attainment of a robust optimization methodology in 1973. In order to find a solution immune to data uncertainty in linear programming, he injected the worst-case value of each uncertain parameter into the mathematical programming model; however, the model was deemed too conservative for practical implementation by many business practitioners. Then, in the late 1990s, significant progress in tackling the issue of over-conservatism was made by Ben-Tal and Nemirovski [14–16], El-Ghaoui and Lebret [45] and El-Ghaoui et al. [46]. These papers provided the foundation for modern robust optimization. (Note that the earlier paper of Mulvey et al. [91] uses a different concept also called robust optimization that builds upon the stochastic programming problem and optimizes a weighted combination of the traditional Stochastic Programming (SP) objective and a feasibility penalty function, which penalizes violations of the control constraints. This different definition for robust optimization will not be explored here.) The focus was mainly on constructing models more relevant to practitioners by controlling the degree of conservatism in uncertain linear problems with ellipsoidal uncertainty sets centered at the nominal value of the parameters. These problems were reformulated as second-order cone problems [15]. A drawback is that the resulting model is computationally less efficient than its nominal counterpart due to the added nonlinearity. This makes extensions to integer decision variables challenging from a computational standpoint.

In a milestone work, Bertsimas and Sim [25] investigated novel ways to decrease over-conservatism by tackling what they call the *Price of Robustness* using polyhedral uncertainty sets, which they connect to probabilities of constraint guarantees. Their approach offers full control on the level of conservatism for each constraint through a parameter called the budget of uncertainty that is adjusted by the decision maker. The interpretation of this budget of uncertainty is that it limits the number of uncertain parameters that can deviate from their nominal value. In this approach, the robust counterpart of a linear program remains linear, so that the robust model retains the advantages of a linear optimization model in terms of computational efficiency. Further, it can be readily generalized to discrete optimization, so that the robust counterpart of a integer linear program remains an integer linear problem.

For a comprehensive book treatment and survey on robust optimization, the reader is referred to Ben-Tal et al. [18] and Ben-Tal and Nemirovski [17], respectively. Also, Gorissen et al. [56] provide a practical guide to robust optimization that should be of significant interest to researchers attempting to immunize their problems against parameter ambiguity. Gabrel et al. [50] present an overview of recent advances in robust optimization between 2007 and 2012.

The present chapter focuses on studies indexed in Web of Science and published between 2011 and 2015 included, belonging to the area of Operations Research and Management Science, and having "robust" and "optimization" in their title. We narrowed the list of papers to over one hundred we deemed most significant by taking into account the research area, citation number, authors' track record in robust optimization and the journal's impact factor. This was necessarily a subjective process and some recent papers not listed here will certainly go on to have substantial impact on the field; however, we hope that this survey provides a good starting point into robust optimization today. Related book treatments and milestone works are also presented for reference. Papers are grouped by theme; within each theme they are listed in alphabetical order.

5.2 Theory of Robust Optimization

Since robust static (single-objective) linear programming is now well understood, current research efforts have mostly focused on (1) developing a stronger connection with stochastic optimization, (2) incorporating robust optimization to ambiguous probability distributions of random parameters rather than to ambiguous parameters of unknown but fixed value, (3) studying robust static nonlinear optimization, (4) considering multiple objective criteria, leading to the theory of robust Pareto efficiency, and (5) investigating robust dynamic decision-making. Note that Sniedovich [106] cautions against attempts to tackle severe uncertainty, characterized by a poor point estimate, a likelihood-free quantification of uncertainty and a large uncertainty space, using local robustness models based on the "radius of stability" concept.

5.2.1 Connection with Stochastic Optimization

In Stochastic Optimization, the uncertain data is assumed to be random. In the simplest case, these random parameters have a known probability distribution, while in more advanced settings, this distribution is only partially known. While robust optimization first emerged as a deterministic (worst-case) alternative to stochastic programming, each arising from different models of uncertainty, in recent years increasing numbers of researchers have strived to connect the robust optimization and stochastic optimization paradigms so that the models can be best tailored to the available information.

5.2.1.1 Foundational Work

The most important developments have led to a greater connection between the robust and stochastic optimization descriptions of uncertainty. They have been: (1) an argument that uncertainty sets, approached through robust optimization, should serve as the primitive for stochastic systems and (2) the design of safe tractable approximations of chance constraints to obtain guarantees of constraint violation and their Robust Counterpart representations and (3) a connection between linear problems with uncertain probabilities and uncertainty sets constructed as confidence sets using phi-divergences, with a size of the uncertainty set being controlled by the confidence level of the confidence set. Finally, a fourth work tackles robust nonlinear inequalities and thus develops tractable robust counterparts for new, previously unstudied classes of optimization problems.

Bandi and Bertsimas [8] investigate tractable stochastic analysis in high dimensions via robust optimization. They propose a new approach for stochastic systems based on robust optimization, to address the issue of computational tractability that arises when stochasticity is modeled using probabilities in areas such as queueing networks or multi-bidder auctions. Their framework relies on replacing the Kolmogorov axioms and the concept of random variables as primitives of probability theory, with uncertainty sets derived from some of the implications of probability theory like the central limit theorem. Performance analysis of stochastic systems in this new paradigm leads to linear, semidefinite or mixed integer optimization problems for which efficient algorithms capable of solving problems in high dimensions are available. Further, Bandi and Bertsimas [9] develop an optimal design framework for multi-item auctions based on robust optimization where they adopt an uncertainty set based model instead of using probability distributions.

