Trends in the History of Science

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From Classical to Modern Algebraic Geometry

Corrado Segre's Mastership and Legacy

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Preface

At the origin of the Italian School of Algebraic Geometry, the figure of Corrado Segre (Saluzzo 1863–Torino 1924), celebrated for the excellence of his geometric investigations and his exemplary style of scholar, still offers today an enduring model of scientific education to new generations and an outstanding scientific legacy to contemporary geometry.

Corrado Segre played the role of leader of the above-mentioned School in the decades around the beginning of the twentieth century, for scientific, historical, and biographical reasons. The great British geometer Henry Frederick Baker affirmed that "He could probably be said to be the father of the wonderful, Italian School which has achieved so much in the birational theory of algebraical loci."[1]

The times were favorable for several reasons. Corrado Segre, as well as his students and scientific companions, belonged to the first generations growing up in the new, unified Italy. These scholars and scientists could be described, in some sense, as builders of the nation. They were adding the moral and concrete task of building new scientific institutions for a new, modernized country, to their own scientific interest.

In a more general sense, Segre's School, though rooted within a specific discipline of the domain of science, was nevertheless an open and inclusive community of persons. Through scientific exchange and debate, they were interacting on a larger series of cultural issues: amongst themselves and both with society and the rest of the world. Certainly, in this period the ideas of science, education, and progress had many opportunities to meld; indeed, the model of the Italian School of Algebraic Geometry is a fine example of the consequences of this melding.

Federigo Enriques, a former disciple of Segre, was also certainly aware of this atmosphere and of the significance of Segre's leadership. In 1938, while speaking about mathematical schools and the progress and evolution in mathematics, he

[1]Baker, Henry Frederick, Corrado Segre, *Journal of the London Mathematical Society*, 1 (1926): 263–271, on p. 269.

probably referred back to his personal experiences and memories. His words are the most appropriate to understand the mood and soul of Segre's School:

> Actually the progress of mathematics doesn't depend exclusively on the efforts of individual research, but also on the relationship between researchers and the cultural environment from which they originate. In order to correctly understand what history teaches us, it is necessary to underline the importance of the school in forming the mathematician […]. The experiences and inspirations, together with unsuccessful attempts or glimpsed results and problems, as well as different types of research criteria formulated for practical purposes can only be communicated verbally in the intimacy of conversations between colleagues and friends or even better between master and pupil. The pupil continues the master's thoughts and ideas even after he has more or less knowingly reworked them into a new form. […] Schools have a tendency to grow beyond their original conception and at that point the student will be influenced by the new and different ideas which nurture him. The development of mathematical schools […] gains new life passing from one country to another, almost as if the spirit of the world could participate on a larger scale in this collective work.[2]

The School led by Segre flourished so much as to be directly associated with his name. In the same decades, Corrado Segre was a world-renowned master of geometrical sciences and author of fundamental achievements in the study of algebraic varieties.

On the occasion of the 150th anniversary of his birth, the Academy of Sciences, the University and the Polytechnic of Turin, in collaboration with several other scientific institutions organised the international conference *Homage to Corrado Segre (1863–1924)* and a series of initiatives to commemorate Corrado Segre and to reconstruct in a unified view the different aspects of Segre's scientific legacy.[3]

As a consequence, the conference brought together scholars in different fields, mainly from history of mathematics and algebraic geometry.

This volume recollects the refereed contributions of most of the participants in the conference and a few more invited papers, and naturally relies on two sections, reflecting the historical and the geometrical character of the international meeting.

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[2]Enriques, Federigo, *Le matematiche nella storia e nella cultura*, Bologna: Zanichelli 1938, pp. 180–181: *Invero i progressi delle matematiche non dipendono soltanto dallo sforzo della ricerca individuale, sì anche dai rapporti dei ricercatori fra loro e coll'ambiente di cultura da cui traggono origine. Per bene comprendere questo insegnamento della storia, conviene rilevare l'importanza che ha nella formazione del matematico la scuola […] Le esperienze e le suggestioni che si legano a tentativi non riusciti o a resultati e problemi appena intravisti, tanti criterii di ricerca che non sono formulati in maniera astratta, si comunicano soltanto a voce nell'intimità delle conversazioni fra colleghi ed amici o meglio fra maestro e scolaro. Lo scolaro riprende e continua il pensiero del maestro anche quando più o meno consapevolmente lo ricrea in una nuova forma […] La scuola tende ad allargarsi al di fuori del proprio ambiente di origine, ed allora l'influenza sullo scolaro viene a comporsi con altri motivi diversi che la fecondano […] Lo sviluppo delle scuole matematiche […] si ravviva passando da una nazione ad un'altra, quasi a far partecipare più largamente all'opera comune lo spirito del mondo.*

[3]The Conference was held in Turin from November 28 to 30, 2013. See: http://ricerca.mat.uniroma3.it/GVA/Segre150/segre150.html.

The title of the volume contains the words *from classical to modern algebraic geometry*. They put in evidence the extraordinary influence of Segre and his importance today. This is generally visible in all the contributions, offered by the main specialists on subjects related to the life and work of Corrado Segre.

Historians propose to reconstruct how Segre's leadership became recognised in Italy and abroad taking also in account a great number of unpublished and unknown documents.

The first essay by Alberto Conte and Livia Giacardi offers a picture frame for the following papers. The authors, running through the 36 years (1888–1924) of teaching higher geometry in Turin, show how Segre's courses were a veritable forge for future researchers. The forty handwritten notebooks of his university lectures and other unpublished sources allow them to understand how he stimulated and closely interacted with his Italian and foreign students, and to identify the most salient features of his scientific leadership. Erika Luciano and Silvia Roero in their paper, relying on a very rich documentation, illustrate the complex dynamic of scientific exchanges with the international mathematical community, as well as some aspects of the scientific and personal biography of Segre, related to his institutional, political, and editorial role. The remaining essays are dedicated to a thorough analysis of less studied aspects of Segre's work relating to three different stages of his life. David Rowe shows how line geometry was an excellent starting point for both Segre and Italian algebraic geometry, concentrating his attention on two of Segre's papers dating back to the beginning of the eighties. Paola Gario focuses on the relationship between Segre, Guido Castelnuovo, and Federigo Enriques, referring to the period (1887–1897) of their collaboration on the problem of the resolution of singularities of algebraic surfaces, without overlooking the interpersonal dynamics emerging from their rich correspondence. Finally, Aldo Brigaglia shows the genesis and the historical and scientific relevance of Segre's important works concerning the complex projective geometry and comments on how a genuine recognition of it arrived only later, with Julian Coolidge's work, and above all that by Elie Cartan.

The section is completed by the biographical timeline of Segre livened up by quotations and enriched by the portraits of the mathematician at different ages.

The section dedicated to contributions from the field of algebraic geometry confirms the continuity and the presence of the research themes considered by Corrado Segre. Classical algebraic geometry is sometimes used today as a name for a large active area of research within algebraic geometry. This area appears to be connected in a more direct way to the language, themes, and problems (often concerning concrete examples or special projective varieties), which were familiar to algebraic geometers of Segre's times. The above-mentioned contributions largely fit in this area. The picture emerging from them enlightens a very interesting series of nice geometric problems and new results. Several subjects of classical flavor are touched by this picture, with the use of modern techniques and new methods. All the contributions have a correspondence to Segre's work. We can partially summarize them as follows.

Hyperquadrics are considered by Laura Costa, Maria Rosa Mirò-Roig, and Joan Pons-Llopis in order to generalize and study, on an odd dimensional hyperquadric, instanton bundles, and their families. A special attention is payed to three-dimensional quadrics and to Hooft bundles on them. The paper by Luca Chiantini and Duccio Sacchi aims to introduce a notion of Hilbert function for subvarieties of Segre products, that is, products of projective spaces. This notion is more sophisticated than the natural one, defined via Segre embedding, and appears to be a starting point for a new theory and advances in the study of the complexity of a general tensor, with applications to several fields. Nodal cubic threefolds with isolated singularities were classified by Segre. In particular, the cubic threefold with maximal number of nodes bears his name as the Segre primal. Igor Dolgachev takes up the study of six nodal cubic threefolds, which is a case of special beauty and interest. The split surface of lines of a six nodal cubic threefold is described in all details. The Segre primal and its ubiquity in geometry are also revisited. The classical, and modern, subject of algebraic surfaces of general type and their moduli is well represented by the paper of Margarida Mendes Lopes and Rita Pardini. In it, some famous surfaces appear, namely Enriques surfaces and Godeaux surfaces. In particular, the authors construct the family of those Enriques surfaces which are quotient of a Godeaux surface by an involution, proving several properties related to this construction. As mentioned, algebraic curves and their linear series are very well present in this volume, due to the lecture notes of Corrado Segre reproduced here. Moreover, a paper by Edoardo Sernesi offers to the reader a self-contained and very interesting reconstruction of the proof of Riemann–Roch theorem for curves, which is essentially due to Castelnuovo and reflects ideas and observations of both Segre and Castelnuovo. Line geometry is a further very important theme where Corrado Segre played a leading role. This theme is taken up by Emilia Mezzetti in the paper "Geometry of lines and degeneracy loci of morphisms of vector bundles." The title reflects the modern point of view and the modern use of vector bundles techniques in view of several applications. Nevertheless, this paper is also an original survey where the deep connections between new methods and Segre's ideas are pointed out. Finally, the Cremona group of birational automorphisms of a projective space is obviously present in this volume. On this subject, new and important progress was made very recently. Moreover, very interesting connections to other fields, for instance, in the different fields of complex dynamics and of algebraic statistics, were deepened. Two papers on the Cremona group complete the series of geometric contributions to this volume. They reflect very well some of the recent changes of the "state of the art" on this subject. One is the paper by Jeremy Blanc on the set of the algebraic elements of the Cremona group of any projective space, which is proved to be a non-closed countable union of closed sets. The other paper, due to Ciro Ciliberto, Maria Angelica Cueto, Massimiliano Mella, Kristian Ranestad, and Piotr Zwiernik, introduces an effective method, with applications, to linearize suitable rational varieties by a sequence of Cremona transformations.

Notably, papers from historical or from geometric sections often converge on the same geometric theme or question, offering different but complementary points of view. The reader can profit of both. This appears to be an interesting feature of this book and an achievement of the goals of the conference *Homage to Corrado Segre*.

Furthermore, a third part of the volume is enriched by the anastatic print of the unpublished Segre's manuscript *Introduzione alla geometria sugli enti algebrici semplicemente infiniti*. This document, with an introduction by Alberto Conte, is one of the forty manuscripts which recollect the notes of Segre's courses during the academic years. It corresponds to the year 1890–1891. Interestingly, it appears as a preliminary version of the famous memoir on algebraic curves published by Segre in 1894.[4]

This volume is completed by the list of documents of the Segre Archives, due to Livia, Giacardi, Erika Luciano, Chiara Pizzarelli and C. Silvia Roero. The bibliography of the works by Corrado Segre, including all the reports on the papers of his disciples and collaborators, and the list of the handwritten notebooks, closes the volume.

Turin, Italy — Livia Giacardi
Rome, Italy — Alessandro Verra

[4]Segre, Corrado, Introduzione alla geometria sopra un ente algebrico semplicemente infinito, *Annali di Matematica pura ed applicata*, 2, 22 (1894a): 41–142.

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At the end of this work, which has involved the combined collaboration of mathematicians and historians with the intention of giving us a fuller profile of Segre's work, we wish to thank all those who have provided their collaboration. First of all, heartfelt thanks go to the authors and referees who made it possible to carry out this work, and to Birkhäuser for having included the volume in the series *Trends in the History of Science.* Our most wholehearted thanks also go to the President of the Academy of Science of Turin, Alberto Conte, to the Heads of the Department of Mathematics of the University of Torino, Catterina Dagnino and then Alessandro Andretta for according permission to reproduce the portraits of Segre and the handwritten Segre's notebook, and to all those who in various ways have provided their help, Ciro Ciliberto, Sloan Despeaux, Judith Goodstein, Steven and Beverly Kleiman, Antonio Salmeri, Rosanna Roccia, Norbert Schappacher, and KimWilliams.

Special thanks go to the heirs of Corrado Segre, Daniele, Lorenzo and Silvano Fuà for allowing us to access to the unpublished documents in their possession and for donating them to the University of Turin. Our thanks also go to the directors and personnel of the various archives we explored, Paola Novaria and Giuliana Maria Borghino Sinleber (Archivio Storico, University of Turin), Manuel Onjugaren (Archives, Losanna), Elena Borgi and Lavinia Iazzetti (Accademia delle Scienze, Turin), Laura Bitossi (Biblioteca di Scienze, Firenze), Anna Dagnese, Laura Garbolino, Orietta Piccini, Giulia Scarcia, Giuseppe Semeraro, and Antonella Taragna (Biblioteca Speciale di Matematica "Giuseppe Peano", University of Turin), Angharad Gwilym, Kate Hawke, and Edd Mustill (Special Collections and Archives, University of Liverpool), Barbara Gilbert (The University of Chicago Library) and Tom McCutchon (Columbia University Rare Book and Manuscript Library).

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Contents

Abbreviations

ANL-Castelnuovo	*Archivio Castelnuovo*, Accademia Nazionale dei Lincei, Roma. It can be accessed at the Web site [Gario 2010]: http://operedigitali.lincei.it/Castelnuovo/Lettere_E_Quaderni/menu.htm
ANL-Levi-Civita	*Archivio Levi Civita*, Accademia Nazionale dei Lincei, Roma
ANL-Volterra	*Archivio Volterra*, Accademia Nazionale dei Lincei, Roma
ASUT	Archivio storico dell'Università di Torino
AUL-Young	*Papers of Grace and William Young*, Archives, University of Liverpool
BMFI- Montesano	Fondo Domenico Montesano, Biblioteca del Dipartimento di Matematica, Università di Firenze
BMP	Biblioteca Speciale di Matematica "Giuseppe Peano", Università di Torino
BMP- Fano	*Fondo Fano*, Biblioteca Speciale di Matematica "Giuseppe Peano", Torino
BMP-Segre	*Fondo Segre,* Biblioteca Speciale di Matematica "Giuseppe Peano", Torino It can be accessed at the Web site [Giacardi 2013]: http://www.corradosegre.unito.it/
BMP-Terracini	*Fondo Terracini,* Biblioteca Speciale di Matematica "Giuseppe Peano", Torino
DES papers	David Eugene Smith Professional Correspondence, Columbia University Libraries, New York
EJWP	Ernest Julius Wilczynski Papers, The University of Chicago Library, Chicago
EPFL	École Polytechnique Fédérale de Lausanne
MCT-Mary Cytron Treves	*Fascicolo di Mary Cytron Treves,* H123, *Notizie relative a singoli internati o elenchi di internati*, in Archivio Centrale dello Stato, Ministero dell'Interno, Divisione Generale di Pubblica Sicurezza, Affari Generali e Riservati, A4bis (Stranieri internati), b. 85

Segre *Opere*	Corrado Segre, *Opere*, Roma: Ed. Cremonese, 4 vols., 1957–1963
UMI-Archivio	Archivio Storico della Unione Matematica Italiana, Bologna
UTo-ACS	*Archivi Corrado Segre*, Università di Torino. Most of the documents can be accessed at the Web site: http://users.mat.unimi.it/users/gario/Elenco-Segre.html
//	End of page of the manuscript
[…]	Omitted text
[ ]	Addition of editor
f.	*Folium*
fols.	*Folios*
n. d.	No date
n. e.	No publishing house
n. p.	No place
n. n. p.	Not numbered page
op. cit.	Cited reference
r.	*Recto*
rev.	Reviewer
Tr.	Translation
transl.	Translator
v.	*Verso*

Part I
Corrado Segre' Leadership: From Turin to the International Scene

Segre's University Courses and the Blossoming of the Italian School of Algebraic Geometry

Alberto Conte and Livia Giacardi

> *"Anyone wishing to evaluate the work of Segre properly cannot disregard the contributions made by his School, whose value is in large part due to him"* (*Chi voglia rettamente valutare l'opera del Segre non può far astrazione dai contributi portati dalla sua scuola, ché del merito di questi spetta a lui una parte notevole.*
>
> (Castelnuovo 1924b, 358).

Abstract

One of the greatest architects of the "geometric Risorgimento" (Coolidge 1927, 352) in Italy, Corrado Segre provides a shining example of the role of mentor in the history of mathematics. His university courses were a veritable forge for future researchers. The years between 1891 and the beginning of the 20th century witnessed the launch in Turin, under his guidance, of the Italian School of algebraic geometry, which in a short time would assume an internationally recognised role. Undoubtedly decisive in the formation of the School was the fact that Segre had fostered the onset and consolidation of the following lines of research: hyperspatial projective geometry; research in the foundations of the hyperspatial projective geometry; birational algebraic geometry; enumerative geometry; projective differential geometry; and projective geometry in the complex domain. However, also significant was the role of his university teaching, a valuable record of which is conserved in the forty handwritten notebooks of his university lectures. These not only make it possible to reconstruct the genesis and developments of his scientific research, but also allow us to understand how his own research stimulated and closely interacted

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with that of his students, thus showing the importance of his teaching in the flourishing of the Italian School. Here, after a brief overview of Segre's scientific contributions, we intend to show the close interaction between teaching and research, in order to bring out the influence of his university courses on his students and to pinpoint the moment when his role as leader began to be acknowledged; and make evident the enthusiasm, the nature of the collective work of his group of researchers, at least up to the early years of the twentieth century. We also intend to show how his vision of mathematics education—closely linked to his idea of the objectives of mathematics—was transmitted to the future teachers who attended his courses at the *Scuola di Magistero* (Teachers College) of the University of Turin. Finally we will try to identify the most salient features of his scientific leadership.

1 The Roots of the Italian School of Algebraic Geometry in Segre's Works

Segre's scientific work, as is well known, began when the innovating impulse given to geometric studies by Luigi Cremona (1830–1903) had begun to wane, and is linked to the important research in hyperspatial geometry carried out by Enrico D'Ovidio (1843–1933) and above all Giuseppe Veronese (1854–1917).[1] Research in geometry, as Federigo Enriques wrote, "had appeared to be completely absorbed in the contemplation of objects that it created itself, without any apparent connection with the *great problems*".[2] It was Segre who changed the course of things. A "fertile, ingenious precocity, joined to an amazing maturity of intellect and vastness of culture"[3] were the traits that distinguished the young mathematician. He received his degree before he was 20 years old, in August 1883, with a dissertation assigned to him by Enrico D'Ovidio, entitled "Studio sulle quadriche in uno spazio lineare ad *n* dimensioni ed applicazioni alla geometria della retta e specialmente delle sue serie quadratiche", which was published that same year in two memoirs of the Accademia delle Scienze di Torino.[4] As Castelnuovo wrote:

[1]See for example Scorza (1932), Loria (1924), Terracini (1926), Castelnuovo (1929), Boffi (1986), (Severi 1957), Terracini (1958), B. Segre (1961), Togliatti (1963), Menghini (1986), Brigaglia and Ciliberto (1995), Conte and Ciliberto (2004). More information on Segre's scientific biography can be found in Giacardi (2001a) and Brigaglia (2013).

[2]*Geometría parecía completamente absorta en la contemplación de los objetos formados por ella misma, sin connexión visible con los* grandes problemas (Enriques 1920, 5).

[3][…] *feconda, geniale precocità, accomunata ad una sbalorditiva maturità d'ingegno e vastità di cultura* (B.Segre 1963–64, 8). See also, for example, Pascal (1924, 461), Fano (1930, 43).

[4]In citing Segre's writings (articles and reports), reference will always be made to the *Bibliography of the works by Corrado Segre*, in this present volume. The thesis was published in two notes: (Segre 1883b, 1883c). The manuscript of the thesis is conserved in the BMP-Segre Scritti. 1. On Segre's early works, see the essay by David Rowe in this present volume. See also Ghione and Ottaviani (1992). For details on Segre's academic career see Conte et al. (2013).

> Those who still read today […] the two works, intimately connected, remain surprised by the confidence and vastness of views and means with which that young man, Corrado Segre, treats the broad subject. The dissertation seems due, not to a beginner, but to an experienced mathematician.[5]

In his thesis Segre proposed to study hyperspatial projective geometry, basing it solidly on linear algebra, in contrast to the common approach in Italy, in the conviction that "it opens to mathematicians a limitless field of research full of interest".[6] What was new was the idea of developing it systematically as a geometric science and not as a disguised form of analysis; also new was the application to it of the methods of projection that had shown themselves to be fruitful in ordinary projective geometry.

His geometric approach to the research emerges clearly in these first papers. Segre illustrated it to Leopold Kronecker, who appeared not to have understood it correctly. He confidently explained what he meant by the 'geometric interpretation' of some of Kronecker's and Weierstrass' results:[7]

> I have also seen from your letter that I did not explain myself clearly regarding the geometrical interpretation of your research and that of Mr. Weierstrass on the theory of bilinear and quadratic forms. Perhaps should I not say "geometric interpretation" as these words suggest (and you were thinking, it appears to me) a work that consists only in changes of words. Now I would consider ridiculous a scholar who occupied himself with merely changing analytical terms into geometric terms in analytical results already known. But that's not what I meant to say in my last letter. To explain more clearly, consider the theorems on the conditions under which two quadratic forms can be transformed into two other quadratic forms.
>
> In putting it into geometric terms, one can say that these theorems provide the conditions by which two pairs of second-degree surfaces (in an *n*-dimensional space) are identical from the standpoint of projective geometry. But these conditions remain analytical, because they involve elementary divisors, etc.; what is the geometric meaning of elementary divisors? If to a pair of second-degree surfaces there corresponds a double, triple, etc. root of the determinant of their bundle, these two surfaces touch each other at one or more points, but what difference will there be between these contacts according to different degrees of elementary divisors, that is to say, what *singularities* will the intersection of these two surfaces have for a given system of elementary divisors? That is one of the questions that I set out to resolve and which at first presented me with some difficulties. Thus I was able to establish a geometric *classification* of the intersections of two 2nd-degree surfaces. Similarly, analytical results on bilinear forms gave me, through a geometric study, the classification of *homographies* or *collineations* in any linear space. […] But I tell you again, and

[5] *Chi legge anche oggi* […] *i due lavori, strettamente collegati resta sorpreso della sicurezza e vastità di vedute e di mezzi con cui quel giovane, Corrado Segre, tratta l'ampio soggetto. La dissertazione sembra dovuta non già ad un principiante, ma ad un matematico provetto* (Castelnuovo 1924b, 353).

[6] [...] *essa apre ai cultori della matematica un campo sconfinato di ricerche piene di interesse* (Segre 1883b, *Opere* 3, 27).

[7] Segre's aim was to give a geometric interpretation of the elementary divisors and it was not new. This was the subject of Klein's thesis in 1868, but Segre introduced the new notion of "characteristic", which would be used hereinafter (until now); see Brechenmacher (2006, . 475–477). See also Segre (1883–84c).

> I hope to convince you fully when my work is printed, that it is not just a change of terms that produces these geometric results but rather they are the result of more difficult reasonings. (C. Segre to L. Kronecker, Turin 25 December 1883)[8]

Another aspect that deserves to be underlined is the abstract formulation that Segre gives to his treatment, which emerges from the very beginning, when he emphasises how the geometry of a space of arbitrary dimension had by that time assumed an important place in mathematics, even when a point of such a space is considered as an entity whose nature remains undetermined. For the pure mathematician, he noted, the lack of a meaningful representation of the objects under study is irrelevant (Segre 1883b, *Opere* 3, 26). After Segre, the concept of abstract geometry became 'an ordinary tool for working' for the Italian School of algebraic geometry. As Federigo Enriques would write many years later:

> In fact nothing is more fruitful than the multiplication of our intuitive powers deriving from this principle: it is almost as though beside the mortal eyes, which we are given to examine a figure in a certain respect, we had got a thousand spiritual eyes to contemplate many different transformations [of this figure]; while the unity of the object shines in our mind so enriched, that it allows us to pass easily from one form to another.[9]

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[8] *J'ai vu aussi par votre lettre que je ne m'étais pas expliqué clairement à propos de l'interprétation géométrique de vos recherches et de celles de M. Weierstrass sur la théorie des formes bilinéaires et quadratiques. Peut-être ne devrais-je pas dire « interprétation géométrique » car ces mots font penser (et vous ont fait penser, à ce que je vois) à un travail qui consiste seulement dans de changements de mots. Or je considérerais comme ridicule un savant qui ne s'occupât que de changer les mots analytiques en mots géométriques dans de résultats analytiques déjà connus. Mais ce n'est pas là ce que j'entendais dire dans ma dernière lettre. Pour m'expliquer avec plus de clarté, prenons les théorèmes sur les conditions afin que deux formes quadratiques puissent se transformer dans deux autres formes quadratiques. En mettant des mots géométriques, on peut dire que ces théorèmes donnent les conditions pour que deux couples de surfaces du* 2[e] *degré (dans un espace à n dimensions) soient identiques du point de vue de la géométrie projective. Mais ces conditions restent analytiques, car il y entre des diviseurs élémentaires, etc.; quelle est donc la signification géométrique des diviseurs élémentaires? Si à un couple de surfaces du* 2[e] degré correspond une racine double, triple, etc. *du déterminant de leur faisceau, ces deux surfaces se toucheront mutuellement en un ou plusieurs points, mais quelle différence y aura-t-il entre ces contacts suivant les divers degrés des diviseurs élémentaires, c'est-à-dire quelles* singularités *aura l'intersection de ces deux surfaces pour un système donné de diviseurs élémentaires? Voilà l'une des questions que j'ai taché de résoudre et qui n'a pas laissé de me présenter au premier abord des difficultés. C'est ainsi que j'ai pu établir une* classification *géométrique des intersections de deux surfaces du* 2[e] *degré. De même les résultats analytiques sur les formes bilinéaires m'ont donné par un étude géométrique la classification des* homographies *ou* collinéations *dans un espace linéaire quelconque.* […] *Mais je vous le répète et j'espère vous en convaincre à peine mes travaux seront imprimés, ce n'est pas un simple changement de mots qui donne ces résultats géométriques mais bien une suite de raisonnements plus difficiles.* The letter can be accessed at http://users.mat.unimi.it/users/gario/Segre-Ancona/lettereAscienziati.pdf, and it is also transcribed in the essay by Luciano and Roero in this present volume.

[9] *Infatti nulla è più fecondo che la moltiplicazione dei nostri poteri intuitivi recata da codesto principio: pare quasi che agli occhi mortali, con cui ci è dato esaminare una figura sotto un certo rapporto, si aggiungano mille occhi spirituali per contemplarne tante diverse trasfigurazioni; mentre l'unità dell'oggetto splende alla ragione così arricchita, che ci fa passare con semplicità dall'una all'altra forma* (Enriques 1922, 140).

This was an 'abstraction' supported by geometrical intuition, which was at that time a necessary tool to overcome technical difficulties and to obtain new results.

Segre's correspondence of the years 1883–1884 with internationally acclaimed mathematicians such as Klein,[10] Arthur Cayley, Thomas A. Hirst, Theodor Reye, Leopold Kronecker, Karl Weierstrass, and Oscar Schlömilch,[11] show us a young man who enthusiastically read the most recent publications, asked for explanations, made comments, but also illustrated his own research, explaining his approach and seeking to make his results known internationally. For example, he wrote to Kronecker: "I am young and I have great need and desire to learn: I am thus quite grateful to the scholars who are willing to be my teachers for a time" (*Je suis jeune et j'ai beaucoup de besoin et d'envie d'apprendre: je suis donc bien reconnaissant aux savants qui veulent pour quelques moments être mes maîtres.* C. Segre to L. Kronecker, Turin 25 December 1883).

In these two years he wrote no fewer than fifteen articles and notes, of which four were accepted by the *Mathematische Annalen* and two by the *Journal für die reine und angewandte Mathematik*, very high-level journals. His 'precocious maturity' (Fano 1930, 43) and the innovative character of these early research works earned him the award in 1884 of the Mathematics Prize of the Società Italiana delle Scienze. The jury praised Segre "for the generality of the research contained […], for the refinement of various theories treated previously by diverse eminent geometers, for the many and important new results obtained, and finally for the very broad field of research to which he opened the way".[12]

Winner of a competitive examination (*concorso*), in 1888 Segre was called to the chair of higher geometry at the University of Turin, a chair that he would hold until his death. It was his *maestro* D'Ovidio who gave up the professorship of higher geometry and moved to that of higher analysis leaving the field open for his student (Conte et al. 2013, 29–30, 101). His first works regarded above all hyperspatial geometry. Skilfully making use of the recent algebraic results of Weierstrass and Ferdinand Georg Frobenius (Hawkins 2013), Segre was able to provide a geometric and analytical formulation for hyperspatial projective geometry, developing it to such a level as to make it a tool for further research for the Italian School of algebraic geometry.

As his student Gino Fano affirms in the handwritten notes, probably for a lecture given in 1923 in Aberystwyth:

> He [Segre] became so, just in the moment in which Cremona's scientific activity had completely ceased, the new leader of Ital. geometry the founder of a new school. He was also able to learn, to

[10]See Luciano and Roero (2012).

[11]The letters of these mathematicians constitute part of the archives of the Fuà Family (Gario 1989a) and can be accessed online at http://users.mat.unimi.it/users/gario/Elenco.html. They are kept now in the *Archivi Corrado Segre*, University of Turin. See also the essay by Luciano and Roero.

[12][…] *per la generalità delle ricerche contenute* […], *pel perfezionamento* […] *a varie teorie trattate precedentemente da diversi valenti Geometri, per i molti ed importanti risultati nuovi ottenuti, ed infine per il campo vastissimo di ricerche cui esso apre la via*. (RAPPORTO relativo al conferimento del premio nelle Matematiche, dalla Società Italiana delle Scienze, per l'anno 1884, *Memorie di Matematica e Fisica della Società Italiana delle Scienze* (3), VII, 1890, XXXIV–XXXVI, at p. XXXVI. The members of the jury were three great mathematicians of the Italian Risorgimento: Enrico Betti, Eugenio Beltrami and Giuseppe Battaglini.

> make his own, and to let estimate by his pupils all that, for the development of his programme, was to be got from the most important foreign mathematicians (Klein, Noether, Lie, Cayley, Zeuthen, Darboux,…); and by means of his 35 years of teaching, about all most various branches of geometry, diff. and enumerative geom. (abzähl. Geom.) included, he had a very great influence on the development of all geometry in Italy.[13]

Even many years later, in a lecture given in Lausanne in 1944, Fano would say:

> Fortunately the Italian School reacted well against this possible trend, which Ovidio called 'tick-tock geometry', while *C. Segre* (1863–1924), one of the great Italian masters, always warned his students against these kind of works, which lead to the degeneration of scientific development.[14]

Beginning in 1886 Segre's works show a broadening of his horizons under the influence, on the one hand, of the new approach of the German School of Alexander Brill and Max Noether, and on the other hand, of the ideas expounded by Klein in his celebrated *Erlangen Program*. With regard to Brill and Noether, in 1904 Segre wrote, "An entire school of Italian geometers recognises in the Memoir by Brill and Noether [Über die algebraischen Funktionen und ihre Anwendung in der Geometrie, 1873] its point of departure".[15] As for Klein, Segre encouraged the translation into Italian of Klein's *Erlangen Program*, entrusting it to his student Fano, "for the benefit of the Italian geometers who know very little of it".[16] We will say more about this below. Presenting the translation, he underlined the importance of transformation groups for the development of geometrical research, which he himself had realised as early as 1885.[17] Therefore in Segre's research we can see his progressive detachment from a restricted projective view in order to arrive at the study of the properties invariant under birational transformations. The first signals of this shift of interest can be traced back to a note of 1886 on "Remarques sur les transformations uniformes des courbes elliptiques en elles-mêmes" (Segre 1886a), but it is above all in the memoir on algebraic ruled surfaces, published in two parts in the *Mathematische Annalen*,[18] that the line of research becomes clearer. In the second of these memoirs, "Recherches générales sur les courbes et les surfaces réglées algébriques" Segre takes time to explain to the international scientific community the advantages of the Italian approach:

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[13]Cf. Fano n. d., [*Appunti vari*], f. 63r. This is Fano's original English text. See also Fano (1923) and the typewritten notes kept in the Archives of the University of Liverpool, that the authors intend to publish.

[14]*Heureusement l'Ecole Italienne a bien réagi contre cette possible tendance, que d'Ovidio a appelée 'tic tac géométrie' tandis que* C. Segre (1863–1924) *un des grands maîtres Italiens, a toujours mis-en garde ses élèves contre ces productions, conduisant à une dégénération du développement scientifique* (Fano n. d., [*Appunti vari*], f. 58r).

[15]*Tutta una scuola di geometri italiani riconosce nella Memoria di BRILL e NOETHER [Über die algebraischen Funktionen und ihre Anwendung in der Geometrie, 1873] il suo punto di partenza.* (Segre 1905a, *Opere* 4, 462). See also the presentation by Segre of the important memoir of Brill and Noether (Segre 1894–95).

[16][...] *pour l'avantage des géomètres italiens qui ne le connaissent presque pas*. See Segre's letter to Klein dated Turin 19 November 1889 in: Luciano and Roero (2012, 151). All the letters from Segre to Klein cited in what follows are found in Luciano and Roero (2012).

[17]See Hawkins (1994, 187), Hawkins (2000, 251–260).

[18]See Segre (1887b, 1889a).

> It is not a question (I add this for those who may not be aware of the progress that this branch of mathematics is making, especially in Italy) of easy extensions to higher spaces of results for ordinary space that are already known. […] in introducing the spaces of all dimensions one has not only the advantage of greater generality, but can avail himself of all the power of a tool that one who limits himself to ordinary space does not have: that is to say, the consideration of entities of a space as a projection of those of higher spaces.[19]

In a very brief note of 1887, "Sui sistemi lineari di curve piane algebriche di genere *p*" (Segre 1887d) emerges one of the fundamental concepts of classic algebraic geometry, that of *characteristic series*[20] of a linear system of plane curves. With regard to this, Fano wrote:

> I may say that, in studying geometry on an algebraic manifold, *it is the fundam. concept of what we may call the Italian or geometrical method,* to study a (simple) lin. syst. of M_{k-1} on M_k by reducing it to the hyperplanar sections of a new (?) M_k. It involves necessarily more-dims. methods. Particularly: for lin. series of groups of points on a curve, a new curve, on which the groups are determined by hyperplanes, etc. It was stated firstly by *Segre* on the case of lin. syst. of planes curves & rational surfaces; and shortly afterwards applied by himself & *Castelnuovo* to groups of points on alg. curves – later still, in other cases.[21]

The culminating and synthesising work of this period is the important memoir entitled "Introduzione alla geometria sopra un ente algebrico semplicemente infinito" (Segre 1894a), which also includes the research carried out in Turin by Castelnuovo, and which, according to Severi, contains the 'roots' of Italian algebraic geometry. Here:

> […] the geometry of linear series on a curve is expounded according the hyperspatial method, underlining the fact that it needs neither considerations of functions nor algebraic developments, and that the algebraic nature of the entities comes into play only through Chasles's principle of correspondence! The synthesis in this area has achieved its utmost effectiveness. For example the proofs of the Riemann-Roch theorem and of the Cayley-Brill principle of correspondence are admirable.[22]

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[19] *Il ne s'agit pas (j'ajoute cela pour celui qui ne serait pas au courant des progrès que cette branche des mathématiques est en train de faire, surtout en Italie) de faciles extensions aux espaces supérieurs de résultats qui pour l'espace ordinaire soient déjà connus.* […] *en introduisant les espaces de toutes les dimensions on n'a pas seulement l'avantage de la plus grand généralité, mais encore celui de pouvoir se servir dans toute sa force d'un instrument que ne possède pas celui qui veut se borner à l'espace ordinaire: c'est-à-dire la considération des êtres d'un espace comme projection de ceux des espaces supérieurs.* (Segre 1889a, *Opere* 1, 125–126).

[20] It would be Castelnuovo who gave it this name in (Castelnuovo 1892).

[21] Cf. Fano n. d., [*Appunti vari*], f. 80r. This is Fano's original English text. See also the paper by Edoardo Sernesi in this present volume.

[22] [...] *la geometria delle serie lineari sopra una curva viene appunto esposta secondo il metodo iperspaziale, sottolineando che non occorrono in essa né considerazioni funzionali né sviluppi algebrici e che l'algebricità degli enti interviene soltanto attraverso il principio di corrispondenza di Chasles! La sintesi in questo terreno ha raggiunto la sua efficienza massima. Mirabili ad esempio le dimostrazioni del teorema di Riemann-Roch e del principio di corrispondenza di Cayley-Brill* (Severi 1957, X). See also in this present volume the presentation by Alberto Conte of the handwritten notebook that forms the basis of the memoir itself.

This memoir, in fact, served as a spur for important research by his students Castelnuovo, Enriques and Severi, research in which Segre participated rather laterally, as will be shown below, leaving the field open for his students.[23]

However, of particular importance is the introduction in 1896 (Segre 1895–96) of one of the most important invariants of an algebraic surface, today known as the 'Zeuthen-Segre invariant', which Segre had already presented to his students in the course of 1893–94. No less significant is the research carried out on questions related to the singularity of algebraic surfaces (Segre 1897b).

Projective geometry, which had aroused Segre's youthful enthusiasm and which constitutes a sort of *leitmotiv* that runs through his entire scientific research, again attracted his attention between 1889 and 1891; in particular, it was K.G.C. von Staudt's theory of imaginary elements that reawakened his interest. In 1887 Segre had invited Mario Pieri[24] to translate von Staudt's *Geometrie der Lage*; the translation came out in 1889, prefaced by a valuable bio-bibliographical essay written by Segre himself.[25] Extending the German mathematician's field of research, he enlarged the group of projective transformations by adding those that he called 'anti-projectivities', that is, correspondences in which the cross-ratios of four elements are transformed in their conjugate. Segre developed a complete theory of such correspondences and opened the way to a new field of geometric research, that of hyperalgebraic entities. His results, presented in four articles in the *Atti della R. Accademia delle Scienze di Torino*—and later, from a different point of view, in a memoir published in the *Mathematische Annalen*—[26]would be reprised and used many years later, above all by Elie Cartan[27]even if Baker considered them only "an interesting exercise in algebra" (Baker 1926, 270). In the memoir appeared in the *Mathematische Annalen* (Segre 1891d) Segre constructed various real 'models' of a projective space defined in the complex domain, the most simple of these is Segre's variety.[28] This research was completed in 1898 by Gerrit Mannoury and became the starting point of significant works by Wilhelm Wirtinger and William Hodge.[29] In the same paper Segre presented a careful study on anti-projectivity and hyperalgebraic entities and introduced the bicomplex numbers. On this research, in 1906 Eduard Study, who based himself on Segre's results, commented:

[23]See for example Castelnuovo (1924b, 355), Terracini (1961, 12–13), B. Segre (1963-64, 15), Brigaglia and Ciliberto (1995, Sect. 1.3).

[24]See the letter of C. Segre to M. Pieri, Turin 11 October 1887, in Arrighi (1997, 113), and the letter of C. Segre to F. Klein, Turin 14 October 1887.

[25]See the correspondence between C. Segre and A. Papellier and between C. Segre and K. Rudel about the life of von Staudt , in UTo-ACS, VII and http://users.mat.unimi.it/users/gario/Segre-Ancona/lettereRicevute.pdf.

[26]See Segre (1889–90, 1890–91, 1891d).

[27]Hawkins (1994, 200–204). See also the chapter by Aldo Brigaglia in this present volume.

[28]In a brief note of 1891 (Segre 1891c) Segre defined for the first time the product of two projective spaces, now referred to as 'Segre variety', a concept "that had important repercussions for the geometry of the twentieth century" (Severi 1957, XI).

[29]See B. Segre (1963–64, 16).

> SEGRE, however, restricted himself to the consideration of algebraic entities, and made large use of geometric reasoning. (It appears that for this reason the memoir by Segre is not as well-known as it should be).[30]

In letter addressed to Adolf Hurwitz in 1894 Segre himself complained that the importance of this research had not yet been grasped.[31]

Aware of the importance of consolidating his relationships with the European scientific milieu, in the summer of 1891 Segre undertook a journey to Germany with the aim of visiting the principal institutes and libraries in a country that was, at the time, on the cutting edge of mathematical research, and meeting in person those who had influenced his research. He visited Frankfurt a. M., Göttingen, Berlin, Leipzig, Dresden, Nuremberg, and Munich, and was in contact, among others, with Kronecker, Weierstrass, Nöther, Reye, Karl Rohn, Rudolf Sturm, Moritz Cantor, and Klein, some of whom he had been in correspondence with since 1883. His enthusiasm emerges from what he wrote to Castelnuovo:

> No one who hasn't been here can imagine what breed of man Klein is, and what kind of organisation he was able, with a skill that no one else possesses, to impose on mathematical studies in this university: it is something that has made an extraordinary impression on me. And I have already had many extremely vivid impressions of scientists during this journey![32]

In the same year Segre published in the *Rivista di matematica* the article "Su alcuni indirizzi nelle investigazioni geometriche. Osservazioni dirette ai miei studenti" (Segre 1891a), which, as it is well known, became the starting point of the dispute[33] with the director of the journal, Giuseppe Peano, regarding the way scientific research should be conceived. In this paper Segre offers a vivid picture of the recent achievements of algebraic geometry and of the open questions, underlining the importance of using both synthetic and analytic approaches. The English translation (Segre 1904) contributed to make Italian geometry widely known. With regard to this article Lucien Godeaux wrote that "Studying it was a revelation for me, and that's how I came to know Italian geometry". (*Son étude fut pour moi une révélation et c'est ainsi que je connus la Géométrie italienne,* Godeaux 1964, 24).

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[30] *Il* SEGRE *però si è ristretto alla considerazione degli enti algebrici, ed ha fatto largo uso di ragionamenti geometrici. (Pare che per queste ragioni la Memoria del Segre non sia conosciuta quanto meriterebbe).* (Study 1906, 345). See also Castelnuovo (1924b, 356).

[31] C. Segre to A. Hurwitz, Turin 29 June 1894, in Luciano and Roero (2012, 166).

[32] *Chi non è stato qui non può immaginare che razza d'uomo è Klein e che specie d'organizzazione egli ha saputo, con abilità che nessun altro può avere, imporre agli studi matematici in questa Università: è una cosa che m'ha fatto un'impressione straordinaria. E sì che d'impressioni vivissime da parte degli scienziati ne ho già avute parecchie in questo viaggio!* (C. Segre to G. Castelnuovo, Göttingen 30 June 1891, in ANL-Castelnuovo, in Gario (2010). All the letters from Segre to Castelnuovo cited in what follows, are conserved in ANL-Castelnuovo and can all be accessed on the website (Gario 2010); if they have been published, the specific reference will be indicated.

[33] This debate has been discussed several times. See, for example, Manara and Spoglianti (1977), Giacardi (2001a, Sect. 3), Avellone et al. (2002, Sect. 3), Roero (2004, 138–144), Luciano (2006, 65–71).

By the beginning of the 1890s Segre had already acquired a notable reputation even outside Italy. In 1893 he received a letter from Chicago asking him to promote participation at the International Congress of Mathematicians, which was to take place at the end of August in conjunction with the World's Columbian Exposition in that city; on that occasion he invited Castelnuovo to present a brief report on hyperspatial geometry in Italy, something which, however, he did not do.[34] In 1897 Segre was invited to be vice-president of the geometry section of the International Congress of Mathematicians in Zurich. On that occasion he wrote to his wife Olga: "The nomination pleased me, because […] there were many other geometers older than me who could be nominated".[35] His student Fano gave one of the six talks during the section, while Enriques gave one in the section of algebra. Just before leaving for Zurich Segre wrote to Volterra: "I believe that if I were unable to go, later I would feel regretful, as having missed the occasion to see men of value, and singular encounters".[36]

The following year, 1898, the jury commission for the Royal Prize for mathematics awarded by the Accademia dei Lincei, composed of Eugenio Beltrami, Luigi Bianchi, Valentino Cerruti, Luigi Cremona and Enrico D'Ovidio, assigned Segre half of the prize, shared equally with Vito Volterra, with a very flattering statement in which they cited, along with the novelty and importance of the results, the elegance of the method that associates "with rare ability geometric procedures with analytic procedures, grasping their intimate relationships", and explicitly acknowledging his role as leader of a School.[37]

At the basis of a group of Segre's works regarding problems of differential projective geometry dating to the years 1907–1913, there is the first volume of Gaston Darboux's *Leçons sur la théorie générale des surfaces* (1887), which Segre used in his course of 1903–04, as well as his contacts and the encounter with Ernest Wilczynski.[38] The first paper devoted expressly to differential projective geometry of hyperspaces dates back to 1907 (Segre 1906–07b); it is, however, in a later work of 1910 (Segre 1910a) that Segre lays the foundations for a systematic construction of such a geometry, which would receive a great impetus first from his student Alessandro Terracini and later from Enrico Bompiani. The brief note of 1908, "Complementi alla teoria delle tangenti coniugate di una superficie" (Segre 1908), which refers instead to ordinary space, marks a noteworthy step forward in the general theory of surfaces. Here Segre, generalising the concept of conjugate

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[34] C. Segre to G. Castelnuovo,Turin 28 June 1893.

[35] C. Segre to Olga Michelli Segre, [Zurich] 10 August 1897, UTo-ACS, II: *La nomina mi ha fatto piacere perché* […] *vi sarebbero stati tanti altri geometri più anziani di me da nominare.*

[36] *Io credo che se non potessi andarci, dopo ne proverei rammarico, come d'un'occasione perduta di vedere uomini di valore, e riunioni singolari* (C. Segre to V. Volterra, Ancona 31 July 1897, ANL-Volterra).

[37] [...] *con rara abilità i procedimenti geometrici agli analitici, cogliendone le intime relazioni.* (Relazione sul concorso al Premio Reale per la Matematica, pel 1895, 1901, 367).

[38] See Coolidge (1927, 355). Cf also Sect. 4 in Ciliberto and Sallent (2012), and the essay by Luciano and Roero in this present volume.

tangents, is led to introduce, among other things, the particular triad of tangent lines coming out of a point on a surface, today known as 'Segre tangents', the differential equations of which would be set forth by Guido Fubini. Also worthy of note, in the context of differential geometry, is the invariant, known as Wölffing–Mehmke-Segre invariant, relative to a pair of mutually tangent curves.

While it is true that these scientific contributions of Segre's provided the stimulus for the future research of the Italian School of algebraic geometry, it is also true that, in parallel with these, a significant role was played by his university teaching. We have valuable testimony of this in the forty handwritten notebooks[39] in which each summer he carefully developed the topics of the course that he would teach the following autumn. A shining example of the profound interaction between research and teaching, the notebooks not only allow us to reconstruct the genesis and developments of Segre's scientific research, of which they sometimes constitute 'a preliminary stage', sometimes a 'reflection' (Terracini 1953a, 261), but also allow us to understand the importance of his teaching in the birth of the Italian School of algebraic geometry. In his courses Segre presented his students with the most recent research, suggested topics to be studied, addressed problems that were still unsolved, all in the principal aim of directing the most promising young people towards scientific research; sometime, preparing classes, he himself arrived at posing new problems.[40]

2 The 'Geometric Orgies' of Turin

In the 1880s and 1890s Segre was able to create around him a climate of wildly enthusiastic work, of friendly collaboration, of scientific dialogue, lively and fertile. The fruit of this atmosphere would be felt throughout Italy. Castelnuovo recalls the period that he spent in Turin by speaking of 'geometric orgies' of Turin,[41] while Fano, speaking of the research in hyperspatial projective geometry, which represented the core of the 1880–1890 research, wrote: "it was *indispensable* that everything be treated and digested, that it became the blood of our blood, that we

[39]The handwritten notebooks of Segre's university lectures begin with 1888–89, the year in which Segre held the chair in higher geometry, and conclude in 1924, the year of his death. Of these, thirty-four develop topics of higher geometry, three are of mathematical physics and correspond to the years 1895–1897 in which Segre was charged with teaching that subject, and the two remaining lesson notebooks contain respectively brief mentions of various questions of analysis and of geometry, and the lectures given at the Scuola di Magistero. The last one includes, among other things, the lists of the students who attended Segre's courses from 1883 to 1892, with indications of the scores given to them.

[40]See, for example, C. Segre to G. Castelnuovo, Turin 1 June 1899.

[41]G. Castelnuovo to F. Amodeo, 6 February 1893, in Palladino F. & N. (2006, 304).

had it at our fingertips in order to be able to use it in the most advanced research … Fecundity!”[42]

Segre’s university courses became a veritable breeding ground for future researchers. Many students wrote their dissertations on the most advanced topics of research under his supervision; among the most brilliant were Gino Fano (1892), Beppo Levi (1896), Alberto Tanturri (1899), Severi (1900), Giovanni Zeno Giambelli (1901), Alessandro Terracini (1911) and Eugenio Togliatti (1912). There were also many newly-graduated mathematicians, Italian and foreign, who, attracted by his fame, came to Turin to attend his lectures and carry out their post-graduate studies. These included Castelnuovo (1887–1891), Federico Amodeo (1890–1891), Federigo Enriques (November 1892, November 1893–January 1894), Gaetano Scorza (1899–1900), as well as the English husband and wife William H. Young and Grace Chisholm Young (1898–99), Julian Coolidge (1903–04), and some years later, Charles Hershel Sisam (1908–09) from the United States, and others. There was also a group of young mathematicians who, after having earned their degrees under Segre, or having collaborated with him as assistants, or having simply attended his lectures, published their first articles under his influence, such as Francesco Palatini, Umberto Perazzo, and Pilo Predella.[43] Some of the young people in Segre’s entourage later turned their interest to topics that were characteristic of the School of Peano, or collaborated with him. The best known among these are Mario Pieri and Beppo Levi, but there were others as well, such as Luigia Viriglio and Matteo Bottasso, who exerted a certain influence in mathematics education.[44]

The relationship that was closest and most fertile was undoubtedly that with Guido Castelnuovo (Venice 1865–Rome 1952). The two first came into contact in July 1885 when Castelnuovo, who had not yet received his degree, sent Segre an article to read. In the letters that followed, Segre gave advice, proposed topics for research, and suggested articles to read. He came to appreciate his correspondent to such an extent that in October 1887 he offered him the position of assistant in the course taught by D’Ovidio, a position of ‘an honorific nature’ because it was assigned each year to the most outstanding graduate.[45] Segre himself held it in 1883–84. In that same letter, Segre suggested that Castelnuovo study linear systems of curves, and indicated a line of research.[46] As Segre put it, they were “two young men who placed their ideals of goodness, honesty and love of science before any

[42][…] *era* indispensabile *che fossero trattate e digerite, che diventassero sangue del ns sangue, averle sulla p. delle dita, p. valersene in ricerche + elevate … Fecondità!* (Fano n. d., [*Appunti vari*], fls 69r and 69v).

[43]See Appendix 5 at the end of this article, also in Giacardi (2013).

[44]The connections between the two Schools of Peano and Segre deserves a thorough treatment.

[45]C. Segre to G. Castelnuovo, Turin 6 October 1887.

[46]*The important problem to solve is to find all the systems of a given genus p of minimum order to which all the others can be reduced with Cremona transformations; up to now only the cases of p = 0, 1 have been discussed completely* (*La gran questione da risolvere è lo stabilire tutti i sistemi di dato genere p d’ordine minimo ai quali tutti gli altri posson ridursi con trasformazioni cremoniane; solo i casi di p = 0, 1 sono stati finora discussi completamente.* C. Segre to Castelnuovo, Turin 6 October 1887 and 28 July, 13 August 1890).

Philistine egotism";[47] at the time, Segre was 24 years old, Castelnuovo 22. Castelnuovo accepted the position of assistant[48] and remained in Turin until 1891, when he obtained the professorship in Rome. It was Segre who, in August 1890, had convinced him to take part in the competition for the chair, telling him also of the favourable opinions of Eugenio Bertini and D'Ovidio, and at the same time encouraging him to pursue the research on linear systems of curves of a given genus, suggesting points to develop and letting him know that the journal *Mathematische Annalen* was amenable to accepting work in Italian.[49] In 1892 Castelnuovo published his important paper entitled "Ricerche generali sopra i sistemi lineari di curve piane" (Castelnuovo 1892), the germ of which, as Castelnuovo himself affirms, was drawn from a paper in which Segre mentioned the advantages that the theory of linear systems could derive from the geometry of the curve (Segre 1887d). In his long and detailed report Segre underlined the importance of Castelnuovo's paper for the results, the new lines of research that it opened, the new concepts, and the questions that it highlighted.[50] The two entities used by Castelnuovo to translate the results from one theory to the other were 'characteristic series' and 'adjoint system'.

During the period he spent in Turin, Castelnuovo published no fewer than sixteen articles, and the long and intense dialogues with Segre led to the creation of the Italian line of research on the theory of curves, and laid the foundations for all of Italian algebraic geometry. After Segre's death, Castelnuovo wrote:

> To him I owe a good part of what I know; in those long conversations, which we had two or three times a day during my stay of 4 years in Torino, I learned more than in my university courses. To him I owe the incentive for my first works, and the advice and aid arising from his experience and his knowledge.[51]

After leaving Turin Castelnuovo maintained a close, uninterrupted correspondence with Segre,[52] who never ceased to be generous with advice on research and on teaching; for example, he encouraged him to write a work on plane involutions

[47][…] *due ragazzi che al di sopra dell'egoismo dei filistei ripongono i loro ideali di bontà, di onestà e di culto della scienza* (C. Segre to G. Castelnuovo, Turin 12 November 1891).

[48]In Turin Castelnuovo also taught, in an extracurricular setting, a course in projective geometry from 1889–90 to 1893–94, and in the academic year 1889–90 he taught at the military academy as well.

[49]C. Segre to G. Castelnuovo, Turin 13 August 1890.

[50]See Segre's report in the *Atti dell'Accademia delle Scienze di Torino* (1890–91, 602), available in the section *Relazioni* of the website (Giacardi 2013).

[51]*A Lui devo buona parte di quel che so; in quelle lunghe conversazioni, che avevamo due o tre volte al giorno durante la mia permanenza di quattro anni a Torino, ho imparato più che nei miei corsi universitari. A Lui devo l'incitamento ai miei primi lavori, e i consigli e gli aiuti della Sua esperienza e della Sua sapienza* (G. Castelnuovo to Olga Michelli Segre, Rome 25 May 1924, UTo-ACS, II).

[52]There are 255 existing letters from Segre to Castelnuovo from 1885 al 1905, in Gario (2010). Part of these have been published in Bottazzini et al. (1996, 669–678). The words that Segre addressed to his friend immediately after Castelnuovo's transfer to Rome show the profound friendship and scientific solidarity that had arisen between them (C. Segre to G. Castelnuovo,

(28 August 1892) and reread the lecture notes for the course that he was giving in Rome (21 February 1892 and 22 June 1892). Thanks to the letters that he sent to his friend (an average of thirty a year at first) it is possible to follow not only the thread of the scientific work of the two mathematicians, but also Segre's relationships with other collaborators or students, and the academic world in general, as well as the most important events of his personal life.[53] There emerges the figure of a teacher concerned about the training and the future of the young researchers under his guidance, and of the prestige of his faculty: a teacher who devoted time and energy to the preparation of his courses, to the revision of the works of his students, and to promoting Italian research on the international scene,[54] but also a teacher who was lavish with advice on teaching methods.[55] When the need arose, he could be severe, and selective.

This would soon be well understood by the Neapolitan Federico Amodeo (Avellino 1859–Naples 1946), who, as the winner of several competitive examinations for a teaching position in secondary schools, chose the Istituto Tecnico Sommeiller in Turin because he was attracted by Segre's growing fame; he intended to attend Segre's lectures. With a letter of introduction from his teacher Achille Sannia, who was already in correspondence with Segre,[56] Amodeo arrived in Turin in December 1890. He became part of the group of young mathematicians who orbited around Segre and Peano, and which had given rise to a kind of scientific community that was baptised *Pitareide*.[57] Their gathering point was first "Bergia's little rooms",[58] at the corner of Via Lagrange and Corso Vittorio Emanuele II and then the American Bar in the Galleria Nazionale. Amodeo had been in correspondence with Segre since May 1888, and their epistolary relationship would continue[59] even after, in the summer of 1891, Amodeo returned to

(Footnote 52 continued)
Turin, 12 November 1891). On the relationship between Segre and Castelnuovo, see the chapter by Paola Gario in this present volume.

[53]Particular attention is given to his marriage to Olga Michelli (25 and 26 March 1893), and to the birth of his daughters Elena (14 March 1894) and Adriana (28 October 1897). See *Corrado Segre. Biographical Timeline* in this present volume.

[54]See, for example, C. Segre to G. Castelnuovo, 15 July, 29 July, 8 August, 7 September, 23 November 1891; 8 June, 4 September, 28 January, 16 November 1892; 4 February, 27 May, 28 June, 26 September, 7 December, 11 December 1893; 29 January 1894; 5 December 1896; 26 February, 6 June 1899; 12 April 1901, etc.

[55]See, for example, C. Segre to G. Castelnuovo, 23 November 1891, 10 February, 21 February, 24 February, 22 June 1892.

[56]The correspondence between Sannia and Amodeo shows how Sannia continually asked Segre's opinions for the second edition of his *Lezioni di geometria proiettiva*; see Palladino F. and N. (2006); see also the chapter by Aldo Brigaglia in this present volume.

[57]See, for example, the following letters: C. Segre to F. Amodeo, Turin 24 November 1891, G. Castelnuovo to F. Amodeo, Rome 30 November 1891, in Palladino F. and N. (2006, 185 and 283), and C. Segre to G. Castelnuovo, Turin 28 November 1891.

[58]See C. Segre to G. Castelnuovo, Turin, 28 November 1891.

[59]The correspondence between Segre and Amodeo consists in thirty-four letters from 1888 to 1893, published in Palladino F. and N. (2006).

Naples to teach in the Istituto Tecnico Nautico Della Porta there. Segre was a severe scientific referee for Amodeo. He read his works, corrected, them, suggested articles to read and topics to research, encouraged him to avoid questions that were too difficult and to find subjects for research in the writings of the great authors, and recommended that he not occupy himself with questions of scant relevance.[60] Segre wrote:

> if I sometimes seem to you to be a little severe in my judgments […] be persuaded that for me severity is a general principle, which I even use against myself, and which derives from the higher motives related to the seriousness of science and teaching.[61]

And again:

> In my actions […] there has always been complete sincerity and loyalty […] Friendship must not cast a veil over the justice and seriousness and severity of scientific ideals.[62]

In Turin Amodeo attended the famous course of 1890–91, which he recalled thus:

> In the 1890–91 academic year, Segre replicated with D'Ovidio in Turin the excellent experiment carried out by Brioschi, Casorati and Cremona in 1869 in Milan. While D'Ovidio gave a course of lectures on *Functions of a complex variable and Abelian integrals*, he [Segre] taught *Geometry on a simply infinite algebraic variety* with respect to three aspects, *hyperspatial, algebraic and functional*.[63]

Five students were officially enrolled in the course, as shown in Segre's Notebook 38, in which he registered the names and scores of his students;[64] among these was Gino Fano (Mantua 1871–Verona 1952), while Amodeo was among those who audited the course. Both of them undertook to solve the problem that Segre posed

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[60]See for example the following letters: C. Segre to F. Amodeo, Turin 17 August 1890, 24 August 1891, 19 February 1892, in Palladino F. and N. (2006, 176, 180 and 191–192). In his letters to Castelnuovo Segre refers to F. Amodeo as 'Simplicio', and not infrequently shows his impatience for the Neapolitan's naïveté and errors. See, for example, the letters dated Turin 8 August 1891 and Turin 28 January 1892.

[61][…] *se qualche volta io posso esserti sembrato un po' severo nei miei giudizi* […] *sii persuaso che per me la severità è un principio generale, che uso anche contro me stesso, e che deriva da ragioni elevate relative alla serietà della scienza e dell'insegnamento* (C. Segre to F. Amodeo, Turin 4 September 1891, in Palladino F. & N. (2006), 181).

[62]*Nei miei atti ... vi è sempre stata e vi sarà sempre completa sincerità e lealtà... L'amicizia non deve far velo alla giustizia od alla serietà e severità d'ideali scientifici* (C. Segre to F. Amodeo, Turin 19 February 1892, in Palladino F. and N. (2006), 192).

[63]*Nell'anno scolastico 1890–91 Segre ripetette con D'Ovidio a Torino la eccellente prova fatta da Brioschi, Casorati e Cremona nel 1869 a Milano. Mentre D'Ovidio faceva un corso di lezioni sulle* Funzioni di variabile complessa e sugli integrali abeliani, *egli [Segre] esponeva la* Geometria su di una varietà algebrica semplicemente infinita *sotto il triplice aspetto* iperspaziale, algebrico e funzionale (Amodeo 1945, 245).

[64]Archival documents indicate that only Fano and Giacomo Maida took the final examination, which Fano passed with a score of 30/30 cum laude and Maida with 27/30. *Registro dei verbali degli esami di Geometria superiore, 1882–1901* ASUT, *Facoltà di Scienze fisiche, matematiche e naturali, Verbali degli esami speciali, Esame di Geometria superiore* X.D 145.

during a lecture: "Define the space S_r not by means of coordinates, but rather by a series of properties from which the representation with coordinates can be deduced as a consequence".[65] In spite of being encouraged to work together, each published his own article,[66] and Segre did everything he could to eliminate the friction arisen between them.[67] Segre's careful and severe work of revision is documented in his correspondence with Castelnuovo.[68] The two successive works by Enriques and Fano[69] are also part of this line of research; Segre himself cites these in the notebook regarding his course of 1893–1894, with a note added at a later moment,[70] and again in the course of 1898–1899, where he presents them at greater length, with reference also made to an article by Pieri devoted to the same subject.[71]

At Bertini's friendly insistence, in a first moment, Segre thought about preparing a lithograph of his lectures, and to this end had begun to revise the notes taken by Fano during the course, but considering these 'greatly inadequate', he later abandoned the idea, in spite of Castelnuovo's offer to help.[72] It was only in 1894 that the *Annali di matematica* would publish the important paper entitled "Introduzione alla geometria sopra un ente algebrico semplicemente infinito", where Segre illustrates solely the geometric method.[73] The paper was intentionally preceded by that of Bertini, where the topic was addressed by means of the algebraic-geometric method of Brill and Noether, and refined in accordance with the geometric thinking of the Italian School. In this way, Segre wrote, "the readers can see a single topic treated simultaneously with two quite different methods".[74]

The importance of this course of 1890–1891 lies not only in the scientific relevance of Segre's approach to the study of the geometry of an algebraic curve, but because this can be considered the moment in which his role as the leader of the School actually began. This is due not only to the fact that he suggested readings

[65]*Definire lo spazio S_r non già mediante coordinate, ma con una serie di proprietà dalle quali la rappresentazione con coordinate si possa dedurre come conseguenza*. In his article Fano (1892), on p. 107 Fano recalls the problem proposed by Segre during his lecture; see also BMP-Segre, Quaderni. 3, p. 27 and Segre (1891a, *Opere* 4, 407). Segre's notebooks cited in what follows can all be accessed at the website (Giacardi 2013).

[66]See Amodeo (1890–1891) and Fano (1892), an article that is particularly interesting due to several developments that are connected with finite geometries, destined to attract the attention of mathematicians many years later.

[67]See, for example, C. Segre to G. Castelnuovo, Turin 24 and 30 September 1891, and C. Segre to F. Amodeo, Turin 15 August 1892 (Palladino F. and N. 2006, 198).

[68]See the previous note.

[69]See Enriques (1894), Fano (1895); See also F. Enriques to G. Castelnuovo, 22 February 1894, in Bottazzini et al. (1996, 77).

[70]See BMP-Segre, Quaderni. 5, p. 13, in Giacardi (2013).

[71]See Grace and William Young, Notebook 3, p. 16–24 (Appendix 3 at the end of this article). See Pieri (1899).

[72]C. Segre to G. Castelnuovo, Turin 8 August 1891 and Turin 30 September 1891.

[73]See Alberto Conte's introduction to Segre's Notebook 3 in this present volume.

[74][…] *i lettori potranno vedere in pari tempo un argomento trattato con due metodi ben diversi* (Segre 1894a, *Opere* 1, 200).

and topics of research to Castelnuovo, Fano and Amodeo, or because Eugenio Bertini (Forlì 1846–Pisa 1933) consulted him to learn how to "treat curves and ruled surfaces of hyperspace with our works",[75] but above all because he was fully aware of the significance of the new line of research, of the existence of a group of researchers who shared the new approach and also of the importance of disseminating their ideas and methods. For example, Segre wrote to Castelnuovo:

> Thinking just yesterday about my course, [...], I saw so clearly the necessity of introducing as soon as possible the notion of "multiply infinite linear variety", I saw the indispensability, the utility, the convenience, and I don't know what else, of this notion everywhere, and at every step. […] We really must think about writing treatises, lithographing lessons, spreading our ideas widely.[76]

A change of attitude is also palpable in Segre's letters to Klein: their correspondence now becomes a dialogue between peers, in which Segre also gives particular emphasis to the research of his students (Luciano and Roero 2012, *Introduction*).

3 Towards the Conquest of an International Dimension

At the end of 1889 Segre had entrusted to Fano, then still a student, the Italian translation of Felix Klein's *Erlangen Program*, because, as he himself explained:

> This work is not, in my opinion, well enough known to the *young Italian geometers*; and it is especially for them that I wished it to be reprinted. So many general and ingenious ideas are to be found in these pages, such as the substantial *identity* among various mathematical disciplines (and in particular between analytical and geometrical disciplines!) which can be seen to overlap one another when account is taken of the *transformations groups* that form the basis of them.[77]

[75]In July 1890 Eugenio Bertini came to Turin to spend a few days with Segre; see C. Segre to G. Castelnuovo, Turin 28 July 1890. He then carefully read the writings of Segre and Castelnuovo, in particular (Castelnuovo 1892) asking questions and making comments; see for example E. Bertini to G. Castelnuovo, Gromo, 4 August 1891 and E. Bertini to G. Castelnuovo, n. p., n. d. [August 1891], ANL-Castelnuovo, in Gario (2010). In this second letter Bertini announces his intention to present Castelnuovo's results during his classes. On Bertini's work see Kleiman (1998).

[76]*Pensando ieri appunto al mio corso, [...], io vedevo così chiaramente la necessità di introdurre al più presto la nozione di varietà lineare più volte infinita, vedevo l'indispensabilità, l'utilità, la comodità e che so io, di questa nozione dovunque, ad ogni passo* […] *Bisogna proprio pensare a far trattati, a litografare lezioni, a divulgare con estensione le nostre idee.* (C. Segre to G. Castelnuovo, Turin 6 July 1890).

[77]Corrado Segre, [Nota] to (Klein 1890, 307–308): *Questo lavoro non è, a mio avviso, abbastanza noto ai* giovani geometri italiani; *ed è specialmente per essi che ho desiderato si facesse questa ristampa. Tante idee generali ed ingegnose che si trovano in queste pagine, come l'*identità *sostanziale fra varie discipline matematiche (ed in particolare fra discipline analitiche e geometriche!) che si rappresentano l'una sull'altra quando si tenga conto dei* gruppi di trasformazioni c*he in esse si pongono a base.*

The translation appeared in the *Annali di matematica* in 1890, and very soon works began to appear that were inspired by it. The first was by Federigo Enriques (Leghorn 1871–Rome 1946), who came to Turin in November 1892[78] for a few weeks to meet Segre, and who published the article "Le superficie con infinite trasformazioni proiettive in sé stesse" (Enriques 1893a); this was followed by other important articles by Fano of 1896 on the problem of determining hypersurfaces in P^4 (C) which are left invariant under a group of projective transformations (Hawkins 1994, 186–191).

Fano finished his university studies in 1892[79] with a dissertation on hyperspatial geometry that was published as a lengthy memoir by the Accademia delle Scienze of Torino,[80] a work that openly reflected the influence of Segre, but also that of Castelnuovo, who, as the correspondence shows, stood beside Segre in advising and guiding the young mathematician.[81] After a year as an assistant (1892–93) of Enrico d'Ovidio at the University of Turin, Fano spent a period of post-graduate work in Göttingen.[82] In his letter recommending Fano to Klein, Segre wrote:

> He is gifted with a great memory, and has a lively mind. But his tendencies are essentially geometric, for pure geometry. And even though I have repeatedly invited him to cultivate analysis too, and in my courses I have shown not only the synthetic methods but also analytical methods, he has remained up to now too exclusively a geometer […] I believe that it is possible to strengthen him a great deal as a geometer if you can make him acquire fully the tools of analysis.[83]

Fano arrived in Göttingen in mid-October to attend Klein's lectures.[84] Fano attended two of his seminars, in the winter semester of 1893–94, and in the summer semester of 1894. In the first seminar Fano did not make a presentation, but he was one of the twelve speakers in the second, which dealt with topics involving *Kugelfunktionen*: Fano spoke on *Allgemeine Bemerkungen über Fourierische*

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[78]See F. Enriques to G. Castelnuovo, Turin 9 November 1892, in Bottazzini et al. (1996, 4).

[79]Fano earned a score of 90/90 with honours. See his degree certificate in ASUT, *Verbale dell'esame di laurea di Gino Fano. Torino, 22 giugno 1892*. Facoltà di Scienze matematiche, fisiche e naturali, Verbali degli esami di laurea, X D 193, p. 36.

[80]See Fano (1894a) and the report by Segre in *Atti dell'Accademia delle Scienze di Torino* (1892–93): 865–866. Segre would review three other memoirs by Fano, see Segre's report (1895–96, 623–624; 1897–98c, 796–797; 1900–01a, 278–279) in Appendix 2.2 in this present volume.

[81]See the 31 letters from Fano to Castelnuovo from 1889, when he was still a university student, to 1900, ANL-Castelnuovo, in Gario (2010).

[82]For more on this, see Fano (1894b).

[83]C. Segre to F. Klein, Turin 4 October 1893: *È dotato di molta memoria ed ha un ingegno vivace. Ma le sue tendenze sono essenzialmente geometriche, per la pura geometria. E quantunque io l'abbia eccitato ripetutamente a coltivare anche l'analisi, e nei miei corsi gli abbia fatto vedere non solo i metodi sintetici ma anche quelli analitici, egli finora è rimasto troppo esclusivamente geometra* […] *credo che si possa rinforzarlo di molto come geometra se si riesce a fargli acquistare pienamente gli strumenti analitici.*

[84]G. Fano to F. Amodeo, Göttingen, 30 October 1893 (Palladino F. and N. 1893, 372).

Reihen.[85] In Göttingen he also gave several much appreciated lectures at the Mathematische Gesellshaft, illustrating, among other things, the research and results of the Italian School of geometry, thus fostering their dissemination.[86] The works of 1896 inspired by the *Erlangen Program* were in all likelihood behind the invitation to Fano to collaborate on Klein's *Encyklopädie der mathematischen Wissenschaften*. He did so by contributing two articles, the second of which,[87] on "continuous geometrical groups" would exert an influence on Élie Cartan (see Hawkins 1994 and 2000, Chap. 8, Sect. 3). Moreover, in 1899 Klein, who had had the opportunity to appreciate Fano's methods of work, which aimed at exploiting geometric intuition, in the style of Segre's School, offered him a chair in geometry in Göttingen, writing:

> I conceive the chair essentially as a professorship of *geometry*, that is, I wish the one who holds it to exalt geometric intuition and develop geometric studies in all directions […] I have reached the conclusion that *you* are precisely the man for us![88]

Fano replied very diplomatically that he was honored by such an offer, but preferred a chair in an Italian university. In fact, in 1899, following a competition, he obtained a professorship at the University of Messina, but in 1901, again following a competition, he returned to Turin as a full professor of 'projective and descriptive geometry with drawing'. It was here that he carried out all of his teaching activities, except for a forced hiatus following the racial laws, that in 1938 banned the Jewish professors from Italian universities, a period that Fano spent in Lausanne.[89] The *leitmotiv* of his scientific research would be the study of three-dimensional algebraic varieties, an area in which he would carry out a work that was truly pioneering.[90] In particular the problem of the rationality or not of the

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[85]The authors thank David Rowe for this information, drawn from Klein's seminar protocol books, available in http://www.uni-math.gwdg.de/aufzeichnungen/klein-scans/klein/ (1893–94, 33–40, 24 June 1894).

[86]See Terracini (1953b, 704). The Mathematical Library "Giuseppe Peano" keeps a valuable collections of Klein's lithographies, among these the following belonged to Fano: *Nicht-Euklidische Geometrie*, I. *Vorlesung gehalten während des Wintersemesters, 1889–90.* Ausgearbeitet von Fr. Schilling, Göttingen 1893; *Nicht-Euklidische Geometrie*, II. *Vorlesung gehalten während des Sommersemester 1890.* Ausgearbeitet von Fr. Schilling, Göttingen 1893; *Einleitung in die höhere Geometrie, I. Vorlesung gehalten im Wintersemester 1892–93*. Ausgearbeitet von Fr. Schilling, Göttingen, 1893; *Anwendung der Differential-und Integralrechung auf Geometrie, eine Revision der Principien. Vorlesung gehalten während des Sommersemesters 1901*. Ausgearbeitet von Conrad Müller, Leipzig: Teubner, 1902.

[87]Fano (1907a, b).

[88]F. Klein to G. Fano, 5 February 1899: *Ich fasse die Professur wesentlich als eine* geometrische *Professur, d. h. ich wünsche, dass der Neuzuberufende die geometrische Anschauung hervorkehrt und nach allen Richtungen die geometrischen Studien belebt. Nun kennen* Sie *aber den Niedergang der Geometrie in der jüngeren deutschen Generation. Ich bin also auf den Gedanken gekommen, ob nicht Sie der geeignete Mann für uns wären!"*, BMP-Fano, Lettere 9, in Giacardi (2013).

[89]See Giacardi (2011, 111–115).

[90]See Murre (1994), Collino et al. (2014).

V_3 in S_4 has been already mentioned by Segre in Segre (1889c, footnote 36). In 1923 when he was invited to hold classes and two lectures on Italian geometry at the University College of Wales in Aberystwyth, after having illustrated the work of Luigi Cremona, he retraced the history of the heroic period of the Italian School, highlighting Segre's role and the contribution of his students. What is striking is Fano's emphasis on what it meant to belong to a School with methods and areas of research of its own, and on the nature of the work carried out within it, that is a "collective researches—Segre-Castelnuovo: 1890–91 in Turin—Castelnuovo Enr (1896–900) afterwards Severi for surfaces (irreg. 1904–05). Energies of investigators are summed. Their discoveries follow each other rapidly".[91]

In effect, in the mid-1890s Segre and his team—Fano, Castelnuovo and Enriques—already enjoyed the high esteem of the international community: for example, it is significant that they were invited to collaborate on the *Encyklopädie der mathematischen Wissenschaften*[92] and that in 1894 Brill and Noether, in the preface to their *Bericht, Die Entwicklung der Theorie der algebraischen Functionen in älterer und neuer Zeit* (Brill and Noether 1894, IV) made reference to the new line of research (*neuer Weise*) of the Italians, a thing that Segre did not fail to point out to Castelnuovo (C. Segre to G. Castelnuovo, Turin 28 November 1894).

In autumn 1892 Enriques, who, attracted by Segre's research, had asked to do post-graduate work in Turin, was sent instead to Rome,[93] where he began his fruitful collaboration with Castelnuovo; the two would remain friends for life. However, as mentioned, he came to Turin anyway in November 1892 to meet Segre in person[94] and, in November 1893, came a second time in hopes of obtaining a position of some kind and thus be able to work with Segre as well,[95] even though the possibility of becoming an assistant to Luigi Berzolari (Naples 1863–Pavia 1949) had vanished. Berzolari had been called that very year to the chair of

[91]This is Fano's original English text, in Fano n. d., [*Appunti vari*] f. 84v; see also fls. 1v, 17bisr, 69r, 69v, 80r, 103r, 135v; on Segre's role, see fls. 17r, 54r, 58r, 63r, 135r.

[92]The articles by the Italian geometers that appeared in the *Encyklopädie der mathematischen Wissenschaften* in the first two decades of the twentieth century are the following: Luigi Berzolari, *Allgemeine Theorie der höheren ebenen algebraischen Kurven* (III.2.1, June 1906, 313–455); F. Enriques, *Prinzipien der Geometrie* (III.1, March 1907, 1–129); G. Fano, *Gegensatz von synthetischer und analytischer Geometrie in seiner historischen Entwicklung im XIX. Jahrhundert* (III.1, May 1907, 221–288); *Kontinuierliche geometrische Gruppen. Die Gruppentheorie als geometrisches Einteilungsprinzip* (III.1, July 1907, 289–388); C. Segre, *Mehrdimensionale Raume* (III.2.2 C 7, end of 1912, 769–972); G. Castelnuovo, F. Enriques, 6a *Grundeigenschaften der algebraischen Flächen* (III.2.1, 1908, 635–673); 6b *Die algebraischen Flächen von gesichnen aus* (III.2.1 December 1914, 675–767); G. Loria, *Spezielle ebene algebraischen Kurven von einer Ordnung hoher als den vierten* (III.2.1, September 1914, 571–634). The dates cited for the texts refer to the date of submission.

[93]See, for example, C. Segre to G. Castelnuovo, Turin 18 July 1892.

[94]In a letter to Volterra, he speaks about "the damage resulting from the failure to stay in Torino with Segre" (*il danno per non poter stare a Torino con Segre*); see F. Enriques to V. Volterra, Turin 15 November 1892, in Simili (2000, 266).

[95]See C. Segre to G. Castelnuovo, Turin 5 November 1893 and F. Enriques to G. Castelnuovo, Florence 20 November 1893 (Bottazzini et al. 1996, 673 and 39).

projective and descriptive geometry, and would remain in Turin until 1899; he too benefitted from the stimulating contact with the Turinese geometers.[96]

The meeting between the austere, rigorous character of Segre and the impetuous nature of the young Enriques, who had a volcanic creative mind, was not an easy one. Segre invited him repeatedly to think more carefully about his work in order to avoid making mistakes,[97] and Enriques was almost afraid of him.[98] Nevertheless, the months in Turin between November 1893 and January 1894 were very intense for Enriques's scientific research in algebraic geometry, and stimulating for his reflections on the foundations as well. In fact Segre reported on one of the first works of Enriques on algebraic surfaces for the publication in the *Memorie della R. Accademia delle Scienze di Torino*, highlighting both the continuity with his own and Castelnuovo's scientific project and the importance of this kind of research and also its difficulty.[99] On the other hand, it should be recalled that Segre had been behind Fano's 1890 translation of Klein's *Erlangen Program*, and had urged Fano and Amodeo to study the foundations of projective geometry. Enriques also dealt with the same problem,[100] but chose another point of view:

> The route followed by them [Fano and Amodeo] is quite different from that we intend to take, especially in that, while the two esteemed authors propose to establish an arbitrary system of hypotheses capable of defining a linear space to which the results of ordinary geometry may be applicable, here we will seek to establish the postulates derived from experimental intuition of the space that appear to be the simplest for defining the object of projective geometry.[101]

Enriques' thinking emerges more clearly from the lithograph of his lessons on projective geometry of 1894–1895, in which he addresses, among other things, the problem of what geometry is, and explains the relationships between abstract geometry and intuition, thus tying into the guiding thread provided by Segre:

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[96]See Berzolari (1924). Segre presented four papers by Berzolari for publication in the *Atti dell'Accademia delle Scienze di Torino*, see Appendix 5 at the end of this article, also in Giacardi (2013).

[97]For example, after having begun to read a paper by Enriques, Segre wrote to Castelnuovo: *Raccomando poi caldamente il rigore, il rigore, il rigore* […] *Pesi bene ciò che scrive: e se incontra qualche intoppo non ci passi sopra.* (*I heartily recommend rigour, rigour, rigour* [...] *Ponder carefully over your writings and if you meet with some obstacle, not to overlook this*), see C. Segre to G. Castelnuovo, Turin 27 May 1893.

[98]Bottazzini et al. (1996, 61; see also p. 46 and 67)

[99]See Segre's report in the *Atti dell'Accademia delle Scienze di Torino* (1892–93c, 868).

[100]See footnotes 65 and 66 Avellone et al. (2002, Sect. 6).

[101]*L'indirizzo da essi seguito è alquanto diverso da quello a cui noi intendiamo attenerci, specialmente per ciò che, mentre i due egregi autori si propongono di stabilire un qualunque sistema di ipotesi capace di definire uno spazio lineare al quale siano applicabili i risultati dell'ordinaria geometria, noi cerchiamo qui di stabilire i postulati desunti dall'intuizione sperimentale dello spazio che si presentano più semplici per definire l'oggetto della* geometria *projettiva* (Enriques 1894, 551).

> The importance that we attribute to abstract geometry is not (as may be believed) opposed to the importance attributed to intuition: rather, it lies in the fact that *abstract geometry can be interpreted in infinite ways as a concrete (intuitive) geometry by fixing the nature of its elements: so in that way geometry can draw assistance in its development from an infinity of different forms of intuition.*[102]

Enriques in fact asserted, as noted above, that, thanks to Segre, the concept of abstract geometry had undergone a great development, becoming "an ordinary tool for working in the hands of contemporary Italian geometers".[103]

Enriques left Turin in January 1894 for Bologna, where he was assigned to teach projective and descriptive geometry. Three years later, at only 26 years old, he obtained the professorship. As shown by his correspondence with Castelnuovo, he remained in contact with Segre, sending him works to read and accepting his suggestions for readings,[104] but his true point of reference and collaborator became Castelnuovo. Together they constructed the theory of algebraic surfaces, offering a meaningful example of the collective work mentioned by Fano. This work, with "a unity of aim and sharing of concepts" (Bottazzini et al. 1996, 115) would push the two mathematicians to compete jointly for the Academia dei Lincei's Royal Prize in mathematics.[105] In May 1894 Enriques wrote to Castelnuovo that he intended to turn his attention to the study of surfaces,[106] and at the end of 1894, as the letters exchanged among the three friends show,[107] Castelnuovo and Enriques submitted the problem of how to resolve the singularities of algebraic surfaces to Segre.[108]

Segre thus proposed to dedicate his course of 1894–95[109] to the theory of algebraic surfaces, but he quickly abandoned the project of a course of a broad nature because, as he wrote to Castelnuovo, "you (and Enriques) will render obsolete within a few months all that can be done now […] Thus some study of the singularities of surfaces might perhaps be useful … if only to see how far it is

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[102]*L'importanza che attribuiamo alla Geometria astratta non è (come si potrebbe credere) da contrapporsi all'importanza attribuita all'intuizione: essa sta invece nel fatto che la* Geometria astratta si può interpretare in infiniti modi come una Geometria concreta (intuitiva) fissando la natura dei suoi elementi: sicché in tal modo la Geometria può trarre aiuto nel suo sviluppo da infinite forme diverse d'intuizione (Enriques 1894–1895, 9–10).

[103]*Il concetto di geometria astratta ha ricevuto un grande sviluppo, divenendo (dopo Segre) un ordinario istrumento di lavoro nelle mani dei geometri italiani contemporanei*. See Enriques (1922, 139). See the essay by Aldo Brigaglia in this present volume.

[104]From what Enriques told Castelnuovo, his relationship with Segre was most intense until 1898, and then became almost exclusively academic. In 1898 Segre reviewed his *Lezioni di Geometria projettiva* in the *Bollettino di Bibliografia e Storia delle Scienze Matematiche*, 1 (1898): 11–15.

[105]On the lively debate engendered by this joint participation in the competition see Bottazzini et al. (1998).

[106]C. Segre to G. Castelnuovo, Turin 7 May 1894.

[107]See Segre's letters to Castelnuovo during the months October – December 1894, in particular that of 24 December 1894, and that of Enriques to Castelnuovo of 26 December 1894 (Bottazzini et al. 1996, 160).

[108]For this, and on the relationships between the three mathematicians, see the chapter by Paola Gario in this present volume, see also Conte (1994) and Gray (1994).

[109]See BMP-Segre, Quaderni. 6.

possible get rid with them".[110] He in fact limited the program of his course to the singularities of plane curves, while the case of surfaces is mentioned only in the preliminaries, and in a general way:

> As to my course, as long as we are dealing with ordinary or easy not ordinary singularities, I will put curves and surfaces together. But when we go to higher singularities I will have to make a distinction: first curves and then surfaces. And to tell you confidentially, I am not sure that there will be enough time for the last of these![111]

The course was officially attended by only two students, Beppo Levi (Turin 1875–Rosario 1961) and Alessandro Padoa.[112] Levi in particular would benefit from these lessons.

In the meantime, in order to disseminate their findings, in 1896 Castelnuovo and Enriques, who had been awarded the Gold Medal of the Società dei XL in 1895 and 1896, precisely for their work on algebraic surfaces, published an account of their results, entitled "Sur quelques récent résultats dans la théorie des surfaces algébriques" in the *Mathematische Annalen* (Castelnuovo and Enriques 1896). This provided them with the chance for an epistolary exchange with Noether, one of the collaborators of editorial staff of the journal, who in reality did not seem to grasp the innovative nature of the Italian methods.[113] Instead, it was the reading of this account that allowed Émile Picard[114] to become familiar with the Italian geometers' methods, as he wrote to Castelnuolvo in March 1897:

> Your very noteworthy account in the Math. Annalen is extremely useful to me, and now after careful reflection I can admire your research, which on the whole is truly splendid. You have done, both of you, a considerable work.[115]

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[110]C. Segre to G. Castelnuovo, Turin 13 September 1894: *tu (e l'Enriques) renderai in pochi mesi antiquato ciò che ora si potrebbe fare* […]. *Così sarebbe forse utile un po' di studio delle singolarità delle superficie… non fosse che per vedere fino a qual punto si possono cacciare via.*

[111]C. Segre to G. Castelnuovo (Gario 1991, 166): *Quanto al mio corso, finché si tratta di singolarità ordinarie o straordinarie facili, metto insieme le curve e le superficie. Ma quando passerò alle singolarità superiori dovrò far la separazione: prima le curve e poi le superficie. E a dirtela in confidenza non son sicuro che avanzi poi tempo per queste ultime!* The first part of this course was published with a few additions and modifications in Segre (1898).

[112]Archival documents show that Beppo Levi earned a score of 30/30 and Alessandro Padoa 22/30. See *Registro dei verbali degli esami di Geometria superiore, 1882–1901* ASUT, *Facoltà di Scienze fisiche, matematiche e naturali, Verbali degli esami speciali, Esame di Geometria superiore* X.D 145.

[113]See Gario (1997) and *Corrispondenza con Max Noether,* ANL-Castelnuovo, in Gario (2010).

[114]Actually, Picard had already shown an earlier interest in the research of the Italian geometers when he published the article "Sur la théorie des surfaces algebriques", which appeared in the *Revue générale des sciences pures et appliquées*, 5 (1894): 945–959. See E. Picard to G. Castelnuovo, Paris 7 July 1894 (Bottazzini et al. 1996, 659).

[115]*Votre résumé si remarquable des Math. Annalen m'est extrêmement utile, et je puis maintenant admirer en connaissance de cause vos recherches dont l'ensemble est vraiment splendide. Vous avez fait là, tous les deux, une oeuvre considérable.* See E. Picard to G. Castelnuovo, Paris 23 March 1897 (Bottazzini et al. 1996, 662–663).

Summer 1897 brought the release of the first volume of the treatise written by Picard and Georges Simart *Théorie des fonctions algébriques de deux variables indépendantes* (1897, 1906), who in the Introduction cite the important research of Castelnuovo and Enriques "who have renovated an entire portion of the Theory of Surfaces" (*qui ont renouvelé toute une partie de la Théorie des surfaces*. I, p. VI). The second volume was released in 1906 and, as Enriques wrote in 1900 on the publication of the first issue, "The volume is going to give us a lot of publicity';[116] in fact an entire section (II, Note V, pp. 485–522), written by the two Italian mathematicians, was included. This fact is significant not only because of Picard's great appreciation of the Italian geometers' results, as witnessed also by his correspondence with Castelnuovo (Bottazzini et al. 1996, 659–668),[117] but also because of the difficulty that foreign mathematicians had in understanding the Italian geometric language. This difficulty emerges also, for example, from the correspondence with Gösta Mittag-Leffler, who in the same period invited Enriques and Castelnuovo to publish in the *Acta Mathematica.*[118]

The theory of singularities of algebraic surfaces would be treated by Segre in depth in the final part of the course in higher geometry of 1896–97, "Lezioni sulle singolarità delle curve e superficie algebriche"[119] following the publication in 1897 of his important paper entitled "Sulla scomposizione dei punti singolari delle superficie algebriche".[120] Here, extending one of Max Noether's results, he defined in a general and rigorous way the notion of 'infinitely near multiple points' of a surface, and used it to investigate the problem of the resolution of singularities of algebraic surfaces, but there were still points in the treatment that remained to be clarified. Segre involved his student Beppo Levi in this research.[121] Levi had earned his degree in 1896 with honours discussing a brilliant thesis on "Singolarità delle curve algebriche sghembe (iperspaziali)",[122] and Segre announced his student's results in the paper just cited (Segre 1897b, footnote 42).

In two articles of 1897, the second of which (Levi 1897–98) was presented by Segre himself, Levi proves the resolution theorem of the singularities of surfaces. The proof he gives starting from Segre's work was held to be satisfactory until 1935, when Oscar

[116]F. Enriques to G. Castelnuovo, s.l. 15 March 1900 (Bottazzini et al. 1996, 451): *è destinato a farci una grande réclame.*

[117]Segre corresponded with Picard and presented an excerpt of a letter by him for the publication in the *Atti dell'Accademia delle Scienze di Torino.* See Appendix 5 at the end of this article.

[118]F. Enriques to G. Castelnuovo, [Bologna] 14 April 1899 (Bottazzini et al. 1996, 409).

[119]The course was attended by seven students, three of whom sat for the examination that year; see *Annuario accademico per l'anno 1896–97*, Università di Torino, 1897, p. 184, and *Registro dei verbali degli esami di Geometria superiore, 1882–1901*, op. cit.

[120]Segre (1897b). Here Segre also makes some observations on the two works of 1888 and 1889 by Pasquale Del Pezzo, observations that became the starting point for a rather lively polemic between the two mathematicians, see (Gario 1989b). See also Ciliberto and Sallent (2013).

[121]For more on this, see the letter that Segre addressed to both Enriques and to Castelnuovo, Turin 30 December 1896 (Gario 1991, 176–180) and the essay by Gario in this present volume.

[122]ASUT, *Verbale dell'esame di laurea di Beppo Levi. Torino, 6 luglio 1896*. Facoltà di Scienze matematiche, fisiche e naturali, Verbali degli esami di laurea, X D 193, p. 97.

Zariski (1899–1986) published the critical revision of the various proofs of resolution theorem. Zariski expressly attributes to Segre the merit of having laid down the foundations of a geometric theory of singularities of algebraic surfaces (Zariski 1971, 13).

The aspiration of systematically disseminating the geometric research of the Italian School led Segre to express more than once to his friend Castelnuovo his desire to write a treatise on higher geometry. When Enriques joined them in research work, his desire seemed closer to becoming a reality: in December 1896 he thought about how to structure the subject, how to best use the notebooks of his university courses and the articles on hyperspaces and algebraic surfaces that he (Segre 1921c) and Castelnuovo were to write for the *Encyklopädie der mathematischen Wissenschaften*, and who might possibly publish it.[123] Sometime later, in August 1899, together with the publisher Teubner he decided on the title for the treatise: *Vorlesungen über höhere algebraische Geometrie, mit besonderer Berücksichtigung der mehrdimensionalen Räume.* He also sketched an outline of the subjects he intended to treat.[124] Ultimately, however, the treatise never came to light.[125]

Segre's course of 1897–1898[126] on continuous transformation groups is particularly interesting, because he expounds Lie's theory, attempting, however, to rendering it 'more luminous', as he wrote to his friend Castelnuovo in 1897:

> I will not occupy myself with other than studying groups, and making a program for the course. I find that the subject is vast, while the lessons are but few: and I find myself with embarrassing wealth of options. As for the method, I sometimes have difficulty in seeing Lie's reasoning and calculations clearly; and I would like to render the treatment more luminous. There are some fine theorems also in the final chapters, as well as in the first and the third volume of Lie's work: if I push myself to expound those, will I still have time to illustrate also the things on the *algebraic* groups of surfaces, varieties, …? I would like to be a little briefer in the treatment of projective, affine, … groups, because to deal with them extensively would take the whole year: I could sometimes state the results after having given the ways of obtaining them.[127]

[123]C. Segre to G. Castelnuovo, Turin 30 December 1896.

[124]C. Segre to G. Castelnuovo, Ancona 9 August 1899; see also C. Segre to G. Castelnuovo, Turin 13 February 1900 and C. Segre to V. Volterra, Ancona 11 August 1899, ANL-Volterra. In regard to this, see also Terracini (1961, 12), Gario (1991, 178), Giacardi (2001a, 150–151).

[125]On the reasons why this treatise was never carried out, see the essay by Gario in the present volume and that by Luciano and Roero.

[126]See BMP-Segre, Quaderni. 11. There were 19 students enrolled in the course, of whom 10 sat for the examination, including Alberto Tanturri, Umberto Perazzo, Tommaso Boggio, Luigia Viriglio and Modesto Panetti; see *Annuario accademico per l'anno 1898–99*, Università di Torino, 1898, p. 258 and *Registro dei verbali degli esami di Geometria superiore, 1882–1901*, op. cit.

[127]C. Segre to G. Castelnuovo, Turin 22 October 1897: *Non mi occupo d'altro che di studiare i gruppi, e di farmi un programma del corso. Trovo che la materia è molta, mentre le lezioni sono poche: e mi trovo nell'imbarazzo della scelta. Quanto al metodo, trovo difficoltà qualche volta nel veder chiaro nei ragionamenti e calcoli di Lie; e vorrei rendere la trattazione più luminosa. Vi sono dei bei teoremi anche negli ultimi capit*[i] *sì del 1° che del 3° vol. dell'opera di Lie: se mi spingo fino ad esporre quelli, farò in tempo ad esporre anche le cose sui gruppii* algebrici *di superficie, varietà …? Vorrei, essere un po' breve nella trattazione dei gruppi projettivi, affini,…, perché a farla completa se ne andrebbe tutto l'anno: potrei qualche volta enunciare i risultati dopo d'aver dato il modo di ottenerli.*

In the months that followed Segre kept Castelnuovo up to date on the progress of the course:

> In the course on groups I am still in the introductory part, that is, on the *general* properties, of continuous and discontinuous groups. I have also devoted a few lessons to Klein's Program. And I must even waste some time with hyperspaces.[128]

And again:

> I treated in my course the finite groups of projectivities in a way that appears to me to be satisfactory for its simplicity and elegance. Now the young people know a little about groups in general. I am talking about hyperspaces and their projectivities, etc., then I will consider particularly Lie.[129]

The approach that Segre intended to adopt was geometric: "We will attempt to give a rather geometrical guise to the theory; because many arguments are more clearly expressed in a geometric way".[130] This was in some way a return to the geometric origins of Lie's theory.[131] This became much more evident in the successive course of Segre's dedicated to the same topic, "Gruppi continui di trasformazioni (1911–12)" (BMP-Segre, Quaderni. 25).[132]

The studies of the connections between Lie's group theory and geometry took their first steps only after 1890, and as Thomas Hawkins wrote, "Italian geometers played an especially important role in bringing this about".[133] The leading figures were Enriques, with an article of 1893 (Enriques 1893a), and above all Fano, who in a first memoir of 1896, "Sulle varietà algebriche con un gruppo continuo non integrabile di trasformazioni proiettive in sé", (Fano 1896) refereed by Segre, considered hypersurfaces in four-dimensional space rejecting the methods employed by Lie and Enriques in the case $n = 3$. In his report Segre writes:

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[128] C. Segre to G. Castelnuovo, Turin 19 December 1897: *Nel corso sui gruppi sono ancora alla parte introduttoria cioè sulle proprietà* generali, *dei gruppi continui e discontinui. Ho anche dedicato qualche lezione al Programma di Klein. E dovrò pur perdere qualche tempo cogl'iperspazi.*

[129] C. Segre to G. Castelnuovo, Turin 26 January 1898: *Ho trattato nel mio corso i gruppi d'ordine finito di projettività in modo che mi pare soddisfacente per semplicità ed eleganza. Ora i giovani conoscono un po' in generale i gruppi. Sto parlando d'iperspazi e loro projettività ecc.: poi verrò in special modo a Lie*. See also C. Segre to G. Castelnuovo, Turin 17 May 1898.

[130] *Noi cercheremo poi di dare veste piuttosto geometrica alla teoria; perché molti ragionamenti sono più chiari espressi in modo geometrico*. (Youngs' Notebook 1, p. 3, see Appendix 3 at the end of the present article).

[131] See Hawkins (2000), Chap. 1, "The geometrical origins of Lie Theory". Segre refers to the following works: Lie (1888, 1890, 1893) and Lie (1893).

[132] See the *Introduzione* by Simonetta Di Sieno to the two Segre's notebooks Quaderni. 11 and Quaderni. 25 in Giacardi (2013). See also Raspanti 2014.

[133] Hawkins (1994, 186).

> The illustrious author of the general theory of transformation groups, […] LIE, solved that problem for the first cases that occur showing how his methods can be applied to projective groups of any higher space. However, if those methods, which had already required a somewhat detailed research for ordinary space, should they be applied to spaces of four or more dimensions, would lead to calculations of excessive length. Opportune geometric considerations can abbreviate this and even shed greater light on it. This idea inspired recent works of Italian geometers, and in particular the present memoir by Fano.[134]

As Thomas Hawkins observes:

> It was rather remarkable for that time period because it involved what appears to have been the first – and only – serious application of the theory of the structure and representation of Lie algebras to geometry before the work of Cartan some fourteen years later (Hawkins 2000, p. 254).

In addition to those officially enrolled, Segre's courses in higher geometry were also attended by various unregistered students, including non-Italian mathematicians, especially from England and North America, as recalled, for example, by Berzolari, Elisa Viglezio, and Terracini.[135]

The course of 1898–1899 on algebraic curves of projective spaces of various dimensions[136] was attended by the English couple William H. Young (London 1863–Lausanne 1942) and Grace Chisholm Young (Haslemere 1868–Croydon 1944). Grace had earned her doctorate in Göttingen with Felix Klein in 1895, and William would later become president of the London Mathematical Society and of the International Mathematical Union (Grattan-Guinness 1972, 2006). The Youngs had written to Segre of their wish to come to Turin "to live a lifetime of mathematics", as Segre himself told Vito Volterra.[137] The period spent in Turin (October 1898–May 1899) was a particularly happy one for the couple, and it was during this time that they began their long, fruitful scientific collaboration. Conserved today in the archives of the University of Liverpool are the evidences of their interactions with Segre, in the form of a few letters, Grace's *Diario scolastico*, the biographical timeline of William for the years 1897–1899, and five notebooks of handwritten

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[134]Segre's report in the *Atti dell'Accademia delle Scienze di Torino* (1895–96, 623–624): *L'illustre autore della teoria generale dei gruppi di trasformazioni,* […] *LIE, ha risolto quel problema pei primi casi che si presentano, facendo vedere come i suoi metodi si possano applicare ai gruppi proiettivi di qualunque spazio superiore. Però se quei metodi, che già per lo spazio ordinario hanno richiesto dal* LIE *una ricerca un po' minuta, si dovessero applicare agli spazi di quattro o più dimensioni, essi porterebbero a calcoli di una lunghezza eccessiva. Opportune considerazioni geometriche possono abbreviare ed anche illuminarla meglio. A questo concetto s'ispirano recenti lavori di geometri italiani, e in particolare la Memoria attuale del Fano*. See also Rogora 2012.

[135]Berzolari (1924, 532), Viglezio (1924, 2), Terracini (1968, 13).

[136]See BMP-Segre, Quaderni. 12. There were 12 students enrolled in the course, of whom 9 sat for the examination, including Francesco Severi; see *Annuario accademico per l'anno 1898–99*, Università di Torino,1899, p. 258 and *Registro dei verbali degli esami di Geometria superiore, 1882–1901*, op. cit.

[137]C. Segre to V. Volterra, Ancona 9 August 1898, ANL-Volterra; see also F. Enriques to G. Castelnuovo 14 April 1898 (Bottazzini et al. 1996, 365–368) and C. Segre to G. Castelnuovo, Turin 23 October 1898.

notes, almost all in Italian, of Segre's lessons that the Youngs attended in Turin.[138] These documents show that Segre gave a lecture each week to address topics in addition to those treated during the course, even without giving all the proofs, so that by the end of the year the students were able to master all that "everyone must know".[139] The notebooks seem also to indicate that Segre gave lectures expressly for the young couple, something that appears to be confirmed by a card by Segre, dated 11 March [1899], in which he writes: "If you and your husband wish for me to come one of these days and speak to you on some geometric topic that interests you, please write me, indicating the topics to me".[140]

The notes taken by Grace and William are particularly interesting because they illustrate Segre's efficacious way of teaching even better than his own notebooks, compiled in preparation for the courses. In fact the individual lessons are detailed, and the treatment is much more extensive;[141] it is possible to see the changes made with respect to what Segre had planned to do;[142] applications of the theory illustrated, and exercises proposed by Segre are included;[143] mention is made of the new research he submitted to his students,[144] and of the extra-curricular lectures (even evening lectures);[145] the most recent works to be consulted are indicated, and advice for study or research also appear. Here we provide a quote that is significant of Segre's commitment to training young people:

> Today when a young person chooses a topic of advanced science he can no longer use the excuse that he doesn't understand English or German, not to mention French or Latin, he must set himself to study. […] When any topic is being studied, it is necessary to consult the *Jahrbuch der Fortschritte der Mathematik*. This is divided according to topic. It requires patience to take the volumes of 10 years and consult them.[146]

[138]See the Appendices 1, 2, 3 at the end of this article.

[139]Notebook 3, p. 1; see the Appendix 3 at the end of this article, footnote 324.

[140]See the Appendix 1 at the end of this article: *Se lei e Suo marito desiderano che uno di questi giorni io venga a parlar Loro su qualche argomento geometrico di Loro interesse, favorisca scrivermi, indicandomi gli argomenti*.

[141]For example, this is particularly evident in Notebook 1: while the introductory part on groups occupies thirteen pages in Segre's notebook of 1897–1898 (BMP-Segre, Quaderni.11), it occupies eighteen lessons and 104 pages of Notebook 1 of the Youngs.

[142]See, for example, the Notebook 1, *Determinazione dei gruppi d'ordine finito di proiettività binarie, o di sostituzioni lineari in una variabile* (p. 104–134) and BMP-Segre, Quaderni.11, p. 13 and 14.

[143]See, for example, Notebook 1, p. 11, 34, 85; Notebook 2, p. 101, 105; Notebook 3, p. 238, 239, 331; Notebook 4, p. 35, 105; Notebook 5, p. 42, 59, 72, 88, 99, 110, 137.

[144]See, for example, Notebook 4, p. 105.

[145]See Notebook 3, p. 300, 301, [304], Notebook 4, p. 47 (evening lecture), p. 210.

[146]*Oggi quando il giovane prende un argomento della scienza alta non può mettere la scusa che non capisce l'inglese o il tedesco, non parlo del francese e del latino, bisogna mettersi a studiare* [...] *Quando si sta studiando un argomento qualsiasi bisogna consultare* das Jahrbuch der Fortschritte der Mathematik. *Questo è diviso secondo l'argomento. Ci vuole pazienza di prende[re] i volumi di 10 anni e consultarli* (Notebook 3, p. 11, 13).