Nemirovski [93] provides safe tractable approximations of chance constraints when data uncertainty is incorporated through randomly perturbed constraints. He reviews several simulation-based and simulation-free computationally tractable approximations of chance constrained convex programs, primarily, those found in chance constrained linear, conic quadratic and semidefinite programming, when the data is affinely parametrized by a random vector of partially known distribution. The models considered include Conditional Value-at-Risk and Bernstein approximations of the chance constraint. Robust counterpart representations of the approximations are also described.

Ben-Tal et al. [21] investigate robust linear optimization problems where the uncertain parameters with uncertainty regions defined by phi-divergences, which arise in settings involving moments of random variables and expected utility, and applications to asset pricing and the multi-item newsvendor problem. Phi-divergences refer to families of functions that measure "distance" between two vectors. The authors first derive confidence sets that are only asymptotically valid and then describe ways to improve the approximation by considering a modified statistic that uses correction parameters. They finally describe the robust counterpart with phi-divergence uncertainty and study its tractability. This is a special case of distributional robust optimization, which we review in more details below. (The reader is also referred to Bayraksan and Love [11] for a tutorial on data-driven stochastic programming using phi-divergences.)

Finally, Ben-Tal et al. [22] present a model to formulate the robust counterpart of a nonlinear uncertain inequality that is concave in the uncertain parameters, using convex analysis and in particular Fenchel duality. Hence, robust models can be formulated for new classes of optimization models, for which tractable reformulations were not previously available. With respect to tractability, the authors further show that many robust counterparts can be written as linear, quadratic or conic quadratic constraints, or admit a self-concordant barrier function, which implies that the optimization problem can be solved in polynomial time.

5.2.1.2 Distributionally Robust Optimization and Chance Constraints

Ben-Tal et al. [19] consider chance constrained uncertain classification and investigate the problem of constructing robust classifiers when the training is plagued with uncertainty. They also discuss methodologies for classifying uncertain test data points and error measures for evaluating classifiers robust to uncertain data.

Dupacova and Kopa [42] consider stochastic programs whose set of feasible solutions depends on probability distributions that are not fully known, and adopt a contamination technique to study the robustness of results to perturbations on the probabilities. They suggest a robust efficiency test with respect to the second order stochastic dominance criterion.

With motivation drawn from data-driven decision making and sampling problems, Xu et al. [115] study the probabilistic interpretations of robust optimization by showing the connection between robust optimization and distributionally robust stochastic programming, and utilize this result to construct robust optimization formulations for sampled problems.

Zymler et al. [121] develop tractable approximations based on semidefinite programming for distributionally robust chance constraints where only the first- and second-order moments and support of the uncertain parameters are given. They investigate Worst-Case Conditional Value-at-Risk (CVaR) approximations and show that the approximation is tight for robust individual chance constraints with quadratic or concave constraint functions. For joint chance constraints, they show that the Worst-Case CVaR is provably tighter than two benchmark approximations. Further, a distributionally robust joint chance constrained optimization model for the case of the dynamic network design problem under demand uncertainty is developed by Sun et al. [108]. They propose an approach to approximate a joint chance-constrained cell transmission model based system optimal dynamic network design problem with only partial distributional information of uncertain demand.

Wiesemann et al. [113] consider Markov Decision Processes (MDP) with uncertain parameters when an observation history of the MDP is available. They derive a confidence region that contains the unknown parameters with a prespecified probability and obtain a policy that attains the best worst-case performance over this confidence region, using the solution of conic programming problems of moderate size. Further, Wiesemann et al. [114] suggest a unifying framework for modeling and solving distributionally robust convex optimization problems based on standardized ambiguity sets that contain all distributions with prescribed conic representable confidence sets and encompass many ambiguity sets from the literature as special cases. They also model information about statistical indicators that have not yet been considered in the robust optimization literature, such as higher-order moments and the marginal median. The authors determine sharp conditions under which distributionally robust optimization problems based on their approach are computationally tractable, and tractable conservative approximations otherwise.

Alvarez-Miranda et al. [4] presents a note on the Bertsimas and Sim algorithm for robust combinatorial optimization problems with interval uncertainty, where they describe a method to solve fewer deterministic problems to obtain a robust solution. Long and Qi [85] investigate discrete optimization under the distributionally robust framework where they optimize the Entropic Value-at-Risk, a coherent risk measure that serves as a convex approximation of the chance constraint. They propose an approximation algorithm to solve the problem as a sequence of nominal problems and show in computational experiments that the number of nominal problems required is small for various distributional uncertainty sets.

Duzgun and Thiele [43] study 0-1 linear programming with uncertain objective coefficients using a safe tractable approximation of chance constraints, when the decision maker only knows the first two moments and the support of the random variables. They obtain a series of 0-1 linear programming problems parametrized by only one additional variable and show in numerical experiments that their model solves significantly faster than the benchmark robust model.

Zhen [119] investigates a variant of the task assignment problem under uncertainty based on stochastic programming and robust optimization. He develops both a stochastic programming model that tackles the issue of arbitrary probability distributions for the tasks' random workload requirements, and a robust optimization model which can cope with limited information about probability distributions.

Further, Duzgun and Thiele [44] bridge descriptions of uncertainty based on stochastic and robust optimization by considering multiple ranges for each uncertain parameter and setting the maximum number of parameters that can fall within each range, in a model reminiscent of histograms. The corresponding optimization problem can be reformulated in a tractable manner using the total unimodularity property of the uncertainty set and allows for a finer description of uncertainty while preserving tractability.

5.2.2 Nonlinear Optimization

We have already mentioned the work by Ben-Tal et al. [22], which presents a model to formulate the robust counterpart of a nonlinear uncertain inequality concave in the uncertain parameters. In this section, we list additional work pertaining to robust nonlinear optimization.

A specific case of nonlinear problems that are linear in the decision variables but convex in the uncertainty when the worst-case objective is to be maximized is investigated in Kawas and Thiele [76] in the context of portfolio management with uncertain continuously compounded rates of return. In that setting, exact and tractable robust counterparts can be derived. The authors extend their approach to short sales in [77], where they examine a class of non-convex robust optimization problems where the decision variables can be negative, leading to a non-convex problem in the uncertainty.

Ben-Tal and den Hertog [13] immunize conic quadratic optimization problems against ellipsoidal implementation errors. They prove that the robust counterpart of a convex quadratic constraint with ellipsoidal implementation error is equivalent to a system of conic quadratic constraints. They then extend the result to the case in which the uncertainty set is the intersection of two convex quadratic inequalities and show that the robust counterpart for this case is also equivalent to a system of conic quadratic constraints.

Doan et al. [41] build upon the fact that current successful methods for solving semidefinite programs are based on primal-dual interior-point methods and they approach robustness from an algorithmic perspective in order to address ill-conditioning and instability issues. Houska and Diehl [63] present a convex bilevel programming algorithm for the nonlinear min-max problems in semi-infinite programming. A conservative approximation strategy and optimality conditions are provided along with an analysis about strong global and locally quadratic convergence properties.

Poss [98] develops a robust combinatorial optimization model where the uncertain parameters belong to the image of multifunctions of the problem variables. A mixed-integer programming reformulation for the problem, based on the dualization technique is proposed since the feasibility set of the problem is non-convex. Jeyakumar and Li [69] focus on the trust-region problem, which minimizes a nonconvex quadratic function over a ball, and utilize the properties of the problems such as semi-definite linear programming relaxation (SDP-relaxation) and strong duality.

Finally, Suzuki et al. [109] investigate surrogate duality for robust nonlinear optimization and they prove surrogate duality theorems for robust quasiconvex optimization problems and surrogate min-max duality theorems for robust convex optimization problems. They provide necessary and sufficient constraint qualifications for surrogate duality and surrogate min-max duality, and give some examples at which such duality results are used effectively.

5.2.3 Multiple Objectives and Pareto Optimization

A large branch of Robust Optimization focuses on single-objective problems; however, multiple objectives are sometimes considered as well. Hu and Mehrotra [64] studies a family of models for multiexpert multicriteria decision making. Those models utilize the concept of weight robustness in order to identify a (robust) Pareto decision that minimizes the worst-case weighted sum of objectives over a given weight region. The model is then extended to include ambiguity or randomness in the weight region as well as the objective functions. A multi-objective, multi-mode, multi-commodity, and multi-period stochastic robust optimization model is considered by Najafi et al. [92] where the purpose is to achieve the best possible emergency relief for earthquake response. Their method use hierarchical objective functions.

Fliege and Werner [49] consider general convex parametric multiobjective robust optimization problems under data uncertainty. They also present a characterization of the location of the robust Pareto frontier with respect to its nominal counterpart and illustrate their approach on a mean-variance problem. Robust optimization for interactive multiobjective programming with imprecise information is investigated by Hassanzadeh et al. [61] where there are clashing objectives and uncertainty occurs in both objective functions and constraints. They use an iterative procedure to capture the tradeoffs between the objectives.

Fang et al. [47] develop a multiobjective robust optimization model in order to enhance the performance and the robustness simultaneously. The multiobjective particle swarm optimization (MOPSO) algorithm is utilized for producing a set of non-dominated solutions over the entire Pareto space for a non-convex problem, which provides designers with more insightful information. Koebis [79] studied the relation between Scalar Robust Optimization and Unconstrained Multicriteria Optimization with a finite uncertainty set and showed that a unique solution of a robust optimization problem is Pareto optimal for the unconstrained optimization problem.