During their stay in Turin, in 1899 the English couple also attended various lectures that Segre seems to have given especially for them on the theory of the singularities of algebraic curves and surfaces, as shown by their Notebook 5, which has connection with Segre's courses of 1894–1895 and 1896–1897 (BMP-Segre, Quaderni. 6 and 8) on the same topic, with more examples, applications and additions.[147]

In April 1899, Klein stopped in Turin, celebrating his fiftieth birthday with the Italian mathematicians. William Young recalls that day with the following words: "He arrived, on his 50th birthday, in Turin, and was fêted by the mathematicians of that city, where the present writer and his wife […] were then studying" (Young 1928, xiii).

In that same year Segre, as he was accustomed to do for his Italian students, presented two papers written by the English couple for publication in the *Atti della R. Accademia delle Scienze di Torino*.[148] Further, thanks to his interest, an Italian translation of the book by the Youngs entitled *A First Book of Geometry* (1905) was published. The translator was Luigia Viriglio, who had attended Segre's course in 1897–1898.[149]

The course of 1899–1900 was devoted to enumerative geometry,[150] to which Segre had turned his attention above all in the years immediately preceding the beginning of the course,[151] and in which he had open problems to discuss and propose. In July 1899 Alberto Tanturri (Scanno 1877–Sulmona 1924)[152] received his degree with honours, defending a thesis on enumerative geometry assigned by Segre. It was published the following year in the *Annali di matematica pura ed applicata*, in the long memoir "Ricerche sugli spazi plurisecanti di una curva algebrica" (Tanturri 1900), an important work that would be cited with some comments in the article by Zeuthen-Pieri in the *Encyclopédie des sciences mathématiques* (III.1, 1915, p. 277), together with another article on enumerative geometry, also by Tanturri, presented by Segre in February 1900 for publication in the *Atti della R. Accademia delle Scienze di Torino* (Tanturri 1899–1900). After a year as assistant to Eugenio Bertini in Pisa, in November 1900, thanks to Segre's interest, Tanturri transferred to Turin as assistant in projective and descriptive geometry, first with Segre, then with Fano, a position he maintained until 1904–1905, when he switched to secondary teaching. From that time on, his scientific work, which under Segre's guidance had been aimed above all at enumerative geometry, shifted to Peano's sphere of influence.

Other brilliant young people among Segre's students pursued this line of research. The course of 1899–1900 was attended by numerous students, as Segre communicated to the Youngs, many more than the six recorded in the 1899–1900 *Annuario* of the University of Turin:

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[147]See Appendix 3 at the end of this article, Notebook 5.

[148]See Chisholm Young (1898–99), Young (1898–99).

[149]See below the Section 6.

[150]BMP-Segre, Quaderni. 13 and the introduction by Aldo Brigaglia in Giacardi (2013). See also C. Segre to G. Castelnuovo, Turin 4 October 1899.

[151]See Segre (1897–98, 1898, 1900).

[152]ASUT, *Verbale dell'esame di laurea di Alberto Tanturri. Torino, 8 luglio 1899*. Facoltà di Scienze matematiche, fisiche e naturali, Verbali degli esami di laurea, X D 193, p. 152.

> At the University this year I will deal with […] enumerative geometry. There are from 15 to 20 students, including three ladies. There are also five doctors. I hope that at year's end I find myself content with the profit of these listeners.[153]

Among these was Giovanni Zeno Giambelli (Verona 1879–Messina 1953), who received his degree in 1901 with honours,[154] also presenting a thesis in enumerative geometry. That same year Segre reviewed Giambelli's important work entitled "Risoluzione del problema degli spazi secanti" (Giambelli 1903) for publication in the *Memorie della R. Accademia delle Scienze di Torino*. In his presentation Segre wrote:

> The solution to the problem in general appeared difficult, and was until now still longed for by the geometers. It is given in the memoir by D^r Giambelli, with a formula that, thanks to an ingenious symbolism, and as far as the complicated nature of the problem allows, turns out to be relatively simple and elegant. This result is evidently of such importance that we believe we can dispense with other considerations and propose without hesitation the acceptance of this work in the *Memorie* of the Academy.[155]

In the talk given in Heidelberg during the third International Congress of Mathematicians in 1904, Segre cited the research in enumerative geometry of his "student" (Segre 1905a, 119), along with the most important results of the Italian School.

It was precisely in this area that Giambelli made his most important contributions, becoming involved, among other things, in a dispute with Francesco Severi (Arezzo 1879–Rome 1961) (Laksov 1994), another outstanding student of Segre's. Severi had received his degree with honours in June 1900,[156] defending a thesis, under Segre's supervision, entitled "Sopra alcune singolarità delle curve di un iperspazio", published the following year in the *Memorie della R. Accademia delle Scienze di Torino* (Severi 1901), after receiving a favourable referee report by Segre.[157] Here besides highlighting Severi's new results, Segre also underlined the ingenious tools that he uses:

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[153]Segre to G. and W. Young, 30 November 1899 (see the Appendix 1 at the end of this article): *All'Università quest'anno tratto* […] *la geometria numerativa. Vi sono da 15 a 20 studenti, fra cui 3 signore Vi sono anche 5 dottori. Spero che alla fine dell'anno mi troverò contento del profitto di questo uditorio*.

[154]ASUT, *Verbale dell'esame di laurea di Giovanni Zeno Giambelli. Torino, 4 novembre 1901*. Facoltà di Scienze matematiche, fisiche e naturali, Verbali degli esami di laurea, X D 193, p. 225.

[155]*La risoluzione del problema in generale appariva difficile, e costituiva tuttora un desiderio dei geometri. Essa è data in questa Memoria dal D^r giambelli, con una formola, che, grazie ad un ingegnoso simbolismo, e per quanto lo permette la natura complicata del problema, riesce relativamente semplice ed elegante. La cosa è evidentemente di tale importanza che noi crediamo di poterci dispensare da altre considerazioni e vi proponiamo senz'altro l'accoglimento di questo lavoro fra le* Memorie *dell'Accademia* (Segre's report in the *Atti dell'Accademia delle Scienze di Torino* 1901–02c, 733).

[156]See ASUT, *Verbale dell'esame di laurea di Francesco Buonaccorso Severi. Torino, 30 giugno 1900*. Facoltà di Scienze matematiche, fisiche e naturali, Verbali degli esami di laurea, X D 193, p. 177.

[157]See Segre's report in the *Atti dell'Accademia delle Scienze di Torino* (1900–1901b). Segre would also be the referee of two other papers by Severi published in the *Memorie* (Segre, report 1901–1902b; 1902–1903b).

> Also worthy of mention are the tools used by Mr Severi: these frequently consist in ingenious applications of the principle of correspondence on a curve of genus *p*; and instead, for some question relative to S_5, in the functional method, already used by Cayley in the simplest case of the curves in S_3, and here for higher spaces made easier by the Author by means of the preliminary integration of a broad class of functional equations that are present in various research works on enumerative geometry.[158]

Regarding Severi's work, in 1963 Leonard Roth, who was among other things greatly influenced by the Italian School, wrote:

> Segre was quick to recognise the exceptional talent of his young pupil who, under his master's direction, soon began to produce original work of high quality; in fact Severi's doctoral thesis, written in 1900, still retains interest. (Roth 1963, 282)

Between 1900 and 1903 Segre refereed and presented for publication no fewer than twelve works by his three students, Tanturri, Giambelli and Severi.[159] In particular, Giambelli and Severi published other important articles on enumerative geometry, addressing, with different methods, Hilbert's Problem 15.[160] These articles show two different approaches. The contrast between Giambelli's algebraic-symbolic methods and Severi's geometric-intuitive approach led to a bitter polemic between the two (Brigaglia 1982).

Giambelli remained in Torino as assistant to Fano in the course of projective geometry until 1903, and later, after having taught at the universities of Genoa, Cagliari and Pavia, in 1915 he was named to the position of non-tenured professor of algebraic analysis at the University of Messina. Instead, immediately after receiving his degree Severi was awarded the Premio Ferrati, and for a year was D'Ovidio's assistant for the course in analytical geometry. He obtained his teaching certification (*libera docenza*), and received a contract to teach the course of projective and descriptive geometry from 1902–03 to 1904–05. In 1904, only 25 years old, he obtained the professorship of projective and descriptive geometry at the University of Parma, after having been assistant to Enriques in Bologna and Bertini in Pisa. It was in that year that he published his well-known paper entitled "Sulle superficie algebriche che posseggono integrali di Picard della 2ª specie" (Severi 1904), in which, carrying forward the research of the Italian School, he succeeded in proving that a surface having a Picard integral of the second kind is necessarily

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[158]*Meritano menzione anche gli strumenti adoperati dal signor Severi: i quali consistono spesso in ingegnose applicazioni del principio di corrispondenza sulla curva di genere p; e invece, per talune questioni relative allo S_5, nel metodo funzionale, già usato dal Cayley nel caso più semplice delle curve di S_3, e qui per spazi superiori agevolato dall'A. per mezzo dell'integrazione preliminare di un'ampia classe di equazioni funzionali che si presentano in varie ricerche di geometria numerativa* (Segre, report 1900–1901b, 381).

[159]See *Relazioni* in Giacardi (2013) Appendix 5 at the end of this article.

[160]Here is how Hilbert himself stated Problem 15: *Détermination rigoureuse des nombres de la Géométrie énumérative, et cela en fixant d'une manière plus précise les limites de leur validité, et, en particulier, des nombres que Schubert a trouvés en s'appuyant sur le principe de son calcul énumératif* dit de la position spéciale *ou* de la conservation du nombre, in *Compte Rendu du Deuxième Congrès International des Mathématiciens, tenu à Paris du 6 au 12 août 1900. Procès-Verbaux et Communications*, Paris: Gauthier-Villars, 1902, 58–114, on p. 95.

irregular, thus making an important step forward in the study of algebraic surfaces from the transcendental point of view, and showing his skill in handling the algebraic-geometric tools of the Italian School, as well as the transcendent means of the French School. At the beginning of the twentieth century we are witnessing the "grafting" in Italian School research of methods from outside, a factor which stimulated new investigations.

In 1906 Severi was awared the gold medal by the Società dei XL, in 1907 the Prix Bordin together with Enriques, in 1908 the Medaglia Guccia, and in 1913 the Royal Prize for mathematics offered by the Accademia dei Lincei. These were the beginnings of Severi's dazzling scientific career, which would take him to Rome in 1921. In the 1920s the capital, where Castelnuovo, Enriques and Severi all found themselves, became the new centre of gravity for the Italian School of algebraic geometry, which began a new phase of its history.

The period in Turin and Segre's teaching were particularly important for Severi's future research; from his teacher he had come away with not only remarkable skills in the field of higher-dimensional projective spaces, but also a profound interest in algebraic and enumerative questions. As a sign of his gratitude, Severi dedicated his book *Complementi di geometria proiettiva* (1906) to Segre, calling him an "incomparable Teacher who with assiduous care trained my mind for the serious investigations of science" (*Maestro incomparabile che con assidua cura m'educò l'intelletto alle severe indagini della scienza*) and in 1924 in his lecture on the algebraic geometry, given at the International Congress of Mathematicians in Toronto, he affirmed: "The little that I was able to do in science is the fruit of the wise and impassioned teaching of my direct maestro, Corrado Segre" (*Le peu que j'ai pu faire dans la science est le fruit de l'enseignement savant et passionné de mon maître direct, Corrado Segre*. Severi 1928, 154).

The ample liberty that Segre gave his students to follow their own inclinations and to be open to stimuli from outside the School, together with his capacity for recognising his own limits, fostered the broadening of their scientific horizons and their successes in the international arena. On the other hand, it should be noted that in the early years of the twentieth century that climate of "disinterested collaboration" that led, as Castelnuovo said, to "breaking down too-clean divisions between the work of one and that of another" (Babbit and Goodstein 2011, 70) began to grow weaker. Severi's unbridled ambition was certainly one of the factors that contributed to this. We need only think, for example, of the disagreements with Enriques from the scientific and academic points of view, after an earlier phase of collaboration, of those with Castelnuovo and Enriques over his role in the Commission Internationale de l'Enseignement Mathématique (later ICMI),[161] of the controversy with Giambelli, and that with Michele de Franchis.[162]

[161]See the essays in Pompeo Faracovi (2004) and Giacardi and Tealdi (2015).

[162]See, for example, the long letter of F. Severi to G. Castelnuovo, Padua 15 Juillet 1913. See also several letters of the 1930s which highlight the difference in character between the two mathematicians (Babbit and Goodstein 2009 and 2011).

4 A "Bridge" Between the Schools of Segre and of Peano

During the years that marked the end of the nineteenth century and the beginning of the twentieth, the presence in Turin of Peano, a figure of the highest calibre on the international scene, and of his School of mathematical logic, which put forward concepts and methods that contrasted in many ways with those of the School of geometry, contributed to a climate characterised by lively debates, sometimes in a key that was bitter and controversial,[163] but nevertheless always fertile. While it is undeniable that the two Schools were often in opposition, it is also true that relationships of collaboration and exchange existed, which remain in part to be studied in depth. We will cite just a few examples. Fano collaborated on Peano's *Formulario*; Beppo Levi had profound knowledge, and was sometimes critical, of Peano's work, which influenced his own studies on the theory of functions, set theory, and logic; Tanturri, once he had switched to secondary teaching, approached Peano's School; Angelo Ramorino (1869–?) who in 1893 defended his thesis "Sugli elementi immaginari in geometria con approccio storico",[164] under Segre's supervision, later collaborated on the third and fourth editions of Peano's *Formulario.* Similarly, Matteo Bottasso (1878–1918), who earned his degree in 1901 with a thesis on geometry[165] and of whom Segre presented two works in 1908 and 1909 for publication in the *Atti della R. Accademia delle Scienze di Torino*,[166] collaborating later with Peano's School, worked to disseminate the geometric-vectorial methods. The same is true of Luigia Viriglio (1879–1955), who had been a student of Segre's,[167] but stimulated by secondary school teaching, published research of a historic, didactic and linguistic nature that is more easily collocated in Peano's sphere.[168] Moreover there is a long list of both Segre's and Peano's students who treasured methodological tenets of the two masters, when they became mathematics teachers in secondary schools.[169]

There is, as is well known, an important mathematician who embodied methods and research themes of both schools: Mario Pieri (Lucca 1860–S. Andrea Compito 1913).[170] After earning his degree in 1884 from the Scuola Normale Superiore in Pisa, Pieri arrived in Turin the following year after having won a position in the

[163] These debates have been discussed several times. See, for example, the most recent papers (Avellone et al. 2002, Sect. 3; Luciano 2006, Sect. 5).

[164] See Ramorino (1897) and C. Segre to G. Castelnuovo, Turin 28 January 1893. Ramorino devoted part of the second section of this memoir to Segre's contributions to complex projective geometry.

[165] See C. Segre to V. Volterra, Turin 6 July 1901, ANL-Volterra.

[166] See Appendix 5 at the end of this article.

[167] Viriglio earned her degree under Segre's advisement with a thesis "Sui gruppi di ordine finito di sostituzioni lineari" on 9 December 1904.

[168] Concerning the mathematicians of Peano's entourage, see Luciano and Roero (2010).

[169] See the end of Section 6. See also Segre's [Indirizzario] in Uto-ACS, II.

[170] On Pieri's relationships with the two Schools, see Marchistotto and Smith (2007, Chaps. 1 and 2), Brigaglia (2009–2011).

military academy there. Beginning in 1888 he was also assistant to the chair of projective and descriptive geometry (held first by Giuseppe Bruno and then by Luigi Berzolari) in the Faculty of Sciences, where he also gave open courses (*corsi liberi*) in projective geometry (1891–98) and complements of geometry (1898–1901). He remained in Turin until January 1900, when he moved to Catania, having won a professorship there. His first works in algebraic geometry, and especially enumerative geometry, which intertwined in significant ways with those of Giambelli in particular, date back to the first years of his stay in Turin and the time he spent with Segre and his group.[171] Testimony of the early collaboration between the two young mathematicians is found in the acknowledgements to Pieri that appear in a memoir by Segre (Segre 1889c, *Opere* 4, 100), and from the fact that together they examined some parts of the *Lezioni di geometria proiettiva* by Achille Sannia;[172] further, as early as October 1887 Pieri had accepted Segre's invitation to translate von Staudt's *Geometrie der Lage*.[173] The careful reading of this work, and the time spent with Cesare Burali-Forti, his colleague at the military academy, and with Peano, led him to turn his attention to the study of the foundations of geometry, an area in which he would make his most significant contributions. However, as has been repeatedly noted (Marchisotto and Smith 2007, Brigaglia 2009–2011), even Pieri's studies on the foundations of geometry reflect the influence of Segre who, from the very first courses in higher geometry, was sensitive to this topic, and in particular he was concerned with giving a rigorous foundation to the projective geometry of higher-dimensional spaces. In fact, as we saw earlier, Segre stimulated the work of his students Fano and Amodeo, who even though unable to compete on the level of rigour with Peano and Pieri, nevertheless showed

> a very precise plan for work, indicated in fact by Segre and quite distinct from that of Peano: 'Treat all of geometry of position within itself, without introducing metric concepts that would be foreign to it': so Segre had written in the Introduction to *Geometrie der Lage* and this he had suggested to his students, extending the same idea […] to the geometry of higher-dimensional spaces.[174]

Pieri addressed this very program in his article "Un sistema di postulati per la geometria projettiva astratta degli iperspazii" (Pieri 1896), which was a model of formal rigour. Segre, together with Peano and D'Ovidio, presented and refereed three important works of his on the foundations of geometry, for publication in the

[171]Segre, writing to Castelnuovo about Pieri's works, praised the variety of his methods, and the clarity and rigour of the exposition; see C. Segre to G. Castelnuovo, Ancona 25 October 1896. There exist five letters from Segre to Pieri dating from 1887 to 1911; these are published in Arrighi (1997, 113–116).

[172]C. Segre to V. Volterra, Turin 21 October 1887, ANL-Volterra and footnote 56.

[173]See footnotes 24–25 in this present article, and C. Segre to M. Pieri, Turin 11 October 1887 (Arrighi 1997, 113). See also (Marchisotto and Smith 2007, Chap. 2).

[174][…] *un piano di lavoro ben preciso, indicato appunto da Segre e ben distinto da quello di Peano. «Trattare tutta la geometria di posizione da sé, senza introdurre concetti metrici che le sarebbero estranei», così aveva scritto Segre nell'Introduzione alla* Geometrie der Lage *e così aveva indicato ai suoi allievi estendendo lo stesso concetto* […] *alla geometria degli iperspazi.* (Brigaglia 2009–2011, 27).

Memorie della R. Accademia delle Scienze di Torino.[175] Among these, the one that is most closely connected to the research of Segre himself is Pieri's paper "Nuovi principii di geometria projettiva complessa" (Pieri 1905), aimed at determining the axiomatic structure of complex projective geometry. Here he makes several references to the research of Segre presented from 1889 to 1891 in the series of papers entitled "Un nuovo campo di ricerche geometriche",[176] to which we will return.

5 "No Time to Work Ourselves!"

"In having more young people to put to work, there is the problem of not having any time to work ourselves!" This is what Segre wrote to Pieri in 1901.[177] In fact, the extremely intense research activity carried out in the 17 years since he took his degree, during which he had produced more than sixty works, gradually began to slow down, due to both his new family responsibilities—the birth of his two daughters, Elena in 1894 and Adriana in 1897—and to his new institutional commitments: he was one of the editors of the *Annali di matematica pura ed applicata* from 1904, the head of the Biblioteca speciale di matematica from 1907 and the dean of the Faculty of Sciences of the University of Turin from 1909. This did not, however, weaken his commitment to teaching and training young researchers, or to his creative vein, which led him to open new pathways.

His course of higher geometry of 1902–1903 was devoted to non-Euclidean geometry [BMP–Segre, Quaderni. 16].[178] Segre's interest in this topic certainly dates to before that year, as shown, among other things, by the "Account of writings read" conserved among his scientific papers rediscovered in Ancona[179] and in a letter written to Klein in 1890, in which he asks for information about the course in non-Euclidean geometry that Klein taught that semester.[180] The approach adopted by Segre is a historical one, and with respect to Klein's course of 1890, he attributes

[175]See Segre's reports in the *Atti dell'Accademia delle Scienze di Torino* (1897–98a, 1898–99b, 1904–05a).

[176]See Segre (1889–90), Segre (1890–91). See also the essay by Aldo Brigaglia in this present volume.

[177]*Ad aver più giovani da far lavorare c'è l'inconveniente che non si ha più il tempo per lavorare noi!* (C. Segre to M. Pieri, Turin, 20 November 1901, in Arrighi (1997, 115).

[178]The official course register shows eight students enrolled; see *Annuario della R. Università di Torino, 1903–04*, Stamperia Reale di Torino, 1904, p. 209.

[179]See Gario (1989a, 192) where there are in fact various references to works by German authors such as Bernhard Riemann and Klein, while the Italian authors include Giuseppe Battaglini, the principal disseminator of non-Euclidean geometry in Italy, and Eugenio Beltrami, author of the famous *Saggio di interpretazione della geometria non euclidea* (1868). In particular, in the folder whose cover bears the title "Lettere a scienziati (1882–84)", there are several pages of undated notes that might have been intended to reach Beltrami via D'Ovidio, containing several "considerations on the surfaces of constant curvature and on the non-Euclidean plane' (Gario 1989a, 195).

[180]C. Segre to F. Klein, Turin 30 January 1890.

much greater importance to the forerunners of non-Euclidean geometry, in particular to Girolamo Saccheri and to his influence on the creation of non-Euclidean geometry, presenting, among other things, the results of a historical research published that same year in the article entitled "Congetture intorno all'influenza di Girolamo Saccheri sulla formazione della geometria non-Euclidea".[181] Because of the richness and completeness of the reconstruction of the genesis and developments of non-Euclidean geometry presented by Segre, Enriques's student Roberto Bonola (1874–1911) asked his permission to consult his handwritten notes when he prepared his book *La geometria non euclidea* (1906), the first genuine history of non-Euclidean geometries, which was soon translated into German and English. In the introduction Bonola writes:

> Before entrusting this modest work to the judgment of benevolent readers, I feel the duty to thank most heartily my beloved *maestro*, Prof. FEDERIGO ENRIQUES, for the valuable advice with which he aided me to organise the subject and to develop critical aspects; Prof. CORRADO SEGRE, who kindly placed at my disposal the manuscript of a *Corso di lezioni* on non-Euclidean geometry, given by him 3 years ago at the University of Turin.[182]

In the final part of the notebook Segre also explicitly presents the main directions of research and development of non-Euclidean geometries: projective geometry (Cayley, Klein, …); differential geometry (Riemann, Beltrami,…); the theory of transformation groups (Helmholtz, Lie, …); non-Euclidean mechanics (De Tilly, Genocchi, Lindemann, …), a distinction that would be taken up by Bonola in his book, and in general also by later works on the history of non-Euclidean geometries.

It is worthwhile to note that in the introductory chapter, dedicated to the 'essence of geometry' (p. 3), Segre deals with the relationships between logic and experience, between 'mathematics of precision' and 'mathematics of approximation', a topic to which Segre would return frequently in his courses, especially in the course entitled *Vedute superiori sulla Geometria elementare* [BMP–Segre, Quaderni. 30] and in his lectures at the Scuola di Magistero [BMP–*Segre,* Quaderni. 40], of which we will say more below.

Segre is also acknowledged in the preface (p. 6) to the book *The Elements of Non-Euclidean Geometry* (Coolidge 1909), by Julian Lowell Coolidge (Brookline, Boston 1873–Cambridge 1954). Moreover in this book Segre's memoir entitled "Un nuovo campo di ricerche geometriche" (Segre 1889–90 and 1890–91), which in Italy did not enjoy the attention it deserved, is mentioned, (p. 119). The young American mathematician was, in fact, in Turin in 1903–1904 and attended the course on applications of Abelian integrals in geometry,[183] which Segre opened

[181]See Segre (1902–03). About this see Brigaglia (2010).

[182]Bonola (1906, XVII): *Prima di affidare la modesta opera al giudizio dei benevoli lettori, sento il dovere di ringraziare vivamente il mio amato maestro, prof.* FEDERIGO ENRIQUES, *per i preziosi consigli con cui mi ha soccorso per la disposizione e pel contenuto critico della materia; il prof.* CORRADO SEGRE, *che gentilmente ha posto a mia disposizione il manoscritto di un Corso di lezioni sulla Geometria non-euclidea, da lui dettato, or son tre anni, nell'Università di Torino.*

[183]BMP–Segre, Quaderni. 17. The official register shows eight students enrolled in the course; see *Annuario della R. Università di Torino, 1904–05*, Stamperia Reale di Torino, 1905, p. 201.

with a concise treatment of functions of a complex variable and their representation on a surface. Coolidge recalls the opening of this course with this words:

> I remember being impressed at the beginning of one course of lectures by the fact that the professor put down, as principal works of reference, books in four different languages, and remarked that those of his hearers who could not read English, French and German must certainly make up the deficiency in the course ot the year. (Coolidge 1904, 13)[184]

Stimulated by the lessons, Coolidge published two notes in the *Atti della R. Accademia delle Scienze di Torino,* presented by Segre, where he dealt with the geometrical interpretation of complex numbers and their functions[185] The period that he spent in Turin would influence all of his early scientific output,[186] and he expressed his debt to his Turinese teacher in his treatise *The Geometry of the Complex Domain* of 1924:

> Every student of geometry in the complex domain will find that he is forced to refer continually to the work of two admirable contemporary geometers, Professor Corrado Segre of Turin, and Professor Eduard Study of Bonn. The names of both appear incessantly throughout this book; the author had the rare privilege to be the pupil of each of these masters. (Coolidge 1924, 7)

Several years later, in his commemoration of Segre, Coolidge wrote, "There was no limit to the amount of care and patience which he would bestow on one of his pupils" (Coolidge 1927, 357).

As a result Segre's international reputation was established. The invitation to hold a plenary lecture in 1904 at the International Congress of Mathematicians in Heidelberg (8–13 August 1904) marked the official international recognition of him. Here he met, among others, Eduard Study, Ernest Wilczynski and Samuel Dickstein, who would immediately translate into Polish his lecture "La Geometria d'oggidì e i suoi legami coll'Analisi" (Segre 1905a, transl. 1905b).[187]

That same year, his paper "Su alcuni indirizzi nelle investigazioni geometriche. Osservazioni dirette ai miei studenti" was translated into English [Segre 1891a, transl. 1904] with the title "On some tendencies in geometric investigations" and was published in the *Bulletin of the American Mathematical Society.* The translation was by John Wesley Young (1879–1932), a mathematician who was very interested in questions regarding education and who would later become the president of the American Mathematical Society. Among other things, in the bibliographic updates added in the footnotes, in addition to mentioning the recent studies on the

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[184]See also *Alcune indicazioni bibliografiche,* in BMP-Segre, Quaderni. 17, p. 3.

[185]Coolidge (1903–04, 1904–05). The first note was presented by Segre at the session of 20 December 1903 and the second at that of 8 January 1905. See Appendix 5 at the end of this article.

[186]See Struik (1955, 671). For more on this, see the chapter by Aldo Brigaglia in this present volume and Zappulla (2009).

[187]See C. Segre to Olga Michelli Segre, Airolo, 15 August 1904 (UTo-ACS, II): "it was very satisfying to me to see the attention paid to my lecture by eminent men such as Klein, Noether, Zeuthen and many others I won't name" (*fu molto soddisfacente per me il vedere attenti alla mia conferenza uomini sommi come Klein, Noether, Zeuthen e tanti altri che non ti nomino*).

foundations of geometry, Segre proudly inserted all of the most important results achieved by his students. It is no coincidence that from 1904, nearly every year, the titles of his courses were mentioned in the Bulletin of the American Mathematical Society.

Starting in 1904 (from volume X), Segre was one of the directors (with Luigi Bianchi, Ulisse Dini and Giuseppe Jung) of one of the most important scientific journals of the day, the *Annali di Matematica pura ed applicata*, to which he devoted time and energy, taking an interest not only in scientific aspects but in financial aspects as well, as shown, for example, by his correspondence with Virgil Snyder (1869–1950) of Cornell University.[188]

In 1908 Segre was named a member of the awards commission for the Medaglia Guccia, together with Max Noether and Henri Poincaré. To Segre, who was reluctant to accept the assignment, Giovanni B. Guccia wrote: "I will not name any others, and I beg you not to do so either. You or no one!"[189] The medal was awarded to Francesco Severi during the International Congress of Mathematicians held in Rome from 6 to 11 April 1908 (Noether et al. 1909, 216).

The course of 1907–08 was dedicated to *Capitoli vari di Geometria della retta*.[190] The treatment proceeded according the differential point of view; this was no coincidence, because there appear the W congruences whose focal surfaces are ruled (pp. 160 ff.) that Segre had dealt with in a paper of 1907 (Segre 1906–07a), and the notions of projective higher-dimensional geometry that he developed in the paper of that same year entitled "Su una classe di superficie degl'iperspazi legate colle equazioni lineari alle derivate parziali di 2° ordine" (Segre 1906–07b), a work which, together with the later one of 1910, planted the seeds in Italy for the flourishing of projective differential geometry.[191]

The courses of the 3 years that followed were respectively devoted to a *Rassegna di concetti e metodi della Geometria moderna* (1908–1909), *Superficie del 3° ordine e curve piane del 4° ordine* (1909–1910) and *Le curve e le superficie algebriche, dal punto di vista della Geometria delle trasformazioni birazionali* (1910–11).[192] The lessons on cubic surfaces of 1909–10 are of particular interest, both because they provided a systematic and elegant exposition of the subject, and because, as Segre himself observed in his preliminaries:

> The F_3 have exerted a notable influence on the development of modern algebraic geometry. They lend themselves well to illustrating the methods of this [discipline], along various

[188]See UTo-ACS, VII.

[189]G. B. Guccia to C. Segre, 27 July 1904, quoted in Brigaglia (2013, 473): *Altri nomi non faccio e ti prego di non farne. O te o nessuno!*. See also C. Segre to G. Castelnuovo, 5 November 1905, where Segre expressed his wish that the winner might be an Italian mathematician.

[190]BMP-Segre, Quaderni. 21. The official register for the course show 10 students enrolled; see *Annuario della R. Università di Torino, 1908–09*, Stamperia Reale di Torino, 1909, p. 247.

[191]See for example Terracini (1953a, 257–258), Ciliberto and Sallent (2013, Sect. 3.2.2).

[192]BMP–Segre, Quaderni. 22, 23 e 24: the students officially enrolled in these courses were respectively, 9, 13 and 13: see *Annuario della R. Università di Torino, Stamperia Reale di Torino,* 1909–10 (p. 251), 1910–11 (p. 267) and 1911–12 (p. 275).

> lines of research: configurations, singularities, questions of reality and form, geometric generations, representations over the plane, various algebraic problems.[193]

Alessandro Terracini (Turin 1889–Turin 1968), who attended all three of these courses starting in his second year at university (Terracini 1968, 11), many years later wrote about this notebook:

> This little book on cubic surfaces, in spite of the elementary and particular nature of the subject, seemed to me to be one of the most beautiful, perhaps as well because of the projective line along which it proceeded. […] If I had to choose a notebook for publication, I believe this is one that would be among the top.[194]

Terracini took his degree on 5 July 1911[195] with a score of 100/100 and honours, with a thesis written under Segre's supervision entitled "Sulla teoria delle varietà luoghi di spazi", a part of which was published that same year in the *Rendiconti del Circolo Matematico di Palermo* (Terracini 1911). The following year he published, in that same journal, a paper in which he extended to varieties of an arbitrary number of dimensions the result obtained by Segre for surfaces in the 1907 paper just cited and later extended to three-dimensional varieties by Charles Herschel Sisam (1879–1964) and by Ellis Bagley Stouffer (1884–1965), both of whom came to Italy to meet the Italian geometers: Sisam in 1908–1909 to follow Segre's lectures, and Stouffer in 1926–1927 with a Guggenheim fellowship.[196] Sisam had received his degree from Cornell University in 1905 under the supervision of Virgil Snyder, who at the time was also editor of the *Bulletin of the American Mathematical Society* and who would correspond with Segre regarding the *Annali di Matematica pura ed applicata.*[197] Sisam recalls the Turinese teacher this way:

> I was student at the University of Turin during the year 1908–9. I attended Segre's lectures, which will ever stand out in my mind as models of clearness, force and value.[198]

Segre mentions the stay in Turin of his foreign student at the beginning of the short paper (Segre 1910b, 346).

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[193]BMP–Segre, Quaderni. 23: *Le F_3 hanno avuto una notevole influenza sullo sviluppo della moderna Geom^a alg^a. Si prestano molto bene ad illustrare i metodi di questa, in vari indirizzi: configurazioni, singolarità, quistioni di realtà e forma, generazioni geometriche, rappresentazioni sul piano, problemi algebrici vari* (p. 3).

[194]Terracini (1953a, 259): *Questo libretto sulle superficie cubiche, nonostante il carattere elementare e particolare dell'argomento, a me pare uno dei più belli, forse anche per l'indirizzo proiettivo in cui esso si svolge…Se si dovesse scegliere un libretto per la pubblicazione, penso che questo si segnalerebbe in prima linea*.

[195]ASUT, *Verbale dell'esame di laurea di Alessandro Terracini. Torino, 5 luglio 1911,* Facoltà di Scienze matematiche, fisiche e naturali, Verbali degli esami di laurea, 1902–1921, p. 141.

[196]Stouffer had received his Ph.D. in 1911 from the University of Illinois under the advisement of Wilczynski. See, for example, *The American Mathematical Monthly*, 35 (1928): 331–334 and *The Kansas City Times* (1965, 26 November 1965): 50.

[197]See for example V. Snyder to C. Segre, Ithaca 21 November 1922, UTo-ACS, VII.

[198]See C. Sisam to Olga Michelli Segre, Colorado College 14 July 1924, UTo-ACS, II.

From 1911 to 1913 Segre presented the paper by Sisam entitled "On Algebraic Hyperconical Connexes in Space of *r* Dimensions" (Sisam 1910–11) for publication in the *Atti della R. Accademia delle Scienze di Torino*,[199] and two articles by Terracini on topics related to his doctoral thesis, (Terracini 1912–13, 1913–14),[200] Terracini's successive scientific papers ranged through various areas of mathematics, but the most important group of his works regarded projective differential geometry, a field that he had been led to by Segre. As Enrico Bompiani wrote, one might say that "the mentality and methods of Corrado Segre spread to and dominated the research of Terracini, even when he found other fertile fields of investigation in the lines of research and questions studied by Luigi Bianchi and Guido Fubini".[201] Just one year after his degree, in 1912 Terracini took part in the International Congress of Mathematicians in Cambridge,[202] where he met Enriques, Severi, Castelnuovo and Bompiani. With these last two mathematicians, and in particular with Bompiani, who was his same age, he formed lasting friendships.

Terracini would devote several articles to Segre, and would be one of the editors of Segre's collected works, the *Opere*, together with Severi, Togliatti and Beniamino Segre. As his master Terracini too recorded his university lessons in notebooks, that in many cases show evidence of Segre's influence.[203] Among the others it is worth mentioning the two devoted to a detailed analysis of Segre's work: *L'opera geometrica di Corrado Segre e alcuni suoi ulteriori sviluppi (1934–1935)* (Terracini 1934–1935) and *L'opera geometrica di Corrado Segre (1957–1958)* (Terracini 1957–58), in which Terracini acknowledges his mentor most of all the fertility of the lines of research he opened, rather than of the method, because "those who in Italy have persisted from that time in the study of algebraic geometry (to name only these, Enriques, Castelnuovo and Severi) have had to avail themselves of methods that were different and more hidden".[204]

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[199]See Appendix 5 at the end of this article.

[200]After receiving his degree, Terracini was assistant to Gino Fano and after a parenthesis due to World War I, he thaught at the University of Modena. In 1923–24 he was assigned to teach the course in analytical geometry at the University of Turin. The contract was renewed for the following year, but having won a competition, he transferred to Catania in March 1925. In 1925–26 he transferred back to Turin, to assume the chair in analytical geometry, which he would hold until he was forced to abandon it following the adoption of the racial laws in 1938. He also taught a course in higher geometry. He was reinstated to the chair in 1947–48.

[201]Bompiani (1970, 11): *la mentalità e i metodi di Corrado Segre si propagano e dominano nella ricerca del Terracini anche quando questa trova per essi fertile campo d'indagine negli indirizzi e nei problemi coltivati dal Bianchi a dal Fubini*.

[202]Segre, Castelnuovo and Enriques were members of the International Committee, Enriques gave a plenary lecture and Bompiani a communication on "Recenti progressi nella geometria proiettiva differenziale degli iperspazi".

[203]See Giacardi and Rinaldelli (2000) and *Fondo Terracini. Quaderni* in Giacardi (2013).

[204]*Coloro che in Italia hanno proseguito da quel momento nello studio della geometria algebrica (per nominare solo questi, Enriques, Castelnuovo e Severi) hanno dovuto servirsi di metodi diversi e più riposti*. See Terracini (1934–35, 3).

The line of research in projective differential geometry inaugurated by Segre was also carried forward, in addition to Terracini, by Enrico Bompiani (Rome 1889–Rome 1975), who received his degree under Castelnuovo's supervision in 1910 but quickly came into contact with Segre, beginning a fruitful relationship with the Turin School of geometry, Terracini included.[205] Segre presented three works by Bompiani for publication in the *Atti della R. Accademia delle Scienze di Torino*.[206] The letters Segre sent him show the same care and detailed attention he dedicated to his students: he corrected errors, suggested integrations, invited him "to ponder everything a lot, a lot!"[207] In the lengthy, eleven-page letter dated 16 June 1913, in addition to corrections and detailed comments, he also offered editorial suggestions: choose a title such that it is "always as expressive as possible, compatible with brevity"; explain notations, not to imply hypotheses, not to introduce terms that are ambiguous or without sufficient explanation, "draft the works two, three … times: until everything is clear and beautiful! It won't be time wasted!". He concluded with an observation on style:

> But I must add that what is missing here and there is that artistic sense that illuminates the essential things, casting a shadow on what is less important, etc., etc. And once again I apologise for my sincerity.[208]

In 1931 Bompiani, presenting the "Italian contributions to Modern Mathematics" to the American public, especially emphasised various important results of the Italian School of geometry, citing with regard to Segre, in addition to the "very successful combination of the projective and the Cremonian trend, rich in further consequences", "the study of surfaces whose points have as coordinates solutions of a Laplace's equation and of manifold generated by linear spaces". (Bompiani 1931, 91, 94). Segre's role is also highlighted in his lecture at the first congress of the Italian Mathematical Union in Florence (Bompiani 1938).

In the same years, Ernest Lane in his *Projective Differential Geometry of Curves and Surfaces* (1932) cited him a great number of times, and in the section devoted to the historical remarks he emphasized Segre's opening of a new line of research:

> The distinguished Italian geometer C. Segre (1863–1924) began his geometrical researches at the University of Torino in the early eighties of the nineteenth century. His interest in projective differential geometry is said to have been stimulated by Wilczynski at the Heidelberg Congress of 1904. Beginning with a very significant memoir in 1907 Segre

[205]See Terracini's letters to Bompiani in Paoloni (1991, 98–106). See also Sallent and Ciliberto (2012, Sect. 4).

[206]See Appendix 5 at the end of this article.

[207]C. Segre to E. Bompiani, Turin 7 January 1913, in Paoloni (1991, 67): *ponderare tutto, molto, molto!.*

[208]All the quotations come from the letter of C. Segre to E. Bompiani, Turin 16 June 1913, in (Paoloni 1991, 78–88): […] *sempre più espressivo che può, compatibilmente con la brevità* […] *Rediga i Suoi lavori due, tre ...volte: finché tutto sia chiaro e bello…! Non sarà tempo sprecato!* […] *Ma a ciò devo aggiungere che manca pure qua e là quel senso artistico che fa lumeggiare le cose essenziali, mettere in ombra ciò che ha meno importanza, ecc. ecc. E anche questa volta mi scusi della mia sincerità.*

> made important contributions to the subject. He was not only interested in the geometry of ordinary space, to which his contributions of the *tangents of Segre* and the *cone of Segre* have already been studied in this book, but was a leader in studying the projective differential geometry of hyperspace. Segre gave analytic proofs regularly, but was also an outstanding exponent of the synthetic method, making differential properties even in hyperspace appear intuitive. This method has been used with great skill and success by Bompiani. (Lane 1932, 288)

The same line of research was also followed by Eugenio Togliatti (Orbassano 1890–Genoa 1977), who received his degree a year after Terracini, on 3 July 1912, defending a thesis that Segre had assigned to him, entitled "Contributo alla determinazione delle superficie algebriche del 5° ordine con una o più serie infinite di coniche", with a score of 90/90 and honours.[209] The research carried out in his dissertation was elaborated in several important notes published by the Lincei, the first of which (Togliatti 1912) was published the same year he took his degree. These research projects earned him the Steiner Prize from the Berlin Academy of Sciences for the 5-year period 1909–1914. From 1912 to 1924 Togliatti was assistant, contemporaneously at the University and the Politecnico of Turin , to Gino Fano, Enrico D'Ovidio and Guido Fubini. In 1917 Segre presented two papers of his for publication in the *Atti della R. Accademia delle Scienze di Torino*, (Togliatti 1916–17a, 1916–17b). That same year he received his teaching certification in projective and descriptive geometry and as a non-tenured professor taught that course at the University of Turin from 1919 to 1924. His scientific output was almost completely concerned with algebraic and projective differential geometry and had its roots in the work of Segre. He also shared his method of working, which consisted in considering analytical means as a tool to be used without ever losing sight of the geometry. In homage to his teacher, in 1928 he published, completing it, the research that Segre had outlined before he died, whose aim was to express certain geometric properties of a variety by means of second-order partial differential equations (Togliatti 1927–28, 1374).

Moreover evidence of Segre's influence can be found also in his university courses of this period and in his involvement in the training of teachers which he continued even in later years.[210] In 1926 he was named to the chair of analytical geometry at the University of Genoa.

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[209]ASUT, *Verbale dell'esame di laurea di Eugenio Togliatti. Torino, 3 luglio 1912*, Facoltà di Scienze matematiche, fisiche e naturali, Verbali degli esami di laurea, 1902–1921, p. 151.

[210]In fact, in 1923–24 he taught the course in complementary mathematics, and in 1921–22 he taught an extracurricular course on hyperspatial geometry; in 1922–23 he also taught an extracurricular course on non-Euclidean geometry. The topics addressed in the course "Geometria iperspaziale" are the following: "*1. Interpretazioni varie della geometria proiettiva astratta dello spazio a tre dimensioni. Geometria delle sfere. 2. Lo spazio* S_n *ad n dimensioni. Cenni storici. 3. La geometria metrica in* S_n*. 4. La geometria proiettiva in* S_n*. Corrispondenze proiettive. 5. Quadriche. 6. Curve razionali. 7. La geometria metrica in* V_k *in generale, specialmente* V_k *algebriche. 8. La* V_4^2 *di* S_5 *e la geometria della retta in* S_3*. 9. Geometria differenziale in* S_n*. Spazi a curvatura Riemanniana costante. 10. Determinazione metrica in una varietà a più dimensioni.*" The topics addressed in the course "Geometria non euclidea" are the following: *1. Prime ricerche sull'indipendenza del postulato delle parallele. L'opera di Saccheri, Lambert, Legendre. L'opera*

The last course taught by Segre in 1923–24 was dedicated to the differential geometry. It was the second one he devoted to that subject,[211] but from the comparison of the two related notebooks some differences emerge: in the first the subject was framed in a scheme that is better suited to the needs of someone working in projective differential geometry, as Segre was, while in the second, richer in results and applications, he showed preference to metric considerations. Further, in the notebook of 1923–24 he made use of vector calculus, which in those years was only accepted with difficulty: this is a further evidence of Segre's open-mindedness towards new methods.

Many other young people either graduated under Segre's supervision or came to Turin, or in some other way were stimulated by contacts with him. For most of these Segre presented various papers for publication in the *Atti della R. Accademia delle Scienze di Torino*.[212] Here we will only recall Umberto Perazzo (Nizza Monferrato 1878–Nizza Monferrato 1865) who received his degree in 1899 and from 1899–900 to 1907–08 was assistant to the course of projective and descriptive geometry[213]; Gaetano Scorza (Morano Calabro 1876–Rome 1939), who came to Turin as assistant in projective and descriptive geometry in 1899–900, but attended Segre's lectures on enumerative geometry, conserving an indelible memory of them (Scorza 1932, 134–135); Luigi Berzolari, who came to Turin in 1893 as an associate professor of projective and descriptive geometry; Francesco Palatini (Bassano del Grappa 1865–Castell'Azzara 1940), who came to Turin in 1899 to teach at the Istituto tecnico Sommeiller, a secondary school, but who also wrote works on enumerative geometry; Emilio Artom (Turin 1888–Turin 1952) who graduated in 1909 and later became assistant to Enriques in Bologna[214]; David Cytron (Bialystok Turin 1887–1982) who after his degree in 1910[215] was assistant to Beppo Levi in Cagliari; finally Beniamino Segre (Turin 1903–Frascati 1977), who received his degree under Segre's supervision a year before Segre died, and who then became Severi's assistant in Rome. Beniamino, who was Corrado's cousin,

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(Footnote 210 continued)
di Gauss, Lobacefskj, Bolyai. 2. Costruzione della Geometria non Euclidea secondo l'indirizzo elementare di Lobacefskj e Bolyai: a) teoria generale delle parallele; b) geometria nella stella impropria; c) geometria e trigonometria iperbolica; d) geometria e trigonometria ellittica. 3. Interpretazione della geometria piana non Euclidea sulle superficie a curvatura costante non nulla. Spazî a curvatura costante. Ricerche di Beltrami, Riemann, Helmholtz. 4) L'indirizzo proiettivo di Cayley e Klein: subordinazione delle geometria metrica alla geometria proiettiva; determinazioni metrico-proiettive sulle forme fondamentali di 1^a e 2^a specie e nello spazio.5) La Geometria non Euclidea e la rappresentazione geometrica del mondo fisico. 6) Le metriche non Euclidee negli spazî a più dimensioni. ASUT Corrispondenza, Carteggio classificato, 1921, fasc. III.1 *Programmi dei corsi liberi Corrispondenza, e Carteggio classificato, 1922, fasc. III.1 Programmi dei corsi liberi.*

[211]See BMP–Segre, Quaderni. 29 (1915–16) and Quaderni. 37 (1923–24).

[212]See Appendix 5 at the end of this article.

[213]See ASUT, 10 D 193, p. 155, and *Annuari* of the University of Turin.

[214]ASUT, *Facoltà di scienze MFN. Verbali degli esami di laurea e di Magistero dal 27.10.1902 al 16.11.1925*, p. 114.

[215]*Ibidem,* p. 127.

took his degree on 14 July 1923,[216] with a score of 90/90 with honours with a dissertation on "Genere della curva doppia per la varietà di S_4 che annulla un determinante simmetrico, etc." That same year Corrado Segre still managed to present a part of this thesis for publication in the *Atti della R. Accademia delle Scienze di Torino* (B. Segre 1922–23). Beniamino was able to reconcile the legacy of his teachers with the new approach to algebraic geometry that began to develop in the 1930s.

From this extraordinary weave of research and didactics, of which we have tried to present a detailed picture, emerges the important role played by Segre's university courses in the process of setting up and flourishing of the Italian School of algebraic geometry. As Severi wrote:

> The Teacher and the Scholar, which in him were blended harmoniously, played a most notable part in the daily flourishing development of Italian geometry, as much in algebraic geometry as in projective differential geometry.[217]

Witnessing to this are the research works of numerous students that grew out of his teaching, which in their turn would produce further investigations in a sort of expanded and "collective research", as Fano referred to it.[218] Further testimony is provided by various works that refer to his "little notebooks". Amodeo, for instance, reproduced, an entire portion of the notebook of 1890–91, although without citing it.[219] Enriques used Segre's notebooks to write his *Conferenze di Geometria* (1894–95*)*: "I still have the notes from the lessons of Segre that Fano loaned me; please ask him to forgive me and ask if I may keep them until I have finished preparing the final lithographs of my lectures".[220] Bonola consulted them to prepare his book *La geometria non euclidea* (Bonola 1906, XVII). In the preface to his treatise *Introduzione alla geometria proiettiva degli iperspazi* (1907) Bertini wrote that he had consulted "the extensive handwritten summaries that Segre himself compiled annually for his courses".[221] Severi used them in his *Trattato di Geometria algebrica* (1926), especially in the chapter on geometry of algebraic curves (Severi 1926). Enriques and Oscar Chisini cite Segre's notebooks in their *Lezioni sulla teoria geometrica delle equazioni e delle funzioni algebriche* (Enriques and Chisini 1915–1934, II, 541, III, 154).

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[216]*Ibidem,* p. 214.

[217]*Il Maestro e lo Scienziato che in Lui armonicamente si fondevano hanno avuto parte notevolissima nell'odierno rigoglioso sviluppo della geometria italiana, tanto nel campo algebrico che in quello proiettivo-differenziale.* (Onoranze a Corrado Segre 1930, 57).

[218]See footnote 91.

[219]See C. Segre to G. Castelnuovo, Ancona 20 July 1893. Segre is referring to (Amodeo 1893).

[220]See the letter of F. Enriques to G. Castelnuovo, 30 May 1895, in Bottazzini et al. (1996, 195): *Tengo sempre gli appunti delle lezioni di S[egre] prestatimi dal Fano; ti prego di scusarmi con lui e domandargli se posso ancora tenerli finché non abbia ancora ultimato di redigere le ultime litografie delle mie conferenze*. See also Enriques (1894–95): Segre is cited on pages 32, 37, 45, 87, 104, 105, 109, 129, 130.

[221]Bertini (1907, V): *gli estesi sunti manoscritti che il Segre stesso elabora annualmente per i suoi corsi.*

However, Segre's role as master and mentor also went on another level, that of seminars following, to some extent, Klein's model (Rowe 2004)[222] and of the conversations with his students or collaborators. In fact one of the principal aims of Segre's lessons was to keep his students up-to-date regarding the latest steps made in scientific progress, and to stimulate them to undertake research of their own, suggesting significant problems to be studied, but it was not the only one. In addition to the usual course of 3 hours per week, it was also his custom to dedicate an extra hour to his students, in order to invite them to discuss articles or parts of books of the best authors. The purpose was two-fold: to accustom them to reading and understanding scientific texts on their own, and to train them in effectively communicating what they had studied (Boggio 1927–1928, 317–318). Particular care was taken by him with the dissertations that he assigned in writing, with a long, detailed summary of the status of the question that the undergraduates were expected to deal with. He examined their drafts often as they prepared their theses, providing them each time with a written account of his criticisms and advice regarding possible additions. Equal care and meticulousness was devoted to revising his students' papers for the publication.[223] It was for this reason that he always sought—except in cases of extreme need in the Faculty[224]—to teach the only course of higher geometry. He wrote to Volterra:

> But you already know why […] I want to limit myself to a single course, so that I can dedicate myself to that one course with that intensity, with that zeal that is required in order for advanced teaching to be effective.[225]

Proof of this is found in the recollections of his students, and in the rich card index prepared by Segre,[226] which includes cards with topics for seminars, subjects for research, themes for dissertations, and a very rich bibliography organized alphabetically by topic, with hundreds of titles regarding not only geometry, but also other branches of mathematics as well as secondary teaching.[227]

As his student Fano wrote:

> He poured infinite care and treasures of knowledge into his 36 courses of higher geometry, the topics of which he himself set out in writing, with his clear, neat handwriting, in small booklets well known to his students old and new, always composed with great precision and numerous bibliographical citations, with additional remarks that came to him over time,

[222]See, for example, the footnotes 51 and 139.

[223]See for example footnotes, 207–208 and (Terracini 1968, 13).

[224]See *Corrado Segre. Biographical Timeline* in this present volume.

[225]C. Segre to V. Volterra, Turin 4 November 1897 (ANL-Volterra): *Ma sai già perché … io voglio limitarmi ad un corso solo per potermi dedicare a quell'unico corso con quella intensità, con quello zelo che occorrono negl'insegnamenti superiori perché riescano efficaci.*

[226]This refers to 561 cards conserved in the BMP-Segre, Scritti. 17.

[227]The list of the sections of the card index is available in Appendix 4 at the end of this article.

often containing original ideas and insights, with indications for topics for further research, from which he drew themes to propose for dissertations.[228]

Segre's care in preparing his courses and the attention he also devoted to aspects of didactics constituted a living example for his students, some of whom would adopt the same system of composing their own courses in specially dedicated notebooks. Among these Castelnuovo[229] and Terracini[230] are especially worthy of note. The didactic approach that they adopted drew on some of Segre's pedagogical assumptions, in particular concerning the role of intuition in the processes of learning, the care taken to vary the methods of exposition used, to include examples, references to the history of mathematics, and sometimes even practical advice aimed at future teachers.[231] Thus Fano too, for the course in higher geometry that he taught in 1924–1925, immediately following the death of his teacher, compiled a notebook in which Segre's influence is very evident from many points of view: in his preference for courses on a single topic, in the importance he gives to the historic genesis of mathematical problems, and in his making evident the connections between geometry and analysis (Fano 1924–25b, 1, 2–3).

6 Segre and the Teacher Training Course

As we have seen, problems related to mathematics teaching were very important to Segre. It is also significant that he often reminded his students not to neglect university teaching,[232] and that he took due account of the gifts for teaching as well as those for scientific research during the competitions for professorships. His point of view emerges clearly in the celebrated article of 1891 entitled "Su alcuni indirizzi nelle investigazioni geometriche. Osservazioni dirette ai miei studenti" (Segre 1891a), where he highlights the following qualities a good teacher must have: he should invite their pupils to deal only with important problems and teach them to distinguish the significant questions from the sterile and useless ones; advise them to study, along with theories, their applications; urge them not to be "the slave of a single method"; "broaden as much as possible their own knowledge", so as to be

[228]Fano (1924–25a, 219–225): *Egli profuse cure infinite e tesori di sapere nei suoi 36 corsi di geometria superiore, i cui argomenti venivano da lui stesso esposti per iscritto, colla sua calligrafia chiara e nitida, in libretti ben noti ai suoi allievi antichi e recenti, redatti sempre con gran precisione e con numerose citazioni bibliografiche, con complementi che man mano gli sovvenivano, spesso con idee e vedute originali, coll'indicazione di argomenti di ulteriori ricerche, dai quali traeva i temi da proporre per dissertazioni di laurea.*

[229]See *Lettere e Quaderni dell'Archivio di Guido Castelnuovo*, ANL-Castelnuovo, in Gario (2010).

[230]See BMP-Terracini, in Giacardi (2013).

[231]See for example the notebook of Terracini *Metodologia (1940)*, BMP-Terracini, Quaderni. 19, 1–9 and 49–57.

[232]See for example the letters: C. Segre to G. Castelnuovo, Turin 23 November 1891; Turin 10 February 1892; Ancona 25 October 1896.

able to look at things "from a higher point of view"; take due account of the didactic needs; and suggest to read and study the works of the "great masters" (Segre 1904, 444, 445, 453).

Similar suggestions reappear, suitably adapted for the case of secondary teaching, in the lessons Segre gave at the Scuola di Magistero (Teachers College) of the University of Turin.

In fact, in addition to his university courses in higher geometry, for a full 18 years (from 1887–88 to 1890–91 and from 1907–08 to 1920–21) Segre also taught the course of mathematics at the Scuola di Magistero and in 1916 he was appointed director. Following the suppression of these schools in 1920 by the Minister for Education Benedetto Croce, in 1921–23 Segre taught the course of complementary mathematics (*matematiche complementari*) for the "combined degrees" (*lauree miste*) in physical and mathematical sciences.[233]

Segre's courses at the Scuola di Magistero are documented especially by two notebooks: *Vedute superiori sulla geometria elementare* (1916–17)[234] and [*Appunti relativi alle lezioni tenute per la Scuola di Magistero*].[235] The first of these actually refers to the course in higher geometry taught in 1916–17, but as Segre himself points out in the introduction and in various other places in the notebook, there is a very close connection to the lessons at the teacher training school. The second is instead devoted to questions that are purely methodological and didactical; it is untitled and undated, but shows evidence of later additions made up to 1924. The first who attracted attention to it was Francesco Tricomi in a lecture given in Turin on 22 February 1940, 2 years after racial laws were enacted in Italy. That same year he published Segre's notebook, excluding the long and important bibliography (Tricomi 1838–40). It was a significant homage to a Jewish mathematician. From the very beginning, this notebook shows that Segre attributed as much importance to methodological and pedagogical aspects in teacher training as he did to aspects that were strictly scientific. In fact he makes reference to a regulation enacted by the Minister of Education Vittorio Emanuele Orlando on 6 December 1903, and transcribes articles 1 and 7 underlining the fact that the principal aim of these schools is to make the students expert in the art of teaching the different disciplines. However, immediately after this he quotes article 1 of the Faculty regulations, which refers to the objective of maintaining and developing the scientific culture of the Italian nation.

The two notebooks represent precisely these two aspects of his teaching.

The notebook entitled *Vedute superiori sulla geometria elementare* (1916–17), clearly inspired by Klein, is devoted to the development of various topics of elementary mathematics from an advanced standpoint that Segre believed to be important for teacher training: non-Euclidean geometry; foundations of geometry; elementary geometry and projective geometry; geometrical constructions; linkages;

[233]ASUT, *Fascicolo personale di Corrado Segre*; see also Giacardi (2003), Furinghetti and Giacardi (2012).

[234]BMP-Segre, Quaderni. 30.

[235]BMP-Segre, Quaderni. 40.

problems that can be solved with straightedge and compass; algebraic equations that can be solved by extracting square roots; the cyclotomy problem; and the problem of squaring the circle.

The other notebook [*Appunti relativi alle lezioni tenute per la Scuola di Magistero*], which deals with educational issues and teaching practice, opens with some reflections on the nature of mathematics, the aims of teaching, the importance of intuition and rigour. The topics addressed are the following: mathematics and experience; mathematics in relation to applications; mathematics as an exclusively logical science; the aim of mathematical teaching in secondary schools; intuition and the postulates; rigour; method; exercises; and school reforms. The notebook concludes with a rich bibliography divided up into sections.

Segre's educational approach openly reflects his epistemological vision of mathematics and his own particular way of conceiving scientific research where intuition plays the leading role in the phase of discovering, but also rigour is important in the logical development of a theory. According to Segre there are two ways of addressing mathematics: considering it in relation to applications, as Klein does, or considering it from an exclusively logical point of view, in keeping with the ideas of Peano and Hilbert. After having shown his students the difference between the two approaches by means of well-chosen examples (p. 13), Segre expresses his preference for the first approach, which is characterised by three phases: gathering information derived from experience, putting the data obtained into mathematical form and proceeding to a purely mathematical treatment of the problem, and finally, translating the mathematical results into the form most suitable for the applications. With regards to the second approach, Segre observes:

> Let us say immediately that this second approach is of great importance, philosophically as well. It has made it quite clear what pure mathematics is; and has contributed to making various parts of mathematics more rigorous. But, by detaching itself from reality, there is a risk of ending up with constructions, which, even while logical, are too unnatural, and cannot be of lasting scientific importance.[236]

According to Segre, the aim of mathematics is to accustom students "to reason well; not to be satisfied with empty words; to draw conclusions from the hypothesis, to reflect and discover on one's own; […] to speak precisely" (p. 42). Moreover he was convinced that in secondary schools the first approach to mathematics should be experimental and intuitive, so that the student learns "not only to demonstrate truths already known, but to make discoveries as well, to solve the problems on his own" (p. 16). Consequently, the objective of mathematics teaching should be to develop not only the powers of reasoning, but equally those of intuition.

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[236]*Diciamo subito che questo 2° indirizzo ha una grande importanza, anche filosofica. Esso ha messo bene in evidenza che cosa è la matematica pura; ed ha contribuito molto a porre il rigore in varie parti della matematica. Ma, collo staccarsi dalla realtà, vi è il pericolo di finire con costruzioni, che pur essendo logiche, hanno troppa artificiosità, non possono avere importanza scientifica duratura* (BMP-Segre, Quaderni. 40, 13–14).

About what intuition is, Segre himself explains that it is "perceiving a truth spontaneously, without reasoning and without experiences, but it is the fruit of unconscious reasoning or experiences" (p. 15),[237] based on previous knowledge that unconsciously is chosen and combined in new ways or that suggest analogies; it is freedom of creative imagination and freedom of choice of methods. How he understood the relationship between intuition and rigour in the practice of teaching emerges especially from various considerations on the role of postulates, definitions, proofs and from his comparison of different teaching approaches. According to Segre, the postulates must be intuitive and not necessarily independent. Teachers should avoid very obvious postulates (for example, "the successor to a number is a number"), because "a young person cannot understand the purpose of a series of such statements!" (p. 20). With regard to proofs, he observes that it is not necessary to prove propositions that are intuitively evident and that it can be useful to provide sketches of proofs rather than proofs that are rigorous but long and heavy. Concerning the definitions, Segre affirms that "to define for the young person the things that he already knows with a long discourse, is to bore him" (p. 18).[238] From the analysis of textbooks he gathers significant examples of the different approaches. Because is not possible here to enter into further details, we will limit ourselves to a few general considerations.[239]

Segre's lessons to future teachers were the result of various factors: the knowledge of the reform movements in Germany, France and England, and of the debates within the International Commission on Mathematical Instruction; the knowledge of books and articles concerning mathematical education, both from Italy and abroad, in particular from Germany (Klein, Lietzmann, Simon, Treutlein, etc.) and France (Borel, Hadamard, Laisant, Poincaré, etc.); his keeping himself up to date about the most recent studies in the foundations of mathematics; and finally the circulation of ideas within the School (Castelnuovo, Enriques), but also outside of it, with reference, for example, to Rodolfo Bettazzi and Giovanni Vailati from Peano's School.

The very rich annotated bibliography, which offers future teachers literature concerning didactics in general and didactics of the individual disciplines, textbooks, foundations of mathematics, history of mathematics and mathematical games, is significant in two respects: it shows Segre's wide ranging readings and the care in preparing his lessons, and also his policies as director of the mathematics

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[237][…] *a ragionar bene; a non contentarsi di parole vacue; a trarre conseguenze dalle premesse, a riflettere e scoprire da sé;* [...] *a parlare con precisione.*(*Ibidem,* p. 42). […] *non solo* a dimostrare *le verità già note, ma anche a fare le* scoperte*, a risolvere da sé i* problemi; (*Ibidem,* p. 16). […] *significa lo scorgere una verità spontaneamente, senza ragionamenti e senza esperienze, ma è frutto d'incoscienti ragionamenti od esperienze* (*Ibidem,* p. 15).

[238][…] *non può un ragazzo capire lo scopo di una serie di tali enunciati!* (*Ibidem,* p. 20); *definire al ragazzo con un lungo discorso delle cose che egli crede già di conoscere è annojarlo* (*Ibidem,* p. 18).

[239]For more on this, see Giacardi (2003) and a book in preparation by Alberto Conte and Livia Giacardi, where the text of the notebook [*Appunti relativi alle lezioni tenute per la Scuola di Magistero*] is also reproduced with notes and comments.

library (1907–1924). In fact, Segre often added, alongside the books listed in the bibliography, some personal observations and the books' location in the mathematics library. The books acquired during the long period of his direction provide evidence of his great attention not only to research, but also to teaching.[240]

The whole notebook ultimately shows that Segre shared Klein's methodological tenets. He believed that it was important to bridge the gap between secondary and university teaching by introducing, beginning in secondary schools, the concepts of function and transformation; he favored a 'genetic' teaching method; he was convinced that the first approach to mathematics must therefore be experimental and intuitive: he suggested highlighting some applications of mathematics to other sciences (physics, astronomy, political economy, geography,…) in order to make the subject more interesting and stimulating; he believed in looking at the subject from a historical perspective; finally, he thought that elementary mathematics from an advanced standpoint should play a key role in teacher training.

With regard to the number of students attending Segre's courses at the Scuola di Magistero, we can derive information from the notebook [*Elenco e valutazione degli studenti dal 1883 al 1892*],[241] and from archival documents, which supplement the information reported in the *Annuari della R. Università di Torino*.[242] Further, the registry of those who sat for the mathematics examinations at the Scuola di Magistero, which covers a period from 1907–1922, in addition of providing us with information about who actually passed the examinations,[243] illustrates several other significant aspects: the practical nature of the tests assigned by Segre, which required a lesson to be prepared; the variety of types of schools concerned, which ranged from 'normal schools' for training primary teachers to technical schools and institutes, from *ginnasio* and *liceo* to the *liceo moderno*[244] (beginning in 1914); the range of topics, from arithmetic to Diophantine analysis, from geometry to algebra, from trigonometry to the first concepts of infinitesimal

[240]See for example Giacardi and Roero (1999).

[241]BMP-Segre, Quaderni. 38.

[242]From BMP-Segre, Quaderni. 38 and Quaderni. 39, and from ASUT, *Facoltà di Scienze matematiche, fisiche e naturali, Scuola di Magistero, Registri delle lezioni e relazioni finali*, we obtain the following data: 1887–1888: 9 men; 1888–1889: 12 men, 1 woman; 1889–1890: 12 men, 2 women; 1890–1891: 12 men, 1 woman; 1907–1908: 4 men, 13 women; 1921–1922: 2 men, 11 women; see also Conte et al. (2013, 92–94). From the *Annuari* we derive these data: 1908–1909: 1 man, 5 women; 1909–1910: 5 men, 1 woman; 1910–1911: 2 women; 1911–1912: 2 men; 1912–1913: 1 man, 7 women; 1913–1914: 5 men, 12 women; 1914–1915: 1 man, 1 woman; 1915–1921: no data.

[243]Here are the numbers regarding those who passed the examinations: 1907: 1 woman; 1908: 2 men, 5 women; 1909: 1 man, 5 women; 1910: 5 men, 1 woman; 1911: 2 women; 1912: 2 men; 1913: 1 man, 6 women; 1914: 6 men, 12 women; 1915: 1 man, 2 women; 1916: 13 women; 1917: 1 man, 4 women; 1918: 2 women; 1919: 11 men, 8 women; 1920: 5 men, 9 women; 1921: 1 man, 4 women; 1922: 1 woman, in ASUT, *Facoltà di Scienze MFN. Verbali degli esami di laurea e di Magistero dal 27.10.1902 al 16 .11.1925*, pp. 19–131.

[244]*Ginnasio* and *liceo* were lower and upper secondary schools in which humanities played a key role; *liceo moderno* was an upper secondary school in which Greek was replaced with a modern language (German or English) and more attention was dedicated to the scientific subjects.

analysis. Moreover among those who passed the examination we find various followers of Peano, such as Margherita Peyroleri, Maria Gramegna, Vincenzo Mago, Paolina Quarra, and Ugo Cassina, and various students of Segre's course of advanced geometry such as Emilio Artom, Davide Cytron, Eugenio Togliatti, Vittorina and Annetta Segre, and others with whom he remained in contact.[245]

Although Segre never published anything specifically regarding mathematics education, his lessons at the Scuola di Magistero constitute a significant contribution, because as Fano observes: "he gave […] to secondary schools many good teachers".[246] In fact, for 18 years he trained the mathematics teachers who came out of the University of Turin, contributed to the spread of Klein's vision of mathematics teaching among his students, and to the diffusion of a good knowledge of the literature related to education and of the most significant textbooks. Above all, by his very example he transmitted a certain way of teaching which encouraged creativity and the use of more than one method, and established connections between different sectors of mathematics, in a unitary vision.

7 Conclusion: Recognition of Segre's Role as Leader of a School and the Main Features of His Leadership

What shines forth clearly from the interweaving of teaching and research that we have described is Segre's role as leader of a School,[247] a role that was acknowledged by the Italian mathematical community as early as 1898. This is confirmed in the statement of the award of the Royal Prize for Mathematics that year:

> Segre's scientific work is of the most admirable kind. He has left traces of his strong mind and his great and uninterrupted efforts in a vast range of fields in part as yet unexplored, [...] Nor should another, principal merit of Segre's be overlooked: that of having given the start to the scientific line of Italian research in geometry on a curve and on a surface, to which he himself has contributed effectively.[248]

That Segre himself (see Sects. 1 and 2) and his students saw themselves as belonging to a School is evident even before that date, in the early 1890s, as can be seen by the frequent references to Segre in the articles published by his students, as

[245]See Segre's [Indirizzario] in UTo-ACS, II. At the moment of preparing this volume for print, Erika Luciano has just begun to study this community of teachers who were influenced by Segre's educational thinking.

[246]Fano n. d. [Appunti vari], f. 63v.

[247]See for example Castelnuovo (1929), B. Segre (1933), Menghini (1986), Brigaglia and Ciliberto (1995), Giacardi (2001a), Brigaglia (2001) Brigaglia and Ciliberto (2004), Conte and Ciliberto (2004).

[248]*L'opera scientifica del Segre è delle più ammirevoli. Egli ha lasciato tracce del suo forte ingegno e della sua grande e continua operosità in vasti campi, in parte ancora inesplorati.* […] *Né è da tacersi un altro e principale merito del Segre: di avere, cioè, avviato il presente indirizzo italiano degli studi di Geometria sopra una curva ed una superficie, contribuendovi egli stesso efficacemente* (Relazione sul concorso al Premio Reale per la Matematica, pel 1895, 1901, 367).

well as and above all in the correspondence, some of which is still unpublished. Ettore Caporali (1855–1886), professor of higher geometry in Naples since 1878, addressed Segre, then newly graduated, as a peer.[249] Riccardo De Paolis (1854–1892) was another who greatly esteemed and accepted his advice and suggestions.[250] Castelnuovo recalled that when Segre was still very young, he had taken over the leading role in the School by unanimous consensus:

> He was truly a Teacher in the highest, most noble sense of the word […] In the first years of his career, when he had no other cares outside of science and teaching, he kept up an extremely vast correspondence and stayed up to date with everything that was produced in his field of research both in Italy and abroad; he suggested problems, indicated methods, pointed out errors, always impartial in both his praise and his criticism. At that time, Segre, quite young, had assumed, by unanimous consensus, the role of directing the Italian school of geometry, succeeding Cremona.[251]

Fano[252] and Enriques[253] made similar statements as did Severi, who wrote:

> The Cremonian tradition […] had degenerated almost everywhere, in many of the imitators lacking in creative genius, in tick-tock geometry, to use Enrico D'Ovidio's the picturesque witticism […] In constituting the desired School […] the most important thing of all was precisely the powerful work of Corrado Segre.[254]

Analogous sentiments were expressed by Berzolari, Scorza, Beniamino Segre,[255] and others. Terracini, in his memoirs, declared: "Corrado Segre had been and was a most excellent geometer; one of the uncontested founders of the so-called

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[249]See the letters of Caporali to Segre 1884–1885, UTo-ACS, VII.

[250]See Gario (1989a, 195, 196) e R. De Paolis a C. Segre, Pisa, 4 March 1887, Pisa 11 January 1888, Pisa 8 February 1892, UTo-ACS, VII and http://users.mat.unimi.it/users/gario/Segre-Ancona/lettereRicevute.pdf.

[251]*Maestro egli fu veramente nel più alto, nel più nobile senso della parola* […] *Nei primi anni della sua carriera, quando non aveva altre cure fuori della scienza e dell'insegnamento, egli teneva una corrispondenza estesissima e seguiva tutto ciò che in Italia e all'estero si produceva in campi affini al suo; suggeriva problemi, indicava metodi, segnalava errori, equanime sempre negli elogi e nelle critiche. In quell'epoca il Segre, giovanissimo, aveva assunto per unanime consenso, funzioni direttive nella scuola geometrica italiana, succedendo al Cremona*. (Castelnuovo 1924b, 358).

[252]"It is here that C. Segre enters the field, destined to acquire positions of eminence, even above all as maestro and head of a school" (*È qui che entra in campo C. Segre, destinato acquistare posiz[e] eminente, anche soprattutto come maestro e caposcuola*); (Fano 1930, 44). See also Sect. 1, footnotes 14, 21, 91; (Fano 1924–1925a, 220, 225; Fano 1924–25b, 75; Fano n. d. [Appunti vari], f. 1v).

[253]See, for example, Enriques (1920, 3–4, 10). In their survey on the history of geometry from the antiquity to the first decades of the twentieth century, Enriques and Fano mention Segre's School of geometry four times (Enriques and Fano 1932).

[254]*La tradizione cremoniana* […] *era quasi dovunque degenerata, presso parecchi degli epigoni deficienti di genio creativo, in tic tac geometria, secondo il pittoresco mot d'esprit di Enrico D'Ovidio* […] *A costituire la desiderata scuola* […] *valse appunto in primissima linea l'opera possente di Corrado Segre* (Severi 1957, VI).

[255]See for example Berzolari (1924, 532), Scorza (1932, 130–131), B. Segre (1961), B. Segre (1964, 18–19).

Italian School of geometry".[256] All of them underlined the precociousness, the opening new lines of research, the re-founding of the school of geometry in Italy, the care and enthusiam in addressing his students, the effectiveness of his courses, the 'malia'[257] (enchantment) of his lessons, his moral rectitude and generosity. This unanimous and shared picture of Segre as a master and a leader does not have its origin only in the recognition and affection shown by students and followers who often tend towards representing an idealised image of their teacher, but, as we have attempted to show, is justified by the results that Segre's teaching produced. They are matters of fact, as are the fervid activity and enthusiasm that animated those within his entourage at the end of the 1800s. Actually Castelnuovo and Terracini[258] had no hesitation in pointing out that Segre's merit was above all that of having inaugurated a geometric method of working, and of having opened many new lines of research, although he himself was not always concerned in developing them to their very end. However, they also acknowledged that "while imposing limits on his work, in any case extremely extensive, he had immensely fostered the school named for him".[259] Vito Volterra and Tommaso Boggio as well, who were colleagues of Segre's, underlined his significant role in renewing and developing Italian geometric research.[260]

Further, Segre's role as a leader and the existence of a group of Italian geometers with a common project was affirmed not only in Italy but also abroad. One of the first proofs of this is the change in the nature of his correspondence with Klein at the end of the eighties (Luciano and Roero 2012, 25): Segre no longer appears as a young eager to learn, but as a scholar who is carrying on a precise research project with the confidence of a leader. Additional evidence of this new role is the large number of young people from abroad who came to Turin to attend his university courses, or wrote to him to seek his advice.[261] Coolidge, for example, in his account of his mathematical studies in Italy (Coolidge 1904, 12) wrote that for students, who wished to pursue their post-graduate studies abroad, Italian universities offered well-stocked libraries, graduate-level courses, and the personal advisement of a professor. He cited in particular the University of Turin, where in that year he attended the course of Segre, who would become his teacher and friend.[262] Even before, in 1894, as we have said, Brill and Noether, in their *Bericht*, mentioned the

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[256]*Corrado Segre era stato ed era un grandissimo geometra; uno degli incontrastati fondatori della cosiddetta scuola geometrica italiana* (Terracini 1968, 14). He also wrote: *Con Corrado Segre la geometria algebrica trovò in Italia un grande maestro e un nuovo caposcuola* (Italian algebraic geometry had in Corrado Segre a great Maestro and a new leader) (Terracini 1961, p. 11). See also Terracini (1926, 244).

[257]See B. Segre (1964, 18).

[258]Terracini (1934–1935, 1–3).

[259]*Ma se ha posto dei limiti alla sua opera, del resto vastissima, ha immensamente favorito l'attività della scuola che da lui prende il nome* (Castelnuovo 1924b, 358).

[260]See Volterra (1909, 64), Boggio (1927-28, 318–319).

[261]See for example (Berzolari 1924, 532), Fano (1824–25a, 225), Boggio (1927–28, 319), Fuà Segre (1952, 125), Terracini (1968, 13).

[262]See J. Coolidge to O. Michelli Segre, Cambridge 20 September 1924, UTo-ACS, II.

leading position of Italy in geometrical research, a citation that Segre hastened to point out to Castelnuovo (see Sect. 3). At the end of the 1890s, the Italian geometers were asked to write important articles about geometry for the *Encyklopädie der mathematischen Wissenshaften* and at the beginning of the century eight had been completed.[263] In 1923, Franz Meyer and Hans Mohrmann, in the introduction to the volume of that encyclopedia, which presents the state of the art of international scientific research in the field of geometry, emphasized the fact that in the space of a few years at the end of the century Italy had attained a "leading position" (*führende Stellung*) (III.I$_1$, p. VI). Henry Frederick Baker devoted his 1912 (Baker 1913) London Mathematical Society presidential address to the work on the theory of algebraic surfaces of the Italian School of geometry and that same year at the International Congress of Mathematicians in Cambridge Baker presided over the first meeting of the geometry section. His brief address is mainly dedicated to illustrate the "very remarquable" results of Enriques, Castelnuovo and Severi,[264] and in 1926 he wrote, "He [Segre] may probably be said to be the father of that wonderful Italian school which has achieved so much in the birational theory of algebraical loci" (Baker 1926, 269). In his recollection of Segre, Coolidge counted him among the greatest architects of the "geometric *Risorgimento* in Italy" (Coolidge 1927, 352). Further evidence of Segre's reputation can be found in the fact that numerous foreign mathematicians came to Turin to meet him: Friedrich Schur, Ferdinand Lindemann, Eduard Study, Hieronymus Zeuthen, Klein, Snyder, Cyparissos Stephanos, to name only the most renowned.[265]

The analysis that we have presented makes it possible to identify the reasons why Segre was successful in creating a School that was recognised on an international level, as well as to clarify the appropriateness of the word School to describe the group of geometers grown up around him. The characteristics of Segre's leadership are crucial to answer the aforementioned questions. They can be synthesised in the following points:

- First of all, he proposed an innovative scientific project, well-defined and restricted to the field of geometry, and helped important lines of research to emerge and become established: higher-dimensional projective geometry; foundations of higher-dimensional projective geometry; birational algebraic geometry; enumerative geometry; projective differential geometry; and projective geometry of complex domain. Moreover, Segre devised a style with canons for method as well as for aesthetics: a geometric way of reasoning that was elegant yet simple and clear and a treatment of the subject inspired by "that artistic sense that

[263]See footnote 92.

[264]Proceedings of the Fifth International Congress of Mathematicians (1913, 49–50).

[265]See C. Segre to F. Klein, 16 April 1884; C. Segre to D. Montesano, Turin 17 May 1888 (BMFI-Montesano); C. Segre to V. Volterra, 27 October 1897 (ANL-Volterra); C. Segre to O. Michelli Segre, Heidelberg 10 August 1904 (UTo-ACS, II); C. Segre to O. Michelli Segre, Rome 27 September 1906 (UTo-ACS, II); E. Study to O. Michelli Segre, Umhausen Oetzthal 6 August 1924 (UTo-ACS, II); C. Segre to V. Volterra, n. d. (ANL-Volterra).

illuminates the essential things, and leaves what is less important in the shadows".[266] It was, therefore, a style that involves a particular heuristic process, a particular organisation of the research, a particular form of exposition, and particular methods that originated in the charismatic figure of Segre and became affirmed as it cascaded through his circle of students. Terracini wrote:

> Segre's method (if it can be called that, given that it consisted in not following an established analytical method) had advantages of its own: above all that of constantly leading to results that were geometrically interesting without running the risk that the algorithm might prevail over the aim to which it was dedicated.[267]

Beniamino Segre added:

> The method constantly followed by Corrado Segre in his research rested on an extremely skilful, elegant and evocative weave of synthetic considerations and algebraic developments, the latter being reduced to a minimum and used so as to reveal fully the geometric content of the results, sometimes even of single steps, and to provide suitable checks at the most delicate points.[268]

This capacity for intuition permitted Segre to render geometric objects almost tangible, as he himself had explained to Klein:

> What you tell me about the effects that synthetic reasoning about geometry of *n* dimensions have on you, does not surprise me; it is only by living in S_n, and always thinking about it, that we become familiar with these arguments.[269]

This attitude of the Italian geometers towards the geometric objects that they studied would be also highlighted by Oscar Zariski's review of Beniamino Segre's treatise *The Non-singular Cubic Surfaces* (1942) (Zariski 1943).

The characteristics of Segre's method are thus clarity, rigour and a remarkable geometric aptitude, in which a strong synthetic intuition is joined to an abstract conception of geometry, a union that allowed him, as Enriques wrote, to

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[266]C. Segre to E. Bompiani, Turin 16 June 1913, quoted in (Paoloni 1991, 78–88): *quel senso artistico che fa lumeggiare le cose essenziali, mettere in ombra ciò che ha meno importanza*. All the cited articles and books concerning Segre and the Italian School of algebraic geometry highlight the specificity of the Italian style, but already earlier Eugenio Beltrami had underlined the "Italianness" of a certain manner of exposition of the subject: see E. Beltrami to E. Cesaro, Rome 22 January 1894, in Palladino and Tazzioli (1996, 350). On the questions of style in mathematics see Mancosu (2009), Cogliati (2015).

[267]*Il metodo di Segre (se così lo si può chiamare, dato che esso consisteva nel non seguire un metodo analitico determinato) ha i suoi propri vantaggi: soprattutto quello di condurre costantemente a risultati geometricamente interessanti senza correre il pericolo che l'algoritmo possa prevalere sullo scopo al quale esso è dedicato* (Terracini 1958, VI–VII).

[268]*Il metodo costantemente seguito da Corrado Segre nelle Sue ricerche poggia su di un abilissimo, elegante e suggestivo intreccio di considerazioni sintetiche e di sviluppi algebrici, questi ultimi essendo ristretti al minimo e condotti in guisa da rilevare appieno il contenuto geometrico dei risultati, alle volte perfino dei singoli passaggi, e da fornire opportuni controlli nei punti più delicati* (B. Segre 1961, VIII–IX).

[269]*Ce que Vous me dites sur l'effet que Vous font les raisonnements synthétiques de géométrie à n dimens. ne me surprend pas; c'est seulement en vivant dans* S_n, *en y pensant toujours, qu'on devient familier avec ces raisonnements*. See C. Segre to F. Klein, Turin 11 May 1887.

contemplate many different transfigurations of the same figure “with a thousand spiritual eyes”.[270]

- He achieved a perfect symbiosis between teaching, his own personal research and introducing young people to research. He was acute, far-sighted, demanding, but impartial and generous in directing the research of his students, trying to correct defects and encouraging them not to be slave to a single method in dealing with a problem,[271] and most of all, suggesting lines of research according to their own aptitudes.[272] In this way Segre encouraged the free circulation of ideas within the School itself, and even outside it, without narrow-mindedness. Segre was able to recognise his own limits and allow his students to take flight.
- Segre was a very demanding teacher, and rewarded excellence: “better one result fit to live than a thousand doomed to die at birth!”. (Segre 1904, 444). As he wrote, “to be demanding is a general principle, which I apply to myself, and which derives from the higher motives related to the seriousness of science and teaching”.[273] Above all, Segre taught his students that the goal of scientific research is that of “up-building of the great edifice” of mathematics, and that the simple exercises and useless generalisations (Segre 1904, 465) produce a “real encumbrance in the science and an embarrassment for more serious investigators” (Segre 1904, 443).
- Another significant aspect of his role of *Teacher* is the care he took to assure academic positions for his students,[274] and to give the Italian tradition, and his School, national and international scientific visibility. As we have seen, Segre took various routes to achieve this goal: personal contacts, scientific correspondence with a range of scholars, periods of stay abroad, participation in international congresses, translations and publishing activities. Concerning this last aspect, it deserves to be mentioned that in 1884–1885 he wrote 35 reviews of Italian papers, “which up to then had been neglected in that journal or at least treated with scant distributive justice”,[275] for the *Jahrbuch über die Fortschritte*

[270]See Sect. 1 and footnote 9. On the role of geometric intuition in the Italian School of algebraic geometry see, for example, Severi (1950), Gray (1994), Mumford (2011), Schappacher (2015), Rogora (2015).

[271]See, for example, C. Segre to F. Klein, Turin 28 November 1889; Turin 17 October 1890, (Segre 1891a, 52; Segre 1904, 453).

[272]See for example C. Segre to F. Klein, Turin 4 October 1893; C. Segre to G. Castelnuovo, Turin 20 July 1891.

[273]*La severità è un principio generale, che uso anche contro di me stesso, e che deriva da ragioni elevate relative alla serietà della scienza e dell'insegnamento* (C. Segre to F. Amodeo, Turin 4 November 1891, in F. and N. Palladino (2006, 181).

[274]Particular evidence of this fact emerges from his letters to Volterra and to Castelnuovo. ANL-Castelnuovo and ANL-Volterra.

[275][…] *lavori che fin ora erano in quel periodico molto sacrificati o almeno trattati con poca giustizia distributiva*: E. Caporali to C. Segre, 13 September 1885, in UTo-ACS, VII and http://users.mat.unimi.it/users/gario/Segre-Ancona/lettereRicevute.pdf.

der Mathematik; from 1889 to 1924 he presented or refereed no fewer than 189 works to be published in *Atti* or *Memorie della R. Accademia delle Scienze di Torino*, in particular he saw to the immediate publication of original results of his pupils' dissertations.[276] For 20 years, from 1904 to 1924 Segre was one of the editors of one of the most important scientific journals of the day, the *Annali di Matematica pura ed applicata*, and starting in 1888 he was a member of the editorial board of the *Rendiconti del Circolo Matematico di Palermo*: he himself and many of his students published important works in both of these journals.[277] The *Atti* and the *Memorie della R. Accademia delle Scienze di Torin*o often served as a launch pad for the early research of his students. This multifaceted activity shows a perfect balance between a feeling of national identity and internationalism. In 1893 he wrote to Castelnuovo, inviting him to publish his results in the *Annali di Matematica pura ed applicata* so that his most important research works would be connected to an Italian journal:

> Why when your discovery must be cited, should it be necessary to name a foreign periodical? Foreigners should accustom themselves to reading *our* collections.[278]

But he was also a strong supporter of the importance of scientific internationalism, both in the sense of making Italian work known abroad, and in the sense of scientific cooperation beyond any political barriers. Symptomatic of this is the letter he addressed in 1916 to the rector of the University of Turin to avoid the block at customs of scientific publications coming from Germany (Conte et al. 2013, 45–46), as well as that he wrote in 1922 to Salvatore Pincherle expressing his opposition to the exclusion of the mathematicians of the countries of the former Central Powers from the International Mathematical Union (see Nastasi and Tazzioli (2013, 387–390) and in this present volume *Corrado Segre. Biographical Timeline*).

- Finally, as we have seen, Segre attributed great importance to the quality of university teaching, taking particular care with the advanced courses, without, however, overlooking compulsory university courses and the teacher training schools. Numerous and significant considerations on this are found in his letters to his students. For example, he wrote to Castelnuovo:

> I fully agree with you on the main criterion of teachers: that of making themselves understood by their listeners. With the passing years I am increasingly persuaded, and this

[276]See Appendix 5 in this present article and the section *Relazioni* in Giacardi (2013). In particular, he presented 16 papers by Fano, 13 by Severi, 5 by Giambelli, 10 by Beppo Levi, 5 by Giambelli, 4 by Tanturri, 7 by Terracini, 3 by Togliatti, as well as those of other members of the School and those of visiting foreigners, 3 by the Youngs, 2 by Coolidge and 1 by Sisam.

[277]About 90 articles by his Italian students appeared in the journals of the Academy of Lincei, more than 50 in the *Rendiconti del Circolo Matematico di Palermo*, 25 in the *Annali* and not counting those that were published in other Italian and foreign journals.

[278][…] *perché quando si dovrà citare la tua scoperta si dovrebbe nominare un periodico estero? Gli stranieri si abituino a leggere le* nostre *raccolte.* C. Segre to G. Castelnuovo, Turin 26 September 1893.

> criterion is my guide […] And, since these are pupils from the engineering school you are also right not to make them study many things that are not absolutely necessary. If one day I were to go back to teaching in the first two years, I would very much restrict my *mandatory* program; but (here is the difference with what Peano does) would also have various complementary, optional lessons, especially for pure mathematics students.[279]

And again:

> I am very happy about what you write me regarding the impression that you have made on your students: this is a great victory for you: to show that your teaching skills are not inferior to your intellect.[280]

It is significant that Segre took due account of the gifts for teaching, as well as those for scientific research, of the candidates in the competitions for professorships.[281]

Segre's notebooks offer a vivid example of his own teaching skills. There, as we have attempted to show, he also recorded the modifications that he made to the lessons during the course of the year, the additions, the refinement of certain proofs, the changes in the way certain subjects were presented, ideas for new exercises or research, questions to pose to the students, additions of bibliographic entries with the most recent articles and books, and even advice to himself about how to best explain a topic.[282]

In addition to these factors that favoured the flourishing of the Italian School of algebraic geometry, others were particularly important in affirming the School on the national and international levels: the unitary character of the research, all concentrated in the field of geometry, although in different areas; the outstanding quality of Segre's students, the awards they won and the important academic and institutional positions that they held; the participation in international publishing enterprises such as the *Encyklopädie der Mathematischen Wissenschaften*; the collaboration with foreign journals such as the *Mathematische Annalen*, and the large network of scientific relationships. Last but not least, in the period we considered, there was, according to us, a broad way of understanding the School: as a group of researchers who from a shared *teacher* gleaned topics to investigate, methodologies, approaches to research, and a particular scientific style, but who were also capable of blazing new trails independently; as a place where talents were developed, contacts were

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[279]*Sono pienamente del tuo avviso sul criterio principale dell'insegnante: quello di farsi capire dagli uditori. Col passare degli anni me ne son sempre più persuaso: e su questo criterio mi guido* […] *E trattandosi poi di allievi-ingegneri hai anche ragione a non volerli obbligare a studiare tante cose che non sono di prima necessità. Se un giorno io ritornassi ad insegnare nel 1° biennio limiterei di molto il mio programma* obbligatorio*; ma (e qui sta la differenza con quanto fa Peano) farei pure varie lezioni complementari, facoltative, specialmente per gli studenti di matem. pure*. (C. Segre to G. Castelnuovo, Turin, 10 February 1892).

[280]*Sono molto contento di quanto mi scrivi sull'impressione che hai fatto nei tuoi studenti: quella è la tua gran vittoria* […]: *mostrare che le tue attitudini didattiche non sono inferiori all'ingegno!*. (C. Segre to G. Castelnuovo, Turin 23 November 1891). See also, for example C. Segre to G. Castelnuovo, Turin 12 November 1902 and Turin 24 December 1904.

[281]See for example C. Segre to G. Castelnuovo, Ancona 25 October 1896.

[282]See for example BMP-Segre, Quaderni. 6, p. 112; Quaderni. 17, p 105; Quaderni. 34, p. 55.

made, and a shared vision of the transmission of knowledge matured; as a group that had a national origin and identity, but was open and ready to engage in a dialogue with the international community, a School, as Enriques would write, "that tends to spread itself beyond the milieu where it originated".[283]

Acknowledgments We are very grateful to Daniele and Silvano Fuà, for allowing us to access to the unpublished documents in their possession and to the directors and personnel of the various archives we explored, Paola Novaria and Giuliana Maria Borghino Sinleber (Archivio Storico, University of Turin), Manuel Onjugaren (Archives, Losanna), Angharad Gwilym, Kate Hawke, Edd Mustill, (Special Collections and Archives, University of Liverpool), Elena Borgi and Lavinia Iazzetti (Accademia delle Scienze, Turin), Anna Dagnese, Laura Garbolino, Orietta Piccini, Giulia Scarcia, Giuseppe Semeraro, Antonella Taragna (Biblioteca Matematica "Giuseppe Peano", University of Turin). A special thanks to Steven and Beverly Kleiman for reading our text. Our most wholehearted thanks also go to all those who in various ways, have provided their help, Aldo Brigaglia, Ciro Ciliberto, Paola Gario, Judith Goodstein, Thomas Hawkins, Elena Anne Marchisotto, David Rowe, Antonio Salmeri and Norbert Schappacher.

Appendices

Appendix 1. Unpublished Letters from Segre to Grace Chisholm Young and William H. Young

AUL—*Papers of Professor William Henry Young and Grace Chisholm Young*

C. Segre to Grace Chisholm Young, 11 March [1899]
C. Segre to G. and William Young, 30 November 1899
C. Segre to Grace Chisholm Young, 10 October 1899
C. Segre to Grace Chisholm Young, Turin 23 December 1899
Segre's letter of introduction for William Young, Turin 20 January 1908

C. Segre to Grace Chisholm Young, [Turin], 11 March [1899][284]
Autograph card, n.y.

11 Marzo

Gentil.ma Signora,

Ella ha ben ragione che quella dimostrazione per p qualunque si può fare come per $p = 2$.

Anzi, anche quella che avevo esposto in lezione per $p = 2$[285] modificherei ora leggermente così (generalizzando):

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[283][…]*che tende ad allargarsi fuori dal proprio ambiente d'origine*- (Enriques 1938, 181). On the process of internationalisation of the mathematical community in the period 1800–1945 see Parshall and Rice (2002).

[284]The letter is contained in the Youngs' Notebook 3 (D.140/19/5), p. 225. See the following Appendix 3.

[285]See BMP-Segre, Quaderni. 12.

Si prendano $p + 2$ punti generici $A_1A_2 \ldots A_pBC$; e si consideri una g'_{p+1} contenente il gruppo $A_1 \ldots A_pB$ ed un'altra g'_{p+1} contenente il gruppo $A_1 \ldots A_pC$. Non accade allora che due gruppi di queste due serie aventi un punto in comune abbian *sempre* di conseguenza in comune un altro punto. In fatti, se ciò accadesse quel gruppo della 2ª serie che contiene B dovrebbe contenere anche uno dei punti A. Ma allora esso coinciderebbe col suddetto gruppo $A_1 \ldots A_pC$: il che è assurdo, perché B non è tra questi punti.

Se Lei e Suo marito desiderano che uno di questi giorni io venga a parlar Loro su qualche argomento geometrico di Loro interesse, favorisca scrivermi, indicandomi gli argomenti.

Tanti cordiali saluti

Suo C. Segre

C. Segre to Grace Chisholm Young e William Young, Turin 30 November 1899[286]

Autograph postcard (D.140/9/179)

Torino 30 nov. 99

Cari Signori,

Con molto piacere ricevo ora gli estratti delle Loro Note torinesi[287] e altri due lavori della sig.ª Young, alla quale sono particolarmente grato per essersi ricordata d'inviarmi la sua dissertazione di laurea. A tutti e due i miei cordiali ringraziamenti.

Scriverei Loro un po' più a lungo se fossi ben sicuro del loro indirizzo. Inviai Loro una cartolina a Gottinga un mese fa: l'hanno ricevuta? Avrei molto piacere che mi dessero notizie di Gottinga, dei loro studi, dei corsi che seguono, dei professori che vedono (del Prof. Klein specialmente). E Franceschino[288] continua a crescere bene?

Noi stiamo bene. Adriana[289] si fa sempre più vivace ed intelligente.

All'Università quest'anno tratto (come forse già scrissi Loro) la geometria numerativa.[290] Vi sono da 15 a 20 studenti, fra cui 3 signore. Vi sono anche 5 dottori. Spero che alla fine dell'anno mi troverò contento del profitto di questo uditorio.

Li saluto cordialmente; e Li prego di fare i miei saluti al prof. Klein

Loro aff^mo C. Segre

C. Segre to Grace Chisholm Young, Turin, 10 December 1899[291]

Autograph letter (D.140/9/180)

Torino 10 Dic. 99

Gentilissima Signora,

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[286]This postcard is addressed to "Herren W.H. Young und Frau, 11 Bergstrasse (oder Mathem. Seminar der Universität, Göttingen (Germania)".

[287]See Young (1898–99), Chisholm Young (1898–99).

[288]Franceschino is Frank one of the Youngs' sons, born in June 1897.

[289]Adriana is Segre's second daughter, born on 28 October 1897.

[290]See BMP-Segre, Quaderni. 13.

[291]On the first page, William wrote, "Keep this or send it back".

La Sua cartolina del 30 nov.[e] si è incrociata con una mia che Le ho inviato in quegli stessi giorni.

Ho avuto piacere delle Loro buone notizie. E son contento di sapere che tanto Lei quanto Suo marito si stanno occupando di ricerche matematiche. Le sarò grato se mi terrà informato di queste Loro ricerche: e se per esse io potrò servirli in qualche modo, tanto meglio!

Fano mi scrisse che si trova bene a Messina: vi è un sufficiente numero di studenti, che lavorano abbastanza. Tanturri ha spedito la sua tesi di // laurea agli Annali di matematica;[292] e continua a lavorare. Bertini[293] mi scrive che è soddisfattissimo di Tanturri come assistente. La signora Bertini ha sofferto molto in questi ultimi tempi (mal di fegato), ora sta meglio. Enriques lavora per *tre* libri! Castelnuovo per l'articolo dell'Enciclopedia. Qui a Torino l'insegnamento della Geometria projettiva e descrittiva, che prima era fatto dal Berzolari,[294] ora si fa dal Pieri, fino a che la Facoltà decida di fare il concorso per quella cattedra. Dei miei allievi dell'anno scorso da Lei conosciuti, la signorina Viriglio[295] ha dovuto sospendere di studiare, perché soffriva il // suo sistema nervoso; Severi (quel giovane alto che mi domandava sempre spiegazioni) lavora per la tesi, su certi caratteri delle curve algebriche iperspaziali, e mi pare che finora faccia bene.

Al mio corso di quest'anno assistono molti giovani. Nella geometria numerativa incontrerò certo delle nuove questioni importanti da risolvere. Per esempio pare che non sia ancora risolta la seguente, che pure si presenta molto naturale. Nello spazio S_{mn-1} quale è l'ordine della varietà che si rappresenta annullando tutti i determinanti d'ordine i estratti dalla matrice

$$\left|\begin{matrix} x_1 & x_{2\ldots} & x_n \\ x_{n+1} & x_{n+2\ldots} & x_{2n} \\ x_{2n+1} & x_{2n+2\ldots} & x_{3n} \\ \ldots & \ldots & \ldots \\ x_{(m-1)n+1} & x_{(m-1)n+2} & x_{mn} \end{matrix}\right|$$

// Ciò si sa solo per $i = 2$ e per $i = m$ (se $m \leq n$); ma non si conosce l'ordine per i qualunque.

Per oggi mi fermo; ma mi riservo di riscriverle quanto Ella mi avrà comunicato qualcosa delle ricerche che Lor due stan facendo. Non manchi d'informarmi anche dei corsi a cui assistono.

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[292]See Tanturri (1900).

[293]Eugenio Bertini (1846–1933) taught at the University of Pisa from 1892. Thanks to Segre's interest Tanturri became his assistant in 1899–900.

[294]Luigi Berzolari (1863–1949) taught projective geometry and descriptive geometry with drawing at the University of Turin from 1893–1894 to 1898–1899.

[295]Luigia Viriglio (1879–1955) received her degree under Segre's advisement on 9 December 1904.

Tanti cordiali saluti a Loro, anche da parte della mia Signora: Anche Elena[296] vuole essere ricordata a Franceschino, ma Adriana non lo ricorda più!

Suo dev.$^{\text{mo}}$ C. Segre

C. Segre to Grace Chisholm Young, Turin, 23 December 1899
Autograph letter (D.140/9/181)

Torino 23 Dic.$^{\text{e}}$ 99

Gentilissima Signora,
mi scusi del ritardo a risponderle. Son stato molto occupato nei giorni scorsi dalla preparazione delle lezioni; ma ora sono in vacanza, e posso finalmente scriverle.

La Sua lettera mi ha interessato. Ho visto quali sono le difficoltà che Ella ha incontrato; e non ho nessun dubbio che Ella riescirà a superarle. Se io non mi sono accinto io stesso a risolverle è perché: (1) credo meglio lasciare a Lei questa soddisfazione; (2) avrei dovuto tardare ancora qualche giorno a risponderle ed io desidero inviarle oggi stesso i miei auguri per Natale e Capo d'anno! La ricerca infatti esige anzi // tutto un esame accurato della Sua trasformazione (1, 4) di S_6, e specialmente degli elementi fondamentali della trasformazione(*). Se nella Sua M_3^4 dello spazio (x) sostituiamo le formole $\rho x_1 = y_2 y_3$, … $\rho x_y = y_y^2$ otteniamo

$$\left\| \begin{matrix} y_y^2 - y_2 y_3 + y_3 y_1, + y_1 y_2, \cdots \\ y_5 y_6, \cdots \end{matrix} \right\|$$

Una M_3 che contiene evidentemente l'S_3 $y_2 = y_3 = y_y = 0$, ed altri 5 analoghi. Ella evidentemente toglie via questi S_3. Ma dopo ciò, se applico la trasformazione alle cose della Sua Nota di Torino,[297] non vedo subito che a $\lambda = 0$ e $\lambda = \infty$ corrispondano i due piani di cui Ella mi scrive, ognun dei quali prenda il posto di una F_2^{16}. Sarà forse da aggiungere // a ciascuno di quei piani un *resto* (proveniente da elementi fondamentali della trasformazione) per completare una F_2^{16}? Così pure altre cose che Ella asserisce io non posso ben giudicare, perché non ho fatto uno studio minuto della trasformazione.

La superficie F^6 di S_6, di cui Ella mi chiede, non si può chiamare "di Del Pezzo"; perché la si trova già in *G. Bordiga* "Di alcune superficie del 5° e del 6° ordine" Atti Istituto Veneto (6) 4 1886.[298]

Non badi, La prego, alla brevità di questa mia lettera: e continui Lei a scrivermi diffusamente.

Invio a Lei, a Suo marito e al Loro caro bimbo i più cordiali auguri per queste feste e pel nuovo anno. Gradisca i miei più cordiali saluti.

Suo C. Segre

(*)Non credo di poterle fare nessuna citazione utile su trasformazioni multiple di S_n.

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[296]Elena is Segre's eldest daughter, born in March 1894.

[297]Segre refers to Chisholm Young (1898–99).

[298]Bordiga, Giovanni, Di alcune superficie del 5° e del 6° ordine che si deducono dallo spazio a sei dimensioni, *Atti del R. Istituto Veneto di Scienze, Lettere ed Arti*, (6) 4 (1885–1886): 1461–1501.

Segre's letter of introduction for William Young, Turin, 20 January 1908[299]
Typewritten letter (D.140/9/182)

Torino, 20 Gennajo 1908.

Il sig. Dr. W.H. Young, che fu già mio allievo a Torino vari anni or sono, si è creato col suo assiduo intelligente lavoro una solida posizione scientifica.

Va rilevato in particolare il suo libro "The Theory of Sets of Points" (Cambridge, 1906).[300] In esso si trovano riunite, insieme con le cose altrui, parecchie ricerche originali dell'autore, in un campo di Matematiche, che è molto importante, ma anche molto difficile, per la grande delicatezza dei ragionamenti, e per l'acume che essi esigono.

Il Dr. Young ha superato assai bene queste difficoltà, dando alla scienza un libro che sarà sempre consultato da chiunque voglia occuparsi coscienziosamente della teoria degli aggregati di punti.

Con quest'opera, come con altre varie memorie, fra cui anche alcune più recenti, il sig. Young ha dato prova di profondi studi negli argomenti che prende a trattare, e di valida capacità nella ricerca originale.

Egli è certamente adatto per istruire gli studenti d'università nelle discipline matematiche, ed eccitarli col suo esempio e coll'entusiasmo suo al lavoro ed alla ricerca.