Iancu and Trichakis [66] incorporate Pareto efficiency to robust linear optimization problems and they present a characterization of Pareto robustly optimal solutions. Specifically, they argue that the classical RO paradigm may not produce solutions that possess the associated property of Pareto optimality, leading to potential inefficiencies and they propose practical methods that generate Pareto robustly optimal solutions by solving optimization problems that are of the same complexity as the underlying robust problems. Their numerical experiments are drawn not only from portfolio optimization—the best-known application area for Pareto optimal solutions—but also inventory management and project management. Hu and Mehrotra [65] consider robust decision making over a set of random targets or risk-averse utilities. In their setting, the random target has a concave cumulative distribution function or a risk-averse manager's utility is concave. Finally, Tong and Wu [111] investigate robust reward-risk ratio optimization models based on the positive homogenous and concave/convex measures of reward and risk.

5.2.4 Multi-Stage Decision-Making

While the main focus of robust optimization was static decision making when it was first investigated in the 1990s (following Soyster's 1973 work), multi-stage robust decision making has garnered substantial attention in recent years. In this setting, uncertainty is revealed in stages and the decision maker adjusts his strategy based on the new information. The ability to take recourse action also allows the decision maker to tackle over-conservatism issues that affect static robust optimization when applied over multiple time periods. Delage and Iancu [40] provide an excellent tutorial on robust multi-stage decision-making.

5.2.4.1 Two Stages

Due to the difficulty inherent in multiple stages, many works have focused on two-stage robust optimization. The most notable works in this category are Bertsimas et al. [28], Hanasusanto et al. [60] and Ben-Tal et al. [23].

Bertsimas et al. [28] analyze the performance of static solutions for two-stage adjustable robust linear optimization problems with uncertain constraint and objective coefficients. They show that for a fairly general class of uncertainty sets, a static solution is optimal for two-stage adjustable robust linear optimization, which is quite counter-intuitive since static policies are generally believed to be conservative. Further, they develop a tight characterization of the adaptivity gap when no static solution is optimal. Their results lead to new geometric intuition about the performance of static robust solutions for adjustable robust problems, based on a certain transformation of the uncertainty set which helps highlight properties of the set when static robust policies do not perform well. Hence, the paper provides guidance in selecting the uncertainty set such that the adjustable robust problem can be well approximated by a static solution.

Hanasusanto et al. [60] extends the robust optimization methodology to problems with integer recourse, by approximating two-stage robust binary programs by their corresponding K-adaptability problems, in which the decision maker pre-commits in the first stage to K second-stage policies and implements the best of these policies once the uncertain parameters are realized. The authors study the quality of their approximation and the computational complexity of the K-adaptability problem. Further, they propose two mixed-integer linear programming reformulations that can be solved with off-the-shelf software.

Ben-Tal et al. [23] develop a method for approximately solving a robust optimization problem using tools from online convex optimization, where at every stage a standard (nonrobust) optimization program is solved. They find an approximate robust solution using a number of calls to an oracle that solves the original (nonrobust) problem that is inversely proportional to the square of the target accuracy. Their approach yields significant computational benefits when finding the exact solution of the robust problem is a NP-hard problem, for instance in the case of robust support vector machine with an ellipsoidal uncertainty set.

Additional work includes the following. Minoux [89] introduces a new subclass of polynomially solvable two-stage robust optimization problems with uncertainty on the right-hand side coefficients. Remli and Rekik [101] investigate the problem of combinatorial auctions in transportation services under uncertain shipment volumes and develop a two-stage robust formulation where they use a constraint generation algorithm. Chan et al. [33] propose a computationally tractable and dynamic multi-stage decision methodology that can hedge against uncertainty by utilizing information from the previous stage iteratively, with an application to IMRT (intensity-modulated radiation therapy) treatment planning for lung cancer. Bo and Zhao [118] solve two-stage robust optimization problems by developing a column-and-constraint generation algorithm and compare their approach with the existing Benders-style cutting plane methods.

5.2.4.2 Optimal and Approximate Policies

We have already mentioned Bertsimas et al. [28], where the authors investigate the performance of static policies in two-stage robust linear optimization. Further, Bertsimas et al. [27] analyze the effect of geometric properties of uncertainty sets, such as symmetry, in the power of finite adaptability in multi-stage stochastic and adaptive optimization. They investigate finitely adaptable solutions, which generalize the notion of static robust solutions in the sense that a small set of solutions is specified for each stage and the solution policy implements the best solution from the set, depending on the realization of the uncertain parameters in past stages. In particular, they show that a class of finitely adaptable solutions is a good approximation for both the multistage stochastic and the adaptive optimization problem.

Kuhn et al. [80] consider primal and dual linear decision rule policies in stochastic and robust programming, and compute the loss of optimality due to this policy. They show that both approximate problems are equivalent to tractable linear or semidefinite programs of moderate sizes. Shapiro [104] considers the adjustable robust approach to multistage optimization, derives related dynamic programming equations and connects the problem to risk-averse stochastic programming. He also shows that, as in the risk-neutral case, a basestock policy is optimal.

Supermodularity and affine policies in a particular class of dynamic robust optimization problems are investigated by Iancu et al. [67]. They aim to provide a connection between dynamic programming and decision rules, and solve tractable convex optimization problems. Bertsimas and Goyal [24] consider adjustable robust versions of convex optimization problems where the constraints and objectives are uncertain and they show that a static robust solution yields a good approximation for these problems under general assumptions.