Prof. C. Segre

Appendix 2. Unpublished Documents of Grace Chisholm Young and William H. Young

AUL—*Papers of Professor William Henry Young and Grace Chisholm Young*

[299]A letter by Segre in the same vein, dated 9 November 1901, is reproduced in Italian with an English translation in [Recueil d'articles de périodiques publiés par William Henry Young et Grace Chisholm Young, constitué à partir de tirés à part de leurs écrits], [United Kingdom], 1863–1944, 5 vols, vol. I (EPFL, Bibliothèque de Mathématiques, 01YOU). Here we reproduce the English translation: *Mr W. H. YOUNG spent the academic year 1898–1899 at Turin, and attended my course of Lectures on Higher Geometry. I had the pleasure in this way of making his personal acquaintance, and of having frequent mathematical discussions with him. I convinced myself that he is a very studious man, full of enthusiasm for the science; and that his scholarship (coltura) is not confined to a single branch but embraces the whole range of mathematics. He has acquired an extended knowledge of Italian Geometry thanks to his mastery of the Italian language, and to the relations he has formed with various mathematicians. In his scientific researches he proceeds, I consider, with rigour, and with seriousness of purpose: and l have recorded some of his results in my article on the geometry of hyperspaces, destined for the Encyklopädie der Mathematischen Wissenschaften. I am of opinion that Mr Young would be of great service to a University, both by his lectures, and by the assistance he would give to students in research work in Geometry.*

[300]Young, William Henri, Chisholm, Grace, *The Theory of Sets of Points*, Cambridge: at the University Press, 1st Ed. 1906.

Academic Diary of Grace Chisholm Young

Outline of the lessons of Segre attended by Grace and William in 1898–1899 (D.140/9/183)

«(Printed label) R. Ginnasio Massimo d'Azeglio.
Torino Anno Scolastico 1898–99
Diario Scolastico della Signora Grace Chisholm Young Phil Doc. Gött.
Via Saluzzo 32^{II}
Lezioni del Prof. Corrado Segre
Curve algebriche …
All in G e Y.

p. 23 3rd lect. Nov. 23d Wednesday

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p. 299 Note to p. 86 in English

p. 300 Conferenza 1a Prof. Segre[301] Linear Systems in Reciprocation etc. in English to 205 then Italian to end + *LETTER from C. Segre 11 marzo* "Ella ha ragione"…

N.B. p. 36 Nov 30[302]		26 Dec. 21		
p. 38 Nov. 30th 6th lecture		82 16th lecture:	Jan. 4	25 Jan 30
p. 39 7th lecture Dec. 2nd		88 17th	Jan. 11	Feb. 1
p. 43 8th lecture Dec. 5th		94 18th	13	Feb. 20
48 9th	7	104	16th	
53 10th	9		18th	20-22
56 11th	13th	21	20	
62 12th	15	214	23	
67 (13th)	15-16	222 23	25	
71 14th	19			

2nd notebook[303]
Lez.e 31 Feb. 24
" 57 Maggio 1
" 58 Aprile 3[304]
Has large chunk W. H. pp. 213 seg. »[305]

Excerpt from "Dates in the Life of W.H.Y. and Notes on His Career by C.T".[306] (n. d.)

(D140/11/7a–c)

March 1897 Klein at Cambr. 20 Forsyth, Hon. degree.

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[301]See Notebook 3, in the following Appendix 3.

[302]In Notebook 3, on page 36 it is written "Nov. 30th" and on page 38 "Nov. 30th, 6th lecture".

[303]See Notebook 4, in the following Appendix 3.

[304]In the Notebook 4 on page 122 we find "Aprile 3", later corrected to "Maggio 3".

[305]See Notebook 4 in the following Appendix 3.

[306]C.T. is for Cecilia Tanner. Rosalind Cecilia Hildegard Tanner (1900–1992) was the eldest daughter of Grace and William. She was in the habit of using the name Cecily.

His family at St Paul's Rd. n^r^. Youngs.

April 1897 W.H.Y. Welsh Intermediate Schools examiner application (May failed)

June 1897 birth of F.C.Y.[307]

July 1897 Cambridge given up as home

Sept. 1897 Göttingen [G. C. Y. typed notes. Oct. to Febr. First ideas on *n*-dimensions written down for W.H.Y's 1st paper (to L.M.S. May 1898)[308]

GRASSMANN Ausdehnungslhere[309]

March 1898 On way to Italy. Joint Nature (5th.3.98) Section on SCHERING[310]

April 1898 Italy

[Oct. 1898 Klein in London as G. delegate, Commission for Intern. Catalogue (R. S.)

G.C.Y a/c of 1899 //

Italy 1898

Apr. Padua VERONESE[311] Venice
Bologna M^rs^ Fiorelli
Florence Miss Zimmern

May Joint Nature article on VOIGT book (proofs)[312]
W.H.Y.'s FIRST SIGN OF ORIGINAL MATH^L^ IDEAS
Paper in L.M.S. coming out August (G.C.Y. letters/12.5.98)[313]

June W.H.Y.'s 2nd paper finished, 3rd begone [sic]

July Badia P.C. SEGRE Letter from SEGRE.

ENRIQUES family (13th)

Aug. "prof. Segre is thinking of giving a particular course of lectures specially for us (17th)

Oct. Florence CASTELNUOVO[314]

Levanto BIANCHI at Pisa[315]

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307 See footnote 288.

308 See Young, William Henry, On systems of one-vectors in space of *n*-dimensions, *Proceedings of the London Mathematical Society*, XXIX (1897–1898): 478–487.

309 Graßmann, Hermann Günther, *Die Lineale Ausdehnungslehre, ein neuer Zweig der Mathematik*, Leipzig: Otto Wiegand, 1844.

310 Young, William Henry, Chisholm, Grace, Ernst Christian Julius Schering, *Nature* 57.1479 (1898):416.

311 Giuseppe Veronese (1854–1917) at that time was teaching analytic geometry and higher geometry at the University of Padua. His most important research concerns the hyperspatial geometry and non-archimedean geometry.

312 Young, William Henry, Chisholm, Grace, The physical properties of crystals, *Nature*, 57.1492 (1898): 99–100.

313 See Young, William Henry, On flat-space coordinates, *Proceedings of the London Mathematical Society*, XXX (1898–1899): 54–69.

314 Guido Castelnuovo.

315 Luigi Bianchi (1856–1928) is well-known for his research in differential geometry.

Rapallo
Genoa
Savona
Moncatini[316] n^r Turin SEGRE
Turin for 6 month
Nov. Lectures attended:
Segre. Curve algebriche M. n. F. notes by GCY.[317]
Dec. W.H.Y. to England to collect Exam papers

Proc. L.M.S. 29 pp. 428–87 On systems of one-vectors in space of *n* dimensions
30 pp. 54–59 On flat-space coord.^tes[318]

Italy 1899

March 19 Klein's visit to Turin[319]
Sept G. + F. at Kleins while W.H. prepares house
Sept. Bergstrasse Göttingen
Oct. Klein put G. on piece of work to do with unpublished Gauss
Dec. W.H. examining at Rugeley Galsgow, Edinborough
Cambridge.
Rendiconti dell'Acc. R. di Torino. Sulle Sizygie che legano le relazioni…[320]

Appendix 3. Indexes of the Notebooks of Young's Handwritten Notes Referring to the Courses of Segre

AUL—*Papers of Professor William Henry Young and Grace Chisholm Young*. Notebooks D.140/19/1–5

The notebooks cover the period from November 1898 to May 1899. They are written in Italian with some parts in English by Grace Chisholm Young with some insertions by William Young. Only one contains a true title sheet, thus the incipit of each is given here. Since the individual sections of the notebooks are not always given a title, in order to compile the index reference has been made to notes in the margin, most of which are in English.

The page numbers indicated are those stamped on the pages by the Youngs.

Note that the archival numbering of the Notebooks does not correspond to the dates indicated in the Notebooks. These dates are given in what follows, and the notebooks are listed according these dates. It should be noted that the Notebooks 1

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[316]Probably Young refers to Moncalieri, a town near Torino.
[317]See Appendix 3 that follows, Notebooks 3 and 4.
[318]See footnotes 308 and 313.
[319]See Young (1928, xiii).
[320]See Young (1898–99).

and 2 likely have been copied by the Youngs from the notes taken by some of Segre's students who listened the lectures of the 1897–98 course, in fact the Youngs came to Italy only in March 1898[321]; this hypothesis seems to be confirmed by the use of the Italian language which is more accurate than in the remaining Notebooks. The Notebook 5 in all likelihood contains the supplementary lessons given by Segre to the Youngs.

Notebook 1, ?–March 1898[322]—AUL D140/19/4
Incipit: ***Prime nozioni sui gruppi di trasformazioni***
Manuscript by Grace Chisholm and William Young, 226 numbered pages, in Italian with some English.
Contents:
Prime nozioni sui gruppi di trasformazioni (p. 3); *Prodotti di trasformazioni* (p. 10); *Gruppi di trasformazioni* (p. 17); *Distinzione dei gruppi in speci*e (p. 47); *Cristallog.*[y] (p. 54); *Discont.*[s] *Groups in Theory of Functions* (p. 55); *Invariants in Geometry* (p. 59); *Projective geometry on the line* (p. 72); *Stereographic projection* (p. 74); *Alcune proprietà e nozioni generali sui gruppi* (p. 84); *Determinazione dei gruppi d'ordine finito di proiettività binarie, o di sostituzioni lineari in una variabile* (p. 104); *Cenno sulla geometria a* n *dimensioni* (p. 134); *Omografie in uno spazio a* n *dimensioni* (p. 137); *Sistemi completi di equazioni alle derivate parziali* (p. 197); *Note 1* (p. 226); *Note 2 p. 18* (p. 226).

Notebook 2, March–May 1898—AUL D140/19/1[323]
Incipit: ***Sulle trasformazioni di*** S_n ***e sui loro ampliamenti***
Manuscript by Grace Chisholm and William Young, 193 numbered pages, in Italian with some English.
Contents:
Sulle trasformazioni di S_n *e sui loro ampliamenti* (p. 1); *Prime nozioni sui gruppi continui di trasformazioni* (p. 17); *Transitività, Invarianti, Primitività* (p. 29); *Equazioni differenziali fondamentali dei gruppi continui finiti (*p. 51); *Gruppi monomî e trasformazioni infinitesime* (p. 57); *Trasformazioni infinitesime e gruppi monomî contenuti in un gruppo continuo finito qualunque* (p. 70); *Funzioni e varietà invarianti di un gruppo* (p. 105); *Il Teorema principale* (p. 114); *Gruppi nei campi ad una dimensione* (p. 130); *Gruppi nel piano* (p. 139); *Composiz. dei gruppi. Gruppo aggiunto* (p. 157); *Struttura dei gruppi binomî e trinomî* (p. 181); *Teoria dei Gruppi di Tras.*[i] *(Vivanti)* (p. 191).

[321]See Appendix 2 here above.

[322]The numbers of the lessons are indicated in the notebooks. The first are not dated, and the last, no. 44, is dated 3 March; the year 1898 can be deduced from the fact that, for example, on p. 173 we find written "26 Feb. 98 [Lezione 32[a]]".

[323]This Notebook is a continuation of the previous one. Some lessons are dated: the first lesson is dated March 16th 1898. The two Notebooks present an extension of the contents of the course of 1897–98 (See BMP-Segre, Quaderni. 11).

Notebook 3, November 1898–February 1899—AUL D140/19/5
[*Lezioni sulle curve algebriche dei vari spazi*][324]
Manuscript by Grace Chisholm and William Young, 228 numbered pages, in Italian with some English.
Contents:
[Introductory class] (p. 1); *Veronese* (p. 13); *Pieri* (p. 18); blank pages (pp. 28–35); *Projezione* (p. 39); *Homology* (p. 44); *Reciprocity* (p. 46); *Linear complex* (p. 48); *Prime nozioni sopra le varietà algebriche* (p. 49); *Intersections* (p. 50); *Tangent hyperplane* (p. 52); *Cono tangente* (p. 55); *Polarity* (p. 57); *Circumscribed cone* (p. 59); *Parabolic Variety of a V* (p. 60); *Sistemi lineari di forme* (p. 62); *Varietà algebriche* (p. 69); *Skew Curves* (p. 69); *Parametric representation of a Variety* (p. 75); *Trasformazioni quadratiche* (p. 86); *Cremonian Transformations* (p. 88); *Decomposition of singolarities* [sic] (p. 102); *Branches* (p. 108); *Caracters* [sic] *of curves* (p. 223); *Ranks of a curve* (p. 227); *Serie lineari* ∞^1 *su una curva. Genere di una curva algebrica* (p, 238); *Theorem of Zeuthen and Halphen about correspondences* (p. 263); *Applicazione alle curve piane* (p. 268); *Rappresentazione particolare degli enti generici dei primi generi. Curve ellittiche. Curve iperellittiche* (p. 274); *Note to page 86* (p. 299); *Nov. 30th*. *Conferenza 1*^{a}. *Prof. Segre. Linear Systems in Reciprocation* (p. 300);[325] 2 *Conferences on the rational normal curve* C^r *in* S^r (p. 301)[326]; *Conferenza Jan. 21th. On the double tangents of the plane quartic* (s. n.); *(Segre). Literature.* (p. 205); *(Feb. 4th)* (p. 209); *Feb. 28th* (*Notes of* S^r *Tanturri*) (p. 210); Letter by Segre, 11 marzo (p. 225).[327]

Notebook 4, February–May 1899–AUL D140/19/2[328]
Incipit: *Feb. 24. Lezione 31*
Manuscript by Grace Chisholm and William Young, 239 numbered pages, in Italian with some English.
Contents:
Moduli of Hyperelliptic curves (p. 3); *Linear series* (p. 11)[329]; *Dimension of system* (p. 16); *Trasform. Cremoniane e birazionali* (p. 31); *Order of Evolute* (p. 35)[330]; *Estensione del teorema di Poncelet* (p. 52); *Relazioni fra i caratteri di una curva*

[324] Although the Notebook is without a title page, its title can be deduced from the first lines, which say, "*Visto la vastità della materia bisogna studiare molto. Proporrò di dare delle idee fuori di quelle della corso in una conferenza ogni settimana svilluppando le idee senza demonstrazioni così in un modo o l'altro, al fine del anno spero così di farle conoscere molte cose che tutti devono sapere, o almeno avere sentito. … Posso intitulare l'argumento delle lezioni che darò in quest'anno:* Lezioni sulle curve algebriche dei vari spazi" (p. 1). N.b. The orthographic errors are contained in the original text.

[325] See Appendix 2.1 here above.

[326] By mistake, the page numbers go from 301 to 202 ff.

[327] See Appendix 1 here above.

[328] This notebook is a continuation of the previous one. See also here above, the *Academic Diary of Grace Chisholm Young*.

[329] On p. 11 appears the note: "*Lie è morto*". Sophus Lie died in February 1899.

[330] On p. 47 there is a reference to an evening lesson on 6 March.

qualunque (p. 54); *Segre's Formulae* (p. 67); *Iperpiani stazionari* (p. 68); *Curve, intersezioni complete di r-1 forme* (p. 75); *Principio di corrispondenza sugli enti razionali, e sua applicazione* (p. 88); *Applicazione a trovare il numero di punti comuni a 2 curve che giacciono su una data rigata alg.* (p. 97); *Una ricerca* (p. 105); [*Problemi dei gruppi con punti multipli sulle curve razionali, grazie a una memoria di Jonquières*] (p. 114); *Cenno sull'appl. del principio di corrispondenza su una curva algebrica qualunque* (p. 133); *Le Serie Lineari su un dato Ente algebrico. Teorema del Resto, ecc.* (p. 143); *Applicazioni* (p. 178); *Determinaz. delle serie lineari esist.*[i] *su un dato ente alg. di gen p. Numero delle costanti da cui dipendono le curve di dato ord. e dato genere in un dato spazio* (p. 198); *Conferenza. Marzo 4°. Notes of. S*[r] *Tanturri.*[331] *Complesso quadratico (Segre)* (p. 200); *Continuaz. da pag. 199* (p. 213).

Notebook 5, 1899—AUL D140/19/3[332]

Teoria delle singolarità delle curve e superficie algebriche

Manuscript by Grace Chisholm and William Young, 231 numbered pages, in Italian with some English.

Contents:

Indice (p. 3); *List of references* (p. 7); *Le formole di Plücker* (p. 22); *Cenni sulle trasformazioni birazionali piane* (p. 27); *Risoluzione e studio delle singolarità mediante trasformaz.*[i] *quad.*[e] (p. 31); *Rami di una curva passante per un punto* (p. 38); *Genere* (p. 50); *Equivalenti plückeriani* (p. 60); *Studio delle singolarità* (p. 62); *Rappresentazione del ramo come inviluppo; classe, ecc.* (p. 68); *Forma dei rami reali* (p. 70); *Calcolo degli sviluppi in serie* (p. 82); *Metodo di Newton* (p. 90); *Applicazione degli sviluppi in serie alla determinazione dei caratteri di una curva* (p. 102); *Multiplicità nelle intersezioni di due curve* (p. 113); *I numeri analoghi pei rami considerati come inviluppi* (p. 121); *Applicazione ad una formola di Zeuthen* (p. 128); *Applicazione alla multiplicità dell'intersezione di 2 curve ecc.* (p. 129); *Applicazioni agli equivalenti plückeriani di una data singolarità*; (p. 130); *Superficie razionali e loro rappresentazioni piane* (p. 137); *Generalità sui sistemi lineari e sulle superficie imagini* (p. 140); *Curve corrispondenti sul piano e sulla superficie* (p. 151); *Su alcuni caratteri dei sistemi lineari* (p. 156); *Sistemi triplamente infiniti* (p. 165); *Trasformazioni Cremoniane* (p. 174); P*roprietà delle trasformazioni Cremoniane* (p. 180); *Jacobiana della rete omaloidica di* π (p. 182); *Trasformaz.*[i] *quadratiche* (p. 186); *Applicaz.*[e] *di una trasf.*[e] *quadratica ad una* γ^n (p. 188); *Singolarità superiori* (p. 190); *Sistemi lineari con punti fond.*[li] *qualunque* (p. 192); *Le due proposizioni di* Del Pezzo *sopra ottenute* (p. 208); *Teorema di Bertini* (p. 214); *Continuazione da pag. 238 delle lezioni sulle curve algeb.* (p. 217).

[331]Alberto Tanturri, who received his degree in July 1899 defending a thesis on enumerative geometry written under Segre's advisement.

[332]The title page, in addition to the title, says, "W. H. and Chisholm Young, Turin 1899". This in all likelihood contains the supplementary lessons that Segre gave to the Youngs, with reference to the courses of 1894–1895 and 1896–1897 (See BMP-Segre, Quaderni. 6 and Quaderni. 8) on the same topic, with new examples, applications and additions.

Appendix 4. Segre's Card Index[333]

BMP—Segre—Scritti. 17

The card index is divided in two sections: the first consist of 515 not numbered folios without date which list topics for seminars, subjects for research, themes for dissertations, and a very rich bibliography organized alphabetically by topic. The second section is made up of 46 not numbered folios without date including bibliographical references related to secondary teaching, organized alphabetically by topic.

The items of the first section are the following:

Discorsi e simili per un eventuale discorso inaug.; Abel (Teorema di); Affinità; Aggregati I, II; Analysis situs I, II, III; Analysis situs (superficie), II; Antiprojettività; Apolarità; Area; Asintotiche; Bilineari (Forme); Biografie, II, III, IV; Birapporti; Carte geografiche; Cerchi I, II, III; Collineazioni I, II, III; Collineazioni (pr.^{à} metriche); Combinatoria; Complessi di linee; Complessi di rette I, II, III; Complesso lineare di rette; Complessi quadratici di rette; Configurazioni I, II, III; Congruenze I, II; Congruenze di rette, II, III, IV, V; Congruenze di rette del 1° ordine; Congruenze di rette del 2° *ordine; Congruenze di rette di 3° ordine e sup.^{i}*; *Coniche I, II, III, IV; Connessi; Connessione; Contatti; Coordinate; Corrispondenze; Corrispondenze alg.^{e}; Corrispondenze plurilineari; Corrispondenze tra curve alg., II; Corrispondenze tra superficie; Costruzioni geometriche I, II, III; Covarianti; Cristalli; Cubiche piane I, II, III, IV, V; Cubiche sghembe I, II, III; Curvatura (raggi, linee), II; Curvatura delle linee; Curvatura delle superf.; Curve I, II, III, IV, V; Curve algebriche; Curve algebriche piane (teoria generale) I, II, III, IV; Curve algebriche particolari, II, III; Curve algebriche con trasformazioni in sé; Curve di dati ordini > 6; Curve di dati generi; Curve ellittiche; Curve razionali; Curve sghembe (geom. diff.^{le}); Curve sghembe alg.^{e}, II; Curve sghembe particolari; Curve su superficie; Deformazione delle superficie; Derivazione; Determinanti (Matrici ecc.); Differenziali totali algebr.^{i}; Dinami (Screws ecc.); Dirichlet (Principio di); Discriminante; Distanze; Divisori elementari; Eliminazione; Ellittiche (Funzioni); Enti iperalgebrici; Equazioni algebriche; Equazioni algebriche (geometria); Equazioni alg.^{e} lineari; Equazioni differenziali (geometria) I, II, III, IV; Equazioni differenz.^{li} algebriche; Equivalenza, II; Esagoni; Evolute; Fasci di curve; Fasci di superficie; Fondamenti della geom. II, III, IV; Fondamenti della geom.^{a} projettiva I, II; Forma delle curve I, II, III; Forma delle curve e superficie; Forma delle superficie; Forme; Forme (aritmet.); Forme bilineari; Forme binarie (geom.); Forme canoniche; Forme differenziali; Forme di Hermite (a variabili complesse coniugate; Forme quadratiche; Funzioni (di var.^{i} reali o complesse); Funzioni; Funz.^{i} Abeliane (Funzioni); Funzioni algebriche, II;*

[333]The list of the sections of the card index can be accessed at Giacardi (2013).

Fuochi; Generazioni di enti algebrici; Genere delle curve; Generi delle superficie; Geodetiche I, II; Geom.^{a} affine; Geom.^{a} conforme; Geometria della retta; Geom.^{a} descrittiva, II; Geometria differ.^{le} (questioni generali); Geom. Differ. ^{le} intrinseca; Geom.^{a} differenz.^{le} projettiva; Geom. elementare; Geom. non-euclidea, II, III, IV, V, VI; Geometria numerativa (delle curve e superficie) ecc.; Geometria projettiva (varia); Geometria sopra una curva I, II; Geometria sopra una superficie I, II, III; Geom. su superficie algebriche IV, V, VI; Geometrie varie (Antiproj$^{\grave{a}}$ Geom.^{e} di Study. ecc.) I, II; Gnomonica; Grandezze; Gruppi I, II, III; Gruppi (continui); Gruppi (d'ord. finito); Gruppi algebrici; Gruppi di movimenti; Gruppi di trasform.^{i} biraz.^{li}, I, II; Gruppi di trasform.^{i} projettive, II, III, IV; Hessiana; Imaginari (num.^{i} complessi ecc.), II; Infiniti I, II; Integrali; Integrali Abeliani; lntersezioni di curve (e forme), II; Intersezioni di superficie e var.$^{\grave{a}}$; Invarianti, II; Invarianti differenziali I, II; Invarianti integrali; Inversione; lnviluppi di linee; Inviluppi di superficie; Involuzioni sulla retta; Involuzioni piane I, II; Involuzioni sulle curve; Involuzioni spaziali; Involuzioni sulle superf.; Iperellittici (enti); Isofote; Jacobiana; Massimi e minimi I, II; Matrici; Meccanica; Meccanica non-euclidea; Moduli di forme ecc.; Movimenti I, II, III, IV; Normali alle curve e sup.; Normali alle quadriche; Ottica; π; Pascal (teor. di); Pentagono, pentaedro; Piani doppi e multipli; Nullsystem, II; Osculazione; Pohlke (teorema di); Polarità; Poliedri, II, III; Poligoni; Poligoni di Poncelet e simili, II; Postulati; Postulazione; Principi di corrispondenza; Probabilità; Projettività; Projettività binarie, Projezioni; Projezione centrale; Punti multipli di Superf.; Quadriche I, II, III, IV, V; Quadriche (sistemi) I, II; Quadrilateri; Quartiche piane I, II; Quartiche piane speciali I, II, III; Quartiche sghembe, II; Quaternioni; Quintica binaria; Quintiche piane I, II; Quintiche sghembe; Rappresentazioni (geom. projettiva); Rappresentazioni (geom. infinit.^{le}) I; Rappresentaz^{i} (geom. gener^{e}) II; Rappresentaz.^{i} conformi, II; Reciprocità; Relatività (geom.); Riemanniane (sup.); Rigate I, II, III; Rigate di 4° grado; Rigate di 5° grado; Rigate di 6° grado, ecc.; Risultanti; Secanti multiple delle curve; Serie (lineari o no) su una curva; Sestica binaria; Sestiche piane; sestiche sghembe; Sfere I, II; Similitudini; Simmetrie; Singolarità; Singolarità d. curve I, II, Singolarità d. curve sghembe; Singolarità d. superficie I, II; Sistemi (lineari o no) in gen.^{e}; Sistemi di curve, II; Sistemi di curve definiti da equaz.^{i} diff.^{li}; Sistemi di enti partic.^{i}; Sistemi di superficie; Sistemi lineari di curve piane; Sistemi lineari di forme quadr ^{e}; Sistemi lineari di superficie; Sistemi nulli super.^{i}; Sistemi ortogonali; Sistemi punti-piani (Punkt-Ebenensysteme); Sostituzioni; Sostituzioni lineari; Sostituzioni ortogonali; Steiner; Superfice (geom. infinites.) I; Superfice (geom. generale) II, III, IV, V; superficie (classi speciali) I, II, III, IV, V; Superficie algebriche (Questioni generali proj.^{e}) I; Superf. Algebriche (gener^{i}) II; Superficie algebriche (Questioni gener.^{i} metriche); Superficie algebriche particolari I, II, III, IV; Superficie autoprojettive; Superficie cubiche I, II, III, IV; Superfice del 4° ordine; Superficie del 4° ordine particolari I, II, III, IV, V; Superficie del 5°ord. ^{e}; Superficie del 6° ordine; Superficie del 7° ordine, 8° ecc.; Superficie minime e generalizzazioni I, II; Superficie razionali; Sviluppabili; Tangenti; Tangenti principali ecc.; Tavole numeriche; Teoria dei numeri (geometr.); Terne di rette sghembe; Tetraedri I, II; Topologia; Torsione delle curve; Trasformazioni I, II, III;

Trasformaz. ^{i} biraz.^{li} delle curve alg. I, II; Trasformaz.^{i} birazionali delle superficie; Trasform.^{i} conformi piane; Trasformazioni Cremoniane I, II; Trasformaz.^{i} Cremoniane piane I, II; Trasformaz. ^{i} Cremoniane periodiche; Trasformaz. ^{i} Cremoniane dello spazio I, II; Trasformaz. ^{i} di contatto I, II; Trasformaz. ^{i} di superf.^{e}; Trasformaz. ^{i} piane multiple; Trasformazioni piane multiple; Trasformaz. ^{i} spaz. ^{li} multiple; Trasversali; Triangoli I, II; Trigonometria; Trilineari (Corrisp.^{e}); Trisezione dell'angolo; Varie; Vettori (e simili); Volumi; Collineazioni in S_n; Complessi di piani, o spazi; Corrispondenze; Curvatura delle varietà; Curve iperspaziali, Curve alg. ^{e} ipersp.; Deformazione delle varietà; Fondamenti (S_n); Geom.^{a} a più dim.^{i}, in generale; Geom.^{a} delle rette; Geom.^{a} descrittiva a più dim.^{i}; Geom. differ. proj. ^{a} delle varietà; Geom. ^{a} elem.^{e}; Geom. numerativa; Geom. ^{a} proj. ^{a} (varia); Geom. ^{a} su una varietà I, II; Grandezze; lntersezioni; Iperspazi (fondamenti); Ipersuperficie algebr.; Iperspazi (Quest.^{i} metriche); Meccanica iperspaz.^{le}; Movimenti; Politopi; Punti multipli di varietà, Quadriche iperspaz.^{li}; Reciprocità; Rigate; Rigate algebriche; Sfere (Geom. delle); Sistemi di rette; Sistemi di spazi; Sistemi di varietà; Spazi a ∞ dimens.^{i}; Spazi subordinati di S_n; Superficie (in S_n) (geom. infin.); Superficie speciali (geom. generale); Superficie algebriche degl'iperspazi; Superf. alg.^{e} (particolari); Superf. dei primi ordini; Sviluppabili (iperspazi); Trasform.^{i} di var.^{à}; Trasformaz.^{i} in S_n; Trasformaz.^{i} iperspaz.^{li} biraz.^{li} I, II; Varietà (geom. infinites.) I, II; Varietà (geom. gener.^{e}) speciali; Varietà (questioni diverse); Varietà algebriche; Varietà algebriche partic.^{i}, II; Varietà cubiche; Varietà luoghi di spazi; Varietà quadratiche, Varietà del 4° ordine, Varietà algebriche di ordini superiori al 4°; Varietà di rette negl'iperspazi; Varia.

The items of the second section are the following:

Analisi indet.^{a} di 1° grado; Approssimazioni numeriche; Calcoli numerici; Didattica; Didattica (sulla geom.^{a} elementare); Divisibilità; Equazioni; Equivalenza; Euclide; Fondam.^{i} dell'Aritmetica; Formole; Funzioni; Fusione (geom.); Frazioni; Geometria; Grandezze (lungh., aree, vol.^{i}); Impossibilità; Logaritmi; Numerazione; Numeri primi; Numeri trascendenti; Pitagora (teor.^{a}); Poligoni; Probabilità; Proporzioni; Quadratura del Circ.; Teoria dei numeri; Triangoli; Trigonometria piana; Volumi.

Appendix 5. Papers Presented by Segre for the Publication in *Atti dell'Accademia delle Scienze di Torino*

	Volume	Page
1890		
(9 February)* *M. PIERI, Sulla corrispondenza algebrica fra due spazi rigati*	1889–90, 25	341
(11 May*) C. FABRI, Sopra alcune proprietà generali delle funzioni che dipendono da altre funzioni e da linee*	1889–90, 25	653
(16 November) *M. CHINI, Sopra alcune deformazioni delle superfici rigate*	1890–91, 26	2
(30 November) *E. BERTINI, Intorno ad alcuni teoremi della Geometria sopra una curva algebrica*	1890–91, 26	117
(28 December) *G. PEANO, Sopra alcune curve singolari*	1890–91, 26	267
1891		
(22 March) *V. REINA, Della compensazione del problema di Hansen*	1890–91, 26	570
(22 November) *V. REINA, Sull'errore medio dei punti determinati nei problemi di Hansen e di Marek*	1891–92, 27	3
1892		
(3 January) *P. PREDELLA, Sulla teoria generale delle omografie*	1891–92, 27	266
(19 June) *D. MONTESANO, Su una congruenza di rette di secondo ordine e di quarta classe*	1891–92, 27	972
1893		
(11 June) *G. CASTELNUOVO, Sulla linearità delle involuzioni più volte infinite appartenenti ad una curva algebrica*	1892–93, 28	718
(19 November) *S. KANTOR, Les correspondances dans les courbes elliptiques déduites géométriquement*	1893–94, 29	3
1894		
(28 January) *F. ENRIQUES, Sulla massima dimensione dei sistemi lineari di curve di dato genere appartenenti ad una superficie algebrica*	1893–94, 29	273
(1° April) *G. FANO, Sulle congruenze di rette del terzo ordine prive di linea singolare*	1893–94, 29	469
1895		
(3 February) *V. VOLTERRA, Sulla teoria dei moti del polo terrestre*	1894–95, 30	283
(17 February) *E. BERTINI, Sugli spazi lineari delle quadriche a numero pari di dimensioni*	1894–95, 30	349
(3 March) *V. VOLTERRA, Sul moto di un sistema nel quale sussistono moti interni stazionari*	1894–95, 30	371
(6 June) *V. VOLTERRA, Sulla teoria dei moti del polo nell'ipotesi della plasticità terrestre*	1894–95, 30	727
(6 June) *L. BIANCHI, Sulle superficie a curvatura nulla negli spazi di curvatura costante*	1894–95, 30	727

*The dates in brackets refer to the session of the Academy of Science of Turin, in which the papers were presented. The numbers in the third column refer to the pages of the *Atti* where related minutes of those sessions appear.

	Volume	Page
(6 June*) L. BERZOLARI, Sopra un problema che comprende quello di trovare il numero degli ombelichi d'una superficie generale d'ordine n*	1894–95, 30	727
(17 November) *T. LEVI CIVITA, Sull'inversione degli integrali definiti nel campo reale*	1895–96, 31	2
1896		
(9 February) *R. BETTAZZI, Sulla catena di un ente in un gruppo*	1895–96, 31	437
(23 February) *L. BERZOLARI, Sulle curve piane che in due dati fasci hanno un semplice o doppio contatto oppure si osculano*	1895–96, 31	474
(23 February) *A. LEVI, Sulle singolarità della jacobiana di quattro superficie*	1895–96, 31	474
(23 February) *R. BETTAZZI, Gruppi finiti ed infiniti di enti*	1895–96, 31	474
(26 April) *G. FANO, Sulle congruenze di rette del terzo ordine prive di linea singolare*	1895–96, 31	692
(11 November) *E. BERTINI, Le tangenti multiple della Cayleyana di una quadrica piana generale*	1896–97, 32	2
1897		
(5 December) *B. LEVI, Risoluzione delle singolarità puntuali delle superficie algebriche*	1897–98, 33	65
1898		
(27 February) *G. FANO, I gruppi continui primitivi di trasformazioni cremoniane dello spazio*	1897–98, 33	448
(15 May) *L. BERZOLARI, Sulla curvatura delle varietà tracciate sopra una varietà qualunque, Nota I e II*	1897–98, 33	691
1899		
(26 February) *G. FANO, Sulle equazioni differenziali lineari, che appartengono alla stessa specie delle proprie aggiunte*	1898–99, 34	362
(12 March) *G. FANO, Sulle equazioni differenziali lineari del 5° e del 6° ordine, le cui curve integrali sono contenute in una quadrica*	1898–99, 34	414
(16 April) *A. BEMPORAD, Complessi di 2° grado costituiti dalle normali ad una serie di curve piane*	1898–99, 34	480
(30 April) *GRACE C. YOUNG, Sulla varietà razionale normale di* M_4^3 *di* S_6 *rappresentante della trigonometria sferica*	1898–99, 34	561
(30 April) *W.H. YOUNG, Sulle sizigìe che legano le relazioni quadratiche fra le coordinate di retta in* S_4	1898–99, 34	562
(14 May) *B. LEVI, Dell'intersezione di due varietà contenute in una varietà semplicemente infinita di spazi*	1898–99, 34	704
(19 November) *C. ROSATI, Sulle superficie di Veronese e di Steiner*	1899–900, 35	2
(19 November) *B. LEVI, Sulla trasformazione dell'intorno di un punto per una corrispondenza birazionale fra due spazi*	1899–900, 35	2
1900		
(11 February) *A. TANTURRI, Un problema di geometria numerativa sulle varietà algebriche luogo di spazi*	1899–900, 35	403
(11 February) *G. SCORZA, Sopra le corrispondenze (p, p) esistenti sulle curve di genere p e moduli generali*	1899–900, 35	404

	Volume	Page
(27 May) *G. SCORZA, Sulle curve canoniche di uno spazio lineare qualunque e sopra certi loro covarianti quartici*	1899–900, 35	764
(27 May) *F. SEVERI, Ricerche sulle coniche secanti delle curve gobbe*	1899–900, 35	764
(18 November) *F. SEVERI, Sopra le coniche che toccano e secano una o più curve gobbe*	1900–01, 36	5
1901		
(13 January) *G. FANO, Sopra alcune particolari congruenze di rette del 3° ordine*	1900–01, 36	325
(24 February) *G.Z. GIAMBELLI, Prodotto di due condizioni caratteristiche relative ai piani di un iperspazio*	1900–01, 36	442
(10 March) *G. SCORZA, Aggiunta alla Nota "Sulle corrispondenze (p,p) nelle curve di genere p"*	1900–01, 36	576
(14 April, 28 April) *E. PICARD, Sur les systèmes linéaires du genre zéro (estratto di lettera)*	1900–01, 36	682
(15 May) *E. DANIELE, Sulla deformazione infinitesima delle superficie di 2° grado*	1900–01, 36	836
(15 May) *G. CASTELNUOVO, Le trasformazioni generatrici del gruppo cremoniano nel piano*	1900–01, 36	836
(26 May) *U. PERAZZO, Sopra una forma cubica con nove rette doppie dello spazio a cinque dimensioni e i corrispondenti complessi cubici di rette nello spazio ordinario*	1900–01, 36	886
(17 November) *F. ENRIQUES, Intorno ai fondamenti della geometria sopra le superficie algebriche*	1901–02, 37	4
1902		
(9 March) *A. TANTURRI, Intorno ad alcune semplici infinità di spazi, e sopra un teorema del prof. Castelnuovo*	1901–02, 37	321
(13 April) *A. TANTURRI, In qual modo alcuni numeri, relativi ad infinità ellittiche di spazi, si deducano dagli analoghi, relativi ad infinità razionali*	1901–02, 37	392
(11 May) *G. FANO, Le congruenze di rette del 3°ordine composte di tangenti principali di una superficie*	1901–02, 37	468
(15 June) *F. SEVERI, Il genere aritmetico e il genere lineare in relazione alle reti di curve tracciate sopra una superficie algebrica*	1901–02, 37	584
1903		
(25 January) *F. SEVERI, Sulle superfici che rappresentano le coppie di punti di una curva algebrica*	1902–03, 38	183
(8 March) *G. FUBINI, Sulla teoria degli spazi che ammettono un gruppo conforme*	1902–03, 38	382
(26 April) *L. BIANCHI, Intorno alle superficie applicabili sui paraboloidi ed alla loro trasformazione*	1902–03, 38	513
(10 May) *L. BIANCHI, Sui gruppi continui finiti di trasformazioni che conservano le aree od i volumi*	1902–03, 38	596
(24 May) *L. BIANCHI, Sui gruppi continui finiti di trasformazioni proporzionali*	1902–03, 38	700

	Volume	Page
(14 June) *G. Z. GIAMBELLI, Alcune proprietà delle funzioni simmetriche caratteristiche*	1902–03, 38	818
(20 December) *L. COOLIDGE, Les congruences isotropes qui servent à représenter les fonctions d'une variable complexe, Nota I*	1903–04, 39	120
1904		
(31 January) *L. BIANCHI, Sulla rappresentazione di Clifford delle congruenze rettilinee nello spazio ellittico*	1903–04, 39	380
(28 February) *O. NICCOLETTI, Su alcune applicazioni del teorema di Sturm*	1903–04, 39	454
(28 February) *F. SEVERI, Osservazioni sui sistemi continui di curve appartenenti ad una superficie algebrica*	1903–04, 39	454
(28 February) *A. TANTURRI, Alcune equazioni funzionali ed il numero dei gruppi neutri di seconda specie in una serie lineare*	1903–04, 39	454
(27 March) *G. FANO, Sulle superficie algebriche contenute in una varietà cubica dello spazio a quattro dimensioni*	1903–04, 39	563
(1 May) *G. FANO, Sul sistema* ∞^2 *di rette contenuto in una varietà cubica generale dello spazio a quattro dimensioni*	1903–04, 39	768
(18 December) *B. LEVI, Punti doppi uniplanari delle superficie algebriche*	1904–05, 40	124
1905		
(8 January) *J. L. COOLIDGE, Les congruences isotropes qui servent à représenter les fonctions d'une variable complexe, Nota II*	1904–05, 40	201
(22 January) F. SEVERI, *Sulla differenza tra i numeri degli integrali di Picard, della 1ªe della 2ª specie appartenenti ad una superficie algebrica*	1904–05, 40	287
(16 April) *G. FUBINI, Alcuni nuovi problemi che si presentano nella teoria delle equazioni alle derivate parziali*	1904–05, 40	615
(7 May) *L. BIANCHI, Sulla deformazione delle superficie flessibili ed inestendibili*	1904–05, 40	662
(7 May) *F. SEVERI, Sul teorema di Riemann-Roch e sulle serie continue di curve appartenenti ad una superficie algebrica*	1904–05, 40	662
(31 December) *F. SEVERI, Su alcune proprietà dei moduli di forme algebriche*	1905–06, 41	203
(31 December) *R. TORELLI, Sopra certe estensioni del teorema di Noether*	1905–06, 41	203
(31 December) *G. Z. GIAMBELLI, Alcune estensioni del 'Fundamentalsatz' di Noether negli iperspazi*	1905–06, 41	203
1906		
(11 February) *G. GUARESCHI, Sulla geometria di una forma quadratica e di una forma di Hermite a variabili coniugate*	1905–06, 41	404
(8 April) *F. PALATINI, Sulle superficie algebriche i cui* $S_h(h+1)$*-seganti non riempiono lo spazio ambiente*	1905–06, 41	633
(29 April) *B. LEVI, Saggio per una teoria aritmetica delle forme cubiche ternarie, Nota I*	1905–06, 41	689
(13 May) *B. LEVI, Saggio per una teoria aritmetica delle forme cubiche ternarie, Nota II*	1905–06, 41	840

	Volume	Page
(27 May) *U. PERAZZO, Sopra la geometria descrittiva di uno spazio ad un numero qualunque di dimensioni*	1905–06, 41	921
(18 November) *W. YOUNG, G. CHISHOLM YOUNG, Note on Bertini's transformation of a curve into one possessing only nodes*	1906–07, 42	5
(18 November) *R. TORELLI, Sui sistemi algebrici di curve appartenenti ad una superficie algebrica*	1906–07, 42	5
1907		
(10 March) *H. LEBESGUE, Sur les transformations ponctuelles, transformant les plans en plans, qu'on peut définir par des procédés analytiques*	1906–07, 42	524
(26 May) *P. PREDELLA, Ricerche sulle coppie di quadriche di uno spazio ad n dimensioni*	1906–07, 42	935
(16 June) *G. SCORZA, Lettera al prof. Eugenio Bertini intorno alle corrispondenze (p,p) sulle curve di genere p e ad alcune loro applicazioni*	1906–07, 42	1003
(29 December) *E. E. LEVI, Sulla deformazione delle superficie flessibili ed inestendibili*	1907–08, 43	291
(29 December) *B. LEVI, Saggio per una teoria aritmetica delle forme cubiche ternarie, Nota III*	1907–08, 43	411
1908		
(8 March) *B. LEVI, Saggio per una teoria aritmetica delle forme cubiche ternarie, Nota IV*	1907–08, 43	647
(10 May) *L. TONELLI, Sulla rettificazione delle curve*	1907–08, 43	765
(24 May) *E. BERTINI, F. SEVERI, Osservazioni sul Restsatz per una curva iperspaziale*	1907–08, 43	845
(14 June) *G. FANO, Sopra alcune varietà algebriche a tre dimensioni aventi tutti i generi nulli*	1907–08, 43	958
(15 November) *E. BERTINI, Sulle serie segnate sopra una curva iperspaziale dalle sue ipersuperficie aggiunte e da tutte le ipersuperficie dell'iperspazio*	1908–09, 44	3
(15 November) *M. BOTTASSO, I caratteri di un piano multiplo ciclico la cui curva di diramazione è irriducibile e generale nel suo ordine*	1908–09, 44	3
1909		
(24 January) *M. BOTTASSO, Alcune singolarità elementari di un piano multiplo ciclico la cui curva di diramazione è irriducibile*	1908–09, 44	254
(21 February) *F. PALATINI, Sulle varietà algebriche per le quali sono di dimensione minore dell'ordinario, senza riempire lo spazio ambiente, una o alcune delle varietà formate da spazi seganti*	1908–09, 44	338
(7 March) *M. PANNELLI, Sul genere aritmetico di una varietà completa intersezione di forme*	1908–09, 44	398
(9 May) *G. FANO, Sulle varietà algebriche che sono intersezioni complete di più forme*	1908–09, 44	631
(9 May) *F. SIBIRANI, Su la rappresentazione approssimata delle funzioni di più variabili reali e delle loro derivate per polinomi trigonometrici*	1908–09, 44	631

	Volume	Page
(5 December) *G. SANNIA, Sull'inviluppata media di una congruenza di rette*	1909–10, 45	55
(19 December) *G. SCORZA, Sulle varietà di Segre*	1909–10, 45	118
1911		
(5 March) *C. H. SISAM, On Algebraic Hyperconical Connexes in Space of r Dimensions*	1910–11, 46	438
1912		
(14 January) *F. ENRIQUES, Sui moduli d'una classe di superficie algebriche e sul teorema d'esistenza per le funzioni algebriche di due variabili*	1911–12, 47	280
(14 April) *E. DANIELE, Sui centri di librazione in un problema più generale di quello ristretto dei tre corpi*	1911–12, 47	575
(16 June) *L. TONELLI, Sulla lunghezza di una curva*	1911–12, 47	980
(1° December) *G. SANNIA, Equazione differenziale delle congruenze W*	1912–13, 48	108
1913		
(26 January) *E. BOMPIANI, Sopra alcune estensioni dei teoremi di Meusnier e di Eulero*	1912–13, 48	374
(26 January) *A. TERRACINI, Sulle varietà di spazi con carattere di sviluppabili*	1912–13, 48	374
(30 March) *F. SEVERI, Sopra alcune proprietà aritmetiche delle corrispondenze fra i punti di una curva algebrica*	1912–13, 48	608
(16 November) *E. BOMPIANI, Sistemi di equazioni simultanee alle derivate parziali a caratteristica*	1913–14, 49	2
(14 December) *A. TERRACINI, Alcune questioni sugli spazi tangenti ed osculatori ad una varietà, Nota I*	1913–14, 49	177
(14 December) *C. POLI, Sugli integrali estesi al contorno di un campo qualunque*	1913–14, 49	177
1914		
(22 March), *S. CHERUBINO, Sulle curve iperellittiche con trasformazioni birazionali singolari in sé e sui loro moduli algebrici*	1913–14, 49	699
(5 April) *G. FUBINI, Definizione proiettivo-differenziale di una superficie*	1913–14, 49	756
(26 April) *R. TORELLI, Un criterio di equivalenza per le curve di una superficie algebrica*	1913–14, 49	824
1915		
(10 January) *G. FUBINI, Esiste un corpo pesante a densità sempre nulla?*	1914–15, 50	268
(7 February) *C. BONFERRONI, Sui sistemi lineari di quadriche la cui Jacobiana ha dimensione irregolare*	1914–15, 50	413
(7 February) *A. COMESSATTI, Sulle trasformazioni Hermitiane delle varietà di Jacobi*	1914–15, 50	413
(21 February) *F. CHELLI, Osservazioni del passaggio di Mercurio sul disco del Sole il 6-7 novembre 1914*	1914–15, 50	460
(7 March) *C. ROSATI, Sugli integrali abeliani riducibili*	1914–15, 50	588

	Volume	Page
(21 March) *E. LAURA, Sopra il problema della propagazione di moto all'esterno di una sfera in un mezzo elastico isotropo indefinito, Nota II*	1914–15, 50	726
(25 April) *G. BOCCARDI, Saggio sulla costante di aberrazione*	1914–15, 50	880
(9 May) *G. FANO, Osservazioni sopra alcune varietà non razionali aventi tutti i generi nulli*	1914–15, 50	1035
(28 November) *G. VITALI, I teoremi della media e di Rolle*	1915–16, 51	58
(12 December) *A. VERGERIO, Sull'equazione integrale di Fredholm di seconda specie*	1915–16, 51	171
1916		
(6 February) *G. BOCCARDI, Saggio sulla costante di aberrazione, Nota II*	1915–16, 51	500
(20 February) *A. TERRACINI, Sulla rappresentazione delle forme quaternarie mediante somme di potenze di forme lineari*	1915–16, 51	561
(5 March) *A. TERRACINI, Alcune questioni sugli spazi tangenti e osculatori ad una varietà, Nota II*	1915–16, 51	677
(16 April) *G. FUBINI, I teoremi di Bernstein e Pringsheim per lo sviluppo in serie di Taylor*	1915–16, 51	891
(16 April) *C. POLI, Un teorema d'esistenza per equazioni integrali non lineari*	1915–16, 51	891
(7 May) *C. ROSATI, Sulle corrispondenze plurivalenti fra i punti di una curva algebrica*	1915–16, 51	944
1917		
(25 February) *M. PICONE, Formole razionali per la correzione del tiro*	1916–17, 52	429
(15 April) *E. TOGLIATTI, Sui fasci di reciprocità degeneri tra spazi ad n dimensioni*	1916–17, 52	618
(29 April) *E. TOGLIATTI, Su alcune classi di sistemi lineari di reciprocità degeneri tra spazi ad n dimensioni*	1916–17, 52	735
(18 November) *C. ROSATI, Sulle valenze delle corrispondenze algebriche fra i punti di una curva algebrica*	1917–18, 53	4
(18 November) *G. COLONNETTI, Un singolare esempio di lesioni in un anello di acciaio temprato*	1917–18, 53	4
1918		
(12 May) *F. GERBALDI, Simmetria e periodicità nelle frazioni continue di Halphen, Nota I*	1917–18, 53	765
(26 May) *F. GERBALDI, Simmetria e periodicità nelle frazioni continue di Halphen, Nota II*	1917–18, 53	845
(16 June) *G. SCORZA, Sopra alcune notevoli matrici riemanniane*	1917–18, 53	1007
(16 June) *E. LAURA, Sopra i moti quasi-liberi di un fluido elastico*	1917–18, 53	1007
(16 June) *G. FUBINI, Fondamenti della Geometria proiettivo-differenziale di una superficie*	1917–18, 53	1007
(17 November) *G. FUBINI, I differenziali controvarianti*	1918–19, 54	4

	Volume	Page
1919		
(23 February) *C. RACITI, Sopra una classe di varietà abeliane a tre dimensioni*	1918–19, 54	442
(30 November) *M. PICONE, Sul cambiamento della variabile di integrazione nell'integrale di Lebesgue*	1919–20, 55	6
1920		
(11 April) *A. TERRACINI, Alcune questioni sugli spazi tangenti e osculatori a una varietà, Nota III*	1919–20, 55	478
1921		
(4 December) *E. TOGLIATTI, Sulle varietà a k dimensioni contenenti k rette, Nota I*	1921–22, 57	48
(4 December) *G. GHERARDELLI, Intorno ad un teorema di Picard relativo alle curve di un complesso lineare di rette*	1921–22, 57	48
1922		
(25 June) *M. LOMBARDINI, Considerazioni geometriche per l'analisi periodale*	1921–22, 57	574
1923		
(28 January), B. SEGRE, *Genere della curva doppia per la varietà di* S_4 *che annulla un determinante simmetrico*	1922–23, 58	161
1924		
(20 January) *A. TERRACINI, Sui punti di flesso delle quartiche piane generali*	1923–24, 59	167
(27 April) *E. BOMPIANI, Determinazioni proiettive, differenziali, relative ad una superficie dello spazio ordinario*	1923–24, 59	372
(27 April) *A. TERRACINI, Sulle superficie con un sistema di asintotiche in complessi lineari*	1923–24, 59	372

References

Amodeo, Federico, Quali possono essere i postulati fondamentali della Geometria proiettiva di uno S_r, *Atti della R. Accademia delle Scienze di Torino*, 26 (1890–1891): 741–770.

Amodeo, Federico, Serie residue nella serie canonica delle curve aggiunte di ordine m - 3 - α, *Atti della R. Accademia Nazionale dei Lincei. Rendiconti*, 5, 2 (1893): 528–532.

Amodeo, Federico, *Sintesi storico-critica della geometria delle curve algebriche*, Napoli: Conte, 1945.

Arrighi, Gino, *Lettere a Mario Pieri (1884–1913)*, Quaderni Pristem 6, Milano: Università Bocconi, 1997.

Avellone, Maurizio, Brigaglia, Aldo and Zappulla, Carmela, The Foundations of Projective Geometry in Italy from De Paolis to Pieri, *Archive for History of Exact Sciences* 56, 5 (2002): 363–425.

Babbit, Donald, and Goodstein, Judith, Guido Castelnuovo and Francesco Severi: Two Personalities, Two Letters, *Notices of the AMS*, 56 (7), (2009): 800–808.

Babbit, Donald and Goodstein Judith, Lettere e controversie, *Sapere*, 77.4 (2011): 66–73, Italian translation of Babbitt, Goodstein (2009), with a few minor changes.

Baker, Henry Frederick, On Some Recent Advances in the Theory of Algebraic Surfaces. *Proceedings of the London Mathematical Society,* (2) 12 (1913): 1–40.
Baker, Henry Frederick, Corrado Segre, *Journal of the London Mathematical Society*, 1 (1926): 263–271. Italian translation by G. Loria, *Bollettino della Unione Matematica Italiana*, 6 (1927): 276–284.
Bertini, Eugenio, La geometria delle serie lineari sopra una curva piana secondo il metodo algebrico, *Annali di matematica pura ed applicata*, 2, 22 (1894): 1–40.
Bertini, Eugenio, *Introduzione alla geometria proiettiva degli iperspazi*, *con appendice sulle curve algebriche e loro singolarità*, Pisa: Spoerri, 1907.
Berzolari, Luigi, Corrado Segre, *R. Istituto Lombardo di Scienze e Lettere. Rendiconti*, s. 2, 57 (1924): 528–532.
Boffi, Giandomenico, On Some Trends in the Italian Geometric School in the Second Half of the 19th Century, *Rivista di Storia della Scienza*, 3 (1986): 103–112.
Boggio, Tommaso, Nel 4° anniversario della morte di Corrado Segre, *Atti della R. Accademia delle Scienze di Torino*, 63 (1927–1928): 303–320.
Bompiani, Enrico, Italian Contributions to Modern Mathematics, *The American Mathematical Monthly*, 38 (1931): 83–95.
Bompiani, Enrico, Moderni indirizzi di geometria differenziale, *Atti del primo Congresso della Unione Matematica Italiana, tenuto in Firenze nei giorni 1-2-3 aprile 1937–XV*, Bologna: Zanichelli, 1938, 88–108.
Bompiani, Enrico, Alessandro Terracini, in *Accademia Nazionale dei Lincei. Celebrazioni Lincee*. 36, Roma: Accademia Nazionale dei Lincei, 1970, 3–22.
Bonola, Roberto, *La geometria non euclidea*. *Esposizione storico-critica del suo sviluppo con 69 figure*, Bologna: Nicola Zanichelli, 1906.
Bottazzini, Umberto and Dahan Dalmedico, Amy (eds.), *Changing Images of Mathematics. From the French Revolution to the New Millenium*, London: Routledge, 2001.
Bottazzini, Umberto, Conte, Alberto and Gario, Paola (eds.), *Riposte armonie. Lettere di Federigo Enriques a Guido Castelnuovo*, Torino: Bollati Boringhieri, 1996.
Bottazzini, Umberto, Conte, Alberto and Gario, Paola, La relazione di Castelnuovo ed Enriques. Documenti inediti per il Premio reale di matematica del 1901, Supplemento. *Rendiconti del Circolo Matematico di Palermo*, s. 2, 55 (1998): 75–156.
Brechenmacher, Frédéric, *Histoire du théorème de Jordan de la décomposition matricielle (1870–1930): formes de représentations et méthodes de décompositions, Thèse de doctorat en Histoire, sous la direction de Jean Dhombres*, 2006 in: https://tel.archives-ouvertes.fr/tel-00142786/file/these_brechenmacher.pdf.
Brigaglia, Aldo, La geometria algebrica italiana di fronte al 15° problema di Hilbert, *Atti del Convegno La storia delle matematiche in Italia*, *Cagliari 1982,* 1982, 469–481.
Brigaglia, Aldo, The Creation and Persistence of National Schools: the Case of Italian Algebraic Geometry, in *Changing Images of Mathematics. From the French Revolution to the New Millenium*, Bottazzini, Umberto and Dahan Dalmedico, Amy (eds.), London: Routledge, 2001, 187–206.
Brigaglia, Aldo, Mario Pieri e la Scuola di Corrado Segre, *Quaderni di Storia dell'Università di Torino*, 10 (2009–2011): 19–34.
Brigaglia, Aldo, Saccheri vu par Corrado Segre en Italie et par Mansion et Bosmans en Belgique, *Bulletin de la Classes des Sciences*, *Académie Royale de Belgique*, s. 6, 21 (2010): 83–104.
Brigaglia, Aldo, Per una biografia scientifica di Corrado Segre, *La Matematica nella Società e nella Cultura*, 1, 6 (2013): 415–474.
Brigaglia, Aldo and Ciliberto, Ciro, *Italian Algebraic Geometry between the two World Wars,* Queen's Papers in Pure and Applied Mathematics, vol. 100, Kingston, Ontario: Queen's University, 1995.
Brigaglia, Aldo and Ciliberto, Ciro, Remarks on the relations between the Italian and American schools of algebraic geometry in the first decades of the 20th century, *Historia Mathematica* 31 (2004): 310–319.

Brill, Alexander and Noether, Max, Bericht, Die Entwicklung der Theorie der algebraischen Functionen in älterer und neuer Zeit, *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 3 (1894): 107–566.

Castelnuovo, Guido, Ricerche generali sopra i sistemi lineari di curve piane, *Memorie dell'Accademia delle Scienze di Torino*, (2), 42 (1892): 3–43.

Castelnuovo, Guido, [Cenni necrologici sul socio C. Segre], *Atti della R. Accademia dei Lincei, Rendiconti*, s. 5, 33 (1924a): 460.

Castelnuovo, Guido, [Commemorazione del socio C. Segre], *Atti della R. Accademia dei Lincei, Rendiconti*, s. 5, 33.2 (1924b): 353–359.

Castelnuovo, Guido, La geometria algebrica e la scuola italiana, *Atti del Congresso internazionale dei matematici*, *Bologna 3-10 settembre 1928*, Bologna: Zanichelli, I, 1929, 191–201.

Castelnuovo, Guido and Enriques, Federigo, Sur quelques récent résultats dans la théorie des surfaces algébriques, *Mathematische Annalen,* 48 (1896): 241–316.

Castelnuovo, Guido and Enriques, Federigo, Sur quelques résultats nouveaux dans la théorie des surfaces algébriques, in Picard, Emile and Simart, Georges (eds.), *Théorie des fonctions algébriques de deux variables indépendantes*, Paris: Gauthier-Villars, t. 2, 1906, 485–522.

Chisholm Young, Grace, Sulla varietà razionale normale di M_3^4 di S_6 rappresentante della trigonometria sferica, *Atti della R. Accademia delle Scienze di Torino*, 34 (1898–99): 587–596.

Ciliberto, Ciro and Sallent Del Colombo, Emma, Enrico Bompiani: the Years in Bologna, in S. Coen (ed.) *Mathematicians in Bologna 1861–1960*, Basel: Springer 2012, 143–177.

Ciliberto, Ciro and Sallent Del Colombo, Emma, Pasquale del Pezzo, Duke of Caianello, Neapolitan mathematician, *Archives for History of Exact Sciences* 67 (2013): 171–214.

Cogliati, Alberto, Questioni di stile, in *Conferenze e Seminari 2014–2015*, ed. by F. Ferrara, L. Giacardi and M. Mosca, Associazione Subalpina Mathesis, Torino: Kim Williams Book, 2015, 81–92.

Collino, Alberto, Conte, Alberto and Verra, Alessandro, On the life and scientific work of Gino Fano, *La Matematica nella Società e nella Cultura,* s.1, VII (2014): 99–137. Also in http://www.bdim.eu/item?fmt=pdf&id=GM_Fano_P_1.

Conte, Alberto, La scoperta e i primi tentativi di classificazione delle superficie di Enriques nella corrispondenza inedita di Federigo Enriques a Guido Castelnuovo, in Ciliberto C. and Sernesi E. (eds.), *Algebra e geometria (1860–1940): il contributo italiano*, *Rendiconti del Circolo matematico di Palermo.* Supplemento, s. 2, 36 (1994): 103–116.

Conte, Alberto and Ciliberto, Ciro, La Scuola di geometria algebrica italiana, in *Storia della Scienza*, Roma: Istituto della Enciclopedia Italiana, vol. VIII, 2004, 92–104.

Conte, Alberto, Giacardi, Livia and Novaria, Paola, *Corrado Segre (1863–1924). A 150 anni dalla nascita, Catalogo delle Mostre documentarie, Torino, Novembre 2013,* Torino: Kim Williams Books, 2013.

Coolidge, Julian Lowell, Les congruences isotropes qui servent à représenter les fonctions d'une variable complexe, Nota I, *Atti della R. Accademia delle Scienze di Torino*, 39 (1903–04): 175–186.

Coolidge, Julian Lowell, The opportunities for mathematical study in Italy, *Bulletin of the American Mathematical Society*, 11 (1904): 9–17.

Coolidge, Julian Lowell, Les congruences isotropes qui servent à représenter les fonctions d'une variable complexe, Nota II, *Atti della R. Accademia delle Scienze di Torino*, 40 (1904–05): 202–218.

Coolidge, Julian Lowell, *The Elements of Non-Euclidean Geometry*, Oxford: Oxford University Press, 1909.

Coolidge, Julian Lowell, *The Geometry of the Complex Domain*, Oxford: Clarendon Press, 1924.

Coolidge, Julian Lowell, Corrado Segre, *Bulletin of the American Mathematical Society*, 33, (1927): 352–357.

Dieudonné, Jean, The beginnings of Italian algebraic geometry, in Symposia Mathematica, XXVII, London and New York: Academic Press, 1986, 245–263.

Enriques, Federigo, Le superficie con infinite trasformazioni proiettive in sé stesse, *Atti del R. Istituto Veneto di Scienze, Lettere ed Arti*. s. 7 51 (1893a): 1590–1635.

Enriques, Federico, Ricerche di geometria sulle superficie algebriche, *Memorie della R. Accademia delle Scienze di Torino*, s. 2 44 (1893b): 171–232.
Enriques, Federigo, Sui fondamenti della geometria proiettiva, *Rendiconti dell'Istituto Lombardo di scienze, lettere ed arti*, s. 2 XXVII (1894): 550–567.
Enriques, Federigo, *Conferenze di Geometria*, Bologna: Litografia, 1894–95.
Enriques, Federigo, La evolución del concepto de la geometria y la escuela italiana durante los últimos cincuenta años, *Revista matemática Hispano-Americana*, II, n. 1–2 (1920): 1–17.
Enriques, Federigo, *Per la storia della Logica,* Bologna: Zanichelli, 1922.
Enriques, Federigo, *Le matematiche nella storia e nella cultura*, Bologna: Zanichelli 1938.
Enriques, Federigo and Chisini, Oscar, *Lezioni sulla teoria geometrica delle equazioni e delle funzioni algebriche*, Bologna: Zanichelli, vols. I–IV, 1915–1934.
Enriques, Federigo and Fano, Gino, Geometria, in *Enciclopedia Italiana di Scienze, Lettere ed Arti*, vol. XVI: 623–637.
Fano n.d. [*Appunti vari*], BMP, *Fondo Fano*, Scritti 4, available at http://www.corradosegre.unito.it/fondofano/scritti4.pdf.
Fano, Gino, Sui postulati fondamentali della geometria proiettiva in uno spazio lineare a un numero qualunque di dimensioni, *Giornale di Matematiche*, 30 (1892): 106–132.
Fano, Gino, Sopra le curve di dato ordine e dei massimi generi in uno spazio qualunque, *Memorie R. Accademia delle Scienze di Torino*, s. 2 44 (1894a): 335–382.
Fano, Gino, Sull'insegnamento della matematica nelle Università tedesche e in particolare nell'Università di Gottinga*, Rivista di Matematica*, 4 (1894b): 170–188.
Fano, Gino, Sui postulati fondamentali della geometria proiettiva (Due lettere al Prof. Enriques)*, Rendiconti del Circolo matematico di Palermo*, 9 (1895): 79–82 and 84–85.
Fano, Gino, Sulle varietà algebriche con un gruppo continuo non integrabile di trasformazioni proiettive in sé, *Memorie della R. Accademia delle Scienze di Torino*, s. 2, 46 (1896): 187–218.
Fano, Gino, Gengensatz von synthetischer und analytischer Geometrie in seiner historischen Entwicklung im XIX Jahrhundert, *Encyklopädie der mathematischen Wissenschaften*, III.1, 1907a, 221–288.
Fano, Gino, Kontinuierliche geometrische Gruppen. Die Gruppentheorie als geometrisches Einteilungsprinzi*p, Encyklopädie der mathematischen Wissenschafte* III.1, 1907b, 289–388.
Fano, Gino, *A preface to a series of special lectures on Italian Geometry, and 2 general lectures. Intuition in mathematics. All geometry is theory of Relativity*, Aberystwith: University of Aberystwith, 1923.
Fano, Gino, Corrado Segre (1863–1924), *Annuario della R. Università di Torino*, (1924–25a): 219–228.
Fano, Gino, [*Geometria superiore*], handwritten notebook, BMP-Fano, 1924–25b.
Fano, Gino, Discorso commemorativo del Prof. Gino Fano, in Onoranze a Corrado Segre, *Rendiconti del Circolo matematico di Palermo*, Supplemento 15 (A. 1926–28) 1930: 42–50.
Fuà Segre, Elena, Un grande geometra ebreo: Corrado Segre, *La rassegna mensile di Israel*, 18: 125–127.
Furinghetti, Fulvia and Giacardi, Livia, Training of secondary school teachers in pre- and post-unity Italy (1810–1920), *ZDM—The International Journal on Mathematics Education*, 44, 4 (2012): 537–550.
Galuzzi, Massimo, Geometria algebrica e logica tra Otto e Novecento, in *Storia d'Italia, Annali 3,* Gianni Micheli (ed.), Torino: Einaudi, 1980: 1001–1105.
Gario, Paola, Su alcune carte di Corrado Segre recentemente rinvenute, *Atti dell'Accademia delle Scienze di Torino*, 123 (1989a): 187–198.
Gario, Paola, Resolution of singularities of surfaces by P. Del Pezzo. A mathematical controversy with C. Segre, *Archive for History of Exact Sciences,* 40 (1989b): 247–274.
Gario, Paola, Singolarità e geometria sopra una superficie nella corrispondenza di C. Segre a G. Castelnuovo, *Archive for History of Exact Sciences*, 43 (1991): 145–188.
Gario, Paola, Enriques e le comunità matematiche europee tra Otto e Novecento, *Seminari di geometria 1996–1997,* (1997): 279–306.

Gario, Paola (ed.), *Lettere e Quaderni dell'Archivio di Guido Castelnuovo*, http://operedigitali.lincei.it/Castelnuovo/Lettere_E_Quaderni/menu.htm (2010).

Ghione, Franco and Ottaviani, Giorgio, A tribute to Corrado Segre, in G. Ellingsrud et al. (eds.), *Complex projective geometry*, London Mathematical Society. *Lecture Note Series* 179 (1992): 175–188.

Giacardi, Livia, Corrado Segre maestro a Torino. La nascita della scuola italiana di geometria algebrica, *Annali di storia delle università italiane*, 5 (2001a): 139–163: http://www.cisui.unibo.it/annali/05/testi/09Giacardi_frameset.htm.

Giacardi, Livia, Educare alla scoperta. Le lezioni di Corrado Segre alla Scuola di Magistero, *Bollettino della Unione Matematica Italiana. La Matematica nella società e nella cultura,* (8), VI-A (2003): 141–164.

Giacardi, Livia, Testimonianze sulla Scuola italiana di geometria algebrica nei fondi manoscritti della Biblioteca "Giuseppe Peano" di Torino, in S. Montaldo, P. Novaria (eds.), *Gli Archivi della scienza. L'Università di Torino e altri casi italiani*, Milano: Angeli, 2011, 105–119.

Giacardi, Livia, (ed.), *Corrado Segre e la Scuola Italiana di Geometria Algebrica*: http://www.corradosegre.unito.it/, 2013. The website reproduces (with some integrations) the Cd-Rom Livia Giacardi (ed.) 2002, *I Quaderni di Corrado Segre,* CD-ROM, Dipartimento di Matematica, Università di Torino.

Giacardi, Livia and Rinaldelli, Lucia, I Fondi Fano e Terracini della Biblioteca Speciale di Matematica "Giuseppe Peano" di Torino*, Quaderni di Storia dell'Università di Torino,* 4 (2000): 381–413.

Giacardi, Livia and Roero, C. Silvia, La Biblioteca speciale di matematica 'Giuseppe Peano' in C. S. Roero (ed.), *La Facoltà di Scienze Matematiche Fisiche Naturali di Torino, 1848–1998,* 2 vols., Torino: Deputazione Subalpina di Storia Patria, 1999, vol. I, 437–458.

Giacardi, Livia and Tealdi, Alice, Francesco Severi and mathematics teaching in secondary schools. Science, politics and schools in the first half of the twentieth century, in *"Dig where you stand" 3. Proceedings of the Third International Conference on the History of Mathematics Education*, K. Bjarnadóttir, F. Furinghetti, J. Prytz and G. Schubring (eds.), Uppsala: Uppsala University, 2015, 187–202.

Giacardi, Livia and Varetto, Tiziana, Il Fondo Corrado Segre della Biblioteca 'G. Peano' di Torino, *Quaderni di Storia dell'Università di Torino* 1 (1996): 337–370.

Giambelli, Giovanni Zeno, Risoluzione del problema degli spazi secanti, *Memorie della R Accademia delle Scienze di Torino* (2) (1903): 171–211.

Godeaux Lucien, Hommage à Corrado Segre, *Rendiconti del Seminario Matematico Università e Politecnico di Torino*, 23 (1963–64): 23–24.

Grattan-Guinness, Ivor, A Mathematical Union: William Henry and Grace Chisholm Young, *Annals of Science* 29 (1972): 105–186.

Grattan-Guinness, Ivor, Un'unione matematica. William Henry Young e Grace Chisholm, in: *Scienza a due voci*, Raffaella Simili (ed.), Firenze: Olschki, 2006, 217–246.

Gray, Jeremy, German and Italian Algebraic Geometry, in Brigaglia A., Ciliberto C., Sernesi E. (eds.), *Algebra e geometria (1860–1940): il contributo italiano*, *Rendiconti del Circolo matematico di Palermo*. Supplemento, s. 2, 36 (1994): 151–183.

Hawkins, Thomas, *Lie groups and geometry: the Italian connection*, in Brigaglia A., Ciliberto C., Sernesi E. (eds.), *Algebra e geometria (1860–1940): il contributo italiano*, *Rendiconti del Circolo matematico di Palermo.* Supplemento, s. 2, 36 (1994): 185–206.

Hawkins, Thomas, *The emergence of the theory of Lie groups. An essay in the history of mathematics 1869–1926*, New York: Springer, 2000.

Hawkins, Thomas, *The Mathematics of Frobenius in Context. A Journey Through 18th to 20th Century Mathematics*, New York: Springer, 2013.

Kleiman, Steven, Bertini and his two fundamental theorems, *Rendiconti del Circolo matematico di Palermo*. Supplemento, s. 2, 55 (1998): 9–37.

Klein, Felix, Considerazioni comparative intorno a ricerche geometriche recenti (Translated by Gino Fano), *Annali di matematica pura ed applicata*, s. 2, 17 (1890): 307–343.

Laksov, Dan, Remarks on Giovanni Zeno Giambelli's work and life, in A. Brigaglia, C. Ciliberto, E. Sernesi (eds.), *Algebra e geometria (1860–1940): il contributo italiano*, *Rendiconti del Circolo matematico di Palermo.* Supplemento, s. 2, 36 (1994): 207–218.

Lane, Ernest, *Projective Differential Geometry of Curves and Surfaces*, Chicago: The University Chicago Press, 1932.

Levi, Beppo, Risoluzione delle singolarità puntuali delle superficie algebriche, *Atti della R. Accademia delle Scienze di Torino*, 33 (1897–98): 66–86.

Lie, Sophus. *Theorie der transformationsgruppen. Unter Mitwirkung von ... Friedrich Engel,* 3 vols, Teubner: Leipzig, 1888, 1890, 1893.

Lie, Sophus, *Vorlesungen über continuirliche Gruppen mit geometrischen und anderen Anwendungen. Bearbeitet und herausgegeben von Dr. Georg Scheffers*, Leipzig: Teubner, 1893.

Loria, Gino, L'opera geometrica di Corrado Segre, *Annali di Matematica pura ed applicata,* s. 4, 2 (1924): 1–21.

Luciano, Erika, At the Origins of Functional Analysis: G. Peano and M. Gramegna on Ordinary Differential Equations, *Revue d'histoire des mathématiques* 12 (2006): 35–79.

Luciano, Erika and Roero, C. Silvia, La Scuola di Giuseppe Peano, in C. S. Roero (ed.), *Peano e la sua Scuola fra matematica, logica e interlingua*, Torino: Deputazione Subalpina di Storia Patria, 2010.

Luciano, Erika and Roero C. Silvia, From Turin to Göttingen: Dialogues and Correspondence (1879–1923), *Bollettino di Storia delle Scienze Matematiche*, 32, 1 (2012): 7–232.

Manara, Carlo Felice and Spoglianti, Maria, La idea di iperspazio. Una dimenticata polemica tra G. Peano, C. Segre e G. Veronese, *Memorie dell'Accademia Nazionale di Scienze, Lettere ed Arti di Modena*, 19 (1977):109–129.

Mancosu, Paolo, *Mathematical Style*, Stanford Encyclopedia of Philosophy, 2009: http://plato.stanford.edu/entries/mathematical-style/.

Marchisotto, Elena Anne and Smith, James, *The Legacy of Mario Pieri in Geometry and Arithmetic*, Boston: Birkhäser, 2007.

Menghini, Marta, Sul ruolo di Corrado Segre nello sviluppo della geometria algebrica italiana, *Rivista di Storia della scienza*, 3(3) (1986): 303–322.

Mumford, David, Intuition and rigor and Enriques's quest, *Notices of the AMS*, 58 (2), (2011): 250–260.

Murre, Jacob P., On the work of Gino Fano on three-dimensional algebraic varieties, in A. Brigaglia, C. Ciliberto, E. Sernesi (eds.), *Algebra e geometria (1860–1940): il contributo italiano*, *Rendiconti del Circolo matematico di Palermo*. Supplemento, s. 2, 36 (1994): 219–229.

Nastasi, Pietro and Tazzioli, Rossana, I matematici italiani e l'internazionalismo scientifico (1914–1924), *La Matematica nella Società e nella Cultura, Rivista della Unione Matematica Italiana* (1) VI (2013): 355–405.

Noether, Max, Poincaré, Henri and Segre, Corrado, Relazione del concorso internazionale per la «Medaglia Guccia», in Castelnuovo, Guido (ed.), *Atti del IV Congresso Internazionale dei Matematici,* I, Roma: Accademia dei Lincei, 1909, 209–216.

Onoranze a Corrado Segre, *Supplemento ai Rendiconti del Circolo matematico di Palermo*, XV (1926–1928) 1930: 40–73.

Palladino, Franco and Palladino, Nicla, *Dalla 'moderna geometria' alla 'nuova geometria italiana'. Viaggiando per Napoli, Torino e dintorni*, Firenze: Olschki, 2006.

Palladino, Franco and Tazzioli, Rossana, Le lettere di Eugenio Beltrami nella corrispondenza di Ernesto Cesaro, *Archive for History of Exact Sciences* 49 (1996): 321–353.

Paoloni, Giovanni, *Il Fondo Enrico Bompiani*, Quaderni PRISTEM 2, Milano: Università Bocconi, 1991.

Parshall, Karen Hunger and Rice, Adrian, *Mathematics Unbound: the Evolution of an International Mathematical Research Community 1800–1945*, American Mathematical Society, 2002.

Pascal, Ernesto, Personale accademico, *Atti della R. Accademia Nazionale dei Lincei. Rendiconti*, 5, 33 (1924): 459–461

Pieri, Mario, Un sistema di postulati per la geometria projettiva astratta degli iperspazii, *Rivista di matematica* (1896): 9–16.

Pieri, Mario, I principii della Geometria di Posizione composti in sistema logico-deduttivo, *Memorie R. Accademia delle Scienze di Torino*, 48 (1899): 1–62.

Pieri, Mario, Nuovi principii di geometria projettiva complessa, *Memorie della R. Accademia delle Scienze di Torino*, 55 (1905): 189–235.

Pompeo Faracovi, Ornella (ed.) *Enriques e Severi matematici a confronto nella cultura del Novecento*. Centro Studi Enriques. La Spezia: Agorà Edizioni, 2004 (with essays of Edoardo Vesentini, Ciro Ciliberto, Aldo Brigaglia, Giorgio Bolondi, Ornella Pompeo Faracovi, Sandra Linguerri, Giorgio Israel, Paolo Bussotti).

Proceedings of the Fifth International Congress of Mathematicians (Cambridge, 22–28 August 1912), E. W. Hobson, A. E. H. Love (eds), Cambridge: at the University Press, 1913.

Ramorino, Angelo, Gli elementi immaginari nella Geometria. Monografia storica, *Giornale di Matematiche*, 35 (1897): 242–258, 317–345.

Raspanti, Marianna, *The diffusion of Sophus Lie's theory of continuous transformation groups in Italy. The role of Corrado Segre and his followers*, Trento 2014: http://www.science.unitn.it/cirm/Abstracts-MIRIPC2014.pdf.

Relazione sul concorso al Premio Reale per la Matematica, pel 1895*, Atti della R. Accademia dei Lincei, Rendiconti delle sedute solenni*, 1 (1892–1901) 1901: 354–374.

Roero, C. Silvia, Giuseppe Peano, geniale matematico, amorevole maestro, in *Maestri dell'Ateneo torinese dal Settecento al Novecento*, (ed. R. Allio), Torino: Centro Studi di Storia dell'Università di Torino, 2004, 115–144.

Rogora, Enrico, Developement of the theory of Lie groups in Bologna (1884–1900), in S. Coen (ed.), *Mathematicians in Bologna 1861–1960*, 2012, Basel: Springer, 415–426.

Rogora, Enrico, *Lie e i matematici italiani e l'intuizione geometrica*, in XIV Congresso della Società Italiana di Storia delle Matematiche: http://www.sism.unito.it/node/congressi/palermo/sunti.pdf.

Roth, Leonard, Francesco Severi, *Journal of the London Mathematical Society*, 38 (1963): 282–307.

Rowe, David, Making mathematics in an Oral Culture: Göttingen in the Era of Klein and Hilbert, *Science in Context*, 17 (2004): 85–129.

Schappacher, Norbert, Remarks about Intuition in Italian Algebraic Geometry, in *History of Mathematics: Models and Visualization in the Mathematical and Physical Sciences*, Oberwolfach Report no. 47 (2015): 2805–2807.

Scorza, Gaetano, La scuola geometrica italiana, in *L'Italia e la scienza*, G. Bargagli Petrucci (ed.), Firenze: Le Monnier, 1932, 130–135.

Segre, Beniamino, Genere della curva doppia per la varietà di S_4 che annulla un determinante simmetrico, *Atti della R. Accademia delle Scienze di Torino*, 58 (1922–23): 162–170.

Segre, Beniamino, La geometria in Italia, dal Cremona ai giorni nostri, *Annali di Matematica pura ed applicata,* s.4, 11 (1933): 1–16.

Segre, Beniamino, Prefazione, in *Corrado Segre, Opere*, vol. III, Roma: Cremonese, 1961, V–IX.

Segre, Beniamino, Nel primo centenario della nascita di Corrado Segre, *Rendiconti del Seminario Matematico, Università e Politecnico di Torino*, 23 (1963–64): 7–21.

Segre, Corrado. Studio sulle quadriche in uno spazio lineare ad un numero qualunque di dimensioni, *Memorie della R. Accademia delle Scienze di Torino*, 2, 36 (1883b): 3–86. (*Opere*, 3, 25–126).

Segre, Corrado. Sulla geometria della retta e delle sue serie quadratiche, *Memorie della R. Accademia delle Scienze di Torino*, 2, 36 (1883c): 87–157. (*Opere*, 3, 127–217).

Segre, Corrado, Remarques sur les transformations uniformes des courbes elliptiques en elles-mêmes, *Mathematische Annalen*, 28 (1886a) : 296–314. (*Opere*, 1, 36–55)

Segre, Corrado, Recherches générales sur les courbes et les surfaces réglées algébriques (I partie, Courbes algébriques), *Mathematische Annalen*, 30 (1887b) : 203–226. (*Opere*, 1, 80–104)

Segre, Corrado, Sui sistemi lineari di curve piane algebriche di genere p (Estratto di lettera al dott. G.B. Guccia), *Rendiconti del Circolo Matematico di Palermo*, 1 (1887d): 217–221. (*Opere*, 1, 105–109)

Segre, Corrado, Recherches générales sur les courbes et les surfaces réglées algébriques (II partie, Surfaces réglées algébriques), *Mathematische Annalen*, 34 (1889a): pp. 1–25. (*Opere*, 1, 125—151)

Segre, Corrado, Sulle varietà cubiche dello spazio a quattro dimensioni e su certi sistemi di rette e certe superficie dello spazio ordinario, *Memorie della R. Accademia delle Scienze di Torino*, 2, 39 (1889c): 3–48. (*Opere*, 4, 99–159)

Segre, Corrado. Un nuovo campo di ricerche geometriche, Note I-III, *Atti della R. Accademia delle Scienze di Torino*, 25 (1889–90): 180–205; 290–317; 376–396. (*Opere*, 2, 237–303).

Segre, Corrado. Un nuovo campo di ricerche geometriche, Nota IV, *Atti della R. Accademia delle Scienze di Torino*, 26 (1890–91): 35–71. (*Opere*, 2, 303–337).

Segre, Corrado, Su alcuni indirizzi nelle investigazioni geometriche. Osservazioni dirette ai miei studenti, *Rivista di Matematica*, 1 (1891a): 42–66. (*Opere*, 4, 387–412).

Segre, Corrado. Sulle varietà che rappresentano le coppie di punti di due piani o spazi, *Rendiconti del Circolo Matematico di Palermo*, 5 (1891c): 192–204. (*Opere*, 1, 173–184).

Segre, Corrado, Le rappresentazioni reali delle forme complesse e gli enti iperalgebrici, *Mathematische Annalen*, 40 (1891d): 413–467. (*Opere*, 2, 338–395)

Segre, Corrado, Cenni del Socio Corrado Segre su "Die Entwickelung der Theorie der algebraischen Funktionen in älterer und neuerer Zeit", von A. Brill und M. Noether, *Atti della R. Accademia delle Scienze di Torino*, 30 (1894–95): 109–111. (*Opere*, 4, 438–440)

Segre, Corrado, Introduzione alla geometria sopra un ente algebrico semplicemente infinito, *Annali di Matematica pura ed applicata*, 2, 22 (1894a): 41–142. (*Opere*, 1, 198–304)

Segre, Corrado. Intorno ad un carattere delle superficie e delle varietà superiori algebriche, *Atti della R. Accademia delle Scienze di Torino*, 31 (1895–96): 485–501. (*Opere*, 1, 312–326).

Segre, Corrado. Sulla scomposizione dei punti singolari delle superficie algebriche, *Annali di Matematica pura ed applicata*, 2, 25 (1897a): 2–54. (*Opere*, 1, 327–379).

Segre, Corrado, Su alcuni punti singolari delle curve algebriche, e sulla linea parabolica di una superficie, *Atti della R. Accademia Nazionale dei Lincei. Rendiconti*, 5, 6 (1897b): 168–175. (*Opere*, 2, 1–8)

Segre, Corrado, Su un problema relativo alle intersezioni di curve e superficie, *Atti della R. Accademia delle Scienze di Torino*, 33 (1897–98): 19–23.

Segre, Corrado, Le molteplicità nelle intersezioni delle curve piane algebriche con alcune applicazioni ai principii della teoria di tali curve, *Giornale di Matematiche*, 36 (1898): 1–50. (*Opere*, 1, 380–429)

Segre, Corrado, Gli ordini delle varietà che annullano i determinanti dei diversi gradi estratti da una data matrice, *Atti della R. Accademia Nazionale dei Lincei. Rendiconti*, 5, 9 (1900): 253–260. (*Opere*, 4, 179–187)

Segre, Corrado, Congetture intorno all'influenza di Girolamo Saccheri sulla formazione della geometria non-Euclidea, *Atti della R. Accademia delle Scienze di Torino*, 38 (1902–03): 535–547. (*Opere*, 4, 444–455)

Segre, Corrado. On some tendencies in geometric investigations, *Bulletin of the American Mathematical Society*, 2, 10 (1904): 442–468. English translation by John Wesley Young of Segre (1891a).

Segre, Corrado. La Geometria d'oggidì e i suoi legami coll'analisi, in *Verhandlungen des dritten internationalen Mathematiker-Kongresses in Heidelberg vom* 8 bis 13 August 1904, pp. 109–120; also in *Rendiconti del Circolo Matematico di Palermo*, 19 (1905a): 81–93. (*Opere*, 4, 456–468).

Segre, Corrado. Geometrya dzisiejsza i jej związki z Analizą, *Wiadomości Matematyczne*, Warszawa, 9 (1905b): 7–21. Polish translation by Samuel Dickstein of Segre (1905a).

Segre, Corrado, Le congruenze rettilinee W aderenti a due superficie rigate, *Atti della R. Accademia delle Scienze di Torino*, 42 (1906–07a): 539–550. (*Opere*, 2, 9–19)

Segre, Corrado, Su una classe di superficie degl'iperspazi, legate con le equazioni lineari alle derivate parziali di 2° ordine, *Atti della R. Accademia delle Scienze di Torino*, 42 (1906–07b): 1047–1079. (*Opere*, 2, 20–49)

Segre, Corrado, Complementi alla teoria delle tangenti coniugate di una superficie, *Atti della R. Accademia Nazionale dei Lincei. Rendiconti*, 5, 17 (1908): 405–412. (*Opere*, 2, 62–70)

Segre, Corrado. Preliminari di una teoria delle varietà luoghi di spazi, *Rendiconti del Circolo Matematico di Palermo*, 30 (1910a): 87–121. (*Opere*, 2, 71–114).

Segre, Corrado, Aggiunta alla memoria: Preliminari di una teoria delle varietà luoghi di spazi, *Rendiconti del Circolo Matematico di Palermo*, 30 (1910b): 346–348. (*Opere*, 2, 115–118)

Segre, Corrado. Discorso inaugurale. Le previsioni, *R. Università di Torino* 1918–1919. *Inaugurazione dell'anno accademico. Discorsi e dati statistici*, Torino: Stamperia Reale, 1918–19, pp. 11–25.

Segre, Corrado. Mehrdimensionale Räume, in *Encyklopädie der Mathematischen Wissenschaften*, III.2.2 C 7, 1921c , pp. 769–972.

Segre, Corrado, *Opere*, Roma: Ed. Cremonese, 4 vols., 1957–1963, http://www.bdim.eu/item?id=GM_Segre

Severi, Francesco, Sopra alcune singolarità delle curve di un iperspazio, *Memorie della R. Accademia delle Scienze di Torino*, s. 2, 50 (1901): 81–114.

Severi, Francesco, Sulle superficie algebriche che posseggono integrali di Picard della 2[a] specie, *Atti della R. Accademia Nazionale dei Lincei. Rendiconti,* s. 5, 13 (1904): 253–258.

Severi, Francesco, *Trattato di Geometria algebrica*, Bologna: Zanichelli, 1926.

Severi, Francesco, La Géométrie Algébrique, in *Proceedings of the International Mathematical Congress held in Toronto, August 11–16, 1924*, vol. I, Toronto: The University of Toronto Press, 1928, 149–154.

Severi, Francesco, La géométrie algébrique italienne: sa rigueur, ses méthodes, ses problèmes, in *Centre Belge de Recherches Mathématiques, Colloque de Géométrie algébrique. Tenu à Lièges les 19, 20 et 21 décembre 1949*, Liège: George Thone, 1950, 9–55.

Severi, Francesco, Prefazione, in *Corrado Segre, Opere*, Roma: Cremonese, vol. I, 1957, V–XII.

Simili, Raffaella, *Per la scienza. Scritti editi e inediti*, Napoli: Bibliopolis, 2000.

Sisam, Charles Herschel, On Algebraic Hyperconical Connexes in Space of *r* Dimensions, *Atti della R. Accademia delle Scienze di Torino*, 46 (1911–12): 481–487.

Struik, Dirk, Obituary. Julian Lowell Coolidge, *The American Mathematical Monthly*, 62 (1955): 669–682.

Study, Eduard, Sugli enti analitici, *Rendiconti del Circolo Matematico di Palermo*, 21 (1906): 345–359.

Tanturri, Alberto, Un problema di geometria numerativa sulle varietà algebriche luogo di ∞^1spazi, *Atti della R. Accademia delle Scienze di Torino*, 35 (1899–1900): 427–442.

Tanturri, Alberto Ricerche sugli spazi plurisecanti di una curva algebrica, *Annali di matematica pura ed applicata* 3, 4 (1900): 67–121.

Terracini, Alessandro, Sulla teoria delle varietà luoghi di spazi, *Rendiconti del Circolo Matematico di Palermo*, 31 (1911): 392–396.

Terracini, Alessandro, Sulle varietà di spazi con carattere di sviluppabili, *Atti della R. Accademia delle Scienze di Torino*, 48, (1912–13): 297–319.

Terracini, Alessandro, Alcune questioni sugli spazi tangenti ed osculatori ad una varietà, Nota I, *Atti della R. Accademia delle Scienze di Torino*, 49 (1913–14): 215–250.

Terracini, Alessandro, *Corrado Segre (1863–1924),* Jahresbericht der Deutschen Mathematiker-Vereinigung, 35 (1926): 209–250.

Terracini, Alessandro, *Geometria superiore 1934–1935. L'opera geometrica di Corrado Segre e alcuni suoi ulteriori sviluppi*. 1934–35, BMP - Terracini, Quaderni.13, now in Giacardi (2013).

Terracini, Alessandro, I quaderni di Corrado Segre, in *Atti del IV Congresso della Unione Matematica Italiana*, Roma: Ed. Cremonese, vol. I, 1953a, 252–262.

Terracini, Alessandro, Commemorazione del Socio Gino Fano, *Atti della Accademia Nazionale dei Lincei. Rendiconti*, s. 8 14 (1953b): 702–715.

Terracini, Alessandro, *Geometria superiore 1957–58. L'opera geometrica di Corrado Segre*, BMP - Terracini, Quaderni. 34, 1957–58, now in Giacardi (2013).

Terracini, Alessandro, Prefazione, in *Corrado Segre, Opere*, Roma: Cremonese, vol. II, 1958, V–XVII.

Terracini, Alessandro, Parole del Prof. Alessandro Terracini, in *Atti del Convegno internazionale di geometria algebrica*, *Torino, 24-27.5.1961*, Torino: Rattero, 1961, 9–14.

Terracini, Alessandro, *Ricordi di un matematico. Un sessantennio di vita universitaria*, Roma: Ed. Cremonese, 1968.

Togliatti, Eugenio, Sulle superficie algebriche del 5° ordine, irriducibili, con un fascio di coniche, *Atti della R. Accademia Nazionale dei Lincei. Rendiconti,* s. 5, 21.2 (1912): 35–37.

Togliatti, Eugenio, Sui fasci di reciprocità degeneri tra spazi ad *n* dimensioni, *Atti della R. Accademia delle Scienze di Torino*, 52 (1916–17a): 628–645.

Togliatti, Eugenio, Su alcune classi di sistemi lineari di reciprocità degeneri tra spazi ad *n* dimensioni, *Atti della R. Accademia delle Scienze di Torino*, 52 (1916–17b): 759–773.

Togliatti, Eugenio, Sulle V_3 di S_5 con coincidenze di tangenti principali, *Reale Istituto Veneto di Scienze, Lettere ed Arti. Atti,* 87 (1927–28): 1373–1421.

Togliatti, Eugenio, *Prefazione*, in *Corrado Segre*, *Opere*, Roma: Cremonese, vol. IV, 1963, V–XIII.

Tricomi, Francesco, Essenza e didattica delle Matematiche in un manoscritto inedito di Corrado Segre, *Rendiconti del Seminario matematico dell'Università e del Politecnico di Torino*, 7 (1938–40): 101–117.

Viglezio, Elisa, In memoria di Corrado Segre, *Rassegna di Matematica e Fisica* 5, 1–2 (1924): 1–2.

Volterra, Vito, Le matematiche in Italia nella seconda metà del secolo XIX, in *Atti del IV Congresso Internazionale dei Matematici (Roma 6-11 aprile 1908)*, Castelnuovo, Guido (ed.), Roma: Tip. R. Acc. Lincei, v. 1, 1909, 55–65.

Young, William, Sulle sizigìe che legano le relazioni quadratiche fra le coordinate di retta in S_4, *Atti della R. Accademia delle Scienze di Torino*, 34 (1898–99): 596–599.

Young, William, Christian Felix Klein, 1949–1925, *Proceedings of the London Royal Society*, Series A 121 (1928): 1–xix.

Zappulla, Carmela, *La Geometria Proiettiva Complessa. Origini e sviluppi da von Staudt a Segre e Cartan*, Ph.D. Thesis, Palermo University, 2009.

Zariski, Oscar, Review of: *B. Segre, The Non-singular Cubic Surfaces*. Oxford: University Press, Oxford, 1942, in *Mathematical Reviews* 1943.

Zariski, Oscar, Algebraic surfaces, *Ergebnisse der Mathematik und ihrer Grenzgebiete*, 61, Second Supplemented Edition, New York-Heidelberg-Berlin: Springer, 1971.

Corrado Segre and His Disciples: the Construction of an International Identity for the Italian School of Algebraic Geometry

Erika Luciano and Clara Silvia Roero

Abstract

It is well known that the construction of an identity for the Italian School of Algebraic Geometry directed by C. Segre was the result of a complex dynamic of scientific exchanges with the international mathematical community. In particular, Felix Klein was a reference interlocutor for Segre, Fano, Enriques and Castelnuovo, and he exerted a notable influence on mathematical studies, on the teaching of mathematics, on publishing activity, and on the organization of cultural and academic life. In this paper, in light of the correspondence that Segre

Abbreviations adopted for archives and journals: ACS = Archivio Centrale dello Stato; AMS = American Mathematical Society; ANL-Castelnuovo = Accademia Nazionale dei Lincei Fondo Castelnuovo, Roma; ANL-Levi-Civita = Accademia Nazionale dei Lincei Fondo Levi-Civita, Roma; ANL-Volterra = Accademia Nazionale dei Lincei Fondo Volterra, Roma; ASUT = Archivio Storico dell'Università di Torino; BDMI Florence = Biblioteca del Dipartimento di Matematica e Informatica di Firenze; BMP Turin = Biblioteca Speciale di Matematica, Università di Torino; CUP = Cambridge University Press; DESPC = David Eugene Smith Professional Correspondence, Columbia University Libraries, New York; DSSP = Deputazione Subalpina di Storia Patria; EJWP = Ernest Julius Wilczynski Papers, The University of Chicago Library; EMSH = Editions de la Maison des Sciences de l'Homme; ICM = International Congress of Mathematicians; IMLSA = Institut Mittag-Leffler Stockholm Archive; JDMV = *Jahresbericht der Deutschen Mathematiker Vereinigung*; JFM = *Jarhbuch über die Fortschritte der Mathematik*; Klein GMA = F. Klein *Gesammelte Mathematische Abhandlungen*, 3 vols. 1921–1923; MIT = Massachusetts Institute of Technology; Mss. = Manuscripts; nnp = not numbered page; n. p. = no publisher; RS = *Revue semestrielle des publications mathématiques redigée sous les auspices de la Société mathématique d'Amsterdam*; rev. = reviewer; *Tr.* = *Translation*; transl. = translator; UMI = Unione Matematica Italiana; UTo-ACS = Università di Torino Archivi di Corrado Segre (cf. Appendix 1 in this volume).

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carried on with Italian and foreign colleagues, we will illustrate the international relationships of the Italian School of Algebraic Geometry in the period of formation (1883–1891). Further, through the reconstruction of Segre's partnerships with American scholars, we will show how he took on the role of a 'Maestro a distanza' or 'distance *Maestro*', for foreign geometers up to the first years of 20th century. From these letters and other archival sources there will emerge the cosmopolite features of the research activities pursued by the Segre's team, the diffusion of Italian geometric methods and results abroad, as well as some aspects of the biography of Segre, related to his institutional, political and editorial committment.

1 The Italian Style in Algebraic Geometry

Since the beginning of the twentieth century, historiography has underlined the existence of a School of Algebraic Geometry led by Corrado Segre, a School distinguished by a precise identity, with "its own impress, an Italian one" (Fano 1924–25, 220).[1] Built up over a time span that coincided with the period of most intense scientific activity of its leader, this identity consists both in a tendecy for the development of particular areas of research: projective hyperspace geometry, geometry on an algebraic curve, geometry of the surfaces, etc., and in the adoption of distinctive methods (synthetic-hyperspace, algebraic, differential-projective), languages and ways of thinking.[2]

The existence and characteristics of the Italian School of Algebraic Geometry were specified by the members of the team and by their Italian colleagues, with singular stability in time and in various circumstances: in obituaries and commemorations of Segre, a few years after his death,[3] in lectures held in Italy and

[1]On the Italian School of Algebraic Geometry and on the leading role played in it by C. Segre, cf. Menghini (1986), Conte (1993), Gario (1997), Brigaglia (2001), Giacardi (2001a, 2004, 2006–2007), Luciano and Roero (2012, 67–73).

[2]On the *Italian style* in research on Algebraic Geometry cf. Enriques (1920, 1924, 1928), Scorza (1932), B. Segre (1933), Severi (1950), Galuzzi (1980, 1033, 1058–1085), Boffi (1986), Bottazzini (1988), Boi (1990, 31, 38–43, 56–62, 71–73), Gray (1994), Brigaglia and Ciliberto (1995, 2004), Brigaglia (2001), Brigaglia et al. (2004), Mancosu (2009).

[3]Cf. Volterra (1924, 459), Castelnuovo (1924a, 460), Fano (1924, 460), Castelnuovo (1924b, 353, 354, 359), Berzolari (1924, 530, 531), Fano (1924–25, 220–223, 227), Terracini (1926, 209–250), Viglezio (1924, 1), Boggio (1928, 310, 313). The leadership of the Italian School of Algebraic Geometry had already been ascribed to Segre in the *Relazione sul concorso reale per la Matematica pel 1895*, Atti della R. Accademia dei Lincei, Rendiconti delle sedute solenni, I, 1892–1901, 367.

abroad,[4] in prefaces to essays and volumes,[5] in inaugural addresses and courses on Higher Geometry,[6] etc. This identity was soon recognized by foreigners. Julian L. Coolidge for instance affirmed:

> There is a pronounced rise and fall in the tide of mathematical interest in different countries and at different times. […] As for the modern birational geometry, that is almost a monopoly of *Italian mathematicians* and few others like Macaulay and Snyder, who have been strongly influenced by Italian thought (Coolidge 1927, 352).

Henry F. Baker, in the same line of thought, noticed[7]:

> He [Segre] may probably be said to be the father of that *wonderful Italian school* which has achieved so much in the birational theory of algebraic loci (Baker 1926, 269).

In the first phase of rereading of their activity, in a historical-mathematical retrospective, Italian algebraic geometers underlined not only and not so much the *national* landmark of their tradition of studies, but above all the international cultural roots and the contributions from exchanges with foreign colleagues.[8]

Recognition of a cosmopolitan dimension in geometry research was weakened, however—and eventually disappeared—during the fascist period, when some members of the School of Segre, with clear ideological intents, stressed the *Italianness* of their style of researches, to the point of distorting the evolution and identity of the School to which they belonged to make it an *emblem* of the 'Latin-Aryan genius'.[9]

Apart from this latter drift, there appears to be no doubt of the fact that Segre made a decisive contribution to the construction of an *international* identity for the *Italian* School of Algebraic Geometry, to its definition *au fil du temps* and to its affirmation at a world level. Some disciples of his rather emphasised that Segre's scientific activity had been affected—especially in quantitative terms—by the energies that he had devoted to this purpose (Castelnuovo 1924b, 369). In this connection, in the initial phase of his scientific production, between 1882 and 1884,

[4]Cf. for example Volterra (1907, 1909, 64), Fano (1923), Castelnuovo (1929), Severi (1932), Fano n.d., conferenze del 4 e 11 maggio (1942, f. 53).

[5]Cf. for example Bertini (1907, V), Enriques and Chisini (1915–34, v. 2, 541, v. 3, 154), Severi (1926, 141–144, 240–243, 289–296), Enriques (1937, 9–12).

[6]Cf. for example Terracini (1934–35, fols. 1, 3, 5), Terracini (1957–58, fols. 1, 3, 5, 7, 9, 11), Enriques (1929, 107–109).

[7]Cf. also Fehr (1923).

[8]Cf. Castelnuovo and Enriques (1896), Enriques (1906), Segre (1905, 1921c), Enriques (1926, 129–134), Castelnuovo (1929, v. 1, 191–201), Loria (1930[4], nnp).

[9]Cf. for instance Severi (1935, 581–589, 1941, 139), Conforto (1939, v. I, 133–134). On these nationalistic and anti-Semitic tendencies in historiography regarding Segre's School cf. Israel and Nastasi (1998, 286–307), in particular 300–302 and Israel (2010, 274–287), in particular 280–282.

he had written no fewer than 16 articles, and had entertained a wide network of correspondences with mathematicians from various countries like A. Voss, F. Klein, L. Kronecker, O. Schlömilch, G. Darboux, C.F. Geiser, T.A. Hirst, K. Weierstrass, A. Cayley, T. Reye[10] and J. Rosanes[11]. However after 1887 there followed a less intense period of work. This second period could be properly defined as that of 'Segre Maestro' because of the fact that, though continuing to cultivate his personal studies and interests, he devoted a large part of his time and activity to pushing his direct and distance disciples to produce original researches,[12] which in a sense he ended up considering "as his own."[13]

In this outlook it thus becomes important to stress that phase in Segre's life which was entirely unwinded in the Short Century, a phase traditionally somewhat overshadowed by the golden period (1883–1888),[14] in order to reconstruct the strategies worked out by him to stabilize the identity of his School and 'give colour and solidity' to it. A *histoire par traces* according to Bloch springs, which, investigating some less well-known aspects of Segre's scientific biography like the epistolary relationships and conversations with foreign colleagues, the publishing activity, the attendance at international congresses of mathematicians, the intentions and actions in the period of the Great War and in the early Twenties, shows that contributions were made to the construction of this identity not only by strictly technical aspects (mathematical ideas, contents and methods), but also by a set of paradigms and socio-cultural behaviours. It was *also* the sharing of certain civic values and particular stances, proper to many intellectuals of the first generation after the Risorgimento, that helped to make the team of Italian algebraic geometers a real School, internationally recognized, endowed with an appearance and cohesion of intents that went beyond the existence of a common research project, and

[10]For this purpose see the correspondences between Segre and foreigners in Annexes 1–16 to this article. Thanks to the discovery of new archival sources, today it is possible to reconstruct Segre's international relationships with more precise details, in regard to Terracini (1926) and B. Segre (1963–64) had done. In this connection it is significant that Segre's wife Olga and daughters Elena and Adriana made these letters available to his alumni, so that they could illustrate the international dimension of Segre's culture, at the beginning of his research on algebraic geometry.

[11]Cf. the manuscript by C. Segre *Sur les invariantes simultanés de deux formes quadratiques. Lettre à M. J. Rosanes par Corrado Segre à Turin*, 11 April 1884, described in Giacardi and Varetto (1996, 363).

[12]Cf. the correspondences by E.J. Wilczynski, C.H. Sisam, V. Snyder, E. Study and J.L. Coolidge to C. Segre, and to his wife: Annexes 28–31, 44–47 and 52–56 to this article.

[13]Cf. Segre to Pieri, 20 November 1901: Arrighi (1997, 115) and Segre to Castelnuovo, 3 April 1901: "From years and years I don't have the time to make researches that are quite substantial!" (ANL-Castelnuovo, Segre to Castelnuovo, 3 April 1901 "Sono anni ed anni che io non ho il tempo di far ricerche un po'lunghe!").

[14]It is difficult to establish the chronology of the 'golden period'in Segre's activity. Baker for example stated: "as to his work, one's judgement is dark formed by what he did before he was twenty-five years of age than by the publications of the remaining thirty-five years." (Baker 1927, 263). To us it seems nevertheless that this timeline is too narrow and that the most prolific period in Segre's research should be extended at least to 1895. For a study of Segre's youthful contributions, and particularly of the essays of the 1880s, cf. Ghione and Ottaviani (1992).

which remained strong, at least until the divergent political and cultural choices by some members, such as F. Severi, E. Bompiani and F. Conforto, during the fascist dictatorship.[15]

2 Circulation of Knowledge Between Italy and Europe

The first tiele in the construction of an international identity for this School was represented by the efficient network of circulation of texts and readings, put in place by Segre through his masterly teaching, documented by his handwritten *Notebooks* (Giacardi 2001a, 2013), and through the correspondences with some Italian and foreign colleagues.[16]

In effect, if it is true that the School of Segre identified in the writings of the Italians L. Cremona and E. D'Ovidio, and in some other renowned works by E. Bertini, G. Veronese, G. Battaglini and R. De Paolis, its *points de repère*, it is likewise undeniable that the cultural horizons of this School were much broader and took up the legacy of a wide range of influences and references to more or less famous foreign authors.

It was Segre himself that looked to international production for a profound renewal of the development of the three fields mainly cultivated by him. Regarding projective hyperspace geometry he referred, from his first works, to the results attained by F. Klein, F.G. Frobenius and K. Weierstrass (Giacardi 2001a, 141, 143–144, 148; Zappulla 2009; Luciano and Roero 2012). In particular, during the period in which he wrote his degree dissertation (spring-summer 1883) and published his first papers (Segre 1883a, b), he 'reinterpreted' some recent algebraic results obtained by Weierstrass, for the purpose of giving a geometrical and analytical pattern to projective hyperspaces, as we see from his correspondence with L. Kronecker (16 November 1883, 10 December 1883, 25 December 1883, Annexes 3, 4 and 5), with Weierstrass (28 March 1884, Annex 13) and with A. Cayley (14 May 1884, Annex 14). Aware of the difference of approach in the language adopted by the Italian and German Schools, in those circumstances Segre intensely stressed the utility of geometrically 'translating' some elements of the theory of bilinear and quadratic forms, developed by Weierstrass and by Kronecker himself (25 December 1883, Annex 5):

> Peut-être ne devrais-je pas dire «interprétation géométrique», car ces mots font penser (et vous ont fait penser à ce que je vois) à un travail qui consiste seulement dans des changements de mots […]. Mais ce n'est pas là ce que j'entendais dire dans ma dernière lettre.

[15]On the disagreements inside the Italian School of Algebraic Geometry, due to different political and ideological choices, cf. Vesentini (1990), Israel and Nastasi (1998, 156–160, 317–327), Pompeo Faracovi (2004), Guerraggio and Nastasi (2005, 83–118, 243–268), Babbitt and Goodstein (2009), Goodstein and Babbitt (2012).

[16]For illustrating the Italian milieu we have used above all some correspondences of Segre's with Castelnuovo and with Volterra. The former has been partially published (Gario 1991; Bottazzini et al. 1996, 669–678), while the second will be edited by L. Giacardi and P. Nastasi.

Some years later he again emphasised the true coincidence between the results obtained by Italian and German geometers, in spite of the different languages used, writing to Castelnuovo:

> Certain results can perhaps already be found in the analytical works (Kronecker, Weierstrass and alumni), but, because of their style of exposition, which is completely different from ours, comparisons are difficult. (ANL-Castelnuovo, Segre to Castelnuovo, 17 May 1894: "Certe cose poi può darsi si trovino già in lavori analitici (Kronecker, Weierstrass e scolari), ma per il loro modo d'esposizione, completamente diverso dal nostro, i riscontri son difficili").

Regarding the second favourite field of studies by Segre (the geometry on curves and surfaces), it was instead above all the contacts with Max Nöther that were fundamental,[17] while for differential geometry and that of imaginary quantities Segre's letters and conversations with G. Darboux,[18] with the American E.J. Wilczynski[19] and with the Czechoslovak E. Čech were important.[20]

Besides these direct contacts with the international community, it must also be remembered that Segre, starting from the third year of his university studies, frequented Turin bookshops looking for texts by famous authors and was wont to consult in the University Library the collections of German journals like the *Mathematische Annalen*,[21] the *Journal für die reine und angewandte Mathematik*[22] and the Berlin *Monatsberichte*.[23]

[17]On the relationships between M. Nöther and the Italian School of Algebraic Geometry cf. Segre (1921–22b, 161–163), Castelnuovo et al. (1925), Castelnuovo (1922), Berzolari (1921), and the correspondence between Nöther and Castelnuovo in the years 1889–1921 (ANL-Castelnuovo).

[18]Segre to Darboux, 11 March 1884, Annex 9.

[19]Segre to Wilczynski: 18 March 1904, Annex 29; 2 July 1904, Annex 31; 15 May 1906, Annex 48; 4 June 1906, Annex 49; 20 April 1908, Annex 56; 16 April 1916, Annex 59. Very interesting, on this subject, is the interpretation that Guido Fubini gave of the differences between the line of Italian research in projective geometry and that of E. J. Wilczynski, P. Sperry and G. M. Green (cf. EJWP, G. Fubini to E. J. Wilczynski, 27 May 1919).

[20]See below, Sect. 4.

[21]Cf. Segre to Voss, 1882, Annexes 1 and 2. Some years later he wrote to Castelnuovo: "I will certainly remain here for another month and perhaps for all the holidays. I go dipping into a few selected articles in the *Mathematische Annalen.*" (ANL-Castelnuovo, Segre to Castelnuovo, 15 August 1889: "Io rimarrò qui certo per un mese ancora e forse per tutte le vacanze. Vado leggicchiando poche cose scelte nei *Mathematische Annalen*"). Hence this journal was to remain a reference beacon to perceive the most advanced and promising lines of research.

[22]Cf. Segre to Voss, 1882, Annexes 1 and 2.

[23]Cf. Segre to Kronecker, 16 November 1883 and 10 December 1883, Annexes 3 and 4.

In the first years of his scientific career, there was an evident influence on Segre of the German mathematicians, as can be inferred from his *Resoconti di Scritti letti*[24] and from examination of his correspondences with A. Voss, L. Kronecker, O. Schlömilch, C.F. Geiser and K. Weierstrass. It was in this period that Segre, showing a great entrepreneurial spirit, turned to scholars that were already prominent, expounding his research projects, asking for suggestions on how to pursue his studies on complexes of lines and on the geometry of the straight line, and asking them for offprints, essays and books. This allowed him to contextualize his results in an international perspective and thus to enter in his own right the arena of the most authoritative scholars in the discipline.

The helpfulness shown by these foreign mathematicians towards a young researcher aroused Segre's gratitude and induced him to ask those people that he was accustomed to define his "*Maestri* for a moment",[25] for hospitality for his own articles. Hence his essays on the classification of second-order and fourth-order complexes, on the Kummer surface, on linear complexes, on binary homographies, on curves and ruled algebraic surfaces appeared in the *Journal für die reine und*

[24]Cf. UTo-ACS. *Appunti e Resoconti. Resoconti di scritti letti*, f. AP1 (F. Klein), AP3 (A. Clebsch, Salmon), AP6 (E. Malus), AP7 (J. Sylvester), AP9 (L. Schläfli), AP12 (B. Riemann), AP14 (K. Weierstrass), AP15 (L. Schläfli), AP16 (R. Sturm, M. Chasles), AP17 (G. Halphen), AP19 (R. Sturm, E. Genty), AP21 (G. Kobb), AP22 (M. Nöther), AP23 (A. Clebsch, S. Lie), AP24 (E. Study), AP25 (H. Schwarz), AP27 (E. Kummer), AP28 (G. Halphen, G. Fouret, C. Jordan), AP29 (A. Cayley), AP31 (F. von Lindemann), AP32 (M. Chasles), AP33 (M. Chasles), AP35 (D. Hilbert), AP36 (D. Hilbert), AP37 (H. Schubert), AP38 (J. Bischoff), AP39 (H. Schubert), AP40 (M. Nöther), AP41 (M. Nöther), AP42 (M. Nöther, L. Kraus), AP43 (G. Hanck), AP45 (T. Reye), AP46 (A. Hurwitz), AP47 (H.G. Zeuthen), AP48 (Harnack, Klein), AP49 (Brill), AP50 (Chasles), AP51 (Clifford), AP52 (H.G. Zeuthen), AP53 (C. Harnack, A. Ameseder, E. Weyr), AP54 (E. Weyr), AP58 (A. Voss), AP59 (F. Frobenius), AP60 (R. Sturm, T. Hirst), AP61 (H. Schröter), AP64 (E. Kummer), AP68 (J. Rosanes), AP69 (F. Frobenius); AP70 (L. Stickelberger, F. Frobenius); AP71 (C. Jordan, L. Kronecker); AP73 (L. Kronecker), AP75 (G. Halphen, E. Laguerre, C. Jordan), AP76 (G. Halphen), AP 77 (H. Picquet), AP78 (R. Sturm), AP80 (C. Stéphanos), AP81 (J. Lüroth), AP83 (F. Franklin), AP84 (K. Rohn), AP86 (A. Cayley), AP87 (H. Durrande, E. Laguerre), AP88 (G. Darboux), AP89 (T. Moutard), AP90 (E. Kummer), AP91 (E. Kummer), AP92 (E. Kummer), AP93 (F. Klein), AP94 (F. Klein), AP95 (F. Klein), AP97 (T. Reye, F. Auerbach, F. Neumann, F. Schur), AP98 (T. Weddle), AP100 (F. Schur), AP101 (A. Schumann), AP106 (A. Cayley), AP107 (G. Fouret), AP108 (F. Meyer), AP109 (H. Durège), AP111 (F. Klein, S. Lie), AP114 (M. Chasles), AP115 (G. Halphen, M. Chasles), AP116 (A. Cayley), AP117 (J. Plücker), AP119 (F. Schur), AP124 (S. Lie, F. Klein, Liouville), AP126 (R. Sturm), AP127 (L. Painvin), AP128 (J. Steiner), AP129 (F. Seydewitz), AP130 (J. Gergonne), AP131 (A. Voss). By contrast few are the reports on papers by Italian authors. Cf. UTo-ACS. *Appunti e Resoconti. Resoconti di scritti letti*, f. AP2 (L. Cremona), AP8 (L. Cremona), AP10 (P. Del Pezzo), AP26 (E. Beltrami), AP55 (F. Aschieri), AP56 (F. Aschieri), AP57 (E. Padova), AP62 (G. Battaglini), AP63 (G. Battaglini), AP65 (A. Sannia), AP66 (S. Dino); AP74 (L. Cremona), AP79 (G. Veronese), AP82 (G. Loria), AP85 (L. Cremona), AP96 (E. Bertini), AP102 (F. Ruffini, G. Battaglini), AP103 (F. Siacci), AP104 (F. Siacci), AP105 (E. D'Ovidio), AP110 (E. D'Ovidio), AP112 (F. Siacci), AP113 (G. Battaglini), AP117 (L. Cremona), AP118 (E. Caporali), AP120 (D. Boccella), AP121 (G. Veronese), AP122 (G. Veronese), AP123 (E. D'Ovidio), AP125 (G. Battaglini).

[25]Segre to Kronecker, 25 December 1883, Annex 5.

angewandte Mathematik (Segre 1884a, c, g), directed by Kronecker, and in *Mathematische Annalen* (Segre 1883a, b, 1884d, e, 1886a, 1887b, c, 1889a, 1891d), coedited by Klein.[26] By contrast did not succeed Segre's aspiration to publish a paper in the German journal *Zeitschrift für Mathematik und Physik,* edited by O. Schlömilch. He sent his article in French: "Sur les droites qui ont des moments donnés par rapport à des droites fixes", only later realising that only texts in German were accepted in that periodical. It was Schlömilch himself that forwarded Segre's paper to Klein, proposing that he publish it in *Mathematische Annalen*, but in the end it was printed in the *Journal für die reine und angewandte Mathematik* (Segre 1884c).[27]

As is well known, it was above all Klein that, starting from August 1883, assumed the role of Segre's 'distance *Maestro*' (Luciano and Roero 2012, 18–27, 81–148). Klein stimulated the young disciple to develop particular lines of research, carefully reread his first works, extended the sphere of his readings and involved him in international publishing initiatives, for instance enrolling him as a reviewer for the *Jahrbuch über die Fortschritte der Mathematik.* This proposal was immediately welcomed by Segre. In the two years 1883–1885 he wrote 35 reviews of works by Italian mathematicians (with a single exception[28]), and his comments were then translated into German by E. Lampe (Togliatti 1963, XII).

Furthermore it was Klein that widened the circle of Segre's interlocutors, putting him in contact with various European mathematicians (A. Hurwitz, F. Schur, J. Rosanes and T. Reye) that dealt with the same research themes as him,[29] and inviting Segre to vulgarize his results abroad by sending offprints to foreign colleagues. This suggestion too was attentively followed up by Segre, as is proved by

[26]Cf. Segre to Kronecker, 10 December 1883 and 25 December 1883, Annexes 4 and 5.

[27]On the behind the scenes of this publication cf. Segre to Schlömilch, 17 January 1884 and 8 February 1884, Annexes 6 and 7; Segre to Kronecker, 18 February 1884, Annex 8, and Luciano and Roero (2012, 104–106).

[28]Cf. the review by Segre of J.S. and M.N. Vaneček, Sur la génération des surfaces et des courbes gauches par les faisceaux de surfaces, *Atti della R. Accademia dei Lincei, Rendiconti*, 4, 1 (1885): 130–133, published in *Jahrbuch über die Fortschritte der Mathematik* 17 (1885): 667–668.

[29]Cf. Luciano and Roero (2012, 24).

his promotion of his essay on metric geometries of linear complexes and the spheres (Segre 1883–84a), offered to Darboux, Cayley, Geiser, Hirst and Mittag-Leffler.[30]

The capacity of Segre's to move into the European context of research on algebraic geometry—at first as a 'distance disciple' and then as an equal partner—was greatly appreciated at the national level and put him in the condition to become, in turn, a leader for the new generation of Italian geometers. For instance, one of the aspects noticed by the examination board in the competitions for the qualification for university teaching (*libera docenza*) and for the chairs in Higher Geometry, in which Segre participated (in Turin, Catania and Naples), was precisely the fact that "the excellent young scholar" had built up relationships of scientific collaboration with illustrious foreign mathematicians and yet had an excellent reputation abroad.[31] Aware of the weight, in evaluation of his curriculum, of the flattering judgments received in the international parterre, a few years later Segre suggested to Castelnuovo that he adopt analogous strategies for himself:

> By the way sending some printed copies of Nöther's letter to the examinatory board would always be a good precaution! (ANL-Castelnuovo, Segre to Castelnuovo, 8 August 1891: "Ad ogni modo l'invio al concorso di alcune copie stampate della lettera di Nöther sarebbe sempre una buona precauzione!").

[30]Cf. Segre to Schlömilch, 8 February 1884, Annex 7; Segre to Darboux, 11 March 1884, Annex 9; Segre to Cayley, 15 March 1884, Annex 10; Segre to Geiser, 23 March 1884, Annex 11; Segre to Hirst, 23 March 1884, Annex 12; Segre to Weierstrass, 28 March 1884, Annex 13; Segre to Cayley, 14 May 1884, Annex 14; C. Segre to G. Mittag-Leffler, 29 January 1885, IMLSA, *C. Segre n. 3*: "Monsieur le Professeur, Je vous envoi une copie d'un travail à moi, qui est paru à-présent, et qui contient des recherches géométriques et mécaniques sur lesquelles je prends la permission d'appeler votre attention. J'espère que les travaux, que je vous ai envoyés en pli chargé quelques jours après que j'avais eu le bonheur de faire votre connaissance, vous seront parvenus. Bien qu'ils versent surtout sur la géométrie à plusieurs dimensions, je pense qu'ils peuvent aussi intéresser un peu vous, qui êtes analyticien, car cette branche de la géométrie est liées très-étroitement à l'analyse. Je suis en train d'étudier le travail, dont vous m'avez fait cadeau, et je vous en suis vraiment reconnaissant. Vous me ferez aussi beaucoup de plaisir en me tenant au courant de vos découvertes futures. Votre très dévoué Corrado Segre".

[31]Cf. Relazione della Commissione pel concorso alla detta cattedra di professore straordinario, *Bollettino Ufficiale dell'Istruzione. Atti e documenti scolastici*, XIII (maggio 1887): 342.

Moreover, Castelnuovo, who had been Segre's assistant in Turin, and then became his friend and confidant, was a witness to the frequent requests for volumes, offprints[32] and lithographed polycopies of courses[33] that Segre made to his outlander colleagues. Hence, immediately after his death, as one of the principal merits of the Piedmont algebraic geometer he stressed his having enriched "with vital nourishment the culture of the Italian geometric School" through his activity of direction of Italian research and thanks to the international opening of his teaching (Castelnuovo 1924b, 354).

As a matter of fact, the manuscript *Notebooks* of Segre's courses in Higher Geometry (1888–1924) gradually grew fuller and fuller of references to foreign scientific literature. For example, as regards his own use of the texts by M. Nöther in his university lectures, Segre wrote to Castelnuovo:

> Have you seen the *Bericht* by Brill and Noether on algebraic functions? I am enthusiastic about it; although, only having received it 2 days ago, I have not been able to examine it thoroughly yet; and although of the Italian style there is only the name (for which the A [Authors] apologise in the preface, and Nöther also apologised to me in the short letter that he sent with the volume). Apart from this lacuna and the one relating to the arithmetic approach, in this treatise there is a historical exposition and critique, minute, meticolous, detailed, of all the theory, inclusive of all the methods and all the connections. If I am not to be disanchanted during the reading (which I do not believe), here there is the true story, which I had been yearning for so long, of this field. If it had come out a few months before it would have saved me – with the chapter on singularities – a large part of the work that I have done to structure my course for this year! (ANL-Castelnuovo, Segre to Castelnuovo, 28 November 1894: "Hai visto il *Bericht* di Brill e Noether sulle funzioni algebriche? Io ne sono entusiasta; quantunque, avendolo ricevuto da soli due giorni, non abbia ancor potuto esaminarlo a fondo; e quantunque dell'indirizzo italiano non vi sia che il nome (di che gli A [Autori] si scusano nella prefazione, ed il Nöther si scusò pure con me nelle poche righe con cui accompagnò l'invio del volume). Tolta questa lacuna e quella relativa all'indirizzo aritmetico, vi è in quest'opera un'esposizione storica e critica, minuta, coscienziosa, particolareggiata, di tutta quanta la teoria, in tutti i metodi, con tutte le connessioni. Se non incontrerò delusioni nella lettura (il che non credo) vi è qui la vera storia, quale io vagheggiavo da tanto tempo, di questo campo. Se usciva qualche mese prima mi

[32]Segre commented, for example to Castelnuovo, on the benevolence of Cayley for the gifts of his complete works (ANL-Castelnuovo, Segre to Castelnuovo, 6 January 1892) and later enthusiastically told him: "Humbert has sent me a big parcel of his offprints." (ANL-Castelnuovo, Segre to Castelnuovo, 16 February 1893: "Humbert m'ha inviato un grosso pacco di suoi lavori").

[33]For example Segre obtained and studied lithographs of the courses of Klein, H. Weber, etc. (Luciano and Roero 2012, 43, 181) as one can see from the following excerpts from the correspondence with Castelnuovo: "I have received those Lectures by Klein, and I have found a lot of interesting elements in them, including … my name (an allusion to the *Nuovo Campo*!)." (ANL-Castelnuovo, Segre to Castelnuovo, 12 November 1891: "ho ricevuto quelle Lezioni di Klein, e vi ho trovato molte cose interessanti, fra cui … il mio nome (un'allusione al *Nuovo Campo*!)"). Segre's interest was not only limited to geometry courses. For instance in 1906 he asked Volterra for a copy of the lectures held in Stockholm (ANL-Volterra, Segre to Volterra, 24 April 1906 and 28 December 1906). The lithograph he received from his friend was much appreciated: "In the last days I completed the reading of your Stockholm conferences and I am full of admiration!" (ANL-Volterra, Segre to Volterra, 24 June 1907: "Ho finito nei giorni scorsi le tue Lezioni di Stoccolma, e ne son rimasto ammirato!").

risparmiava – col capitolo sui punti singolari – una gran parte del lavoro che ho fatto per architettare il mio corso di quest'anno!").

Similarly while preparing the 1897–98 course entitled *Lezioni sui Gruppi continui di trasformazioni* (Segre's *Notebook* 11 in Giacardi 2013) Segre confided the difficulties that he met in studying the work by S. Lie[34]:

> I only deal with studying groups, and drawing up a program for the course. […] As for the method, I sometimes find it difficult to understand Lie's reasonings and calculus; and I would like to make the treatment clearer. There are also some beautiful theorems in the last chapters of both the 1st and 3rd vols. of Lie's work. (ANL-Castelnuovo, Segre to Castelnuovo, 22 October 1897: "Non mi occupo d'altro che di studiare i gruppi, e di farmi un programma del corso. […] Quanto al metodo, trovo difficoltà qualche volta a veder chiaro nei ragionamenti e calcoli di Lie; e vorrei rendere la trattazione più luminosa. Vi sono dei bei teoremi anche negli ultimi capitoli sia del 1° che del 3° vol. dell'opera di Lie").

Indeed, in 1899 reflecting on the use of the papers by David Hilbert for his lectures on Enumerative Geometry, Segre again addressed Castelnuovo for advice:

> Tell me if you have ever studied the memoir by Hilbert *M[athematische] A[nnalen]* 36 "*Ü[ber] die Theorie d[er] alg[ebraische] Formen*" in which there is a general formula for postulation of an M_k of S_r for forms of suitable high order. I would like to expound it this year in my course: but I'm held back by the complicatedness of that part in which it is proved that the number of certain relationships is finite (and at most = $r + 1$): 3rd theorem of that Memoir. Do you have any suggestion to make me known on this subject? (ANL-Castelnuovo, Segre to Castelnuovo, 4 October 1899: "Dimmi se hai mai studiato la memoria di Hilbert M.A. 36 "U. die Theorie d. alg. Formen" in cui si trova una formola generale di postulazione di una M_k di S_r per forme di un ordine abbastanza alto. Io vorrei esporla quest'anno nel mio corso: ma mi trattiene la complicazione di quella parte in cui si dimostra che è finito (e al più = $r + 1$) il numero di certe relazioni: teorema 3° di quella Memoria. Hai tu qualche suggerimento da darmi in proposito?").

Equally numerous were the requests for purchases and exchanges of foreign offprints and volumes, personally made by Segre, and presented from 1907 to 1924 in his capacity of director of the Mathematical Library in Turin.[35] These

[34]Regarding the use of Lie's writings in Segre's courses cf. also Segre to Castelnuovo, 26 January 1898, 17 May 1898 (ANL-Castelnuovo) and Hawkins (2000, 243–244, 251–260, 305–316).

[35]Here are some examples: "Give *me* the Note by Hilbert and Hurwitz (the Library of the Teachers Training School already has it in the *Acta Mathematica*, I think). As for the other two papers by Hilbert, the Library and I already hold them in the *Mathematische Annalen*: so you can donate them to some Roman geometer" (ANL-Castelnuovo, Segre to Castelnuovo, 12 January 1892: "Regala *a me* la Nota di Hilbert e Hurwitz (la Biblioteca di Magistero l'ha già negli *Acta mathematica*, credo). Quanto alle altre due di Hilbert, la Biblioteca ed io le abbiamo già nei *Mathematische Annalen*: sicché puoi regalarle a qualche geometra romano"); "If you didn't need Lüroth's memoir any more, at least for now, the Library of the Teachers Training School would take it back." (ANL-Castelnuovo, Segre to Castelnuovo, 7 December 1893: "Se la memoria Lüroth non ti occorresse più, almeno per ora, la Biblioteca di magistero la riprenderebbe."); "Please send us those volumes by Helmholtz and Klein, because our young scholars often need them." (ANL-Volterra, Segre to Volterra, 9 December 1900: "Favorisci inviarci quei tali volumi di Helmholtz e Klein, perché occorrono spesso ai nostri giovani."). Cf. also ANL-Castelnuovo, Segre to Castelnuovo, 6 January 1892 and 11 December 1893 and ANL-Volterra, Segre to Volterra, 30 December 1900.

acquisitions highly contributed to increasing the scientific collections of the University of Turin (Giacardi and Roero 1999, 442–447).

The range of Segre's readings, and consequently the international dimension of the exchanges of manuscripts and printed texts, was broadened above all after 1890 when, as an authentic leader, Segre devoted himself systematically to "importing to Italy ideas that had been developing elsewhere." (Terracini 1961, 11; Fano 1924, 225). Moreover he was one of the first Italian mathematicians to be associated to the *Deutsche Mathematiker Vereinigung* in 1897,[36] and from its journal (JDMV) he got news on the scientific activities in universities of Germany and other countries.

As we deduce from examination of his *Card Index*[37] and his *Library*,[38] Segre was an attentive, untiring and meticulous reader, whose horizons, initially limited to the German area, with the passing of time were extended towards world production, not only in geometry, but also in other disciplines, like analysis, abstract algebra, set theory, foundational studies, mathematical physics and maths education.[39]

In turn, Segre repeatedly encouraged students and colleagues, in both Turin and other places in Italy, to look to international publications to enter into flourishing lines of research and to renew their university teaching. In 1886, for instance, he suggested to Castelnuovo, at the time a student, to read the articles on *n*-dimensional geometry by W.F. Meyer, W.K. Clifford and H. Schubert, published in *Mathematische Annalen* and *Acta Mathematica*,[40] and he recommended that he conveniently quote them in his works. A few years later he again urged

[36]Cf. *Mitglieder-Verzeichnis*, JDMV 6 (1897): 20; JDMV 21 (1912): 44.

[37]Fano was the first to mention Segre's habit of 'gradually collecting and cataloging precious bibliographical references, over a period of 30 years' (Fano 1924–25, 222). The handwritten cards that make up his personal *Card Index*, built up and updated over time, are stored in BMP Turin (Giacardi and Varetto 1996, 367–368; Giacardi 2001b, 296).

[38]Segre's personal library was sold in 1924 to Guido Toja and bequeathed to the Mathematical Institute of Florence University (cf. Annexes 74 and 75 to this article).

[39]UTo-ACS. *Elenchi di opere, articoli e Schede bibliografiche*. The analysis of the cards devoted to the themes *Aggregates I*, *Aggregates II*, *Foundations of geometry*, *Elementary Geometry*, *Postulates*, *Foundations* (S_n) at numbered fols. 48, 49, 173, 174, 175, 176, 211, 300, 456, 461, and *Euclid and the Foundation of Arithmetic* (second part of Segre's *Card Index*, not numbered cards fols. 9, 10) is particularly interesting, both because it documents Segre's great culture, also in the realm of logical-foundational studies, and because it makes it possible to compare his point of view with that of G. Peano, who worked on the same themes with quite antithetical visions of foundational stances in respect of both research and teaching (Luciano and Roero 2012, 28–37, 66–73; Luciano 2016). From a cross exam of the *Card Index*, the *Library* and Segre's correspondence, it emerges for example that Hilbert's *Grundlagen der Geometrie*, received by Segre in 1902, were cited in his 1902–03 course on *Geometrie non-Euclidee* (Segre's *Notebook* 16 in Giacardi 2013) and in his *Lectures at the Magistero School* (Segre's *Notebook* 40 in Giacardi 2013). The book was then commented on with Castelnuovo and, through an exchange of letters between Castelnuovo, Enriques, Klein, Minkowski and Hilbert, it became known to Enriques, who referred to it in the chapter on *Foundations of Geometry* for the *Encyklopädie der mathematische Wissenschaften* (Bottazzini 2001a, 295–304; Luciano and Roero 2012, 33–37).

[40]ANL-Castelnuovo, Segre to Castelnuovo, 27 July 1886, 29 July 1886 and 30 December 1886.

Castelnuovo, who had become a colleague of his, to examine in depht into the M. Nöther's memoirs, and shared his intention to study F. Neumann's *Vorlesungen*.[41] Similarly Segre advised Loria and Castelnuovo to refer to the international literature for their courses on Higher Geometry in Genoa and Rome, respectively.[42]

Further, during all the period of his direction of the Italian School of Algebraic Geometry, Segre circulated offprints and texts by foreign geometers in Italy[43]; with his team of coworkers he commented on the most recent bibliography[44] and even

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[41]ANL-Castelnuovo, Segre to Castelnuovo, 10 October 1888 and 15 July 1889.

[42]For example Segre wrote to Castelnuovo: "Loria asked for my advice about basing his course on geometry of the straight line upon Sturm's text and a long time ago I answered him 'not', more or less explaining what you say now: but this was completely useless; later Loria wrote to me that the 2nd vol. (which has already been published, but I don't have it yet) is much more important than the 1st." (ANL-Castelnuovo, Segre to Castelnuovo, 27 December 1892: "A Loria che mi chiese consiglio sul prendere lo Sturm a testo per un corso di geometria della retta risposi tempo fa di no, adducendo all'incirca quel che dici tu: ma ciò non valse; più tardi Loria mi scrisse che il 2° vol. (già uscito, ma che io non ho ancora) è molto più importante del 1°"). Segre also recommended to Castelnuovo the texts by A. Wrigley, *Collection of Examples and Problems in Pure and Mixed Mathematics* (1865) and J. Wostenholme, *A Book of Mathematical Problems on subjects included in the Cambridge Course* (1867) for his Geometry course in Rome.

[43]In July 1900 for instance Segre sent Volterra the article by F. Klein, Riemann und seine Bedeutung für die Entwicklung der modernen Mathematik, *Amtlicher Bericht der Naturforscherversammlung zu Wien* (26 September 1894, JDMV, 4 (1894–95): 71–87—Klein GMA 3 (1923): 482–497); cf. also ANL-Volterra, Segre to Volterra, 19 July 1900, 30 December 1900 and 7 March 1901.

[44]Segre announced for example to Castelnuovo that he had "devoured" a new volume by Lie in just 1 month (ANL-Castelnuovo, Segre to Castelnuovo, 29 August 1891). The following summer, about to prepare his famous course *Introduzione alla geometria sugli enti algebrici semplicemente infiniti*, Segre announced to Castelnuovo his intention to pivot on the Memoir by S. Kantor (ANL-Castelnuovo: Segre to Castelnuovo, 28 August 1892). In September he added: "The 1st vol. of Sturm's *Liniengeometrie in synthetischer Behandlung* has been published (I don't know if I have already told you). It is a good book; I believe you will like it. By the way, do you read Kirckhoff? I am also receiving, this very day, the 2nd vol. of Klein-Fricke." (ANL-Castelnuovo, Segre to Castelnuovo, 14 September 1892 and 18 September 1892). To structure the course on the *Teoria della singolarità delle curve e superfici algebriche*, Segre "grants a lot of time" to *Continuirliches Gruppen* by S. Lie and L. Scheffers (ANL-Castelnuovo, Segre to Castelnuovo, 13 September 1894 and 23 September 1894). By contrast, of the book by P. Appell and E. Goursat, *Théorie des fonctions algébriques et de leurs intégrales* (1895) he did not get a "very good impression, browsing through it." On Segre's comments related to international literature, and in particular on the essays of T. Reye, K. Dochlemann, E. Kötter, cf. ANL-Castelnuovo: Segre to Castelnuovo, 27 September 1892 and ANL-Volterra, Segre to Volterra, 27 October 1897.

"researches not yet published",[45] and kept him up to date the epistolary contacts of his disciples with foreign authors.[46]

Finally, to Segre's desire to afford international cultural roots for his School there are connected two publishing enterprises: the Italian translations of *Geometrie der Lage* by C. von Staudt and F. Klein's *Erlangen Program*, commissioned by him from his students M. Pieri and G. Fano (Luciano and Roero 2012, 37–45).

3 Promotion of the Italian Style Abroad

If the construction of a national identity for the School of Algebraic Geometry could not aside from the recourse to sources and comparaisons with international models, reciprocally, for Segre, it was the task of a leader to channel towards other countries the flow of the best contributions by his disciples. This was an action of diffusion of mathematical knowledge that he took on seriously and tenaciously, both in interwining a ramified network of epistolary dialogues and in working inside examining boards for attribution of prestigious prizes like the International

[45]In this connection the question of the quotations of the unpublished works of S. Kantor was at the origin of a series of disagreements and claims of priority by the German mathematician against Castelnuovo and Segre, which induced Segre to advise Castelnuovo to use prudence in his answers to the attacks (cf. ANL-Castelnuovo, Segre to Castelnuovo, 6 October 1887, 30 September 1891, 5 March 1892, 11 March 1892, 11 March 1892, 30 March 1892, 6 April 1892, 19 April 1892, 22 April 1892, 25 April 1892, 21 October 1893, 23 November 1893, 6 January 1894). In general, Segre was always careful to draw the attention of members of his School on external contributions, recommending to quote them. For instance he wrote to Castelnuovo: "The announcement of the work by Kronecker, if I remember rightly, can be found in the *Rendiconti Lincei* of 3–6 years ago; it seems to me that he precisely proved (and, according to what he told me, it was no simple matter from the algebraic and rigorous point of view) that a surface whose section with π° [plane] tangent splits, it is ruled or it is a Steiner's surface. However, I don't know how Brioschi or Cremona in the same sitting mentioned a Roman surface of Kronecker's that would have corresponded to Steiner's. It is a vague souvenir that I have. You check there. You would do well to question Cremona (if he doesn't become a minister) on what he knows on the subject, before publishing (and you will do well to publish) your proof." (ANL-Castelnuovo, Segre to Castelnuovo, 30 November 1893: "L'annunzio del lavoro del Kronecker, se ben ricordo, si trova nei *Rendiconti Lincei* di 3 a 6 anni sono; mi pare che egli precisamente dimostrasse (e, a quanto mi diceva, non è cosa semplice dal punto di vista algebrico e rigoroso) che una superficie la cui sezione col π° [piano] tangente si spezza, è rigata o superficie di Steiner. Però non so come il Brioschi od il Cremona nella stessa seduta accennarono ad una superficie romana di Kronecker che avrebbe fatto riscontro a quella di Steiner. È un ricordo confuso che io ho. Tu riscontra colà. Faresti bene ad interrogare Cremona (se non diventa ministro) su ciò che sa in proposito, prima di pubblicare (e farai bene a pubblicare) la tua dimostrazione."). Cf. also ANL-Castelnuovo, Segre to Castelnuovo, 23 November 1893 and 4 March 1894.

[46]For instance he wrote to Castelnuovo: "Tell me if Nöther and Rohn have supplied you interesting answers" (ANL-Castelnuovo, Segre to Castelnuovo, 31 July 1891: "Dimmi se Nöther e Rohn ti hanno fatto risposte interessanti"). "Keep me informed about the issues of curves and surfaces which you are working on with the French" (*Ibidem*, 12 March 1895: "Tienimi informato delle questioni su curve e superficie, di cui ti occupi coi francesi"). "Then tell me what Zeuthen answers you." (*Ibidem*, 29 November 1897: "Dimmi poi che cosa ti risponderà Zeuthen.").

Bressa Prize and the Guccia Medal. It was above all this last award, attributed to Severi in the IVth International Congress of Mathematicians (Rome 1908),[47] that Segre intended to 'exploit' for illustrating, at international level, the relevance and value of the researches of his School. So, since 1905 he recommended to Castelnuovo:

> If you or your brother-in-law should have an opportunity to write to Severi, you should incite him (as I have already done) to work for the Guccia medal. You know that Valentiner aspires to it. It would now have a flattering meaning for Italy and one corresponding to the leading position that Italy currently holds in Geometry, if an Italian was proclaimed the winner at the Rome congress! It would be enough for Severi to work on *the curves existing on a given algebraic surface*, profiting by his knowledge of surface geometry, to stick to the proposed theme, without straying too far away from his favourite field! (ANL-Castelnuovo, C. Segre to G. Castelnuovo, 5 November 1905: "Se tu, o tuo cognato, aveste occasione di scrivere a Severi, dovreste eccitarlo (come già feci io) a lavorare per la medaglia Guccia. Sai che vi aspira il Valentiner. Ora avrebbe un significato lusinghiero per l'Italia, e corrispondente al primato che questa ha attualmente in Geometria, il fatto che nel congresso di Roma fosse proclamato vincitore un italiano! Basterebbe che Severi lavorasse su le curve esistenti su una data superficie algebrica, approfittando della sua conoscenza della geometria sulla superficie, per essere nell'ambito del tema proposto, senza allontanarsi troppo dal suo campo preferito!")

A short time later, returning to the matter, Segre asked his friend:

> Speaking of twisted curves, after I wrote to you some time ago telling you to incite Severi to work for the Guccia prize, I wondered whether you yourself or your brother-in-law would not like to compete. Severi has written to me, also recently, that he will think about it. And you? For me it would be a great satisfaction to be able to judge that competition in favour of an Italian! Is it impossible for it to be you? (ANL-Castelnuovo, Segre to Castelnuovo, 25 December 1905: "A proposito di curve sghembe, dopo che qualche tempo fa t'avevo scritto di eccitar Severi a lavorare pel premio Guccia, mi son chiesto se tu stesso o tuo cognato non vorreste concorrere. Severi mi ha scritto, anche recentemente, che vi penserà. E voi? Sarebbe per me una grande soddisfazione il poter giudicare quel concorso in favore di un italiano! È escluso che tu possa esser quello?").

Above all it is the study of Segre's correspondence that allows us to retrace his strategy to promote abroad the Italian style in algebraic geometry and to show a clear difference in tone among the relationships he entertained with his outlander colleagues. From the letters of his youthful years (1883–1887) with F. Klein, Kronecker, Weierstrass and others, full of details on his research projects and on their links with international production, he moved on to letters and postcards, whose purpose was to announce to F. Klein, A. Hurwitz, M. Nöther (Luciano and Roero 2012, 25–27, 151–161, 164–165; ANL-Castelnuovo, Segre to Castelnuovo, 25 February 1897 and 10 March 1897), E. Picard[48] and E.J. Wilczynski[49] his own

[47]Cf. Segre et al. (1909, 145–151). On the role played by Segre in the attribution of the Guccia Prize, cf. also Segre to O. Michelli Segre, 25 July 1904 (Annex 34), 30 July 1904 (Annex 36), 4 August 1904 (Annex 38), 5–7 April 1908 (Annexes 53 and 54).

[48]On the contacts between Picard and the Italian School of Algebraic Geometry cf. ANL-Castelnuovo, Segre to Castelnuovo, 9 May 1901 and 13 May 1901.

[49]Cf. Segre to Wilczynski, 16 April 1916, Annex 59.

important results and, subsequently, those of Castelnuovo, Enriques, Fano, B. Levi and Fubini. For instance, Segre encouraged Castelnuovo to get into epistolary contact with Picard to tell him about the recent essays on the theory of surfaces published by Italians:

> I am writing to you from the room of the same [examinations] to congratulate you on the card from Picard (at the appropriate time you will tell me about the noteworthy points in it), and to tell you … that I have nothing to tell you: the recent Italian works on the surfaces, which I would referenced, are only yours and those of Enriques … Perhaps (think about it) it will also be possible to mention the surfaces encountered by Fano in his works, which fit into our line of research. I sent my memoir to Picard some time ago. Before answering him, see in the last issue of the *Intermédiaire des mathématiciens* the question signed by Poincaré and Automne on curves with trisecant chords; and think whether it is appropriate to mention to Picard that in Italy the question … is not such. (ANL-Castelnuovo, Segre to Castelnuovo, 13 July 1894: "Ti scrivo dalla sala dei medesimi [esami] per congratularmi teco per la letterina del Picard (della quale all'occasione mi comunicherai i punti salienti), e per dirti … che non ho nulla da dirti: i lavori italiani recenti sulle superficie, ai quali io penserei, sono solo i tuoi e quelli di Enriques … Forse (pensaci) si potran nominare anche le superficie incontrate da Fano nei suoi lavori, che rientrano nel nostro indirizzo. La mia Memoria l'ho inviata a suo tempo al Picard. Prima di rispondere a questo, vedi nell'ultimo fascicolo dell'*Intermédiaire des mathématiciens* la questione firmata Poincaré e Automne sulle curve le cui corde son trisecanti; e pensa se sia opportuno accennare al Picard che in Italia la questione … non è tale").[50]

Aware of the influence of Castelnuovo and Enriques, Segre did not hide from his friends the satisfaction that he had felt reading the treatise by E. Picard and G. Simart, *Théorie des fonctions algébriques de deux variables indépendantes* (cf. ANL-Castelnuovo, Segre to Castelnuovo, 7 September 1897). On the other hand the gift of the very same volume allowed Segre to face the French colleague on the theme of the resolution of singularities:

> I had written a long letter to Picard (…). He answered me very politely recognising the difficulty "… je vous remercie bien d'avoir appelé mon attention sur l'inadvertance que j'ai commise, et que je tâcherai de réparer dans une note du second volume. En réalité, j'ai dans ma rédaction passé, je le reconnais, un peu vite sur ces théorèmes de réduction, au sujet desquels nous n'avons tous aucun dout pour le fond, et j'avais hâte d'arriver à des questions ayant pour moi plus de nouveauté. Si vous, ou vos élèves trouvaient quelque chose de tout-à-fait définitif sur ce sujet, j'en serais bien heureux." (ANL-Castelnuovo, Segre to Castelnuovo, 22 October 1897: "Avevo scritto al Picard una lunga lettera …. Mi ha risposto molto gentilmente riconoscendo la cosa …").

Besides, during the period of his leadership of the Italian School of Algebraic Geometry Segre recommended to his former students, who had become colleagues of his, that they transmitted their articles abroad[51]; informed them about the

[50]Cf. also ANL-Castelnuovo, Segre to Castelnuovo, 8 August 1894 and 23 September 1894.

[51]ANL-Castelnuovo, Segre to Castelnuovo, 11 May 1892: "Loria charges me to tell you he believes you once sent to the *Jahrbuch* your important Memoir on the systems of curves, to be reviewed. Yes, it is always a good idea to forward one's works to the *Jahrbuch*." ("Loria m'incarica di dirti che ritiene tu abbia mandato a suo tempo al *Jahrbuch* la tua gran Memoria sui sistemi di curve, perché ne sia fatta la recensione. Già, è bene mandar sempre i propri lavori al *Jahrbuch*.").

quotations of their contributions that he was founding, in international journals[52]; enlightened about their exchanges with Brill,[53] Nöther,[54] Rohn[55] and Hurwitz[56] and was ready to put his disciples in touch with scholars like F. Macaulay[57] and the Swede A. Wiman.[58]

Moreover, on several occasions Segre made every effort to avoid clashes or controversies between Italian and foreign mathematicians. For instance, in 1903, he suggested to René Baire that he privately point out to Beppo Levi some errors that he had found in notes "Sur la résolution des points singuliers des surfaces algébriques" and "Sur la théorie des fonctions algébriques de deux variables" (*Comptes Rendus de l'Académie des Sciences de Paris*, 134, 1902, 222–225, 642–644).[59] In

[52]For example (ANL-Castelnuovo, Segre to Castelnuovo, 26 September 1892): "In Klein's lithographed lectures for the last semester (*Riem. Flächen* II, just published, your name is repeatedly and honourably mentioned. In particular he analyzed, as really instructive, the method with which you determined the number of the special series (4 pages of analysis). Remember (if it has not been done yet) to get them to purchase all those Lectures by Klein for your library as soon as possible. Do you perhaps have a manuscript copy of Nöther's Memoir on double planes (*Medic. Erlangen*)? I may need it later: and then I will ask you for it. It is for my course." ("Nelle lezioni litografate di Klein dell'ultimo semestre (*Riem. Flächen* II) uscite ora si trova ripetutamente ed onorevolmente citato il tuo nome. In particolare vi è analizzato, come particolarmente istruttivo, il metodo con cui tu hai determinato il numero delle serie speciali (4 pag. di analisi). Ricordati (se ancora non s'è fatto) di far acquistare al più presto tutte quelle Lezioni di Klein alla vostra biblioteca. Hai forse tu una copia manoscritta della Memoria di Nöther sui piani doppi (*Medic. Erlangen*)? Può darsi che mi occorra più tardi: e allora te la chiederò. Si tratta del mio corso."). Cf. also ANL-Castelnuovo, Segre to Castelnuovo, 6 December 1892 and 29 October 1894: "Have you seen in the latest issue of Crelle [journal] a work by Kantor (extracted from the Naples one) with a mention of you?" ("Hai visto nell'ultimo fascicolo del [giornale di] Crelle un lavoro del Kantor (estratto da quello di Napoli) con citazione di te?").

[53]ANL-Castelnuovo, Segre to Castelnuovo, 28 July 1890: "Make me known Brill's answer; also tell me the praises that he will certainly bestow on you." ("Comunicami la risposta di Brill; riportami anche gli elogi che egli certo ti farà").

[54]ANL-Castelnuovo, Segre to Castelnuovo, 15 July 1891: "Then Nöther praised your latest works to me: he sees with pleasure that you have already dealt fruitfully with surface geometry, and wishes you to continue in this direction." ("Nöther poi mi ha lodato i tuoi ultimi lavori: vede con piacere che tu ti sei già occupato con qualche frutto di geometria sulla superficie, e desidera che tu continui in questo indirizzo"); 31 July 1891: "Let me known if Nöther and Rohn have given you interesting answers." ("Dimmi se Nöther e Rohn ti hanno fatto risposte interessanti"); 19 March 1896: "By the way, Enriques wrote to me some time ago that Nöther has explained to you that in his opinion the resolution of singularities of higher order in the sense that we pursue doesn't need demonstrating anymore." ("A proposito: l'Enriques mi scrisse tempo fa che il Noether t'ha scritto che a suo avviso la risoluzione delle singolarità superiori delle superficie nel senso che ci occorre non ha più bisogno di dimostrazione"). Cf. also ANL-Castelnuovo, Segre to Castelnuovo, 13 May 1901.

[55]Cf. ANL-Castelnuovo, Segre to Castelnuovo, 31 July 1891 and 5 September 1891.

[56]ANL-Castelnuovo, Segre to G. Castelnuovo, 23 August 1890: "I wrote to Hurwitz our proof for ∞^1 curves univocal correspondences: I will show you his answer." ("Scrissi ad Hurwitz la nostra dimostrazione per le curve ∞^1 corrispondenze univoche: te ne mostrerò la risposta.").

[57]Cf. ANL-Castelnuovo, Segre to Castelnuovo, 19 March 1896.

[58]Cf. ANL-Castelnuovo, Segre to Castelnuovo, 4 June 1901.

[59]Cf. Baire to Borel, 12 November 1903 (in Dugac 1990, 65).

1906 Segre again acted as an intermediary between H. Lebesgue and B. Levi, trying to abstain from a controversy that would have damaged the international reputation of his School:

> Mes théorèmes invoqués par Fatou sont mis en doute actuellement par Beppo Levi dans les *Rendiconti dei Lincei*. Beppo Levi n'a pas su rétablir quelques raisonnements intermédiaires simples et il s'est cassé le nez sur une faute de rédaction grave que Montel m'a jadis signalée et qu'il est facile de réparer. Naturellement j'ai commencé par rédiger une note où je l'attrapais comme du poisson pourri puis, sur une lettre de Segre, et parce que ce n'est pas le moyen d'acquérir une réputation mondiale que d'attraper ceux qui s'occupent de mes histoires, j'ai été moins dur. (H. Lebesgue to E. Borel, 1 June 1906, in Bru and Dugac 2004, 148–149).[60]

In the context of the international promotion of the Italian geometric style an essential role was also played by Segre's project, several times postponed and finally abandoned, to publish a volume of *Lectures on Algebraic Geometry*. It was E. Bertini that suggested that initiative to him, as an ideal way to increase the prestige of the Italian School and to convey knowledge of the production of its members to other countries. Aware of the advantages of similar publishing entreprises, in the summer of 1890 Segre wrote to Castelnuovo:

> It is really necessary to thinking about writing treatises, lithographing lectures, extensively popularizing our ideas (I refer to yours and mine, we being perhaps the only ones in Italy, in all modesty, and without offending Peano, that have the right views on the subject). We will resort to Fano, who has already written me a long letter telling me to greet you, and with news on his readings and his researches (on cubic varieties), which always show his activity is prodigious. (ANL-Castelnuovo, Segre to Castelnuovo, 6 July 1890: "Bisogna proprio pensare a far trattati, a litografare lezioni, a divulgare con estensione le nostre idee (parlo di quelle di te e di me, che forse forse siamo i soli in Italia, modestia a parte, e senza offender Peano, che la pensiamo rettamente in proposito). Ricorreremo a Fano, che m'ha già scritto una lunga lettera coll'incarico di salutarti, e con notizie sulle sue letture e sue ricerche (sulle varietà cubiche), che lo mostrano sempre di un'attività prodigiosa.").

Segre therefore entrusted to his young student the task of compiling the *Summaries* of the lectures in Higher Geometry in the 1890–91 academic year. It was the famous course devoted to the *Introduzione alla geometria sopra un ente algebrico semplicemente infinito* (Segre's *Notebook* 3, in Giacardi 2013). Nevertheless,

[60]The dialogue between Segre and Lebesgue continued until 1910, as we see from the letter of Lebesgue to Borel, 20 May 1910 (Bru and Dugac 2004, 238): "La recherche des transformations ponctuelles transformant tout plan en plan a été ramenée par M. Segre, grâce à un résultat de M. Darboux, à la résolution des équations fonctionnelles $f(x_1+x_2)=f(x_1)+f(x_2), f(x_1x_2)=f(x_1)f(x_2)$. M. Segre m'a demandé mon avis sur ce problème analytique. J'ai montré que si l'on raisonne comme M. Zermelo on doit admettre l'existence d'une infinité de solutions autres que les solutions connues, mais que nommer effectivement une telle transformation revient à nommer une fonction échappant à tout mode de représentation analytique. Autant dire qu'il n'y a pas d'autres solutions".

starting from the first revision, Segre was not satisfied of the notes by Fano since they were "very careless"[61] and predicted that they would have required months of patient work, before he could send them to be printed.[62]

For their part, the colleagues Castelnuovo and Bertini urged the time to be shortened, and they offered Segre their help in the work of correction,[63] with the purpose not to procrastinate the appearance of these *Summaries*. Reluctant to delegate to others the responsibility of the revision, in the autumn of 1896 Segre went back to the project of editing his *Lectures*. Fundamental changes were made to the previous plan, for the aim of arriving at a more ambitious publication that would constitute a true treatise, and not only a volume of *Summaries*. The text should have organically recapitulated his most important and most appreciated monographic courses so as to reach a broad readership. Therefore Segre abandoned the idea of lithography to turn, instead, to well-known and 'powerful' publishing houses like Gauthier-Villars or Teubner. Segre also changed the nature of the *authorat* of these *Lectures* and the intention to circulate the summaries of a specific university course gave way to a project for a collective publication, that of a 'master work', 'harmonious and original', the expression of a School, realized thanks to collaboration among Segre, Castelnuovo and Enriques. The project so began to set up in its structure and organization:

[61]ANL-Castelnuovo, Segre to Castelnuovo, 8 August 1891: "Two days ago I started the revision of Fano's summaries of my lectures. And I find them very careless! So I don't know if I will have the patience to spend about 3 months on them (which will be necessary if the rest corresponds to the beginning) to make them suitable for lithographing! Yet Bertini insists on lithographing!" ("Da due giorni ho cominciato la revisione dei sunti di Fano delle mie lezioni. E li trovo molto trascurati! Sì che non so se avrò la pazienza di perdervi tre mesi attorno (come ci vogliono se il seguito corrisponde al principio) per renderli litografabili! Eppure Bertini insiste per la litografazione!"); 29 August 1891: "This month (…) I have corrected 1/9 of Fano's summaries (really not at all accurate!, and I have now left them aside to return to that work on hyper-algebraic entities that I suspended in January because of the paper for the *Rivista*." ("In questo mese ho corretto 1/9 dei sunti di Fano (poco accurati davvero!, ed ora li ho piantati per riprendere quel tal lavoro sugli enti iperalgebrici che in Gennaio avevo sospeso per causa del lavoro della *Rivista*."). cf. also Segre to Amodeo, 16 August 1891 (Palladino and Palladino 2006, 179–180).

[62]In any case, Segre was not favourable to the practice of *Summaries*. While many of his colleagues every year released notes on courses, both preparatory and advanced, it is actually curious that a single volume of *Summaries* of his lectures appeared, the one relating to the course on *Projective Geometry* for the academic year 1885–86, no less than 989 pages (Segre C. *Geometria proiettiva*, Torino: Litografia dell'Università, 1886).

[63]ANL-Castelnuovo, Segre to Castelnuovo, 30 September 1891: "I thank you for the very kind offer to help me with the lithographs of my lectures: but it would be difficult, and besides I would be sorry to involve you in very boring jobs. Besides, I have not in the least given up the idea of dealing myself, perhaps in November, with this lithographing." ("Ti ringrazio dell'offerta gentilissima di aiutarmi per le litografie delle mie lezioni: ma sarebbe cosa difficile, e d'altronde avrei rimorso di occuparti in cose molto nojose. Non ho punto rinunziato, del resto, ad occuparmi io stesso, in Novembre forse, di questa litografazione."). Cf. also ANL-Castelnuovo, Segre to Castelnuovo, 18 and 23 September 1892, 9 October 1892, 9 July 1893, 20 July 1893, 17 August 1893, 30 November 1893, 25 December 1893, 25 February 1894, 17 April 1894.

> Let's now speak of the grand treatise on Higher Geometry. Castelnuovo will remember that here in Turin we have sometimes spoken about it together as of a work that we would gladly have done *together later*. Now that a new member, an active one, moves in, the initiative becomes much easier. Besides, I must point out that precisely for Enriques it would be better to defer the enterprise a few more years: because, for good or bad it is, the regulation requires him to prepare titles for his promotion to full professor; so for some time it is better for him to devote his scientific industriousness only to original researches. Then there is the usual difficulty about the publisher: if it were possible to turn to Gauthier-Villars or Teubner, it would be all right! For the rest, I completely approve of the idea, and I believe that among us three it would be possible to write a good treatise, harmonious, original. Whatever the form of collaboration to assume, I already have a sure way to take part in the work: and that is with the summaries (by me) of various courses of mine: that is to say of some general courses on curves, algebraic surfaces, etc. and of special courses on higher singularities, on geometry on a curve, on rational surfaces and linear systems of plane curves, etc. If the undertaking is delayed a few years, I will do some new courses which can also be useful. I must also prepare for the German Encyclopaedia the article on hyperspaces (algebraic varieties), and this can also serve for the respective chapters of the treatise (likewise for the article on algebraic surfaces promised by Guido [Castelnuovo] for the Encyclopaedia). Then I fully approve of a treatise being done by us three and no others: because we three share the same ideas; and others would perhaps upset the harmony of the work. I will be pleased if you write something more to me on this matter. (ANL-Castelnuovo, Segre to Castelnuovo and Enriques, 30 December 1896: "Parliamo ora del grande trattato di Geometria Superiore. Castelnuovo ricorderà che qui a Torino ne abbiam parlato insieme qualche volta come di un lavoro che avremmo fatto volentieri insieme *più tardi.* Ora che un nuovo elemento, energico, s'introduce, la cosa diventa molto più facile. Per altro fo notare che appunto per Enriques converrebbe differire ancora di qualche anno l'impresa: perché, buono o cattivo che sia, il regolamento vuole che egli si prepari titoli per la sua promozione ad ordinario; sicché per qualche tempo è meglio che egli dedichi la sua operosità scientifica solo a ricerche originali. Vi è poi la solita difficoltà dell'editore: se fosse possibile ricorrere a Gauthier-Villars oppure a Teubner, andrebbe bene! Del resto io approvo completamente l'idea, e credo che fra noi tre si potrebbe fare un buon trattato, armonico, originale. Qualunque sia la forma di collaborazione da adottare, io ho già un modo sicuro di prender parte all'opera: e cioè coi sunti (da me fatti) di vari miei corsi: cioè di qualche corso generale sulle curve, superficie ecc. algebriche, e di corsi speciali sulle singolarità superiori, sulla geometria sopra una curva, sulle superficie razionali e sistemi lineari di curve piane, ecc. Se l'impresa verrà ritardata di qualche anno, qualche nuovo corso farò che potrà pur servire. Inoltre io debbo preparare per l'Enciclopedia tedesca l'articolo sugl'iperspazi (varietà algebriche), e ciò potrà pur servire per capitoli corrispondenti del trattato (analogamente per l'articolo sulle superficie algebriche promesso da Guido [Castelnuovo] all'Enciclopedia). Approvo poi pienamente che il trattato sia di noi tre e non d'altri: perché noi tre siamo in piena uniformità d'idee; ed altri forse turberebbe l'armonia dell'opera. Avrò piacere se mi scriverete altro su questo argomento.").

Nevertheless, this second project too was postponed, both because of Segre's recurring hesitations, and because of the fear of damaging Enriques, busy preparing his dossier in view of the competition for the Bologna chair.[64] In April 1898,

[64]Cf. ANL-Castelnuovo, Segre to Castelnuovo, 26 January 1897: "As Ghigo [Federigo Enriques] will have written to you, following your letter I decided to propose to him that we postpone the grand project of our Treatise to later. However, it is right to have spoken of it: perhaps the idea will be realized!" ("Come t'avrà scritto Ghigo, in seguito alla tua lettera io mi son deciso a proporgli di rimandare a più tardi il grandioso progetto del nostro Trattato. Però è bene che ne abbiamo parlato: forse l'idea avrà seguito!").

however, the project seemed to be becoming concrete and Segre committed himself with Teubner to bring out the treatise in his name, in German:

> I am working on that blessed article for the *Encyclopaedia*. As if this were not enough, I am negotiating with Teubner for the publication of my *Vorlesungen* on higher geometry, based on some courses that I have already delivered, but enriched, etc. It is another enterprise that will then occupy me for several years, and I hesitated a lot before deciding. (ANL-Volterra, Segre to Volterra, 23 April 1898: "Lavoro attorno a quel benedetto articolo per l'*Enciclopedia*. Come se questo non bastasse, sto trattando col Teubner per la pubblicazione di mie *Vorlesungen* di geometria superiore, tolte da alcuni corsi che ho già fatto, ma arricchite, ecc. È un'altra impresa che mi occuperà poi per vari anni, ed ho esitato molto prima di decidermi.").

The outline of the text, the distribution of the contents and the financial aspects were illustrated and discussed with Castelnuovo and Enriques between 1899 and 1900.[65] Segre's *Vorlesungen* would have chained the classical texts by Salmon-Fiedler (1874) and Clebsch (1876) to the volume on algebraic surfaces, commissioned by Teubner from Castelnuovo and Enriques. The *Vorlesungen* were intended to offer a balanced overview, extensive but not encyclopaedic, of recent researches on algebraic geometry conducted by Italians.

Against the quite advanced schedule of this 'collective publication', it is strange that after 1900 no traces of the manuscript of these *Vorlesungen* have been spotted, except for a mention of their imminent publication in the *Bulletin of the American Mathematical Society*[66] and the advertising announcement which appeared "for several years in the Catalogues of the Teubner publishing house" (see Fig. 2.1; Loria 1924, 13; Terracini 1961, 12).

It is quite possible that other commitments distracted Segre's attention from this task, above all the one connected to his collaboration, direct and indirect, on the *Encyklopädie der mathematische Wissenschaften*, a collaboration that can be ascribed to the same strategy of international diffusion of the results attained by the Italian School of Algebraic Geometry that he had put in place.[67] As far as concerns

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[65]Cf. ANL-Castelnuovo, Segre to Castelnuovo, 9 August 1899 and 13 February 1900, Annexes 26 and 27 to this article.

[66]Notes, *Bulletin of the AMS*, 6 (1900): 305–312.

[67]The *Encyklopädie der mathematischen Wissenschaften,* printed by Teubner in Leipzig, published the followings essays by members of the School of Segre: F. Enriques, Prinzipien der Geometrie, Bd. III.1, W.F. Weber and H. Mohrmann (eds., 1907: 1–129); G. Fano, Gegensatz von synthetischer und analytischer Geometrie in seiner historischen Entwicklung im XIX Jahrhundert, *Ibidem*: 221–288; G. Fano, Kontinuierliche geometrische Gruppen. Die Gruppentheorie als geometrisches Einleitungsprinzip, *Ibidem*: 289–388; Segre (1921c); G. Castelnuovo and F. Enriques, 6a Grundeigenschaften der algebraischen Flächen, Bd. III.2, W.F. Weber and H. Mohrmann (eds., 1915: 635–673, chapter sent in 1908); G. Castelnuovo and F. Enriques, 6b Die algebraischen Flächen vom Gesichtspunkte der birationalen Transformationen aus, *Ibidem*: 675–767, chapter sent in 1908; G. Loria, Spezielle ebene algebraischen Kurven von einer Ordnung hoher als den vierten, *Ibidem*: 571–634.

28 Mathematik, techn. u. Naturwissenschaften. [Mitteilungen

F. Dingeldey, Kegelschnitte und Kegelschnittsysteme.
F. Dingeldey, Sammlung von Aufgaben zur Anwendung der Differential- und Integralrechnung.
F. Enriques, Prinzipien der Geometrie.
J. Harkness, elliptische Funktionen.
G. Kohn, rationale Kurven.
A. Krazer, Handbuch der Lehre von den Thetafunktionen.
R. v. Lilienthal, Differentialgeometrie.
G. Loria, spezielle, algebraische und transcendente Kurven der Ebene. Theorie und Geschichte.
R. Mehmke, über graphisches Rechnen und über Rechenmaschinen, sowie über numerisches Rechnen.
E. Netto, Kombinatorik.
E. Pascal, Determinanten. Theorie und Anwendungen. [Unter der Presse.]
S. Pincherle, Funktional-Gleichungen und -Operationen.
A. Pringsheim, Vorlesungen über Zahlen- und Funktionenlehre. (Elementare Theorie der unendlichen Algorithmen und der analytischen Funktionen einer komplexen Veränderlichen.) Bd. I. Zahlenlehre. Bd. II. Funktionenlehre.
C. Segre, Vorlesungen über ~~Algebraische~~ Geometrie [der algebraischen Gebilden,] mit besonderer Berücksichtigung der mehrdimensionalen Räume.
D. Seliwanoff, Differenzenrechnung.
M. Simon, Elementargeometrie.
P. Stäckel, Differentialgeometrie höherer Mannigfaltigkeiten.
O. Staude, Flächen und Flächensysteme zweiter Ordnung.
O. Stolz und **J. A. Gmeiner,** theoretische Arithmetik.
R. Sturm, Theorie der geometrischen Verwandtschaften.
R. Sturm, die kubische Raumkurve.
K. Th. Vahlen, Geschichte des Fundamentalsatzes der Algebra.
K. Th. Vahlen, Geschichte des Sturmschen Satzes.
A. Voss, Abbildung und Abwicklung der krummen Flächen.
E. v. Weber, Vorlesungen über das Pfaffsche Problem u. die Theorie der partiellen Differentialgleichungen 1. O. [Erscheint Ende März 1900.]
A. Wiman, endliche Gruppen linearer Transformationen.
H. G. Zeuthen, die abzählenden Methoden der Geometrie.

In Aussicht genommen:

W. Wirtinger, algebraische Funktionen und ihre Integrale.
W. Wirtinger, partielle Differentialgleichungen.

☞ Mitteilungen über weitere Bände werden baldigst folgen.

Mitteilungen der Mathematischen Gesellschaft in Hamburg. Band III, Heft 10. Ausgegeben im Februar 1900. Redigiert von Repsold, Schröder und Busche. [19 S.] gr. 8. geh. n. *M.* —.80.

Inhalt: Michael Stifels handschriftlicher Nachlafs. Von *Edmund Hoppe.* — Über Reduktion linearer Modulsysteme. Von *Otto Pund.* — Verzeichnis der Abhandlungen des 1890 verstorbenen Mitgliedes der Mathematischen Gesellschaft Wilhelm Lazarus. Nach *Orthmann* und nach dem Katalog der Mathem. Gesellschaft. — Bericht über das Gesellschaftsjahr 1899/1900. Zusammengestellt vom Adjunkten Dr. *J. Schröder.*

—— —— —— Band III. 1891—1900. [IV u. 429 S.] gr. 8. geh. n. *M.* 10.60.

Fig. 2.1 Teubner's announcement of Segre's *Vorlesungen* with autograph corrections by Segre (UTo-ACS) this aspect, we may remind that the first Italian mathematicians to be asked by F. Klein and F. Meyer to join the authors of the third volume of the *Encyklopädie* were Castelnuovo, Enriques and Fano, in September 1895.[68] Segre indeed supervised the progress of the work, from the very beginning, readily advising these 'disciples' on the structure to give their own essays, suggesting the suitable content extension of their chapters, and offering to revise himself their manuscripts:

> I am pleased that your *Bericht* is going ahead. Regarding what Meyer has asked Fano for, it seems to me that the latter would do well to agree: of course, provided that the theme is specified better than it is by the *Jahrbuch*: that is to say "algebraic transformations of algebraic entities in 2 and higher dimensions." For my part, while I could not myself accept such an assignment, I will willingly help, for what I am capable, whoever does it. As you say, it cannot be a heavy labour! (ANL-Castelnuovo, Segre to Castelnuovo, 23 September 1895: "Ho piacere che il vostro *Bericht* vada avanti. Quanto a quello che il Meyer domanda al Fano, mi pare che questi farebbe bene ad accettare: s'intende, purché il tema fosse meglio precisato di quel che non sia dal *Jahrbuc*h: vale a dire "trasformazioni algebriche degli enti algebrici a 2 e più dimensioni". Per parte mia, mentre non potrei accettare io stesso un tal lavoro, ajuterò volontieri, per quel che valgo, chi lo farà. Come tu dici, non dev'essere poi una gran fatica!").

Taken up with numerous teaching and institutional responsibilities, at last Segre reluctantly forewent the revision of the first article by Castelnuovo and Enriques (ANL-Castelnuovo, Segre to Castelnuovo, 7 January 1896) and, in actual fact, until the first months of 1896 he limited himself to marginally contributing to the *Encyklopädie*. In May, however, he was entrusted with writing the article on hyperspace geometry. As he announced to Castelnuovo, it was:

> about 2 sextodecimos; deadline for presentation 1st of 1899. It seems to me that the general programme of the work is very solid: and I have decided to accept. (ANL-Castelnuovo, Segre to Castelnuovo, 14 May 1896: "circa 2 fogli di stampa; termine per la presentazione 1° del 1899. Mi è parso che il programma complessivo dell'opera sia molto serio: e mi son deciso ad accettare.").[69]

From that moment on, Segre became in actual fact the Italian 'delegate' for the third volume of the *Encyklopädie*.[70] In addition to collecting sources for his chapter, to the writing of which he devoted himself for years, with immense

[68]On the *Encyklopädie der mathematischen Wissenschaften* and on the partnership that were established between the German and Italian mathematical communities cf. von Dyck (1909), Gispert (2001), Luciano and Roero (2012, 32–37, 51–53, 64, 185, 199–200, 204, 213), Luciano and Roero (2017).

[69]Segre is mentioned as being in the team of the *Encyklopädie* in Notes, *Bulletin of the AMS*, 5 (1898): 150–157; 6 (1900): 213–219.

[70]Cf. ANL-Castelnuovo, Segre to Castelnuovo, 14 May 1896 and 11 June 1896: "You are very right to accept the theme of algebraic surfaces for the *Encyclopaedia*. Indeed, I would probably have written about it myself if Meyer hadn't preceded me." ("Fai benissimo ad accettare per l'*Enciclopedia* il tema delle superficie algebriche. Anzi, probabilmente te ne avrei scritto io stesso se il Meyer non mi preveniva.").

zeal,[71] he was charged by Klein to make contact with other contributors, coordinated the work of the members of his School and, in order to avoid overlaps and omissions, discussed with Fano, Castelnuovo and Enriques the distribution of the contents among the essays on projective hyperspace geometry, on linear differential equations, and on the theory of algebraic surfaces:

> For a while I have been collecting sources for my article in the *Encyclopaedia*. Now I am concerned to specify my task clearly; and, among other things, to have from you and Enriques the assurance that in your article IIIC8 "Algebraische Transformationen und Correspondenzen" which comes *after* mine, you will also deal with the algebraic transformations of S_n (excluding projective ones), that is to say that you, not me, will do what Noether expounded in the *M*[*athematische*] *A*[*nnalen*] *II* on such transformations, etc., and Del Pezzo and S. Kantor in some recent works, etc. It seems appropriate to me that that article of yours, as it encompasses together transformations of the plane and space, should also contain those of hyperspaces: and this (I repeat) also because, otherwise, I should speak of hyperspace transformations before those of the plane and of S_3 are explained. I will also have to find out whether Zeuthen in IIIC10 will present the numerative geometry of S_n in addition to that of S_3. (ANL-Castelnuovo, Segre to Castelnuovo, 19 February 1898: "Da qualche tempo raccolgo materiali pel mio articolo dell'Enciclopedia. Ora m'importa precisare bene il mio compito; e fra l'altro, avere da te e da Enriques l'assicurazione che nel vostro articolo IIIC8 "Algebraische Transformationen und Correspondenzen" il quale vien *dopo* del mio, tratterete anche le trasformazioni algebriche di S_n (escluse le projettive) cioè farete voi, e non io, quel tanto che su tali trasformazioni dà Noether nei M. A. II, ecc., Del Pezzo e S. Kantor in alcuni recenti lavori, ecc. Mi pare opportuno che quel vostro art[icolo], come abbraccia insieme le trasformazioni del piano e dello spazio, così contenga anche quelle degl'iperspazi: e ciò (ripeto) anche perché, in caso contrario, io dovrei parlare di

[71]Way back the summer of 1897 Segre announced to Castelnuovo that he was dealing "exclusively with the future lectures and the article on hyperspaces for the *Encyclopaedia*." (ANL-Castelnuovo, Segre to Castelnuovo, 7 September 1897: "esclusivamente delle future lezioni e dell'articolo sugl'iperspazi per l'*Enciclopedia*"). Writing this essay induced Segre to return to his youthful studies, and to consult a broad collection of articles and volumes, so much so that he confessed to Castelnuovo as long ago as 4 October 1899: "I'm sending you the papers I wrote many years ago on that subject. I remember that I stopped that research because I didn't have sufficient knowledge of the theory of algebraic numbers. … Reading the article by Hilbert and seeing cleverly exploited in it the properties of fields of algebraic numbers confirmed to me my old idea that this was my weak point. I'm going on working for the article for the *Encyclopaedia* but now I despair of finishing it for the end of the year!" (ANL-Castelnuovo: "T'invio le carte che avevo scritto tanti anni fa su quell'argomento. Mi ricordo che mi ero fermato nella ricerca perché non avevo sufficiente cognizione della teoria dei numeri algebrici. … Leggendo l'articolo di Hilbert e vedendovi sfruttate abilmente le proprietà dei corpi di numeri algebrici, mi confermavo nella mia idea antica che lì era il mio punto debole. Continuo a lavorare per l'articolo dell'*Enciclopedia*: ma ormai dispero di condurlo a termine per la fine dell'anno!"). The chapter, completed in 1912, was only published after the end of the Great War (Segre 1921c). On this overview of hyperspatial geometry different judgments were expressed: "The longest, and by far the most important of these was his article in the *Mathematical Encyclopedia* on the geometry of *n*-space. The thought that he must one day complete this, depressed his spirits at times for a good many years, for he was one who took his responsibilities seriously, and he felt in honor bound to put the thing through. Complete it he finally did, thus earning the gratitude of geometers for many years to come." (Coolidge 1927, 355). By contrast (Baker 1926, 271): "For completeness of detail, breadth of view, and generous recognition of the work of a host of other writers, this must remain for many years a monument of the comprehensiveness of the man." Among the alumni in the School of Segre, many stigmatized the 'injustice'of Coolidge's comment (Terracini 1961, 12; Togliatti 1963, XI).

trasformazioni iperspaziali prima che sian state trattate quelle del piano e di S_3. Dovrò anche informarmi se Zeuthen in IIIC10 darà la geometria numerativa di S_n oltre a quella di S_3").[72]

4 Study Trips and Sojourns

In the activity of a mathematical School an important component is the oral tradition, that is to say the habit of verbal exchanges, which contribute in a decisive way to the collective construction, the sharing and transmission of knowledge between the *Maestro* and the disciples. From this point of view we can state that the Italian team of algebraic geometers was an authentic School, linked to a very precise local milieu, constituted by the University of Turin and some 'satellite' contexts like the Academy of Sciences, in whose meetings Segre assiduously participated, cultural cafes like *Giaccardi*, *Bergia* and the *American Bar*, where the so-called *Pitareide* met,[73] and Segre's house. In his 'little studio' various disciples and colleagues were entertained, both Italians and foreigners, cf. (Terracini 1968, 9, 13 and Annexes 29, 30, 31, 49, 57, 58, 71 and 72).

[72]Cf. also ANL-Castelnuovo, Segre to Castelnuovo, 28 August 1898: "I have continued to accumulate material for my article on hyperspaces. In extracting from a Note of yours presented at the Accademia dei Lincei the number of secant spaces of a curve in certain cases, I thought about asking you if you don't have the number (when it is finite) of the secant spaces of a curve in all cases, and if you couldn't immediately publish it, so that I can give it in my article. We also need to agree on *surfaces*, to separate clearly what is given by me and what comes from your and Enriques'article on surfaces. For reasons of uniformity it seems appropriate to me to leave all the properties of geometry on the surface, even if obtained by means of hyperspaces; I would limit myself in general, regarding the surfaces of hyperspaces, to projective properties. In this way I will reproduce your results on surfaces with elliptic sections, hyper-elliptic, of type 3, etc.: not those on canonical curves, on the bigeneric (on surfaces considered in the work on the rationality of plane involutions, etc.; or, if I need to, I will only mention them. Tell me if you approve, and if Enriques approves." ("Ho continuato ad accumulare materiali pel mio articolo degli iperspazi. Nell'estrarre da una tua Nota lincea il numero degli spazi secanti di una curva in certi casi, ho pensato di domandarti se tu non possiedi il numero (quand'è finito) degli spazi secanti di una curva in tutti i casi, e se non potresti pubblicarlo subito, affinché io lo possa dare nel mio articolo. Occorre anche che ci accordiamo riguardo alle *superficie*, per separare bene quel che spetta a me e quel che spetta all'articolo tuo e di Enriques sulle superficie. Per ragioni di uniformità mi pare conveniente di lasciare a voi tutte quante le proprietà di geometria sulla superficie, anche se ottenute col mezzo degl'iperspazi; io mi limiterei in generale, quanto alle superficie degl'iperspazi, alle proprietà projettive. Così io riprodurrò i risultati tuoi sulle superficie a sezioni ellittiche, iperellittiche, di genere 3, ecc.: non quelli su curve canoniche, sul bigenere (sulle superficie considerate nel lavoro sulla razionalità delle involuzioni piane, ecc.; od, occorrendomi, li citerò soltanto. Dimmi se approvi tu, e se approva Enriques.").

[73]Cf. ANL-Castelnuovo, Segre to Castelnuovo, 28 November 1891; Segre to Amodeo, 24 November 1891 and Castelnuovo to Amodeo, 30 November 1891 (Palladino and Palladino 2006, 186, 282–283). The group of young mathematicians that met in these Turin cafes, at least until 1891, colloquially called the *Pitareide*, included members of both Peano's and Segre's Schools. Castelnuovo, Fano, B. Levi, Pieri, Amodeo, A. Tanturri and P. Predella were used to frequent the *Pitareide*. On the events that led to the break-up of the initial group cf. Giacardi (2001a, 147) and Roero (2004, 139).

That algebraic geometry on the curve was born of dialogues between Segre and Castelnuovo under the porticos of Via Po is not an hyperbole (Conte 1993, 438). The Piedmont geometer, in effect, assigned great importance to conversation, and generally to all those vectors of scientific sociability (private teaching, seminars, reunions, working lunches, etc.) that were set alongside institutional education and, allowing greater freedom of expression and debate between the interlocutors, proved particularly useful in the creative phase of research activity.

Significant in this connection is the role of study trips and sojourns by Italian geometers abroad, and, reciprocally, those of foreigners in Italy. Heir to the tradition of the *Grand Tour* of the Baroque age and the Enlightenment, the practice of going abroad for a period of master-class is certainly not a prerogative of the School of Algebraic Geometry. Back in the Risorgimento period, mathematicians like C.I. Giulio, L.F. Menabrea, L. Cremona and Q. Sella had grasped the advantage of offering international training to their best alumni. Therefore they had sent them to complete their studies in French and German *Ecoles*, or in the most dynamic environments from the cultural, industrial, manufacturing, agricultural, mining and technological points of view. In turn, these scholars had known how to exploit to the best the meetings with outstanding scholars visiting our country for work, amusement or health, and had derived fruitful stimuli from conversations with promising scientists.[74]

Having trained in contact with *Maestri* like F. Faà di Bruno, E. D'Ovidio, F. Siacci and A. Genocchi, who had been capable of treasuring international interplays, Segre entertained the idea of a trip to Germany since 1884. He chose as his privileged destination the University of Göttingen, an emblem, at a world level, of the culture of orality, toward which he would then oriented many of his alumni (Luciano and Roero 2012, 47–49).

In the summer of 1891, accompanied by Gino Loria, Segre stayed in Göttingen a few days (29–30 June and 11 July). However, for the rest of his life, he was to preserve an indelible memory of the 'School of Klein', and above all of the 'luminous conversations' with the German *Maestro*. In their itinerary the two Italians touched on Frankfurt am Main (25 June), Berlin (4–8 July), Dresden, Leipzig, Nuremberg (15 July) and finally Munich (16–17 July), before returning homeland on 18 July.[75]

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[74]Cf. Dröscher (1992), Bottazzini (1994), Gouzévithc (1993), Gouzévitch and Gouzévitch (2002), Becvárová and Becvár (2006), Gouzévitch (2006), Brianta (2007), Lacaita (2009), Dhondt (2008), Luciano and Roero (2012, 45–55), Pepe (2012), Ferraresi and Signori (2012), Roero (2013, 383–388, 414–439, 485–507, 514–543), Israel (2016).

[75]ANL-Castelnuovo, Segre to Castelnuovo, 20 July 1891: "In Munich before leaving I received your postcard of 16th, and now I've got yesterday's. I came here yesterday afternoon and found on my table such a massive amount of works to read, matters to sort out, that I already foresee that I won't find the time to write to Veronese." ("Ebbi a Monaco prima di partire la tua cartolina del 16, ed ora ricevo quella di ieri. Io sono giunto qui ieri nel pomeriggio e mi son trovato sul tavolo una tal mole di cose da leggere, di faccende da sbrigare, che vedo già che non troverò il tempo di scrivere a Veronese.").

Although partially spoiled by the controversy between G. Peano and G. Veronese, which in the meantime had broken out in the pages of the *Rivista di Matematica*, the experience of the 'German journey' was fully exploited by Segre. He visited libraries and scientific institutes; he assisted, on an invitation from Kronecker and Weierstrass, at a meeting of the Berlin Academy; he attended the *Mathematische Verein* in Göttingen; he promoted the output of the Italian geometers, and above all that of Castelnuovo, publicizing it to Klein, Nöther and Rohn; and he strengthened the friendship and the scientific collaborations that he had already opened with Reye and Sturm (Boggio 1928, 305). The notes he took in Germany, the cutting-edge literature consulted in the Göttingen Library and the 'impressions' of his 1891 trip spangled the letters to Castelnuovo and F. Amodeo, even months after Segres's return to Turin.[76]

The journey by Segre and Loria on one side consolidated the tradition of the study sojourns in Germany, inaugurated by A. Tonelli (1874–75) and continued by A. Abetti (1876), E. Caporali (1877–78), C. Romaniello (1877–78), S. Pincherle (1878), G. Ricci Curbastro (1878–79), L. Bianchi (1879–80), A. Capelli (1879–80), G. Veronese (1880–81), F. Gerbaldi (1882–83), G. Morera (1883–84), E. Pascal (1888) and R. Marcolongo (1888–89). On the other side it launched the trend of scientific 'pilgrimages' to Göttingen. In the ensuing years, numerous Italian mathematicians, went to the 'School of Klein': V. Volterra (1891, 1904, 1914), G. Vailati (1899, 1906), Castelnuovo (1903), Enriques (1903), Severi (6–8 January 1937) and E. Bompiani (summer semester 1913). In his conversations with Klein and Hilbert in Göttingen, Enriques, for example, several times dealt with the issues regarding mathematics education and the new developments in algebraic geometry. Klein and Enriques also planned some joint publishing ventures, seeking to establish an agreement between the Teubner and Zanichelli companies, in view of special sale prices of volumes, treatises, textbooks and journals.

In addition to geometers that spent study sojourns in foreign countries on their own accord, there were disciples of Segre's that—urged by the *Maestro*—sojourned in Germany and France with ministerial scholarships. Segre for instance carefully monitored the works of the Junta of the Higher Council that assigned posts for master-class abroad. In particular he attended its meetings in the academic year

[76]ANL-Castelnuovo, Segre to Castelnuovo, 8 August 1891: "You have more things to tell me than I have you … if you except the various impressions of the trip to Germany that I intend to recount you, illustrating them with the photos of Reye, Sturm, Nöther, etc. etc. (…) Poor Simplicio [Amodeo]! Speaking with him, I indeed had immediately to delve in all possible details on my German trip … He even wanted me to describe what Klein's house was like, how his studio was set out, how many chairs there were, etc. etc.: in which I was not able to satisfy him." ("Tu hai più cose da raccontarmi che non ne abbia io per te … Se eccettui le varie impressioni del viaggio in Germania che mi riservo di esporti a voce, illustrandole con le fotografie di Reye, Sturm, Nöther, ecc. ecc. (…) Povero Simplicio [Amodeo]! A lui sì che ho dovuto dare subito tutti i dettagli possibili sul mio viaggio germanico … Voleva persino che io gli descrivessi com'era fatta la casa di Klein, come era disposto il suo studio, quante sedie vi erano, ecc. ecc.: nel che non l'ho potuto soddisfare.").

1893–94, when G. Fano was given a grant to spend the winter semester in Göttingen, and then at the time of his return to Italy.[77]

Moreover the summer holiday periods by members of the School of Segre in famous places in the Alps, frequented by foreign colleagues like Hurwitz, Hilbert and Klein, afforded unexpected opportunities to establish scientific and personal bonds and opened up new scenarios for international promotion of the team.[78]

The relevance of oral exchanges, in the framework of the activity of a mathematical School and for its success on a supranational scale, was fully acknowledged by Segre's disciples: the homes of the Enriques and Castelnuovo families, in Bologna and Rome, in turn became known for being a 'cosmopolite theatre of debates' among the most eminent scholars of the period (Enriques 1983; Parikh 1991; De Benedetti 2001; Linguerri and Simili 2008). And likewise, the charm and brilliant conversations of Fano, Severi, Terracini and B. Levi were to be important in promoting knowledge of Italian algebraic geometry abroad, not only in Great Britain (1923), Japan (1936) and Switzerland (1940), but also in peripheral or developing countries, like Argentina (Luciano 2016).

If many members of Segre's School crossed the borders to go to foreign countries, no less numerous were the outlanders that chose to stay in Turin to attend Segre's lectures, or to get acquainted with him in Rome, Ancona, Naples, and Engadina. We can for instance mention C. Hermite in 1892,[79] F. Mouton in 1895,[80] F. Lindemann in 1897,[81] G. Mittag-Leffler in 1899, H.G. Zeuthen in June 1900,[82]

[77]ANL-Castelnuovo, Segre to Castelnuovo, 3 October 1893: "Fano wrote to me that between the position as assistant offered again to him and the opportunity to complete his training in Göttingen he opted for the latter." ("Fano mi scrisse che fra l'assistentato ripropostogli e il perfezionamento a Gottinga s'è deciso per quest'ultimo."). 30 May 1894: "In the Easter holidays Fano visited me, and he told me that after this stay in Germany he greatly desires to return to the Italian Universities, and that is to say to Turin or Rome." ("Nelle ferie pasquali il Fano mi venne a trovare, e mi disse che dopo questo soggiorno in Germania desidera vivamente rientrare nelle Università italiane, e cioè a Torino od a Roma."). 22 August 1894; "If Fano will choose to go to Paris it will be *faute de mieux:* from what he told me and wrote to me, his ambition seemed to be to come to study under your direction." ("Se il Fano andrà a Parigi sarà *faute de mieux*: da quanto mi disse e scrisse, il suo ideale pareva che fosse di recarsi presso di te"). In Göttingen Fano was particularly appreciated by Klein, so much so that Klein offered him an opportunity to be appointed there as professor in 1899 (Luciano and Roero 2012, 49–50).

[78]ANL-Castelnuovo, Segre to Castelnuovo, 29 August 1891: "A pity you didn't advise me you were travelling to Pontresina. Precisely in this period (about the 20th and afterwards) I learnt from Hurwitz that he would spend 2 weeks there and I could have organized an encounter between you there!" ("Peccato che non m'hai avvertito che andavi a Pontresina. Appunto in questo periodo (verso il 20 e seguenti) sapevo da Hurwitz che egli ci sarebbe stato per passarvi 14 giorni ed avrei potuto farvi incontrare colà!").

[79]ANL-Castelnuovo, Segre to Castelnuovo, 6 January 1892: "I congratulate you on the visit from Hermite. It was a serious lacuna not knowing him!" ("Mi congratulo teco per la visita ricevuta dall'Hermite. Era una grave lacuna il non conoscerlo!"); cf. also *ibidem*, 5 March 1892.

[80]Cf. ANL-Castelnuovo, Segre to Castelnuovo, 2 October 1895 and 11 October 1895.

[81]Cf. ANL-Volterra, Segre to Volterra, 11 August 1899.

[82]Cf. ANL-Volterra, Segre to Volterra, (s.d.) and Segre to Volterra, 13 June 1900.

C. Stéphanos, E. Study and J. Coolidge in 1903,[83] E.J. Wilczynski in 1904 and 1906,[84] L.W. Dowling in 1907,[85] V. Snyder in 1922[86] and E. Čech in 1921–22, alongside with F. Klein, who travelled round Italy several times: in the summer of 1874, in the 1878 Easter holidays, again in March 1899 and in the spring of 1900.[87]

Further, among the professors at the University of Turin, Segre was one of the few that could boast of an international audience of students in his classrooms. As alumni, colleagues and family members recalled, starting from the end of the 19th century:

> the fame of his skill as a *Maestro* went far beyond the borders of our nation, and more or less every year scholars from other countries flocked to listen to his lectures, especially ones from England and North America, who from what they learnt in Italy often drew the inspiration for fine publications (Berzolari 1924, 532).[88]

The first to sojourn in Turin to take Segre's courses and to carry out studies on higher geometry, under his direction, were Grace Chisholm (1868–1944) and his husband William Henry Young (1863–1942) (Grattan-Guinness 1972, 105–185; Conte and Giacardi in this volume). Having arrived in Turin in October 1898, they stayed until March of 1899 and attended the Higher Geometry course devoted to the *Curve algebriche dei vari spazî* (Segre's *Notebook* in 12 Giacardi 2013). Showing great friendliness, Segre not only integrated his university teaching with private lessons and domestic lectures imparted to them, but on 30 April 1899 presented to the Academy of Sciences two notes, respectively by Chisholm, "Sulla varietà

[83]Cf. Study to O. Michelli Segre, 6 August 1924 and Coolidge to O. Michelli Segre, 20 September 1924, Annexes 71 and 72.

[84]Cf. Segre to Wilczynski, 18 March 1904, 27 April 1904 and 4 June 1906, Annexes 29, 30, 49. On the influence of Segre's results on Wilczynski cf. Wilczynski (1911): 3, 7.

[85]Cf. Sisam to Wilczynski, 12 December 1908, Annex 57.

[86]Cf. Snyder to Segre, 21 November 1922 and 19 February 1923, Annexes 61 and 65.

[87]On the verge of designing volumes 4 and 5 of the *Encyklopädie der Mathematische Wissenschaften*, related to applied mathematics, Klein scheduled a sojourn in Turin from 21 to 23 March 1899 and asked Volterra to help to contact potential collaborators for this enterprise. It was Volterra himself that organized Klein's stay, planning his meetings at the Academy of Sciences with the members of the various Schools: that of Electrotechnics founded by Galileo Ferraris (L. Lombardi, R. Arnò, L. Ferraris, G. Grassi, F. Lori), that of Geodesy and Practical Geometry directed by N. Jadanza (V. Baggi, C. Aimonetti), that of Statics (C. Guidi, E. Ovazza), and that of Applied Mechanics (G. Bertoldo, S. Cappa). As far as the mathematical community concerns, Volterra drafted the detailed programme of Klein's rendezvous with E. D'Ovidio, N. Jadanza, C. Segre, G. Peano, L. Berzolari, F. Porro, A. Garbasso, O. Zanotti Bianco, M. Pieri, R. Bettazzi and B. Levi, as well as those with the Youngs, at that time in Turin (ANL-Volterra, Klein to Volterra, 27 January 1899; Volterra to Klein, 29 January 1899, 10 March 1899, 11–13 March 1899; Klein to Volterra, 14 March 1899). Klein returned to Turin in March 1900, during the Easter holidays, to continue to discuss with Volterra and Tedone the agenda of the *Encyklopädie* (ANL-Volterra, Klein to Volterra, 30 December 1899, 12 February 1900, 1 March 1900, 15 March 1900; Volterra to Klein, 23 March 1900). The correspondence between F. Klein and V. Volterra in the 1892–1912 period is being edited: Luciano and Roero (2017).

[88]Cf. also Fano (1924–25, 225), Viglezio (1924, 2), Boggio (1928, 319), B. Segre (1963–64, 18), Fuà Segre (1952, 125).

razionale normale di M_3^4 di S_6 rappresentante della trigonometria sferica", and by Young, "Sulle sizigie che legano le relazioni quadratiche fra le coordinate di retta in S_4".[89] Furthermore Segre put the Youngs in contact with Castelnuovo:

> Until 18 November we won't start our lectures. In my audience I will have the Youngs, whom I believe know you. Yesterday I had a visit from the husband, who gave me a brief Note of his, quite good. If you can then send him some offprints of yours (especially on M^3, geometry of the straight line in S_4, numerative geometry) it will be a good thing. (ANL-Castelnuovo, Segre to Castelnuovo, 23 October 1898: "Fino al 18 Novembre non cominceranno le nostre lezioni. Avrò fra i miei uditori i coniugi Young, che credo ti conoscano. Ieri ho avuto la visita del marito, che mi ha dato una sua breve Nota, abbastanza buonina. Se potrai inviargli poi qualcosa di tuo (specialmente su M^3, geometria della retta in S_4, questioni numerative) farai bene").[90]

Chisholm and Young preserved a good memory of their Turin stay, together with deep gratitude for the generosity of the *Maestro* to them (cf. Chisholm Young to O. Michelli Segre, 19 June 1924, Annex 68).

From the United States there then came Julian Lowell Coolidge (1873–1954) to complete his training under Segre's guidance. Engaged in a tour of European universities between 1902 and 1904, the American geometer, together with his wife Theresa Reynolds, spent in Turin the winter semester from October 1903 to the spring of 1904. In this period he published the first part of the essay "Les congruences isotropes qui servent à répresenter les fonctions d'une variable complexe" in the *Atti* of the Academy of Sciences, under the presentation by Segre in the meeting of 20 December 1903 (Struik 1955, 671–672; Hammond et al. 1955; Dauben 1999).[91] Coolidge's experience 'at Segre's School' was shortly afterwards described as follows in the article "The Opportunities for Mathematical Study in Italy":

> At the same time such headings as Higher Analysis, Higher Geometry are so comprehensive as to leave to the teacher the greatest discretion in the choice of material. Some fortunate professors give a new course each year, others run through a cycle including a greater or less number of subjects. Americans are sure to find lectures on subjects that will interest them, and they will have the French, rather than the German standard of clearness and elegance. They will also be struck by the eclecticism of the instructor, for Italian mathematicians read widely. I remember being impressed at the beginning of one course of lectures by the fact that the professor put down, as principal works of reference, books in four different languages, and remarked that those of his hearers who could not read English,

[89]Chisholm (1899), Young (1899). Cf. Segre to Chisholm, 11 March 1899, Conte and Giacardi 2017. Some years later, in the sitting of 18 November 1905, Segre presented to the Turin academy the article by Young and Chisholm (1907), *Note on Bertini's transformation of a curve into one possessing only nodes*.

[90]Cf. also ANL-Castelnuovo, Segre to Castelnuovo, 9 November 1898: "When you want to send some papers to the Youngs, address it to me, because they have not found a lodging to their satisfaction yet." ("Quando vuoi inviare qualcosa ai signori Young, dirigi a me, perché essi non hanno ancora trovato un alloggio di loro soddisfazione.").

[91]Coolidge (1904a). The second part Coolidge (1905) was presented by Segre in the sitting of 8 January 1905. Regarding the influence of Segre on the American scholar see Brigaglia's paper in this volume.

> French and German, must certainly make up the deficiency in the course of the year (Coolidge 1904b, 13).

In fact, that year Segre began his lectures on *Applicazioni degli integrali abeliani alla Geometria*, giving references to writings by 23 foreign authors, among them W.F. Osgood, W. Wirtinger, A. von Brill and M. Nöther, B. Riemann, K. Weierstrass, A. Clebsch and P. Gordan, F. Neumann, E. Picard, C. Jordan, A. Forsyth, C.A. Briot and C. Bouquet, F. Klein, P. Appell and E. Goursat, H.F. Baker, K. Hensel, G. Landsberg, N. Abel and A. Hurwitz, and only quoting three Italian authors: L. Bianchi, F. Casorati and S. Pincherle (Segre's *Notebook* 17: 3–6, in Giacardi 2013).

The synergy between Coolidge and Segre was not interrupted at the end of the Turin sojourn. Although there were no further opportunities for encounters, the two mathematicians continued to correspond,[92] and in 1924, on Segre's death, Coolidge evoked his scientific and ethic stature in a long obituary (Coolidge 1927) and with these heartfelt words sent to Segre's wife:

> Je ne me flatte pas que vous vous souviendrez d'un Américain errant qui est arrivé à // Turin avec sa petite famille au moins d'Octobre 1903, pour suivre les cours de l'université, et surtout pour profiter de l'enseignement de votre illustre mari. Pour lui, pourtant, ça a été un évènement d'importance capitale. Non seulement a-t-il trouvé une impulsion scientifique dont il n'a cessé de profiter énormément depuis, mais, chose beaucoup plus précieuse, il a eu le privilège de nouer de liens d'amitié avec son maitre, que chaque année depuis n'a que rendu plus forts. Je ne saurais vous exprimer, madame, ni l'estime que je ressentis pour votre mari comme savant, ni l'affection qui me lié à lui. Toujours je serai fier d'avoir été à la fois de ses élèves et de ses amis (J. Coolidge to O. Michelli Segre, Annex 72).

Very effective in attracting disciples from America was the fact that between 1904 and 1920 Segre sent the outlines of his courses on Higher Geometry to be published in the *Bulletin of the American Mathematical Society*.[93] As a consequence from the United States in 1908 two other students of Segre's: Charles Herschel Sisam (1879–1964) and Clarence Lemuel Elisha Moore (1876–1931) hastened to Turin to hear his lectures.

An alumnus of V. Snyder, Sisam completed his Ph.D. in 1905 at Cornell University with the thesis *Classification of Scrolls of Order Seven Having a Rectilinear Directrix*. He then decided to refine his studies in Europe, with post-doc fellowship in Göttingen and Turin, before returning home, to Colorado College, where the rest of his brilliant career was played out. Having reached Piedmont at the

[92] Cf. Coolidge (1947^2, 219, 223–227, 247, 272–277, 390–391, 401).

[93] *Notes*, in *Bulletin of the AMS* 10 (1904): 321–324; 13 (1906): 87–94; 13 (1907): 522–530; 14 (1908): 504–508; 16 (1909): 40–48; 17 (1910): 47–55; 18 (1911): 34–42; 19 (1912): 33–43; 21 (1914): 43–50; 22 (1915): 41–47; 23 (1916): 46–54; 25 (1918): 39–45; 26 (1919): 87–93; 27 (1920): 39–46.

beginning of September 1908, Sisam attended Segre's course *Rassegna di concetti e metodi della Geometria moderna* (Segre's *Notebook* 22, in Giacardi 2013) and was a guest of the professor for private lessons, which proved very stimulating for him, as he confided to Wilczynski:

> In his lectures he speaks very distinctly and I have had no difficulty whatever in following them. In his lectures this year he is covering, in a general way, the entire field of Geometry. In our private conferences at his home he is very approachable, makes me feel entirely at liberty to come where I want to, and is very stimulating. I am working on some properties of triply infinite varieties in five dimensions, with references to line geometry. I am very much pleased with the results I have obtained thus far (Sisam to Wilczynski, 12 December 1908, Annex 57).[94]

During the months he spent in Turin, Sisam developed his researches under the guidance of Segre, who presented them at the Academy of Sciences, in the sitting of 5 March 1911.[95] Between Segre and Sisam a relationship *inter pares* was maintained, despite the professional, cultural and age difference between the two. On one side, thanks to Segre, Sisam corrected a significant mistake which he, and before him Wilczynski, had made:

> In a paper, published in the *Bulletin* for June, 1904, page 440, Mr. C. H. Sisam gives a proof for a theorem previously enunciated and proved by me in the *Mathematische Annalen*, volume 58, page 256. Unfortunately, however, he follows me in giving an inexact formulation of the theorem in question. I have used the word self-dual in a more restricted sense than is usual, without having properly called attention to the fact. As others may be misled also, a few words of explanation seem to be in order. A dualistic transformation may have the property of converting a ruled surface into itself without interchanging its generators, so that every generator of the surface is transformed into itself. It is merely of scrolls, for which such a transformation exists, that I wish to assert the theorem that they belong to a non-special linear complex. It is only to this case that Mr. Sisam's demonstration applies. There actually exist ruled surfaces, self-dual in the general sense, which do not belong to a linear complex. The following example of such surfaces is due to Professor Corrado Segre, who first called my attention to the fact that my theorem was badly formulated (Wilczynski 1904, 8).

At his turn, Segre was prompted by the sodality with Sisam to develop some studies of his own, and expressly thanked the American disciple in the article "Aggiunta alla memoria: Preliminari di una teoria delle varietà luoghi di spazi":

> Mr. C.H. Sisam, of the University of Illinois, has kindly pointed out to me that there are exceptions to the theorem (no. 21) enounced in the middle of p. 107 [here at pp. 96–97]. As far as the content of that no. 21 concerns, I also want to emphasize that, since the beginning of last year (1909), Dr. Sisam, who at that time was studying with me in Turin, presented a work of his to me on the V_3varieties that satisfy four or more homogeneous linear partial differential equations of order 2. I hope that that research is published soon (Segre 1910b, 346).

[94]Cf. also Sisam to Wilczynski, 29 March 1909, Annex 58 and Sisam to O. Michelli, 14 July 1924, Annex 70.

[95]C.H. Sisam, On Algebraic Hyperconical Connexes in Space of *r* Dimensions, *Atti R. Acc. Scienze di Torino* 46 (1911): 481–487.

In Sisam's case too, Segre's role as a *Maestro* was not limited to the Turin period[96] but went on until the twenties. Though having a single opportunity to return to Italy in 1928, as invited lecturer in Bologna, at the VIII International Congress of Mathematicians, Sisam kept in touch with Segre by letter.

By contrast, the recollection by Terracini, that the American Ellis Bagley Stouffer (1884–1965) also spent a study sojourn in Turin as a disciple of Segre's, is unfounded (Terracini 1968, 13). An alumnus of Wilczynski's, Stouffer took his Ph. D. in 1911, defending the thesis *Invariants of Linear Differential Equations, with Applications to Ruled Surfaces in Five-Dimensional Space*. Having become an Instructor in Mathematics at Drake University and the University of Illinois, he pursued most of his career at the University of Kansas, where he was an Assistant Professor (1914–1917), and Associate Professor (1917–1921), and finally a Full Professor and Dean of the Graduate School (1921–1955). An eminent and prolific scholar, Stouffer obtained a fellowship of the John Simon Guggenheim Memorial Foundation in 1926, for Mathematics, reserved for American and Canadian citizens. The grant was to finance a study sojourn of 10 months, beginning from 1 August 1926, to conduct comparative studies on the three general methods of projective differential geometry. Went in company with his wife Anna Lucile Shepard and their children, Stouffer spent most of his time in Italy, where he attended the courses of E. Bompiani at the University of Bologna (Fitch 1928, 331; Ciliberto and Sallent 2012, 156). He also sojourned in Turin, and returned to Italy in 1928, on the occasion of the International Congress of Mathematicians in Bologna, during which he strengthened his contacts with B. Segre, Fano and Severi, but not with Corrado Segre, who passed away four years before.

Quite the contrary, although till now it has not been noticed, the pool of Segre's American disciples included Clarence Lemuel Elisha Moore (1876–1931). An expert in algebraic and Riemannian geometry, he obtained a Ph.D. at Cornell University, with a thesis on the classification of the surfaces with singularities of the quadratic spherical complex, with Virgil Snyder as advisor. Moore improved his geometric culture by going to study in Göttingen, in Turin with C. Segre, and in Bonn with E. Study. In 1904 he entered the Department of Mathematics of MIT, at first as an instructor, then as an assistant, an associate professor and finally a full professor (Franklin 1933). Reaching Turin in March 1908, Moore attended Segre's course devoted to *Capitoli vari di Geometria della retta* (Segre's *Notebook* 21 in Giacardi 2013) and entered in friendly relations with the Italian geometer.[97] For example, Segre advised him and helped him in drafting the report of the IV International Congress of Mathematicians,[98] which Moore had been entrusted with for the *Bulletin of the American Mathematical Society* (C.L.E. Moore 1908).

[96]Both Baker (1926, 263) and Coolidge (1927, 357) remember the care and promptness with which Segre maintained his correspondence with young English and American scholars.

[97]Cf. C.L.E. Moore (1911, 350, 355–356).

[98]Cf. DESPC: C.L.E. Moore to D.E. Smith, n.d. (1908) and C.L.E. Moore to D.E. Smith, 31 March 1908.