5.3 Application Areas of Robust Optimization

5.3.1 Classical Logistics Problems

5.3.1.1 Newsvendor Problem

The newsvendor problem is the building block of modern inventory theory. While robust newsvendor problems were first studied long before the time window for publication of interest in this review, they continue to be the focus of significant research. Jiang et al. [71] consider robust newsvendor competition under asymmetric information about future demand realizations. They devise an approach based on absolute regret minimization and derive closed-form expressions for the robust optimization Nash equilibrium solution for a game with an arbitrary number of players. Qiu et al. [100] investigate the robust inventory decision-making problem faced by risk-averse managers with incomplete demand information with ellipsoidal uncertainty in a newsvendor setting. Three basic models are developed: expected profit maximization, Conditional Value-at-Risk (or CVaR)-based profit maximization, and a combination of these two.

Finally, Hanasusanto et al. [59] consider multi-item newsvendor problems from a distributional robust optimization perspective when the demand distributions are multimodal. The products considered are subject to fashion trends that are not fully grasped at the time when orders are placed. Spatially separated clusters of probability mass lack a complete description. The decision-maker minimizes the worst-case risk of the order portfolio over all distributions compatible with the modality information. The authors show the robust problem admits a conservative, tractable approximation using quadratic decision rules, which achieves a high level of accuracy in numerical tests.

5.3.1.2 Combinatorial Optimization Problems

Remli and Rekik [101] study the robust winner determination problem for combinatorial auctions in transportation services when shipment volumes are uncertain and propose a two-stage robust formulation solved using a constraint generation algorithm.

Poss [98] extends the Bertsimas-and-Sim model for robust combinatorial optimization using variable budgeted uncertainty, which is less conservative than (traditional) budget of uncertainty for vectors with few non-zero components. The author uses a mixed-integer programming reformulation for the problem and compare his approach with that of Bertsimas and Sim on the robust knapsack problem, where variable budgeted uncertainty achieves a reduction of the price of robustness by an average of 18 %.

Chassein and Goerigk [36] propose a new bound for the midpoint solution in minmax regret optimization, which evaluates a solution against the respective optimum objective value in each scenario and aims to find robust solutions that achieves the lowest worst-case difference between the two. Heuristics with performance guarantees have potentially great value in this context because most polynomially solvable optimization problems have strongly NP-hard minmax regret counterparts. One of these approximations is the midpoint solution, obtained when the decision maker approximates the uncertain parameters by the average of their lower and upper bound and solves that problem. They derive an instance-dependent performance guarantee for the midpoint solution of at most 2 and apply their methodology to the robust shortest path problem.

5.3.1.3 Scheduling

Robust berth scheduling at marine container terminals where vessel arrival and handling times are uncertain is studied by Golias et al. in [55]. They propose a bi-objective optimization problem and a heuristic algorithm, and test the results using simulation.

Varas et al. [112] focus on production scheduling for a sawmill where the uncertainty arises from the supply of logs and the finished product orders. Using a two-stage adaptive robust optimization approach, Lima et al. [84] investigate weekly self-scheduling, forward contracting, and pool involvement for an electricity producer operating a mixed power generation station.

Che et al. [37] study the cyclic hoist scheduling problem with processing time window constraints. The uncertainty comes from the perturbations or variations of certain degree in the hoist transportation times. The authors propose a method to measure the robustness of a cyclic hoist schedule and develop a bi-objective mixed integer linear programming model to optimize cycle time and robustness.

5.3.2 Facility Location

Facility location is concerned with the optimal placement of facilities to minimize the design and transportation costs while considering factors such as customer satisfaction, covering/serving a certain area, or avoiding placing hazardous materials near housing. Baron et al. [10] applied robust optimization to a capacitated multi-period fixed-charge network location problem in a network under uncertain demand over multiple periods. Their goal is to determine the number of facilities, their location and capacities, as well as the production amount and allocation of demand to facilities.

Another network design problem has been studied by Li et al. [83], for the planning of network infrastructure such as roads, pipelines and telecommunication systems. Uncertainty originates from the demand, and maintenance related issues such as operating costs, degradation rates. They propose an efficient and tractable approach for finding robust optimum solutions to linear and quadratic programming problems with interval uncertainty using a worst case analysis.

Robust hub location problems are studied in Alumur et al. [2] where the uncertainty arises due to the set-up costs for the hubs and the demands to be transported between the nodes. The authors analyze the changes in the solutions driven by the different sources of uncertainty when considered either in isolation or in combination.

Guelpinar et al. [58] consider a stochastic facility location problem in which multiple capacitated facilities serve customers with a single product, given uncertain customer demand and a constraint on the stock-out probability. Robust optimization strategies for facility location appear to have better worst-case performance than non-robust strategies.

Gabrel et al. [51] investigate a robust version of the location transportation problem with an uncertain demand using a two-stage formulation. The resulting robust formulation is a convex (nonlinear) program, and the authors apply a cutting plane algorithm in order to solve the problem exactly. Finally, Ghezavati et al. [53] investigate the optimization of reliability for a hierarchical facility location problem under disaster relief situations by a chance-constrained programming, with the aim of rapidly bringing the appropriate emergency supplies to the affected villages.