To complete the panorama of Segre's international disciples, it is also necessary to mention those outlanders that, like René Baire, sojourned in Turin to specialize in other domains of mathematics, and nevertheless assisted to some lectures on Higher Geometry. Among Segre's other foreign alumni, we can mention David Cytron (1887–1982), a Jew from Bialystok, afterwards naturalized, who came to study in Italy because he was forbidden to do so in his homeland. In the years 1908–1910 he attended two courses of Segre's (Segre's *Notebooks* 22 and 23 in Giacardi 2013) and his lectures at the Teacher Training School (Segre's *Notebook* 40 in Giacardi 2013), with good profit. After the degree examination, on 2 July 1910, at which he was given the maximum vote, Cytron became an assistant of B. Levi at the University of Cagliari. Appreciated by Segre for his mathematical talent and diligence, Cytron then devoted himself to finance and trading. Interned with his wife Ida Tyktin, in the province of Chieti, he was detained at Pizzoferrato on 15 October 1940 and subsequently at Villa Santa Maria (10 November 1940–1942) and, after the liberation, returned to Turin.[99]

Finally, from analysis of some notebooks preserved at the Special Library of the Turin Department of Mathematics, in which the alumni were registered that took examinations with Segre in Projective, Descriptive and Higher Geometry, many other names of foreigners have emerged, for whom it is difficult to retrieve information, since often they did not take up a university career.

5 Annali di Matematica: A Journal for the Italian School of Algebraic Geometry?

In the years 1850–1880, many Italian geometers, among them F. Brioschi, L. Cremona, G. Battaglini, E. Beltrami, E. D'Ovidio, E. Bertini, R. De Paolis, L. Bianchi and G. Veronese published essays and articles in foreign journals, first of all *Mathematische Annalen*, but also *Bulletin des Sciences Mathématiques* and *Journal für die reine und angewandte Mathematik,* to reach a broader and more specialized readership, both quantitatively and qualitatively, compared to that afforded by the national context.[100]

This strategy of self-promotion of his own results on an international scale was soon appropriated by Segre, ever since the years of his university studies, thanks, as already mentioned, to the example of D'Ovidio, Faà di Bruno and Genocchi. It was soon effective: it is sufficient to consider the reviews and quotations of Segre's

[99]Cf. ASUT, *Registri di carriera scolastica n. 31*, *1906–07*, p. 13; Fonti H40, *Nominativi di ebrei internati in provincia di Chieti e di Pescara, tratti dall'Archivio Arcivescovile di Chieti, Carteggio dell'arcivescovo Giuseppe Venturi (1931–1947)*; MCT-Mary Cytron Treves H123-*Notizie relative a singoli internati o elenchi di internati*, in Ministero dell'Interno, Divisione Generale di Pubblica Sicurezza, Affari Generali e Riservati, A4bis (Stranieri internati, b. 85); Terracini 1968, 11.

[100]Brioschi (1855a, b, c, 1856, 1857a, b, 1858, 1861, 1864, 1869, 1870, 1871, 1877a, b, 1878, 1879a, b); Cremona (1861, 1862a, b, 1864a, b, 1865, 1868, 1871a, b, 1876, 1878); Battaglini (1868); Beltrami (1869, 1877); D'Ovidio (1877); Bertini (1877, 1878); De Paolis (1878); Bianchi (1880a, b); Veronese (1881, 1882).

works, dotted around, from 1883 on, in repertoires like *Revue Semestrielle*[101] and *Jahrbuch über die Fortschritte der Mathematik*,[102] or the reports of conferences and activities in mathematical societies.[103]

However, as the identity of the Italian School of Algebraic Geometry was consolidated, it became important to have a journal that could confirm, at a national and international level, the 'command position' attained by this School. Thus increasing weight was taken on by Segre's active presence in the editorial board of the *Annali di Matematica pura ed applicata*[104]—alongside with L. Bianchi, G. Jung and U. Dini—from 1904 onwards.[105]

Publishing activity—to which Segre devoted himself with fervour and with a scrupulousness that was universally appreciated and sometimes feared (Segre to Wilczynski, 16 April 1916 and 2 March 1917, Annexes 59 and 60)[106]—was rather

[101]Cf. RS I.1 (1893): 86; III.1 (1895): 98; III.2 (1895): 115, 125; IV.1 (1896): 105, 115-116; IV.2 (1896): 104-105; V.1 (1896): 115; V.2 (1897): 98; VI.1 (1898): 56, 97-98, 107; VI.2 (1898): 135-136; VII.2 (1899): 113; VIII.1 (1900): 112, 118, 132, 134; IX.2 (1901): 70, 115; X.1 (1902): 114, 115, 117; X.2 (1902): 127.

[102]Cf. JFM 16.0716.02 (rev. Lampe); JFM 16.0716.01 (rev. Aachen); JFM 16.0703.02 (rev. August); JFM 16.0596.01 (rev. August); JFM 16.0096.01 (rev. F. Meyer); JFM 17.0784.01 (rev. W. Stahl); JFM 17.0773.01 (rev. August); JFM 18.0795.01 (rev. G. Loria); JFM 18.0523.01 (rev. G. Loria); JFM 19.0682.01 (rev. Krazer); JFM 19.0676.02 (rev. G. Loria); JFM 19.0676.01 (rev. G. Loria); JFM 24.0640.01 (rev. G. Loria).

[103]Cf. for example Meyer (1894: 279, 283) translated into Italian by G. Vivanti, see F. Meyer, Rapporto sullo stato presente della teoria degli invarianti, *Giornale di Matematiche* 33 (1895): 261, 36 (1898): 313; E. Study, Ein neuer Zweig der Geometrie, Hamburg 23 September 1901, JDMV 11 (1902): 97–123, in particular 122; JDMV 14 (1905): 81, 287; JDM 15 (1906): 467; JDMV 18 (1909): 159; JDMV 21 (1912): 44; JDMV 25 (1917): 113 (Mathematische Gesellschaft in Wien, 15 December 1916 Gustav Kohn, *Über einige Arbeiten von Segre*); JDMV 30 (1921): 69; JDMV 31 (1922): 79, 93.

[104]Segre critically monitored the publishing policies of the *Annali*, long before becoming a member of their editorial committee. For instance, he wrote to Castelnuovo in the winter of 1894 (ANL-Castelnuovo, Segre to Castelnuovo, 8 February 1894): "Regarding the *Annali*, nowadays they have been given new stimuli: but they still leave something to be desired … Perhaps, if (as seems likely) the journal soon has to change its editorial board, it will fulfill that role which is now performed by the periodicals published by our bosom friend [Guccia] and the porcupine [Peano]: if the editing fell into good hands …" ("Riguardo agli Annali, in questi ultimi tempi hanno ricevuto un po' di spinta: ma ancora lasciano qualcosa a desiderare … Forse, se (come sembra probabile) il Giornale dovrà fra non molto cambiar direzione, potrebbe esso venir ad adempire a quegli uffici a cui servono ora i periodici pubblicati dall'amicone [Guccia] e dall'istrice [Peano]: se la direzione cadesse in buone mani …").

[105]On the history of the *Annali di Matematica* during Brioschi's direction cf. Bottazzini (2000, 71–84), Martini (2003, 171–198) and Lacaita (2012); on the period 1880–1930, Roero (2015).

[106]Cf. also ANL-Castelnuovo, Segre to Castelnuovo, 29 July 1886, 16 May 1893, 27 May 1893, 7 June 1893, 18 June 1893, 28 November 1894, 9 May 1901; ANL-Levi-Civita, Segre to Levi-Civita, 11 May 1905, 9 April 1923; Enriques to U. Amaldi, (1901) and Segre to Amaldi, 20 April 1905, (Nastasi and Rogora 2007, 12–13, 53). Segre's severity as a reviewer was proverbial, in both the Italian and the international mathematical sphere. Before accepting a work for publication, presented by a young beginner or an eminent colleague like S. Kantor, he checked the manuscript in every detail, appraised its contents and form and even the spelling. If his "conscience" was not fully satisfied, he did not hesitate to refuse it for publication, or to ask for it to be completely rewritten.

onerous. It was not by chance that in 1922 he declined the invitation to become a member of the editorial board of the *Bollettino dell'Unione Matematica Italiana*, giving as quite a reason the desire to dedicate himself entirely to *Annali* (S. Pincherle to C. Segre, 27 January 1923, Annex 64).

As co-editor in chief of this illustrious journal, Segre participated in the rapid changes taking place in the twentieth-century world of mathematics publishing, in the decline of many academic collections and in the resulting process of specialization and internationalization, undertaken by the most successful journals, in Italy above all *Rendiconti del Circolo Matematico di Palermo* by G.B. Guccia (Brigaglia 2014, 165–178).

Though he continued to choose to publish most of his works in the series of the *Atti* and *Memorie* of the Turin Academy of Sciences, or in the *Rendiconti* of the Lincei Academy, Segre's attitude regarding the most effective tactic to be adopted for promoting Italian algebraic geometry gradually varied, adapting to the new scientific demands and practices. The letters to F. Klein of the 1880s, where the young Segre and his colleagues and disciples (Veronese, Loria, Amodeo, Fano, Pieri, etc.) asked for hospitality in *Mathematische Annalen*, certain that this journal could truly afford an international showcase for their production, gave way, in the 1890s, to the correspondences of Segre '*Maestro*', careful to acquire for *Annali di Matematica pura ed applicata* the best essays by Castelnuovo, Enriques, Fano, B. Levi, Severi, etc.:

> If you are still in time, I would beg you to consider whether, out of national respect, you might say, if you think it is better not to give your important work to the *Mathematische Annalen*, but instead to *Annali di Matematica* (or another Italian journal). I am sorry not to have thought about it before, but I believe that you are still in time. For when your discovery has to be quoted why should a foreign periodical be named? Foreigners must get used to reading our collections. About other works, for instance the one that Klein asked you for, I will not say anything. But precisely for this one I would prefer *Annali di Matematica,* in which it seems that at the moment there is not too much material; so perhaps the delay would not be great. (ANL-Castelnuovo, Segre to Castelnuovo, 26 September 1893: "Se ne sei ancora in tempo, ti pregherei di considerare se, per un riguardo, dirò, nazionale, non ti paja meglio di non dare ai *Mathematische Annalen*, ma bensì agli *Annali di matematica* (od altra raccolta italiana) il tuo importante lavoro. Mi rincresce di non averci pensato prima, ma credo che tu sia in tempo ancora. Poiché quando si dovrà citare la tua scoperta si dovrebbe nominare un periodico estero? Gli stranieri s'abituino a leggere le nostre raccolte. Per altre cose, ad esempio per quella che ti chiese il Klein, non dico. Ma per questa proprio preferirei gli *Annali di matematica*, nei quali pare che ora non vi sia troppa materia; sicché il ritardo forse non sarebbe grande").[107]

This action was not devoid of effects: after 1891 Segre published no other works in *Mathematische Annalen.* By contrast, in *Annali di Matematica* seven articles of his appeared, four papers by Castelnuovo (against two edited in *Mathematische*

[107]Cf. also ANL-Castelnuovo, Segre to Castelnuovo, 3 October 1893, 12 October 1893 and 16 November 1893.

Annalen), three essays by Enriques (against five appeared in *Mathematische Annalen*), and five memoirs by Fano (who only addressed one work to *Mathematische Annalen*).

Nonetheless, despite Segre's intentions, the publishing activity continued to be the weakest and most marginal element of his leadership of the Italian School of Algebraic Geometry. On one side we have to recognize, as has often been done (Brigaglia and Ciliberto 1995, 14–16), that the volume XXII of *Annali,* containing the two famous essays by Bertini "La geometria delle serie lineari sopra una curva piana secondo il metodo algebrico" (Bertini 1894) and by Segre "Introduzione alla geometria sopra un ente algebrico semplicemente infinito" (Segre 1894a), perfectly reflected the interplay in national research on algebraic geometry and the *Italian style* that characterized it. On the other hand it should not be denied that *Annali di Matematica* never became the 'journal of a School', with a role analogous to this taken on by the *Rivista di Matematica*, edited by Peano, for the promotion of studies on logic and foundations of mathematics (Roero 2015).

The task of retrieving and coordinating the articles to publish in *Annali* was undertaken by Segre in a fragmentary and occasional way. In actual fact, he limited himself to examining the works that through various vectors were submitted to him for publication, without pursuing any specific cultural line. In his role of leader of a School, Segre did not clearly show he had a preference for *Annali.* Indeed, he sometimes advised Castelnuovo, Enriques and Fano—for most disparate reasons, not all of a scientific nature—to use the *Rendiconti* of the Accademia dei Lincei, *Memorie di Matematica e di Fisica della Società Italiana delle Scienze detta dei XL*, the *Giornale di Matematiche ad uso degli studenti delle Università Italiane*,[108] etc. Equally inconsistent it was the strategy pursued by Segre to place his own publications, so much so that it was only in one circumstance, when he thought he was about to complete the essay "Sulla scomposizione dei punti singolari delle superficie algebriche" (Segre 1897a), that he requested Castelnuovo and Enriques to "get the place reserved for him" in *Annali*, "because he would have been very pleased to appear in their company. In this way, precisely, their School in its entirety would have been represented!" (ANL-Castelnuovo, Segre to Castelnuovo,

[108]Cf. ANL-Castelnuovo, Segre to Castelnuovo, 24 September 1891: "Publication in that journal [*Giornale di Matematiche*] would remove any idea of hostility to Amodeo's work edited in the *Atti* of Turin Academy of Sciences. I would like your opinion." ("La stampa in quel giornale [*Giornale di Matematiche*] toglierebbe ogni idea di ostilità contro il lavoro di Amodeo pubblicato negli Atti di Torino. Desidero il tuo parere.") Cf. also ANL-Castelnuovo, Segre to Castelnuovo, 13 September 1891.

9 February 1895: “perché avrei molto piacere di uscire in vostra compagnia. Così appunto sarebbe stata tutta la nostra scuola rappresentata!”).[109]

Besides, continuing to be anchored to ‘old-style’ publishing practices, Segre constantly asked disciples and colleagues to ‘donate’ works to the Turin Academy of Sciences, especially in the periods in which “in our series it seemed that the geometers were silent.” (ANL-Castelnuovo, Segre to Castelnuovo, 21 March 1893 and 25 December 1893). On manifold occasions he ‘reproached’ Castelnuovo in a friendly way for “always turning to the Accademia dei Lincei and not the Turin Academy” for the publication of his most interesting results (ANL-Castelnuovo, Segre to Castelnuovo, 17 April 1894), and reminded him that—as a member of this latter Society—he was ‘morally’ bound to ensure effective collaboration to its periodicals:

> I will be pleased to present Enriques’ work, because I like our Academy to publish the important geometric researches that you two are doing. Therefore I need not repeat to you that this year too you must give our Academy some works of yours. (ANL-Castelnuovo, Segre to Castelnuovo, 25 December 1893: “sarò lieto di presentarlo [il lavoro di F. Enriques], perché mi piace che la nostra Accademia pubblichi le importanti ricerche geometriche che si vanno facendo da voi. Non occorre quindi ch’io ti ripeta che anche quest’anno tu devi dare alla nostra Accademia qualche tuo lavoro”).

However, though naive and heterogeneous, Segre’s commitment within the editorial board of *Annali di Matematica* appears to have been marked by two aspects: the defence, in a Risorgimento and patriotic key, of the Italian language as one of the international languages of mathematics, equal in dignity and breadth of use to French, German and English; and the battle for those journals that, so to speak, had ‘made the history’ of Italian scientific culture. It is therefore not surprising that, though having a full expertise in both French and German, after 1906

[109]Cf. also ANL-Castelnuovo, Segre to Castelnuovo, 16 January 1895, 9 March 1895, 12 March 1895 and 18 March 1895: “I believe that the prize will be awarded, and it will be well deserved! Naturally, if for this the President deems it necessary that your works should be published by the XLs there is no doubt that you have to accept. Then you and Enriques judge whether the filiation between your works is such as to make it more useful to publish them together in a learned collection that is not very widespread, rather than using *Annali* (for Enriques) and so to spread your ideas more widely (through two journals). Concerning my Note, I happened, near the end of the final version, to notice a mistake of some gravity that I had committed! I hope to correct it: but, being occupied with my course, I cannot do it very soon. So you two mustn’t wait any longer for me! Before you correct your proofs you will know whether and how to quote my paper. Probably between the *Memorie dei XL* and *Annali* I would prefer the latter: but for now I don’t have to decide anything on the matter.” (“Ritengo che la premiazione avrà luogo, e sarà ben meritata! Naturalmente, se per questo il Presidente ritiene necessario che i tuoi lavori sian pubblicati dai XL, non v’è dubbio che tu devi accettare. Tu e l’Enriques giudicate poi se il legame fra i vostri lavori sia tale da rendere più utile il pubblicarli insieme in una raccolta poco diffusa, anzi che valersi anche degli *Annali* (per l’Enriques) e così diffondere maggiormente (per mezzo di due raccolte) le vostre idee. Riguardo alla mia Nota, m’è accaduto, pressoché al termine della redazione definitiva, di accorgermi di una svista di una certa gravità che avevo commessa! Spero di ripararla: ma, con le occupazioni del corso, non potrò farlo tanto presto. Sicché non aspettatemi più voi due! Prima che correggiate le vostre bozze saprete se e come citarmi. Probabilmente tra le *Memorie dei XL* e gli *Annali* preferirei questi ultimi: ma per ora non occorre che io decida nulla in proposito”).

Segre did not submit any more work in a foreign language and that as long ago as 1890 he informed Castelnuovo, with a vein of satisfaction, of the possibility of also using Italian for publications abroad:

> The problem on the $g_2^{(1)}$ of particular curves that you wrote to me saying you had solved already seems to me very general and important; and you will do well to publish it. If you decide on *Mathematische Annalen* I must warn you that the editorial board of the latter is really determined also to accept works in Italian (also in the competition of the Berlin Academy for the Steiner prize they now also admit papers in our language). (ANL-Castelnuovo, Segre to Castelnuovo, 13 August 1890: "Il problema sulle $g_2^{(1)}$ di curve particolari che tu mi scrivi di aver risolto mi par già molto generale ed importante; e farai bene a pubblicarlo. Se ti decidi pei *Mathematische Annalen* t'avverto che la redazione di questi è proprio decisa ad accogliere anche lavori in italiano (anche nel concorso dell'Acc [ademia] di Berlino pel premio Steiner adesso accolgono anche i lavori scritti nella nostra lingua)").

However, defence of the use of Italian was typical of the period. One need only think that in an other prestigious context too, that of the Accademia dei Lincei Prizes for Mathematics, candidates could only compete with essays written in Italian and Latin. Segre himself, who won the prize in 1898, together with Volterra, was forced to devote the 1895 Christmas holidays to the "very boring occupation" of "summarizing, in Italian" some works of his published in French (ANL-Castelnuovo, Segre to Castelnuovo, 7 January 1896).

In the same line of thought and action was the campaign of international mobilization for the survival of *Annali di Matematica*, in which Segre was involved in the years after World War One. Though sorry to have to turn to foreigners, which "it is always regrettable to have to ask for money",[110] in that circumstance he was willing to sacrifice patriotic pride, whilst it should be guaranteed the life of the journal that, since 1857, had represented the 'voice' of Italian mathematical community. To face the serious economic crisis of *Annali*, Segre asked the Americans for help, expressing his worry over the destiny of the journal to Snyder, at the time in Turin for a study sojourn.[111] Through the *Bulletin of the American Mathematical Society* Snyder released an invitation to his American colleagues,[112] which was echoed by 'distance disciples' of Segre's like C.L. Moore, as well as by many estimators of the Italian geometer (E.B. Stouffer, S. Lefschetz, S. Lipka, etc.).[113] The excellent results of this subscriptions campaign, an "effect of Segre's work",[114] succeeded in avoiding the 'death' of *Annali* and indeed projected the journal—with the opening of the fourth series, edited by L. Bianchi, C. Segre, S. Pincherle and T.

[110] ANL-Volterra, Segre to Volterra, 5 July 1912: "L'estendere la sottoscrizione agli stranieri va bene sotto un aspetto. D'altro lato rincresce sempre chieder denari, noi italiani agli stranieri." Cf. also Segre to Volterra, 9 July 1912 and 27 December 1912.

[111] Cf. Snyder to Segre, 21 November 1922, Annex 61.

[112] Cf. Notes, *Bulletin of the AMS*, 28 (1922): 370; Notes, *Bulletin of the AMS*, 29 (1923): 41.

[113] Cf. Snyder to Segre, 21 November 1922, 8 December 1922, 5 January 1923 and 19 February 1923, Annexes 61, 61.1 and 61.2, 62, 63 and 65.

[114] Pincherle to Segre, 27 January 1923, Annex 64.

Levi-Civita—towards a "new season of Italian mathematics" (Pincherle to Segre, 27 January and 3 November 1923, Annexes 64 and 67).

6 Segre's School at International Congresses of Mathematicians

For Mathematical Schools in the 19th and 20th centuries international congresses constituted very propitious occasions to promote their research *style*. At the same time, they favoured exchanges and interactions among the participants, creating the premises for new collaborations among scholars in different geographical areas or renovating ancient relationships (Albers et al. 1986; Lehto 1998; Curbera 2009).

Segre, probably influenced by Klein, soon appeared aware of the importance of that form of sociability for spreading and vulgarizing his results and projects. In this connection, since 1893, on the occasion of the Evanston Colloquium (28 August–9 September 1893), he commented to Castelnuovo on the usefulness of participating in international conferences:

> They write to me from Chicago that I should promote a report from my Italian friends at that congress, especially with "brief critical review of the development during the last 20–25 years of the definite small subdivisions of the science." Would you like to make such a brief account on geometry *of hyperspaces* (algebr[aic] entities; proj[ective] geom[etry], etc.) in Italy? I believe that it would cost you little work (it being a matter of *briefly* characterizing the progress due to the various main contributions, which you already know) and it would be a useful survey. The congress is being held *from 21 to 28 August*. Let me know if you accept this task. If the answer is affirmative you could then send *me* your text, authorizing me to retouch it. (ANL-Castelnuovo, Segre to Castelnuovo, 28 June 1893: "Mi scrivono da Chicago di promuovere un resoconto dai miei amici italiani a quel congresso, specialmente con "brief critical reviews of the development during the last 20–25 years of the definite small subdivisions of the science." Avresti tu voglia di fare un breve rapporto sulla geometria *degl'iperspazi* (enti algebr.; geom. proj. ecc.) in Italia? Credo che ti costerebbe poca fatica (trattandosi di caratterizzare *brevemente* i progressi dovuti ai vari principali lavori, che tu già conosci) e sarebbe una cosa utile. Il congresso si tiene *dal 21 al 28 d'Agosto*. Avvertimi se accetti quest'incarico. In caso affermativo potresti poi inviare *a me* il tuo scritto con facoltà di ritoccarlo.").[115]

However, the first opportunity to present themselves collectively, as a true School, came to the Italian geometers from the International Congress of Mathematicians in Zurich (9–11 August 1897). The previous summer, Segre had already told his closest friends of his intention of attending it:

[115] Cf. also ANL-Castelnuovo, Segre to Castelnuovo, 9 July 1893: "To Chicago I will send some works by others; nothing of mine probably. I am sorry that you cannot draft the little paper I proposed to you. Do you have some other small paper?" ("A Chicago manderò lavori altrui; nulla di mio probabilmente. Mi rincresce che tu non possa fare il lavoretto che io ti proponevo. Hai qualche altra cosetta?"). On the outcomes of the Chicago congress cf. ANL-Castelnuovo, Segre to Castelnuovo, 3 March 1894, 8 August 1894 and 22 August 1894.

> Your letter gave me great pleasure because of the hope that it gives me that you yourself will decide to come to Zurich. […] I believe that if I could not go there, later I would regret it, like an opportunity missed to meet scholars of high value and to attend at special meetings […]. I would devote Saturday evening and part of Sunday to the mathematicians that are already in Zurich. At Göschenen Fano will join me; it seems, instead, that Enriques will come later. (ANL-Volterra, Segre to Volterra, 31 July 1897: "Molto piacere m'ha fatto la tua lettera per la speranza che mi dà che tu ti decida a venire a Zurigo. […] Io credo che se non potessi andarci, dopo ne proverei rammarico, come di un'occasione perduta di vedere uomini di valore, e riunioni singolari […]. La sera di sabato e parte della domenica li dedicherei ai matematici che già si trovano a Zurigo. A Göschenen si unirà a me il Fano; Enriques, invece, pare che verrà più tardi.").

Regarding personal contacts, the Zurich Congress proved profitable for Segre, who exploited his time there to establish a dense network of partnerships, which he was to preserve for the rest of his life. Besides being acquainted with E. Borel, T. Reye, H. Zeuthen and the American C.A. Scott,[116] Segre met F. Kraft, who after the end of the Congress went to Italy.[117] On an invitation from Klein, moreover, Segre was appointed vice-president of the *Geometry* section, directed by Reye, in which appreciated talks were given by F. Gerbaldi, C. Burali-Forti and G. Fano; Enriques spoke in the *Arithmetic and Algebra* session; Loria in the *Mechanics and Mathematical Physics* one, but on a historical theme.[118] Although C. F. Geiser pressed him to submit a paper, Segre firmly declined the invitation, fearing the judgment of mathematicians of the standing of M. Nöther and F. Klein:

> Alongside these there will be, it is true, others for whom my words might perhaps have been of some utility. But meanwhile wouldn't the former have branded me superficial, wouldn't they have said that I was expounding ideas that were already known, and partly already developed by me in that article of mine in *Rivista di Matematica* …? (ANL-Castelnuovo, Segre to Castelnuovo, 12 June 1897: "Accanto a questi vi saranno, è vero, altri pei quali le mie parole potevan essere forse di qualche utilità. Ma intanto quei primi non m'avrebbero tacciato di superficiale, non avrebbero detto che io esponevo cose già note, ed in parte già svolte da me in quel tale mio articolo della *Rivista di matematica* …?").

Despite some important defections like those of Castelnuovo, Guccia and P. Del Pezzo,[119] the Italian delegation in Zurich was quite large, with about twenty participants, including, in addition to the speakers mentioned above, Volterra, F.

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[116]Cf. Segre to O. Michelli Segre, 8 August, 8–9 August and 10 August 1897, Annexes 21, 22 and 23.

[117]ANL-Volterra, Segre to Volterra, 19 August 1897.

[118]Cf. Gerbaldi (1898), Sul gruppo semplice di 360 collineazioni piane, *Verhandlungen des Ersten Internationalen Mathematiker-Kongresses in Zürich vom 9 bis 11 August 1897*, F. Rudio (ed.), Leipzig: Teubner, 1898: 242–246; Burali-Forti (1898), Postulats pour la géométrie d'Euclide et de Lobatschewsky, *Ibidem*, 247–250; Fano 1898, Über Gruppen, insbesondere continuierliche Gruppen von Cremona Transformationen der Ebene und des Raumes, *Ibidem*, 254–255; Enriques (1898), Sur les problèmes qui se rapportent à la résolution des équations algébriques renfermant plusieurs inconnues, *Ibidem*, 145–146; Loria (1898), Aperçu sur le développement historique de la théorie des courbes plane, *Ibidem*, 289–298; cf. also Segre to O. Michelli Segre, 10 August 1897, Annex 23.

[119]Cf. Annex 22 and ANL-Castelnuovo, Segre to Castelnuovo, 7 September 1897: "In Zurich there was great regret over your absence … Ghigo [F. Enriques] will have told you!" ("A Zurigo s'è rammaricata tanto la tua assenza … Ghigo te l'avrà detto!").

Brioschi, S. Pincherle, G. Ricci Curbastro, G. Veronese and T. Levi-Civita. In particular, one of the four plenary lectures was delivered by G. Peano, who illustrated to the world scholarly community the second edition of his *Formulaire de Mathématiques*, showing how this ambitious encyclopaedia had been achieved thanks to the creation of a specific logical-ideographic language (Peano 1898, 299). Segre confided to Volterra the fear that the choice of such a theme might have harmed the image of Italian mathematics abroad:

> I want to believe that nothing disagreeable for us will happen there: something *comical* perhaps, but if there will be cause for laughter, it won't be such a bad thing! (ANL-Volterra, Segre to Volterra, 31 July 1897: "Io voglio credere che nulla abbia da accadere là di spiacevole per noi: di *comico* forse sì, ma se vi sarà da ridere, non sarà un gran male!").

The following October, Segre 'found on his desk' a report of the Congress, published by Borel in *Revue générale des Sciences pures et appliquées* (Borel 1897, 783–789) and regretfully ascertained that the French colleague too had not appreciated Peano's plenary lecture.[120]

A sort of 'rivalry' between the two Turin Mathematical Schools, and consequently between their leaders Segre and Peano, was perceived in a more evident way in Paris at the International Congresses of Philosophy (1–5 August) and of Mathematicians (6–12 August), which opened the Short Century. Segre did not go to Paris but asked Volterra to convey him his impressions, without "waiting to tell him all of them face to face" on his return to Italy, and reminding him:

> At present there is a philosophy congress there, with the active participation of the Peanians with their leader: Peano will be speaking of mathematical logic in general, Padoa and Pieri of its applications to arithmetic and geometry. I am warning you about it, so that you can attend this event if you are still in time! (ANL-Volterra, Segre to Volterra, 3 August 1900: "In questi giorni si fa costì un congresso di filosofia, al quale prendon parte attiva i peaniani col loro duce: questi parlando di logica matematica in genere, Padoa e Pieri delle sue applicazioni all'aritmetica e alla geometria. Te ne avverto, affinché tu possa andare se ne sei ancora in tempo!").

Volterra did not fail to send Segre, from Paris, a letter containing detailed accounts of the two congresses together with an abstract of the famous talk by Hilbert "Mathematische Probleme" (ANL-Volterra, Segre a Volterra, 11 September 1900).

In Paris, the Italian School of Algebraic Geometry was certainly not well represented. Only two of Segre's alumni presented talks: F. Amodeo, who traced a "Coup d'oeil sur les courbes algébriques au point de vue de la gonalité" (Amodeo 1902), and A. Padoa, who, though having had Segre as the advisor of his degree dissertation, cannot be properly considered a disciple of Segre. Padoa presented two papers, the first one on the theme "Un nouveau système irréductible de postulats pour l'Algèbre" and the second one on a hypothetical-deductive system for

[120]ANL-Volterra, Segre to Volterra, 27 October 1897: "Borel's article [is] equal to the spirit that he shows in talking: there are also some barbs … which I liked a great deal." ["L'articolo di Borel [è] pari allo spirito che questi dimostra nel discorrere: vi è anche qualche stoccata … che m'è piaciuta assai"). For an analysis of the reception of Peano's logic by Borel and his coworkers R. Baire, H. Lebesgue, cf. Luciano (2016).

Euclidean geometry (Padoa 1902a, b).[121] To the absences of Segre, Castelnuovo and Enriques, there should be added the circumstance that Veronese spoke in the section devoted to *Teaching and methods* and that Fano did not present any paper.

The plenary lecture by Volterra himself (Volterra 1902), which Segre had hoped "would have been very beautiful and would have doubly honoured the name of Italy both for the value of the eminent scholars mentioned and for the expertise and the competence of the speaker" (ANL-Volterra, Segre to Volterra, 3 August 1900) proved a little disappointing for him. In fact Volterra concentrated on the study traditions opened up by E. Betti, F. Brioschi and F. Casorati and only marginally hinted at the evolution of Italian geometry.

While the congresses in Paris, and above all that of Philosophy, for Peano's 'phalanx' represented the key moment of its affirmation, for the Italian algebraic geometers the apex of internationalism was reached in 1904. During the third Congress, which was held in Heidelberg from 8 to 13 August of that year, Segre, together with F. Morley, chaired the *Geometry* session (10 August), with talks by F. S. Macaulay, C. Guichard, E. Study, F. Meyer, K. Rohn and G. Scheffers and he took part, together with L. Automne, in the relevant debates. He was also entrusted with one of the four plenary lectures, on an indication by F. Klein. The topic chosen, "La Geometria d'oggidì e i suoi legami coll'Analisi", gave Segre an excellent opportunity to recap the progress of the researches of the Italian School and to highlight the *style* that marked them:

> A whole Italian *school* of geometers recognizes its starting point in the Memoir by Brill and Noether! Those concepts became even more fertile when, thanks precisely to this *school*, they took on a more abstract and more general character, being referred to algebraic curves, especially with the methodical introduction of the important notion of the sum of two linear series (corresponding to that of product in the field of rationality defined by an algebraic irrational). With these tools Castelnuovo obtained major new results on algebraic curves, for example regarding the issue of postulation, which I have already mentioned. More important still is the way in which it has been possible to apply that theory, or to extend it, by analogy, to surface geometry! (Segre 1905, 115).

Preparing the lecture, which Segre held in Italian on Saturday 13 August, for him was a source of some worry and constant commitment, during the summer stay in the Swiss Alps. From Airolo, where some colleagues joined him (Guccia and G. Morera), he updated his wife on the revisions that he was broughting to the text, on the decision not to have it printed before the Congress and on the programme of the events in which he would participate in Heidelberg (cf. Segre to O. Michelli Segre, letters from 17 July to 8 August 1904, Annexes 32–39). His presence at the Congress, together with Castelnuovo, Loria and Fano, certainly allowed the team of algebraic geometers to broaden the horizon of their scientific links. We nevertheless notice the unusual fact that, against the large number of members of this School (four out of eleven Italians that attended the Congress), only Segre gave a plenary

[121]These are two very important works from the logical-foundational point of view. For in-depth examinations of these essays the reader is referred to Borga (2005, 3–24), (2011, 89–114), Borga et al. (2009, 233–254), Lolli (2010, 47–66), Pasini (2010, 327–367), Luciano (2012, 49–52).

lecture and only Loria a talk, not on a geometrical theme but on a historical one, as he had already done in Zurich in 1897.

From correspondences and other testimonies we infer that in Heidelberg Segre made conversations with Klein, M. Nöther, G. Mittag-Leffler, E. von Weber, H. Zeuthen, A. von Brill, P. Stäckel, S. Dickstein, E. Study and E. Wilczynski.[122] Further, to his wife he confided:

> This is the great pleasure of conferences: to entertain with so many people that one only knew through their works, and to talk together of so many topics (8 August 1904, Annex 39).

1904 was also a year of very great importance for Segre from the point of view of the internationality of publications. In this connection, in June the *Bulletin of the AMS* published the English translation, by John Wesley Young (1879–1932), of the article "Sulle investigazioni geometriche, Osservazioni dirette ai miei studenti", which, as is well known, had been at the origin of a bitter controversy with Peano in *Rivista di Matematica* (Segre 1891a). The English version "On Some Tendencies in Geometrical Investigations" of his 'old' text[123]—whose salient points were taken up in the Heidelberg plenary lecture—gave Segre an opportunity to illustrate at an international arena how he conceived the role of *Maestro*, giving details of the modalities with which he structured the higher teachings and with which he trained young scholars towards original production. The fundamental convictions on the balance between intuition and rigor, on the alternation of methods and on the distinction between elementary mathematics, higher mathematics and elementary mathematics from an advanced standpoint, expressed by Segre in the 1904 article, and strongly permeated by the ideas of Klein, echoed the assumptions that he had defended since long time in his lectures at the university and at the Teacher Training School.[124] Filtered by the network of exchanges intertwined by Segre and

[122]Cf. C. Segre to O. Michelli Segre, 9 August 1904, 10 August 1904, 12 August 1904, 13 August 1904 and 15 August 1904, Annexes 40, 41, 43, 44, 45. For Segre's meetings with Study and Wilczynski, in Heidelberg, we also have the testimonies of Coolidge (1927, 354–355). He wrote: "he [Segre] thereby established contact with Study, whom he subsequently met at the Heidelberg Congress in 1904, and with whom he retained cordial relations for many years." On Wilczynski, instead, Coolidge erroneously affirmed that Segre had met him in Heidelberg, while actually the two had met in Turin, for the first time, in the spring of 1904 (cf. C. Segre to E. Wilczynski, 18 March 1904, Annex 29, and 27 April 1904, Annex 30).

[123]Re-examining the drafts of the translation in the spring of 1904, Segre made some additions and changes to it, above all regarding bibliography (Segre 1891a, 442, 443, 446, 447, 448, 449, 450, 452, 455, 458, 459, 461, 462, 463, 465 and 467). For example, he pointed out that in 1888 F. Schütte translated the essay by G. Loria *The past and the present of the main geometric theories*; he added some references to the recent works by R. Ball, S. Lie, R. K. Fricke, F. Klein, A. R. Forsyth, E. Kötter, E. Picard, G. Simart and W. Killing, to the translations into Italian of the *Erlangen Program*, edited by G. Fano in 1890, and in English, edited by W. Haskell in 1892, and to his own publications and those of his disciples, which appeared in *Annali di Matematica* in 1890, 1893, 1895 and 1897.

[124]Cf. UTo-ACS. *Quaderni e Documenti relativi all'attività didattica*: *Lezioni di Geometria non euclidea (1902–03)*, Segre's *Notebook* 16: 1–22; *Vedute superiori sulla Geometria elementare (1916–17)*, Segre's *Notebook* 30: 7–27 and *Lezioni per la Scuola di Magistero*, Segre's *Notebook* 40, in Giacardi 2013.

his disciples with D. E. Smith and others American scholars, sensitive to mathematical instruction and education, these instances achieved an international resonance even before the publication of Segre's *Notes for the lectures at the Teacher Training School* (Tricomi 1940). This led foreign mathematicians to perceive Segre's School as a team that not only shared a particular research project but also precise assumptions and convictions regarding teaching and methodology.

Further the opportunity to publish the article "On Some Tendencies in Geometrical Investigations" (Segre 1904) in the United States was grasped by Segre to illustrate the changes—compared to 1891—in the conditions of research on and teaching of higher geometry in Italy, and to maintain that, precisely thanks to the more recent studies on the foundations of mathematics, it was now possible to consider as universally accepted the abstract and logical-deductive character of geometry:

> Since the appearance of the present paper multi-dimensional geometry has spread more and more, so that now (among mathematicians!) its opponents have become rare, who at one time were so common. […] In regard to the foundations of geometry, the books by Pasch and by Peano, and since the publication of this article the book by Veronese and the papers of Pieri, Hilbert, and others have led mathematicians in recent years more and more to consider geometry from an abstract, purely logical or deductive point of view, detaching it entirely from every physical consideration. […] Following this method the points of a space of 4, or 5, … dimensions are treated as above stated in the same way as those of S3, the system of postulates being slightly modified (Segre 1904, 459, 462, 463).

If for Segre the Heidelberg Congress marked the acme of his international prestige, the next symposium, held in Rome in April 1908[125] had as its protagonists above all his disciples. Castelnuovo was the General Secretary and Fano the deputy-secretary; Enriques, B. Levi and Severi delivered plenary lectures and talks (Enriques 1909; Levi 1909; Severi 1909). With the exception of Severi, nevertheless, none of the members of Segre's School spoke about algebraic geometry or higher geometry themes.

Entrusted, with L. Bianchi, with organizing and introducing the *Geometry* session, Segre proposed sending a best wishes telegram to his friend and colleague Reye, who had not been able to come to Rome for family reasons, and suggested that the meetings of 7 and 10 April should be chaired, respectively, by two 'great *Maestri*' from abroad: the Danish Hieronymus Georg Zeuthen and the German Issai Schur. However, the *Geometry* session was almost entirely animated by foreigners. There were talks by the French J. Andrade, the Croatian V. Varićak, the Dane Zeuthen, the Romanian G. Tzitzeika and the Ukrainian G. F. Pfeiffer; the only Italian to lecture was D. Montesano.

In Rome Segre's School saw its importance fully recognized, in the international arena, in relation to three main aspects: the awarding to Severi of the Guccia Medal, the praise of the contributions of this team to the *Encyklopädie der mathematischen Wissenshaften* and the recognition of the Italian tradition in algebraic geometry by Volterra in his plenary lecture "Le matematiche in Italia nella seconda metà del secolo XIX" (Volterra 1909).

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[125]On the history of the International Congress of Mathematicians in Rome cf. Guerraggio and Nastasi (2008).

Regarding the first aspect, Segre, who had been charged with delivering the final report for the attribution of the Guccia Medal, succeeded in "capturing the attention of the immense public" present,[126] particularly underlining the connections between Severi's researches on geometry on algebraic surfaces, the algebraic-geometrical methods of Enriques and Castelnuovo and the transcendent ones of Picard (Segre 1909, 212). He also listened with satisfaction to the presentation of the *Encyklopädie der mathematischen Wissenshaften* by W. von Dyck, who, speaking of the third volume, emphasised the input by Italian geometers as follows:

> I would like to particularly single out another field, that of algebraic curves and surfaces and their integrals, in connection with the *Analysis situs*. The latter first arose in Germany, and then in Italy, through the problems promoted by the life work of Cremona, recently the rivalry of French and Italian geometers has operated successfully and – I refer to the latest presentation by Mr Segre about the Guccia Prize – produced new studies, rich in surprising results. We owe to this intense interest in geometric researches if the volume of the *Encyclopaedia* devoted to geometry benefited from the excellent collaboration of our Italian colleagues (von Dyck 1909, 128).

Lastly, the Italian *style* in geometrical research was "honourably" mentioned (Segre to his wife, 6–7 April 1908, Annex 54) by Volterra, who, referring to his friend Segre, in his *lectio magistralis* affirmed:

> The further development of these studies in Italy and the new direction that they have taken is mainly to the credit of Segre with the first line of his researches, and to him there should be added Del Pezzo, Fano and others. Then, in the second phase of his scientific career, in which he drawn on the great essay of Noether, Segre was responsible for the beginning of that patrimony of works with which Castelnuovo, Enriques, Severi and De Franchis achieved their important results on the theory of the surfaces, the most recent of which are connected to the discoveries of Picard on algebraic functions and hence are in the framework of the theory of functions (Volterra 1909, 63–64).

The Rome Congress was also very fruitful from the point of view of conversations and exchanges with foreign colleagues. In addition to seeing Borel, Nöther and Mittag-Leffler again, Segre on that occasion got to know Henri Poincaré and met the young mathematician Emmy Nöther, Max Nöther's daughter.[127] By contrast, he regretfully noticed the absence of many American correspondents of his, including Coolidge, Wilczynski, Sisam and Stouffer.

In the ensuing years the participation of Segre's School in International Congresses of Mathematicians became increasingly episodic. The team continued to be represented by a remarkable group of members at the Cambridge Congress (22–28 August 1912). The new ranks of scholars like F. Severi, E. Bompiani and A. Terracini were present alongside the algebraic geometers of the first generation: Castelnuovo and Enriques. Fano and Segre, although enrolled, did not attend the conference. However, in Cambridge the themes of the talks reflected the new interests, not of a geometric type, cultivated by some members of Segre's équipe.

[126]Cf. Segre to O. Michelli Segre, 6–7 April 1908, Annex 54.

[127]Cf. Segre to O. Michelli Segre, 5 April 1908 and 5–6 April 1908, Annexes 52 and 53.

Enriques, for instance, presented a talk "Sul significato della critica dei principii nello sviluppo delle Matematiche" (Enriques 1913, 67–79) and Castelnuovo—as one of the Italian delegates of ICMI—limited his activity to the session on mathematics education (Giacardi and Furinghetti 2008). The first *Geometry* meeting (23 August), chaired by H.F. Baker, elected Bompiani as Assistant Secretary, together with W. Blaschke. As had already happened in Rome, the session was dominated by foreigners: L. E. J. Brouwer, F. Morley, L.P. Eisenhart, E. Neville, M. Brückner, C. Stéphanos and A. Martin. The only Italian speaker was Bompiani, who dealt with a topic in line with the favourite research interests of Segre, like projective hyperspace geometry:

> In a series of works published from 1906 to 1910 Prof. Segre resumed this branch of hyperspace geometry and took it to a high degree of perfection. The contribution made to it, under Segre's impulse, by young geometers in Italy and abroad, can therefore be characterized as truly Italian (Bompiani 1913, 23).

At the following session too (24 August), chaired by Severi, there were interventions by W. Esson, M. Grassmann, P.H. Schoute, E. Kasner and G. Tzitzeica, while there were no Italian lecturers.

The presence of Segre's School, seen as a community at International Congresses of Mathematicians was further reduced after the Great War. Indeed, Segre refused to take part in the Strasburg conference (22–30 September 1920), an event that was only deemed international 'in name', seeing the exclusion of German, Austrian, Hungarian and Bulgarian mathematicians. Agreeing with Segre's stance, the School of Algebraic Geometry deserted the conference *en masse*, submitting no paper (cf. Klein to Enriques, 13 August 1920 and Enriques to Klein, 18 January 1921, in Luciano and Roero 2012, 214–217).

After Segre's death, in a context that was radically changed from both the cultural and the political points of view, it was Severi and Castelnuovo that at the International Congresses in Toronto (11–16 August 1924), Bologna (3–10 September 1928) and Zurich (5–12 September 1932) represented the Italian School and inherited that directional role that Segre[128] had maintained for thirty years:

> Nous devons cet esprit à nos maitres italiens Cremona, Betti, Bertini, Veronese, Segre, aux savants allemands Riemann, Clebsch, Klein, Brill et Noether, au danois Zeuthen, aux anglais Cayley, Sylvester et Salmon, et aux travaux, si profondément géométriques dans leur esprit, des analystes français, de Galois à Poincaré, à Picard, à Painlevé, à Humbert. Comme le peu que j'ai pu faire dans la science est le fruit de l'enseignement savant et passionné de mon maitre direct, Corrado Segre, que la mort nous a prématurément ravi, le 18 mai dernier, qu'il me soit permis d'envoyer à son souvenir les hommages du disciple affectionné et reconnaissant et ceux, bien plus hauts, du Congrès (Severi 1929, 154).
>
> A current of thought that was different from Cremona's and, through Klein, spread in our country between 1880 and 1990, led projective geometry to be extended to hyperspaces. Giuseppe Veronese and Corrado Segre were the greatest representatives of this trend. Segre in particular, an eclectic spirit, an insuperable *Maestro*, prematurely taken away from our affection and our admiration, foresaw the applications that could be made of hyperspace geometry to the theory of algebraic curves (Castelnuovo 1929, 192).

128Severi (1932, 216): "Pour l'espace projectif on a ainsi une variété remarquable, qui a été découvert par Corrado Segre, mon regretté et éminent maître.".

In the light of this overview, it seems pertinent to affirm that Segre's School only partially succeeded in exploiting (and to a lesser extent with respect to the 'rival' School of Peano) Mathematical Congresses as showcases to build and validate its own identity at an international level, and to give resonance to the Italian *style* in algebraic geometry. It was mainly two elements that prevented the maximum profit from being derived from this strategy: in the first place, the decision of several members of Segre's School not to present papers, or to devote their talks to themes that were not strictly mathematical, but rather historical, philosophical or methodological; in the second place, the fact that Segre himself was not able, and/or did not choose to coordinate the involvement of his disciples in the various Congresses, and often was not informed on the intentions of Castelnuovo, Enriques and Fano.[129] Very effective, by contrast, was the policy of exchanges developed by the Italian geometers within these symposia, a policy that led them to build up a network of long-lasting relationships destined to be maintained until the First World War.

7 Interventionism and Pacifism: Segre's School and the First World War

The web of supranational partnerships shaped by Segre's School underwent an abrupt, though temporary, interruption following the outbreak of the First World War. Even in the months around the Sarajevo assassination, the letters from Castelnuovo and Enriques to their German colleagues document a rich agenda of commitments, regarding both research and teaching. Castelnuovo and Klein, for example, planned to send out a questionnaire of the International Commission on Mathematical Instruction devoted to the training of secondary school teachers; the final report was to be presented by Loria, the Italian delegate, at the 1916 Stockholm International Congress of Mathematicians (Castelnuovo to Klein, 3 March 1914, in Luciano and Roero 2012, 208–209). At the same time, several members of Segre's School, like Castelnuovo, Enriques, Loria and Segre himself, were dealing with the translation and correction of the proofs of their chapters for the *Encyklopädie der mathematische Wissenshaften*, while Klein was already thinking about recruiting yet another Italian, Berzolari, for the essay on transformations and correspondences (Klein to Castelnuovo, 4 March 1915 and Castelnuovo to Klein, 10 March 1915 (Luciano and Roero 2012, 209–213).

[129]For example (ANL-Castelnuovo, Segre to Castelnuovo, 23 May 1897): "Please *immediately* write to me if you intend to take part in the congress in Zurich (9, 10, 11 August). I intend to. I have received a letter today from Geiser inviting me to give a talk in the Geometry session. I had not thought about it. Before thinking about it I would like to know who is coming of the young Italian geometers: you, Enriques, Fano, … and also if coming you are going to give talks and what on. If you agree, talk to Fano about it, and tell him to write to me immediately." ("Ti prego di scrivermi *subito* se tu hai intenzione di prender parte al congresso di Zurigo (9, 10, 11 Agosto). Io ne avrei intenzione. Ho ricevuto oggi una lettera del Geiser che m'invita a fare qualche comunicazione nella sezione di Geometria. Non avevo ancora pensato a ciò. Prima di pensarvi vorrei sapere chi viene dei giovani geometri italiani: tu, Enriques, Fano, … ed anche se venendo avete comunicazioni da fare e su che cosa. Se credi, parlane a Fano, e digli che mi scriva subito.").

However, even before Italy entered the war, the contacts became increasingly difficult: a part of the Italian mathematical community—and first of all Volterra—took up interventionist stances, while postal censorship hindered the circulation of letters and books. Klein signed the *Aufruf an die Kulturwelt,* denying the war crimes committed by German army in Belgium, and this alienated him from many of his 'distance disciples'.[130]

In this situation, Segre put in place a series of concrete initiatives, to limit the effects of ostracism towards colleagues from the central powers. The pacifist beliefs of the Turin algebraic geometer—which dated back to his youth, though never paraded—led him to keep in touch with his friends Zeuthen and Reye, for example painfully assisting at the epilogue of the life of Reye, who died shortly after, having repaired to Würzburg at his daughter's home:

> On the eve of our entry into war, I received a postcard, dated "Strassburg Els., 18-5-15", approved by the German censorship, which said: *Lieber Freund und College, Bewahren Sie mir Ihre freundschaftlichen Gesinnungen, wie ich die meinigen Ihnen bewahren werde, auch wenn Italien, wie ich fürchte, in den Weltkrieg hineingerissen wird. Herzlich grüsst Sie Ihr. Th. Reye.* This kind act, which touched me and shows the delicacy of feeling of our dear departed colleague, came to my mind when, some days ago, from the President of the Academy I received an invitation to commemorate Theodor Reye (…) whom I began to admire when I was a student, reading his classic *Geometrie der Lage*; and with whom I did not wait for enter into a scientific relationship, and also a personal one. I was able to appreciate not only his value as a mathematician, but also the real goodness of the man: a true gentleman! […] Strasburg being occupied by the French, Reye and his wife in March 1919 were expelled from that city, in which they had lived no less than 47 years, without any respect for their advanced age (Segre 1922, 492–493).

Further, in his quality of national member of the Turin Academy of Sciences, Segre made every possible effort to send the volumes of the academic collections to hostile or neutral countries, as to the Sweden through Mittag-Leffler.[131] Moreover, as the dean of the Turin Faculty of Sciences, in the war years he actively worked for students who were enlisted to help them in catching up on their examinations. He also interceded so that lecturers engaged at the front in research activity of military engineering, meteorology or ballistics would receive the instruments and books necessary for their studies (ANL-Volterra, Segre to Volterra, 13 January 1917). As far as this latter aspect is concerned, Segre insisted that the Special Library of Mathematics should go on receiving German publications, as they were "essential for scientific institutes", despite the customs block on commodities coming from the Central Empires (Giacardi and Roero 1999, 444). Finally, in relation to an extraordinary Agenda, concerning participation in chairs by foreign scholars, voted

[130]Cf. Klein to Enriques, 13 August 1920 and Enriques to Klein, 18 January 1921 in Luciano and Roero (2012, 214–217).

[131]C. Segre to G. Mittag-Leffler, 27 February 1920, IMLSA, C. Segre n. 8: "Seulement aujourd'hui j'ai pû avoir une réponse à votre question. On m'a dit que le tome 64 des *Memorie* vous a été envoyé en juillet 1914, et le tome 65 en juillet 1916: tous les deux au moyen des échanges internationaux. Maintenant on m'a promis de vous en envoyer un nouvel exemplaire. Après la 1ère Partie du t. 66 nous n'avons plus pû publier les *Memorie*, mais seulement les *Atti.*".

by the Faculties of Letters and Sciences in Rome and by that of Letters and Philosophy in Turin, Segre affirmed, in one of the last meetings held while he was dean:

> Personally [I] do not feel to wholly adhere to the votes formulated in the said Agenda. They are delicate matters, which it is necessary to deal with and decide on serenely, and ones which should not be subjected to deliberations taken irrationally, moved by feeling alone, as happens in some manifestations occasioned by the present war.[132] ("[Segre] personalmente, non si sente di aderire in tutto e per tutto ai voti formulati in detti Ordini del Giorno. Sono questioni gravi, che occorre trattare e decidere con serenità, e alle quali non convengono deliberazioni prese affannosamente, mosse dal solo sentimento, come avviene in talune manifestazioni occasionate dalla guerra presente.").

The ensuing discussion saw many colleagues in the Faculty align with Segre's opinion. In particular, D'Ovidio declared he was "favourable to admitting full Professors of those countries which granted full reciprocity of treatment."[133]

These various actions caused Segre to be branded filo-German, and to be criticised by some foreign and Italian scholars (Mazliak and Tazzioli 2009, 23; Aubin and Goldstein 2014, 189–192).

Actually, his conduct in the war years cannot be easily liquidated, since it took on distinct tones according to the role that he was playing. In this connection, if on one side Segre—as a private citizen—contributed to charity initiatives for ex-combatants and disabled war veterans,[134] on the other side—as a functionary—deemed it right to applaud the heroism of students and colleagues fighting on the battlegrounds.[135] As a scientist, he did not betray the ideals of brotherhood and cosmopolitism, which he had believed in since his youth, when he read the volumes by A. Thiers on the history of the French Revolution,[136] or when he attended the free course on Criticism of socialistic doctrines held by Salvatore Cognetti de Martiis. As a man, Segre always respected the deeds of those co-workers that, faithful to the ideals of the Risorgimento, 'made the nation illustrious with their mind and arms' (Luciano 2013, 307–309, 335–345). Therefore it is not surprising that to his friend Volterra he freely manifested patriotic enthusiasms, even ending letters with the exclamation: "Long live Italy!" (ANL-Volterra, Segre to Volterra, 27 May 1915). Nor does it appear contradictory that Segre acted in favour of A. Terracini and M. Picone[137] so that the young mathematicians could receive the authorizations, the books and the sources

[132] ASUT, *Verbali del Consiglio della Facoltà di Scienze*, 13 May 1916.

[133] *Ibidem*: "favorevole ad ammettere nelle nostre Università Insegnanti di quelle Nazioni le quali accordino piena reciprocità di trattamento."

[134] Cf. the newspaper *La Stampa*, Turin 1st June 1915: 6; 5 June 1915: 5; 24 July 1915: 5; 7 November 1915: 5; 19 April 1916: 4.

[135] Cf. the praise written by Segre for F. Vercelli, a lecturer at the University of Turin and an officer of the 3rd Army on the Carso, for his *Analisi armonica dei barogrammi e previsione della pressione barometrica.* In that case "collaboration between meteorology and war was really effective, especially in the most serious and decisive impasses." (Segre 1919–20, 20).

[136] Cf. UTo-ACS. *Appunti e Resoconti. Zibaldone di appunti*, fols. 1r–5r, incipit: "È cosa ammessa da tutti che l'umanità progredisce …" and *La Rivoluzione Francese*.

[137] This intervention was entirely spontaneous and not prompted by G. Fubini, or by Picone himself, or by Segre's cousin Roberto, a general in the 5th Artillery Corps.

indispensable to lead their studies on mountain artillery, "to the advantage of our war, which is at the core of our thoughts".[138]

Segre's international relationships and those of his disciples took on a new impulse after the end of the war. He revived his links with Hilbert and Klein, expressing regret at not having been able to participate in their jubilees (Segre to Hilbert, 20 October 1919 and Segre to Klein, 24 February 1921, Luciano and Roero 2012, 213, 218). Further, in the inaugural lecture for the 1919–20 academic year[139] he expressed his feelings with these words full of *pathos*:

[138]Cf. ANL-Volterra, Segre to Volterra, 13 January 1917: "Dr Picone, as you will read in a sheet that I enclose, resuming the good results that his calculations gave last summer in the raids against the *Alpe di Cosmagnon* and against the *Dente di Pasubio*, was charged with compiling, here, in the winter, new tables for firing in mountains with medium and big calibres, using the facilities that are found in the nearby shooting range at Cirié. This appointment was given with all possible formalities: Supreme Command, Ministry of War, etc. And the work must be completed within the end of March. So at the Cirié shooting range, after many flatteries, Picone: 1st) was warned that for the experiences of rectification and control of the calculations he would have, *for every shot*, to wait for a special ministerial authorization; 2nd) he was even forbidden to consult the old experiences, without a new permit by the Ministry; 3rd) he could not even have books, except with the indirect intervention, in Turin, by General Arlorio. I will say nothing about the fact that he was not even given a draftsman or a calculator! Fubini took an interest in Picone's problems, and also indicated to him some modifications of his method. By the way he found a formula for adjusting the fire, valid in general, to replace the usual correction coefficients, not efficient in firing with large differences in height. (…) Colonel Bianchi from here (a professor of Ballistics, on whom it seems everything that concerns artillery depends) objected to him that this formula was too complicated, and that there was nothing better than a coefficient like Parodi's! (…) In a few words, your work, that of Fubini, etc. is going to sleep. Obstructionism at Cirié, obstructionism in Turin with Colonel Bianchi. They don't want help from those who could give it! And it is our Italy!" ("Il D[r] Picone, come leggerai in un foglio che qui ti unisco, in seguito agli ottimi risultati che i suoi calcoli avevano dato l'estate scorsa nelle azioni contro l'*Alpe di Cosmagnon* e contro il Dente del Pasubio, ebbe l'incarico di compilare, qui, nell'inverno, nuove tavole di tiro per il tiro in montagna dei medii e dei grossi calibri, valendosi dei mezzi che si trovano nel vicino poligono di esperienze d'artiglieria di Cirié. Questo incarico è dato con tutte le formalità possibili: Comando supremo, Ministero della Guerra, ecc. E il lavoro deve essere ultimato entro la fine di marzo. Orbene al poligono di Cirié, dopo molti complimenti, il Picone: (1°) fu avvertito che per le esperienze di rettifica e di controllo dei calcoli avrebbe dovuto, *per ogni colpo*, attendere di avere un'apposita autorizzazione ministeriale; (2°) ebbe persino il divieto di consultare le vecchie esperienze, senza una nuova corrispondenza col Ministero; (3°) nemmeno libri poté avere, se non quando ricorse all'intervento indiretto, a Torino, del generale Arlorio. Non parlo poi del fatto che non gli si diede nemmeno un disegnatore o un calcolatore! Fubini s'interessò ai problemi del Picone, e gl'indicò anche qualche modificazione nel metodo. Trovò, fra l'altro, una formola per l'aggiustamento dei tiri, valida in generale, da sostituire ai soliti coefficienti di correzione non validi nei tiri con forti dislivelli. (…) Il colonnello Bianchi di qui (prof. di Balistica, da cui pare dipenda tutto ciò che riguarda l'Artiglieria …) gli obbiettò che era cosa troppo complicata, e che non c'era niente di meglio di un coefficiente come quello del Parodi! (…) In poche parole la cosa tua, quella di Fubini, … tutto si mette a dormire. Ostruzionismo a Cirié, ostruzionismo a Torino col colonnello Bianchi. Non vogliono aiuti da chi potrebbe darli! E si tratta della nostra Italia!"). On Segre's role in this matter cf. also Terracini (1968, 84–88).

[139]Segre devoted particular care to writing this *lectio inauguralis.* After some hesitations on the theme, he several times corrected the manuscript. In this case too the literature quoted was almost all international. Alongside classical texts like *Conférences scientifiques et allocutions* by W. Thomson (Lord Kelvin 1893) and *La valeur de la Science* by H. Poincaré (1905), Segre referred to essays on astronomy by J. Sageret (1913), on meteorology by Moreux (1910) and A. Angot (1916), and on atomic physics, radiochemistry and cosmology by B. Brunhes (1908), S. Arrhenius (1909), F. Soddy (1912), A. Berget (1912) and E. S. Grew (1914).

> More than twenty years ago, a work dedicated to the subject of the war that caused a great stir, by the Russian De Bloch, contained these prophetic words: “It is impossible to guarantee that the emperor Wilhelm II, in one of those fits of passion, vibrant with partiality, that he is wont to have, may not be capable of violating a treaty and taking on himself the responsibility … of provoking a war, whose consequences are impossible to predict.” Others too foresaw the huge tragedy that was to break out. But humanity has not been able to prevent it. There is a great difference between ‘foresee’ and ‘prevent’! […] And yet, through uncertainties and corrections, Science progresses; it becomes more and more capable of foreseeing; and to its practitioners it appears more and more beautiful. O young people, about to take up new studies, acquire new knowledge, or competences that will serve you in life; and you, who are distant, and to whom our thought always went out in these years, full of love and gratitude, students that have fought gloriously for our great mission, and have won! There, the day has come when humanity, freed of the arrogant, can start works of peace again with greater safety than it ever had. And we will be able in these rooms, without that shadow of remorse that during the war we seemed to feel, to resume all together to cultivate Science: not only that which is applied to bringing material wellbeing to men, but also that other kind of Science whose only aim is the satisfaction of our spirit. And Science will give you—allow me, in concluding, this prevision—the highest, the purest joys! (Segre 1918–19, 11, 24).

It was, however, above all in the context of learned Societies, that Segre continued to be the messenger of convictions and actions of authentic supranational landmark. He did this, for instance, when in 1922 he told Pincherle that he would resign from UMI if ostracism persisted towards mathematicians from Central Empires.[140]Segre’s internationalism also shines through the penetrating commemorations of Reye (22 April 1922), Zeuthen,[141] Schwarz and Nöther, who were “glories not only of Germany, but of the whole civilized world”.[142] In these he charmingly illustrated to what extent their works had conditioned the evolution of Italian algebraic geometry. Lastly, we shall remember, what Segre wrote to Klein, following the publication of his *Gesammelte Mathematische Abhandlungen*, which he presented at the Accademia dei Lincei together with Castelnuovo and Enriques:

> After so many vicissitudes, after so many sorrows, I have always preserved for you the affection and the veneration that I started to profess when I was a university student, when I grew excited reading your geometric works. What a deep influence those readings had on

[140]Cf. Segre to Pincherle, 19 May 1922, in Nastasi and Tazzioli (2013, 387–390).

[141]Cf. Segre (1919–20, 327–328): “Because of the affection that linked me to him for many years, because of the gratitude that I felt for him because of everything that I have learned from him, I feel the duty to draw your attention, though briefly, to our great loss … Very much devoted to Italian geometers, he was wont to express very flattering judgments on our geometry, also publicly. And he loved Italy: to which (particularly in Turin) he had repeatedly come. It is not one month since he wrote to me saying this; and he joined me in deploring the recent loss of two other illustrious geometers who were his contemporaries: T. Reye and R. Sturm.”.

[142]Segre (1921–22b, 161). Speaking of scientific exchanges between the Italian School of Algebraic Geometry and Nöther, Segre annotated (1921–22b, 162, 163): “Nöther’s works were then the starting points for Enriques, Castelnuovo and the other geometers, particularly Italian and French, that so brilliantly erected the edifice of present-day Geometry on algebraic surfaces. (…) Nöther’s work, as I have already mentioned, had a great influence on the modern development of algebraic geometry in Italy. He admired this development; and he was pleased about his personal relationships with the Italian geometers.”.

> my ideas at that time! Now, cutting out the pages of this volume, and seeing those papers again, I seem to feel once again the freshness and the limpidity of those impressions. You were my *Maestro*, though we were so distant from one another! (Segre to Klein, 24 February 1921, in Luciano and Roero 2012, 218).[143]

8 Conclusion

Speaking of the flourishing School of Luigi Cremona, in 1930 Castelnuovo stated:

> Now since the School goes beyond the value of the man and the importance of a given discipline, to affect all scientific activity, it is worth saying a few words on the subject. All of you know what difficulties we meet in our Latin countries, which are prevalently individualist, in constituting a scientific School, that is to say a reunion or I would almost say a family of people collaborating in developing and pursuing a well defined project of research. But you also understand what advantages the School brings with itself. In scientific respects it offers the means accelerating and deepening the exploration of a given field, penetrating every facet of it, illuminating it from various perspectives. But the School also brings advantages as regards individuals, since it makes it possible to exploit in the most effective way the various aptitudes, and also to treasure the work of mediocre scholars, who, if guided, can perform useful services, while if they are abandoned to their own devices they tend to encumber science with contributions of little or no value. Now, to create a School the worth of the *Maestro* is not sufficient, nor is it sufficient that he knows how to trace out such a vast plan of researches as to go beyond his own working capacity. It is also necessary that he succeed in communicating his passion and his faith to his disciples and in demanding and directing their collaboration (Castelnuovo 1930, 615).

This description of what it means the membership to a mathematical School, of what advantages and detriments it introduces in the collective work, but above all of what the prerogatives are that a *Maestro* has to have for creating and directing it, is well suited to the case of the team of Italian algebraic geometers.

The question of the pertinence and effectiveness of the interpretative category of ‘School’ in relation to the group that Segre gathered around himself has already been discussed (Luciano and Roero 2012). However, it was still necessary to explore more closely the extent and the nature of the exchanges that involved the members of Segre’s School, according to the recent literature on the topic ‘internationality and science’ in the *Belle époque*.[144] In light of new archival sources it appears clear that Segre contributed in an essential way to the construction of a specific identity for his School and to the promotion of its image abroad, not only with writings and lectures, but also through other forms of sociability.

The various aspects of twentieth-century internationalism were not, of course, equally present in his scientific activity. If the breadth of Segre’s cultural horizons really was supranational, also thanks to his training in a context like the Turinese

[143] Cf. also Segre to Klein, 2 June 1923, in Luciano and Roero (2012, 219).

[144] Cf. for example Ausejo and Hormigòn (1993), Rasmussen (1995), Bottazzini and Dahan Dalmedico (2001), Dhombres (2004, 81–114), Goldstein et al. (1996), Parshall and Rowe (2002).

one, permeated from the time of the Risorgimento by influences of French and German science, his international vocation was weaker from the publishing point of view. In this sphere his patriotic feelings sometimes ran over, above all in the 20th century, into a policy of almost nationalistic promotion of Italian journals.

Likewise, if it is true that Segre did not reproduce the stereotype of the 'Jew with a suitcase in his hands' (Heims 1980),[145] nonetheless he was capable of developing a strategy of internationalism that was very effective for his School, even in the short term, fully exploiting the dimension of orality, and in particular conversations with his 'distance *Maestri*' and then with his 'distance disciples', during trips, study sojourns and international congresses.

The Short Century may have seen Segre less prolific on the research front. On the other hand, it marked the moment of his greatest commitment to crystallizing the identity of his School, not only in the milieu of Old Europe, but also in that of developing countries, from Americas to Scandinavia and Poland.[146]

From 1890 on, and even more from 1904, Segre would have devoted his best energies and much of his time to drive abroad the Italian *style* in algebraic geometry, recognizing—in the evolution of his School—a shining example of the law of rapprochement highlighted by Klein, according to which:

> The development of mathematical Schools, subject to alternations of progress and decadence in the limits of a nation, is revivified passing from one nation to another, almost making the spirit of the world participate more amply in the common work (Frajese, but Enriques 1938, 181).

Acknowledgments We express our gratitude to the heirs of Corrado Segre, Silvano, Daniele and Lorenzo Fuà, for bequeathing to the University of Turin the personal archives of their outstanding ancestor. We wish to thank Paola Gario, Emma Sallent, Pietro Nastasi and Pasquale Tucci who put at our disposal unpublished sources concerning Segre's scientific activity. Barbara Gilbert (Special Collections Research Center, University of Chicago Library), Tom McCutchon (Public Services Specialist, Columbia University Rare Book and Manuscript Library), and Laura Bitossi (Università degli Studi di Firenze, Biblioteca di Scienze, BDMI) kindly helped and assisted us in archival research. We are also indebted to Sloan Despeaux, Livia Giacardi, Aldo Brigaglia, Ciro Ciliberto, David Rowe and Norbert Schappacher for the interesting exchanges on some critical aspects of our work. At last lovely thanks go to Sergio, Luca, Ivan and Amos for having supported us during this adventure.

Annexes.

1. Corrado Segre to Aurel Voss,[147] [Turin November] 1882

UTo-ACS. *Carteggi, Lettere di Segre* (Gario L3, N. 192, R. 219), f. 1r-v.

[145] From this point of view Segre was more 'sedentary', compared to both his collaborators Castelnuovo, Enriques and Severi, and colleagues like Peano, Volterra and Levi-Civita.

[146] To have a concrete image of the size of the network of international contacts cultivated by Segre, see his *Address Book* (UTo-ACS. *Documenti di famiglia. Indirizzario*, fols. 13–27) where there are listed the addresses of mathematicians from all over the world.

[147] Aurel E. Voss (1845–1931).

A M^{r} le D^{r} A. Voss, professeur de mathématiques Darmstadt.

Monsieur,

Veuillez m'excuser si j'ose vous écrire sans vous être connu, pour vous faire une demande que vous trouverez peut-être indiscrète. Cependant ce n'est pas sans avoir fait tout mon possible pour éviter de vous incommoder que je me résous à vous écrire. J'étudie avec passion cette belle science que vous illustrez, la Géométrie[148]; spécialement les complexes de droites forment aujourd'hui l'objet de mes études. Il me serait donc indispensable non seulement de lire, mais d'approfondir vos mémoires, qui contiennent la résolution de tant de questions importantes et générales de cette théorie si jeune et si belle. Je nomme[149] particulièrement les mémoires publiés dans les tomes VIII, IX et X des «Mathematische Annalen»[150] et[151] spécialement celui du tome IX «Über Complexe und Congruenzen» qui a tant d'importance et qu'en aucune manière je n'ai pu me procurer chez les libraires. Si vous vouliez, Monsieur, m'expédier ces mémoires[152] en m'avertissant de leur prix, ou bien m'écrire par quels moyens je pourrais les acquérir je vous en serais infiniment obligé. Excusez-moi, encore une fois, de mon indiscrétion, mais comme elle est causée par l'amour de la // science, j'espère que vous me compatirez.[153] Quelle que puisse être votre réponse, je vous en remercie auparavant[154] Votre humble serviteur Corrado Segre.

2. Corrado Segre to Aurel Voss, [Turin] 1882

UTo-ACS. *Carteggi, Lettere di Segre* (Gario L4, N. 193, R. 220), fols. 1r-2r.

Monsieur,

J'ai reçu[155] vos mémoires sur la géométrie des espaces de droites[156] en même temps que votre lettre, et je vous en remercie vivement. Vous ne pouvez vous figurer le plaisir qu'elles m'ont fait.[157] Outre la joie[158] d'avoir enfin dans mes mains ces mémoires que j'avais tant désiré, recevoir encore d'un géomètre distingué comme vous une lettre si obligeante comme vous avez écrit, à moi qui ne suis

[148]In the manuscript Segre erased: "et je me trouve à près présentement j'..".

[149]Here Segre erased: "spécialement".

[150]A. Voss, Zur Theorie der windschiefen Flächen, *Mathematische Annalen* 8 (1875a): 54–135; Über Complexe und Congruenzen, *Mathematische Annalen* 9 (1875b): 55–162; Die Liniengeometrie in ihrer Anwendung auf die Flächen zweiten Grades, *Mathematische Annalen* 10 (1876): 143–188.

[151]Here Segre erased: "particulièrement".

[152]Here Segre erased: "et m'avertir".

[153]Segre erased: "Agreez, monsieur, mes remerciments, qu'elle que puisse être votre réponse".

[154]Segre erased: "en vous saluant".

[155]Segre erased: "vos quatre de".

[156]Cf. Voss (1875a, 1875b, 1876) cit.

[157]Segre erased: "; vous le comprendrez peut-être mieux quand vous serez que je n'ai même pas encore le titre de".

[158]Segre erased: "plaisir".

même pas encore docteur (je[159] suis étudiant de la troisième année à cette université) c'était beaucoup plus que je ne méritais.

Je vous remercie aussi de vos conseils. Dans l'étude que[160] je fais avec passion de la géométrie de la droite, après l'ouvrage[161] de Plücker[162] et les mémoires de Battaglini,[163] de Pasch,[164] etc. je suis passé aux recherches de M^r Klein, en tâchant de bien posséder la méthode de coordonnées si féconde qu'il a développée particulièrement dans les mémoires du tome II des Mathematische Annalen «Über die Theorie der Complexe I und II Grades».[165] Seulement alors j'ai pu[166] commencer à lire vos beaux mémoires sur les complexes d'un degré quelconque. Je vous assure que je tâcherai d'en tirer tout les profits possibles, ne fût-ce que pour gratitude envers vous. Du reste comme pour moi l'étude des mathématiques, et particulièrement de la géométrie, est un grand plaisir et non pas une chose ennuyeuse, je n'y aurai pas beaucoup de mérite.

Vous avez déjà eu tant de bonté que peut-être me permettrez-vous de vous entretenir encore un moment sur une question étrange relative à la théorie des complexes. Monsieur Klein dans le mémoire[167] cité donnait une interprétation géométriques de ses cordonnées de la droite fondée sur les théorèmes qu'il énonçait ainsi: «si dans l'équation (en coordonnées plückeriennes p_{ik}) d'un complexe linéaire l'on substitue[168] les coordonnées d'une droite donnée, on aura une expression qui sera proportionnelle au[169] moment de la droite donnée et de la droite qui lui correspond relativement aux complexe linéaire».[170] Mais[171] en tâchant de prouver ce théorème que j'ai avais aussi trouvé dans la II édition allemande de la géométrie

[159]Segre erased: "n'étant qu'un".

[160]Segre erased: "j'ai fait, non seulement à l'école mais pour moi-même sur".

[161]Segre erased: "l'etude de".

[162]Julius Plücker (1801–1868). J. Plücker, Théorie générale des surfaces réglées, leur classification et leur construction, *Annali di Matematica* (2) 1 1867, 160–169; *Neue Geometrie des Raumes,* Leipzig: Teubner 1868.

[163]Giuseppe Battaglini (1826–1894). G. Battaglini, Intorno ai sistemi di rette di primo grado, *Giornale di Matematiche* 6 (1868): 24–37; Intorno ai sistemi di rette di secondo grado, *Giornale di Matematiche* 6 (1868): 239–259; 7 (1869): 55.

[164]Moritz Pasch (1843–1930). M. Pasch, Zur Theorie der linearen Complexe, *Journal für reine und angewandte Mathematik* 75 (1873): 106–152.

[165]Felix Klein (1849–1925). F. Klein, Zur Theorie der Liniencomplexe des ersten und zweiten Grades, *Mathematische Annalen* 2 (1870): 198–226, Die allgemeine lineare Transformation der Liniencoordinaten, *Mathematische Annalen* 2 (1870): 366–370.

[166]Segre erased: "lire".

[167]Segre erased: "que j'ai déjà".

[168]Segre erased: "à la place".

[169]Segre erased: "la racine carrée du".

[170]Klein (1870, 366–370) cit. Segre erased: "Ce théorème a été reproduit dans la géométrie de l'espace de Salmon traduite … Fiedler (version allemande)".

[171]Segre erased: "cette année en faisant le calcul, d'ailleurs très simple, pour le vérifier, je".

de l'espace de Salmon-Fiedler[172] il me semblait qu'au lieu de dire «proportionnelle au moment etc.» il fallait dire «proportionnelle *à la racine carrée*[173] du moment etc.»,[174] lorsque dans la III édition de cette géométrie j'eus le plaisir d'y trouver dans une «errata» précisément cette correction[175]: on y dit qu'elle est due à M. Sturm (Crelle's Journal, Bd 86, pag. 138).[176] Cependant après y avoir encore réfléchis et avoir lu ce que dit M. Sturm dans le lieux cité je suis venu à la conclusion que le théorème même ainsi modifié est faux et je vous prie d'écouter mes raisons. Le soupçon de cette fausseté m'était venu en raisonnant ainsi: soit $(cp) = c_{12}p_{34} + \ldots = 0$ un complexe linéaire; le théorème (modifié) dit que (cp) est proportionnel à la racine carrée du moment des droites $p{\cdot}p$' // qui se correspondent relativement au complexe donné. Mais supposons[177] que $(c \cdot c) = 0$, c'est-à-dire que le complexe soit spécial, alors on sait positivement que (cp) est proportionnel au moment (et non à sa racine carrée) des droites $p{\cdot}p$' (p' étant en ce cas la droite c) il y aurait donc une exception très singulière au théorème pour ce cas-ci. Mais en examinant un peu la démonstration donnée par M. Sturm, et que moi aussi j'avais cru un moment acceptable, il me parait évident qu'elle est insuffisante. M. Sturm dit ainsi (je ne fais que changer légèrement ses notations): l'on a, comme on sait:

$$p'_{ik} = -\frac{1}{2}(cc)p_{ik} + (cp)c_{ik}.$$

De là on tire par une proposition que j'ai déjà citée, que le moment des deux droites $p{\cdot}p$' sera proportionnelle à $(pp') = (cp)^2$ d'où l'on conclut le théorème en question. Mais si M. Sturm au lieu d'écrire

$$p'_{ik} = -\frac{1}{2}$$

avait écrit, comme c'est plus juste:

$$pp'_{ik} = -\frac{1}{2}(cc)p_{ik} + c_{ik}(cp),$$

peut-être se serait-il aperçu que ce raisonnement est insuffisante. Car le facteur p qui[178] reste à la fin du calcul, car l'on obtient $p(pp') = (cp)^2$ peut très bien être

[172]George Salmon (1819–1904, Otto Wilhelm Fiedler (1832–1912). G. Salmon and W. Fiedler, *Analytische Geometrie des Raumes*, 2 Bände, 2 Teil, 2 Aufl., Leipzig: Teubner 1865[2].

[173]Segre underlined "à la racine carrée".

[174]Segre erased: "Peu de temps après étant"; "Etant pour la dernière édition de la géométrie de Salmon–Fiedler".

[175]G. Salmon and W. Fiedler, *Analytische Geometrie des Raumes*, 3 Aufl., Leipzig: Teubner 1874[3].

[176]F.O. Rudolf Sturm (1841–1919). R. Sturm, Darstellung binärer Formen auf der cubischen Raumcurve, *Journal für die reine und angewandte Mathematik* 86 (1879): 116–145.

[177]Segre erased: "pour un moment".

[178]Segre erased: "l'on obtiendra à la fin".

une fonction non seulement des coordonnées c du complexe, mais encore des coordonnées p_{ik} de la droite, pourvu qu'elle soit symétrique en celles-ci. Dès lors, on ne peut plus rien conclure sur le théorème en question.

J'ai même tâché de prouver qu'il est vraiment faux et voici comment. J'ai pris pour coordonnées, non homogènes, d'une droite les rapports p_{ik} des[179] volumes des tétraèdres déterminés par deux sommets i, k du tétraèdre fondamental et deux points de la droite dont la distance soit = 1, au volume V du tétraèdre fondamentale. Alors entre les 6 coordonnées d'une droite passent les *deux* relations caractéristiques:

$$(p \cdot p) = 0, \quad \sum a_{mn,rs} p_{mn} p_{rs} = 1$$

où m, n, et r, s sont des combinaisons binaires de 1234, et où $a_{mn,rs}$ sont des quantités constantes qui dépendent seulement de la forme du tétraèdre fondamental. Cela posé l'on trouve facilement (de la même manière que pour les coordonnées projectives) que la droite p' que coorrespond à p relativement au complexe $(c \cdot p) = 0$ a des coordonnées p'_{ik} telles que:

$$pp'_{ik} = (cc)p_{ik} - 2(cp)c_{ik}$$

où

$$p^2 = 4(cp)^2 \sum a_{mn,rs} c_{mn} c_{rs} - 4(cc)(cp) \sum a_{mn,rs} c_{mn} p_{rs} + (cc)^2.$$

On voit bien ainsi ce qu'est le facteur p. Maintenant comme dans ces cordonnées-ci le moment de deux droites p, p' est précisément égal à $6V(pp')$ l'on aura:

$$\text{mom } (pp') = -12V \frac{(cp)^2}{p}$$

d'où l'on voit que ce moment n'est pas du tout proportionnel à $(cp)^2$, car p est une fonction des c et des p. Lorsque c est un complexe spécial $(cc) = 0$,[180] alors p^2 devient égal à moins d'un facteur indépendant de p (c'est-à-dire $4 \sum a_{mn,rs} c_{mn} c_{rs}$) à $(cp)^2$ et la formule précédente donne que le // moment de p et p' est proportionnelle à (cp), comme cela devrait être. Il parait ainsi qu'en général lorsqu'on fait usage de coordonnées homogènes il serait mieux de préciser un peut plus le sens de la parole «proportionnel».

Pardonnez-moi, Monsieur, cette longue lettre, mais c'était une belle occasion pour moi pour soumettre mes doutes à un savant. Je vous répète encore une fois mes remerciements pour votre bonté et en même temps je vous prie[181] de m'écrire

[179] Segre erased: "au volume du te".

[180] Segre erased: "c'est-à-dire une droite".

[181] Segre erased: "encore".

encore pour me dire[182] ce que je vous dois pour les mémoires que vous m'avez envoyé,[183] car je désire beaucoup le savoir. Vous avez déjà été trop courtois de me les envoyer sans me connaitre et je[184] ne voudrais absolument pas que vous y perdiez. Même après vous avoir expédié leur prix, ce sera toujours moi qui aurai gagné au charge. Vous me ferez aussi infiniment plaisir en me disant un mot sur ces doutes que je vous ai soumis. Votre très-dévoué Corrado Segre.

3. Corrado Segre to Leopold Kronecker,[185] Turin 16 November 1883

UTo-ACS, *Carteggi, Lettere di Segre* (Gario L16, N. 206, R. 233), fols. 1r-2r

Herr Prof. Leopold Kronecker, Professor der Mathematik an der Universität zu Berlin

Turin, le 16 Novembre 1883

Monsieur,

Sans avoir le plaisir de vous être connu, je prends la permission de vous écrire pour vous demander quelques renseignements sur la théorie des formes bilinéaires, théorie dont je m'occupe depuis quelque temps, spécialement en vue de ses applications géométriques.[186] J'ai tant de confiance dans votre courtoisie que,[187] malgré mon hardiesse, je suis sûr que vous me donnerez ces renseignements.

En 1868 votre savant ami, M. Weierstrass,[188] a donné dans un mémoire des *Monatsberichte* de Berlin un théorème fondamental sur la théorie des formes bilinéaires et quadratiques, c'est-à-dire il a démontré que la condition nécessaire et suffisante pour que deux séries (*Schaaren*, suivant votre expression) de formes quadratiques

$$u\Phi + v\psi, u\Phi' + v\psi'$$

puissent se transformer linéairement l'une dans l'autre, est que les déterminants de

$$u\Phi + v\psi, u\Phi' + v\psi'$$

aient les mêmes diviseurs élémentaires (*Elementartheiler*).[189] Mais M. Weierstrass avait exclu les cas dans lequel ces déterminants sont nuls identiquement. Or dans

[182]Segre erased: "ce qui vous pensez de mes doutes et pour le prix".

[183]Segre erased: "Il ne serait pas juste que je ne satis".

[184]Segre erased: "je vous en serai reconnaissant même".

[185]Leopold Kronecker (1823–1891).

[186]Segre erased: "Je suis sûr que bien que je soi pour".

[187]Segre erased: "bien que".

[188]Karl Weierstrass (1815–1897).

[189]K. Weierstrass, Zur Theorie der bilinearen und quadratischen Formen, *Monatsberichte der königlichen Preussischen Akademie der Wissenschaften zu Berlin* 18 Mai 1868 (1869): 310–338, in particular 326–326. Cf. also Segre (1883b), (1883–84c).

les remarques que vous, Monsieur, avez fait suivre à ce mémoire,[190] vous considérez précisément les séries pour lesquelles se présente ce fait, et vous avez démontré l'importante proposition qu'une telle série peut toujours être représentée par

$$u\left(f_1 x'_{m+1} + f_2 x'_{m+2} + \cdots + f_m x'_{2m} + F\right) + v\left(\varphi_1 x'_{m+1} + \varphi_2 x'_{m+2} + \cdots + \varphi_m x'_{2m} + \Phi\right)$$

où F et Φ sont des formes quadratiques de

$$x'_{2m+1}, x'_{2m+2}, \ldots x'_{n-1}$$

et de f_i, φ_i sont des formes linéaires de toutes les n variables $x'_0, x'_1, \ldots x'_{n-1}$. Mais ce que je ne trouve pas dans // votre note c'est la condition nécessaire et suffisante pour que deux séries de formes quadratiques de cette espèce particulière puissent se transformer linéairement l'une dans l'autre, c'est-à-dire qu'elles soient *équivalentes.* Il me semble que cette condition doit être la suivante: que la série $uF + v\Phi$ et son analogue (qui sont deux séries pour lesquelles les déterminants ne sont pas nuls identiquement)[191] soient elles-mêmes équivalentes; de manière qu'en appliquant le théorème de M. Weierstrass à ces dernières séries à un nombre moindre de variables on aura précisément la condition cherchée. En d'autres termes on aurait ainsi ce qu'il m'importerait beaucoup d'avoir, c'est-à-dire tous les cas que peut présenter du point de vue de l'algèbre des transformations linéaires une série de formes quadratiques dont le déterminant soit identiquement nul, et par conséquent aussi tous les invariants absolus d'une telle série des formes. Mais comme ce n'est là qu'un résultat que ne suis pas encore réussi à prouver rigoureusement pour tous les cas, je désire vivement de savoir si vous vous êtes déjà occupé de cette question et quels sont les résultats que vous avez obtenus. Peut-être dans le cas contraire pourrez-vous la résoudre avec peu de mots[192] et m'éclairer sur une chose qui m'intéresse beaucoup.[193]

Permettez-moi encore une autre question. Dans un autre mémoire très-importante et très-connu des *Monatsberichte*, *Über die Congruenten Transformationen der bilinearen Formen*[194] vous avez démontrez, Monsieur, que si f et f' sont deux formes bilinéaires conjuguées, dans le déterminant de $uf + vf'$ aura les

[190]L. Kronecker, Zur Theorie der bilinearen und quadratischen Forme: Bemerkungen (Weierstrass 1868) *Monatsberichte der königlichen Preussischen Akademie der Wissenschaften zu Berlin* 1868 (1869): 339–346.

[191]Segre erased: "puissent se transformer linéairement l'une dans l'autre (c'est-à-dire soient *équivalentes*)".

[192]Segre erased: "cette question qui".

[193]Segre erased hereafter: "mais qui en même temps me présente quelque difficulté".

[194]L. Kronecker, Über die Congruenten Transformationen der bilinearen Formen, *Monatsberichte der königlichen Preussischen Akademie der Wissenschaften zu Berlin* 23 April 1874 (1875): 302, 397–447.

diviseurs élémentaires satisfaisant à la condition d'être deux-à-deux du même degré et correspondants à des valeurs réciproques // de $u : v$, exceptés ceux de la forme $(u + v)^{2x+1}$ et $(u - v)^{2x}$. Or il m'importerait beaucoup, pour mes recherches, de savoir si celles-ci sont les seules conditions auxquelles satisfassent les diviseurs élémentaires du déterminant de $uf + vf'$, c'est-à-dire si réciproquement, étant donné un système de diviseurs élémentaires satisfaisant à ces conditions, on peut trouver deux formes conjuguées f, f' telles que le déterminant de $uf + vf'$ ait précisément ces diviseurs élémentaires.

J'aurais quelques raisons pour en douter, mais je ne suis pas encore assez fort analyste pour trouver les autres conditions, s'il y en a. En conséquence je vous prie,[195] Monsieur, de me dire si vous avez étudié cette autre question, qui me semble très-intéressante, et quels sont les résultats auxquels vous êtes parvenu. Et s'il arrive que vous ne vous en soyez pas encore occupé et que ma lettre vous y fasse penser, je serai doublement content de l'avoir écrite, car non seulement moi, mais aussi la science y aura gagné.

En vous demandant encore excuse pour la liberté que je me suis prise, et en attendant avec impatience votre réponse pour laquelle je vous remercie déjà, je me déclare, Monsieur, avec la plus grande estime Votre très-dévoué D[r] Corrado Segre (presso la R. Università di Torino, Italia).

4. Corrado Segre to Leopold Kronecker, Turin 10 December 1883

UTo-ACS. *Carteggi, Lettere di Segre* (Gario L17, N. 207, R. 234), fols. 1r-2v.

Herr Professor Dr. L. Kronecker, Berlin W. Bellevuestrasse 13

Turin, le 10 Décembre 1883

Monsieur,

Je vous remercie vivement soit pour les explications que vous avez bien voulu me donner sur les deux questions que je vous avais faites, soit pour le cadeau, que vous avez eu la bonté de me faire de votre mémoire *Über die congruenten Transformationen der bilinearen Formen*,[196] que je conserverai précieusement. J'ai retardé à vous écrire pour étudier vos mémoires de Janvier, Février et Mars 1874,[197] mémoires que je ne connaissais pas encore, et qui, avec les nouveaux détails contenus dans votre lettre et dans votre carte postale, m'ont parfaitement éclairé sur la question fondamentale des conditions pour l'équivalence de deux faisceaux quelconques de formes quadratiques. J'étais arrivé par mes propres recherches à un système d'invariants d'un tel faisceau, mais je n'avais pas encore vu l'importance de cet autre système d'invariants que vous avez introduit: les degrés des équations

[195] Segre erased: "encore".

[196] Kronecker (1874) cit. The letter by Kronecker to Segre is lost.

[197] L. Kronecker, Über Schaaren von quadratischen Formen, *Monatsberichte der königlichen Preussischen Akademie der Wissenschaften zu Berlin* 19 Januar 1874 (1875): 59–76, 16 Februar 1874 (1875): 149–156, 16 Marz 1874 (1875): 206–232.

qui lient les dérivées partielles de[198] la forme[199] générale du faisceau. Je préfère, à vous dire le vrai,[200] cette dernière méthode que vous avez adoptée pour exprimer l'équivalence de deux faisceaux, ou le système complet des invariants d'un faisceau donné (c'est-à-dire la méthode dont vous avez fait usage, à ce que vous me dites, dans votre cours de l'été 1882), à celle qui faisait usage de // la considération des faisceaux élémentaires. Je sais bien qu'au fond les résultats sont les mêmes, mais j'aime à éviter l'usage des formes particulières[201] c'est-à-dire des formes *canoniques* pour l'étude de formes générales. Si parfois on ne possède pas une analyse si puissante qu'on puisse éviter tout-à-fait ces[202] formes canoniques dans une[203] recherche, il faut du moins qu'elles ne paraissent plus dans les résultats. En me rappelant ce que vous avez écrit à propos des recherches sur les formes bilinéaires de M. Jordan,[204] je crois que vous pensez aussi comme moi.

Et à ce propos je ne sais si vous avez remarqué[205] que la plus grande partie des écrivains paraissent au contraire s'occuper plutôt des[206] formes canoniques que de la condition générale d'équivalence établie par le moyen du système complet des invariants. Dans ma thèse pour le doctorat (qui paraitra bientôt dans les mémoires de l'Académie de Turin)[207] je me suis occupé précisément, entre autres choses, de l'interprétation géométrique du théorème de M. Weierstrass sur le système des invariants d'un faisceau des formes quadratiques dont le déterminant ne soit pas identiquement nul (le cas d'un faisceau de formes bilinéaires est traité dans un mémoire présenté à la R. Académie des Lincei),[208] et j'ai remarqué précisément que plusieurs géomètres ayant voulu appliquer ces recherches à différents sujets de géométrie (et on peut en faire effectivement des applications géométriques très-importantes) avaient toujours fait usage, non pas de ces théorèmes mêmes, mais des équations canoniques dont M. Weierstrass[209] s'était servi pour établir son théorème, mais qui n'ont certainement pas l'importance (ni analytique ni géométrique) de[210] celui-ci: de cette manière on était contraint à changer la forme des équations quadratiques pour chaque système de degrés des diviseurs élémentaires! Maintenant, // à la suite de votre lettre je m'occupe de l'interprétation géométrique

[198]Segre erased: "une".

[199]Segre erased: "quelconque".

[200]Segre erased: "la mani…".

[201]Segre erased: "d'équations".

[202]Segre erased: "équations".

[203]Segre erased: "démonstration".

[204]Camille Jordan (1838–1922). C. Jordan, Mémoire sur les substitutions *Comptes Rendus hebdomadaires des séances de l'Académie des Sciences Paris* 22 Décembre 1873, 66 (1873): 952–954; Kronecker (1874) cit.: 71–76.

[205]Segre erased: "que la majorité".

[206]Segre erased: "équations".

[207]Segre (1883–84d).

[208]Segre (1883–84c).

[209]Segre erased: "avait cherché".

[210]Segre erased: "qui est bien étrange".

de vos résultats sur les faisceaux dont le déterminant est nul (avec les subdeterminants d'un certain ordre). La considération d'un espace (*Mannigfaltigkeit*) linéaire à $n - 1$ dimensions permet de représenter par un point chaque système de valeurs de n variables x_i, de sorte qu'un faisceau de formes quadratiques $u\Phi + v\Psi$ est représenté par un faisceau de surfaces du 2^e^ ordre. Si le déterminant de toutes ces formes est nul, ces surfaces sont toutes des cônes. Quel est le lieu des sommets de ces cônes? Supposons qu'entre les dérivées partielles de $u\Phi + v\Psi$ passe une seule équation linéaire, dont le degré par rapport à u, v soit m, alors le lieu des sommets des cônes du faisceau est une courbe (*Normalcurve*) de l'ordre m, qui est contenue dans un espace linéaire à m dimensions. C'est ainsi que dans l'espace ordinaire à 3 dimensions il y a deux espèces des faisceaux de cônes; les faisceaux de cônes ayant le même sommet ($m = 0$) et les faisceaux dans lesquels les sommets ont pour lieu une droite ($m = 1$); dans l'espace à 4 dimensions il y a encore, outre ces deux espèces de faisceaux une espèce composée de faisceaux dont les sommets des cônes forment une conique ($m = 2$), etc., etc.. Cette interprétation géométrique du degré m de la relation, qui lie les dérivées de $u\Phi + v\Psi$ me parait remarquable. S'il y a plusieurs relations entre ces dérivées, c'est-à-dire si les surfaces du 2^e^ ordre du faisceau ont, non pas seulement un point, mais une droite, un plan, etc. … double, alors les degrés en u, v de ces relations dépendent des degrés des lieux géométriques de ces espaces doubles; mais je n'ai pas encore fini ces recherches. Lorsqu'je l'aurai finies et complétées j'en ferai une note *Sur les faisceaux de cônes quadriques*, que je vous demanderai la permission de vous envoyer, pour qu'il // soit publié dans votre *Journal für Mathematik*.[211]

En attendant, je vous remercie encore, Monsieur, de votre courtoisie envers moi, et vous envoie mes salutations les plus respectueuses. Votre très-dévoué Corrado Segre.

5. Corrado Segre to Leopold Kronecker, Turin 25 December 1883

UTo-ACS. *Carteggi, Lettere di Segre* (Gario L18, N. 208, R. 235), fols. 1r-2v

Herr Prof. D^r^. L. Kronecker, Berlin W. Bellevuestrasse 13

Turin, le 25 Décembre 1883

Monsieur

Vous me ferez beaucoup de plaisir en continuant à m'écrire dans votre langue maternelle, que je comprends parfaitement. Mais permettez-moi de ne pas vous écrire en italien, car comme cette langue vous est probablement moins familière que

[211]This paper was published in Segre (1883–84d). In the manuscript of this letter Segre erased here: "Quant à la deuxième demande que je vous avais faite, j'attendrai impatiemment que vous poussiez vous en occuper. Il me semble que la question si les particularités que vous avez trouvées pour le système des invariants d'un système de deux formes conjuguées (*f*, *f*') sont les seules qu' il présente, c'est-à-dire si étant donné un tel système l'invariant on puisse trouver les formes correspondantes *f*, *f*', ne peut-être résolue qu'avec difficulté en composant additivement ces formes par l'addition des formes élémentaires qui correspondent à ces invariants.".

la langue française, mes lettres pourraient vous fatiguer et je crains qu'elles ne vous ennuient déjà que trop.[212] A peine avais-je fini d'écrire ma dernière lettre,[213] je pensai que[214] peut-être j'avais fait confusion en considérant vos faisceaux élémentaires de formes quadratiques comme n'ayant autre importance que celle des formes canoniques et votre lettre[215] m'a montré qu' effectivement il n'y a pas de relation nécessaire entre l'une chose et l'autre, c'est-à-dire que l'on peut, il est vrai, donner aux faisceaux élémentaires des représentations[216] simples ou *canoniques*, mais que[217] l'idée des faisceaux est indépendante de ces représentations particulières.

J'ai vu aussi par votre lettre que je ne m'étais pas expliqué clairement à propos de l'interprétation géométrique de vos recherches et de celles de M. Weierstrass sur la théorie des formes bilinéaires et quadratiques. Peut-être ne devrais-je pas dire «interprétation géométrique», car[218] ces mots font penser (et vous ont fait penser à ce que je vois) à un travail qui consiste[219] seulement dans des changements de mots[220] or je considérais comme // ridicule un savant qui ne s'occupât que de changer les mots analytiques en mots géométriques dans des résultats analytiques déjà connus. Mais ce n'est pas là ce que j'entendais dire dans ma dernière lettre.[221] Pour m'expliquer avec plus de clarté, prenons les théorèmes[222] sur les conditions afin que deux formes quadratiques puissent se transformer dans deux autres formes quadratiques. En mettant des mots géométriques, on peut dire que ces théorèmes donnent les conditions pour que deux couples de surfaces du 2^{e} degré (dans un espace à *n* dimensions) soient identiques du point de vue de la géométrie projective. Mais ces conditions restent analytiques, car il y entre des diviseurs élémentaires, etc.; quelle est donc, par exemple, la signification géométrique des diviseurs élémentaires? Si à un couple des surfaces de 2^{e} degré correspond une racine double, triple, etc. du déterminant de leur faisceau,[223] ces deux surfaces se toucheront mutuellement en un ou plusieurs points, mais quelle différence y aurait-il entre ces contacts suivant les divers degrés des diviseurs élémentaires, c'est-à-dire quelles *singularités* aura l'intersection de ces deux surfaces pour un système donné de diviseurs élémentaires? Voilà l'une des questions que j'ai tâché de résoudre et qui n'a pas laissé de me présenter au premier abord des difficultés. C'est ainsi que j'ai pu établir une *classification* géométrique des intersections de deux surfaces du 2^{e}

[212]Segre erased: "mais comme je n'ose … Il n'y a que quelques mois que j'ai cessé d'être étudiant pour devenir professeur, mais je sens encore".

[213]Segre to Kronecker, 10 December 1883, Annex 4.

[214]Segre erased: "probablement".

[215]Segre erased: "me confirme dans cette". This letter by Kronecker to Segre is lost.

[216]Segre erased: "plus".

[217]Segre erased: "la concept".

[218]Segre erased: "cela".

[219]Segre erased: "uniquement".

[220]Segre erased: ", qui d".

[221]Segre refers to his letter to Kronecker, Annex 4. Then Segre erased: "Pour m'expliq".

[222]Segre erased: "de M. Weierstrass et de vous".

[223]Segre erased: "on sait".

degré. De même les résultats analytiques sur les formes bilinéaires m'ont donné par une étude géométrique la classification des *homographies* ou *collinéations* dans un espace linéaire quelconque.

Et votre théorème que "la condition afin qu'une forme bilinéaire *f* puisse être transformée dans une autre f par une substitution congruente est que les deux formes conjuguées *f*, *f*' puissent se transformer dans les deux autres formes conjuguées // *f*, *f*' me donnerai la classification des *corrélations* ou *correspondances dualistiques*. Mais je vous le répète et j'espère vous en convaincre à peine mes travaux seront imprimés, ce n'est pas un simple changement de mots qui donne ces résultats géométriques mais bien une suite de raisonnements plus ou moins difficiles.

J'ai vu avec plaisir soit par votre lettre soit par quelques travaux où vous parlez de géométrie à *n* dimensions, que vous n'êtes pas de ces savants (et il y en a beaucoup) qui n'attachent d'importance à cette science qu'on ce qu'elle peut s'appliquer à l'espace ordinaire: la géométrie à *n* dimensions (comme toutes les branches des mathématiques, si abstraites qu'elles soient) a le droit d'être étudiée en dehors de ses applications. Cependant lorsqu'on peut en faire des applications à l'espace ordinaire il est bon de les faire: or[224] on ne peut se figurer combien d'applications on peut faire de la classification géométrique de l'intersection de deux surfaces du 2^{e} degré à plusieurs dimensions. Toute la classification des complexes de droites du 2^{e} degré et celle des congruences du 2^{e} degré (et en conséquence aussi des surfaces de Kummer[225] à 16 nœuds) en découlent immédiatement; la classification des surfaces du 3^{e} ordre, celle des surfaces du 4^{e} ordre à conique double,[226] puis aussi (comme je m'en suis aperçu seulement[227] il j a deux ou trois jours) la classification des surfaces du 4^{e} ordre ayant au moins deux points doubles et un point uniplanaire ou bien une droite double. Cette méthode donne lieu à des rapprochements curieux entre ces différentes surfaces, qui viennent à être représentées les unes sur les autres d'une manière fort intéressante.

Est-ce-que vous, Monsieur, n'avez jamais abordé le problème sur la condition pour l'équivalence de deux systèmes de plus que deux formes quadratiques de 3, de 4, de *r* formes quadratiques? Vous me faisiez plaisir en faisant aussi la même demande à M. Weierstrass: il me semble improbable que ni vous ni votre amis n'y ayez pas pensé et ne l'ayez pas résolue. Probablement // c'est encore la considération du déterminant de

$$\lambda\varphi + \mu\psi + \nu\chi + \cdots$$

et des[228] diviseurs communs à ses subdéterminants qui donnera tous les invariants du système des formes φ, ψ, χ, … Et est-ce-que vous n'avez pas pensé à étendre cette même méthode au cas de plusieurs formes d'un même degré quelconque en

224Segre erased: "vous ne pouvez vous".

225Segre erased: "du 4^{e} ordre".

226Segre erased: "et des surfaces du 3^{e} ordre".

227Segre erased: "hier".

228Segre erased: "plus".

considérant encore le *discriminant* de leur série et les invariants ou covariants qui en s'annulant indiquent que la forme a plusieurs points singuliers? Je suis bien indiscret en vous faisant toutes ces demandes, mais vous avez répondu si obligeamment aux premières que je vous ai faites que je me sens entrainé à vous en faire des autres. Le système des invariants de deux formes quadratiques est établi d'une manière satisfaisante par les recherches de M. Weierstrass et de vous, qu'il me semble très-désirable d'étendre la même méthode à des cas plus généraux.

Quant à la note dont je vous avais parlé sur les faisceaux de surfaces du 2^e degré à points singuliers (ou cônes) je ne l'ai pas encore écrite,[229] et si mes recherches ne me conduiront pas à en faire travail assez bon je ne vous l'enverrai certainement pas, car je connais trop bien le respect que l'on doit à votre journal, à l'ancien *Journal de Crelle*. Cependant, même dans ce cas, je vous remercie de l'avoir acceptée.[230] Et je vous remercie aussi de la bonté avec laquelle vous voulez bien répondre à mes lettres. Je suis jeun, et j'ai beaucoup de besoin et d'envie d'apprendre: je suis donc bien reconnaissant aux savants qui veulent pour quelques moments être mes maîtres.

Croyez-moi Votre très-dévoué Corrado Segre.

6. Corrado Segre to Oscar Schlömilch,[231] Turin 17 January 1884

UTo-ACS. *Carteggi, Lettere di Segre* (Gario L14, N. 204, R. 231), f. 1r-v.

Herr Prof. D^r Oscar Schlömilch, professor der Mathematik Kgl. Sächs. Geh. Schulrath Leipzig

Turin, le 17 Janvier 1884

Monsieur,

je vous envoie avec cette lettre en un pli chargé une note *Sur les droites qui ont des moments donnés par rapport à des droites fixes*, que je vous prie de publier dans votre *Zeitschrift für Mathematik und Physik*.[232] Dans cette note je m'occupe surtout du complexe des droites qui ont un moment donné par rapport à une droite fixe, complexe quadratique très-remarquable dont la surface singulière se décompose en un cylindre droit et dans le cercle imaginaire à l'infini. Dans la 1^e partie[233] j'étudie élémentairement et par voie synthétique ce complexe; dans la 2^e je me sers des coordonnées pour en trouver d'autres propriétés, qui regardent surtout la *série homofocale* de ce complexe, série qui se compose de complexes quadratiques

[229] He is referring to the paper mentioned in his letter to Kronecker, 10 December 1883, Annex 4.

[230] Segre erased: "sans reserves".

[231] Oscar Schlömilch (1823–1901).

[232] This article by Segre was published in the *Journal für die reine und angewandte Mathematik* 97 (1884): 95–110. Segre (1884c).

[233] Segre erased: " de ma note".

généraux de la classe [(22)11] de M. Weiler,[234] et de ces complexes je trouve ainsi une définition géométrique remarquable. Enfin dans la 3[e] partie je m'occupe brièvement de la congruence, de la surface réglée et du groupe de droites qui ont des moments donnés par rapport resp. à 2, 3, 4, droites fixes.

Dans un mémoire qui va paraitre dans le 2[ième] cahier du nouveau volume des *Mathematische Annalen*, je me suis occupé (avec mon ami D[r] Loria) // de trouver tous les complexes quadratiques qui peuvent s'obtenir comme complexes des droites coupant harmoniquement des surfaces du second ordre.[235] Le complexe des droites qui ont moment donné par rapport à une droite fixe appartient aussi à cette catégorie: je l'ai simplement affirmé dans ce mémoire, et je le prouve dans la note que je vous envoie. C'est pour cela que je désirais que celle-ci fut publiée le plus tôt possible. Vous m'obligeriez infiniment en lui donnant un peu de place dans le prochain cahier de votre *Zeitschrift*. Mais si cela ne vous est pas possible, je ne vous serai pas moins reconnaissant pour la publication. Agréez, Monsieur le Professeur, mes salutations les plus respectueuses. Votre D[r] Corrado Segre.

7. Corrado Segre to Oscar Schlömilch, Turin 8 February 1884

UTo-ACS. *Carteggi, Lettere di Segre* (Gario L15, N. 205, R. 232) postcard.

Turin, le 8 Février 1884

Monsieur le Professeur

Je reçois votre lettre,[236] dont je vous remercie: j'espère que vous serez tout-à-fait guéri de votre maladie. Lorsque j'avais écrit ma note en français, plutôt qu'en italien, je n'avais pas réfléchi à ce que vous me faites justement remarquer, c'est-à-dire que dans votre *Zeitschrift* n'ont jamais paru que des travaux écrits en allemand. Aussi je regret de vous avoir incommodé en vous envoyant mon manuscrit: le faire traduire en allemand vous coûterait aussi plus de peine que le mémoire ne vaille. J'accepte donc volontiers l'offre que vous me faites de le renvoyer aux *Mathematische Annalen*, et je vous remercie de la bonté que vous avez de vous charger vous-même de cet envoi.[237] J'écris tout-de-suite à M. Klein, avec lequel j'ai aussi le plaisir d'être en relation, pour l'en aviser.[238]

En attendant recevez encore, Monsieur, mes remercîments et mes salutations les plus cordiales et respectueuses. Votre très-dévoué D[r] Corrado Segre.

PS. Je prends la permission de vous envoyer une copie d'un travail qu'est paru il y a quelques jours dans le *Giornale*.[239]

[234] Adolf Weiler (1851–1916).

[235] Segre (1883a).

[236] This letter is lost.

[237] Segre (1884c) was published in *Journal für die reine und angewandte Mathematik*.

[238] Cf. Segre to F. Klein, 8 February 1884, in Luciano and Roero (2012, 104–105).

[239] Segre (1883d).

8. Corrado Segre to Leopold Kronecker, Turin 18 February 1884

UTo-ACS. *Carteggi, Lettere di Segre* (Gario L19, N. 209, R. 236), postcard.

Herr Prof. L. Kronecker, Berlin W. Bellevuestr. 13

Turin, le 18 Février 1884

Monsieur,

Je prends la permission de vous envoyer en un pli chargé un manuscrit d'une note que je désirerais voir imprimée dans votre *Journal für Mathematik*. Cette note regarde la géométrie de la droite: j'y étudie d'une façon assez élémentaire quelques complexes quadratiques dont les propriétés me semblent assez remarquables pour mériter d'être étudiées à part.[240] Bien que ce travail puisse se rattacher à un autre imprimé dans les *Mathematische Annalen*[241] (dont[242] j'espère que vous aurez reçu une copie), cependant j'ai fait en manière[243] qu'il se suffit à lui-même de sorte qu'il n'est pas besoin d'avoir lu l'autre pour le comprendre.

Je vous remercie en avance de la publication et en attendant votre réponse[244] (dans laquelle j'espère trouver les explications sur la théorie des formes bilinéaires que je vous avais demandées dans ma dernière lettre,[245] et que sans doute vos occupations vous[246] ont empêché jusqu'à-présent de me donner), croyez-moi Votre très-dévoué D[r] Corrado Segre.

9. Corrado Segre to Gaston Darboux,[247] Turin 11 March 1884

UTo-ACS. *Carteggi, Lettere di Segre* (Gario L12, N. 202, R. 229), f. 1r.

A M. le prof. G. Darboux—Paris

Turin, le 11 Mars 1884

Monsieur,

Je vous envoie deux copies d'une note sur les géométries métriques des complexes linéaires et des sphères[248] qui, j'espère, ne vous déplaira pas tout-à-fait, car elle se lie à des recherches que vous avez fait autrefois, et elle montre (du moins si je dois croire ce que m'en écrit ce juge si compétent qu'est M. Klein) des analogies

240Segre (1884c).

241Segre (1884b).

242Segre erased: "j'espère vous aurez … me suis empressé à vous faire hommage d'une copie".

243Segre erased: "que ce petit travail".

244Segre erased: "j'attends encore avec impatience".

245Segre to Kronecker, 25 December 1883, Annex 5.

246Segre erased: "empêchent de me donner. En vous remerciant par anticipation des deux faveurs croyez-moi".

247Gaston Darboux (1842–1917).

248Segre (1883–84a).

très-intéressantes entre ces deux géométries métriques. Comme M. G. Koenigs,[249] qui a été votre élève, s'est aussi occupé de ces analogies, et comme je ne connais pas son adresse, je vous prie de vouloir lui faire parvenir une de ces deux copies de ma note, si cela ne vous incommode pas trop. Excusez-moi mon hardiesse et recevez mes remercîments et mes respects. Votre D^r^ C. Segre.

10. Corrado Segre to Arthur Cayley,[250] Turin 15 March 1884

UTo-ACS. *Carteggi, Lettere di Segre* (Gario L11, N. 200, R. 227), f. 1r.

Mr. Arthur Cayley, F. R. S., Sadlerian Professor of Mathematics in the University of Cambridge

Turin, le 15 Mars 1884

Monsieur,

je vous envoie une copie d'une note sur les géométries métriques des complexes linéaires et des sphères,[251] en souhaitant que vous vous daigniez-y jeter un regard, car les analogies que j'y découvre entre ces deux géométries métriques (par des méthodes qui vous sont dues en partie) me paraissent fort remarquables. Agréez mes salutations les plus respectueuses D^r^ Corrado Segre.

11. Corrado Segre to Carl Friedrich Geiser,[252] Turin 23 March 1884

UTo-ACS. *Carteggi, Lettere di Segre* (Gario L9, N. 198, R. 225), f. 1r

Herr D^r^ C.F. Geiser. Docent am Polytechnikum Zürich

Turin, le 23 Mars 1884

Monsieur,

Je vous envoie une copie d'un travail sur les géométries métriques des complexes linéaires et des sphères, qui j'ai fait paraitre en ces jours-ci.[253] Si vous pourrez y jeter un regard, vous me ferez plaisir en m'en faisant la critique.

Je vous demanderai encore un autre plaisir: en lisant votre mémoire *Über die Flächen vierten Grades, welche eine Doppelcurve zweiten Grades haben* (Crelle 70)[254] je y vois cité une note *Über eine geometrische Verwandtschaft des zweiten*

[249]Gabriel Koenigs (1858–1931).

[250]Arthur Cayley (1821–1895).

[251]Segre (1883–84a).

[252]Carl Friedrich Geiser (1843–1934).

[253]Segre (1883–84a).

[254]C. Geiser, Über die Flächen vierten Grades, welche eine Doppelcurve zweiten Grades haben, *Journal für die reine und angewandte Mathematik* 70 (1869): 249–257.

Grades (*Mittheil. der Berner naturforsch. Gesellsch.* 1865),[255] qui traite de l'inversion de l'espace par rapport à une quadrique. Serais-je indiscret en vous priant de m'envoyer une copie de cette dernière note, ou bien de me donner des détails sur ce qu'elle contient? Je dois envoyer dans quelques semaines aux *Mathematische Annalen* un travail sur les surfaces du 4^{e} ordre à conique double ou cuspidale dans lequel je fais usage de ces inversions[256] (et de plusieurs de leurs cas particuliers) et j'aurais besoin de connaitre tout ce qui a été écrit là-dessus.

Excusez-moi mon hardiesse et agréez mes remercîments et mes salutations D^{r} Corrado Segre.

12. Corrado Segre to Thomas a. Hirst,[257] Turin 23 March 1884

UTo-ACS. *Carteggi, Lettere di Segre* (Gario L8, N. 197, R. 224), f. 1r.

Prof. Thomas Archer Hirst, F. R. S. Professor of Mathematics in the University of London

Turin, le 23 Mars 1884

Monsieur,

Dans un très long travail que j'écris en ce jours-ci pour les *Mathematische Annalen* j'ai besoin[258] d'appliquer la théorie des inversions de l'espace par rapport à une quadrique (générale ou décomposée en un cône ou en un couple de plans).[259] Permettez que je vous demande si vous avez développé cette théorie quelque part car dans ce cas je pourrais me borner à une citation sans développer moi-même cet argument. Je connais seulement votre mémoire sur les inversions du plan par rapport à une conique (*Proceedings*, Mars 1865),[260] mais il me semble probable que vous vous soyez aussi occupé de l'inversion dans l'espace, laquelle présente encore plus d'intérêt. M. Geiser[261] s'en est occupé en 1865 et je lui écris en ce moment-même pour lui demander des détails sur ses résultats.[262]

Permettez encore que je profite de cette occasion pour rappeler votre attention sur[263] le contenu de quelques travaux que j'ai eu l'honneur de vous envoyer dernièrement et surtout sur l'un d'eux qui montre les liens étroits qu'il y a entre la

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255 C. Geiser, Über eine geometrische Verwandtschaft des zweiten Grades, *Mittheilungen der Berner naturforschenden Gesellschaft* 22 April 1865, Nr. 592 (1866): 97–107.

256 Segre (1884e).

257 Thomas Archer Hirst (1830–1892).

258 Segre erased: "de considérer".

259 Segre (1884e).

260 T.A. Hirst, On the Quadric Inversion of Plane Curves, *Proceedings of the Royal Society of London* 14 (1865): 91–106.

261 Segre erased: "en a dit quelques mots dans un mémoire du".

262 Segre to Geiser, 23 March 1884, Annex 11.

263 Segre erased: "quelques".

géométrie métrique des complexes linéaires et celle des sphères, liens qui n'avaient pas encore été aperçus.[264]

Agréez,[265] Monsieur le Professeur, mes remercîments anticipés et mes salutations les plus respectueuses. D^{r} Corrado Segre.

13. Corrado Segre to Karl Weierstrass, Turin 28 March 1884

UTo-ACS. *Carteggi, Lettere di Segre* (Gario L10, N. 199, R. 226), f. 1r-v.

Herr Prof. Weierstrass Berlin

Turin, le 28 Mars 1884

Monsieur,

J'ai l'honneur de vous envoyer une copie d'un mémoire, dont une partie au moins vous est due,[266] car elle contient des recherches sur la signification géométrique d'un théorème analytique sur la théorie des formes bilinéaires qui est à vous.[267] Je parle du théorème fondamental pour cette théorie: «la condition nécessaire et suffisante pour que deux couples de formes bilinéaires puissent se transformer linéairement l'une dans l'autre est que les déterminants de leurs faisceaux aient les mêmes diviseurs élémentaires».[268] A vrai dire dans ce mémoire et dans un autre qui le suivra bientôt je n'ai appliqué à la géométrie que la partie de ce théorème qui regarde les formes quadratiques,[269] mais dans un mémoire que j'ai envoyé à l'Académie des Lincei j'ai aussi étudié le cas de formes bilinéaires quelconques.[270] Est-ce que vous n'avez jamais pensé à étendre votre méthode et votre théorème[271] en considérant, au lieu de deux, plusieurs formes quadratiques,[272] ou même en considérant au lieu de deux formes quadratiques[273] plusieurs formes de degrés quelconques (auquel cas on pourrait substituer au déterminant et à ses subdéterminants le *discriminant* et certaines autres fonctions[274] qui en s'annulant indiquent que la forme a plusieurs points singuliers)? On résoudrait avec cela[275] les principaux problèmes de la théorie des invariants des formes: et je comprends que par cette raison même l'extension dont je parle doit présenter des difficultés;

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[264]Segre (1883–84a).

[265]Segre erased: "recevez".

[266]Segre (1883b).

[267]Segre erased: "vous est dû".

[268]Segre (1883b, 59; *Opere* 3, 29).

[269]Segre (1883c).

[270]The mentioned reference is Segre (1883–84c). In the manuscript Segre erased: "je vous prie de saluer M. Kronecker".

[271]Segre erased: "soit".

[272]Segre erased: "soit".

[273]Segre erased: "deux".

[274]Segre erased: "des coefficients".

[275]Segre erased: "tous".

cependant[276] s'il est possible de résoudre ces problèmes généraux par quelque méthode, il me semble que ce doit être par la vôtre convenablement étendue. // M. Kronecker,[277] avec lequel je suis entré en relation précisément à propos de la théorie des formes bilinéaires, mais j'ignore s'il vous en a parlé. Agréez, Monsieur, mes salutations respectueuses Votre dévoué D^r^ Corrado Segre.

14. Corrado Segre to Arthur Cayley, Turin 14 May 1884

UTo-ACS. *Carteggi, Lettere di Segre* (Gario L13, N. 203, R. 230), f. 1r-v.

M^r^ Arthur Cayley Cambridge

Turin, le 14 Mai 1884

Monsieur,

en feuilletant le tome 50 du *Journal de Crelle* j'y trouve avec surprise à la pag. 317 dans votre note de 1854 *Recherches sur les Matrices dont les termes sont des fonctions linéaires d'une indéterminée* (pag. 313–317) une idée importante et[278] dont je ne pensais pas qu'elle fût connue depuis si longtemps: c'est-à-dire,[279] que toute la théorie et la classification des homographies[280] se rattachent aux propriétés d'une[281] matrice, ou,[282] comme l'on dit d'après Weierstrass, aux *diviseurs élémentaires* d'un certain déterminant.[283] Vous comprendrez tout l'intérêt que je porte à cette matière lorsque je vous dirai que j'ai justement étudié (dans l'automne passé) cette théorie et cette classification des homographies (dans un espace à *n* dimensions) par la méthode que vous indiquez dans un mémoire qui paraitra dans les Atti de la R. Académie des Lincei[284] et dont vous pourrez voir dans les derniers *Transunti* de cette Académie le résumé contenu dans la relation[285] qu'en a fait M. Cremona.[(1)]

Dans la[286] page, dont je vous parlais je vois clairement que vous aviez déjà il y a trente années une idée[287] complète de cette théorie (il faudrait seulement corriger en disant que dans votre symbolique on a pour l'homologie[288] le symbole 2 1 et non

[276]Segre erased: "le caractère de votre méthode".

[277]Segre to Kronecker, 16 November 1883, 10 December 1883, 25 December 1883, Annexes 3–5.

[278]Segre erased: "que".

[279]Segre erased: "l'idée".

[280]Segre erased: "dépenden".

[281]Segre erased: "des diviseurs".

[282]Segre erased: "c'est-à-dire".

[283]A. Cayley, Recherches sur les Matrices dont les termes sont des fonctions linéaires d'une seule indéterminée, *Journal für die reine und angewandte Mathematik* 50 (1855): 313–317.

[284]Segre (1883–84c).

[285]L. Cremona and F. Siacci, Relazione sulla Memoria di C. Segre, Sulla teoria e sulla classificazione delle omografie in uno spazio lineare ad un numero qualunque di dimensioni, *Atti della R. Accademia Nazionale dei Lincei Transunti* (3) 281 1883–84 (1884): 212–214.

[286]Segre erased: "dernie".

[287]Segre erased: "tout-à-fait".

[288]Segre erased: "on a, dans votre symbo".

pas $\frac{2}{1}$); mais vous ne la développez pas. Cepen-//dant comme vous finissez votre note en disant que vous reviendrez à cette théorie dans une autre occasion, je voudrais vous prier de me dire si (et dans quel travail) vous y êtes réellement revenu; soit par le vif intérêt que je prends à cela, soit aussi parce-que, ayant encore à corriger les épreuves du mémoire dont je vous parlais je tiendrais beaucoup à[289] avertir[290] avec plus de détails que je puisse que vous m'avez précédé. Je vous envoi quelques travaux que j'ai publiés dernièrement. Veuillez m'excuser si je vous dérange et recevez[291] mes salutations et mes remercîments anticipés. D^r Corrado Segre.

(1) Voyez aussi un mémoire sur le même argument de mon ami le D^r Gino Loria, dans le dernier cahier du *Giornale di matematiche.*[292]

15. Draft of Corrado Segre to Theodor Reye,[293] Turin 15 October 1884[294]

UTo-ACS. *Carteggi, Lettere di Segre* (Gario L6, N. 195, R. 222), f. 1r-v.

Dimostrazione sintetica di un teorema del Reye sulle curve assintotiche della superficie di Kummer.

Il teorema consiste in questo che ogni piano tangente della superficie di Kummer considerata la tocca nel centro di uno dei due fasci di rette di un determinato Q dei complessi quadratici di cui quella superficie è singolare, quando il piano stesso è pure tangente alla superficie singolare di quel complesso quadratico che contiene la congruenza singolare di Q ed è infinitamente vicino a Q. Da esso si trae poi, per una proposizione di Klein, il risultato del Reye che una curva delle tangenti principali della superficie di Kummer è base di un fascio di superficie del 4° ordine.[295]

Per dimostrare quel teorema si consideri un piano singolare π di Q: siano A, B i centri dei due fasci di rette di π contenute in Q e sia P quel punto della retta singolare A, B in cui π tocca la superficie singolare di Q, e P' il punto coniugato armonico di P rispetto ad A, B. La congruenza singolare di Q è base di un fascio di

[289]Segre erased: "ne pas".

[290]Segre erased: "que avec des".

[291]Segre erased: "par anticipation".

[292]G. Loria, Sulle corrispondenze proiettive fra due piani e fra due spazi, *Giornale di Matematiche* 22 (1883): 1–16.

[293]Theodor Reye (1838–1919).

[294]This draft is the first part of the letter sent by C. Segre to T. Reye on 24 October 1884 which was published in French, Segre (1884g), with the title: Sur les courbes de tangentes principales des surfaces de Kummer, Extrait d'une lettre adressée à M. Th. Reye par M. Corrado Segre, *Journal für die reine und angewandte Mathematik* 98 (1885): 301–302 (*Opere* 3, 545–546). The second part is edited in Annex 16.

[295]T. Reye, Über die Singularitätenflächen quadratischer Strahlencomplexe und ihre Haupttangentencurven, *Journal für die reine und angewandte Mathematik* 97 (1884): 242–260.

complessi quadratici (tra cui è Q) le cui coniche poste su π formano una schiera di coniche tangenti tutte alla retta A B in P' (per un teorema generale di Pasch): in questa schiera una conica degenera nella coppia di punti A, B appartenente a Q ed esiste solo più una conica degenerata nel punto P' ed in un altro punto, conica appartenente ad un certo complesso Q' del fascio rispetto a cui π è piano singolare. Ora quando Q' è infinitamente vicino // a Q, e solo allora, accadrà che la seconda conica degenere dovendo essere infinitamente vicina alla prima, il punto P' dovrà venire a coincidere con A (o con B), e quindi anche P coinciderà con A. Il teorema è dunque dimostrato.

Torino, 15 Ottobre 1884.

16. Draft of Corrado Segre to Theodor Reye, Turin 24 October 1884[296]

UTo-ACS. *Carteggi, Lettere di Segre* (Gario L7, N. 196, R. 223), f. 1r-v.

Nuove conseguenze del detto teorema del Reye.

Si ha immediatamente, coi metodi esposti nella mia memoria *Sulla geometria della retta e delle sue serie quadratiche*,[297] la proposizione seguente:

I complessi quadratici passanti per la congruenza delle rette singolari di un complesso quadratico hanno tutti la stessa caratteristica e gli stessi complessi lineari fondamentali (ed in particolare le stesse rette doppie) di questo.

Siccome d'altra parte il teorema di Reye vale per ogni specie particolare di superficie di Kummer, così applicando anche quella proposizione avremo:

Le linee asintotiche della superficie [21111], cioè della *Complexfläche* generale sono (intersezioni di essa con superficie quartiche aventi la stessa // retta doppia, cioè) curve d'ordine e classe 12 aventi 4 punti singolari sulla retta doppia della superficie (e non appartenenti a superficie cubiche).

Le linee asintotiche della superficie [2211], cioè della superficie a due rette doppie secantisi e 4 punti doppi, le curve asintotiche sono d'ordine e classe 8 ed appartengono a superficie cubiche passanti per le due rette doppie (ma non a quadriche).

Le linee asintotiche della superficie cubica a 4 punti doppi (reciproca di quella di 4° ordine di Steiner) sono intersezioni di essa con superficie cubiche contenenti le stesse 3 rette (non passanti per punti doppi) e sono per conseguenza curve del 6° ordine intersezioni della superficie con delle quadriche (e di 4ª classe).

Le linee asintotiche della superficie di Steiner di 4° ordine a 3 rette doppie sono intersezioni di essa con altre superficie di Steiner aventi le stesse rette doppie: ne segue tosto che esse sono linee razionali di 4° ordine (2ª specie) e 6ª classe aventi

[296]This draft is the second part (see footnote 294) of the paper Segre (1884g), published in *Journal für die reine und angewandte Mathematik*, 98 (1885): 302–303 (*Opere* 3, 546–547).

[297]Segre (1883c, 36, *Opere* 3, 127).

quelle 3 rette per corde e le cui sviluppabili osculatrici inviluppano quadriche. Ecc. ecc. Le superficie singolari rigate non danno risultati interessanti su questa via.

Torino,[298] 24 Ottobre 1884.

17. Extract of Corrado Segre to Guido Castelnuovo,[299] Frankfurt am Main 25 June 1891

ANL-Castelnuovo, postcard.

Spero di trovare tuoi scritti a Göttingen (*poste restante*), ove saremo lunedì 29 e martedì 30; e poi a Berlino ove saremo nei primi giorni di luglio. Ho visto ed imparato un'infinità di cose. Ti farò poi leggere i pochi appunti che ho preso e che prenderò ancora. Intanto ti posso dare la buona notizia che Reye (gentilissima persona che ci fece accoglienze cordialissime) sta pubblicando la nuova ediz. della 2ª parte della G.d.L. [*Geometrie der Lage*]: la quale viceversa, per le copiose aggiunte fattevi (come pentaedro ed esaedri della F^3; calcolo simbolico con le omografie ecc., si comporrà ora di una 2ª e di una 3 parte.[300] M. Cantor[301] poi (altra persona gentilissima!) ha compiuto a stampa il 2° vol.ᵉ della sua Storia della matematica[302] (dillo a D'O [D'Ovidio]).[303] Salutami Porro[304] e digli che, seguendo il suo consiglio, siamo andati (con Reye) a visitare minutamente l'Osservatorio di Strassburgo, guidati colla massima cortesia dall'astronomo Becker.[305] È un osservatorio splendido!

18. Extract of Corrado Segre to Guido Castelnuovo, Göttingen 30 June 1891

ANL-Castelnuovo, postcard.

Chi non è stato qui non può imaginare che razza d'uomo è Klein e che specie di organizzazione egli ha saputo, con un'abilità che nessun altro può avere, imporre agli studi matematici in quest'Università. È una cosa che m'ha fatto un'impressione straordinaria. E sì che di impressioni vivissime da parte degli scienziati ne ho già avute parecchie in questo viaggio! Ho trovato qui la tua lettera di cui ti ringrazio tanto tanto. Ti prego di ringraziare e salutare, anche a nome di Loria, il prof. D'O.

[298] Segre erased: "23".

[299] Guido Castelnuovo (1865–1962).

[300] T. Reye, *Die Geometrie der Lage*, 3 vols., Leipzig: Baumgärtner 1886–1893.

[301] Moritz Cantor (1829–1920).

[302] M. Cantor, *Vorlesungen über Geschichte der Mathematik*, Band II, Leipzig: Teubner 1892.

[303] Enrico D'Ovidio (1843–1933).

[304] Francesco Porro de' Somenzi (1861–1937), astronomer at the Turin university.

[305] Ernst Becker (1843–1912) astronomer.

[D'Ovidio] per la sua gentilissima. Queste lettere che si ricevono dagli amici quando si è in paese straniero fanno un grandissimo piacere. Ti sarò quindi grato se mi scriverai ancora a Berlino ove saremo sabato 4 Luglio e ci tratterremo fino all'8 almeno. Con Klein s'è parlato oggi di te e delle tue ricerche: m'incarica di ripeterti che avrà molto caro che tu gli dia quel tal lavoro. Vorrebbe pure che si risolvesse la questione: quali g_n^r, $g_{n'}^{r'}$, … eccezionali son possibili in enti (particolari) di genere *p*.

19. Extract of Corrado Segre to Guido Castelnuovo, Göttingen 11 July 1891

ANL-Castelnuovo, postcard.

Mio ottimo Castelnuovo,

Non ho parole per ringraziarti di quanto hai fatto per me nei giorni scorsi. Debbo limitarmi a dirti un semplice "grazie"! Ma tu mi conosci e capisci i miei sentimenti se anche non li manifesti. Ieri ho ricevuto insieme la tua cartolina e la tua lettera dell'8, unitamente ad un telegramma di mio fratello che mi diceva di telegrafare subito al Rettore che presentasse la mia domanda alla Facoltà ecc. […] Che la mia faccenda scorra liscia mi par molto difficile … Solo stamane ho letto in questo Seminario matem[atico] le parole del Direttore.[306] È roba da cani! Non trovo altro modo di qualificarla. È peggio, od almeno mi fece peggior impressione ancora che quella prima redazione che P. [Peano][307] m'aveva fatto vedere! Cosa mostruosa! […] Il 15 sarò a Nürnberg; il 16, 17 a München.

20. Extract of Corrado Segre to Guido Castelnuovo, Nürnberg 15 July 1891

ANL-Castelnuovo, postcard.

Trovo qui la tua car[issi]ma cartolina. Ti raccomando di calmare Veronese.[308] Che non si prolunghino i pettegolezzi in Italia, per carità! Dico ciò, per quanto io sia sempre indignatissimo contro P. [Peano], e per quanto, ora che io taccio assolutamente, potrebbe soddisfarmi il vedere altri a bastonare quell'… animale! Ma è meglio che lascino stare: non c'è nulla da guadagnare, mentre ci si può sempre perdere qualche cosa … Mi è accaduto ripetutamente in questo viaggio, non solo di fare, ma anche di sentir altri a fare i tuoi elogi: puoi figurarti con quanto piacere per me! A Dresda il Rohn[309] mi disse che in ricerche che sta facendo sulle curve

[306]Segre is referring to the polemic note by Peano, Giuseppe, Osservazioni del Direttore sull'articolo precedente, *Rivista di Matematica* 1 (1891: 66–69, published at the end of Segre's paper: Su alcuni indirizzi nelle investigazioni geometriche. Osservazioni dirette ai miei studenti (Segre 1891a).

[307]Giuseppe Peano (1858–1932).

[308]Giuseppe Veronese (1854–1917).

[309]Karl F. Rohn (1855–1920).

sghembe e sulle rigate ebbe occasione di vedere qualche tuo lavoro, specialm[ente] le *Ricerche di geom[etria] sulle curve algebriche*,[310] che pare siano state indicate dal Brill.[311] Faresti bene ad inviargli subito (all'indirizzo: Prof. K. Rohn, Dresden, Werderstrasse 7) quelli fra i tuoi lavori di cui hai ancora copie e che tu più stimi. Il Rohn m'ha pur detto che l'ultima formola delle tue *Ricerche* gli gioverebbe assai, se fosse completata in guisa da dare per ogni caso il vero limite *raggiunto*. Non son però sicuro d'averlo ben capito: se credi, potresti scrivergli domandandogli precisamente qual è il suo desiderio. Nöther[312] poi m'ha lodato i tuoi ultimi lavori: vede con piacere che tu ti sia già occupato con qualche frutto di geom. sulla superficie e desidera che tu continui in questo indirizzo. Mi domandò se tu conoscevi *tutti* i sistemi piani iperellittici (egli si occupò altra volta senza frutto di determinarli): anche quelli li vedrebbe con piacere. È lieto che tu sia riuscito a dimostrare completam[ente] la rappresentabilità delle involuzioni piane (di coppie di punti): crede che sian rappresentabili anche le involuzioni di *n*-ple di punti (di qualche spazio). […] Stassera sarò a Monaco, e la mattina del 18 ne ripartirò per l'Italia. Il 19 sarò a Torino.[313]

21. Extract of Corrado Segre to O. Michelli Segre, Zürich 8 August 1897

UTo-ACS. *Documenti di famiglia. Lettere di Segre alla moglie*, 10710, fols. 1r-v, 2r-v.

Il viaggio da Milano a qui fu ottimo. Nel treno mi sono incontrato col Prof. Ricci[314] di Padova, altro congressista. Poi a Göschenen si unì a me il Fano[315]; e intanto nei cinque minuti di fermata in quella stazione mi son trattenuto coi genitori di Fano. […] Qui alla stazione ero atteso da Volterra[316] e da Lombardi,[317] che è un mio antico discepolo, ora assistente al Politecnico di Zurigo. Venimmo a questo Hotel Central, ove mi era stata assegnata una bella camera, di cui sono pienamente // soddisfatto. Intanto che mi lavavo Volterra mi si *lagnava*, perché tutto gli fa prevedere il fiasco del congresso, perché non verranno alcuni illustri francesi che avevan detto di venire, perché verranno invece tante nullità; e così via. In realtà invece io credo che sarò

[310]Castelnuovo, Guido, Ricerche di geometria sulle curve algebriche, *Atti R. Accademia delle Scienze di Torino* 24 (1889): 346–373.

[311]Alexander von Brill (1842–1935).

[312]Max Nöether (1844–1921).

[313]Cf. also ANL–Castelnuovo, Segre to Castelnuovo, München 17 July 1891, 31 July 1891, 5 September 1891.

[314]Gregorio Ricci Curbastro (1853–1925).

[315]Gino Fano (1871–1952).

[316]Vito Volterra (1860–1940).

[317]Luigi Lombardi (1867–1936). After taking a degree in Civil Engineering in Turin in 1890, he was a lecturer at the Zurich Polytechnic in 1895, graduated in Philosophy at the University of Zurich and in 1897 returned to Turin as a lecturer in Technical Physics.

contento del congresso; perché dall'elenco che ho visto dei congressisti mi pare che, con molte nullità, vi siano pure moltissimi uomini di valore …: ho detto male, dicendo solo uomini; vi sarà anche la signorina Scott,[318] americana, con la quale io sono in relazione scientifica, ma che ancora non conosco personalmente … Intanto che io mi lavavo e che Volterra si sfogava // bussano: era un professore francese, M.^{r} Borel,[319] che abita pure in quest'albergo, ed avendo sentito che io ero arrivato, voleva complimentarmi. Volterra fece le mie veci finché io fui in grado di riceverlo. Poi siamo stati quasi sempre insieme, anche la sera. […] // $_{2v}$ […] Ora andrò alla stazione, ove stanno in permanenza i matematici svizzeri per ricevere i congressisti che arriveranno oggi. Dall'Italia, oltre a quelli che già sapevo, arriveranno Veronese, Levi-Civita,[320] Amodeo,[321] ecc.

22. Extract of Corrado Segre to Olga Michelli Segre, Zürich 8–9 August 1897

UTo-ACS. *Documenti di famiglia. Lettere di Segre alla moglie*, 10720, fols. 1r, 2v.

T'avevo scritto stamattina, prima di mettermi in moto. Dopo d'allora quanta gente ho veduta! Alla stazione, dal lato da cui arrivano i treni, era aperta una sala, a cui invitavano molti cartelloni con la scritta "congrès des mathématiciens". E là mi sono trattenuto delle ore, con qualche matematico svizzero e coi numerosi stranieri che arrivavano alla spicciolata, // coi diversi treni, da tutte le parti. Ogni momento era una nuova presentazione, o fatta da intermediari, o fatta direttamente tra me ed il congressista. Tedeschi, russi, francesi, inglesi, polacchi, americani, italiani: di tutte le nazioni. Con alcuni, simpaticissimi, s'entrava subito in intimità. Tanto che si fissò lì per lì un convegno per la colazione: ove ci trovammo poi ad una tavola in parecchi. Stassera poi sarà una cosa più grandiosa: perché è già fissato nella nostra "Festkarte" una riunione di *tutti* i congressisti per la cena. Stamane, prima di colazione, andammo Volterra, Fano ed io a far visita a Brioschi,[322] che era arrivato poco prima. Nel pomeriggio poi visita, con Fano, al tedesco Reye ed al danese Zeuthen.[323] Sapendo poi che nell'albergo dov'era quest'ultimo alloggiava pure la professoressa americana miss Scott, di cui già ti scrissi, abbiamo fatto visita anche a lei! […] son contento di averle usato questo riguardo, perché ho potuto capire che è tanto una buona donna, ed ho visto che era molto lusingata della nostra visita. […] Ho anche visto oggi Peano, Gerbaldi,[324] Burali-Forti,[325] ecc.; ed ho stretto la mano a tutti quanti, discorrendo con loro solo per

[318] Charlotte Angas Scott (1858–1931).

[319] Émile Borel (1871–1956).

[320] Tullio Levi–Civita (1873–1941).

[321] Federico Amodeo (1859–1946).

[322] Francesco Brioschi (1824–1897).

[323] Hieronymus Georg Zeuthen (1839–1920).

[324] Francesco Gerbaldi (1858–1934).

[325] Cesare Burali-Forti (861–1931).

pochi minuti. Pare che né Del Pezzo,[326] né Guccia[327] non verranno. In complesso sono già arrivati o stanno per arrivare un grande numero di scienziati di vero valore. Sicché per questo lato possiamo essere soddisfattissimi del congresso. […] Ho contato adesso nella lista dei congressisti: siamo poco meno che duecento. D'italiani quasi una ventina.

23. Extract of Corrado Segre to Olga Michelli Segre, Zürich 10 August 1897

UTo-ACS. *Documenti di famiglia. Lettere di Segre alla moglie*, 10730, postcard.

Sono sempre più contento di questo congresso, che a detta di tutti … i non lagnosi è riuscito egregiamente. Ieri mattina lunga seduta con due interessanti letture, e altre cose. A 1 ora pom[eridiana] grande pranzo, elegante, abbondante, allietato dalla musica. Poi lunga gita su questo bel lago, ritornando a Zurigo verso le 9½ tra i fuochi artificiali e la musica. Ma quel che di più interessante, per me e per molti vi era in questa gita, era nella opportunità di discorrere un po' con l'uno un po' con l'altro scienziato: ed io ne ho approfittato largamente, trattenendomi specialmente con quelli verso cui ho più ammirazione. Vi è molta cordialità in tutti, ed io sono contento del modo come vengo accolto. Oggi sarà una giornata di molto lavoro, perché cominciamo le sedute scientifiche alle 8 e proseguiremo forse per tutto il giorno.

24. Extract of Corrado Segre to Olga Michelli Segre, Zürich 10 August 1897

UTo-ACS. *Documenti di famiglia. Lettere di Segre alla moglie*, 10740, postcard.

Carissima, Questa è in aggiunta ad altra cartolina che ho impostato stamane. Ed è per dirti che m'han fatto l'onore di nominarmi vice-presidente per la sezione di geometria. Presidente è il Prof. Reye; poco fa, in sua assenza, ho dovuto presiedere io. La nomina m'ha fatto piacere, perché, oltre al Reye, vi sarebbero stati tanti altri geometri più anziani di me da nominare. Ho un gran da fare, e quindi ti lascio.

25. Corrado Segre to Olga Michelli Segre, Rome 23 April 1899

UTo-ACS. *Documenti di famiglia. Lettere di Segre alla moglie*, 10850, f. 1r-v.

Ti scrivo avendo da un lato un accademico, dall'altro un altro: mi converrà dunque esser prudente, perché non mi vedano a baciarti! Siamo nella gran sala dei Lincei, e un accademico sta esponendo a voce tonante le sue scoperte. È seduta

[326] Pasquale Del Pezzo (1859–1936).

[327] Giovan Battista Guccia (1855–1914).

della classe di scienze storiche ecc., ma siam venuti anche noi matematici per render onore a Lord Kelvin.[328] È venuto il gran Lord insieme colla moglie e colla figlia. È un bel vecchio, dallo sguardo acuto. Il Presidente Beltrami[329] ha aperto la seduta con un bel discorsetto in cui ha parlato della grandezza di quest'uomo, le cui prime pubblicazioni datano dal 1840! Lord Kelvin ha risposto con poche parole in inglese, che nessuno ha capito. Ora mi toccherà rimanere qui chissà per quanto tempo, perché l'oratore attuale non accenna a finire, e dopo di lui chi sa quanti altri parleranno!

26. Corrado Segre to Guido Castelnuovo, Ancona 9 August 1899

ANL-Castelnuovo, fols. 1r-2v.

Carissimo,

Ho ricevuto la tua cartolina del 3, mentre tu forse ricevevi la mia. Ho molto piacere delle buone notizie della tua Signora e dei due bimbi. Oggi ti riscrivo riguardo al libro che intenderei scrivere per Teubner: avendo ricevuto ora la lettera di accettazione di questo. Premetto che nello scrivere al Teubner io avevo indicato come argomenti che ora avrei in mente (fra quelli trattati nelle mie lezioni e da mettersi nel libro) i seguenti:

"Iperspazi. Varietà algebriche più notevoli che si presentano negl'iperspazi. Geometria sopra una curva (serie lineari di gruppi di punti, ecc.) e sue applicazioni alle curve sghembe e iperspaziali. Superficie razionali dei vari spazi, in relazione coi sistemi lineari di curve piane; riduzione di questi sistemi lineari a tipi, ecc. A queste teorie mi riserverei di aggiungerne qualche altra, se mi paresse opportuno, per rendere più armonica o // più completa l'opera. Lo svolgimento dovrebbe farsi secondo i punti di vista più moderni, ed in modo che il mio libro, insieme coi classici trattati di Geometria analitica di Salmon e Clebsch,[330] e con quello che scriveranno Castelnuovo ed Enriques sulle superficie algebriche[331] contribuisca a dare un'idea abbastanza completa dello stato attuale della geometria algebrica. Come titolo si potrebbe dire, all'incirca *Vorlesungen über höhere algebraische*

[328]William Thompson, Lord Kelvin (1824–1907).

[329]Eugenio Beltrami (1835–1900).

[330]Salmon, Georg, *A Treatise on the Higher Plane Curves*, Dublin: Hodges, Foster and Figgis 1852, 3rd ed. 1879; *Traité de Géométrie analytique (Courbes planes*, *suivi de Étude sur les points singuliers)*, éd. G. Halphen, Paris: Gauthier-Villars 1884. Clebsch, Alfred, *Vorlesungen über Geometrie*, Leipzig: Teubner 1876.

[331]Castelnuovo, Guido and Enriques, Federigo *Sur quelques récents résultats dans la théorie des surfaces algébriques*, Lepzig: Teubner 1896.

Geometrie oppure *Vorlesungen über algebraischen Curven und höhere Räumen*, ecc.". Tutto questo io dicevo nella mia lettera del 28 luglio a Teubner.[(*)] Questi mi risponde in data 7 corrente così:

"Für das mir untern 28 v. Mts. freundlichst von Ihnen gemachte Verlagsanerbieten spreche ich Ihnen meinen verbindlichsten Dank aus und erkläre mich gern bereit die von Ihnen geplanten *Vorlesungen über höhere algebraische Geometrie* in Verlag zu nehmen und zwar in einer Reihe // von Handbüchern über die verschiedenen Gebiete der mathematischen Wissenschaften, die ich im Anschluss an die Encyklopädie zu veröffentlichen beabsichtige. Es würde sich wohl empfehlen, wenn Sie sich über die Abgrenzung des von Ihnen zu behandelnden Gebietes mit den Herren Castelnuovo und Enriques in Verbindung setzten. Mit der Fertigstellung würde ich selbstverständlich in keiner Weise drängen. Was die Bedingungen anlangt, so würde ich Ihnen wie den anderen Herren bei einer Ausstattung wie der des beiliegen Blattes die Zahlung eines Honorars von M. 50 pro Bogen bei einer Auflage von 1000 Exemplaren, die sich bei jeder Auflage wiederholen würde und sich bei stärkerer Auflage entsprechend erhöhte, vorschlagen können, wovon freilich noch ein Teil des aufzuwendenden Übersetzungshonorars abgezogen werden müsste. Dasselbe würde voraussichtlich M. 24 pro Bogen betragen, hiervon würde ich bereit sein—womit ich freilich schon den deutschen Mitarbeitern gegenüber Ihnen ein besonderes // Entgegenkommen bewiese—1/3 meinerseits zu übernehmen, während der Rest von Ihnen getragen werden müsste."[332]

Ora io ti prego:

1° di scrivermi subito se queste condizioni, specialmente riguardo alle spese di traduzione, coincidono con quelle fatte a te ed Enriques; e se voi le avete accettate o in qualche modo modificate,

2° di scrivermi, ora e poi, quali argomenti vorresti aggiunti a quelli indicati più sopra pel mio libro, acciocché meglio aiuti il vostro, e qualunque altra cosa ti venga in mente riguardo al *Abgrenzung* dei nostri *Gebiete*, di cui parla il Teubner.

[332]*Tr.* "For the proposal that you kindly sent me on 28th of last month I express my most heartfelt thanks and declare my willingness to print your *Vorlesungen über höhere algebraische Geometrie* (*Lectures in Higher Algebraic Geometry*) and in a series of textbooks on the different spheres of the mathematical sciences that I intend to publish after the Encyclopaedia. It would be very useful for you to get in touch with Messrs Castelnuovo and Enriques for delimitation of the subjects to be dealt with. Regarding the deadlines, naturally I do not intend in any way hurry you. As regards the conditions it is possible to propose to you, as to other authors, a contract like the one enclosed, which contemplates the payment of a fee of 50 Marks per sheet for an edition of 1000 copies, which could be repeated for each edition and with an increase for subsequent reprints, from which however a part would have to be deduced for the translation expenses. Concerning this the expected expense is 24 Marks per sheet and I would be prepared to give you, compared to the German collaborators, a special concession that consists of taking on myself the onus of 1/3 of the costs, while the rest should be met by you.".

Avverti che io vorrei fare un'opera, ampia sì ma non troppo, un'opera armonica, non un'enciclopedia. Poi fammi il favore d'inviare questa mia lettera ad Enriques, al quale è pure diretta. Anche da lui aspetto che mi esprima *tutti i suoi desideri* riguardo alla mia opera, in relazione colla vostra, come considerata da sé, e *tutti i consigli* che gli posson venire in mente.

Ringraziamenti e saluti affettuosi ad entrambi Vostro C. Segre.

(*) Dicevo poi che mi ci vorrebbero alcuni anni per compier tutta l'opera; e che nemmeno potrei cominciare subito, dovendo prima finire l'articolo per l'Enciclopedia.[333]

27. Extract of Corrado Segre to Guido Castelnuovo, Turin 13 February 1900

ANL-Castelnuovo, fols. 1r-1v.

Carissimo,

ti prego di riflettere subito e di scrivermi subito il tuo parere intorno al titolo del mio molto futuro libro. Tempo fa io avevo scritto al Teubner che il titolo poteva essere all'incirca *Vorles*[*ungen*] *ü*[*ber*] *höhere algebraische Geometrie, mit besonderer Berücksichtigung der mehrdimensionalen Räume* (la 2ª parte del titolo l'ho aggiunta perché il T[eubner] ci teneva ai *mehrdim*[*ensionalen*] *Räume* nel titolo). Ora nella bozza che anche tu avrai ricevuto trovo soppresso il *höhere*. E son dubbioso fra metterlo o no. Tu che sai all'incirca gli argomenti principali che saranno esposti (argomenti di corsi di geometria *superiore*) mi sapresti consigliare? Io temo che a sopprimere quell'aggettivo // rimanga un titolo troppo indeterminato.

28. Corrado Segre to Ernest J. Wilczynski,[334] Turin 8 February 1904

EJWP, B2 F2, fols. 1r-2r.

Torino 8 II 04

Pregiatissimo Signore,

nell'ultimo fascicolo dei *Math. Annalen* a pag. 256 trovo un teorema di una Sua Nota che mi stupisce: "If a ruled surface is self-dual, it must belong to a linear complex".[335]

[333] Segre was preparing the essay *Mehrdimensionale Raume* (Segre 1921c) which would be published in *Encyklopädie der Mathematischen Wissenschaften*, III.2A.7, Leipzig: Teubner 1921: 769–972.

[334] Ernest Julius Wilczynski (1876–1932).

[335] E.J. Wilczynski, A fundamental theorem in the theory of ruled surfaces, *Mathematische Annalen* 58 (1904): 249–256, in particular 256.

O io capisco male il significato, oppure la proposizione non è esatta. Infatti se noi abbiamo una reciprocità R, e prendiamo una retta g con tutte le sue trasformate g_1, g_2, … per mezzo di R, R^2 …, possiamo far muovere g in modo che essa con g_1, g_2, … descrivano una stessa rigata: e questa sarà trasformata in sé da R, sarà *self-dual*, senza giacere necessa-//riamente in un complesso lineare.

Per esempio, prendiamo le coordinate di retta (di Klein) x_1 … x_6 tali che

$$\sum_1^6 x_i^2 = 0.$$

Tre equazioni del tipo

$$\sum_1^3 a_{ik}x_ix_k + \sum_4^6 b_{lm}x_lx_m = 0$$

determinano una rigata che è trasformata in sé dalla reciprocità (polarità rispetto a una quadrica) che muta $(x_1 \ldots x_6)$ in $\left(-x_1, -x_2, -x_3, x_4, x_{5,} x_6\right)$. Profittando dei coefficienti liberi si può fare in modo che quella rigata non stia in un complesso lineare. Oppure in coordinate plückeriane p_{ik} consideriamo tre equazioni quadratiche fra queste 6 coordinate, ognuna delle quali sia simmetrica rispetto a p_{12}, p_{34}. Le tre equazioni rappresenteranno una rigata, che non sta in generale in nessun // complesso lineare, ma corrisponde a se stessa nel *Nulsystem* definito dal complesso lineare $p_{12} - p_{34} = 0$.

Anche la proposizione seguente di quella citata, cioè "If however this complexe is special, ecc.", mi lascia dei dubbi analoghi. Se i miei dubbi non hanno ragione, voglia scusarmi e spiegarmene il perché.[336] Cordiali saluti Suo Corrado Segre.

29. Corrado Segre to Ernest J. Wilczynski, Turin 18 March 1904

EJWP, B2 F2, f. 1r-v.

Turin, le 18 III 04

Monsieur et honoré Collègue,

[336]After this letter Wilczynski wrote on the same journal the following addition (Bemerkung zum Aufsatz von E.J. Wilczynski, *Mathematische Annalen* 58 (1904): 584): "The theorem on page 256 is liable to misinterpretation, as Mr. Corrado Segre has kindly pointed out to me. The self-dual surfaces mentioned in the theorem are such, that a dualistic transformation exists which transforms every one of the generators of such a surface into *itself*. There exist ruled surfaces, not belonging to a linear complex, invariant under a dualistic transformation, which does not however have the individual generators unchanged.".

Je Vous remercie pour Votre aimable réponse.[337] Maintenant je suis parfaitement d'accord avec Vous sur le théorème dont je Vous avais écrit.[338] Il est une conséquence de ce fait: que les droites *autoréciproques* d'une réciprocité de l'espace, si elles sont en nombre infini, ne peuvent présenter que les cas suivants: ou bien elles forment deux faisceaux de droites; ou bien elles forment une ou deux *Regelschaaren*, ou enfin elles sont les droites directrices d'un *Nulsystem*, qui définit la réciprocité. Je serai très-heureux de faire Votre // connaissance personnelle. Je serai à Turin presque certainement vers le 5 ou 6 Avril; mais si par hasard je devais éloigner d'ici en ces jours, je Vous en avertirais en Vous écrivant à Florence. Aujourd'hui, après avoir reçu Votre lettre, je n'ai pas vu M. Fano: mais, connaissant ses habitudes, je crois pouvoir supposer qu'il ne sera pas à Turin pendant les vacances de Pâques, mais qu'il reviendra ici pour le 7 Avril. Lui aussi, j'en suis sûr, sera très-heureux de Vous connaître. Agréez mes salutations cordiales. Votre C. Segre.

30. Corrado Segre to Ernest J. Wilczynski, Turin 27 April 1904

EJWP, B2 F2, f. 1r.

Torino, 27 aprile 04

Preg.mo Signore,

La ringrazio vivamente per il gentile invio dei Suoi *Studies in the gen. theory of ruled surfaces*.[339] Non mancherò di ricambiarla alla prima occasione con qualche mio lavoro! Ho avuto molto piacere di averla conosciuta personalmente, quantunque solo per pochi istanti. Spero che ci rivedremo altre volte, e che allora potrò godere di più la Sua simpatica conversazione. Tanti cordiali saluti Suo C. Segre.

31. Corrado Segre to Ernest Wilczynski, Turin 2 July 1904

EJWP, B2 F2, f. 1r.

Torino 2 VII 04

Egregio Collega,

La ringrazio per la Sua lettera. Ella è stata molto gentile a voler citare il mio nome per quella piccola osservazione.[340] Avrò molto piacere di rivederla a Heidelberg,[341] e poi anche (spero più a lungo!) quest'inverno in Italia. Suo C. Segre.

[337] This letter seems to be lost.

[338] Segre to Wilczynski, 8 February 1904, Annex 28.

[339] Wilczynski, Ernest J., Studies in the General Theory of Ruled Surfaces, *Transactions of the AMS*, 5 (1904): 226–252.

[340] Cf. Annex 28 and footnote 336 above.

[341] He is referring to the International Congress of Mathematicians which was held in Heidelberg from 8 August to 13 August 1904.

32. Extract of Corrado Segre to Olga Michelli Segre, Airolo 17 July 1904

UTo-ACS. *Documenti di famiglia. Lettere di Segre alla moglie*, 11320, f. 1r-v.

Ho ricevuto una lettera del Prof. Krazer[342] (del comitato organizzatore del congresso di Heidelberg) in cui è detto: "Gestatten Sie mir // die Benachrichtigung, dass Sie als einer der 4 Vortragenden in den allgemeinen Sitzungen des Kongresses beim Bankett Ihren Platz am Tische Seiner Königlichen Hoheit des Erbgroßherzogs werden angewiesen erhalten, und dass mit Rücksicht darauf Frack und weiße Binde als Toilette notwendig sein dürfte."[343]

33. Extract of Corrado Segre to Olga Michelli Segre, Airolo 20 July 1904

UTo-ACS. *Documenti di famiglia. Lettere di Segre alla moglie*, 11340, fols. 1v-2r.

… Il mio discorso![344] È attorno ad esso che lavoro un'ora o due ogni giorno. Per forza: bisogna pur finirlo! Anzi: in quella tal lettera di cui ti riferii l'altro giorno un brano, il sig. Krazer mi avverte pure che gli altri 3 conferenzieri (non solo l'inglese, ma anche il tedesco ed il francese)[345] fanno stampare e distribuire le loro conferenze a Heidelberg, prima di darne lettura; sicché mi eccita a non tardare l'invio del mio manoscritto per la stampa. Gli ho risposto // che sono stato poco bene, e che il mio discorso non sarà redatto in modo definitivo che quando sarò in procinto di partire per Heidelberg: sicché non si potrà stamparlo prima della lettura!

[342]Adolf Krazer (1858–1926).

[343]*Tr.* "Allow me to inform you that as you are one of the four speakers in the general sessions of the Congress you will be a guest in the banquet at the table of His Royal Highness the Grand Duke and therefore should wear a tailcoat and a white tie.".

[344]Segre is referring here to his lecture in Heidelberg, *La Geometria d'oggidì e i suoi legami coll'analisi* (Segre 1905).

[345]The four guest speakers were Gaston Darboux from Paris, who was replaced by Paul Painlevé: *Le problème moderne de l'intégration des équations différentielles*, Alfred G. Greenhill from London: *The mathematical Theory of the Top considered historically*, Corrado Segre from Turin: *La Geometria d'oggidì e i suoi legami coll'Analisi* and Wilhelm Wirtinger from Wien: *Riemanns Vorlesungen über die hypergeometrische Reihe und ihre Bedeutung*. Cf. *Jahresbericht der Deutschen Mathematiker Vereinigung* 13 (1904): 299–303, 382 and *Verhandlungen des dritten Internationalen Mathematiker-Kongresses in Heidelberg vom 8 bis 13 August 1904*, hrsg. Adolf Krazer, Leipzig: Teubner 1905.

34. Extract of Corrado Segre to Olga Michelli Segre, Airolo 25 July 1904

UTo-ACS. *Documenti di famiglia. Lettere di Segre alla moglie*, 11380, postcard.

Guccia m'ha scritto che vuol aprire un concorso internazionale per un premio di 3000 £ sulle curve algebriche (sai bene che cosa sono!), e che la commissione giudicatrice sia composta di me, Noether[346] e Poincaré.[347] Dopo qualche riflessione, ieri ho risposto che ho già tanto lavoro da fare pei miei corsi e pei manoscritti dei miei discepoli che non posso accettare un ufficio che mi obbligherebbe ad esaminare altri manoscritti. Scelga un altro geometra italiano!

35. Extract of Corrado Segre to Olga Michelli Segre, Airolo 25–26 July 1904

UTo-ACS. *Documenti di famiglia. Lettere di Segre alla moglie*, 11390, f. 1v.

Ricevo, rinviatomi da Torino, un elegante invito litografato ad un pranzo che darà il Prof. Weber,[348] presidente della Società matematica tedesca, la sera del 13 agosto. Sotto sta scritto da un lato: "Bitte im Rock", dall'altro che si prega d'inviare subito la risposta. Quel "Rock" suppongo sia il frac. Domanderò oggi stesso a qualche tedesco di qui.

36. Extract of Corrado Segre to Olga Michelli Segre, Airolo 30 July 1904

UTo-ACS. *Documenti di famiglia. Lettere di Segre alla moglie*, 11400, f. 1v.

Guccia mi ha riscritto insistendo perché io accetti di entrare in quella sua Commissione, ché se no egli manda tutto a monte, che io devo fare questo sacrifizio per la scienza e per amor suo (!), ecc. ecc. ecc. Andrà a Heidelberg e passerà di qui probabilmente mercoledì, fermandosi due ore per stare con me. Gli ho risposto che è inteso che farà qui colazione con me, ma ho insistito nel mio rifiuto, ripetendogli che è facilissimo sostituirmi con un altro italiano. // Quanto ai miei progetti di viaggio, eccoli. Domenica 7 agosto conterei partire per Heidelberg ove arriverei la sera stessa. Viaggerei coll'amico Loria. Là resterei fin verso il 15. Poi tornerei qua per restarvi ancora una diecina di giorni circa, a rinforzare maggiormente la mia salute.

[346]Max Noether (1844–1921).

[347]Henri Poincaré (1854–1912).

[348]Heinrich Weber (1842–1913).

37. Extract of Corrado Segre to Olga Michelli Segre, Airolo 3 August 1904

UTo-ACS. *Documenti di famiglia. Lettere di Segre alla moglie*, 11420, f. 2r.

Ho finito finalmente di ritoccare e di mettere in buona copia il mio discorso di Heidelberg. Se lo finivo due giorni prima, facevo ancora in tempo a farlo stampare prima della seduta in cui dovrò leggerlo. Cioè sabato 13. Ma, come già dicemmo insieme, per quelli che ne capiranno qualcosa è meglio se la lettura sarà una novità. D'altronde, per ragione di prudenza (e sapienza …!) non ho voluto affannarmi. Ora me lo rileggerò alcune volte, per vedere di poterlo poi dire alla seduta fissando spesso l'uditorio senza aver bisogno sempre di guardare le carte!

38. Extract of Corrado Segre to Olga Michelli Segre, Airolo 4 August 1904

UTo-ACS. *Documenti di famiglia. Lettere di Segre alla moglie*, 11430, fols. 1 r-2v.

Annunziato con due lettere da Palermo, con un telegramma da Roma ed altro telegramma da Milano, giunse qui stamane alle 11½ Guccia, per restare ventiquattr'ore a discorrere con me sul famoso premio di 3000 Lire! Che seccatura! E il guaio è che la cosa continuerà anche ad Heidelberg, ove ci troveremo con Noether, altro commissario. Eppure, con tutte le insistenze che m'ha fatto, non mi pareva di poter continuare a rifiutarmi. Aggiungi che col Noether sono in ottime relazioni; e col Poincaré (che è, secondo l'opinione gene//rale, il più grande matematico vivente) sono lieto di entrare ora in relazione, grazie a questo premio.

39. Extract of Corrado Segre to Olga Michelli Segre, Heidelberg 8 August 1904

UTo-ACS. *Documenti di famiglia. Lettere di Segre alla moglie*, 11450, fols. 1 r-2v.

Ieri mattina alle 11½, salutato alla stazione da varie signore dell'albergo con cui avevo stretto intima (!) amicizia, salii nel treno coll'amico Morera.[349] E subito vedo Volterra, che era lì colla Sig[a], provenienti direttamente da Roma. Viaggiammo insieme fin qui. Insieme più o meno, perché loro erano in 1[a] cl., Morera ed io in 2[a]: ma vi era comunicazione, e spesso ci parlavamo. Il viaggio fu abbastanza nojoso e caldo; // Morera e i Volterra non ne potevan più dalla stufezza. […] Poi, ritornato all'albergo, vidi arrivare, e ci unimmo insieme, l'un dopo l'altro, vari scienziati tedeschi, di cui qualcuno conoscevo già e vari altri desideravo conoscere. Questo è il grande piacere dei congressi: trovarsi con tante persone, che si conoscevan solo per i lavori, e discorrere insieme di tante cose. Loria arriverà oggi. Castelnuovo non

[349]Giacinto Morera (1856–1909).

c'è ancora, e non so quando verrà. In quest'albergo il solo italiano son io, perché scrissi molto tempo prima, per fissare la camera; i miei amici italiani che scrissero più tardi non poterono più esser alloggiati qui, perché era già pieno. Oggi, per tutto il giorno, non si farà altro che un cercarsi a vicenda fra // i vari congressisti. Stassera riunione generale al Municipio, dove siamo invitati. E da domandi comincieranno le sedute!

40. Extract of Corrado Segre to Olga Michelli Segre, Heidelberg 9 August 1904

UTo-ACS. *Documenti di famiglia. Lettere di Segre alla moglie*, 11460, postcard.

Ieri fu una giornata piacevolissima. Dal mattino alla sera sempre nuove conoscenze o rinnovazione di antiche: tanti sommi matematici, e tanti "scienziati in erba"! Con parecchi ho parlato, ai quali avevo da tempo desiderio di conferire su l'una o l'altra cosa! Giunsero ieri Loria, Castelnuovo, Capelli,[350] Levi-Civita, Vailati[351] e due o tre altri italiani, oltre quelli che già sai. Dopo questo congresso Volterra e Sig[a] andranno a quello di Cambridge,[352] a cui anch'io ero invitato.

41. Corrado Segre to Olga Michelli Segre, Heidelberg 10 August 1904

UTo-ACS. *Documenti di famiglia. Lettere di Segre alla moglie*, 11470, fols. 1r-2v.

Ieri mattina apertura ufficiale del Congresso, con S.A.R. il granduca ereditario, e una quantità di lunghi discorsi, dopo alcuni dei quali io sono scappato via con altri colleghi, alla spicciolata. Nel pomeriggio si son costituite le 6 sezioni: geometria, analisi, ecc. ecc. Io sono andato alla sezione di geometria e lì si è dato principio alle conferenze. Io sono stato messo dentro al comitato generale del Congresso. Di più m'hanno pregato di essere oggi presidente della sezione di geometria. Ieri per un'ora e mezza // presiedette Zeuthen (che tu hai conosciuto a Torino, colla moglie). […] Verso le 8½ dovrò trovarmi al mio posto per presiedere; e ne avrò per più ore! Più tardi, verso le 16½, partiremo per Schwetzingen (a ½ ora di ferrovia da qui), ov'è il castello con ampio parco del granduca: siamo invitati colà da S.A.R. ad una *garden-party* o qualcosa di simile. Ritorneremo qui con altro treno speciale verso le 9 di sera. Ieri, dopo la seduta, son venuto all'albergo a mettermi // frac, gibus e compagnia bella pel gran pranzo. Giunto il granduca ereditario, vi furono anzitutto alcune presentazioni, fra cui la mia. Il granduca è un ufficiale sui 25 o 30

[350]Alfredo Capelli (1855–1910).

[351]Giovanni Vailati (1863–1909).

[352]On 17 August 1904 there would be a meeting of the British Association for the Advancement of Science.

anni, alto, biondo, simpatico, molto affabile, sorridente con tutti. Mi disse che rammaricava di non conoscere Torino; mi chiese se ero già stato in Germania, e a Heidelberg in particolare, se da molti anni son professore, ecc.; mi espresse la sua soddisfazione perché il bel tempo di questi giorni presentava Heidelberg sotto la miglior luce; e concluse: "conto di vederla domani" (cioè al // castello dove siamo invitati). Mi parlò sempre in francese. Poco dopo prendemmo posto alle tavole. Io (solo fra gl'italiani) alla tavola del granduca cogli altri del comitato, il rettore, il sindaco, qualche ufficiale d'ordinanza ecc. Mi trovavo fra 2 scienziati, Brill[353] e Stäckel,[354] con cui sono in relazione intima: sicché abbiam discorso tutta la sera. Il pranzo, ottimo, durò 3 ore! Come qui si usa, i brindisi cominciarono fin dalla 1[a] portata, e furono quasi una dozzina! Vi fu molta allegria, a cui prese parte anche S. A.R. Poco dopo le 10 questi uscì, e io pure: ma la maggior parte rimase ancora a lungo!

42. Extract of Corrado Segre to Olga Michelli Segre, Heidelberg 11 August 1904

UTo-ACS. *Documenti di famiglia. Lettere di Segre alla moglie*, 11480, postcard.

Ieri dalle 17 alle 20 splendido ricevimento (*garden-party*) del granduca ereditario. Trattamento suntuoso, abbondantissimo. Bellissimo giardino. Compagnia piacevolissima. Ho persino trovato che una sig[a] (di un prof. di Berlino), colla quale ho passeggiato a lungo, è parente di miei parenti (dei Montel)! Sono più che mai soddisfatto del congresso … è vero che finora non ho detto la mia conferenza! Quella è il punto oscuro …: vi sarà qualcuno che la capisca? Piacerà? Stamane vi furon le prime due conferenze generali: quella francese e quella inglese. Ma dopo la 1[a] moltissimi vennero via. Andranno via tutti anche prima della mia?

43. Extract of Corrado Segre to Olga Michelli Segre, Heidelberg 12 August 1904

UTo-ACS. *Documenti di famiglia. Lettere di Segre alla moglie*, 11490, fols. 1r-2v.

I congressisti sono circa 400. Ve n'è di tutte le nazioni, compreso il Giappone. Sono tanti // quelli che mi si vengono a presentare, e tanti quelli che io stesso cerco di conoscere, che finisco per fare confusioni e non sapere più chi è l'uno e chi è l'altro … Del resto ciò succede a tutti! In ogni modo alcune conoscenze più importanti per me mi si sono bene impresse. E ho potuto avere parecchi colloqui interessanti con vari colleghi. A questo fine giovano molto le festicciuole che abbiamo ogni giorno, come pure il trovarsi insieme nei Restaurants, ecc. Ieri al

[353]Alexander von Brill (1842–1935).

[354]Paul Stäckel (1862–1919).

tocco Guccia offrì un pranzo squisito a me, ai Volterra, ai coniugi Noether, a // Castelnuovo e Mittag-Leffler.[355] Tutto molto chic, da gran signore. Iersera alle 6 tutti i congressisti ebbero il divertimento di una scampagnata nei dintorni. Partenza in ferrovia. Ritorno in battello sul fiume Neckar. Giunti davanti al castello di Heidelberg (che hai visto nelle cartoline), questo si illuminò improvvisamente col bengala: spettacolo interessante. Poi fuochi d'artifizio sul fiume, pure bellissimi; serenata, barche illuminate, ecc. Del resto in questi giorni del congresso la città sembra sempre in festa. Dovunque si vedono bandiere, di tutti i colori. È // una città graziosissima; tutte le case sono belle, pulite, da signori! Pare che non debba esservi qui gente povera! [...] Stassera altro divertimento, con musica!

44. Extract of Corrado Segre to Olga Michelli Segre, Heidelberg 13 August 1904

UTo-ACS. *Documenti di famiglia. Lettere di Segre alla moglie*, 11500, fols. 1r-2v.

Due righe in gran fretta, perché alle 17 ho il pranzo dal prof. Weber, e prima vorrei preparare un po' di bagaglio perché partirò domattina alle 8½, arrivando la sera stessa alle 19 ad Airolo (accompagnato di nuovo da Morera). Ti ho telegrafato che la mia conferenza ebbe ottimo successo, e credo di non esagerare. In questi giorni vi era tutti i momenti qualche scienziato straniero che mi domandava se la mia conferenza sarebbe stata distribuita stampata prima della lettura, come s'è // fatto per le altre. E rispondendo io che non avevo fatto in tempo a farla stampare, tutti mi esprimevano il rammarico di non poterla capire, o mi raccomandavano di dirla molto lentamente. E io l'ho detta con grande lentezza, ad alta voce, pronunciando con molta chiarezza le parole, dando colorito alla cosa, come se avessi improvvisato. La lentezza del parlare mi permetteva (dopo un'occhiata al manoscritto, di tanto in tanto) di tenere per lo più gli occhi fissi sull'uditorio. Questo era molto numeroso, e stette attento dalla prima parola all'ultima, // per 40 minuti, fissi gli sguardi su di me, anzi che sulle copie stampate come accadde per le altre conferenze. E poi scrosciarono gli applausi, mentre io salutavo e scendevo dall'alta cattedra per andare a riprendere posto fra i colleghi. E allora tutti a stringermi la mano, a ringraziarmi, a far le meraviglie perché io avevo saputo farmi capire così bene. E gl'italiani tutti mi confermarono che durante la conferenza i loro vicini avevan mostrato di capire; o che dopo di essa avevan fatto dichiarazioni entusiastiche. Quanto alla sostanza, pare anche che sia // piaciuta. Un prof. dell'università di Varsavia[356] m'ha chiesto subito di lasciargliela tradurre in polacco.[357] Guccia vorrebbe riprodurla nei Rendiconti del Circolo matematico di Palermo[358]... Dopo la mia, vi fu ancora la conferenza tedesca: ma con più scarso uditorio; e letta senza

[355] Gösta Mittag-Leffler (1846–1927).

[356] Samuel Dickstein (1851–1939).

[357] Segre (1905a), published in *Wiadomości Matematyczne*, 9 (1905): 7–41.

[358] Segre (1905), published in *Rendiconti del Circolo Matematico di Palermo*, 19 (1905): 81–93.

colorito, riescì poco efficace. Nella Tribuna comparirà, credo, un telegramma sulla deliberazione di fare il nuovo congresso a Roma; e in esso si parla forse anche della mia conferenza.

45. Corrado Segre to Olga Michelli Segre, Airolo 15 August 1904

UTo-ACS. *Documenti di famiglia. Lettere di Segre alla moglie*, 11510, fols. 1r-2v.

Per tutta la giornata di sabato, a Heidelberg, incontrando ora l'uno ora l'altro, ricevevo congratulazioni. Fu molto soddisfacente per me il vedere attenti alla mia conferenza uomini sommi come Klein, Noether, Zeuthen, e tanti altri che non ti nomino, perché a te sconosciuti, quantunque noti a tutti i matematici. Vi erano anche delle signore, mogli o figlie di matematici, venute per sentire l'*italiano*; e poi soddisfatte di averlo capito. Un americano diceva a Loria che egli aveva capito il mio discorso anche meglio che se l'avesse letto stampato. // […] Sabato alle 5 gran pranzo dal Weber, presidente del Congresso e scienziato illustre. D'italiani Volterra ed io. Di stranieri vari sommi, di varie nazioni. Io ero fra un austriaco e uno svizzero, coi quali ero già in relazione. Si è sempre discorso … in tedesco ben inteso. Fra parentesi: per // tutta la settimana ho fatto un grande esercizio di tedesco! Pochi furon quelli con cui potei parlar francese. Era poi un sollievo quando potevamo riunirci fra italiani! Altra parentesi: gl'italiani eran pochi (11), ma fecero buona figura. Oltre alla mia conferenza in seduta plenaria, parlarono nelle varie sezioni Volterra, Levi-Civita, Loria, Vailati, Capelli.[359] Io non potei sentirli, perché dovevo andare nella sezione di Geometria; ma pare che abbian fatto buona figura, specialmente i primi due. In ogni sezione vi furon [da] 4 a 5 sedute, e come per la Geometria io ebbi da presiedere, così Volterra, Levi-Civita, e Loria ebbero da presiedere le loro sezioni. Tutti i congressisti dichiararono la loro grande soddisfazione del modo come si è // svolto il Congresso: sì per le comunicazioni, alcune delle quali veramente importanti, come per le relazioni personali che si fecero, per le belle feste, per il clima propizio, ecc. ecc. Furon 7 bei giorni di cui tutti conserveranno un ottimo ricordo! Ritornando al pranzo Weber, ti dirò che anche quello andò ottimamente, per la cordialità, come pel menù (che ti spedirò). Cominciato alle 5, verso le 8 potemmo Volterra, io e qualche altro congedarci dal Weber. […] Ieri viaggiai con Morera, benissimo. Pare che il mio buon collega non sappia staccarsi da me perché s'è fermato di nuovo qui, e qui starà tutt'oggi. Stamane abbiam passeggiato due ore insieme.

[359] V. Volterra, Sur la théorie des ondes (Volterra's paper was not sent for the publication, cf. Vorvort, *Verhandlungen des dritten Internationalen Mathematiker-Kongresses in Heidelberg*, 1905 cit.: III); T. Levi-Civita, Sur la résolution qualitative du problème restreint des trois corps, 1905 cit.: 402–408; G. Loria, Pour une histoire de la géométrie analytique, 1905 cit.: 562–574, Sur l'enseignement des mathématiques en Italie, 1905 cit.: 594–602; V. Vailati, Intorno al significato della differenza tra gli assiomi ed i postulati nella geometria greca, 1905 cit.: 575–581, A. Capelli, Ein Beitrag zum Fermatschen Satze, 1905 cit.: 148–150, Über die Additionsformeln der Thetafunktionen, 1905 cit.: 272–274.

46. Extract of Corrado Segre to Olga Michelli Segre, Airolo 16 August 1904

UTo-ACS. *Documenti di famiglia. Lettere di Segre alla moglie*, 11200, postcard.

Ti ho spedito ora, raccomandati, i giornali che ho preso a Heidelberg. Il nome che tu cerchi si trova solo in quello di Mittwoch, Mittag. I rendiconti del Congresso si ritrovan facilmente guardando i titoli dei vari articoli. Ancora non ho giornali col rendiconto della seduta di sabato. Ancora non ho finito di lavorare per quella conferenza! Un periodico americano ed uno di Ginevra vogliono subito da me un riassunto, da inserire nei loro rendiconti del congresso![360] Sarà presto fatto, fortunatamente.

47. Extract of Corrado Segre to Olga Michelli Segre, Airolo 18 August 1904

UTo-ACS. *Documenti di famiglia. Lettere di Segre alla moglie*, 11530, f. 1r.

Se non passerà il peso, unirò una lettera di un prof. d'università americana. Il congresso di cui parla è quello di St. Louis.[361]

48. Corrado Segre to Ernest J. Wilczynski, Turin 15 May 1906

EJWP, B2 F2, f. 1r-v.

Torino 15 V 06

Chiarissimo D^{r} Wilczynski,

Ella è stata molto gentile ad inviarmi il Suo bel libro,[362] ed io La ringrazio vivamente! Ho sempre pensato come Lei che la geometria projettiva debba estendersi alle proprietà differenziali degli enti geometrici, senza limitarsi, come di solito si fa, agli enti algebrici. Le Sue ricerche e in particolare il Suo libro, contribuiranno molto—io spero ed auguro!—a questo ampliamento del dominio della geometria projettiva! // Di nuovo La prego di gradire tanti tanti ringraziamenti miei, ed i miei migliori augurî pel Suo avvenire. Suo obbligatissimo C. Segre.

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[360]Cf. H.W. Tyler, The International Congress of Mathematicians at Heidelberg, *Bulletin of the AMS* 11 (1905): 191–205, in particular, on Segre's speech, 202–204; H. Fehr, Le 3^{e} congrès international des mathématiciens, Heidelberg, 1904, *L'Enseignement Mathématique* 6 (1904): 379–400, in particular, 383–384.

[361]Segre is referring to the First International Congress of Arts and Science which would be held in S. Louis from 19 to 25 September 1904. The professor he mentioned could be Edwin Bidwell Wilson (1879–1964) or Ernest Wilczynski who were in Heidelberg at the ICM, or Eliakim Hastings Moore (1862–1932).

[362]E.J. Wilczynski, *Projective Differential Geometry of Curves and Ruled Surfaces*, Lepzig: Teubner, 1906.

49. Corrado Segre to Ernest J. Wilczynski, Turin 4 June 1906

EJWP, B2 F2, f. 1r-v.

Torino 4 VI 06

Caro D^{r} Wilczynski,

È la seconda volta che mi accade d'inviarle una lettera … in California, mentre Lei è in Italia! Ho piacere che Ella sia di nuovo qui, e spero che potrò rivederla e ringraziarla ancora—come già feci nella lettera inviata a Berkeley[363]—per la gentilezza che Ella mi ha usato inviandomi il Suo bel libro.[364] Questo libro m'è parso subito molto attraente, da uno sguardo sommario che vi ho dato. Ma io mi propongo di studiarlo minutamente nelle prossime vacanze estive; e conto di profittarne presto in qualche corso uni-//versitario. La ringrazio anche per la Sua gentile letterina[365]; e La prego di disporre di me, quando potessi servirla in qualche cosa. Suo obbl.^{mo} C. Segre.

50. Corrado Segre to Ernest J. Wilczynski, Brusson 25 August 1906

EJWP, B2 F2, f. 1r.

Brusson, 25 Agosto 1906

Caro Prof. Wilczynski,

Gradisca le mie vive felicitazioni per le Sue nozze! Sono lieto che queste abbiano stretto vieppiù i Suoi legami coll'Italia.[366] Le auguro ogni bene, nella famiglia come nella scienza! Suo C. Segre.

51. Corrado Segre to Ernest J. Wilczynski, Turin 23 January 1907

EJWP, B2 F2, f. 1r-v, All. f. 1r-v.

Torino 23 I 07

Caro Professore,

[363]Segre to Wilczynski, 15 May 1906, Annex 48.

[364]Wilczynski (1906) cit.

[365]This letter seems to be lost.

[366]On 9 August 1906 E.J. Wilczynski married the Italian countess Inez Macola (1876–1975) of Verona, from whom he had three daughters. Cf. E.P. Lane, Ernest Julius Wilczynski In Memoriam, *Bulletin of the AMS* 39 (1933): 7–14 and E.P. Lane, Biographical memoir of Ernest Julius Wilczynski 1876–1932, *National Academy of Science Biographical Memoirs* XVI, 6 (1934): 293–327.

Io non credo che vi sia alcuna difficoltà a ciò che Ella concorra a qualche cattedra universitaria italiana che a Lei possa piacere. S'intende che con ciò Ella verrebbe a sottoporsi al giudizio di una Commissione, la quale dovrebbe confrontare Lei cogli altri concorrenti. In questo momento è aperto solo un concorso a cui Ella potrebbe prender parte: quello per la cattedra (unica) di Algebra e Geometria analitica a *Cagliari*. Non so se a Lei piacerebbe di andare in quell'Università, ove son pochi gli studenti di mate-//matiche, manca il 2° biennio di studi pel dottorato, e mancano biblioteche matematiche. In ogni modo io Le ho trascritto nel foglietto qui unito l'avviso di concorso. Veda Lei quel che Le convenga fare. Io sarei ben lieto che Ella venisse in Italia come professore universitario! Se Ella me ne esprimerà il desiderio, io La terrò informato di altri concorsi che vi fossero più tardi. Gradisca i miei cordiali saluti Suo C. Segre.

Concorso per professore ordinario alla cattedra di Analisi algebrica e Geometria analitica nella *R. Università di Cagliari*. I concorrenti dovranno far pervenire al Ministero dell'Istruzione Pubblica (Divisione per l'Istruz[ione] superiore) la loro domanda in carta bollata da £ 1,20 non più tardi del 30 aprile 1907, e vi dovranno unire:

(a) un'esposizione, in carta libera e *in cinque copie*, della loro operosità scientifica ed eventualmente didattica;
(b) un elenco, in carta bollata e *in sei copie*, dei titoli e delle pubblicazioni che presentano;
(c) i loro titoli e le loro pubblicazioni; queste ultime, possibilmente, *in cinque esemplari*.

Sono ammessi soltanto lavori pubblicati, e fra questi dev'esservi almeno una memoria originale concernente la disciplina che è oggetto della cattedra messa a concorso. I concorrenti che non appartengono all'insegnamento o all'amministrazione governativa, devono inoltre presentare il certificato penale di data non anteriore di un mese a quella del presente avviso. // Non sarà tenuto conto delle domande che perverranno dopo il giorno stabilito, anche se presentate in tempo utile alle autorità …, e non saranno neppure accettate, dopo il giorno stesso, nuove pubblicazioni o parti di esse o qualsiasi altro documento.

52. Extract of Corrado Segre to Olga Michelli Segre, Rome 5 April 1908

UTo-ACS. *Documenti di famiglia. Lettere di Segre alla moglie*, 11920, f. 1r-v.

A Pisa salì soltanto il Bianchi.[367] Bertini,[368] che pure era venuto a salu//tarmi non partì, perché ha il figlio convalescente d'influenza, e non sta ancora pienamente tranquillo. Dice che spera di poter venire domani. Speriamo! Da Pisa, per prudenza,

[367] Luigi Bianchi (1856–1928).

[368] Eugenio Bertini (1846–1933).

ho fatto spedire ancora un telegramma a Castelnuovo, firmato Loria e Segre. In treno, facendo da interprete a un signore francese e famiglia, scoprii che era un matematico congressista di cui conosco le opere. E così si fece una conversazione piacevolissima, a cui presero parte anche Loria e Bianchi. Molto elegante, anche in treno, la sig[a] Loria. // Arrivando a Roma si vide con grande soddisfazione che vi erano i facchini […]. Ho visto Pincherle[369] colla figlia (fidanzata), Volterra, Pizzetti,[370] Maggi,[371] Gerbaldi, Borel, Runge,[372] Levi-Civita, Enriques e Signora, ecc. tutti qui all'albergo. // Dopo la gita fino al telegrafo son ritornato qui, e credo che passerò qui la mattina, per riposare. È un buon posto di osservazione: passano qui, come t'ho detto, tanti congressisti! Fano mi dice che sono iscritti al Congresso circa 700 persone. È molto, trattandosi di matematici. […] Oggi passerò il pomeriggio ai Lincei, ove è il quartier generale del congresso.

53. Extract of Corrado Segre to Olga Michelli Segre, Rome 5–6 April 1908

UTo-ACS. *Documenti di famiglia. Lettere di Segre alla moglie*, 11930, fols. 1v-2v

Stamane sono arrivati D'Ovidio e Morera. Abbiam fatto colazione in parecchi al ristorante Nazionale. […] Poi da Aragno ove si unirono a noi i due Levi,[373] Vailati, ecc. Mi scordavo di dirti che son qui anche i coniugi Severi come i coniugi Enriques. Poi giunse da Aragno Guccia con Poincaré, scienziato sommo che da gran tempo desideravo conoscere. Guccia mi consegnò il *chèque* di 3000 £ e la medaglietta d'oro che domani io dovrò consegnare al vincitore, dopo letta la relazione. Questa medaglietta egli fece già vedere a varie persone; ma sopra è incollata una listina di carta in modo da nascondere soltanto il nome del vincitore, che non si deve conoscere prima della proclamazione! // Una copia in argento di quella medaglia ha regalato a ciascuno di noi tre commissari. Alle 15 avevamo seduta ai Lincei, non del Congresso, ma dell'Accademia. Il Presidente Blaserna[374] cominciò con un saluto ai 5 illustri soci stranieri che eran presenti: Noether, Zeuthen, Gordan,[375] Darwin,[376] Mittag-Leffler. Vi era pure la giovane figlia di Noether,[377] *dottrice* in matematica. Ed assistevano come spettatori vari congressisti stranieri, di cui così ho fatto la conoscenza. […] // 6 mattina. Prima d'andare alla seduta inaugurale al Campidoglio (col Re; ma è prescritta la redingote, non il frac) aggiungo due righe. Iersera alle 18 sono andato dai Volterra, ove ho trovato con lui

[369]Salvatore Pincherle (1853–1926).

[370]Paolo Pizzetti (1860–1918).

[371]Gian Antonio Maggi (1856–1937).

[372]Carl David T. Runge (1856–1927).

[373]Beppo Levi (1875–1961), Eugenio Elia Levi (1883–1917).

[374]Pietro Blaserna (1836–1918).

[375]Paul Gordan (1837–1912).

[376]George Howard Darwin (1845–1912).

[377]Emmy Noether (1882–1935).

Mittag-Leffler e un altro svedese. Ho visto la sig[a] Angelica,[378] che sta bene e ti saluta. Mi son goduto molto i 3 bambini che han subito stretto amicizia con me, e mi han fatto grandi discorsi. La Luisa si ricordava i nomi delle nostre care figliole. Son rimasto fin le 19¼, senza che giungesse la sig[a] Virginia,[379] che era andata ad un concerto colla sig[a] Mittag-Leffler.[380] L'ho vista più tardi al ricevimento dal Rettore, ove ho potuto andare senz'essere insonnito (forse grazie ad una tazza di thé che avevo preso da Volterra). Folla immensa, ottimo trattamento a cui ho fatto onore.

54. Extract of Corrado Segre to Olga Michelli Segre, Rome 6–7 April 1908

UTo-ACS. *Documenti di famiglia. Lettere di Segre alla moglie*, 11940, fols. 1r-2r

Ho letto or ora la mia relazione sul premio Guccia,[381] davanti ad un pubblico immenso. Come già a Heidelberg son riuscito, parlando ad alta voce, lentamente, e con colorito ad incatenare l'attenzione di tutti; quantunque in maggioranza fossero stranieri e non in grado di capire sempre. Il gran segreto fu svelato! La Commissione (Noether, Poincaré e Segre), non riconoscendo degno del premio nes//suno dei concorrenti, ha cercato fra i lavori stampati nel triennio sull'argomento i più meritevoli, ed ha concluso col premiare Severi.[382] Questi assisteva alla seduta, insieme colla Sig[a]. Grandi applausi. Congratulazioni anche a me per la mia relazione! Domani sera gran pranzo all'albergo del Quirinale, offerto da Guccia alla Commissione e al vincitore.

Domattina 1[a] seduta della sezione di Geometria, presieduta da me. Stamani la seduta d'apertura si fece in Campidoglio alla presenza // del Re. Dopo i brevi discorsi di Nathan,[383] Rava[384] e Blaserna[385] vi fu quello, più lungo, di Volterra su la matematica in Italia negli ultimi cinquant'anni.[386] Disse, brevemente per

[378] Angelica Volterra, mother of Vito Volterra.

[379] Virginia Almagià Volterra, wife of Vito Volterra.

[380] Signe Lindfors, wife of Gösta Mittag-Leffler.

[381] M. Noether, H. Poincaré and C. Segre, Relazione del Concorso Internazionale per la Medaglia Guccia, *Atti del IV Congresso Internazionale dei Matematici Roma 6–11 Aprile 1908*, vol. I, Roma: Tip. R. Accademia dei Lincei, 1909: 209–216. This prize was mentioned on the more important journals of the time: JDMV 17 (1908): 65, 72.

[382] Noether, Poincaré and Segre, Relazione del Concorso Internazionale …, *Atti del IV Congresso …1908*, vol. I, 1909 cit.: 211–216.

[383] Ernesto Nathan (1848–1921, Mayor of Rome. Cf. ICM 1908 Opening Speech by E. Nathan, *Atti del IV Congresso Internazionale dei Matematici* ..., vol. I, 1909 cit.: 25–26.

[384] Luigi Rava (1860–1938, Minister of Education. Cf. ICM 1908 Opening Speech by L. Rava, *Atti del IV Congresso Internazionale dei Matematici* ..., vol. I, 1909 cit.: 28–30.

[385] Cf. ICM 1908 Opening Speech by P. Blaserna, *Atti del IV Congresso Internazionale dei Matematici* ..., vol. I, 1909 cit.: 27.

[386] V. Volterra, Le matematiche in Italia nella seconda metà del XIX secolo, *Atti del IV Congresso Internazionale dei Matematici* ..., vol. I, 1909 cit.: 55–66.

ciascuno, qualcosa sui vari indirizzi, nominando quasi tutti i matematici italiani. Citò "il mio amico Segre" molto onorevolmente.[387] Però il tema era troppo spinoso: come si fa a contentare tutti e a dire in pari tempo la verità? Sicché ho già sentito varie critiche!

55. Extract of Corrado Segre to Olga Michelli Segre, Rome 20 April 1908

UTo-ACS. *Documenti di famiglia. Lettere di Segre alla moglie*, 11990, fols. 1v, 2v.

Stamane ho avuto la visita di Fano, che è qui di passaggio per andare a Messina. Abbiam preso qualche accordo, nell'ipotesi (che è quasi certezza) che egli vinca // il concorso. […] Quanto alla lettera di Noether[388] mi ha soddisfatto molto, non solo perché mi dà ragione, ma anche perché riconosce che non è tanto facile risolvere la difficoltà che io ho sollevata: egli stesso non lo può fare.

56. Corrado Segre to Ernest J. Wilczynski, Turin 20 April 1908

EJWP, B2 F2, fols. 1r-2v.

Torino 20 IV 08

Caro Prof. Wilczynski,

Le ho inviato giorni sono una copia che ancora avevo della mia Nota sulla riducibilità delle trasformazioni Cremoniane.[389] Ed ho scritto a Castelnuovo che Le inviasse un esemplare della sua propria, se ancora ne aveva. Quanto all'altra questione che Ella mi faceva nella Sua lettera del 29 III, ho interrogato mio fratello Arturo,[390] che è docente in questa Facoltà di Lettere, e conosce i giovani che si laurearono qui in questi ultimi anni. Egli mi ha detto che potrebbe raccomandare il *D^r Carlo Calcaterra*,[391] dottore in lettere, laureatosi qui // con pieni voti assoluti nell'autunno scorso, con una buona tesi di letteratura italiana. Mi dice mio fratello che questo giovane è serio, accurato nei suoi studi, modesto e colto; e che è stimato anche pei versi che ha pubblicato negli anni 1906–07. Egli invierà presto la sua domanda, corredata dei suoi titoli.

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[387]Volterra (1909) cit.: 63–64.

[388]This letter seems to be lost.

[389]Segre (1900–01).

[390]Arturo Segre (1873–1928), brother of C. Segre, was a teacher at the Liceo M. D'Azeglio in Turin and a historian of the Savoy Kingdom.

[391]Carlo Calcaterra (1884–1952).

Ho ricevuto a suo tempo la Sua Memoria sulle superficie,[392] che ho letto con interesse. Non so se Ella abbia osservato nelle due mie Note, che Le inviai l'anno scorso, una certa tendenza ad occuparmi di geometria differenziale projettiva.[393] Anche nel corso // che ho svolto quest'anno intorno a certe parti della geometria della retta ho dato speciale rilievo alle questioni projettive generali, cioè che non esigono l'algebricità degli enti considerati.[394] Son persuaso che in questo campo della geometria differenziale projettiva vi sia molto da mietere! Ella ha fatto molto bene ad occuparsene; e può esser contento dei Suoi risultati. Da molte parti ho inteso gli elogi del D^r^ Sisam[395]; ed io sarò molto lieto se l'anno venturo lo avrò fra i miei scolari!

Il Congresso di Roma è riuscito bene. Mi spiacque che né Lei, né altri miei amici Americani, // non abbian potuto venire. Ho fatto la conoscenza personale di E.H. Moore.[396] Vi era anche Newcomb,[397] che già conoscevo. Il prossimo congresso si farà a Cambridge (England). La medaglia Guccia fu conferita a Severi, mio antico allievo. Cordiali saluti Suo aff^mo^ C. Segre.

57. Charles H. Sisam to Ernest J. Wilczynski, Turin 12 December 1908

EJWP, B2 F10, fols. 1r-2v.

Turin, Italy 12.XII.08

Dear Professor Wilczynski

[392]E.J. Wilczynski, Projective Differential Geometry of Curved Surfaces, *Transactions of the AMS*, 8 (1907): 233–260, 9 (1908): 79–120, 293–315.

[393]Segre (1884e), (1906).

[394]Cf. Segre's *Notebook* 21, *1907–08 Capitoli vari di Geometria della retta*: 43–48, in Giacard,i 2013.

[395]Charles Herschel Sisam (1879–1964). On his stay in Turin see Sect. 4 above. Born in Cedar Rapids, Iowa, Sisam studied as an undergraduate at the University of Michigan. He received his Ph.D. from the Cornell University in 1906, having submitted, under Snyder's supervision, the dissertation *Classification of Scrolls of Order Seven Having a Rectilinear Directrix*. Then he went to postdoctoral studies in Europe at Göttingen and Turin. Returned to the United States he taught at the Naval Academy and the University of Illinois. In 1918 he obtained the chair of Mathematics at Colorado College and retired in 1948. Sisam spoke at the ICM in Toronto in 1924 and in Bologna in 1928. He was cooperating editor of the *Transactions of the AMS* (1930–1935).

[396]Eliakim Hastings Moore (1862–1932). American mathematician, from 1892 until his death he was head of Mathematics Department at Washington University. After his Ph.D. received from the Yale University, where he submitted the dissertation, supervised by Huber Anson Newton, *Extensions of Certain Theorems of Clifford and Cayley in the Geometry of n Dimensions*, he went to continue his studies in Germany (Parshall 1988; 1989).

[397]Simon Newcomb (1835–1909). Canadian and American astronomer and applied mathematician, he was Professor of Mathematics in the U.S. Navy and at Johns Hopkins University.

I hope you will excuse me for having delayed so long in writing to you. I have been trying for some time to find an opportunity to write to my friends, but the days fly past so rapidly that it has gotten to be December before I knew it. We spent a very pleasant summer in Göttingen where I attended Klein's and Hilbert's lectures and read in the library.[398] I liked it very well there although not so well as I do here. As soon as the semester ended we spent a month travelling up the Rhine and through Switzerland and arrived in Turin on the third of September. // We spent several days looking for suitable apartments and finally settled down in the apartments occupied by Professor Dowling,[399] of Wisconsin, when he was here a couple of years ago. We are very comfortably located and the people in the dining-room—who are mostly students—are very kind about helping us with the language; which meant a good deal to us, especially at first.

While Turin is not considered to be of especial interest to travelers, we have found it a pleasant and interesting place to live in. It is well laid out and maintained with a great deal of civic pride. On clear days the snow-clad Alps in // the distance are very beautiful and form, according to our ideas, one of the chief attractions of the city. There are also, here, several interesting collections of art and antiquities.

I am most highly pleased with Professor Segre. He fully lives up to all the nice things I heard about him before I came here. In his lectures he speaks very distinctly and I have had no difficulty whatever in following them. In his lectures this year he is covering, in a general way, the entire field of Geometry.[400] In our private conferences at his home he is very approachable, makes me feel entirely at liberty to come where I want to, and is very stimulating. I am working on some properties of triply infinite varieties // in five dimensions, with references to line geometry. I am very much pleased with the results I have obtained thus far. Professor Beman[401] of Michigan, has been in Rome all this fall. He is devoting his time primarily to seeing the sights, however I take it, and only secondarily to Mathematics.

We had our first snow-storm here yesterday morning. When we arose, the ground was really covered. It snowed a little more during the morning but by night the snow was all gone.

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[398]On Sisam's stay in Göttingen cf. JDMV 17 (1908): 87.

[399]Linnaeus Wayland Dowling (1867–1929). He graduated from Adrian College in 1890 and obtained the Ph.D. of Philosophy at Clark University in 1895, with the disseration *On the Forms of Plane Quintic Curves*, supervised by William Eduard Story. He went in Turin to attend the course of Advanced Geometry on *I Gruppi in Geometria*, held by C. Segre in 1906–07, cf. Segre's *Notebook* 20 (in Giacardi 2013). Returning to the United States, he gave his entire service of 33 years to the University of Wisconsin, becoming successively Instructor, Assistant Professor, Associate Professor, and Professor of Mathematics. His special field of research was geometry, and he was the author of textbooks on Analytic Geometry, Projective Geometry, and Mathematics of Insurance. In obituary of L.W. Dowling it is underlined his love for Italian culture (Linnaeus Wayland Dowling In memoriam, *Bulletin of AMS* 35 (1929): 123): "His knowledge of the language and literature of Italy, where he studied under the geometer Segre, was exceptional; and he was an ardent reader and lover of poetry.".

[400]Cf. Segre's Notebook 22, *1908–09 Rassegna di concetti e metodi della Geometria moderna,* fols. 239, in Giacardi 2013.

[401]Wooster Woodruff Beman (1850–1922).

Wishing you a Merry Christmas and a Happy New Year, I remain yours truly Char. H. Sisam.

58. Charles H. Sisam to Ernest J. Wilczynski, Turin 29 March 1909

EJWP, B2 F10, fols. 1r-2v.

Turin, Italy March 29, 1909

Dear Professor Wilczynski

Your favor of March 14 was duly received. I was quite surprised to hear of the problem you have given. Mr. Börger[402] and I fear that there may be some chance for a conflict. My problem is not in the least algebraic and depends fundamentally on some invariants of a system of differential equations. It deals with three-spreads in n dimensions which satisfy four, or more, partial differential equations of the second order. The three-spreads in S_5 satisfy four such equations and form an important particular case. I have determined under what conditions a three-spread satisfies more than four // such equations and have shown that, if it satisfies four, it has at each point, four tangents having contacts of the second order with the three-spread. I have determined under what conditions the three-spread has, at each point, an infinite number of such tangents and under what conditions two or more of the four tangents coincide.

As my results are already worked out I will ask Dr. Börger to let me furnish him with the results for this part of the problem insofar as they may be useful to him in building up the other phases of this interesting problem. In return, I shall be glad, if I am informed as to what he is doing, to keep out of his way. I shall write to Dr. Börger, too, in regard to the matter. // I was very glad to hear of yours campaign in regard to the matter of appointments. I have already heard something about your address although I have not read it. There certainly is a "crying need" for such a reform in America and I am glad that you have set your shoulder to the wheel to bring it about. If I can be of any service to you in the matter I shall be glad to do so. I have, in fact, already done a little stirring among my friends. Mrs. Sisam and I are both very sorry to learn of Mrs. Wilczynski's ill health. We hope that her trip next summer will bring her, not only a great deal of happiness but better health as well. I fear that // we shall not see you next summer as we leave about the middle of June for Paris.

[402]Robert Lacey Börger (1873–1932). In 1907 he discussed his Ph.D. Thesis *On the Determination of Ternary Linear Groups in a Galois Field of Order p^2* under Leonard Eugen Dickson's supervision. He was instructor at the University of Illinois and he moved to Ohio University in 1916. Cf. Zitarelli, David, Hilbert in Missouri, *Mathematics Magazine* 84 (2011): 351–364, in particular 357.

Mr. Denton seems to have done very well with his master's thesis[403] and to have conferred a great deal of credit, not only upon himself but upon his teacher.

Have you talked over with Dean Townsend,[404] yet, the matter of revising the courses in Geometry which we discussed together last spring? Yours truly, Charles H. Sisam.

Care of Paolo Longo, via Pio V 16, Torino, Italia.

59. Corrado Segre to Ernest J. Wilczynski, Turin 16 April 1916

EJWP, B2 F2, fols. 1r-2v.

Torino 16 IV 16

Caro Prof. Wilczynski,

Ebbi la Sua lettera del 9 III,[405] e più tardi la memoria Yeaton.[406] Ho dato un rapido sguardo a questa, fidandomi per i particolari nella revisione che Ella già ne ha fatto. E poi, persuaso che essa lo meriti, l'ho inviata per la stampa agli *Annali di matematica*. Potrà essere stampata nell'autunno di quest'anno.[407]

La interesserà sapere, a questo riguardo, che il Prof. G. Fubini[408] in seguito a una questione che gli avevo posto, ha fatto in // questi ultimi tempi delle ricerche di geometria projettiva-differenziale delle superficie, in cui figurano le Sue *Directrix-Curves* (che egli chiama "linee di Wilczynski"),[409] insieme con altre (che egli chiama "di Darboux-Segre")[410] che Ella ha pure incidentalmente nominato in una Sua Memoria.[411] Queste ricerche di Fubini La interesseranno certamente. Una Memoria è in corso di stampa negli *Annali di Matematica*.[412] Una Nota è uscita già nei *Rendiconti Lincei* di febbrajo (vol. 25 fascic.^{o} 3^{o})[413]: e si riferisce a quelle

[403]William Wells Denton (1882–1961). In 1912 he would have discussed his Ph.D. Thesis on *Projective Differential Geometry of Developable Surfaces*, under Edgar Jerome Townsend's supervision.

[404]Edgar Jerome Townsend (1864–1955).

[405]This letter seems to be lost.

[406]Chester H. Yeaton (1886–1970).

[407]Yeaton, Chester H., Surfaces characterized by certain special properties of their directrix congruences, *Annali di Matematica pura ed applicata*, (3) 26 (1917): 1–34.

[408]Guido Fubini (1879–1943).

[409]G. Fubini, Invarianti proiettivo–differenziali delle curve tracciate su una superticie e definizione proiettivo-differenziale di una superficie, *Annali di Matematica pura ed applicata*, (3) 25 (1916): 229–252, in particular 233 and 249–251.

[410]Fubini (1916) cit.: 229, 230, 233, 243–244, 250.

[411]E.J. Wilczynski, Projective Differential Geometry of Curved Surfaces, *Transactions of the AMS* 10 (1909): 279–296, in particular 282.

[412]Fubini (1916) cit.

[413]Fubini, Guido, Su una classe di congruenze W di carattere proiettivo, *Atti R. Accademia dei Lincei Rendiconti Classe di Scienze fisiche, matematiche e naturali* 313, 5, 25 (1916): 144–148.

superficie su cui il sistema delle linee di Wilczynski è un sistema coniugato. (Sono // dunque un caso particolare di queste superficie quelle del D^r^ Yeaton). Fubini dimostra che tali superficie si posson anche definire come quelle che son falde focali di congruenze rettilinee, che fan corrispondere sulle 2 falde focali le linee Darboux-Segre (particolari congruenze W). Un altro lavoro di questa serie è stato inviato da Fubini ai *Rendiconti* di Palermo.[414] Egli non mancherà certo d'inviarle gli estratti, a suo tempo, forse tutti insieme. Per risparmiar tempo, posso pregare Lei di comunicare al D^r^ Yeaton ciò che in questa lettera lo può interessare? Voglia anche // raccomandargli a nome mio di usare molta cura, quando sarà il momento, nella correzione delle bozze del suo lavoro (i giovani principianti non vedono facilmente gli errori di stampa: bisogna che usino più attenzione che noi maturi!). S'intende che poi le rinvierà direttamente alla Stamperia da cui le riceverà. Tanti cordiali saluti a Lei. Auguri al Suo discepolo D^r^ Yeaton. Suo aff^mo^ C. Segre.

60. Corrado Segre to Ernest J. Wilczynski, Turin 2 March 1917

EJWP, B2 F2, f. 1r-v.

Torino 2 III 17

Caro Prof. Wilczynski,

il sig. A. F. Carpenter[415] mi manda un suo scritto per gli *Annali di matematica, Some fundamental relations in the projective differential geometry of ruled surfaces*,[416] dicendomi che me lo invia dietro suggerimento di Lei[417] in conseguenza io prego Lei di dirmi qual è il Suo pensiero sul valore di quello scritto. Quanto a me, ho dato finora solo un'occhiata ai risultati del lavoro, ed ho potuto controllare con rapide considerazioni sintetiche alcune applicazioni che l'Autore fa delle sue formole. Ma m'im-//porta molto avere da Lei un giudizio sulle formole stesse. Scusi del disturbo, e gradisca i miei cordiali saluti Suo aff^o^ C. Segre.

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[414] Fubini, Guido, Applicabilità proiettiva di due superficie, *Rendiconti del Circolo Matematico di Palermo* 41 (1916): 135–162.

[415] Allen Fuller Carpenter (1880–?). Born in Marengo, Iowa on 12 January 1880, he studied at the Department of Mathematics of the University of Nebraska (1908–09). In 1915 he received Ph.D. from the University of Chicago where he submitted his dissertation on differential geometry, under Wilczynski's supervision: Rules Surfaces whose Flecnode Curves have Plane Branches, *Transactions of the AMS* 16 (1915): 509–531. Carpenter was faculty instructor with rank of captain at the University of Washington, then the chairman of the Mathematical Department at the same University.

[416] Carpenter, Allen Fuller, Some fundamental relations in the projective differential geometry of ruled surfaces, *Annali di Matematica pura ed applicata*, (3) 26 (1917): 285–300.

[417] The words "di Lei" are underlined in the manuscript.

61. Virgil Snyder[418] to Corrado Segre, Ithaca 21 November 1922

UTo-ACS. *Carteggi, Annali di Matematica*, 12480, f. 1 typewritten letter with autograph signature.

Department of Mathematics Cornell University Ithaca N.Y.

21 Novembre 1922

Caro Professore Segre

Rispondendo al suo chiesto, voglio dire che il *Bulletin Amer*[*ican*] *Math*[*ematical*] *Soc*[*iety*] per l'anno presente non costa [al]la Biblioteca matematica di Torino nulla.