5.3.3 Supply Chain Management

Supply chain problems deal with the management of the flow of goods and services from the producer to the customer. It includes the movement and storage of raw materials, work-in-process inventory, and finished goods from point of origin to point of consumption in a way that ensures good service level and high profit. There exists uncertainty in many parts of a supply chain especially due to demand uncertainty.

A production planning problem in small-size furniture companies has been studied by Alem et al. [1]. They utilized robust optimization tools to derive robust combined lot-sizing and cutting-stock models when production costs and product demands are uncertainty. Their motivation to adopt robust optimization instead of two-stage stochastic programming was the absence of an explicit probabilistic description of the input data and the incentive of not having to deal with a large number of scenarios in robust optimization.

Aouam and Brahimi [6] considered an integrated production planning problem and order acceptance decisions under demand uncertainty. Orders/customers are classified into classes with respect to the marginal revenue, quantity they are willing to buy and reliability assessment. Their model provides flexibility to decide on the fraction of demand to be satisfied from each customer class and consider production-related constraints as well as factors such as congestion on production lead times. An order acceptance strategy allows the decision maker to maintain an appropriate level of utilization.

Schoenlein et al. [103] investigate the measurement and optimization of the robust stability of multiclass queueing networks with an application to dynamic supply chains. Stability of these networks implies that the total number of customers in the network remains bounded over time. The authors rely on fluid network analysis to quantify robustness using a single number, called the stability radius.

Qiu and Shang [99] study robust multi-period inventory decisions for risk-averse managers with partial demand distribution information for products with a short life cycle. The three inventory models we developed aim respectively to maximize expected profit, maximize conditional value-at-risk-based profit, and balance between the two objectives where the corresponding robust counterparts are presented.

Ashayeri et al. [7] consider a supply chain where a company faces bankruptcy to fulfill its debt obligation with limited financial resources. The uncertainty arises from demands and exchange rates. They formulate a MIP model with specific downsizing features, which maximizes the utilization of resources through a combined operation of demand selection and production assets reallocation. A pulp production planning and supply chain management has been studied in Carlsson et al. [32]. They utilize a robust optimization approach to handle the demand uncertainty and to establish a distribution plan, together with related inventory management. In this setup, they observe that there is no need for explicit safety stock levels and they achieve higher profit. Kawas et al. [78] study a game-theoretic setup of a production planning problem under uncertainty in which a company is exposed to the risk of failing authoritative inspections due to non-compliance with enforced regulations.

Finally, Kang et al. [74] investigate distribution-dependent robust linear optimization with applications to inventory control where every element of the constraint matrix is subject to uncertainty and is modeled as a random variable with a bounded support.

5.3.4 Industry-Specific Applications

In this section, we reference papers on three industry-specific logistics-driven applications that have received substantial attention in the robust optimization literature.

In *warehouse management*, Ang et al. [5] propose a robust storage assignment approach in unit-load warehouses facing variable supply and uncertain demand in a multi-period setting. They assume a factor-based demand model and minimize the worst-case expected total travel in the warehouse with distributional ambiguity of demand.

In *train timetabling operations*, Cacchiani et al. [30] focus on Lagrangian heuristics the application of train time-tabling. Galli [52] describes the models and algorithms that arise from implementing recoverable robust optimization to train platforming and rolling stock planning, where the concept of recoverable robustness has been defined in Liebchen et al. A survey of nominal and robust train timetabling problems in its nominal and robust versions is presented in Cacchiani and Toth [29].

In the sawmill planning problem, in addition to previously-mentioned Varas et al. [112], which focuses on production scheduling for a sawmill where the uncertainty arises from the supply of logs and the finished product orders, Alvarez and Vera [3] consider a related formulation where variability affects the yield coefficients related to the cutting patterns used. Finally, Ide et al. [68] investigate an application of deterministic and robust optimization in the wood cutting industry with the goal of attaining resource efficiency.

5.3.5 Finance

5.3.5.1 General Portfolio Problems

Robust portfolio optimization is studied by Ye et al. [117] in the context of a Markowitz mean-variance model with uncertainty on mean and covariance matrix. They formulate the robust problem as a second-order cone programming problem and show in computational experiments that the portfolios generated by the robust model are not as sensitive to input errors as the ones given by the classical model.

Nguyen and Lo [94] develop robust portfolio optimization models based on investors' rankings of the assets instead of estimates of their parameters such as expected returns, when the ranking is subject to uncertainty. They solve a robust ranking problem using a constraint generation scheme. Marzban et al. [87] study a multi-period robust optimization model including stocks and American style options. The decision maker selects the level of robustness through the length and the type of the uncertainty set.

5.3.5.2 Risk Measures

Chen et al. [38] considers robust portfolio problems where expected utility is maximized under ambiguous distributions of the investment return, while Moon and Yao [90] investigate robust portfolio management when absolute deviation from the mean is used as a risk measure, leading to a linear programming problem. The authors test the robust strategies on real market data and discuss performance of the robust optimization model based on financial elasticity, standard deviation, and market condition such as growth, steady state, and decline in trend.