Il "Council" della Società non ha, fino ad ora, fatto una decisione finale concernente le soscrizioni Europee mentre il cambio è tanto disastroso. Per quest'anno però, io mi sono permesso di presentare il *Bulletin* alla Biblioteca di Torino, in considerazione parziale della gentile accoglienza che ho trovato in Torino. Durante il mio soggiorno a Torino mi ricordo d'una conversazione con Lei concernente gli *Annali* e, in particolare, d'una osservazione che Lei ha fatto: "… va male, lo stato finanziario degli *Annali* è molto difficile". Quest'occasione mi ha data l'opportunità di chiamare l'attenzione dei matematici Americani sugli *Annali* e di pregarli di accrescere il numero delle soscrizioni. Dopo la seduta della Società in Settembre ho mandato una lettera ai soci (una copia di questa lettera si trova in questa busta)[419] pregandoli di prendere gli Annali, o almeno di aiutarli.

Fino ad ora ho ricevuto i seguenti sottoscrittori,[420] e una summa in denaro. Inoltre vi sono alcuni, non so quanti, che hanno mandato le loro soscrizioni direttamente agli editori in Milano.

Le mando due checks, l'uno per Lire 1904, che rappresenta la summa che io stesso ho ricevuto, l'altro viene dal Sig. Moore in Cambridge,[421] pagando per i quindici soscrittori in Cambridge. Per queste summe la prego di pagare tutte le soscrizioni, e di ritenere il resto per le spese generali degli *Annali*.

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[418]Virgil Snyder (1869–1950). Born in Dixon, Iowa in 1869, Snyder attended Cornell University, after graduating from Iowa State College in 1890. Supported by a W. Brooks fellowship from Cornell, in 1892 he went to study mathematics under Felix Klein's guidance in Göttingen, where he received his Ph.D. with the dissertation *Über die Lineare Complexe der Lie'schen Kugelgeometrie* (1895). Then he returned to Cornell where he stayed for his entire career. Snyder was an invited speaker at three International Congress of Mathematicians, he served as president of the Society (1927–28) and supervised over forty Cornell Ph.D., three of whom were speakers at the International Congress of Mathematicians over the years. He was a fellow of the American Academy of Arts and Sciences. He died in Ithaca in 1950.

[419]Cf. Annex 61.1.

[420]Cf. Annex 61.2 *Elenco dei soscrittori agli Annali di Matematica*, sent by Snyder.

[421]Clarence Lemuel Elisha Moore (1876–1931). He had travelled for a year in Europe, where he was influenced by E. Study in Bonn and by C. Segre in Turin (Struik 1989, 166). In 1904 C.L.E. Moore received his Ph.D. in Mathematics from the Cornell University, having submitted the dissertation *Classification of the Surfaces of Singularities of the Quadratic Spherical Complex*, under V. Snyder's supervision. From 1904 on he took a position at the MIT were he stayed until his retirement in 1931. Cf. Sect. 4 above.

Mia moglie ed io noi ricordiamo con gratitudine nostra visita a Torino, ed in particolare, della affabilità di Lei e della sua gentile signora. Speriamo di rivedere loro più tardi. Con cordiali saluti il suo Virgil Snyder.

61.1 Virgil Snyder to the Members of the *AMS*, Ithaca 18 September 1922

UTo-ACS. Carteggi, *Annali di Matematica*, 12470, f. 1 typewritten letter with note "Copia della lettera circolare"

Ithaca, New York September 18, 1922

My dear colleague,

Among the numerous adjustments made necessary by present economic conditions, are several concerning mathematical books and periodicals. The increased price of American publications has afforded at least temporary relief, and various gifts and subventions to a number of European periodicals have supplied them with new life and vigor.

Among those still adversely affected is the *Annali di Matematica*, published by Tipo-Litografia Off Carte Valori Turati Lombardi e C, Via Rovello 16, et Milano, Italy. Its present editors, L. Bianchi, of Pisa, G. Jung, of Milano, E. Pincherle, of Bologna, and C. Segre of Torino, have succeeded in tiding it over thus far, partly by their own personal contributions and partly by advances from public spirited Italian banks, but this relief is at best temporary and of course cannot be continued indefinitely. Notwithstanding this struggle, the standard set by the *Annali* has been maintained, and it continues and will continue to rank among the highest mathematical periodicals.

Neither the editors nor the publishers are making any appeal, but after I learned of the condition I felt it my duty and my privilege to call attention of fellow Americans to it, and personally to invite them to cooperate with me in a step that will help themselves as well as the *Annali*.

My proposals, approved by the Council of the American Mathematical Society is to have a representative in each institution to join with his colleagues in providing for one or more subscriptions, or in making cash contributions for the purpose. Will you kindly serve in that capacity for

The periodical appears quarterly, and costs 40 Lire per volume. Subscriptions may be sent to me or directly to the publishers. Cash contributions should be sent to me. About November 15 I shall send the total amount received to Professor Segre. Sincerely yours, Virgil Snyder.

61.2 List of Subscribers to *Annali Di Matematica*[422]

UTo-ACS. *Carteggi*, *Annali di Matematica*, 12490, f. 1r-v.
Elenco dei soscrittori agli Annali di Matematica

Professor I.A. Barnett,[423] Saskatoon, Saskatchewan, Canada.
Professor S. Lefschetz,[424] 937 Missouri St. Lawrence, Kansas, U.S.A.
Professor E.B. Stouffer,[425] 1019 Maine St. Lawrence, Kansas, U.S.A.
Professor Thomas E. McKinney,[426] 222 North University St. Vermilion, South Dakota, U.S.A. (Sig. McKinney ha pagato per quattro anni, 1923–1926).
Professor John A. Miller,[427] Swarthmore College, Swarthmore, Tenn. U.S.A.
Professor L.P. Einsenhart,[428] Princeton University, Princeton, New Jersey.

[422]Segre added in the ms: "(vol. 1° della 4ª serie)". Not by hand of Snyder, with some corrections by C. Segre.

[423]Isaac Albert Barnett (1894–1974). He received the Ph.D. in Mathematics from the University of Chicago in 1918 with the dissertation *Differential Equations with a Continuous Infinitude of Variables*, advisor Gilbert Ames Bliss.

[424]Solomon Lefschetz (1884–1972). He was a Russian born, Jewish mathematician who from a young age was educated in Paris, where from 1902 to 1905 he attended lectures by E. Picard and P. Appell. In 1905 Lefschetz went to the United States of America and working for Westinghouse Electric Company in Pittsburgh he had the misfortune to lose both his hands and forearms. He received the Ph.D. from Clark University in 1911, submitting the thesis on algebraic geometry entitled *On the Existence of Loci with Given Singularities* (advisor William Edward Story). Then he was appointed an instructor in mathematics at the University of Nebraska in Lincoln (1911–1913) and at the University of Kansas in Lawrence (1913–1915). Promoted to Assistant Professor in 1916 and to Associate Professor in 1919, he became full professor in 1923. For his contributions he was awarded the Prix Bordin (1919) and the Bôcher Memorial Prize (1923). Then Lefschetz went to Princeton as a visiting Professor (1924) and there he accepted a permanent post as Associate Professor, becoming Henry Fine Research Professor in 1933.

[425]Ellis Bagley Stouffer (1884–1965). He was Professor of Mathematics and Dean of the Graduate School at the University of Kansas. After the degrees of Bachelor of Science and Master of Science from Drake University at Des Moines, Iowa, in 1911 he received a Doctor of Philosophy degree from the University of Illinois discussing the thesis *Invariants of Linear Differential Equations, with Applications to Ruled Surfaces in Five-Dimensional Space*, under E.J. Wilczynski's supervision.

[426]Thomas Emery McKinney (1864–1930). Student at J. Hopkins, in 1905 received his Ph.D *Concerning a Certain Type of Continued Fractions Depending upon a Variable Parameter*, under E.H. Moore's supervision. He was Professor of Mathematics and Astronomy at the University of South Dakota.

[427]John Anthony Miller (1859–1946). Pupil of Heinrich Maschke, under his supervision Miller received the Ph.D. at the University of Chicago, discussing the dissertation: *Concerning Certain Elliptic Modular Functions of Square Rank* (1899). He was Professor of Mathematics and Astronomy and vice-president of the Swarthmore College.

[428]Luther Pfahler Einsenhart (1876–1965). He was introduced to differential geometry at Hopkins, after a lecture by Thomas Craig. He received his PhD in 1900 and he went to Princeton, where he took part in founding and leading the American school of differential geometry (Truesdell 1984, 416).

Professor J.W. Alexander,[429] Princeton University, Princeton, New Jersey.
Professor A.F. Carpenter,[430] University of Washington, Seattle, Washington.
Professor C.F. Craig,[431] 311 Ehnwood Ave., Ithaca, New York, U.S.A.
Professor Arthur Ranum,[432] 3 Central Ave., Ithaca, New York, U.S.A.
Professor Virgil Snyder,[433] 214 University Ave., Ithaca, New York, U.S.A.
Dr. J.R. Musselman,[434] Johns Hopkins University, Baltimore, Maryland, U.S.A.
Dr. F.D. Murnaghan,[435] Johns Hopkins University, Baltimore, Maryland, U.S.A.

[429]James Waddel Alexander II (1888–1971). After his studies in mathematics and physics at Princeton, where he obtained the bachelor degree (1910) and the master degree (1911), in 1912 he went to Europe to further his studies at Paris and Bologna. In 1915 he received from Princeton the Ph.D with the dissertation *Functions Which Map the Interior of the Unit Circle Upon Simple Regions*, under Oswald Veblen's supervision. Then he was instructor and lecturer at Princeton and from 1917 served as a lieutenant in the U.S. Army Ordnance Office at the Aberdeen Proving Ground. Returned to Princeton, Alexander was appointed Assistant Professor (1920) and was promoted to Associate Professor in 1926 and full Professor in 1928. He married the Russian Natalia Levitzkaja and they spent time from 1917 until 1937 in Chamonix or in the Swiss Alps, where Alexander used to climb mountains.

[430]See footnote 415 above.

[431]Clyde Firman Craig (1881–1964). In 1908 he received from Cornell University the Ph.D. in Mathematics, submitting the dissertation *On a Class of Hyperfuchsian Functions*, published on the *Transactions of the AMS* 11 (1910): 37–54.

[432]Arthur Ranum (1870–1934). Born at La Crosse, Wisconsin, he received the bachelor degree from the University of Minnesota (1892). Then he studied mathematics at Cornell University (1893–96) and in 1896–97 was a graduate student at the University of Chicago. Then he was Professor of Mathematics and Astronomy at the University of Washington (1897–1904), instructor in mathematics at the University of Wisconsin (1904–05). In 1906 he was awarded his Ph.D. in Philosophy at the University of Chicago, under Leonard E. Dickson's supervision, with the dissertation *The Group of Classes of congruent Matrices with applications to the Group of Isomorphisms of any Abelian Group*, Chicago: University of Chicago, 1906. He was Assistant Professor at the Cornell University from 1907, where he spend the rest of his career.

[433]See footnote 418 above.

[434]John Rogers Musselman (1890–1968). He received his Ph.D. in Philosophy from the Johns Hopkins University in 1916, under the direction of Arthur Byron Coble. He was a teaching assistant at Gettysburg Academy (1910–1912), an instructor in mathematics at the University of Illinois (1916–18) and at Washington University (1920–1928), a Professor of Mathematics at Western Reserve University in Cleveland (1928–1961). Some of his articles are published on the *American Journal of Mathematics*.

[435]Francis Dominic Murnaghan (1893–1976). Irish mathematician who moved to the Johns Hopkins University in 1913, after his first-class honors BA in mathematical sciences. Here he received a Ph.D., under the supervision of Harry Bateman and Frank Morley, presenting the dissertation *The Lines of Electric Force Due to a Moving Electron.* According to C.A. Truesdell, for two decades he had dominated the activity in mathematics and mathematical physics at Johns Hopkins for his contributions to group theory and mathematics applied to continuous mechanics (Truesdell 1984, 60, 406–409, 419–431).

Dr. C.A. Nelson,[436] Johns Hopkins University, Baltimore, Maryland, U.S.A.
Professor D.P. Bartlett,[437] Mass. Institute of Technology, Cambridge Mass.
Dr. F.S. Woods,[438] Mass. Institute of Technology, Cambridge Mass.
Dr. C.L.E. Moore,[439] Mass. Institute of Technology, Cambridge Mass.
Dr. N.R. George,[440] Mass. Institute of Technology, Cambridge Mass.
Dr. L.M. Passano,[441] Mass. Institute of Technology, Cambridge Mass.
Dr. H.B. Phillips,[442] Mass. Institute of Technology, Cambridge Mass.
Dr. J. Lipka,[443] Mass. Institute of Technology, Cambridge Mass.
Dr. F.L. Hitchcock,[444] Mass. Institute of Technology, Cambridge Mass.
Dr. G. Rutledge,[445] Mass. Institute of Technology, Cambridge Mass.
Dr. N. Wiener,[446] Mass. Institute of Technology, Cambridge Mass.
Dr. S.D. Zeldin,[447] Mass. Institute of Technology, Cambridge Mass.
Dr. L.H. Rice,[448] Mass. Institute of Technology, Cambridge Mass.

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[436]Cyril Arthur Nelson (1893–1984). In 1919 he received his Ph.D. from the University of Chicago, where he presented the dissertation *Conjugate Systems with Conjugate Axis Curves*. Nelson taught at different colleges and at the Johns Hopkins University. From 1927 he accepted a position at the New Jersey College for Women and later became affiliated with Rutgers University until his retirement in 1959.

[437]Dana P. Bartlett (1892–1929).

[438]Frederick Shenstone Woods (18??–1934) In 1895 he received the Ph.D. from Göttingen University, discussing the dissertation *Über Pseudominimalflächen*, under F. Klein's supervision. Returned to the United States he played an important role in upgrading mathematics instruction at the MIT and other technical schools (Parshall and Rowe 1989, 15–16).

[439]See footnote 421 above.

[440]Nathan R. George, Jr. (19??–1936).

[441]Leonard Macgruder Passano (1866–1943). Author of mathematical textbooks. On his teaching at the MIT cf. (Struik 1989, 170).

[442]Henry Bayard Phillips (1881–1947), Assistant Professor of mathematics at the Massachusetts Institute of Technology.

[443]The name "J. Lipka" has been added by C. Segre on this list, after the delivery of Snyder's letter dated 5 January 1923. Joseph Lipka (1883–1924) was Polish born, emigrated to America as a child. He was educated at the Columbia University, where he received his Ph.D. in 1912, under Edward Kasner's supervision. In 1921 Lipka went to Italy for study under T. Levi-Civita and he represented the MIT at the 700th anniversary of Padua University in 1922 (Struik 1989, 169).

[444]Frank Lauren Hitchcock (1875–1957). In 1910 he obtained a Ph.D. from Harvard University with a thesis entitled *Vector Functions of a Point.*

[445]George Rutledge (1881–1940). He received his Ph.D. from the University of Illinois in 1915 with the thesis *The Number of Abelian Subgroups of Groups Whose Orders are the Powers of Primes*, under George Abram Miller's supervision. At the MIT he was Instructor (1915–23), Assistant Professor (1923–29); Associate Professor (1929–34) and Professor (1934–1940).

[446]Norbert Wiener (1894–1964).

[447]Samuel Demitry Zeldin (1894–1965). Born in Russia Zeldin obtained the Ph.D. from the Clark University in 1917, with the thesis *On the Structure of Finite Continuous Groups with a Single Exceptional Infinitesimal Transformation*. (Struik 1989, 170).

[448]LePine Hall Rice (1870–1933). At the MIT he was Instructor (1919–29) and Assistant Professor (1929–1933) (Struik 1989, 169).

Dr. R. Douglass,[449] Mass. Institute of Technology, Cambridge Mass.
Dr. K.L. Wildes,[450] Mass. Institute of Technology, Cambridge Mass.
Dr. J.S. Taylor,[451] Mass. Institute of Technology, Cambridge Mass. //
The[452] Library Iowa State College, Ames, Iowa, U.S.A.
Professor W.L.G. Williams,[453] White Hall, Ithaca, New York, U.S.A.

62. Virgil Snyder to Corrado Segre, Ithaca 8 December 1922

UTo-ACS. *Carteggi*, *Annali di Matematica*, 12510, f. 1r.

American Mathematical Society, Virgil Snyder, 214 University Avenue, Ithaca N.Y.

Ithaca, il 8 dec. 1922

Caro Professore Segre,

Dopo averle mandato l'altro "check" per gli *Annali*, ho ricevuto alcuni contributi ritardati. Ecco un piccolo incremento.[454] Speriamo che anche altri seguiranno. Mia moglie ed io desideriamo esprimere i nostri auguri migliori per un felice capo d'anno per Lei e per la sua gentile signora. Con cordiali saluti, suo Virgil Snyder

63. Virgil Snyder to Corrado Segre, Ithaca 5 January 1923

UTo-ACS. *Carteggi*, *Annali di Matematica*, 12580, f. 1r.

Ithaca 5 gennaio 1923

Caro Professore Segre,

[449] Raymond Donald Douglass (1894–1960). He received his Ph.D. from the MIT in 1931 with the thesis *Stirling Expansions Derived by Means of Finite de la Vallèe-Poussin Summation*, under Rutledge's supervision.

[450] Karl Leland Wildes (1895–1986) was Instructor (1920–23) and professor at the Department of Electrical Engineering of the MIT.

[451] The name "J.S. Taylor" has been added by C. Segre on this manuscript, after the delivery of Snyder's letter dated 5 January 1923. James Sturdevant Taylor obtained his Ph.D. from the University of California in 1918 with the thesis *A set of five postulates for Boolean algebras in terms of the operations "exception"*, under the supervision of Mellen Woodman Haskell. Taylor was instructor in Mathematics at the MIT from 1919 to 1924.

[452] These last two names have been added by C. Segre on this manuscript, after the delivery of Snyder's letter dated 19 February 1923.

[453] William Lloyd Garrison Williams (1888–1976). He received his Ph.D. from the University of Chicago under L.E. Dickson's supervision in 1920 with the thesis *Fundamental Systems of Formal Modular Seminvariants of the Binary Cubic*. He joined Cornell University on an instructorship and was promoted to Assistant Professor in 1922.

[454] Snyder wrote "£. 290" in the margin of this manuscript.

Era una vera festa per me d'aver ricevuto la Sua gentile lettera appunto al capo d'anno. La ringrazio cordialmente per i buoni auguri. Sì, anch'io credo che sarà meglior che le sottoscrizioni, che io Le ho mandato, devono cominciare col volume XXXII, piuttosto [che] col volume corrente, di cui un fascicolo doppio è già uscito. In rispetto all'elenco dei sottoscrittori di Cambridge, mi rincresce molto d'aver fatto lo sbaglio di omettere due nomi.[455] Ho fatto una nuova copia che Lei vuole mandare al Professore Jung,[456] per suo uso. Recentemente ho ricevuto due di Suoi scritti, Commemorazione del Reye, e le superficie degl'iperspazi con ∞^2 curve spaziali. Per questi e per tutti gli altri, la ringrazio di nuovo. Per me e per mia moglie sarà difficile aspettare fino al tempo quando possiamo ritornare in Italia! Coi saluti cordiali a Lei e alla Sua Signora Suo Virgil Snyder. //

L'elenco completo dei 15 sottoscrittori agli "*Annali*" di Cambridge, Mass. Tutti collo stesso indirizzo Mass. Institute Technology, Cambridge, Mass., Stati Uniti.

Professor D.P. Bartlett
Professor F.S. Woods
Professor C.L.E. Moore
Professor N.R. George
Professor L.M. Passano
Professor H.B. Phillips
Professor J. Lipka
Professor F.L. Hitchcock
Dr. G. Rutledge
Dr. N. Wiener
Dr. S.D. Zeldin
Dr. J.S. Taylor
Dr. L.H. Rice
Dr. R. Douglass
Dr. K.L. Wildes.

Le sottoscrizioni devono cominciare col volume XXXII, non col volume corrente.

[455]This typescript list (UTo-ACS. Carteggi. *Annali di Matematica*, 12590, f. 1r) was sent in attachment by Snyder, who must have realized that he had forgotten in the previous manuscript list sent on January 21, 1922 the two names Lipka and Taylor among the subscribers of Cambridge Mass. The typescript contains in the upper left the following note in pencil "R. 19.1.23". It probably refers to the date on which Segre sent the same list to Giuseppe Jung, at the time director of *Annali di Matematica pura ed applicata*.

[456]Giuseppe Jung (1845–1926).

64. Salvatore Pincherle to Corrado Segre, Bologna 27 January 1923

UTo-ACS. *Carteggi*, *Annali di Matematica*, 12640, c. 1r-2v, Letterhead of the *Unione Matematica Italiana*.

Bologna, 27.1.23

Carissimo Segre,

ti ringrazio molto delle informazioni che mi dai sull'azienda degli *Annali*; sono lieto dell'interesse che hanno mostrato per questi alcuni matematici americani, e ciò è un effetto dell'opera tua. Del resto mi pare che, nel momento attuale, l'America sia il solo paese del mondo da cui si possa attendere qualche aiuto finanziario.

Il gr. uff. Franchi,[457] direttore della Casa Zanichelli, uomo intelligentissimo, intraprendente, che ha incoraggiata ed incoraggia presso di noi la produzione matematica in un momento assai difficile, assumerebbe la continuazione degli *Annali*; ma avrebbe voluto garantita la copertura di una parte almeno del deficit, che calcola all'ingrosso in £. 10000 annue (cifra massima). Coi sussidi disponibili che sono presso allo Jung e presso di te, con qualche altro che si può spe[rare] // e con una piccola somma che, per il primo anno, l'U.M.I. toglierebbe al suo fondo di riserva[(*)] se il Consiglio direttivo acconsente, forse il Franchi assumerà l'impresa, anche in perdita per il primo tempo. Il tomo 32 diverrebbe, se il nostro progetto viene ad effetto, il 1° della 4ª serie; ciò è suggerito dal Jung medesimo. Il formato non verrebbe mutato, ma si ridurrebbero alquanto i margini, veramente eccessivi. Sugli altri particolari c'intenderemo facilmente; in quanto al rapporto fra il numero dei lavori di pura teoria e quelli di applicazione, quello da me proposto era un semplice modo di dire che i primi dovrebbero avere la prevalenza; del resto, la bontà del lavoro è naturalmente il primo criterio. Il comitato di redazione, anche per ragione di continuità, dovrebbe rimanere quale è; e se tu parli di cedere il posto ad un più giovane cosa dovrebbe dire un // vecchio quale io sono! Tu devi rimanere in tutti i modi, e così lo Jung che rappresenta il legame colla Direzione Brioschi[458] e che tanto ha fatto e fa per il periodico glorioso. Tutto dipende ora da una gita che il Franchi farà fra breve a Milano, dove vedrà la ditta Turati ed il prof. Jung; al suo ritorno, mi comunicherà la decisione, di cui mi affretterò ad informare te ed il Bianchi.

Circa al Comitato di redazione, aggiungo una proposta che vi sottopongo. Poiché vogliamo fare una parte all'applicazione, non sarebbe bene aggiungere un quinto membro appartenente appunto alle Scienze applicate? In quanto alla persona, mi rimetto a voi; uno di Milano forse andrebbe meglio perché gli *Annali* hanno appartenuto per tanti anni a quella città; ma forse anche il Colonnetti[459] andrebbe bene. Su ciò, mi rimetto interamente ai colleghi. Venendo alla votazione per il Consiglio direttivo dell'U.M.I., hai raccolto sul // tuo nome una bellissima

[457] Oliviero Franchi.

[458] Francesco Brioschi.

[459] Gustavo Colonnetti (1886–1968).

maggioranza, per non dire l'unanimità dei votanti. Se vuoi rifiutare, farai a me e a tutti noi un grande dispiacere …, ma non ne vediamo il motivo. Anche per la questione internazionale, hai veduto dai primi numeri quanto siamo stati eclettici, e il fatto che numerosi periodici tedeschi ci hanno già chiesto il cambio dimostra che lo scopo puramente scientifico e pacificatore della nostra impresa è stato pienamente riconosciuto anche dagli ex nemici. D'altra parte, l'occhiata più superficiale data al nostro *Bollettino* dimostra che non fa doppio impiego con nessun altro nostro periodico. Dato tutto ciò, perché non vorresti essere dei nostri, onde aiutarci coi tuoi suggerimenti ed i tuoi consigli in un'opera che non potrà che giovare ai nostri studi prediletti? Scusa la lunga chiacchierata, e ricevi i miei più affettuosi saluti Tuo S. Pincherle.

(*) Costituito da un B[uono] del T[esoro] di £. 10000 dovuto all'elargizione di un grande industriale lombardo.

65. Virgil Snyder to Corrado Segre, [Ithaca] 19 February 1923

UTo-ACS. *Carteggi*, *Annali di Matematica*, 12660, f. 1r, typewritten, letterhead *Department of Mathematics Cornell University Ithaca, New York*

19 Febbraio 1923

Caro Professore Segre,

rispondendo alla domanda che Lei mi ha fatta nella Sua lettera recente, voglio dire che l'ultimo *check*, quello per Lire 290, rappresenta solamente contributi liberi, e non sottoscrizioni. Dopo aver scritto l'ultima volta ho ricevuto ancora altri contributi, e anche due sottoscrizioni, i cui indirizzi sono:

The Library, Iowa State College, Ames, Iowa, U.S.A.

Professor W.L.G. Williams, White Hall, Ithaca, New York, U.S.A.

Solo lieto di poter mandarle, con questa, un nuovo *check*; questo per Lire due cento sessanta (£. 260,00). Ho fatto mettere in nostro *Bulletin* che dice che le nuove sottoscrizioni devono cominciare col volume XXXII, di cui il primo numero sarà probabilmente uscito nella primavera di quest'anno.

Siamo qui in New York nel mezzo d'un inverno molto rigido e duro, ma non ostante, mia moglie ed io non vogliamo, non possiamo dimenticare la calda e simpatica accoglienza che abbiamo trovate a Torino. Desideriamo esprimere i nostri migliori auguri a Lei e alla Sua gentile signora. Con cordiali saluti, Suo Virgil Snyder.

66. Salvatore Pincherle to Corrado Segre, Bologna 15 May 1923

UTo-ACS. *Carteggi, Annali di Matematica*, 12790, fols. 1r-2r.

Addì 15 Maggio 1923

Carissimo Segre,

rispondo subito alla tua lettera d'ieri. Non so come tu abbia pensato che nella circolare—che è stata dattilografata sulla bozza corretta da te—il tuo nome non figuri fra i componenti il Comitato di redazione. Vi sono i nostri tre nomi, in ordine d'alfabeto (Bianchi, Pincherle, Segre) e non mi pare che si potesse fare diversamente. A meno che nella copia a te giunta non sia accaduto qualche errore del copista, non riesco a spiegarmi il tuo dubbio, la cui sola espressione mi è rincresciuta assai! Il Levi-Civita, ufficiato dal Bianchi, e al quale io pure ho scritto facendo il tuo nome, ha accettato di entrare nel Comitato. Il suo nome non figura sulla circolare che era già in distribuzione, e ciò non si sarebbe potuto fare senza un accenno // doveroso al ritiro di Jung e alle sue benemerenze. Di ciò, e della assunzione del Levi-Civita al posto di Jung, sarà fatto cenno in una circolare (a stampa) che invieremo a tutti gli antichi associati e che annunzierà a questi, in modo ufficiale, l'inizio della nuova serie. Questa circolare servirà anche di prefazione al primo fascicolo.[460] Penserei che la redazione di questo documento si potesse combinare fra noi alla fine del mese, quando ci troveremo per le sedute dei Lincei, alle quali ritengo che interverrai senza dubbio. Ti pare che convenga questo modo di procedere? Questa circolare, inviata ai periodici esteri, potrà avvertire che si continuano i cambi. In quanto al 1° fascicolo (per il quale ho attualmente due soli lavori, quello di Sannia[461] e la conferenza di Cipolla[462] sulla metamatematica di Hilbert)[463] pare opportuno di diramarlo solo in Ottobre, quando ripiglia dovunque l'attività accademica. // Questo è l'avviso dell'editore, che del 1° fascicolo vuole fare una larghissima distribuzione. Lo *Jahrbuch*[464] è arrivato a Bianchi, il quale me lo ha ceduto per la nostra biblioteca di Matematica—quel periodico non esistendo nella nostra Biblioteca Universitaria; in quanto ai *Monatshefte*,[465] se giungono qui te li spedirò, e se li ricevi tu, puoi continuare a tenerli; solo ti prego di avvisarmi del loro arrivo. Dei *Math.-Annalen* procureremo di aver due copie come per il passato,

[460]Cf. L. Bianchi, T. Levi-Civita, S. Pincherle, C. Segre, Avvertenza, *Annali di Matematica pura ed applicata*, (4) 1, 1923: I–III.

[461]Gustavo Sannia (1875–1930).

[462]Michele Cipolla (1880–1947).

[463]G. Sannia, Nuova trattazione della geometria proiettivo-differenziale delle curve sghembe, *Annali di Matematica pura ed applicata*, (4) 1, 1923: 1–18; M. Cipolla, Sui fondamenti logici della Matematica secondo le recenti vedute di Hilbert, *Annali di Matematica pura ed applicata*, (4) 1, 1923: 19–29.

[464]The periodical *Jahrbuch über die Fortschritte der Mathematik* began publication in 1869, and was published by Walter de Gruyter until 1943.

[465]The journal *Monatshefte für Mathematik und Physik* was founded by Gustav von Escherich and Emil Weyr in 1890 and was published until 1944.

il Bianchi avendomi detto che ne ricevevate una copia per ciascuno. Ben inteso i condirettori riceveranno, come per il passato, il fascicolo ed è giustissimo che si continui l'invio a Jung. La casa Zanichelli non è ancora riuscita ad avere da Turati né l'elenco dei cessati, né quello degli abbonati! Cordiali saluti dal tuo aff.mo S. Pincherle.

67. Salvatore Pincherle to Corrado Segre, Bologna 3 November 1923

UTo-ACS. *Carteggi, Annali di Matematica*, 12800, fols. 1r-2v.

Bologna, 3.XI.23

Carissimo Segre,

rispondo anzitutto alla tua cartolina del 5.X, con cui mi chiedevi se abbiamo disponibili posti di assistente, autorizzandomi a stare zitto nel caso negativo. Effettivamente, non vi è attualmente qui alcuna vacanza di tali posti. Vengo ora a riscontrare la tua lettera del 30, e vi rispondo punto per punto.

(1°) Il tuo indirizzo è perfettamente noto alla Casa editrice, che ti chiede di scusare l'errore commesso dall'ufficio di spedizione, il quale ha confuso questo indirizzo con quello del destinatario di un recente invio di bozze. L'errore non si ripeterà.

(2°) Il 1867 sfuggito invece del 1897 sulla circolare sarà aggiustato nella ristampa della circolare stessa, che verrà ripubblicata a capo del 1° fascicolo. È poi ben naturale il mutamento della prima parte della circolare in "Col presente fascicolo si inizia" … che ora sottinteso …

(3°) Sulla copertina del fascicolo 1° è posto fascic. 1°, Novembre 1923. In quanto alla numerazione del volume, per maggiore chiarezza ho fatto scrivere così: // *Serie Quarta—Tomo Primo* (in maiuscoletto) (Tomo LVIII della Raccolta) (in carattere piccolo).

Così, mentre rimane l'addentellato col passato, l'indicazione per le citazioni (S. 4, T. I) è data dalla riga in carattere maiuscolo. Va bene?

(4°) Per i cambi, il meglio è, mi pare, che tutto vada alla Casa editrice, la quale farà poi la spedizione ai singoli con-direttori. Perciò, io lascio a te di metterti in corrispondenza con Bianchi e Levi Civita per quella parte dei cambi che desiderate; io mi rimetto a voi e mi darai solo notizie della vostra decisione. Ciò che mi resterà andrà, naturalmente, alla biblioteca del nostro Istituto Matematico. L'unico desiderio—vivissimo—è il conservare il Giornale di Crelle,[466] perché, dopo la guerra, la nostra Biblioteca Universitaria l'ha sospeso, pur troppo! I cambi, attualmente, sono quelli che troverai nell'accluso foglietto che è tale e quale l'ha mandato a noi la Ditta Turati e Lombardi

[466]Pincherle refers here to the *Journal für die reine und angewandte Mathematik* founded in 1826 by August Leopold Crelle and renowned under the name of Crelle's *Journal*.

(abbastanza ricco di errori). Lo puoi comunicare ai colleghi, e poi ti prego di ritornarmelo.

(5°) Ho avuto anch'io, a suo tempo, la lettera del Lichtenstein,[467] ed una uguale // ha avuto anche il Bianchi. Ho disposto di fare conoscere quanto espone il L. [Lichtenstein] nelle *Notizie* del prossimo fascicolo del *Bollettino* dell'U.M.I. per gli *Annali*,[468] vedremo di trovare posto ad un fervorino analogo nel 2^do^ fascicolo, il T[omo] essendo ormai completo. Secondo il tuo desiderio, mando la lettera di L. [Lichtenstein] al De Franchis.[469]

In quanto alla proposizione del mio libro rilevata dal Togliatti,[470] può darsi che il mio enunciato abbia una generalità che giustifica la tua obbiezione: ma in questi giorni sono stato così completamente assorto dagli esami, che non mi è stato possibile di vedere la cosa a mente quieta. Io direi che il Togliatti scrivesse a Picone,[471] e (dopo averne avuta la risposta e dopo che io ne avrò riscritto a te) redigesse una breve nota (simile a quella che mi hai mandata e possibilmente anche più concisa) che si potrebbe pubblicare nel *Bollettino dell'U.M.I.*

Per i "secondi insegnamenti" ormai aboliti, abbiamo, come voi, provveduto con incarichi. La Mat[ematica] Compl[ementare], che facevo io, la farà il Belardinelli[472] sotto la mia direzione, e la Geometria Analitica la farà l'Agostini. Per // la G. proiettiva e descrittiva, abbiamo chiesto il trasferimento di Bompiani, che speriamo ci venga concesso. Chisini[473] va a Cagliari.

Fra breve, spero prima del 15, uscirà il fascicolo degli *Annali*. Mi è costato molto tempo e non poca fatica: lo sanno la Casa editrice e la stamperia! Speriamo di essere riusciti; in quanto a materiale, ne abbiamo in abbondanza; forse per più di un volume. Ho avuto ieri anche una Memoria di Enriques, sulle funzioni algebriche di due variabili. Coi più cordiali saluti del tuo aff^{mo}. S. Pincherle.

Riparto degli scambi[474]

Bianchi
Mathematische Annalen e (n. 5) *Acta math*[*ematica*]
Segre (n^{i} 12, 14, 6, 9)
Jahrbuch über die Fortschritte der Math[*ematik*]

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[467]Leon Lichtenstein (1878–1933). In 1918 he was one of the founders and the first editor of the journal *Mathematische Zeitschrift*. In 1933, as the Nazi party came to power in Germany, Lichtenstein abandoned his chair at Leipzig University and left to Poland, as he would have been dismissed anyway for being Jewish.

[468]*Bollettino dell'Unione Matematica Italiana.*

[469]Michele De Franchis (1875–1946).

[470]Eugenio G. Togliatti (1890–1977).

[471]Mauro Picone (1885–1977).

[472]Giuseppe Belardinelli (1894–1978).

[473]Oscar Chisini (1889–1967).

[474]UTo-ACS. Carteggi, *Annali di Matematica*, 12500, f. 1r.

Monatshefte für Mathematik und Physik
Journal des Math[*ématiques*] *pures et appl*[*iquées*]
Annales de l'Ecole Normale Supérieure

Levi-Civita (n^i 3, 8)
Bulletin (?) *de la Société Math*[*ématique*] *d'Amsterdam*
Kansas University Quarterly

Pincherle (n^i 1, 2, 4, 7, 10, 11, 13, 15)
Bulletin of the American Math[*ematical*] *Society*
Circolo Mat[*ematico*] *di Palermo*
Enseignement Math[*ématique*]
American Journal
Giornale di Matematiche di Battaglini
Fundamenta Math[*ematicae*]
Abhandlungen aus dem Math[*ematischen*] *Seminar zu Hamburg*
Journal für die reine und angewandte Math[*ematik*]

68. Grace Chisholm Young[475] to Olga Michelli Segre, Collouge La Couversion (Vaud) 19 June 1924

UTo-ACS. *Documenti di famiglia. Lettere di Condoglianze*, 28, f. 1r.

Cara Signora,

come scriverle l'impressione triste che ci ha fatto la notizia della morte di nostro vecchio amico e maestro. Tanto era buono e caro. Mi ricorderò sempre di lui. Adesso Lei è sola, senza marito e figlie a Torino, e Le mando nostra simpatia e spero che si troverà conforto. Ma conosco troppo bene quanto è dura la nostra vita e quanto è difficile trovare conforto. Con saluti distinti di noi ambedue, Sua dev.^{ma}
Grace Chisholm Young

P.S. Prego di mandare mia lettera all'Elena e Adriana.[476]

[475]Grace Chisholm Young (1868–1944). On his stay in Turin see Sect. 4 above and the article by Conte and Giacardi in this volume.

[476]Elena and Adriana were the daughters of C. Segre and O. Michelli Segre.

69. Virgil and Margarete Snyder to Olga Michelli Segre, Turin 12 July 1924

UTo-ACS. *Documenti di famiglia. Lettere di Condoglianze*, f. 1r.

Cara Signora Segre

Avendo appunto letto della morte del Suo illustre marito, ci doliamo più di non potere dire. Desideriamo assicurarla della nostra simpatia profonda, e della nostra affezione sincera e cordiale. Il partito vivrà sempre nella nostra memoria. Suoi afflittissimi amici Virgil Snyder [e] Margarete Snyder.

70. Charles H. Sisam to Olga Michelli Segre, Colorado Springs 14 July 1924

UTo-ACS. *Documenti di famiglia. Lettere di Condoglianze*, f. 1r.

Madame C. Segre Turin Italy

July 14, 1924

Dear Madame Segre

It is with the greatest sorrow that I learned a few days ago of the death, on May 19, of valued teacher and adviser, Professor Segre.

I was a student at the University of Turin during the year 1908–9. I attended Professor Segre's lectures, which will ever stand out in my mind as models of clearness, force and value. My fondest memory of that year, however, will be the sympathy, insight and self-effacing zeal with which he aided and guided my research work with him.

In Professor Segre, the world has lost one of its foremost geometricians, one whose numerous investigations are fundamental in several branches of geometry.

Far beyond this great loss to science, however, I feel the personal loss of a valued friend and a sympathetic co-worker. I join with you in your grief for the loss of a great and good man. Sincerely Charles H. Sisam.

71. Eduard Study[477] to Olga Michelli Segre, Umhausen, Oetzthal 6 August 1924

UTo-ACS. *Documenti di famiglia. Lettere di Condoglianze*, 23, fols. 1r-2r.

Gentile Signora,

[477]Christian Hugo Eduard Study (1862–1930). German mathematician, he was Privatdozent in Leipzig, professor at Göttingen in 1894 and full professor at Greifswald in 1897 and from 1904 at Bonn University. On his stays in Turin see Sect. 4 and the article by Brigaglia in this volume.

con profondo rammarico ho ricevuto, già mesi fa, la triste notizia della morte di Suo marito, partecipatami gentilmente dal Sig. G. Fano. Scrivo soltanto ora, dopo la chiusura delle lezioni, ed in viaggio, perché non stavo bene di salute, e prego di voler scusare il ritardo. Ho perduto nel defunto *un mio migliore amico*. L'ho potuto chiamare così da molti anni. Prima c'era qualche corrispondenza scientifica—mi sono sentito riunito // con lui per un comune interesse per la geometria, ho imparato molto dai suoi scritti, e poi, parecchie volte, ho avuto il grande piacere della sua presenza personale, specie in casa loro a Torino, riscontri di cui sempre serberò un grato ricordo, come di tutto che debbo a Loro. Anche nei miei ultimi lavori, scritti prima del triste avvenimento, che stanno in parte ancora per essere pubblicati, ho fatto ampio uso delle nozioni importantissime introdotte da lui, le quali faranno sempre onore al suo nome. // Mi rincresce molto di non potere esprimere in parole più adeguate quel che sento. Il mio italiano, che mai fu degno dell'idioma gentile, ha dovuto molto soffrire della conseguenze della guerra: da dieci anni non ho riveduto la "terra promessa". Spero però che Ella sentirà che queste mie povere parole vengono dal cuore. La prego di voler salutarmi anche le Sue figlie, delle quali serbo un grato ricordo. Gradisca, gentile Signora, i miei migliori auguri per Lei ed i Suoi cari. Sempre Suo E. Study.

72. Julian Coolidge[478] to Olga Michelli Segre, Cambridge Mass. 20 September 1924

UTo-ACS. *Documenti di famiglia. Lettere di Condoglianze*, 8, fols. 1r-2v.

27 Fayerweather St. Cambridge Mass. Stati Uniti 20 IX '24

Chère Madame,

Est-il permis à un ancien disciple de votre mari de vous adresser quelques lignes d'appréciation de son très grande valeur comme savant, comme précepteur et comme ami? Je ne me flatte pas que vous vous souviendrez d'un Américain errant qui est arrivé à // Turin avec sa petite famille au moins d'Octobre 1903, pour suivre les cours de l'université, et surtout pour profiter de l'enseignement de votre illustre mari. Pour lui, pourtant, ça a été un évènement d'importance capitale. Non seulement a-t-il trouvé une impulsion scientifique dont il n'a cessé de profiter énormément depuis, mais, chose beaucoup plus précieuse, il a eu le privilège de // nouer de liens d'amitié avec son maitre, que chaque année depuis n'a que rendu plus forts. Je ne saurais vous exprimer, madame, ni l'estime que je ressentis pour votre mari comme savant, ni l'affection qui me lié à lui. Toujours je serai fier d'avoir été à la

[478]Julian Lowell Coolidge (1873–1954). American mathematician, who received his Ph.D. from Bonn University in 1904, under E. Study's supervision, for a thesis entitled *Die dual-projektive Geometrie im elliptischen und sphärischen Raume.* On his stays in Turin see Sect. 4 and the paper by Brigaglia in this volume.

fois de ses élèves et de ses amis. C'est pourquoi je me permets maintenant de vous adresser // mes hommages les plus respectueuses, et ma sympathie la plus sincère. Agréez, madame, je vous en prie, l'assurance de mes respects les plus profonds. Julian Coolidge.

73. Gino Loria to Arturo Segre, Genua 10 December 1924

UTo-ACS. *Documenti di famiglia. Lettere di Condoglianze*, 15, postcard.

Genova, 10 Dic. 1924

Egregio e caro Professore,

mi vennero spediti da Torino gli estratti dei due lavori postumi del nostro diletto Corrado; non so a chi io debba tale dono gentile, onde penso di ringraziarne Lei che certamente non è estraneo alla cosa. Aspetto di giorno in giorno copie di un volume di *Pagine di storia della scienza* edito dal Paravia e che io ho dedicato alla memoria del caro scomparso;[479] non appena possibile ne invierò una copia a Lei degno rappresentante della famiglia. Da tempo ho anche licenziato un articolo su "l'opera geometrica di C.S." ed appena ne riceva gli estratti non mancherò d'inviarne a Lei qualche esemplare. Quanto al progetto di pubblicare le lezioni[480] del Suo povero fratello, l'idea non è abbandonata ma l'esecuzione è irta di difficoltà, che io mi studio di vincere.

Gradisca i più cordiali saluti dal Suo aff.$^{\text{mo}}$ Gino Loria.

74. Guido Toja[481] to Adriana Segre Morpurgo, Rome 31 December 1924

UTo-ACS. *Documenti di famiglia. Lettere di Condoglianze*, 26, f. 1r.

Roma, 31 dicembre 1924

Gentile Signora Adriana Morpurgo Segre,

[479]G. Loria, *Pagine di storia della scienza e della tecnica*, Torino: Paravia 1924, nnp: "The author of this book dedicates it to the beloved memory of Corrado Segre, whose name was not even erased by the death, so that the works of a man who devoted all his noble life to science and teaching might be remembered in Italian schools."

[480]From the first years of the 20th century the Teubner publishing company in Leipzig advertised the series of treatises on pure and applied mathematics, written by authoritative contemporary mathematicians. For instance, in 1906 poster the titles of the 19 volumes already printed were listed (indicated with two asterisks) and the seven being printed (indicated with one asterisk, as well as those planned in future. The latter included Corrado Segre's, entitled *Vorlesungen über algebraische Geometrie, mit besonderer Berücksichtigung der mehrdimensionalen Raume,* which was never delivered. Cf. B.G. Teubners *Sammlung von Lehrbüchern auf dem Gebiete der Mathematischen Wissenschaften mit Einschluss ihrer Anwendungen*. Cf. Fig. 2.1 above.

[481]Guido Toja (1870–1933).

La Biblioteca del Suo compianto Signor Padre[482] mi è giunta in modo perfetto: ho già scritto al Prof. Arturo Segre e voglio ripetere a Lei la mia riconoscenza. Studioso della matematica e ammiratore devoto di quanti la coltivano con vero successo e con spirito geniale, sono portato a tributare, direi quasi un culto a tutto ciò che ad essi più o meno intimamente appartiene. A Lei sarà, ne sono certo, dolce il pensare che la Biblioteca di Corrado Segre, collaboratrice della Sua opera gloriosa e nella quale Egli vive ancora, sia custodita con amorosa cura e commosso sentimento. Gradisca, Signora, le espressioni del mio distinto ossequio. Suo dev.[mo] G. Toja.

75. Corrado Segre's Library at the Department of Mathematics, Florence University

UTo-ACS. *Elenchi di opere e articoli, Schede bibliografiche*, 1–19 typewritten pages –BDMI Florence, Toja Room.

The engineer Guido Toja, who had been the founder and director of the National Insurance Institute, and then became a lecturer in Financial and Actuarial Mathematics at the Faculty of Economy and Commerce in Florence, in his will left his personal library to the U. Dini Mathematical Institute in Florence (Sansone 1963, 8–11). *It included the mathematics volumes and offprints he had got the National Insurance Institute to purchase in 1924, after the death of Corrado Segre (Annex 74). In this way Segre's rich library was not dispersed and today it is still possible to consult it in Florence, at the BDHI, Toja Room. The volumes and the collections of Memoirs of some Italian and foreigner colleagues of C. Segre in his library have been identified thanks to the ownership initials that Segre put on the frontispiece and to the dedications by the various authors to the Piedmontese algebraic geometer, here pointed out with ('in homage').*[483] *Moreover, thanks to the recent donation of Segre's archives kept in Ancona to the Turin University by*

[482]Cf. Annex 75. In 1924 Toja, as President of the Council of the National Insurance Institute, proposed purchasing the mathematics volumes and offprints in C. Segre's personal library, underlining the need, for every enterprise of an industrial type, to have a library able to provide executives and technicians with tools for investigation and study. In the minutes we read: "Few opportunities have arisen to furnish the library itself with volumes regarding pure mathematics, although it is necessary to prepare an organic plan so that in the space of a few years the library of the Institute in suitable rooms in the new home can reach the desired potentialities and be such as to be fit for purpose with respect for the industrial and cultural technical aims of the institute. (…) However, he desires the unanimous consent of his colleagues." (De Donno 1988, 99: "Poche occasioni si sono presentate per fornire la biblioteca stessa di volumi riguardanti le matematiche pure sebbene sia necessario di preparare un piano organico affinché nello spazio di pochi anni la biblioteca dell'Istituto possa in locali adatti della nuova sede raggiungere le potenzialità desiderate e tale da rispondere allo scopo nei riguardi delle finalità tecniche industriali e culturali dell'istituto. … Egli desidera però il consenso unanime dei colleghi.").

[483]We are grateful to the staff of the Florence Library and, in particular, to Laura Bitossi, for having provided us with the book list and the information necessary to our reconstruction.

Silvano and Daniele Fuà, the grand-nephews of Corrado Segre, we have now available the original lists of books and journals in Segre's personal library at his own home in Turin, corso Vittorio Emanuele 24 (Appendix 1 in this volume). An accurate collation between the lists allowed to identify those books and offprints which have been sent to Toja in December 1924. The latter ones, called Memorie *by Segre, were made binding by Toja. Another collection of offprints from Segre's library have been donated by his wife Olga Michelli Segre to the Mathematical Library of Turin University in May 1924 (Giacardi and Roero* 1999, 446–447). *Moreover, in March 1926 she offered to the same Library the valuable corpus of Segre's* Notebooks (Conte et al. 2013, 46, 69–76).

Abel, Niels H., *Oeuvres*, 2 vols. Christiania: Grøndahl 1881.

Ahrens, Wilhelm, *Scherz und Ernst in der Mathematik*, Leipzig: Teubner 1904.

Amaldi, Ugo and Enriques, Federigo, *Nozioni di Matematica ad uso dei Licei moderni*, 2 vols. Bologna: Zanichelli 1914, 1915.

Amanzio, Domenico, *Aritmetica pratica*, Napoli: Morano 1888[2].

Amanzio, Domenico, *Trattato di aritmetica teorica*, Napoli: Morano 1889[2].

Amanzio, Domenico, *Trattato di algebra elementare*, Napoli: Pellerano 1892.

Amiot, Antoine, *Trattato di Geometria elementare*, (transl. G. Novi) 2 vols. Firenze: Le Monnier 1858.

Amodeo, Federico, *Lezioni sulle omografie binarie a.a. 1887–88*, Napoli: Pellerano 1889.

Amodeo, Federico, *Aritmetica particolare e general*e, vol. 1, Napoli: Pierro 1900.

Amodeo, Federico, *Elementi di algebra*, vol. 2.1, Napoli: Pierro 1903.

Amodeo, Federico, *Vita matematica napoletana, Studio storico, biografico, bibliografico*, Napoli: Tip. Giannini 1905.

André-Fouet, Edouard, *Leçons élémentaires sur la théorie des fonctions analytiques*, 2 vols. Paris: Gauthier-Villars 1907, 1910.

Artom, Emilio, *Elementi di aritmetica ad uso delle scuole tecniche*, Rocca S. Casciano: Cappelli 1911.

Arzelà, Cesare, *Lezioni di Calcolo infinitesimale*, vol. 1.1, Firenze: Le Monnier 1901.

Baker, Henry F., *Principles of Geometry*, 3 vols., *Foundations*, *Plane Geometry*, *Solid Geometry*, Cambridge: Cambridge Univ. Press 1922–1923.

Baire, René, *Leçons sur les théories générales de l'Analyse*, 2 vols. Paris: Gauthier-Villars 1907, 1908.

Ball, Robert Stawell, *The Theory of Screws*, Dublin: Hodges 1876.

Baltzer, Richard, *Theorie und Anwendung der Determinanten*, Leipzig: Hirzel 1875.

Baltzer, Richard, *Die Elemente der Mathematik* 2 vols. Leipzig: Hirzel 1878, 1879.

Baltzer, Richard, *Analytische Geometrie*, Leipzig: Hirzel 1882.

Basset, Alfred B., *A Treatise on the Geometry of Surfaces*, Cambridge: Deighton Bell 1910.

Bassi, Alfredo, *Risoluzione dei triangoli piani: norme ed esempi*, Torino: Paravia 1907.

Beltrami, Eugenio, *Opere matematiche*, 4 vols. Milano: Hoepli 1902–1920.

Berliner, Hans, *Involutionssysteme in der Ebene des Dreiecks*, Braunschweig: Vieweg 1914.

Bertini, Eugenio, *Lezioni di Geometria superiore a.a. 1900–1901*, Lit., Pisa 1901.

Bertini, Eugenio, *Lezioni di geometria descrittiva ad uso degli studenti della R. Università di Pisa*, Lit. Firenze: Tip. Materassi 1907.

Bertini, Eugenio, *Introduzione alla geometria proiettiva degli iperspazi, con appendice sulle curve algebriche e loro singolarità*, Pisa: Spoerri 1923^2.

Bertrand, Joseph, *Trattato di Algebra elementare*, (transl. E. Betti), Firenze: Le Monnier 1862.

Bertrand, Joseph, *Trattato d'Aritmetica*, (transl. G. Novi), Firenze: Le Monnier 1862.

Bertrand, Joseph, *Calcul des probabilités*, Paris: Gauthier-Villars 1889.

Berzolari, Luigi, *Lezioni di geometria proiettiva ... R. Univ. Torino 1894–95*, Torino 1895.

Berzolari, Luigi, *Il metodo delle coordinate*, 2 vols. Milano: Hoepli 1911^2, 1920^2.

Berzolari, Luigi, *Curve e superficie del 2° ordine*, Milano: Hoepli 1922^2.

Betti, Enrico, *Teorica delle forze newtoniane e sue applicazioni ...*, Pisa: Nistri 1879.

Betti, Enrico and Brioschi, Francesco, *Gli elementi d'Euclide con note, aggiunte ed esercizi ...* (Libri 1–2–3, 4–5–6, 11–12, Appendice), Firenze: Le Monnier 1835–1868.

Betti, Enrico, *Opere matematiche*, 2 vols. Milano: Hoepli 1903, 1913.

Bettini, Bettino and Ciamberlini, Corrado, *Elementi di aritmetica e geometria per la 1° classe ginnasiale*, Firenze: Bemporad 1901.

Bianchi, Luigi, *Lezioni di Geometria differenziale fatte nell'Università di Pisa nell'anno 1885–86*,[484] Lit., Pisa: Nistri 1886.

Bianchi, Luigi, *Lezioni sulla teoria dei gruppi di sostituzioni e delle equazioni algebriche secondo Galois*, Pisa: Spoerri 1900.

Bianchi, Luigi, *Lezioni sulla teoria delle funzioni di variabile complessa e delle funzioni ellittiche*, Pisa: Spoerri 1901.

Bianchi, Luigi, *Lezioni di Geometria differenziale*, Pisa: Spoerri vol. 1 1902^3; vol. 3 1909^2.

Bianchi, Luigi, *Lezioni sulla teoria dei Gruppi continui finiti di trasformazioni*, Pisa: Spoerri 1918.

Bianchi, Luigi, *Lezioni sulla teoria dei numeri algebrici ...*, Pisa: Spoerri 1921.

Biermann, Otto, *Theorie der analytischen Funktionen*, Leipzig: Teubner 1887.

Boccardi, Giovanni, *Elementi di Astronomia*, Lit., n.p., n.d. [Torino 1896].

Bôcher, Maxime, *Introduction to higher Algebra*, New York: Macmillan 1907.

[484]This book is missing in BDMI Florence, Toja Room.

Bôcher, Maxime, *Einführung in die höhere Algebra*, Leipzig: Teubner 1910 ('in homage').

Boggio, Tommaso, *Calcolo differenziale con applicazioni geometriche*, vol. 1, Torino: Lattes 1921.

Bolyai, Wolfgang and Johann, *Geometrische Untersuchungen*, (Stäckel, Paul ed.), 2 vols. Leipzig: Teubner 1913.

Bonola, Roberto, *La Geometria non euclidea: esposizione storico-critica del suo sviluppo*, Bologna: Zanichelli 1906.

Boole, George, *A Treatise on Differential Equations*, London: Macmillan 1877.

Bordiga, Giovanni, *Lezioni di Geometria proiettiva 1904–05*, Lit., Padova: Pinzon 1905.

Borel, Emile (ed.), *Leçons sur la théorie des fonctions Monographies sur la théorie des fonctions* Paris: Gauthier-Villars vol. 1–4 (1898, 1900, 1901, 1902) vol. 5–8 (1903–1905), v. 9–12 (1905–1910), v. 13–16 (1910–1913), v. 17–20 (1913–1916), v. 21–24 (1917–1920), v. 25–28 (1922–1924), v. 29–32 (1924), v. 33.

Borel, Emile, *Die Elemente der Mathematik*, (transl. P. Stäckel), 2 vols., Leipzig: Teubner 1908, 1909.

Bortolotti, Ettore, *Lezioni di geometria analitica,* 2 vols., Bologna: Zanichelli 1923.

Brill, Alexander von and Nöther, Max, *Die Entwicklung der Theorie der algebraische Funktionen in älterer und neuerer Zeit,* Berlin: Reimer 1894.

Briot, Charles A. and Bouquet, Jean-Claude, *Théorie des Fonctions elliptiques*, Paris: Gauthier-Villars 1875.

Buffa, Pietro, *Primo studio della geometria piana …*, Torino: Paravia 1901.

Burali-Forti, Cesare, *Teorica elementare-razionale dei numeri commensurabili*,[485] Torino: Petrini 1889.

Burali-Forti, Cesare, *Aritmetica razionale: I numeri razionali*, Torino: Petrini 1892.

Burali-Forti, Cesare and Ramorino, Angelo, *Aritmetica razionale*, Torino: Petrini 1892.

Burali-Forti, Cesare, *Note scientifiche e critiche alle lezioni di aritmetica pratica*, Torino: Petrini 1897.

Burali-Forti, Cesare, *Lezioni di aritmetica pratica*, Torino: Petrini 1897.

Burali-Forti, Cesare, *Introduction à la géométrie différentielle suivant la méthode de H. Grassmann*, Paris: Gauthier-Villars 1897.

Burali-Forti, Cesare and Ramorino, Angelo, *Elementi di aritmetica razionale ad uso della terza classe della scuola tecnica*, Torino: Petrini 1898.

Burali-Forti, Cesare and Ramorino, Angelo, *Elementi di algebra con 570 problemi ed esercizi ad uso della terza classe della scuola tecnica*, Torino: Petrini 1898.

Burali-Forti, Cesare, *Lezioni di geometria analitico-proiettiva*, Lit., Torino: Artigianelli 1909.

[485]This book is missing in BDMI Florence, Toja Room.

Burali-Forti, Cesare and Marcolongo, Roberto, *Elementi di Calcolo vettoriale*, Bologna: Zanichelli 1909.

Burali-Forti, Cesare and Marcolongo, Roberto, *Omografie vettoriali*, Torino: Petrini 1909.

Burali-Forti, Cesare and Marcolongo, Roberto, *Elements de Calcul vectoriel*, Paris: Hermann 1910.

Burali-Forti, Cesare and Ramorino, Angelo, *Elementi di algebra per la scuola normale*, Torino: Petrini 1911, 1920[3].

Burali-Forti, Cesare, *Logica matematica*, Milano: Hoepli 1919[2].

Burali-Forti, Cesare and Boggio, Tommaso, *Meccanica razionale*, Torino: Lattes 1921.

Caldarera, Francesco, *Primi fondamenti della geometria del piano*, Torino: Clausen 1891.

Cameletti, Ignazio, *Geometria pura elementare esposta per dualità*, Torino: Paravia 1893.

Cantor, Moritz, *Vorlesungen über Geschichte der Mathematik*, 4 vols. Leipzig: Teubner 1880–1908.

Casorati, Felice, *Teorica delle funzioni di variabile complessa*, Pavia: Frat. Fusi 1868.

Castellano, Filiberto, *Elementi di algebra ad uso dei licei, istituti tecnici e scuole militari*, Torino: Bocca 1891.

Castellano, Filiberto, *Lezioni di meccanica razionale*, Torino: Candeletti 1894.

Castelnuovo, Guido, *Lezioni di geometria analitica*, Roma: Soc. D. Alighieri 1909.

Castelnuovo, Guido (ed.), *Atti del IV Congresso Internazionale dei Matematici (Roma 6–11 aprile 1908)*, 3 vols. Roma: Tip. R. Acc. Lincei 1909.

Castelnuovo, Guido, *Calcolo delle probabilità*, Milano: Alighieri 1919.

Castelnuovo, Guido, *Spazio e tempo secondo le vedute di A. Einstein*, Bologna: Zanichelli 1923.

Cayley, Arthur, *Trattato elementare delle funzioni ellittiche*, (transl. F. Brioschi), Milano: Hoepli 1880.

Cayley, Arthur, *Collected Mathematical Papers*, 13 vols. and Supplementary, Cambridge, CUP 1889–1898.

Čebyšev, Pafnutij L., *Teoria delle congruenze*, (transl. I. Massarini), Roma: Loescher 1895.

Cesàro, Ernesto, *Corso di analisi algebrica, con introduzione al calcolo infinitesimale*, Torino: Bocca 1894.

Cesàro, Ernesto, *Elementi di calcolo infinitesimale con numerose applicazioni geometriche*, Napoli: Alvano 1897, 1905[2].

Chasles, Michel, *Traité des sections coniques*, Paris: Gauthier-Villars 1865.

Chasles, Michel, *Aperçu historique sur l'origine et le développement des méthodes en Géométrie*, Paris: Gauthier-Villars 1875.

Chasles, Michel, *Traité de Géométrie supérieure*, Paris: Gauthier-Villars 1880.

Chisholm, Grace Young and Young, William H., *The first Book of Geometry*, London: Dent 1905.

Chisholm, Grace and Young, William H., *The Theory of Sets of Points*, Cambridge: CUP 1906.

Chisholm, Grace Young and Young, William H., *Geometria per i piccoli*, (transl. L. Viriglio), Torino: Paravia 1911.

Ciamberlini, Corrado, *Esercitazioni e ricreazioni geometriche ad uso degli alunni delle scuole elementari*, Lanciano: Carabba 1902.

Ciamberlini, Corrado, *Saggi di didattica matematica: raccolta di scritti vari*, Torino: Paravia 1920.

Ciani, Edgardo, *Lezioni di geometria proiettiva ed analitica*, Pisa: Spoerri 1912.

Ciani, Edgardo, *Il metodo delle coordinate proiettive omogenee nello studio degli enti algebrici*, Pisa: Spoerri 1915.

Clausius, Rudolf, *Die Potentialfunction und das Potential*, Leipzig: Barth 1885.

Clebsch, Alfred and Gordan, Paul, *Theorie der Abelschen Funktionen*, Würzburg: Physica-Verlag 1866.

Clebsch, Alfred, *Theorie der Binären algebraische Formen*, Leipzig: Teubner 1872.

Clebsch, Alfred and Lindemann, Ferdinand, *Vorlesungen über Geometrie*, (Lindemann, Ferdinand ed.), 2 vols. Leipzig: Teubner 1876, 1891.

Colonnetti, Gustavo, *Principi di statica dei solidi elastici*, Pisa: Spoerri 1916.

Coolidge, Julian, *The Elements of Non-Euclidean Geometry*, Oxford: UP 1909.

Coolidge, Julian, *A treatise on the Circle and the Sphère*, Oxford: Clarendon 1916.

Coolidge, Julian, *The Geometry of the complex domain*, Oxford: Clarendon 1924.

Cossa, Alfonso, *Prime nozioni fondamentali di elettrochimica*, Milano: Hoepli 1901.

Cremona, Luigi, *Introduzione ad una teoria geometrica delle curve piane*, Bologna: Gamberini Parmeggiani 1862.

Cremona, Luigi, *Grundzüge einer allgemeinen Theorie der Oberflächen*[486] (transl. M. Curtze), Berlin: Calvary 1870.

Cremona, Luigi, *Opere matematiche*, 3 vols. Milano: Hoepli 1914–1917.

Czuber, Emanuel, *Theorie der Beobachtungsfehler*, Leipzig: Teubner 1891.

Darboux, Jean Gaston, *Sur une classe remarquable de courbes et de surfaces algébriques*, Paris: Hermann 1873.

Darboux, Jean Gaston, *Leçons sur la théorie générale des surfaces*, 4 vols. Paris: Gauthier-Villars 1887–1896.

Darboux, Jean Gaston, *Eloges académiques et discours*, Paris: Hermann 1912.

Dedekind, Richard, *Was sind und was sollen die Zahlen?*,[487] Braunschweig: Vieweg 1888.

De Franchis, Michele, *Geometria elementare ad uso dei licei …*, Palermo: Sandron 1909.

[486]This volume is bound with that by Sturm, Rudolf, *Synthetische Untersuchungen über Flächen 3 Ordnung*, 1867.

[487]This book is missing in BDMI Florence, Toja Room.

Del Pezzo, Pasquale, *Principi di geometria proiettiva ... 1910–11*, Napoli: Alvano 1911.

De Paolis, Riccardo, *Elementi di geometria,* Torino: Loescher 1884.

Descartes, René, *Geometria*, (van Schooten, Frans ed.), Lugduni Batavorum: Maire 1649.

Dickson, Leonard E. and Osgood, William F., *The Madison Colloquium 1913*, New York: AMS 1914.

Dini, Ulisse, *Grundlagen für eine Theorie der Funktionen einer veränderlichen reellen Grösse*, (transl. J. Lüroth, A. Schepp), Leipzig: Teubner 1892.

Lejeune-Dirichlet, P. Gustav, *Vorlesungen über Zahlentheorie*, Braunschweig: Vieweg und Sohn 1879.

Lejeune-Dirichlet, P. Gustav, *Vorlesungen über die im ungekehrten Verhältniss des Quadrats der Entfernung wirkenden Kräfte*, (Grube, Franz ed.), Leipzig: Teubner 1887.

Lejeune-Dirichlet, P. Gustav, *Werke*, 2 vols. Berlin: Reimer 1889, 1897.

D'Ovidio, Enrico, *Il libro primo di Euclide*, Napoli: Pellerano 1887.

D'Ovidio, Enrico, *Compendio di algebra complementare: lezioni professate nella R. Università di Torino a.a. 1898–1899*, Torino: Lit. 1899 ('in homage').

D'Ovidio, Enrico, *Geometria analitica*, Torino: Bocca 1903.

Engel, Friedrich and Lie, Sophus, *Theorie der Transformationsgruppen*, 3 vols. Leipzig: Teubner 1888–1893.

Engel, Friedrich and Stäckel, Paul, *Die Theorie der Parallelinien von Euklid bis auf Gauss*, Leipzig: Teubner 1895.

Enriques, Federigo, *Lezioni di Geometria proiettiva 1893–94*, Lit. n.p., bound with *Geometria descrittiva 1893–94*, Lit. n.p.

Enriques, Federigo, *Lezioni di Geometria proiettiva*, (Serrazanetti, Gaetano ed.), Bologna: Lit. R. Università di Bologna 1894–95.

Enriques, Federigo, *Lezioni di Geometria descrittiva* (Schincaglia, Ignazio ed.), Bologna: Lit. R. Università di Bologna 1894–95.

Enriques, Federigo, *Conferenze di Geometria* 1894–95, Bologna: Tip. Lit. L. Pongetti 1895.

Enriques, Federigo and Amaldi, Ugo, *Elementi di geometria ad uso delle Scuole secondarie superiori*, Bologna: Zanichelli 1908[4].

Enriques, Federigo and Amaldi, Ugo, *Elementi di geometria ad uso delle Scuole tecniche*, Bologna: Cacciari 1909.

Enriques, Federigo, *Problemi della Scienza*, Bologna: Zanichelli 1910[2].

Enriques, Federigo, *Scienza e razionalismo*, Bologna: Zanichelli 1912 ('in homage').

Enriques, Federigo, *Lezioni sulla teoria geometrica delle equazioni e delle funzioni algebriche*, (Chisini, Oscar ed.), Bologna: Zanichelli, vol. I 1915, vol. II 1918.

Enriques, Federigo, *Per la storia della logica*, Bologna: Zanichelli 1922 ('in homage').

Euler, Leonhard, *Einleitung in die Analysis des Unendlichen Introductio in analysin infinitorum* (German transl. J.A. Michelsen), 3 vols. Berlin: Hesse and Matzdorff 1788–1791.

Faà di Bruno, Francesco, *Einleitung in die Theorie der binären Formen* (transl. T. Walter), Leipzig: Teubner 1881.

Faifofer, Aureliano, *Elementi di algebra ad uso degli istituti tecnici e dei licei*, Venezia: Tip. Emiliana 1887[7].

Fano, Gino, *Lezioni di Geometria descrittiva*, Torino: Lit. 1892–93.

Fano, Gino, *Lezioni di Geometria della retta*, Roma: Lit. L. Landi 1896.

Fano, Gino, *Lezioni di Geometria non euclidea*, Roma: Lit. L. Cippitelli 1898.

Fano, Gino, *Algebra complementare*, Lit., n.p., n.d.

Fauque de Jonquières, Ernest, *Mélanges de géométrie pure*, Paris: Mallet-Bachelier 1856.

Ferraris, Galileo, *Lezioni di elettrotecnica*, vol. 1, Torino: Roux & Frassati 1899.

Ferrers, Norman M., *An Elementary Treatise on trilinear Co-ordinates*, London: Macmillan 1876[3].

Fiedler, Wilhelm, *Die darstellende Geometrie in organischer Verbindung mit der Geometrie der Lage*, Leipzig: Teubner 1875[2].

Fiedler, Wilhelm, *Die darstellende Geometrie* Part I *Die Methoden der darstellenden und die Elemente der projektivischen Geometrie*, Leipzig: Teubner 1904[4].

Fink, Karl Kurzer, *Abriss einer Geschichte der Elementar-Mathematik*, Tubingen: H. Laupp 1890.

Fisher, George E. and Schwatt, Isaac J., *Text-book of Algebra, Part I with exercises for secondary schools and colleges*, *Algebra*, Philadelphia: Fischer & Schwatt 1898.

Forsyth, Andrew R., *Lectures on the Differential Geometry of Curves and Surfaces*, Cambridge: UP 1912.

Fouret, Georges F., *Notions géométriques sur les complexes et les congruences de droites*, Paris: Gauthier-Villars 1893.

Frenet, Frederic, *Recueil d'exercises sur le calcul infinitésimal*, Paris: Gauthier-Villars 1882.

Frischauf, Johannes, *Die Elemente der absoluten Geometrie*, Leipzig: Teubner 1876.

Fubini, Guido, *Introduzione alla teoria dei gruppi discontinui e delle funzioni automorfe*, Pisa: Spoerri 1908.

Fubini, Guido, *Lezioni di analisi matematica*, Torino: STEN 1915.

Galois, Evariste, *Oeuvres mathématiques*, 2 vols. Paris: Gauthier-Villars 1897.

Gambardella Filippo, *Lezioni di calcolo infinitesimale*, Napoli: Pellerano 1885.

Gazzaniga, Paolo, *Libro di aritmetica e di algebra elementare*, Padova: Stab. Tip. Prosperini 1897.

Gauss, Carl Friedrich, *Allgemeine Flächentheorie (Disquisitiones generales circa superficies curvas)*, (transl. A. Wangerin), Leipzig: Engelmann 1889.

Gauss, Carl Friedrich, *Untersuchungen über höhere Arithmetik*, (transl. H. Maser) Berlin: Springer 1889.

Gauss, Carl Friedrich, *Allgemeine Lehrsätze in Beziehung auf die im verkehrten Verhältnisse des Quadrats der Entfernung wirkenden Anziehungs- und Abstossungs-Kräfte*, Leipzig: Englemann 1889.

Gauss, Carl Friedrich, *Die vier Gauss'schen Beweise für die Zerlegung ganzer algebraischen Funktionen in reelle Factoren ersten oder zweiteen Grades (1799–1849)*, Leipzig: Engelmann 1890.

Genocchi, Angelo and Peano, Giuseppe, *Calcolo differenziale e principii di calcolo integrale*, Torino: Bocca 1884.

Gerbaldi, Francesco and Loria, Gino (eds.), *Scritti matematici offerti ad Enrico D'Ovidio*, Torino: Bocca 1918.

Gordan, Paul, *Über das Formensystem binären Formen*, Leipzig: Teubner 1875.

Gordan, Paul, *Vorlesungen über Invariantentheorie*, 2 vols., Leipzig: Teubner 1885–1887.

Goursat, Édouard, *Leçons sur l'intégration des équations aux dérivées partielles du second ordre*, 2 vols. Paris: Hermann 1896, 1898.

Grassmann, Hermann, *Lehrbuch der Arithmetik*, Berlin: Enslin 1861.

Grassmann, Hermann, *Gesammelte mathematische und physikalische Werke*, 3 vols. Leipzig: Teubner 1894–1911.

Grassmann, Hermann, *Projective Geometrie der Ebene*, Leipzig: Teubner 1909.

Halphen, Georges-Henri, *Traité des fonctions elliptiques et de leurs applications*, 3 vols. Paris: Gauthier-Villars 1886–1891.

Halsted, George Bruce, *Rational Geometry A Textbook for the Science of Space based on Hilbert's Foundations*, New York: Willey 1904.

Hamilton, William Rowan, *Elemente der Quaternionen* (Hamilton, William Edwin ed., transl. P. Glan), 2 vols. Leipzig: Teubner 1882, 1884.

Hamilton, William Rowan, *Lectures on Quaternions*, Dublin: Hodges and Smith 1853 ('in homage').[488]

Hankel, Hermann, *Vorlesungen über die Complexe Zahlen und ihre Funktionen*, Leipzig: Teubner 1867.

Hankel, Hermann, *Die Elemente der projectivischen Geometrie in synthetischer Behandlung*, Leipzig: Teubner 1875.

Hermite, Charles, *Cours de M. Hermite professé pendant le 2^e Semestre 1881–82* (Andoyer ed. élève de l'école normale supérieure seconde tirage, revue par M. Hermite), Lit. Paris: Hermann 1883.

Hermite, Charles, *Note sur la théorie des fonctions elliptiques*, Paris: Gauthier-Villars 1894 ('in homage').

Hesse, Otto, *Vorlesungen aus der Analytischen Geometrie der geraden Linien, des Punktes und des Kreises in der Ebene*, Leipzig: Teubner 1873.

Hesse, Otto, *Vorlesungen über Analytische Geometrie des Raumes*, Leipzig: Teubner 1876.

Hesse, Otto, *Gesammelte Werke*, München: Verlag der Akademie 1897.

Hilbert, David and Wieckert, Emil (eds.), *Festschrift zur Feier der Enthüllung des Gauss-Weber Denkmals in Göttingen*, Leipzig: Teubner 1899.

[488]Inside cover we find the autograph: "From the author".

Hobson, Ernest W. and Love, Augustus E.H. (eds.), *Proceedings of the Fifth International Congress of Mathematicians, Cambridge 22–28 August 1912*,[489] 2 vols. Cambridge: UP 1913.

Höfer, Ferdinand, *Histoire des mathématiques: depuis leurs origines jusqu'au commencement du dix-neuvième siècle*, Paris: Hachette 1879.

Hoüel, Jules, *Essai critique sur les principes fondamentaux de la géométrie élémentaire*, Paris: Gauthier-Villars 1883.

Hoüel, Jules, *Considérations élémentaires sur la généralisation successive de l'idée de quantité dans l'Analyse mathématique*, Paris: Gauthier-Villars 1883.

Hugues, Luigi, *Nozioni di geografia matematica*,[490] Torino: Loescher 1882.

Insolera, Filadelfo, *Elementi di matematica finanziaria e attuariale*, Torino: Lattes 1916.

Jacobi, Carl Gustav, *Gesammelte Werke*, 7 vols. Berlin: Reimer 1881–1891.

Jacobi, Carl Gustav, *Vorlesungen über Dynamik*, Berlin: Reimer 1884.

Joachimsthal, Ferdinand and Natani, Leopold, *Anwendung der differential und integralrechnung auf die allgemeine Theorie der Flächen und der Linien doppelter Krummung*, Leipzig: Teubner 1881.

Jordan, Camille, *Cours d'analyse de l'école polytechnique*, 3 vols. Paris: Gauthier-Villars 1909^2, 1913^2, 1915^2.

Jouffret, Esprit, *Traité élémentaire de géométrie à quatre dimensions*, Paris: Gauthier-Villars 1903.

Jouffret, Esprit, *Mélanges de géométrie à quatre dimensions*, Paris: Gauthier-Villars 1906.

Kantor, Seligmann, Premiers fondements pour une théorie des transformations periodiques, *Atti della R. Accademia delle Scienze fisiche e matematiche di Napoli* (2) 3 1891.

Kiepert, Ludwig, *Grundriss der Differential und Integralrechnung*, 2 vols. Hannover: Helwing 1908–1910.

Killing, Wilhelm, *Nicht Euklidische Raumformen in analytischer Behandlung*, Leipzig: Teubner 1885.

Killing, Wilhelm, *Einführung in die Grundlagen der Geometrie*, 2 vols. Paderborn: Schöningh 1893, 1898.

Kirchhoff, Gustav R., *Vorlesungen über mathematische Physik*, 2 vols. Leipzig: Teubner 1891–1894.

Klein, Felix, *Vorlesungen über das Ikosaeder und die Auflösung der Gleichungen vom fünften Grade*, Leipzig: Teubner 1884.

Klein, Felix and Fricke, Robert, *Vorlesungen über die Theorie der elliptischen Modulfunktionen*, 2 vols. Leipzig: Teubner 1890–1892.

Klein, Felix, *Nicht Euklidische Geometrie*, 2 vols. Leipzig: Teubner 1890–1892.

Klein, Felix, *Ausgewählte Kapitel aus der Theorie der Linearen Differentialgleichungen zweiter Ordnung*, Göttingen: n. p. 1891.

[489] This book is missing in BDMI Florence Toja Room.

[490] Now the book is lost in BDMI Florence Toja Room.

Klein, Felix, *Einleitung in die geometrischen Funktionentheorie*, 2vols. Lit. Göttingen: Univ. 1892.

Klein, Felix, *Riemmansche Flächen*, 2 vols. Lit. Göttingen: Univ. 1892.

Klein, Felix, *Einleitung in die höhere Geometrie*, 2 vols. Lit. Göttingen: Univ. 1893.

Klein, Felix, *The Evanston Colloquium Lectures on Mathematics*, New York: Macmillan 1894.

Klein, Felix, *Vorträge über ausgewählte Fragen der Elementargeometrie* (F. Tägert ed.) Leipzig: Teubner 1895.

Klein, Felix and Sommerfeld, Arnold, *Über die Theorie des Kreisels*, 4 vols. Leipzig: Teubner 1897–1910.

Klein, Felix, *Conférences sur les mathématiques faites au Congrès de mathématiques tenu à l'occasion de l'exposition de Chicago* (Ziwet, Alexander and Laugel, Léonce eds.), Paris: Hermann 1898.

Klein, Felix, *Anwendung der Differential und Integralrechnung auf Geometrie* ..., Leipzig: Teubner 1902.

Klein, Felix, *Gesammelte mathematische Abhandlungen*, 3 vols. Berlin: Springer 1921–1923.

Knoblauch, Johannes, *Allgemeine Theorie der krummen Flächen*, Leipzig: Teubner 1888.

Köhler, Heinrich G., *Manuale logaritmico-trigonometrico*, Lipsia: Tauchnitz, Torino: Clausen 1900.

Koenigsberger, Leo, *Vorlesungen über die Theorie der elliptischen Funktionen*, Leipzig: Teubner 1874.

Koenigsberger, Leo, *Lehrbuch der Theorie der Differentialgleichungen mit einer unabhängigen Variabel*, Leipzig: Teubner 1889.

Kötter, Ernst, *Grundzüge einer rein geometrischen Theorie der algebraischen ebenen Kurven*, Berlin: Reimer 1887.

Krazer, Adolf (ed.), *Verhandlungen des dritten Internationalen Mathematiker-Kongresses in Heidelberg vom 8 bis 13 August 1904*, Leipzig: Teubner 1905.

Kronecker, Leopold, *Über bilineare Formen mit vier Variabeln*, Berlin: Akademie der Wiss. 1884.

Lagrange, Joseph Louis, *Œuvres* ..., 7 vols., Paris: Gauthier-Villars 1867–1877.

Laisant, Charles Ange, *Introduction a la méthode des quaternions*, Paris: Gauthier-Villars 1881.

Lazzeri, Giulio, *Trattato di Geometria analitica* Livorno: Giusti 1893.

Lazzeri, Giulio, *Analisi infinitesimale*, 3 vols. Livorno: Giusti 1917–1918.

Lazzeri, Giulio, *Geometria analitica*, 3 vols. Livorno: Giusti 1918–1919.

Lebesgue, Henri, *Leçons sur les séries trigonométriques*,[491] Paris: Gauthier-Villars 1906.

Legendre, Adrien Marie, *Zahlentheorie*, 2 vols. Leipzig: Teubner 1886.

Lemoine, Emile, *Géométrographie ou art des constructions géométriques*, Paris: Naud 1902 ('in homage').

[491]This book is missing in BDMI Florence Toja Room.

Leoni, Carlo, *La matematica nel suo insegnamento* ..., Milano: Vallardi 1915.

Levi, Beppo, *Lezioni di analisi algebrica a.a. 1912–13*, Parma: Anghinetti 1913.

Levi, Beppo, *Introduzione alla analisi matematica*, I. *Teorie formali*, Parma: R. Università, Paris: Hermann 1916.

Levi-Civita, Tullio and Amaldi, Ugo, *Lezioni di Meccanica razionale*, vol. 1, Bologna: Zanichelli 1923.

Libri, Guglielmo, *Histoire des sciences mathématiques en Italie, depuis la Renaissance*…, 4 vols., Paris: Renouard 1838–1841.

Lie, Sophus and Scheffers, Georg, *Vorlesungen über Differentialgleichungen mit bekannten infinitesimal Transformationen*, Leipzig: Teubner 1891.

Lie, Sophus and Scheffers, Georg, *Vorlesungen über continuierliche Gruppe* ..., Leipzig: Teubner 1893.

Lie, Sophus and Scheffers, Georg, *Geometrie der Berührungstransformationen*, Leipzig: Teubner 1896.

Lipschitz, Rudolf, *Untersuchungen über die Summen von Quadraten*, Bonn: Cohen 1886.

Lobačevskij, Nikolaj I., *Zwei geometrische Abhandlungen*, (transl. F. Engel) 2 vols., Leipzig: Teubner 1898.

Loria, Gino, *Il passato e il presente delle prime teorie geometriche*, Torino: Clausen 1907.

Loria, Gino, *Vorlesungen über darstellende Geometrie*, 2 vols. Leipzig: Teubner 1907–1913.

Loria, Gino, *Spezielle algebraische und transcendenten ebene Kurven*, (transl. F. Schütte), 2 vols. Leipzig: Teubner 1910–1911.

Loria, Gino, *Le scienze esatte nell'antica Grecia*, Lib. 3, 4, 5, Modena: Soliani 1900–1902, Milano: Hoepli 1914.

Loria, Gino, *Guida allo studio della storia della matematica*, Milano: Hoepli 1916.

Loria, Gino, *Newton*, Roma: Formiggini 1920.

Loria, Gino, *Storia della geometria descrittiva*, Milano: Hoepli 1921.

Macaulay, Francis S., *Geometrical Conics*, Cambridge, Univ. Press 1906.

Maccaferri, Eugenio, *Calcolo numerico approssimato*, Milano: Hoepli 1919.

Maggi, Gian Antonio, *Principi della teoria matematica del movimento dei corpi*, Milano: Hoepli 1896.

Marcolongo, Roberto, *Il problema dei tre corpi da Newton ai nostri giorni*, Milano: Hoepli 1919.

Mascart, Éleuthère E.N., *Leçons sur l'électricité et le magnétisme*, 3 vols. Paris: Masson 1882, 1886; *Phénomènes généraux et théorie*, Paris: Masson 1896.

Maxwell, James Clerk, *Lehrbuch der Elektrizität und des Magnetismus* (transl. B. Weinstein), 2 vols. Berlin: Springer 1883.

Miller, George A. and Blichtfeld, Hans F. and Dickson, Leonhard E., *Theory and Applications of Finite Groups*, New York: Wiley 1916 ('in homage').

Minkowski, Hermann, *Geometrie der Zahlen*, Leipzig: Teubner 1910.

Möbius, August F., *Gesammelte Werke*, (Baltzer, Richard and Scheibner, Wilhelm eds.), 4 vols. Leipzig: Hirzel 1885–1887.

Monge, Gaspard, *Applications de l'Analyse à la Géométrie* ..., Paris: Bernard 1809.

Monge, Gaspard, *Géométrie descriptive*, Paris: Courcier 1827.

Monteverde, G. Francesco, *Elementi di geometria proiettiva*, 2 vols. Genova: Beuf 1886.

Moore, Eliakim H. and Wilczynski Ernest and Mason, Max, *The New Haven Mathematical Colloquium*, New Haven: Yale Univ. Press 1910.

Morera, Giacinto, *Lezioni di meccanica razionale*, 1901–02, Torino: Lit. M. Paris 1902.

Mougin, Emilio, *Log 1 ... 10000*, Milano: Bocca n.d.[492]

Müller, Emil, *Lehrbuch der darstellenden Geometrie für technische Hochschulen*, vol. 2 Leipzig: Teubner 1923.

Muth, Peter, *Theorie und Anwendung der Elementartheiler*, Leipzig: Teubner 1899 ('in homage').

Naccari, Andrea, *La vita di Michele Faraday*, Padova: Drucker 1908 ('in homage').

Natucci, Alpinolo, *Il concetto di numero e le sue estensioni*, Torino: Bocca 1923.

Netto, Eugen, *Substitutionentheorie und ihre Anwendung auf die Algebra*, Leipzig: Teubner 1882.

Neumann, Carl, *Vorlesungen über Riemann's Theorie der Abelschen Integrale*, Leipzig: Teubner 1884.

Neumann, Franz, *Vorlesungen über theoretische Optik*, (Dorn, Ernst ed.), Leipzig: Teubner 1885.

Neumann, Franz, *Vorlesungen über die Theorie des Potentials und der Kugelfunktionen*, Leipzig: Teubner 1887.

Nöther, Max, *Zur Grundlegung der Theorie der algebraischen Raumcurven*, Berlin: Akad. Wiss. 1883.

Pagliero, Giuliano, *Come si usano i logaritmi per calolare rapidamente Teoria e pratica*, Torino: Paravia 1917.

Pagliero, Giuliano, *Applicationes de calculo infinitesimale*, Torino: Paravia 1907.

Palatini, Francesco, *Geometria ad uso delle scuole medie superiori*, Torino: Petrini 1920.

Palatini, Francesco, *Aritmetica ed algebra ad uso delle scuole medie superiori*, Torino: Gallizio 1915.

Pascal, Ernesto, *Lezioni di Calcolo infinitesimale,* Parte I *Calcolo differenziale*, Milano: Hoepli 1902; Parte II *Calcolo integrale*, Milano: Hoepli 1903; Parte III *Calcolo delle variazioni e calcolo delle differenze finite*, Milano: Hoepli 1897.

Pascal, Ernesto, *Die Determinanten*, (transl. H. Leitzmann), Leipzig: Teubner 1900 ('in homage').

[492]This booklet is missing in BDMI Florence.

Pascal, Ernesto, *Repertorium der höheren Mathematik*, I_1 (Epstein, Paul ed.), Leipzig: Teubner 1910; $II_{1,2}$ (Epstein, Paul and Timerding, Heinrich E. eds.), Leipzig: Teubner 1910, (Schepp, Arthur ed.), Leipzig: Teubner 1902.

Pascal, Ernesto, *I gruppi continui di trasformazione*, Milano: Hoepli 1903.

Pascal, Ernesto, *I determinanti*, Milano: Hoepli 1923^2.

Pascal, Ernesto, *Lezioni di algebra complementare*, Napoli: Cimmaruta 1923^5.

Pascal, Ernesto, *Le funzioni ellittiche*, Milano: Hoepli 1924^2.

Pasch, Moritz, *Vorlesungen über neuere Geometrie*, Leipzig: Teubner 1882.

Paulus, Christoph, *Grundlinien der neueren ebenen Geometrie*, Stuttgard: Paulus 1853.

Peano, Giuseppe, *Applicazioni geometriche del calcolo infinitesimale*, Torino: Bocca 1887.

Peano, Giuseppe, *Calcolo geometrico secondo l'Ausdehnungslehre di Hermann Grassmann ...*, Torino: Bocca 1888.

Peano, Giuseppe, *I principi di geometria logicamente esposti*, Torino: Bocca 1889.

Peano, Giuseppe, *Gli elementi di calcolo geometrico*, Torino: Candeletti 1891.

Peano, Giuseppe, *Formulaire de Mathématique*, Torino: Bocca 1897^2; 1901^3; *Formulario Mathematico*, Torino: Bocca 1906^5.

Peano, Giuseppe, *Aritmetica generale e algebra elementare*, Torino: Paravia 1902.

Peano, Giuseppe, *Giochi di aritmetica e problemi interessanti*, Torino: Paravia, 1924 ('in homage').

Pensa, Angelo, *Elementi di geometria ad uso delle scuole*, Torino: Gallizio 1914.

Perazzo, Umberto, *Lezioni di geometria descrittiva*, 2 vols. Torino: Artigianelli 1910.

Petersen, Julius, *Methoden und Theorien zur Auflösung geometrischer Constructionsaufgaben*, Kopenhagen: Host 1879.

Picard, Emile, *Traité d'analyse*, 3 vols. Paris: Gauthier-Villars 1891, 1893, 1896.

Picard, Emile and Simart, Georg, *Théorie des fonctions algébriques de deux variables indépendantes*, 2 vols. Paris: Gauthier-Villars, 1897, 1906.

Picard, Emile, *Sur le developpement de l'Analyse et ses rapports avec diverses sciences*, Paris: Gauthier-Villars 1905.

Pincherle, Salvatore, *Lezioni di calcolo infinitesimale*, Bologna: Zanichelli 1919.

Pincherle, Salvatore, *Gli elementi della teoria delle funzioni analitiche*, Bologna: Zanichelli 1922.

Plana, Giovanni, *Théorie du mouvement de la lune*, 3 vols. Turin: Impr. Royale 1832.

Plücker, Julius, *Theorie der algebraischen Kurven ...*, Bonn: Marcus 1839.

Plücker, Julius, *System der analytischen Geometrie ...*, Berlin: Duncker-Humblot 1835; *System der Geometrie des Raumes ...*, Dusseldorf: Scheller 1846.

Plücker, Julius, *Neue Geometrie des Raumes ... Raumelement*, Leipzig: Teubner 1868.

Plücker, Julius, *Mathematische Abhandlungen*, (Schonflies, Arthur ed.), Leipzig: Teubner 1895.

Pockels, Friedrich, *Uber die partielle Differentialgleichung ... mathematischen Physik*, Leipzig: Teubner 1891.

Poincaré, Henri, *Théorie mathématique de la lumière*, Paris: Carré 1889.

Poincaré, Henri, *Les théories de Maxwell et la théorie électromagnétique de la lumière*, v. 1 Paris: Carré 1890.

Poncelet, Jean-Victor, *Applications d'analyse et de géométrie qui ont servi, en 1822, de principal fondement au Traité des proprietés projectives des figures*, 2 vols. Paris: Gauthier-Villars 1864.

Poncelet, Jean-Victor, *Traité des proprietés projectives des figures*, 2 vols. Paris: Gauthier-Villars 1865–1866.

Poncini, Giuseppe, *Elementi sul calcolo delle probabilità*, Milano: Hoepli 1885.

Porro, Francesco, *Astronomia sferica*, Roma: Soc. ed. Dante 1894.

Predella, Pilo, *Lezioni di aritmetica*,[493] Torino: Paravia 1911.

Predella, Pilo, *Lezioni di geometria*,[494] Torino: Paravia 1911.

Predella, Pilo, *Algebra ed aritmetica ad uso dei licei*, Torino: Paravia 1921.

Predella, Pilo, *Geometria ad uso dei licei*, Torino: Paravia 1921.

Predella Longhi, Lia, *Elementi di Algebra*, Torino: Paravia 1909^5.

Predella Longhi, Lia, *Aritmetica*, Torino: Paravia 1910^3.

Prym, Friedrich, *Theorie der Prymschen Funktionen erster Ordnung …*, Leipzig: Teubner 1911.

Raimondi, Emanuele, *Algebra elementare ...*, Napoli: Tip D'Auria 1898.

Ramorino, Angelo, *La Borsa: sua origine, suo funzionamento*,[495] Bari: Laterza 1913.

Rausenberger, Otto, *Lehrbuch der Theorie der periodischen Functionen einer Variabeln*, Leipzig: Teubner 1884.

Rausenberger, Otto, *Die Elementargeometrie des Punktes, der Geraden und der Ebene*, Leipzig: Teubner 1887.

Reidt, Friedrich, *Anleitung zum mathematischen Unterricht an höheren Schulen*, Berlin: Grote 1906.

Reye, Theodor, *Syntetische Geometrie der Kugeln und linearen Kugelsysteme* ..., Leipzig: Teubner 1879.

Reye, Theodor, *Die Geometrie der Lage*, vol. 1 Stuttgart: Kroner 1909, vol. 3 Stuttgart: Kroner 1910.

Rey Pastor, Julio, *Fundamentos de la Geometria Proyectiva Superior*, Madrid: n. p. 1916.

Riemann, Bernhard, *Gesammelte mathematische Werke ...*, (Weber, Heinrich and Dedekind, Richard eds.) Leipzig: Teubner 1876.

[493]This book is missing in BDMI Florence Toja Room.

[494]This book is missing in BDMI Florence Toja Room.

[495]This book is missing in BDMI Florence Toja Room.

Riemann, Bernhard and Weber, Heinrich, *Die Partiellen Differentialgleichungen der mathematischen Physik und deren Anwendung auf physikalische Fragen Vorlesungen*, Braunsweig: F. Vieweg und Sohn, 1876.

Riemann, Bernhard, *Schwere, Elektrizität und Magnetismus*, Hannover: Rümpler 1880.

Ripert, Leon, *La dualité et l'homographie dans le triangle et le tétraèdre*, Paris: Gauthier-Villars 1898.

Roiti, Antonio, *Elementi di Fisica*, 2 vols. Firenze: Le Monnier 1909, 1914.

Rossotti, Marco Aurelio, *Formulario scolastico di matematica elementare*, Milano: Hoepli 1899.

Rudio, Ferdinand (ed.), *Verhandlungen des Ersten Internationalen Mathematiker-Kongresses in Zürich vom 9 bis 11 August 1897*, Leipzig: Teubner, 1898.

Salmon, George, *Analytische Geometrie des Raumes*, (transl. W. Fiedler), 2 vols. Leipzig: Teubner 1874.

Salmon, George, *Vorlesungen über die Algebra der linearen Transformationen*, (transl. W. Fiedler), Leipzig: Teubner 1877.

Salmon, George, *Analytische Geometrie der Kegelschnitten*, (transl. W. Fiedler), Leipzig: Teubner 1878.

Sannia, Achille and D'Ovidio, Enrico, *Elementi di geometria*, 2 vols. Napoli: Pelleramo 1906^6, 1908^6.

Schell, Wilhelm, *Theorie der Bewegung und der Krafte*, 2 vols. Leipzig: Teubner 1879, 1880.

Schlesinger, Ludwig, *De nonnullis absolutae Geometriae ... Johannis Bolyai in memoriam*,[496] Leipzig: Teubner 1903.

Schönflies, Arthur, *Die Entwicklung der Lehre von der Punktmannigfaltigkeiten*, Leipzig: Teubner 1908.

Schotten, Heinrich, *Inhalt und Methode des planimetrischen Unterrichts*, 2 vols. Leipzig: Teubner 1890, 1893.

Schouten, Jan Arnoldus, *Mehrdimensionale Geometrie*, 2 vols. Leipzig: Göschen 1902, 1905.

Schröter, Heinrich, *Theorie der Oberflächen 2ter Ordnung und der Raumkurven 3ter Ordnung*, Leipzig: Teubner 1880.

Schröter, Heinrich, *Die Theorie der ebenen Kurven 3ter Ordnung*, Leipzig: Teubner 1888.

Schröter, Heinrich, *Grundzüge einer rein-geometrischen Theorie der Raumkurven 4 Ordnung*, Leipzig: Teubner 1890.

Schubert, Hermann, *Kalkül der abzählenden Geometrie*, Leipzig: Teubner 1879.

Schwarz, Hermann, *Versuch einer Philosophie der Mathematik*, Halle: Schmidt 1853.

Schwarz, Hermann, *Gesammelte mathematische Abhandlungen*, 2 vols. Berlin: Springer 1890.

Scorza, Gaetano, *Complementi di geometria*, vol. 1 Bari: Laterza 1914.

Scorza, Gaetano, *Corpi numerici e algebre*, Messina: Principato 1921.

[496]This book is missing in BDMI Florence Toja Room.

Serpieri, Alessandro, *Le misure assolute meccaniche, elettrostatiche ed elettromagnetiche*, Milano: Hoepli 1885.

Serret, Joseph Alfred, *Cours de Calcul différentiel et intégral*, 2 vols. Paris: Gauthier-Villars 1879, 1880.

Serret, Joseph Alfred, *Cours d'algèbre supérieure*, 2 vols. Paris: Gauthier-Villars 1885.

Serret, Joseph Alfred, *Trattato di Trigonometria*, Torino: Paravia 1886.

Serret, Paul, *Géométrie de direction*, Paris: Gauthier-Villars 1869.

Severi, Francesco, *Complementi di geometria proiettiva*, Bologna: Zanichelli 1906.

Severi, Francesco, *Lezioni di geometria algebrica*, Lit. Padova 1908.

Severi, Francesco, *Vorlesungen über algebraische Geometrie*, (transl. E. Löffler), Leipzig: Teubner 1921.

Severi, Francesco, *Geometria proiettiva*, Padova: La Litotipo 1922.

Siacci, Francesco, *Lezioni di meccanica razionale*, Torino: Tip. Lit. Giorgi 1892.

Sibirani, Filippo, *Elementi di geometria differenziale*, Milano: Hoepli 1924.

Simon, Max, *Didaktik und Methodik des Rechnens …*, Munchen: Beck 1908.

Staudt, Georg K. C. von, *Die Geometrie der Lage Beiträge zur Geometrie der Lage*, Nürnberg: Korn 1847.

Staudt, Georg C. von, *Geometria di posizione*, (transl. M. Pieri), Torino: Bocca 1889.

Steiner, Jacob, *Vorlesungen über synthetische Geometrie.* 1. *Die Theorie der Kegelschnitte in elementarer Darstellung*, (Carl F. Geiser ed.), Leipzig: Teubner 1875.

Steiner, Jacob, *Die Theorie der Kegelschnitte gestützt auf projectivische Eigenschaften*, (Heinrich Schröter ed.), Leipzig: Teubner 1876.

Steiner, Jacob, *Gesammelte Werke*, 2 vols. Berlin: Reimer 1881–1882.

Stolz, Otto, *Vorlesungen über allgemeine Arithmetik*, vol. 1 Leipzig: Teubner 1885.

Struik, Dirk Jan, *Grundzüge der mehrdimensionalen Differentialgeometrie*, Berlin: Springer 1922.

Study, Eduard, *Methoden zur Theorie der ternären Formen*, Leipzig: Teubner 1889.

Study, Eduard, *Geometrie der Dynamen…*, Leipzig: Teubner 1903.

Study, Eduard, *Vorlesungen über ausgewählte Gegenstände der Geometri*e, Leipzig: Teubner 1911–1913.

Study, Eduard, *Die realistische Weltansicht und die Lehre von Raume*, Braunschweig: Vieweg 1914.

Study, Eduard, *Einleitung in die Theorie der Invarianten linearer Transformationen auf Grund der Vektorenrechnung*, Braunschweig: Vieweg 1923.

Sturm, Rudolf, *Synthetische Untersuchungen über Flächen 3 Ordnung*,[497] Leipzig: Teubner 1867.

[497]This book is bound with that by Cremona, Luigi, *Grundzüge einer allgemeinen Theorie der Oberflächen*, 1870.

Sturm, Rudolf, *Die Lehre von den geometrischen Verwandtschaften*, 4 vols. Leipzig: Teubner 1908–1909.

Sturm, Rudolf, *Die Gebilde ersten und zweiten Grades der Liniengeometrie in synthetischer Behandlung*, 3 vols. Leipzig: Teubner 1892–1896.

Timerding, Heinrich E., *Geometrie der Kräfte*, Leipzig: Teubner 1908.

Tisserand, François Félix, *Recueil complémentaire d'exercices sur le calcul infinitésimal*, Paris: Gauthier-Villars 1877.

Tomaselli, Giulio, *Esercizii sulle equazioni differenziali*, Milano: Hoepli 1883.

Vailati, Giovanni, *Scritti 1863–1909*, Leipzig: Barth, Firenze: Seeber 1911.

Vandermonde, Alexandre T., *Abhandlungen aus der reinen Mathematik*, Berlin: Springer 1888.

Veblen, Oswald and Young, John Wesley, *Projective Geometry*, vol. 1 New York: Ginn & comp. 1910.

Veneroni, Emilio, *Le similitudini*, Livorno: Giusti 1916.

Veronese, Giuseppe, *Fondamenti di geometria a più dimensioni*, Padova: Tip. Seminario 1891.

Veronese, Giuseppe, and Gazzaniga, Paolo, *Elementi di Geometria*, Padova: Drucker 1901.

Veronese, Giuseppe, and Gazzaniga, Paolo, *Nozioni elementari di geometria intuitiva ad uso dei ginnasi*, Padova: Drucker 1901.

Veronese, Giuseppe, and Gazzaniga, Paolo, *Nozioni di geometria intuitiva per le scuole complementari*, Verona, Padova: Drucker 1908.

Veronese, Giuseppe, and Gazzaniga, Paolo, *Elementi di geometria intuitiva ad uso delle scuole tecniche*, Padova: Drucker 1909.

Vleck, Edward B. van and White, Henry S. and Woods, Frederick S. (eds.), *The Boston Colloquium Lectures on Mathematics*, New York: Macmillan 1905.

Volterra, Vito, *Lezioni di Meccanica: prime nozioni di Cinematica*, Livorno: Giusti 1896.

Weber, Heinrich, *Lehrbuch der Algebra*, 2 vols. Braunschweig: Vieweg 1898–1899.

Weierstrass, Karl, *Formeln und Lehrsätze zum Gebrauche der elliptischen Funktionen*, (Schwarz, Hermann ed.), Göttingen: Dieterichsche Universitäts-Buchdruckerei 1885.

Weierstrass, Karl, *Mathematische Werke*: B. *Vorlesungen über die Theorie der Abelschen Transcendenten*, (Knoblauch, Johannes and Hettner, Georg and Rothe, Rudolf E. eds.), Berlin: Mayer & Müller, 1902.

Weyr, Emil, *Beiträge zur Kurvenlehre*,[498] Wien: n. p. 1880.

Weyr, Emil, *Elemente der projectivischen Geometrie*, Wien: Braunmüller 1883–1887.

Weyrauch, Jacob, *Theorie elastischer Körper*, Leipzig: Teubner 1884.

Wilczynski, Ernest, *Projective Differential Geometry of Curves and Ruled Surfaces*, Leipzig: Teubner 1906.

[498]This work is missing in BDMI Florence Toja Room.

Young, William Henri, *The Fundamental Theorems of Differential Calculus*, Cambridge: CUP 1910.

Zanotti-Bianco, Ottavio, *Storia popolare dell'astronomia*, Torino: Sten, 1913.

Zeuthen, Hieronymus G., *Grundriss einer elementar-geometrischen Kegelschnittslehre*, Leipzig: Teubner 1882.

Zeuthen, Hieronymus G., *Die Lehre von den Kegelschnitten im Altertum*, Kopenhagen: Host 1886.

Zeuthen, Hieronymus G., *Histoire des mathématiques dans l'antiquitè et le Moyen Age*, (transl. J. Mascart), Paris: Gauthier-Villars 1902.

Zeuthen, Hieronymus G., *Lehrbuch der abzahlenden Methoden der Geometrie*, Leipzig: Teubner 1914.

Zindler, Konrad, *Liniengeometrie mit Anwendungen*, 2 vols. Berlin: De Gruyter 1902–1906.