Fertis et al. [48] propose the concept of robust risk measure, defined as the worst possible of predefined risks when each among a set of given probability measures is likely to occur. In particular, they introduce a robust version of CVaR and of entropy-based risk measures, and show how to compute and optimize the Robust CVaR using convex duality methods.

Kakouris and Rustem [73] consider robust portfolio optimization with copulas, where copulas are used to describe the dependence between random variables. They provide the copula formulation of the CVaR of a portfolio and extend their approach to Worst Case CVaR (WCVaR) though the use of rival copulas exploiting a variety of dependence structures.

Kapsos et al. [75] investigate the worst-case robust Omega ratio, where the Omega ratio is a performance measure addressing the shortcomings of the Sharpe ratio and is defined as the probability weighted ratio of gains versus losses for some threshold return target. The authors investigate the problem arising from the probability distribution of the asset returns being only partially known and show that the problem remains tractable for three types of uncertainty.

In the most recent body of work, Lagos et al. [81] analyzes the characterizations of the robust uncertainty sets related to coherent and distortion risk measures and aim to mitigate estimation errors of the Conditional Value-at-Risk. Maillet et al. [86] investigate global minimum variance portfolio optimization under some model risk based on a robust regression-based approach. The robust portfolio corresponds to the global minimum variance portfolio in the worst-case scenario and it provides protection against errors in the reference sample covariance matrix. Finally, Bertsimas and Takeda [26] study optimization over coherent risk measures and non-convexities where the relation between coherent risk measures and uncertainty sets of robust optimization is taken into consideration.

5.3.6 Machine Learning and Statistics

The incorporation of Machine Learning and Robust Optimization is a growing field. The reader is referred to Caramanis et al. [31] for an overview of robust optimization in machine learning. Ben-Tal et al. [19] focus on the problem of constructing robust classifiers when the training is subject to uncertainty. The problem is formulated as a chance-constrained program that is relaxed utilizing Bernstein's approximation to yield a second-order cone problem whose solution is guaranteed to be feasible for the original problem. Xu et al. [116] study robust principal component analysis in the presence of contaminated data.

Ozmen et al. [96] utilize Conic Multivariate Adaptive Regression Splines (CMARS) for generalizing the model identification problem including the existence of uncertainty with the aim to increase the trustworthiness of the solution in case of data perturbation. Beliakov and Kelarev [12] study global non-smooth optimization in robust multivariate regression where the objective is non-smooth, non-convex and expensive to calculate. They analyze the numerical performance of several derivative-free optimization algorithms with the aim of computing robust multivariate estimators.

Support vector machine (SVM) classifiers with uncertain knowledge sets via robust optimization are studied by Jeyakumar et al. [70]. They show how data uncertainty in knowledge sets can be handled in SVM classification and provide knowledge-based SVM classifiers with uncertain knowledge sets using convex quadratic optimization duality.

5.3.7 Energy Systems

Another area that has seen significant growth recently is robust optimization in energy. An application of robust optimization to renewable energy, specifically wind energy, is investigated in Jiang et al. [72], with the objective of providing a robust unit commitment schedule for the thermal generators in the day-ahead market that minimizes the total cost under wind output uncertainty.

Classen et al. [39] study a robust optimization model and cutting planes for the planning of energy-efficient wireless networks under demand uncertainty where they apply three different cutting plane methods. Goryashko and Nemirovski [57] study robust energy cost optimization of a water distribution system with uncertain demand with the aim to optimize daily operation of pumping stations based on the concept of Affinely Adjustable Robust Optimization.

Lima [84] works on weekly self-scheduling, forward contracting, and pool involvement for an electricity producer under three different scenarios, corresponding to electricity price forecasts. Sauma et al. [102] adopt a robust optimization approach to assess the effect of delays in the connection-to-the-grid time of new generation power plants over transmission expansion planning where the uncertainty arises from construction times of new power plants. Finally, Zugno and Conejo [120] work on the energy and reserve dispatch in electricity markets where they cast the problem as an adaptive robust optimization problem instead of a stochastic programming problem due to computational efficiency issues.

5.3.8 Public Good

The public good applications aim to improve the health, safety and well-being of the general public. Two main fields are humanitarian relief and health care applications. Examples include determining treatment plans in a hospital, patient transportation among hospitals, patient-doctor scheduling and constructing emergency evacuation routes during a disaster (fire, tsunami, earthquake).

5.3.8.1 Humanitarian Logistics/Emergency Logistics Planning

After a disaster occurs, humanitarian and state organizations gather resources and staff to serve a community's needs in an efficient way. Robust optimization has great relevance in humanitarian relief supply chains since we face data uncertainty during disasters.

Ben-Tal et al. [20] investigate a robust logistics plan generation methodology that can hedge against demand uncertainty. They study the dynamic emergency response assignment and evacuation traffic flow problems. They apply an affinely adjustable robust counterpart approach in order to provide better emergency logistics plans. A multi-objective robust optimization model for logistics planning during earthquake is proposed in Najafi et al. [92]. This paper propose a multi-objective, multi-mode, multi-commodity, and multi-period stochastic model to manage the scarce sources efficiently and they ensure that the distribution plan performs well under the various situations due to robustness.

Tajik et al. [110] adopt a robust optimization approach for the pollution routing problem with pickup and delivery under uncertain data where the aim is to reduce fuel consumption and decrease green house gases emission due to their harmful effects on environment and human health. Their study addresses a new time window pickup-delivery pollution routing problem (TWPDPRP) to deal with uncertain input data.

The most recent developments in robust humanitarian logistics are the following. Lassiter et al. [82] consider the flexible allocation of the workforce after a disaster in order take into account changing (uncertain) needs and volunteer preferences. They use robust optimization to handle the uncertainty in task demands and derive Pareto optimality and allocation decisions for any level of conservativeness. Ghezavati et al. [53] investigate a hierarchical facility location problem under disaster relief situations where robust optimization and chance-constrained programming are applied. Shishebori and Babadi [105] design a robust and reliable medical services network under uncertain environment and system disruptions. Finally, Paul and Wang [97] study the United States Department of Agriculture food aid bid allocations, which aims at providing food aid annually in response to global emergencies and famine.

5.3.8.2 Health Care Applications

Chan et al. [33] consider an adaptive robust optimization approach to IMRT (intensity-modulated radiation therapy) treatment planning for lung cancer. They propose a computationally tractable and dynamic multi-stage decision methodology that can hedge against uncertainty by utilizing the information from the previous stage iteratively. Nha et al. [95] develops a new robust design optimization procedure based on a lexicographical dynamic goal programming approach for implementing time-series based multi-responses for drug formulations in the pharmaceutical industry.

Holte and Mannino [62] study the problem of allocating scarce resources such as operating rooms or medical staff to medical staff when the exact number of patients for each specialty is uncertain and when the allocation is defined over a short period of time such as a week and subsequently repeated over the time horizon. They adopt an adjustable optimization approach and develop a row and column generation algorithm to solve it efficiently.

Chan et al. [34] consider a robust-CVaR optimization approach with application to breast cancer therapy where the loss distribution is dependent on the state of an underlying system and the fraction of time spent in each state is uncertain. Finally, Meng et al. [88] investigate a robust optimization model for managing elective admission in a public hospital, given the priority of emergency patients over elective ones. They propose an optimized budget of variation approach that maximizes the level of uncertainty the admission system can withstand without violating the expected bed shortfall constraint and solve the robust optimization model by deriving a second order conic programming counterpart of the model.

5.4 Conclusions and Guidelines for Implementation

We have provided an overview of recent developments in robust optimization over the past 5 years. As robust optimization is now about 20 years old, it has become a well-established tool to address decision-making under uncertainty but also remains a thriving research area. We remind the reader of the practical guide to implementing robust optimization provided in Gorissen et al. [56]. The researcher interested in implementing robust optimization faces several modeling choices, which will impact the structure of the robust problem, its tractability and the insights the decision maker can gain into the optimal solution.

First, should the uncertainty be on the problem parameters themselves (leading to the classical robust optimization paradigm) or their underlying probabilistic distributions (yielding distributionally robust optimization or DRO)? DRO is particularly suitable if the stochastic programming version of the problem is tractable and the decision maker feels confident that he knows specific attributes of the family of probability distributions, such as their first two moments. If the SP version of the problem suffers from tractability issues, then adding robustness to that formulation will make the problem at least as computationally demanding; hence, it will then be more promising to apply robust optimization to the ambiguous parameters.

Second, what is the type of uncertainty set most suitable for the problem at hand? When the uncertainty is on the ambiguous parameters, the decision maker can then either use polyhedral uncertainty sets, which do not change the complexity of the mathematical programming problems considered but lead to additional constraints and variables in the tractable reformulation, or ellipsoidal uncertainty sets, which do not require any new variable or constraint but introduce non-linearities. When some decision variables are integer, polyhedral uncertainty sets thus seem particularly suitable. When the uncertainty is on the probability distributions, the uncertainty set may for instance incorporate knowledge of support, mean, covariance, directional deviations in the manner of Goh and Sim [54].

Third, is it possible to take corrective action after part of the uncertainty is revealed? If yes, adaptive or adjustable robust optimization will be advisable to address potential over-conservatism issues and lead to decision rules that are easy to implement in practice. The choice of those decision rules and the fine-tuning of their parameters have implications on computational tractability, closeness to optimality and insightfulness of the optimal solution.

In today's fast-changing environment, robust optimization presents an appealing framework that is both intuitive and lends itself to computationally tractable reformulations that either are exact or approximations documented in numerical experiments to perform well against benchmarks. RO is hence expected to keep increasing in relevance and importance in the arsenal of decision making tools of the operations research professional. In the future, researchers are likely to continue investigating improved approaches to multi-stage optimization, and to further connect RO with SP in order to provide an integrated approach to decision-making under uncertainty. Cutting-edge areas of interest include, but are not limited to, complex problems such as adversarial risk analysis, policy design, performance evaluation, optimization with multiple criteria or objectives, alternative models of uncertainty such as fuzzy optimization, new insights into sensitivity analysis and application-specific results on topics that remain of prime relevance today such as job-shop scheduling and portfolio management.

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