Encyklopädie der mathematischen Wissenschaften

Mathematische Annalen, vols. 1–77

Rendiconti del Circolo Matematico di Palermo, vols. 1–47 and *Supplementi*

Esercitazioni matematiche Circolo matematico di Catania

Jahresbericht der Deutschen Mathematiker Vereinigung, vols. 5–23

Rivista di Matematica e Fisica

Bollettino dell'Unione Matematica Italiana

Bulletin des Sciences Mathématiques et Astronomiques (2) 8 (1884)

Rendiconti dell'Accademia Nazionale dei Lincei

Annali di Matematica (2) vols. 10–26; (3) vols. 1–31; (4) Indice

Rivista di Matematica (G. Peano ed.) vols. 1–3 and some files

Journal für reine und angewandte Mathematik (*Crelle's Journal*) vols. 100–120

Bollettino di Bibliografia e Storia delle Scienze matematiche

Bertini, Eugenio, Memorie varie [BDMI: ML 35]

Bianchi, Luigi, Memorie varie, 2 vols.

Brill, Alexander von and Nöther, Max, Memorie varie [BDMI: ML 49]

Caporali, Ettore, Memorie di geometria [BDMI: ML 36]

Castelnuovo, Guido, Memorie varie [BDMI: ML 37]

Castelnuovo, Guido, and Enriques, Federigo, Memorie varie [BDMI: ML 38]

Cremona, Luigi, Memorie varie [BDMI: ML 39]

De Paolis, Riccardo, Memorie [BDMI: ML 40]

Engel, Friedrich and Mayer, Adolf, Memorie varie [BDMI: ML 50]

Fricke, Robert and Ritter, Ernst, Memorie varie [BDMI: ML 51]

Halphen, Georges-Henri, Deux Mémoires [BDMI: ML 53]

Hurwitz, Adolf, Memorie varie [BDMI: ML 54]

Klein, Felix, Memorie varie [BDMI: ML 52]

Kronecker, Leopold Memorie varie, 2 vols. [BDMI: ML 55, ML 56]

Loria, Gino, Memorie varie, 2 vols. [BDMI: ML 41, ML 42]

Montesano, Domenico and Pieri, Mario, Memorie [BDMI: ML 43]

Pascal, Ernesto, Memorie varie [BDMI: ML 44]

Pincherle, Salvatore, Memorie varie [BDMI: ML 45]

Rohn, Karl and Schur, Friedrich, Memorie varie [BDMI: ML 57]

Segre, Corrado, Memorie varie, 2 vols. [BDMI: ML 46, ML 47]
Study, Eduard, Memorie varie [BDMI: ML 58]
Volterra, Vito, Memorie varie [BDMI: ML 48]
Voss, Aurel, Memorie varie [BDMI: ML 59]
Miscellanea [di vari autori], 3 vols. [BDMI: ML 60, ML 61, ML 62]

References

Albers, Donald J. and Alexanderson Gerald L. and Reid, Constance, *International Mathematical Congresses. An Illustrated History 1893–1986*, New York: Springer, 1986.

Amodeo, Federico, Coup d'oeil sur les courbes algébriques au point de vue de la gonalité, in *Compte Rendu du deuxième Congrès International des mathématiciens tenu à Paris du 6 au 12 aout 1900*, Duporcq, Ernest (ed.), Paris: Gauthier-Villars, 1902: 313–326.

AMS *Kansas Section of the Mathematical Association of America 65 Years (1915–1980), February 1980*, prepared by Elaine L. Tatham, Kansas: Johnson County Community College, digitized and reformatted from the original by Tim Flood, Kansas Section Historian, Pittsburg: State University, 2013.

Arrighi, Gino (ed.), *Lettere a Mario Pieri (1884–1913)*, Milano: Quaderni Pristem 6, 1997.

Aubin, David and Goldstein, Catherine, *The War of Guns and Mathematics*, Providence: AMS, History of Mathematics 42, 2014.

Ausejo, Elena and Hormigòn, Mariano (eds.), *Messengers of Mathematics: European Mathematical Journals, 1800–1946*, Madrid: Siglo XXI de Espana Ed., 1993.

Babbit, Donald, and Goodstein, Judith, Guido Castelnuovo and Francesco Severi: Two Personalities, Two Letters, *Notices of the AMS* 56 (7), (2009): 800–808.

Baker, Henry F., Corrado Segre, *Journ. London Math. Soc.* 1 (1926): 263–271, trad. it. G. Loria, *Boll. U.M.I.* 6 (1927): 276–284.

Battaglini, Giuseppe, Sur la géométrie imaginaire de Lobatchefsky, *Nouv. Ann.* (2) VII (1868): 209–265.

Becvárová, Martina and Becvár, Jindrich, Emil Weyr e Luigi Cremona, *Boll. Sto. Sci. Mat.* XXVI, 2 (2006): 1–17.

Beltrami, Eugenio, Zur Theorie des Krümmungsmasses, *Math. Ann*. I (1869): 575–582.

Beltrami, Eugenio, Formules fondamentales de cinématique dans les espaces de courbure constante, *Bull. Sci. Math. Darboux* XI (1877): 233.

Bertini, Eugenio, Système simultané de deux formes biquadratiques, *Math. Ann*. XI (1877): 30–41.

Bertini, Eugenio, Trasformazioni univoche involutorie nel piano, *Bull. Sci. Math. Darboux* (2) II (1878): 212–213.

Bertini, Eugenio, La geometria delle serie lineari sopra una curva piana secondo il metodo algebrico, *Ann. Mat.* (2) XXII (1894): 1–40.

Bertini, Eugenio, Introduzione alla geometria projettiva degli iperspazi con appendice sulle curve algebriche e loro singolarità, Pisa: Spoerri, 1907.

Berzolari, Luigi, Max Noether, *Rend. Ist. Lomb.* (2) 54 (1921): 600–603.

Berzolari, Luigi, Corrado Segre, *Rend. Ist. Lomb.* (2) 57 (1924): 528–532.

Bianchi, Luigi, Über die Normalformen dritter und fünfter Stufe des elliptischen Integrals erster Gattung, *Math. Ann*. XVII (1880a): 234–262.

Bianchi, Luigi, Über die Flächen mit constanter negativer Krümmung, *Math. Ann.* XVI (1880b): 577–582.

Boffi, Guido, On Some Trends in the Italian Geometric School in the Second Half of the 19th Century, *Rivista di Storia della Scienza*, 3 (1986): 103–112.

Boggio, Tommaso, Nel 4° anniversario della morte di Corrado Segre, *Atti Acc. Sci. Torino* 63 (1928): 303–320.

Boi, Luciano, The Influence of the Erlangen Program on Italian Geometry 1880–1890: *n*-dimensional Geometry in the Works of D'Ovidio, Segre and Fano, *Arch. Int. Hist. Sciences* 40 (1990): 30-75.

Bompiani, Enrico, Recenti progressi nella geometria proiettiva differenziale degli iperspazi, in *Proceedings of the Fifth International Congress of Mathematicians*, *Cambridge 22–28 August 1912*, Hobson, Ernest W. and Love, Augustus E.H. (eds.), Cambridge: Univ. Press, v. 2, 1913: 22-27.

Borel Emile, Congrès International des Mathématiciens, Première session: Zurich, Aout 1897, *Rev. Gén. Sci. Pures et Appliquées* 8 (1897): 783–789.

Borga, Marco, Alle origini delle ricerche metamatematiche: coerenza e indipendenza fra Ottocento e Novecento, *Epistemologia* 28 (2005): 3–24.

Borga, Marco, Su alcuni contributi di Alessandro Padoa e Mario Pieri ai Fondamenti della Geometria, *Epistemologia* 34 (2011): 89–114.

Borga, Marco and Fenaroli, Giuseppina and Garibaldi, Antonio Carlo, Un inedito di Alessandro Padoa, *Epistemologia* 32 (2009): 233–254.

Bottazzini, Umberto, I geometri italiani e la geometria algebrica "astratta", in *Storia della scienza moderna e contemporanea* (dir. Paolo Rossi), Torino: UTET, v. III, 1988: 173–195.

Bottazzini, Umberto, *Va' pensiero. Immagini della matematica nell'Italia dell'Ottocento*, Bologna: Il Mulino, 1994.

Bottazzini, Umberto, Brioschi e gli Annali di Matematica, in *Francesco Brioschi e il suo tempo (1824–1897)*, v. I, *Saggi*, Lacaita, Carlo G. and Silvestri, Andrea (eds.), Milano: Franco Angeli, 2000: 71–84.

Bottazzini, Umberto, I geometri italiani e il problema dei fondamenti (1888–1899), *Boll. UMI, La Matematica nella Società e nella Cultura* (8) 4-A (2001): 281–329.

Bottazzini, Umberto and Dahan Dalmedico, Amy (eds.), *Changing Images of Mathematics. From the French Revolution to the New Millenium*, London: Routledge, 2001.

Bottazzini, Umberto and Conte, Alberto and Gario, Paola, (eds.), *Riposte armonie Lettere di Federigo Enriques a Guido Castelnuovo*, Torino: Bollati Boringhieri, 1996.

Brianta, Donata, *Europa mineraria: Circolazione delle élites e trasferimento tecnologico (secoli XVIII-XIX)*, Milano: Franco Angeli, 2007.

Brigaglia, Aldo, The Creation and Persistence of National Schools: the Case of Italian Algebraic Geometry, in *Changing Images of Mathematics. From the French Revolution to the New Millenium*, Bottazzini, Umberto and Dahan Dalmedico, Amy (eds.), London: Routledge, 2001, 187–206.

Brigaglia, Aldo, The *Annali di Matematica* and the *Rendiconti del Circolo Matematico di Palermo*: Two Different Steps in the Dissemination and Progress of Mathematics in Italy, in Gerini, Christian and Verdier, Norbert (eds.) *L'émergence de la Presse Mathématique en Europe au 19ème Siècle*, Milton Keynes: Lightning Source 2014: 155–178.

Brigaglia, Aldo, Segre and the Foundations of Geometry: From Complex Projective Geometry to Dual Numbers, in this volume.

Brigaglia, Aldo and Ciliberto, Ciro, *Italian Algebraic Geometry Between the Two World Wars*, Kingston (Canada): Queen's University 1995: 12–20.

Brigaglia, Aldo and Ciliberto, Ciro, Remarks on the Relations between the Italian and American Schools of Algebraic Geometry in the First Decades of the 20th Century, *Historia Mathematica* 31 (2004): 310–319.

Brigaglia, Aldo and Ciliberto, Ciro and Pedrini, Claudio, The Italian School of Algebraic Geometry and Abel's Legacy in *The Legacy of Niels Henrik Abel*, *The Abel Bicentennial, Oslo 2002*, Laudal, Olav A. and Piene, Ragni (eds.), Berlin: Springer, 2004: 295–347.

Brill, Alexander von and Nöther, Max, *Die Entwicklung der Theorie der algebraische Funktionen in älterer und neuerer Zeit*, Berlin: Reimer 1894.

Brioschi, Francesco, Additions à l'article N° 15, page 239 de ce tome, *J. Reine Angew. Math.* 50 (1855a): 318–321.

Brioschi, Francesco, Sur quelques questions de la géométrie de position, *J. Reine Angew. Math.* 50 (1855b): 233–238.
Brioschi, Francesco, Sur deux formules relatives à la théorie de la décomposition de fractions rationnelles, *J. Reine Angew. Math.* 50 (1855c): 239–242.
Brioschi, Francesco, Sur l'analogie entre une classe de déterminants d'ordre pair et sur les déterminants binaires, *J. Reine Angew. Math.* 52 (1856): 133–141.
Brioschi, Francesco, Sur une nouvelle propriété du résultant de deux équations algébriques, *J. Reine Angew. Math.* 53 (1857a): 372–376.
Brioschi, Francesco, Sur une formule de M. Cayley, *J. Reine Angew. Math.* 53 (1857b): 377–378.
Brioschi, Francesco, Sur l'intégration des équations ultra-elliptiques, *J. Reine Angew. Math.* 55 (1858): 56–60.
Brioschi, Francesco, Développements relatifs au S^3 des Recherches de Dirichlet sur un problème d'Hydrodynamique, vol. 58, pag. 181 et suivantes de ce Journal, *J. Reine Angew. Math.* 59 (1861): 63–73.
Brioschi, Francesco, Note de M. Brioschi relative à la lettre précédente. (Bd. 63), *J. Reine Angew. Math.* 63 (1864): 32–33.
Brioschi, Francesco, Des substitutions de la forme … pour un nombre premier de lettres, *Gött. Nachr* (1869), *Math. Ann.* II (1870): 467–470.
Brioschi, Francesco, Les tangentes doubles à une courbe du quatrième ordre avec un point double, *Math. Ann.* IV (1871): 95–98.
Brioschi, Francesco, La théorie des formes dans l'intégration des équations différentielles linéaires du second ordre, *Math. Ann.* XI (1877a): 401–412.
Brioschi, Francesco, Lettre à M. F. Klein, *Math. Ann*. XI (1877b): 111–113.
Brioschi, Francesco, Über die Auflösung der Gleichungen vom fünften Grade, *Math. Ann*. XIII (1878): 109–160.
Brioschi, Francesco, Sur les équations différentielles linéaires, *Bull. Soc. Math. Fr.* VII (1879a): 105–108.
Brioschi, Francesco, Über die Jacobi'sche Modulargleichung vom achten Grad., *Math. Ann.* XV (1879b): 241–250.
Bru, Bernard and Dugac, Pierre (eds.), *H. Lebesgue. Les lendemains de l'intégrale. Lettres à Émile Borel*, Paris: Vuibert, 2004.
Burali-Forti, Cesare, Postulats pour la géométrie d'Euclide et de Lobatschewsky, in *Verhandlungen des Ersten Internationalen Mathematiker-Kongresses in Zürich vom 9 bis 11 August 1897*, Rudio, Ferdinand (ed.), Leipzig: Teubner, 1898, 247–250.
Castelnuovo, Guido, Max Noether, *Rend. R. Acc. Naz. Lincei* (5), 31 (1922): 404–407.
Castelnuovo, Guido and Corrado Segre, *Rend. R. Acc. Naz. Lincei* (5), 33 (1924): 353–359.
Castelnuovo, Guido, Cenni necrologici sul socio C. Segre, *Rend. R. Acc. Naz. Lincei* (5), 33 (1924): 460.
Castelnuovo, Guido, La geometria algebrica e la scuola italiana, in *Atti del Congresso Internazionale dei Matematici (Bologna 3–10 settembre 1928*), Bologna: Zanichelli, v. 1, 1929: 191-201.
Castelnuovo, Guido, Luigi Cremona nel centenario della nascita, *Rend. R. Acc. Naz. Lincei* (6), 12 (1930): 613–618.
Castelnuovo, Guido and Enriques, Federigo, Sur quelques récents résultats dans la théorie des surfaces algèbriques, *Math. Ann.* 48 (1896): 241–316.
Castelnuovo, Guido and Enriques, Federigo and Severi, Francesco, Max Noether, *Math. Ann.* 93 (1925): 161–181.
Chisholm, Grace, Sulla varietà razionale normale di M_3^4 di S_6 rappresentante della trigonometria sferica, *Atti Acc. Sci. Torino* 34 (1899): 587–596.
Ciliberto, Ciro and Sallent Del Colombo, Emma, Enrico Bompiani: the years in Bologna, in *Mathematicians in Bologna 1861–1960*, Coen, Salvatore (ed.), Basel: Springer 2012: 143-177.
Clebsch, Alfred, *Vorlesungen über Geometrie*, (Lindemann, F. ed.). Leipzig: Teubner 1876.

Conforto, Fabio, Geometria algebrica, in *Un secolo di progresso scientifico italiano 1839–1939*, vol. I, Roma: SIPS 1939: 125-153.

Conte, Alberto, Corrado Segre, in *L'Università di Torino. Profilo storico e istituzionale*, Traniello, Francesco (ed.), Torino: Pluriverso 1993: 437–439.

Conte, Alberto and Giacardi, Livia and Novaria, Paola, *Corrado Segre (1863–1924) a 150 anni dalla nascita. Catalogo delle Mostre documentarie - Novembre 2013*, Torino: KWB 2013.

Conte, Alberto and Giacardi, Livia, *Segre's University Courses and the Blossoming of the Italian School of Algebraic Geometry* in this volume, 2016.

Conte, Alberto and Giacardi, Livia (eds.) *Corrado Segre 1890-91 Introduzione alla geometria sugli enti algebrici semplicemente infiniti, 1903-1921 Lezioni per la Scuola di Magistero*, currently underway, 2017.

Coolidge, Julian, Les congruences isotropes qui servent à répresenter les fonctions d'une variable complexe, *Atti Acc. Sci. Torino* 39 (1904a): 175–186.

Coolidge, Julian, The opportunities for mathematical study in Italy, *Bulletin of the AMS* (2) 11 (1904b): 9–17.

Coolidge, Julian, Les congruences isotropes qui servent à répresenter les fonctions d'une variable complexe, *Atti Acc. Sci. Torino* 40 (1905): 202–218.

Coolidge Julian, Corrado Segre, *Bulletin of the AMS* 34 (1927): 352–357.

Coolidge Julian, *A History of Geometrical Methods*, Oxford: Clarendom Press 1940, 1947[2].

Cremona, Luigi, Sur quelques propriétés des lignes gauches de troisième ordre et classe, *J. Reine Angew. Math.* 58 (1861): 138–151.

Cremona, Luigi, Notes sur les cubiques gauches, *J. Reine Angew. Math.* 60 (1862a): 188–192.

Cremona, Luigi, Sur les surfaces gauches du troisième degré, *J. Reine Angew. Math.* 60 (1862b): 313–320.

Cremona, Luigi, Sur les hyperboloïdes de rotation qui passent par une cubique gauche donnée, *J. Reine Angew. Math.* 63 (1864a): 141–144.

Cremona, Luigi, Sur la surface du quatrième ordre qui a la propriété d'être coupée suivant deux coniques par chacun de ses plans tangents, *J. Reine Angew. Math.* 63 (1864b): 315–328.

Cremona, Luigi, Sur l'hypocycloïde à trois rebroussements, *J. Reine Angew. Math.* 64 (1865): 101–123.

Cremona, Luigi, Mémoire de géométrie pure sur les surfaces du troisième ordre, *J. Reine Angew. Math.* 68 (1868): 1–133.

Cremona, Luigi, Über die Abbildung algebraischer Flächen, Gött. Nachr. (1871): 129–148, *Math. Ann.* IV (1871a): 213–230.

Cremona, Luigi, Observations géométriques à propos de la Note de Mr. Brioschi Sur les tangentes doubles d'une courbe du 4[e] ordre avec un point double, *Math. Ann.* IV (1871b): 99–102.

Cremona, Luigi, Des transformations rationelles dans l'espace, *Bull. Sci. Math. Darboux* VII (1876): 37–48.

Cremona, Luigi, Über die Polarhexaeder bei den Flächen dritter Ordnung, *Math. Ann.* XIII (1878): 301–303.

Curbera, Guillermo, *Mathematicians of the World, Unite! The International Congress of Mathematicians: a Human Endeavor*, Wellesley (Massachusetts): AK Peters, 2009.

D'Ovidio, Enrico, Les fonctions métriques fondamentales dans un espace de plusieurs dimensions et de courbure constante, *Math. Ann*. XII (1877): 403–419.

Dauben, Joseph W., Julian Lowell Coolidge, *American National Biography* 4 (1999): 424–425.

De Benedetti, Andrea, Esplorazioni e viaggi nei ricordi di nonno Ghigo, in *Le città di mare e lo spirito scientifico. Per Federigo Enriques*, Pompeo Faracovi, Ornella (ed.), La Spezia: Agorà, 2001: 54–70.

De Donno, Stefania, La Biblioteca dell'Istituto Nazionale delle Assicurazioni, *Le Carte e la Storia* 2 (1988): 97–100.

De Paolis, Riccardo, Le trasformazioni piane doppie, *Bull. Sci. Math. Darboux* II (1878): 209–211.

Dhombres, Jean, Vicissitudes in Internationalisation: International Networks in Mathematics up until the 1920s, in *Transnational Intellectual Networks. Forms of Academic Knowledge and* *the Search for Cultural Identities*, Charle, Christophe and Schriewer, Jürgen and Wagner, Peter (eds.), Chicago: University Press, 2004: 81–114;

Dhondt, Pieter, Foreign Students at Belgian Universities. A Statistical and Bibliographical Approach, *Revue belge d'Histoire contemporaine / Belgisch Tijdschrift voor Nieuwste Geschiedenis* 38 (2008): 5–44.

Dröscher, Ariane, Die Auslandsstipendien der italienischen Regierung (1861–1894), *Annali dell'Istituto storico italo-germanico in Trento* XVIII (1992): 545–569.

Dugac, Pierre (ed.), Lettres de René Baire à Emile Borel, *Cahiers du séminaire d'histoire des mathématiques*, 11 (1990): 33–120.

Duren, Peter *et alii* (eds.), *A Century of Mathematics in America*, 3 vols., Providence: AMS, 1988–1989.

Enriques, Federigo, Sur les problèmes qui se rapportent à la résolution des équations algébriques renferment plusieurs inconnues, in *Verhandlungen des Ersten Internationalen Mathematiker-Kongresses in Zürich vom 9 bis 11 August 1897*, Ferdinand Rudio (ed.), Leipzig: Teubner, 1898, 145–146.

Enriques, Federigo, Sur quelques résultats nouveaux dans la théorie des surfaces algébriques, in Picard, Emile and Simart, Georg (eds.), *Théorie des fonctions algébriques de deux variables indépendantes*, Paris: Gauthier-Villars, vol. 2, 1906: 485–524.

Enriques, Federigo, Matematiche e filosofia, in *Atti del IV Congresso Internazionale dei Matematici (Roma 6–11 aprile 1908)*, Castelnuovo, Guido (ed.), Roma: Tip. R. Acc. Lincei, vol. 3, 1909: 373–376.

Enriques, Federigo, Il significato della critica dei principi nello sviluppo delle matematiche, in *Proceedings of the Fifth International Congress of Mathematicians*, *Cambridge 22–28 August 1912*, Hobson, Ernest W. and Love, Augustus E.H. (eds.), Cambridge: Univ. Press, v. 1, 1913: 67–79.

Enriques, Federigo, La evolucion del concepto de la geometria y la escuela italiana durante los ultimos cincuenta años, *Revista Mat. Hispano Americana* 2 (1920): 1–17.

Enriques, Federigo, Sur la théorie des équations et des fonctions algébriques d'apres l'Ecole géométrique italienne, *L'Ens. Math.* 23 (1924): 309–322.

Enriques, Federigo, L'Italia nella collaborazione universale della cultura, *Nuova Antologia* (7), 247 (1926): 129–134.

Enriques, Federigo, Arti e studi in Italia nell'ultimo venticinquennio. Gli studi matematici, *Leonardo* IV, n. 5–6 (1928): 132–141.

Enriques, Federigo, Programma del corso sulla Teoria delle superficie algebriche a.a. 1927–28 e 1928-29, *Rendiconti del Seminario matematico dell'Università di Roma* (2) XI, (1929): 107–109.

Enriques, Federigo, Superficie algebriche, in *Enciclopedia Italiana*, vol. XXXIII, 1937: 9–12.

Enriques, Federigo and Campedelli, Luigi, *Lezioni sulla teoria delle superficie algebriche*, Padova: Cedam, 1932–34.

Enriques, Federigo and Chisini, Oscar, *Lezioni sulla teoria geometrica delle equazioni e delle funzioni algebriche*, Bologna: Zanichelli, 1915–34.

Enriques De Benedetti Adriana, Ricordi del Babbo, *Le città di mare e lo spirito scientifico. Per Federigo Enriques*, Pompeo Faracovi, Ornella (ed.), La Spezia: Agorà, 2001: 71–79.

Enriques Giovanni, *Via D'Azeglio 57*, Bologna: Zanichelli, 1983.

Fano, Gino, Über Gruppen, insbesondere continuierliche Gruppen von Cremona Transformationen der Ebene und des Raumes, in *Verhandlungen des Ersten Internationalen Mathematiker-Kongresses in Zürich vom 9 bis 11 August 1897*, Ferdinand Rudio (ed.), Leipzig: Teubner, 1898: 251–255.

Fano, Gino, *A preface to a series of special lectures on Italian Geometry, and 2 general lectures. Intuition in mathematics. All geometry is theory of Relativity*, Aberystwith: University of Aberystwith, 1923.

Fano, Gino, Cenni necrologici del socio C. Segre, *Rend. R. Acc. Naz. Lincei* (5), 33 (1924): 460.

Fano, Gino, Corrado Segre. Cenno necrologico, *Annuario R. Università di Torino* 12 (1924–25): 219–228;

Fano, Gino, [*Appunti vari*], BMP-Fano, *Scritti* 4, available in Giacardi 2013.

Fehr, Henri, Corrado Segre, *L'Ens. Math.* 23 (1923): 332.

Ferraresi, Alessandra and Signori, Elisa (eds.), *Le Università e l'Unità d'Italia*, Bologna: CISUI, 2012.

Fitch, Philip, The twelfth Annual Meeting of the Rochy Mountain Section, *The American Mathematical Montly*, 35, 7 (1928): 331–334.

Frajese, Attilio (but Enriques, Federigo), *Le Matematiche nella Storia e nella Cultura*, Bologna: Zanichelli, 1938.

Franklin, Philip, Clarence Lemuel Elisha Moore (1876–1931), *Proceedings of the American Academy of Arts and Sciences* 67, 13 (1933): 606–608.

Fuà Segre, Elena, Un grande geometra ebreo: Corrado Segre, *La rassegna mensile di Israel*, 18 (1952): 125–127.

Galuzzi, Massimo, Geometria algebrica e Logica tra Otto e Novecento, in *Storia d'Italia, Annali 3,* Micheli, Gianni (ed.), Torino: Einaudi, 1980: 1003–1105.

Gario, Paola, Su alcune carte di Corrado Segre recentemente rinvenute, *Atti Acc. Sci. Torino* 123 (1989): 187–198.

Gario, Paola, Singolarità e Geometria sopra una Superficie nella Corrispondenza di C. Segre a G. Castelnuovo, *Arch. Hist. Exact Sci.* 43 (1991), 145–188.

Gario, Paola. Corrado Segre e i suoi allievi, in *Conferenze e seminari 1996–1997,* Gallo, Elisa and Giacardi, Livia and Roero, Clara Silvia (eds.), Torino: *Ass. Sub. Mathesis*, 1997: 133–148.

Gerbaldi, Francesco, Sul gruppo semplice di 360 collineazioni piane, in *Verhandlungen des Ersten Internationalen Mathematiker-Kongresses in Zürich vom 9 bis 11 August 1897*, Rudio, Ferdinand (ed.), Leipzig: Teubner, 1898: 242–246.

Ghione, Franco and Ottaviani, Giorgio, A Tribute to Corrado Segre, Complex Projective Geometry, *London Math. Soc. Lecture Note* 179 (1992): 175–188.

Giacardi, Livia, Corrado Segre maestro a Torino. La nascita della scuola italiana di geometria algebrica, *Annali di storia delle università italiane* 5 (2001a): 139–163.

Giacardi, Livia, The Corrado Segre Archive, *Historia Mathematica* 28 (2001b): 296–301.

Giacardi, Livia, Il magistero di Corrado Segre a Torino. I quaderni manoscritti delle lezioni universitarie (1888–1924), in *Manuales de enseñanza en la universidad liberal, VII Congreso Internacional sobre la historia de las Universidades Hispánicas*, Biblioteca del Instituto Antonio de Nebrija de estudios sobre la universidad, 13, 2004: 449–476.

Giacardi, Livia, Corrado Segre e le 'orge geometriche' torinesi di fine Ottocento, *Annali del Centro Pannunzio*, 2006–2007: 243–272.

Giacardi, Livia (ed.), Corrado Segre e la Scuola Italiana di Geometria Algebrica, web site http://www.corradosegre.unito.it/docente.php, 2013.

Giacardi, Livia and Furinghetti, Fulvia (eds.), *The First Century of the International Commission on Mathematical Instruction (1908–2008)*, website http://www.icmihistory.unito.it, 2008.

Giacardi, Livia and Roero, Clara Silvia, Biblioteca Speciale di matematica "G. Peano", in *La Facoltà di Scienze Matematiche Fisiche Naturali di Torino 1848–1998*, Roero, Clara Silvia (ed.), v. 1, *Ricerca, Insegnamento, Collezioni scientifiche*, Torino: DSSP, 1999: 437–458.

Giacardi, Livia and Varetto, Tiziana, Il Fondo Corrado Segre della Biblioteca 'G. Peano' di Torino, *Quaderni di Storia dell'Università di Torino* 1 (1996): 337–370.

Gispert, Hélène, The German and French Editions of the Klein-Molk Encyclopedia: Contrasted Images in *Changing Images of Mathematics. From the French Revolution to the New Millenium*, Bottazzini, Umberto and Dahan Dalmedico, Amy (eds.), London: Routledge, 2001: 93–112.

Goldstein, Catherine and Gray, Jeremy and Ritter, Jim (eds.), *L'Europe mathématique: histoires, mythes, identités*, Paris: EMSH, 1996.

Goodstein, Judith and Babbitt, Donald, A Fresh Look at Francesco Severi, *Notices of the AMS* 59, n. 8 (2012), 1064–1075.

Gouzévitch, Irina and Gouzévitch, Dmitri, Les étudiants d'Europe de l'Est à l'Institut électrotechnique de Nancy (1900–1939), in Birck, Françoise and Grelon, André (eds.), *Un siècle de formation des ingénieurs électriciens, ancrage local et dynamique européenne: l'exemple de Nancy (actes du colloque organisé en 2001 à Nancy à l'occasion du centenaire de l'ENSEM)*, Paris: EMSH, 2006: 271–319.

Gouzévitch, Irina and Gouzévitch, Dmitri, Contacts franco-russes dans le monde de l'enseignement supérieur technique et de l'art de l'ingenieur, *Les Cahiers du Monde russe et soviétique (Moscú)* 3, XXXIV (1993): 345–368.

Gouzévitch, Irina and Gouzévitch, Dmitri, Etudiants, savants et ingénieurs juifs originaires de l'Empire Russe en France (1860–1940), *Les Belles lettres Archives Juives* 35 (2002): 120–128.

Grattan-Guinness, Ivor, A Mathematical Union: William Henry and Grace Chisholm Young, *Annals of Science* 29 (1972): 105–185; *Un'unione matematica William Henry Young e Grace Chisholm Young*, in *Scienza a due voci*, Simili, Raffaella (ed.), Firenze: Olschki, 2006: 217–246.

Gray, Jeremy, German and Italian Algebraic Geometry, Studies in the History of Modern Mathematics, *Rendiconti del Circolo matematico di Palermo* (2) 36 (1994): 151–184.

Guerraggio, Angelo and Nastasi, Pietro, *Italian Mathematics between the Two World Wars*, Basel: Birkhauser-Verlag, 2005.

Guerraggio, Angelo and Nastasi, Pietro, *Roma 1908: il congresso internazionale dei matematici,* Torino: Bollati Boringhieri, 2008.

Hammond, Julian Lowell Coolidge, *Harvard University Gazette* (26 February 1955): 136–138.

Hawkins, Thomas, *The Emergence of the Theory of Lie Groups. An Essay in the History of Mathematics 1869–1926*, New York: Springer, 2000.

Heims, Steve *J. John Von Neumann and Norbert Wiener: from Mathematics to the Technologies of Life and Death*, Cambridge Massachusetts: MIT Press, 1980.

Israel, Giorgio, È esistita una 'Scienza Ebraica' in Italia?, in *Cultura ebraica e cultura scientifica in Italia*, Di Meo, Angelo (ed.), Roma: Editori Riuniti, 1994: 29–52.

Israel, Giorgio, *La scienza italiana e le politiche razziali del regime*, Bologna: Il Mulino, 2010.

Israel, Giorgio and Nastasi, Pietro, *Scienza e razza nell'Italia fascista*, Bologna: Il Mulino, 1998.

Israel, Giorgio (ed.), *Correspondence of Luigi Cremona (1830–1903)* , Paris: Brepols, De Diversis Artibus 2016.

Lacaita, Carlo G. and Silvestri, Andrea (eds.), *Francesco Brioschi e il suo tempo (1824–1897)*, vol. I, *Saggi*, Milano: Angeli, 2000.

Lacaita, Carlo G., *Le vie dell'innovazione: viaggi tra scienza, tecnica ed economia (secoli XVIII-XX)*, Milano: Casagrande, 2009.

Lacaita, Carlo G., Francesco Brioschi (1884–1898), in *Scienziati, patrioti, presidenti*, Simili, Raffaella (ed.), Bari-Roma: Laterza, 2012: 43–78.

Lehto, Olli, *Mathematics Without Borders: a History of the International Mathematical Union*, New York: Springer-Verlag, 1998.

Levi, Beppo, Sull'equazione indeterminata del 3°, in *Atti del IV Congresso Internazionale dei Matematici (Roma 6–11 aprile 1908)*, Castelnuovo, Guido (ed.), Roma: Tip. della R. Acc. Lincei, vol. 2, 1909: 173–177.

Linguerri, Sandra and Simili, Raffaella, *Einstein parla italiano: itinerari e polemiche*, Bologna: Pendragon, 2008.

Lolli, Gabriele, 'Peano and the Foundation of Arithmetic', in *Giuseppe Peano between Mathematics and Logic*, Skof, Fulvia (ed.), Milano: Springer, 2011: 47–66.

Loria, Gino, Aperçu sur le développement historique de la théorie des courbes planes, in *Verhandlungen des Ersten Internationalen Mathematiker-Kongresses in Zürich vom 9 bis 11 August 1897*, Rudio, Ferdinand (ed.), Leipzig: Teubner, 1898: 289–298.

Loria, Gino, L'opera geometrica di Corrado Segre, *Annali Mat. pura ed applicata* (4) 2 (1924): 1–21.

Loria, Gino, *Pagine di storia della scienza*, Torino: Paravia, 1924, 1930^{4}.

Luciano, Erika, Mario Pieri e la Scuola di Peano, *Quaderni di Storia dell'Università di Torino* 10 (2012): 35–62.

Luciano, Erika and Roero, Clara Silvia, From Turin to Göttingen: Dialogues and Correspondence (1879–1923), *Boll. Sto. Sci. Mat.* 32 (2012): 7–232.

Luciano, Erika, 'Illustrare la Nazione col senno e colla mano'. Ebraismo e istruzione nel Piemonte risorgimentale, in *Dall'Università di Torino all'Italia unita. Contributi dei docenti al Risorgimento e all'Unità,* Roero, Clara Silvia (ed.), Torino: DSSP, 2013: 307–345.

Luciano, Erika, La diffusione della Logica Peaniana nelle Americhe e il magistero di B. Levi e A. Terracini in Argentina (1938–1948), in *XX Congresso U.M.I., Siena 7-12 settembre 2015, Sunti delle Comunicazioni*, Siena: UMI 2015.

Luciano, Erika, The French 'Analysts' and Peano's Mathematical Logic: Couturat's *Remarques* to Borel, Baire and Lebesgue, in *Images of Italian mathematics in France*, Brechenmaker, Frédéric and Mazliak, Laurent and Tazzioli, Rossana (eds.), Basel: Springer 2016: 181–211.

Luciano, Erika and Roero, Clara Silvia, Dialogues and Correspondence between Felix Klein and Vito Volterra on the *Encyclopädie*, currently underway, 2017.

Mac Lane, Saunders, Mathematics at the University of Chicago, A Brief History, in *A Century of Mathematics in America*, Duren, Peter *et alii* (eds.), vol. 2, Providence: AMS, 1989: 127–154.

Mancosu, Paolo, *Mathematical Style*, Stanford Encyclopedia of Philosophy, 2009 (http://plato.standford.edu).

Martini, Laura, The Politics of Unification: Barnaba Tortolini and the Publication of Research Mathematics in Italy, 1850–1865, in *Il sogno di Galois: Scritti di storia della matematica dedicati a Laura Toti Rigatelli per il suo 60º compleanno*, Franci, Raffaella and Pagli, Paolo and Simi, Annalisa (eds.), Siena: Centro Studi della Matematica Medioevale, 2003: 171–198.

Mazliak, Laurent and Tazzioli, Rossana, *Mathematicians at War. Volterra and His French Colleagues in World War I*, Dordrecht: Springer, 2009.

Menghini, Marta: Sul ruolo di C. Segre nello sviluppo della geometria algebrica italiana, *Rivista di storia della scienza* 3 (1986): 303–322.

Mehrtens, Herbert, Modernism vs. Counter-Modernism, Nationalism vs. Internationalism: Style and Politics in Mathematics, 1900–1950, in *L'Europe mathématique: histoires, mythes, identités*, Goldstein, Catherine and Gray, Jeremy and Ritter, Jim (eds.), Paris: Editions de la Maison des sciences de l'homme, 1996: 518–529.

Meyer, Franz, Bericht über den gegenwärtigen Stand der Invariantentheorie, JDMV 1 (1894): 79–292; it. transl. by G. Vivanti, Rapporto sullo stato presente della teoria degli invarianti, *Giornale di Matematiche* 32 (1894): 319–347, 33 (1895): 260–319, 34 (1896): 290–353, 35 (1897): 284–332; 36 (1898): 306–316; 37 (1899): 186–211.

Moore Clarence L.E., The Fourth International Congress of Mathematicians, *Bull. Am. Math. Soc.* 14 (1908): 481–498.

Moore Clarence L.E., Infinitesimal Properties of lines in S_4 with applications to cirles in S_3, *Proceedings of the American Academy of Arts and Sciences* 46 (1911): 345–362.

Nastasi, Pietro and Rogora, Enrico, *Mon cher ami - illustre professore. Corrispondenza di Ugo Amaldi (1897–1953)*, Roma: Nuova Cultura, 2007.

Nastasi Pietro, and Tazzioli Rossana, I matematici italiani e l'internazionalismo scientifico (1914–1924), *La Matematica nella Società e nella Cultura, Rivista dell'Unione Matematica Italiana* (1) VI (2013): 355–405.

Padoa, Alessandro, Un nouveau système irréductible de postulats pour l'algèbre, in *Compte Rendu du deuxième Congrès International des mathématiciens tenu à Paris du 6 au 12 aout 1900*, Ernest Duporcq (ed.), Paris: Gauthier-Villars, 1902a: 249–256.

Padoa, Alessandro, Un nouveau système des définitions pour la géométrie euclidienne, in *Compte Rendu du deuxième Congrès International des mathématiciens tenu à Paris du 6 au 12 aout 1900*, Duporcq, Ernest (ed.), Paris: Gauthier-Villars, 1902b: 353–363.

Palladino, Franco and Palladino, Nicla, *Dalla moderna geometria alla nuova geometria italiana Viaggiando per Napoli, Torino e dintorni. Lettere di Sannia, Segre, Peano, Castelnuovo, D'Ovidio, Del Pezzo, Pascal e altri a Federico Amodeo*, Firenze: Olschki, 2006.

Parikh, Carol, *The Unreal Life of Oscar Zariski*, New York: Springer, 1991.
Parshall, Karen Hunger, Eliakim Hastings Moore and the Founding of a Mathematical Community in America, 1892–1902, in *A Century of Mathematics in America*, Duren, Peter *et alii* (eds.), vol. 1, Providence: AMS, 1988: 155–175, reprint from *Annals of Science* 41 (1984): 313–333.
Parshall, Karen Hunger, and Rowe, David E., American Mathematics Comes of Age: 1875–1900, in *A Century of Mathematics in America*, Duren, Peter (ed.), vol. 3, Providence: AMS, 1989: 3–28.
Parshall, Karen Hunger, and Rowe, David E., *The Emergence of the American Mathematical Research Community 1876–1900: J.J. Sylvester, Felix Klein, and E.H. Moore*, Providence: AMS, 1994.
Parshall, Karen Hunger and Rowe, David E., *Mathematics Unbound: the Evolution of an International Mathematical Research Community 1800–1945*, Providence Rhode Island: AMS, 2002.
Pasini, Enrico, *La Scuola di Peano e il secondo problema di Hilbert*, in *Peano e la sua Scuola fra matematica, logica e interlingua*, Roero, Clara Silvia (ed.), Torino: CSSUT and DSSP, 2010: 327–367.
Peano, Giuseppe, Logica matematica, in *Verhandlungen des Ersten Internationalen Mathematiker-Kongresses in Zürich vom 9 bis 11 August 1897*, Rudio, Ferdinand (ed.), Leipzig: Teubner, 1898: 299.
Pepe, Luigi (ed.), *Europa matematica e Risorgimento Italiano*, Bologna: CISUI, 2012.
Picard, Emile and Simart, Georg, *Théorie des fonctions algébriques de deux variables indépendantes*, 2 vols. Paris: Gauthier-Villars, 1897, 1906.
Pompeo Faracovi, Ornella (ed.), *Enriques e Severi. Matematici a confronto nella cultura del Novecento, Atti del Convegno Livorno 24–25 ottobre 2002*, La Spezia: Agorà, 2004.
Rasmussen, Anne, *L'internationale scientifique (1890–1914)*, Paris: EHESS, 2 vols., 1995.
Roero, Clara Silvia, *Giuseppe Peano, geniale matematico, amorevole maestro*, in *Maestri dell'Ateneo torinese dal Settecento al Novecento*, CSSUT, *Sesto Centenario*, Allio, Renata (ed.), Torino: Stamperia artistica nazionale, 2004: 115–144.
Roero, Clara Silvia, 'Promuovere l'istruzione e la scienza per l'incremento della pubblica felicità'. Contributi di matematici e fisici, in *Dall'Università di Torino all'Italia unita. Contributi dei docenti al Risorgimento e all'Unità,* Roero, Clara Silvia (ed.), Torino: DSSP, 2013: 367–545.
Roero, Clara Silvia, Peano e Segre, curatori e promotori di riviste matematiche, 1890–1932, in *XX Congresso U.M.I., Siena 7–12 settembre 2015, Sunti delle Comunicazioni*, Siena: UMI 2015.
Rowe, David E. and McCleary, John (eds.) *The History of Modern Mathematics. Proceedings of the Symposium on the History of Modern Mathematics, Vassar College, Poughkeepsie, New York, June 20–24 1989*, Boston: Academic press, 1989-1994.
Rowe, David, Mathematical Schools, Communities, and Networks, in *Cambridge History of Science*, vol. 5, *Modern Physical and Mathematical Sciences*, Nye, Mary J. (ed.), Cambridge: University Press, 2003: 113–132.
Rowe, David, Segre, Klein, and the Theory of Quadratic Line Complexes, in this volume.
Salmon, George, *Analytische Geometrie des Raumes*, (transl. W. Fiedler), 2 vols. Leipzig: Teubner 1874.
Sansone, Giovanni, *L'Istituto Matematico "Ulisse Dini" di Firenze, Discorso inaugurale 2 marzo 1963*, Firenze: Università degli studi, 1963: 5–18.
Scorza, Gaetano, La scuola geometrica italiana, in *L'Italia e la scienza*, Bargagli Petrucci, Gino (ed.), Firenze: Le Monnier, 1932: 130–135.
Segre, Beniamino, La geometria in Italia, dal Cremona ai giorni nostri. Prolusione al Corso di Geometria Superiore, *Ann. di Mat. pura e applicata* (4), 11 (1933): 1–16.
Segre, Beniamino, *Prefazione*, in C. Segre, *Opere*, Roma: Cremonese, vol. III, 1961: V-IX.
Segre, Beniamino, Nel primo centenario della nascita di Corrado Segre, *Rend. del Sem. Mat. e Fis. dell'Univ. e Politecnico di Torino* 23 (1963–64): 7–21.
Segre, Corrado, *Geometria proiettiva*, Torino: Litografia dell'Università, 1886.

Segre, Corrado and Loria, Gino, Sur les différentes espèces de complexes du 2[e] degré des droites qui coupent harmoniquement deux surfaces du second ordre, *Mathematische Annalen*, 23 (1883a): 213–234. (*Opere*, 3, 1–24).
Segre, Corrado, Studio sulle quadriche in uno spazio lineare ad un numero qualunque di dimensioni, *Memorie della R. Accademia delle Scienze di Torino*, 2, 36 (1883b): 3–86. (*Opere*, 3, 25–126).
Segre, Corrado, Sulla geometria della retta e delle sue serie quadratiche, *Memorie della R. Accademia delle Scienze di Torino*, 2, 36 (1883c): 87–157. (*Opere*, 3, 127–217).
Segre, Corrado, Su una trasformazione irrazionale dello spazio e sua applicazione allo studio del complesso quadratico di Battaglini e di un complesso lineare di coniche iscritte in un tetraedro, *Giornale di Matematiche*, 21 (1883d): 355–378. (*Opere*, 3, 234–261).
Segre, Corrado, Sulle geometrie metriche dei complessi lineari e delle sfere e sulle loro mutue analogie, *Atti della R. Accademia delle Scienze di Torino*, 19 (1883–84a): 159–186. (*Opere*, 3, 262–287).
Segre, Corrado, Sulla teoria e sulla classificazione delle omografie in uno spazio lineare ad un numero qualunque di dimensioni, *Atti della R. Accademia Nazionale dei Lincei. Memorie della Cl. Sci. Fis. Mat. Nat.*, 3, 19 (1883–84c): 127–148. (*Opere*, 3, 304–333).
Segre, Corrado, Ricerche sui fasci di coni quadrici in uno spazio lineare qualunque, *Atti della R. Accademia delle Scienze di Torino*, 19 (1883–84d): 878–896. (*Opere*, 3, 485–501).
Segre, Corrado, Teorema sulle relazioni tra una coppia di forme bilineari e la coppia delle loro forme reciproche, *Giornale di Matematiche*, 22 (1884a): 29–32. (*Opere*, 3, 229–233).
Segre, Corrado, Note sur les complexes quadratiques dont la surface singulière est une surface du 2[e] dégré double, *Mathematische Annalen*, 23 (1884b): 235–243. (*Opere*, 3, 218–228).
Segre, Corrado, Sur les droites qui ont des moments donnés par rapport à des droites fixes, *Journal für die reine und angewandte Mathematik*, 97 (1884c): 95–110. (*Opere*, 3, 288–303).
Segre, Corrado, Sur les invariants simultanés de deux formes quadratiques, *Mathematische Annalen*, 24 (1884d): 152–156. (*Opere*, 3, 334–338).
Segre, Corrado, Étude des différentes surfaces du 4e ordre à conique double ou cuspidale (générale ou décomposée) considérées comme des projections de l'intersection de deux variétés quadratiques de l'espace à quatre dimensions, *Mathematische Annalen*, 24 (1884e): 313–444. (*Opere*, 3, 339–484).
Segre, Corrado, Sur les courbes de tangentes principales des surfaces de Kummer (Extrait d'une lettre adressée a M. Th. Reye), *Journal für die reine und angewandte Mathematik*, 98, (1884g): 301–303. (*Opere*, 3, 545–547).
Segre, Corrado, Remarques sur les transformations uniformes des courbes elliptiques en elles-mêmes, *Mathematische Annalen*, 28 (1886a): 296–314. (*Opere*, 1, 36–55).
Segre, Corrado, Recherches générales sur les courbes et les surfaces réglées algébriques (I partie, Courbes algébriques), *Mathematische Annalen*, 30 (1887b): 203–226. (*Opere*, 1, 80–104).
Segre, Corrado, Sur un théorème de la géométrie à n dimensions (Extrait d'une lettre adressée à Mr. F. Klein), *Mathematische Annalen*, 30 (1887c): 308. (*Opere*, 4, 87).
Segre, Corrado, Recherches générales sur les courbes et les surfaces réglées algébriques (II partie, Surfaces réglées algébriques), *Mathematische Annalen*, 34 (1889a): pp. 1–25. (*Opere*, 1, 125–151).
Segre, Corrado, Su alcuni indirizzi nelle investigazioni geometriche. Osservazioni dirette ai miei studenti, *Rivista di Matematica*, 1, (1891a): 42–66. (*Opere*, 4, 387–412).
Segre, Corrado, Le rappresentazioni reali delle forme complesse e gli enti iperalgebrici, *Mathematische Annalen*, 40 (1891d): 413–467. (*Opere*, 2, 338–395).
Segre, Corrado, On some tendencies in geometric investigations, *Bulletin of the American Mathematical Society*, 2, 10 (1904): 442–468. English translation by John Wesley Young of Segre (1891a).
Segre, Corrado, La Geometria d'oggidì e i suoi legami coll'analisi, *Rendiconti del Circolo Matematico di Palermo*, 19, (1905): 81–93. (*Opere*, 4, 456–468).

Segre, Corrado, La Geometria d'oggidì e i suoi legami coll'analisi, in A. Krazer (ed.) *Verhandlungen des dritten internationalen Mathematiker-Kongresses in Heidelberg vom 8 bis 13 August 1904*, Leipzig: Teubner (1905a): 109–120 (*Opere*, 4, 456–468).
Segre, Corrado, Sur la génération projective des surfaces cubiques. (Extrait d'une lettre adressée à M. le Prof. R. Sturm), *Archiv der Mathematik und Physik*, 3, 10 (1906): 209–215. (*Opere*, 4, 188–196).
Segre, Corrado and Nöther, Max and Poincaré, Henri, Relazione del concorso internazionale per la Medaglia Guccia, *Atti del IV Congresso Internazionale dei Matematici, Roma, 6–11, Aprile*, 1908, I, Roma: Tip. R. Accademia dei Lincei (1909): 209–216.
Segre, Corrado, Aggiunta alla memoria: Preliminari di una teoria delle varietà luoghi di spazi, *Rendiconti del Circolo Matematico di Palermo*, 30 (1910b): 346–348. (*Opere*, 2, 115–118).
Segre, Corrado, H.G. Zeuthen. Cenno commemorativo, *Atti della R. Accademia delle Scienze di Torino*, 55, (1919–20): 327–328. (*Opere*, 4, 487–488).
Segre, Corrado, Mehrdimensionale Räume, *Encyklopädie der mathematischen Wissenschaften*, III.2.2 C 7 (1921c): 769–972.
Segre, Corrado, Max Noether e Hermann Schwarz, *Atti della R. Accademia delle Scienze di Torino*, 57 (1921–22b): 161–163. (*Opere*, 4, 489–491).
Segre, Corrado, Commemorazione del socio straniero Carlo Teodoro Reye, *Atti della R. Accademia Nazionale dei Lincei. Rendiconti*, 5, 31 (1922): 269–272. (*Opere*, 4, 492–496).
Severi, Francesco, Di alcuni recenti risultati nella teoria delle superficie algebriche e sopra qualche problema ad essi collegato, in *Atti del IV Congresso Internazionale dei Matematici (Roma 6–11 aprile 1908)*, Castelnuovo, Guido (ed.), Roma: Tip. R. Acc. Lincei, v. 2, 1909: 234–241.
Severi, Francesco, *Trattato di geometria algebrica, 1: Geometria delle serie lineari*, Bologna: Zanichelli, 1926.
Severi, Francesco, La géometrie algébrique, in *Atti del Congresso Internazionale dei Matematici (Bologna 3–10 settembre 1928*), Bologna: Zanichelli, v. 1, 1929: 149–154.
Severi, Francesco, Le role de la géométrie algébrique dans les mathématiques, in *Verhandlungen des Internationalen Mathematiker Kongresses Zürich*, Saxer, Walter (ed.), Zürich: Orell Füssli, v. 1, 1932: 209–220.
Severi, Francesco, Peut-on parler d'un esprit latin mème dans les mathématiques?, *Revue scientifique* 73 (1935): 581–589.
Severi, Francesco, In occasione dell'inizio dell'anno accademico 1940–1941 del Reale Istituto Nazionale di Alta Matematica, *Boll. U.M.I.* (2), 3 (1941): 130–140.
Severi, Francesco, La géométrie algébrique italienne: sa rigueur, ses méthodes, ses problèmes, in *Centre Belge de Recherches Mathematiques, Colloque de Géométrie algebrique. Tenu à Lièges les 19, 20 et 21 Décembre 1949*, Liège et Paris: G. Thone, Masson & Cie, 1950: 9–55.
Struik, Dirk J., Julian Lowell Coolidge (1873–1954): In memoriam, *Amer. Math. Monthly* 62 (1955): 669–682.
Struik, Dirk J., The MIT Department of Mathematics During Its First Seventy-Five Years: Some Recollections, in *A Century of Mathematics in America*, Duren, Peter (ed.), vol. 3, Providence: AMS, 1989: 163–189.
Terracini, Alessandro, Corrado Segre (1863–1924), *Jahresbericht der Deutschen Mathematiker-Vereinigung* 35(1926): 209–250.
Terracini, Alessandro, L'opera geometrica di Corrado Segre e alcuni suoi ulteriori sviluppi, BMP Turin, *Fondo Terracini*, Quaderni. 13, 1934–1935
Terracini, Alessandro, *I quaderni di Corrado Segre*, in *Atti del IV Congresso UMI*, Roma: Cremonese, v. 1, 1953: 252–262.
Terracini, Alessandro, L'opera geometrica di Corrado Segre, BMP Turin, *Fondo Terracini,* Quaderni. 34, 1957–1958
Terracini, Alessandro, Parole …, in *Atti del Convegno Internazionale di Geometria Algebrica, Torino 24–27 maggio 1961*, Torino: Rattero, 1961: 9–14.
Terracini, Alessandro, *Ricordi di un matematico Un sessantennio di vita universitaria*, Roma: Cremonese, 1968.

Togliatti, Eugenio, *Prefazione*, in C. Segre, *Opere*, Roma: Cremonese, vol. IV, 1963: V-XIII.

Tricomi, Francesco G., Essenza e didattica delle Matematiche in un manoscritto inedito di Corrado Segre, *Rend. del Sem. Mat. e Fis. dell'Univ. e Politecnico di Torino* 7 (1940): 101–117.

Truesdell, Clifford A., *An Idiot's Fugitive Essays on Science, Methods, Criticism, Training, Circumstances*, New York: Springer, 1984.

Veronese, Giuseppe, Behandlung der projectivischen Verhaltnisse der Raume von verschiedenen Dimensionen durch das Princip des Projicirens und Schneidens, *Math. Ann.* 19 (1882): 161–234.

Veronese, Giuseppe, Die Anzhal der unabhangigen Gleichungen, die zwischen den allgemeinen Charakteren einer Curve im Raume von *n* Dimensionen stattfinden, *Math. Ann.* 18 (1881): 448.

Vesentini, Edoardo, Il caso della Matematica, in *Conseguenze culturali delle leggi razziali in Italia, Atti del Convegno Roma, 11 maggio 1989*, Roma: Acc. Naz. Lincei, 1990: 97–105.

Viglezio, Elisa, In memoria di Corrado Segre, *Rassegna di Matematica e Fisica* 5, 1–2 (1924): 1–2.

Volterra, Vito, *Betti, Brioschi, Casorati, trois analystes italiens et trois manières d'envisager les questions d'analyse*, in *Compte Rendu du deuxième Congrès International des mathématiciens tenu à Paris du 6 au 12 aout 1900*, Duporcq, Ernest (ed.), Paris: Gauthier-Villars, 1902: 43–57.

Volterra, Vito, Il momento scientifico presente e la nuova società italiana per il progresso delle scienze, *Rivista di Scienza* I (1907): 225–237.

Volterra, Vito, Le matematiche in Italia nella seconda metà del secolo XIX, in *Atti del IV Congresso Internazionale dei Matematici (Roma 6–11 aprile 1908)*, Castelnuovo, Guido (ed.), Roma: Tip. R. Acc. Lincei, v. 1, 1909: 55–65.

Volterra, Vito, Annuncio della morte del Socio Corrado Segre, Cenni necrologici sul socio C. Segre, *Rend. R. Acc. Naz. Lincei* (5), 33 (1924): 459.

von Dyck, Walther, Die Encyklopädie der mathematischen Wissenschaften, in *Atti del IV Congresso Internazionale dei Matematici (Roma 6–11 aprile 1908)*, Castelnuovo, Guido (ed.), Roma: Tip. R. Acc. Lincei, 1909: 123–134.

Young William F. Sulle sizigie che legano le relazioni quadratiche fra le coordinate di retta in S_4, *Atti Acc. Sci. Torino* 34 (1899): 596–599.

Young, William H. and Chisholm, Grace, Note on Bertini's Transformation of a Curve into One Possessing Only Nodes, *Atti Acc. Sci. Torino* 42 (1907): 82–86.

Wilczynski, Ernest, On Self-dual Scrolls, *Bull. Am. Math. Soc.* (2) 11, 1904: 8.

Wilczynski, Ernest, *Projective Differential Geometry of Curves and ruled Surfaces*, Leipzig: Teubner, 1906.

Wilczynski, Ernest, Sur la théorie genérale des congruences, *Académie Royale de Belgique Cl. Sci. Mémoire couronné*, Juin 1911: 1–86.

Zappulla, Carmela, *La Geometria Proiettiva Complessa. Origini e sviluppi da von Staudt a Segre e Cartan*, Ph.D. Thesis, Palermo University, 2009.

Segre, Klein, and the Theory of Quadratic Line Complexes

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Abstract

Two of C. Segre's earliest papers, (Segre 1883a) and (Segre 1884), dealt with the classification of quadratic line complexes, a central topic in line geometry. These papers, the first written together with Gino Loria, were submitted to Felix Klein in 1883 for publication in *Mathematische Annalen*. Together with the two lengthier works that comprise Segre's dissertation, (Segre 1883b) and (Segre 1883c), they took up and completed a topic that Klein had worked on a decade earlier (when he was known primarily as an expert on line geometry). Using similar ideas, but a new and freer approach to higher-dimensional geometry, Segre not only refined and widened this earlier work but also gave it a new direction. Line geometry, as well described by Alessandro Terracini in his obituary for his mentor, proved to be an excellent starting point for both Segre and Italian algebraic geometry. The present account begins by looking back at the early work of Klein and Adolf Weiler on quadratic complexes in order to show how Segre's two papers for Klein's journal represented a new start that reawakened interest in a topic that had been dormant for nearly a decade.

1 Introduction

While it is well known that the torch of algebraic geometry passed from Germany to Italy by the 1880s and that Corrado Segre played a major part in that story, it has seldom been emphasized that line geometry and the theory of quadratic line

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complexes lie at the heart of that story. From today's perspective, this represents a rather special topic quite far removed from mainstream interests in algebraic geometry. Indeed, initially, line complexes were a topic in classical geometry, which meant that they were studied using a variety of methods taken from metrical and projective geometry. Pioneering figures, like Julius Plücker and his pupil Felix Klein, were still working within an older tradition that drew its inspiration from applications to mechanics and geometrical optics.[1] Thus, the torch that Klein passed on to Segre had not only to be relit, it had to be joined with other currents from the time before the distinctive discipline of algebraic geometry would become clearly visible. In the course of that process, it was easy for later generations to overlook the key role line geometry had actually played for Italian algebraic geometry.[2]

Thanks to recent historical studies, we now have a far better vantage point from which to reassess many of the events of the 1880s, a crucial decade for the discipline.[3] With regard to the transition from Germany to Italy, many new details have recently been made available thanks to the research of Erika Luciano and Clara Silvia Roero as presented in Luciano and Roero (2012). Their lengthy article contains not only a full transcription of Segre's letters to Klein and other related correspondence but also a very full account of the special connections that linked Turin's mathematicians to those in Göttingen over many decades.[4] The present contribution has a far more modest goal. By drawing on this recent research I will try to give a close-up view of Segre's earliest work, which immediately revived Klein's interest in line geometry, the topic he had first studied some fifteen years earlier. As we shall see toward the end, Segre's work was particularly important for Klein in connection with his life–long interest in Kummer surfaces and their relation to families of quadratic line complexes, the main topic I will focus on here.

In his obituary for Corrado Segre, Alessandro Terracini stressed the strong influence Felix Klein's ideas had on his early career. With regard to his dissertation (Segre 1883b, c), he wrote: Segre's "entire work was connected with Klein's researches on quadratic line complexes… and yet it clearly differed from these: on the one hand, through its quite different and more skillful classification as well as through its introduction of fruitful new concepts, on the other, through the broader generality of its results and the careful treatment of special cases" (Terracini 1926, 214). For his part, Klein later noted that Segre's dissertation developed the approach to line geometry as a hyperspace—namely a quadric hypersurface in P^5—which he had first alluded to in 1872 (Klein 1921, 112). As for Segre's approach to classification, our main concern here, this reflects a more careful examination of the then current algebraic and geometric methods available for studying this difficult problem.

[1]For a study of this physical tradition, see (Ziegler 1985).

[2]As an example, one can read the remarks about line geometry in Dieudonne (1985, 12–13).

[3]See, for example, (Menghini 1996; Giacardi 2001).

[4]At the time of Segre's death in 1924, Alessandro Terracini was given access to these letters by Klein, and he cited a few passages from them in his obituary article (Terracini 1926).

2 Background Events: Line Geometry ca. 1870

In order to understand, even on a somewhat general level, the nature of Segre's early work on quadratic line complexes, a brief account of some important background events should be mentioned. After Plücker's death in 1868, Klein completed work on his mentor's pioneering study, *Neue Geometrie des Raumes gegründet auf die Betrachtung der geraden Linie als Raumelement* (Plücker 1868, 1869). Although he was only twenty, Klein mastered this new discipline, combining it with new ideas from invariant theory that he came to learn through his second mentor, Alfred Clebsch, who taught in Göttingen. Clebsch was generally familiar with current Italian research in geometry, so by 1869 the stage was already set for Klein to develop the theory of quadratic complexes with ideas that were quite foreign to Plücker's more old-fashioned mathematical style.[5] The events that unfolded at this time would form the background for Segre's works on line geometry during the 1880s.

Clebsch was familiar with the publications of Giuseppe Battaglini on quadratic line complexes (Battaglini 1866, 1868, 1869), work that soon opened Klein's eyes to a much more elegant algebraic approach to this theory.[6] Klein later recalled how reading this work had forced him to learn the new projective geometry, based on algebraic invariant theory, which he found in the textbooks of Wilhelm Fiedler (known as Salmon–Fiedler, since their contents were loosely based on the earlier books of George Salmon) and other literature (Klein 1921, 4). He soon discovered that Battaglini's starting point was a canonical form that contained only 17 parameters, whereas 19 are required for a general quadratic complex.[7]

Klein's dissertation topic thus aimed to show how to derive a completely general canonical form. In the course of solving this problem, however, his thesis advisor in Bonn, Rudolf Lipschitz, insisted that he also deal with the "degenerate" cases as well. The corresponding algebraic problem had, in fact, only just before been solved by Weierstrass, as Lipschitz happened to know because he had received page proofs from the Berlin master. So Klein could now apply this new theory of elementary divisors to count the number of additional cases one would have to study in order to provide a complete theory of the different possible types of quadratic complexes. A few years later in Erlangen, Klein's student, Adolf Weiler,

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[5]Plücker was also quite familiar with the work of Italian geometers, in particular Cremona, with whom he corresponded; see Menghini (1996).

[6]Clebsch already noted the early work of Battaglini on line complexes in his forward to (Plücker 1868), where he contrasted the Italian's modern algebraic approach to the more old-fashioned style of Plücker.

[7]This background information regarding Klein's dissertation seems not to have been widely known until the publication of Klein (1921). His dissertation from 1868 contains only a vague reference to Battaglini's work (Klein 1884, 546), and the notes added when he published (Klein 1884) in *Mathematische Annalen* contain nothing about this; nor does this version contain the theses that Klein had appended to his original dissertation for purposes of his oral defense. The first of these asserted that Battaglini's canonical form did not represent the general case for a quadratic complex (Klein 1921, 49).

wrote his dissertation on this very topic. Weiler was the first to attempt an exhaustive analysis of quadratic complexes, and his results were published in *Mathematische Annalen* in Weiler (1874). Segre and his close friend, Gino Loria, studied this paper carefully while they were working on their own dissertations.[8]

Both Loria and Segre took their doctorates in 1883 in Turin under Enrico D'Ovidio and both wrote on topics that dealt with quadratic line complexes. Much more so than Segre, Loria developed a deep interest in the history of geometry that later gained him his reputation as the foremost Italian historian of mathematics of the day (on Italian traditions in the history of mathematics, see Bottazzini (2002). Loria's dissertation dealt with the theory of tetrahedral line complexes, an important special class of quadratic complexes studied earlier by Reye, Lie, and Klein, among others. These are families of lines that meet the four planes of a tetrahedron in a fixed cross ratio. (Staudt had proved the dual theorem, which states that this cross ratio is identical with that determined by the four planes that pass throught the line and one of the four vertices of the tetrahedron.) The surface of singularity in this case is just the tetrahedron itself, so together with the cross ratio, the complex depends on $4 \times 3 + 1 = 13$ parameters.

Segre and Loria were at the same time interested in the original case that Battaglini had dealt with in the 1860s, but they explored this from a far more geometrical point of view. This possibility arose from an insight of Ferdinando Aschieri. In 1870 Aschieri showed that the Battaglini complex could be regarded as the lines that cut two quadric surfaces harmonically (Aschieri 1870). These complexes were later thus called harmonic, and they came to the attention of Friedrich Schur, who had taken his degree in 1879 under Kummer in Berlin with a dissertation on first- and second-degree line complexes. Schur afterward wrote his *Habilitationsschrift* in Leipzig with Klein, from which he published a paper on harmonic complexes in *Mathematische Annalen* (Schur 1883). It was after reading this paper by Schur that Loria and Segre decided to write to Klein in order to submit their joint paper (Segre 1883a) for publication in his journal (Luciano and Roero 2012, 81).

Loria and Segre pursued the geometric implications of the property of harmonicity and used it as a way to refine the results in Weiler's classification system. At the same time, they brought to Klein's attention numerous errors in Weiler's paper, which Segre then pointed out in Segre (1884). The approach in Segre (1883a) depended on re-investigating all possible cases of harmonic complexes by taking into consideration the mutual position of the two quadric surfaces and by determining the number and position of the double lines contained in the various possible complexes. In certain respects, this type of detailed geometrical analysis brings to mind Plücker's investigation of numerous cases of "complex surfaces," the culminating topic in Plücker (1869). Yet Plücker's whole approach was, on the one hand, wedded entirely to the use of ordinary Cartesian coordinates for Euclidean 3-space, on the other, it was based on a largely static conception that amounted to describing individual cases that arise in a fixed coordinate system.

[8]They also had apparently studied Klein's then unpublished dissertation (Klein 1884); see their letter to Klein, 16 August 1883 (Luciano and Roero 2012, 81).

Klein had already moved away from this static approach in his early work on line geometry, emphasizing how Plücker's complex surfaces naturally arose from Kummer surfaces via a deformation process in which a double line formed. As we shall see below, this shift was accompanied by a far freer interpretation of coordinate systems that enabled Klein to explore the rich geometrical properties of quadratic line complexes and their associated singularity surfaces.

This sensitivity to the underlying geometry was clearly the driving motivation for the new investigations in line geometry of Segre, which were quite in the spirit of Klein, though with far more attention to details. Indeed, the early work of Segre and Loria showed that the dissertation by Weiler, written under Klein's supervision, contained many mistakes that had evidently escaped Klein entirely. Nevertheless, Klein clearly appreciated what these two young Italians had accomplished. After this initial publication with Loria, Segre soon struck up a friendship with Klein that would last until the end of his life. Their mutual interests would also bear considerable fruit for future geometrical research, mainly thanks to Segre's deep grasp of the possibilities Klein's ideas had opened. By building on these he was able to strengthen some of the most distinctive methods of Italian research in algebraic geometry.

3 Plücker's Approach to Quadratic Line Complexes

To appreciate Segre's early work on quadratic complexes, one needs to have a basic understanding of this then popular, but now largely forgotten field of research. In this section I will, therefore, introduce some of the more fundamental notions from line geometry, starting with ideas first presented by Klein's teacher, Julius Plücker, in the late 1860s. Quadratic line complexes are special 3-parameter families of lines in space that satisfy a quadratic equation in line coordinates (see further below for analytic details). Many special cases had been considered in geometrical optics before Plücker launched the general theory, but his work immediately raised the challenging problem of finding a means to classify all possible quadratic complexes. Segre's work from the 1880s gave what came to be regarded as the final solution for that problem, whereas Klein's dissertation from 1868 and his subsequent research papers merely pointed in the general direction that made it possible to give a complete classification.

First-degree (linear) line complexes had been studied earlier by A.F. Möbius and others, in particular to describe force systems and the motions of rigid bodies. The purely geometric theory of linear complexes (earlier called null systems) is relatively simple when compared with the quadratic case: taking a point P in 3-space, the lines through P form a plane pencil; whereas the same holds true when considering the lines in a plane Π. Furthermore, the lines common to two linear complexes, $K_1, K_{1'}$, form a 2-parameter set consisting of the lines that meet two fixed skew lines, ℓ and ℓ', called the directrices of this line congruence. Taking then three complexes, $K_1, K_{1'}, K_{1''}$, we get a 1-parameter family of lines that intersect three mutually skew lines in space, and these form a system of rulings for a quadric surface (for example, a hyperboloid of one sheet if these lines happen to be real). Plücker set forth this theory in considerable detail in Plücker (1868), whereas the theory of quadratic complexes was only sketched in Plücker (1869), the text completed by Klein after Plücker's death in 1868.

The theory of quadratic complexes is closely connected with so-called Kummer surfaces, which are quartics with sixteen singular points and singular planes (Hudson 1905). Such surfaces were first discovered by E.E. Kummer in the mid-1860s within the context of his research on 2-parameter systems of lines, often called ray systems (*Strahlensysteme*). Plücker renamed such ray systems algebraic line congruences, and studied them as intersections of line complexes. Thus, a quadratic complex, when intersected with a linear complex, yields a congruence of the second order and class. This means two lines pass through a generic point and two lines lie in a typical plane (counting imaginary lines and not just those which happen to be real). In a quadratic line complex, the lines through a point will, in general, form a non–degenerate cone; likewise, those in a generic plane will envelope a non-degenerate conic. The exceptional points and planes are of great importance, however. If the cone through a point P collapses into lines that lie in two planes, or if the lines in a plane Π degenerate into two point pencils, then P and Π are singular, and the locus of all such points and planes determines the *singularity surface* of the complex, which is in general a Kummer surface. Thus, the classification of quadratic complexes turns out to be intimately tied to these special types of quartics, and these were much studied objects during the latter third of the nineteenth century.[9]

Kummer's studies were directly motivated by geometrical optics, where caustic surfaces arise naturally as the envelopes of light rays. An algebraic congruence will always envelope an associated surface (*Brennfläche*), and these can be viewed as special types of singularity surfaces if the congruence is derived from a line complex. The earliest and most famous Kummer surface was already introduced by Fresnel in the 1820s as the surface that describes a wave front in bi-axial crystals, which exhibit the phenomenon of double refraction. In this special case, Painvin found a simple way to generate the lines in a complex with this special type of Kummer surface as its singularity surface (Painvin 1872): starting with a nonsingular quadric surface, he considered the family of all lines that arise from pairs of tangents planes that intersect one another orthogonally. These, and many other special cases linking quartic surfaces to quadratic line complexes had already been much studied a decade or more before Segre's first publications.

As Plücker already indicated in his *Neue Geometrie des Raumes* (Plücker 1869, 315–316), the general quadratic complex has a singularity surface of the fourth order and class. This means that lines meet the surface in four points, and through each line pass four planes tangent to the surface. These are, in fact, very special types of quartics with sixteen nodes and tropes (the terms used in Hudson (1905)

[9]See, for example, the classic work on Kummer surfaces, (Hudson 1905), and the references cited therein.

for singular points and planes), the maximum possible. Since Plücker's text contains no references to other literature, it would seem impossible to know whether he was aware of Kummer's concurrent papers. Klein's first publication on line geomtry (Klein 1870), however, drew all the current literature together and planted the study of Kummer surfaces firmly in the soil of quadratic line complexes. Thus, whereas Kummer studied these special quartics as caustics associated with congruences of the second order and class, Klein emphasized how the (16, 6) configuration of nodes and tropes—with 6 nodes lying in each of the 16 tropes and 6 tropes passing through each of the 16 nodes—reflected the underlying properties in a 1-parameter system of co-singular quadratic complexes. Here, he drew on the analogy between the geometry of such a system and the familiar case of confocal systems of quadric surfaces.

Whereas Klein's initial theory aimed to unlock the secrets of these general quadratic complexes by exploiting the symmetries of Kummer surfaces, many examples with fewer parameters reflect a far simpler structure. The role of the singularity surfaces nevertheless remains central even for such special cases. This aspect of the theory can therefore perhaps best be imagined geometrically by considering a highly specialized situation, namely the lines in space tangent to a given quadric surface F_2. Clearly, these form a 3-parameter family, whatever the degree of the fixed surface. Taking any point $P \notin F_2$, the lines through P tangent to F_2 (which may be imaginary) form a quadratic cone. When, on the other hand, $P \in F_2$ the cone collapses into a plane (the tangent plane T_P), since the cone is now just the tangent lines to F_2 though P. The dual case is just as simple, since we only have to distinguish whether the plane Π happens to be a tangent plane or not. In the latter case, $\Pi \cap F_2 = C_2$, a conic, whose tangents in Π are clearly also tangent to F_2. When, on the other hand, the plane $\Pi = T_P$, a tangent plane at $P \in F_2$, then the conic has a singular point at P, which means the lines that formerly enveloped a nonsingular conic in Π have now collapsed into the lines through P that lie in the plane Π.

These elementary considerations can be summarized by saying that the tangents to a fixed nonsingular quadric surface F_2 determine a unique quadratic line complex for which this F_2 will be the associated singularity surface. Since a quadric surface depends on only nine parameters, this special type can be thought of as forming a 9-dimensional hyperspace within the far larger family of all quadratic line complexes, the most general of which have Kummer surfaces as their surfaces of singularity and depend on 19 parameters, not just 9. Furthermore, far from determining a unique quadratic complex, a general surface of singularity will have a one-parameter family of associated quadratic complexes, as Klein showed in 1870. Following Kummer, he was also one of the first to make a model illustrating a Kummer surface for which all sixteen nodes are real and located within a finite region of affine space (see Fig. 1). Attached to a small tetrahedral-shaped piece at the center are four more similar figures. Those attached at two points on opposite sides represented the two halves of three larger tetrahedral shapes that have been cut off within a small region of space. Imagining these as extended throughout

Fig. 1 Klein's model in zinc of a Kummer surface. (C) 2016 Collection of mathematical models, Göttingen University

projective space, they rejoin to produce a symmetric object with eight indistinguishable tetrahedral shapes that hang together along the sixteen nodes. How this structure relates to the geometry of quadratic line complexes was first made clear in Klein's fundamental paper (Klein 1870).

Line complexes are defined as subfamilies of lines that satisfy an algebraic equation in line coordinates, so to begin with Plücker required a way to describe the 4-parameter manifold of lines in 3-space. The simplest way to present this theory analytically is to derive line coordinates from point coordinates. If we begin with two points y and z in projective 3-space with homogeneous coordinates (y_i), (z_j), $i, j \in \{1, 2, 3, 4\}$, these determine a line ℓ which can be expressed by means of the six 2 by 2 subdeterminants: $p_{ij} = y_i z_j - z_i y_j$. These six quantities can then be regarded as homogeneous line coordinates for ℓ, but since the lines in space form a 4-parameter family, these p_{ij} satisfy an additional relation, which can be shown to be quadratic:

$$P \equiv p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23} = 0. \tag{3.1}$$

These p_{ij} are often called *Plücker coordinates*, although Plücker mainly used another less transparent system.[10] Geometrically, one can interpret this coordinate system as defining six *special linear complexes*, namely each $p_{ij} = 0$ can be thought of as a linear complex consisting of just those lines that meet the line whose only non-zero coordinate is k_{ij}. The relation (3.1), then can be conceived as six special linear complexes whose directrices form a tetrahedron (for example $p_{12} = 0$ and $p_{34} = 0$ are opposite edges, since they appear together in the first term but nowhere else, hence they both meet the other four edges). These Plücker coordinates are advantageous in some situations, but less so in others. Soon after Plücker's death, Felix Klein introduced a different system of coordinates that was beautifully adapted to the theory of quadratic line complexes. Klein later also introduced the idea that line geometry could be treated as a metric geometry in projective 5-space

[10]Segre pointed out that Cayley had used these before Plücker.

in which the equation $P = 0$ defines a 4-dimensional quadric hypersurface (Klein 1872). Corrado Segre was the first to explore this possibility, however, in any detail, so it was not until the mid 1880s that this approach gained wide currency.

The remainder of this section and the one following takes up some of the main ideas introduced by Plücker and Klein, highlighting the sharp shift in the direction of research around 1870 when Klein introduced newer algebraic methods. In his well-known address on Plücker's scientific work (Clebsch 1872), Alfred Clebsch noted how Plücker's approach to line geometry did not take advantage of the more recent work on determinants and invariant theory, which were promoted by Salmon, Cayley, Hesse, et al. Clearly, Plücker had lost touch with these developments during the nearly twenty years when he was supervising experiments in physics. Nevertheless, he did have a clear idea of the general notion of an algebraic line complex and, along with Battaglini, he was the first to explore the quadratic case, where the lines satisfy both $P = 0$ and an additional second degree equation $\Omega(p_{ij}) = 0$.

Plücker's strategy for studying these quadratic complexes K_2 of lines in 3-space was to view them in relation to a fixed line $g \notin K_2$, beginning with the case where $g \in E_\infty$, the plane at infinity. The totality of lines that meet g form a special linear complex $K_1(g)$, so Plücker began to investigate K_2 by means of the congruences of lines (here of the second degree and class) in $K_2 \cap K_1(g)$. These were precisely the types of systems of lines that led Kummer to his theory of quartic surfaces as Brennflächen enveloped by such lines, but here the nodal line g forms a double line on the Kummer surface. Plücker called these *complex surfaces*, and he devoted a great deal of attention to the many varied cases that can arise.[11] With the help of a mechanical engineer in Bonn named Epkens, he also built a number of models that were designed to convey what some of these exotic objects really looked like.

The Plücker surface contains a double line g, which absorbs 8 of the 16 singular points of the general Kummer surface. There are four points on g with degenerate cones, so they determine pairs of planes, and correspondingly there are four planes through g that determine pairs of points. The singularity surface of a quadratic complex can have anywhere from zero double lines (the case of a Kummer surface) or one (the Plücker surfaces), up to six, in which case they form a tetrahedron. The tetrahedral complexes, consisting of the lines that meet the four faces of a tetrahedron in a fixed cross ratio, were studied intensively by Reye, Lie, and Klein. A complex with more than six double lines will have infinitely many, and even these can lead to various types of quite complicated structures. One of the simplest, mentioned above, is the case of a quadric surface counted twice as a singularity surface, thus a redundant quartic. Its tangents form a quadratic complex with just 9 parameters, but other cases can have up to 14. As we shall later see, Segre made a careful study of this case, presenting some of his results in (Segre 1884).

[11]For an introduction to these Plücker surfaces, see (Hudson 1905, Chap. VI).

Fig. 2 Plücker's model of an equatorial surface with eight real singular points lying in pairs on two sets of parallel lines. (C) Science Museum/Science & Society Picture Library. Photos of other Plücker models can be found at http://www.lms.ac.uk/content/plucker-collection

Though he was less appreciated in Germany, Plücker had an excellent reputation as an experimental physicist in England. He was a good friend of Michael Faraday, who championed his work; both were physicists who mainly thought with pictures, not formulae. Klein later recalled how Plücker once told him that it was Faraday who gave him the initial impetus to build models illustrating different types of complex surfaces, and these models soon became a centrepiece of his new line geometry (Klein 1922, 7–10). Faraday was by no means the only one in England who took an interest in these exotic spatial artefacts. Thomas Archer Hirst, who had studied under Jakob Steiner in Berlin, was another enthusiast for models like these. He learned about them in 1866, when Plücker delivered a well–received lecture at a meeting in Nottingham at which time he described properties of some of the Plücker surfaces using models he had brought with him (Cayley 1871). Hirst was intent on acquiring copies of these, and so Plücker afterward donated a set made in boxwood to the London Mathematical Society (see Fig. 2).[12]

Interestingly enough, Hirst himself later took up research in line geometry, which may well have led Segre to strike up a correspondence with him.[13] At any rate, Segre was well aware of the results in Hirst (1879), a paper that was reprinted two years later in the memorial volume for D. Chelini edited by Luigi Cremona and Enrico Beltrami. In a letter to Klein from 7 September 1883, Segre confirmed that

[12]The correspondence between Plücker and T. A. Hirst can be found at http://www.lms.ac.uk/content/plucker-collection). Some of the models he sent to England can still be seen today on display in London.

[13]P. Gario has noted that Segre and Hirst corresponded with one another, (Luciano and Roero 2012, 14).

this paper had been important for his work with Gino Loria because Hirst had, indeed, uncovered several errors in the dissertation of Adolf Weiler, a matter we shall take up below.

Not long after Plücker's death, Klein designed four additional models to show the main types of real singularities that can arise in these so–called complex surfaces (Klein 1922, 7–10). These models were, like the original Plücker models, produced and sold by the firm of Johann Eigel Sohn, located in Cologne. Originally made in zinc, they are far heavier than the more familiar plaster models built afterward. For his *Habilitation* lecture in Göttingen, held in January 1871, Klein presented one of these models in order to convey some newly discovered properties of special curves associated with complex surfaces. In Fig. 3, he illustrated how the double line forms in passing from a general Kummer surface to the special types of quartics studied by Plücker.

Later the next year, he and Clebsch presented two new models of cubic surfaces to the Göttingen Scientific Society. It was on this occasion that Clebsch unveiled his famous diagonal surface which illustrates the intricate configuration formed by the 27 lines that lie on a non-singular cubic (Fischer 1986, Kommentarband, 7−14). The original model was built by Adolf Weiler, who was later Klein's doctoral student. At precisely the same time, Klein presented a second model that illustrated a cubic surface with four real singular points, the maximum possible. These two models represent the two extremes in what later became known as the Rodenberg series, a set of 26 models designed and published by Carl Rodenberg in 1881 (Schilling 1911). Thus, much of the original work in Germany on quadratic line complexes and quartic surfaces was strongly connected to a general interest in

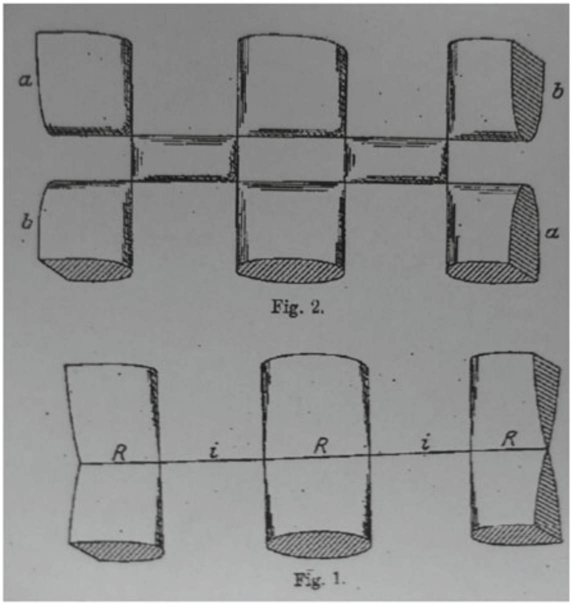

Fig. 3 In passing from a general Kummer surface to a Plücker complex surface, eight of the nodes collapse into four pinch points on the nodal line. The portions of the line marked *R* lie on the real parts of the surface, those marked *i* fall in imaginary regions where the line passes through forming real isolated points (Klein 1922, 8–9)

model making, or what was called *anschauliche Geometrie*. In Klein's case, these would remain lasting interests, but he combined them with the kinds of algebraic methods that Battaglini had introduced for studying canonical forms for quadratic line complexes.

As emphasized above, this tradition drew on strong interest in finding connections with physical phenomena—the Fresnel wave surface being a canonical example—hence the importance of model-making as an aid to studying visualizable geometrical structures. Indeed, the study (Segre 1883a) contains numerous references to earlier investigations from this older tradition. Nevertheless, the program of classifying all quadratic line complexes pointed in new directions that would help shape future geometrical research in the Italian setting. This new trend represented a departure from the viewpoint that geometry should be fundamentally rooted in the study of idealized objects that could be approached by means of models situated in three-space.

4 Klein's Framework for Studying Quadratic Complexes

The turn to a more abstract approach to the study of line complexes can already be seen in Klein's approach to this problem, briefly outlined in this section. Battaglini had assumed that it was always possible to transform the line coordinates p_{ij} so that the quadratic equation $\Omega\,(p_{ij}) = 0$ that defined the complex would be diagonalized while preserving the condition $P = 0$. As Klein noted, however, the equations $\Omega = \sum p_{ij}^2 = 0$ and $P = 0$ both remain unchanged under the 3-parameter group of transformations:

$$p'_{12} = \lambda p_{12}, \quad p'_{34} = \frac{1}{\lambda} p_{34}, \quad p'_{13} = \mu p_{13},$$
$$p'_{42} = \frac{1}{\mu} p_{42}, \quad p'_{14} = \nu p_{14}, \quad p'_{34} = \frac{1}{\nu} p_{34}.$$

In his dissertation (Klein 1884) he therefore went about deriving a general canonical form in Plücker coordinates. Along the way, he also introduced what came to be called Kleinian coordinates $x = (x_i), \quad i = 1, 2, \ldots, 6$ for a quadratic line complex. Using these, the complex will satisfy two equations, and in the general case (where the 6 eigenvalues k_i are all distinct) this will have the simple form:

$$P = \sum_i x_i^2 = 0, \quad \Omega = \sum_i k_i x_i^2 = 0. \tag{4.1}$$

In his dissertation Klein dealt only with this single case, merely noting that there are ten other cases where multiple roots arise. Even so, this ignores that a full investigation using Weierstrass' theory of elementary divisors actually requires an examination of all the different ways multiple eigenvalues can persist among the lower ranking subdeterminants. Such an analysis leads to around fifty cases, a problem first tackled by Klein's student, Adolf Weiler, in Weiler (1874). In several cases, Weiler gave equations for the singularity surfaces of these various complexes, and in all cases he tried to identify the type of quartic surface or degenerate quartic that arose, taking account of the number of double lines and the corresponding forms in order to systematize his classification scheme. After this paper appeared, Klein left the field of line geometry for over a decade.[14] By this time, his friend and early collaborator, Sophus Lie, had also shifted his research interests away from line—and sphere–geometry in order to pursue applications of his theory of transformation groups to differential equations. So it was not until Segre's arrival on the scene that Klein briefly returned to take up his earlier research interests connected with line geometry and Kummer surfaces.[15]

Klein's approach to line geometry exploited algebraic invariant theory, which was totally absent in Plücker's theory. Thus he began by noting that the coefficients in a general linear complex, given by an equation $\sum_i a_i x_i = 0$, lead to a simple invariant when substituted into the form for P. When this invariant vanishes, $\sum_i a_i^2 = 0$, these coefficients can be treated as line coordinates, so the complex is then of the special type corresponding to the lines that meet a fixed line. Taking a second linear complex, $\sum_i b_i x_i = 0$, yields a simultaneous invariant, $\sum_i a_i b_i$, and if this vanishes then Klein refers to complexes in *involution* with one another (Hudson prefers the terminology apolar complexes, which I will use here as it seems better in English).

If both linear complexes happen to be special, then this just says that the two lines they represent intersect one another. If only one is special, then apolarity implies that the distinguished line belongs to the second complex. If neither is special, then the lines common to both determine a congruence of lines with two directrices g, h, which intersect all the lines of the congruence. Hence, taking any plane Π, the complexes determine two points $P_1, P_2 \in \Pi$, as the vertices of the two associated pencils of lines in Π. So the line $\ell = \overline{P_1, P_2}$ lies in the congruence and therefore meets the directrices in two other points, $P_3 = \ell \cap g, P_4 = \ell \cap h$. The complexes are then apolar if for every plane Π the four points on the unique line ℓ in the congruence are harmonically separated. (A dual formulation starting with an arbitrary point P also holds.)

Klein now exploits this framework by introducing six mutually apolar linear complexes that define a special coordinate system. Thus he starts with an arbitrary linear complex, calling this $x_1 = 0$. The condition of apolarity means that we now must chose a second complex from the remaining 4-dimensional space, and this gives us our second fundamental complex $x_2 = 0$. Proceeding accordingly, we are left in the end with but one choice for $x_6 = 0$. Since the coordinates of these six

[14]His last paper on this topic during the 1870s was a short note that followed Weiler's paper, in which he proved that the Plücker complex surface was self-dual as a singularity surface (Klein 1874).

[15]See the papers he published in *Mathematische Annalen* in 1885 and 1886, reprinted in Band I of Klein (1921).

complexes have only one non-vanishing entry, they are clearly mutually apolar. Klein calls these the *fundamental complexes* for this coordinate system, which proves to be ideally adapted for the study of quadratic line complexes.

Taking the first two complexes, $x_1 = 0, \quad x_2 = 0$, he denotes their congruence by (1,2), noting that its directrices have coordinates $\rho\,(1, \pm i, 0, 0, 0, 0)$. But this pair of lines clearly belongs to the other four complexes, and since four linear complexes intersect in two lines, they are so determined. Clearly, the same situation obtains for all the other pairs of these six fundamental complexes: so we have 15 congruences and 30 directrices, and these form a special configuration in space. Taking, for example, the 6 directrices determined by the triple (1, 2), (3, 4), (5, 6), we see that each pair intersects the four other directrices, so these 6 lines form a tetrahedron. Altogether, one easily can count 30 lines which form 15 distinguished tetrahedra in this coordinate system. Each edge is thus shared by three of these 15 tetrahedra. The 60 vertices lie in groups of six in harmonic pairs for each line shared by the three tetrahedra. This entire configuration is thus self-polar with the respect to the six fundamental apolar linear complexes that determine the coordinate system. In addition, each triple of fundamental line complexes, for example (1, 2, 3) representing the intersection of the first three, determines a set of generators on a quadric surface. But since the directrices of the congruences (1, 2), (2, 3), and (3, 1) all belong to the other three complexes, it follows that the lines forming the quadric surface (4, 5, 6) comprise the second set of generators for the same quadric. So altogether there are 10 fundamental quadrics associated with this coordinate system (see Hudson 1905, Chap. 4).

Using these geometric relationships, Klein could now show how a single plane or point, chosen freely, induces an incidence configuration identical with that on a Kummer surface. Thus, by choosing one of the 15 tetrahedra T for such a system of coordinates, each plane E will contain six points, as the vertices of the pencils of lines determined by the six fundamental complexes $x_i = 0,\ i = 1, 2, \ldots, 6$, and these six points will lie on a conic in E. Furthermore, the ten fundamental quadrics determine ten additional points as the poles of E. By the same token, one can take an arbitrary point P, which in the same way generates a system of 16 planes. Thus, an arbitrary point or plane in space, so long as its position is general with respect to the coordinate tetrahedron T, leads to a (16, 6) configuration of points and planes with exactly the same properties that Kummer found for the singularities of the surfaces of fourth order and class named after him. From a geometrical point of view, this coordinate system behaves like a liquid near its freezing point: "dropping" a single point or plane into it causes this special configuration to "freeze" into place, replicating the (16, 6) configuration displayed by the nodes and tropes of a Kummer surface.

Within Klein's framework, the Kummer surfaces arise as the common singularity surface for a 1-parameter family of quadratic line complexes. Starting with a single complex of the form $\Omega = 0$ as in (3.1), he obtains the system:

$$P = \sum_i x_i^2 = 0, \quad \Omega\,(\lambda) = \sum_i \,(k_i - \lambda)^{-1} x_i^2 = 0, \tag{4.2}$$

where the original complex is associated with the parameter $\lambda = \infty$. For an arbitrary line, there will be four quadratic complexes in the family that will contain it. Furthermore, the six fundamental linear complexes (each counted twice) also belong to this family with a common singularity surface Φ.

In Weiler's study (Weiler 1874), the general case in which the eigenvalues k_i are distinct is only briefly mentioned since Klein had discussed it in detail in (Klein 1870). He therefore begins with a discussion of the case in which the original Eq. (3.1) for $P = 0$ still obtains, but two of the eigenvalues are identical, denoted by [1111(11)]. This is the first of six cases in which none of the six fundamental complexes are special. After dealing with these, Weiler then turns to his second type of canonical form:

$$P = \sum_{i}^{4} x_i^2 + 2x_5x_6 = 0, \quad \Omega = \sum_{i}^{4} k_i x_i^2 + 2k_5x_5x_6 + x_5^2 = 0. \tag{4.3}$$

In this case the two fundamental complexes $x_5 = 0$ and $x_6 = 0$ are special and coincide, thus representing a double line in the complex. The most general case here is [11112], in which the singularity surface is a Plücker complex surface and the double line is the nodal line of this special Kummer surface, as discussed above.

In the case of [1111(11)] with the canonical form 3.1, the complex contains two skew double lines and its singularity surface is actually one of the ruled quartics studied by Cremona. This case was discussed in detail by Weiler, as was the one following [111(111)], where the complex has infinitely many double lines that lie on a quadric surface. Here, however, Segre noted that Weiler had misinterpreted the geometric properties of the singularity surface, a topic of discussion in his correspondence with Klein which soon led him to publish (Segre 1884) in order to clarify the true situation.

5 Segre and Loria on Harmonic Complexes

Segre's letters to Klein from the period 1883–85 give a very vivid picture of how their mutual interests in line geometry brought them together, leading to the warm relationship described by Terracini, though in much greater detail in the introduction to Luciano and Roero (2012). Here, we focus only on some of the main issues that concerned the classification of quadratic line complexes. From the letter Segre and Loria wrote to Klein on 16 August 1883 (Luciano and Roero 2012, 81–83), one learns about the circumstances and motivation behind their joint paper, which aims to give deeper insight into the geometry of that special class of quadratic line complexes first studied by Battaglini. Still Battaglini's work, much like Klein's, focused entirely on the generic case derived from the original canonical form, as briefly alluded to in Klein (1870). This theory was also largely algebraic, whereas the context for the investigation of Segre and Loria was the harmonic property for pairs of quadric surfaces, the property first studied by Aschieri. Moreover, all earlier authors (Aschieri, Darboux, and Hirst) had merely dealt with special cases of such complexes. Segre and Loria, on the other hand, made an exhaustive study of the many possibilities that can arise: in (Segre 1883a) they describe 23 different cases that depend on the relative position and type of quadrics involved.

Their paper, in effect, summarizes the results of this investigation. Segre and Loria thus only outline the general method they followed without actually showing any of the calculations needed. At the outset, in fact, they compare two different possible methods, one algebraic and closely related to Weiler's analysis using the theory of elementary divisors. The other approach, which they preferred, is more geometrical and depends on an analysis of the double lines in the complex. These double lines are just the lines in the complex that are self-polar with respect to the two quadric surfaces. Their number and position can then be used to classify the harmonic complex in question. Regarding these two methods, the authors remarked:

> En se servant de ces propriétés caracteristiques d'une droite double d'un complexe *H* on peut trouver géométriquement tous les cas que peuvent présenter deux surfaces du second ordre, par rapport aux droites doubles du complexe *H* qui leur appartient; et puis, du nombre et de la distribution de ces droites doubles on peut déduire, au moyen de la classification de M. Weiler, la classe du complexe *H* que l'on a obtenu pour chaque cas. Ajoutons que c'est seulement par ces considérations géométriques que l'on se rend compte des résultats auxquels la voie analytique précédemment indiquée conduit plus méthodiquement, mais aussi moins lumineusement (Segre 1883a, 215).

In effect, this strategy follows the same general lines of investigation that Weiler had followed, but by doing so Segre and Loria were able to refine his results by adding the condition of harmonicity. Thus, starting from the case [111111], the harmonic condition reduces the number of parameters from 19 (for the general quadratic complex) to 17. Klein had already pointed this out in Klein (1870), where he further noted that the singularity surface is a tetrahedroid. Such a Kummer surface arises when one of the 16 nodes lies in a face of the coordinate tetrahedron. For this case, Segre and Loria pointed out that the two quadrics will be in general position; furthermore, one of them can degenerate to a cone, but not both. If that should occur, then the complex will have only 16 parameters. As in Weiler's study, this paper deals with many types of surfaces that geometers had only recently brought to light. Segre and Loria thus show how these harmonic complexes are closely connected with a number of special types of quartics, which in some cases form the associated singularity surfaces for whole families of a given type, but for many others only a small finite number of complexes occur. Among the more interesting singularity surfaces that arise are Cayley's tetrahedroid, Steiner's Roman surface and its dual the Cayley cubic with four nodes, several of the ruled quartics classified by Cremona, and a few of the cubics first studied by Schläfli, etc.

Klein offered the two authors a number of tips on other literature, which were taken up in some of the footnotes that Segre and Loria added to their original paper. He also asked about recent related work that had been published by the English geometer, Hirst, who had spent a good deal of time around mid-century in Germany. In Berlin he had attended lecture courses taught by Steiner and Dirichlet before going to Marburg to take his doctorate in 1852. In Hirst (1879) he dealt with quadratic complexes that result from collineations between the points in two planes, a special case related to harmonic complexes. Segre and Loria noted that some of his findings revealed errors in Weiler (1874), an observation that naturally drew Klein's attention.

6 On Segre's Second Paper for Klein's Journal

The mistakes in Weiler (1874) were among the many matters addressed by Segre in his reply to Klein from 7 September 1883 (Luciano and Roero 2012, 83–87). Clearly delighted by Klein's positive response, he proceeded to describe how he went about classifying quadratic complexes by means of what came to be known as the theory of projective hyperspaces, an approach Klein had already outlined back in 1872. Segre expressed surprise when he learned from Klein that no one had in the meantime taken up this approach to line geometry in a systematic manner. He also confirmed that Hirst had, indeed, published corrections to errors in Weiler's work, but he went on to explain that the more serious errors were not these, but rather others that concerned those cases where the singularity surface is a double quadric F_2. In view of this he, therefore, decided to submit a second paper for *Mathematische Annalen* dealing with this very topic (Segre 1884).

Segre's letter from September 1883 already explained what had gone wrong in Weiler's paper, so that reading this we gain a clear impression of the main motivation behind (Segre 1884). Weiler's fundamental mistake pertained to his discussion of those cases in which the singularity surface of the complex is a quadric F_2, for which one system of generators $\ell_i \in F_2$ are double lines of the complex. Among such complexes, the most general has the characteristic [111(111)], a case discussed in detail by Weiler as number 2 in his list. What makes this case particularly unusual is that the congruence of singular lines (the lines that join the two singular points in each singular plane), which in general will be irreducible of fourth order and class, here splits into four linear congruences, each with a pair of directrices. These eight lines $r_1, r_{1'}, \ldots, r_4, r_{4'}$ also belong to the singular surface F_2, but to the opposite ruling as the double lines $\ell_i \in F_2$. Each point $P \notin F_2$ then determines four singular lines as those which meet the directrices $r_i, r_{i'}$ and likewise for each plane Π. Weiler emphasized that the caustic surface in this case degenerates into these eight lines, four of which he interpreted as consisting of singular points of the complex, whereas the other four he interpreted as the axes for its singular planes. Segre, however, noted that one must distinguish between primary and secondary directrices in this situation, and that the former alone contained the singular points and planes of the complex.

Felix Klein was clearly impressed by Segre's whole approach to line geometry, but also by the simplicity of the geometrical arguments he presented in (Segre 1884). Many years later, he pointed out how Segre's approach could be used to construct the 1-parameter family of quadratic line complexes with a fixed double quadric F_2 as singularity surface (Klein 1922, 10). Such complexes Q^2 can be identified with a 1-parameter family of conics in a given plane. Thus, making use of Segre's notation as above, if the plane E meets the four generators $r_1, \ldots, r_4$ in the four points $P_1, \ldots, P_4$, there are ∞^1 conics C_2 in E with $P_i \in C_2$. Choosing any one of these determines the four singular lines in E, namely these are the tangents t_i to C_2 through P_i. These four lines then determine four new points $P_i = t_i \cap F_2$, and hence four other generators $r_{1'}, \ldots, r_{4'}$ that belong to the same ruling of F_2 as $r_1, \ldots, r_4$. This follows from the circumstance, as described in (Segre 1884), that the singular lines are determined by these four congruences with directrices $r_i, r_{i'}$. But now taking any other plane Π, the class curve belonging to the complex is completely determined: first by intersecting the plane Π with the four generators r_i, and then by constructing the four singular lines simply by joining these first four points in pairs with the four points $\Pi \cap r_{i'}$. These eight points then fix the conic enveloped by the lines of the complex that lie in Π.

Klein found this particular construction important not only due to its simplicity but also because it seemed to provide a link with the work of his student, Karl Rohn, who in (Rohn 1879) showed how one could start from a double quadric F_2 and deform it to obtain a general Kummer surface. Rohn's idea was to start with the incidence structure determined by eight lines on the quadric, four from each set of generators as shown in Fig. 4. The unshaded areas then correspond precisely to the eight tetrahedra in Klein's model discussed earlier. The nodes can be left fixed, avoiding the edges of the fundamental tetrahedra throughout this deformation which separates the two hyperboloids so that a surface with the same topology as the Kummer quartic emerges.

Klein thought about how this same process could be used to describe the passage from the complex associated with Segre's construction to a general quadratic complex in which the singularities lie on a true Kummer surface (Klein 1922, 10).

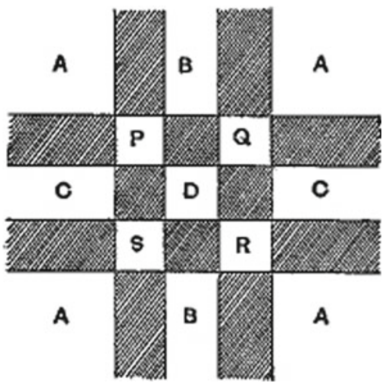

Fig. 4 Rohn's incidence diagram for a degenerate Kummer suface in which the quartic collapses to a doubled quadric surface (Hudson 1905, 98)

Surely there could be no higher praise for the fertility of Segre's methods! Klein also included other references to Segre's work in a few places in his *Gesammelte Mathematische Abhandlungen*, but we can now see from his correspondence with the young Italian that Segre offered much welcome advice back in 1884. Indeed, Klein made good use of this when in that year he decided to republish some of his early work on line geometry in the *Mathematische Annalen*.[16]

Given the polite esteem Segre expressed in these early letters to Klein, one must imagine that he was also attentive to the delicacy of the situation. Not only was he a foreigner who had only recently completed his doctorate (as he dutifully informed Klein, while mentioning that his friend Loria had just attained the same degree), he was now moving into an area of research that Klein had now left behind, after having delegated the details to his student, Adolf Weiler. Segre's dissertation work, while inspired by ideas that Klein had once advocated, represented a far more geometrical standpoint than the one that Weiler had adopted in the course of carrying out ideas Klein had sketched in his own dissertation. Segre was clearly well aware of all these circumstances. Furthermore, Klein had obviously overlooked the mistakes in Weiler's dissertation, a clear reflection that even in the early 1870s he never really acquired a firm control of all the details in this complicated theory.

One can thus well imagine Segre's sense of relief in reading Klein's letters, although we can only form a vague idea of what he wrote from Segre's responses, since unfortunately Klein's letters have not survived. Still, he clearly recognized that Segre was already a leading authority on all matters relating to this domain of geometry, and he reacted in a manner that would become familiar throughout the course of his career. Klein's later success in Göttingen had much to do with his ability to spot young talent and to draw aspiring young mathematicans into his network. His friendly relations with the mathematicians of Segre's generation included, of course, Hurwitz and Hilbert, but also somewhat older figures, including Luigi Bianchi. Thus, Klein's early interest in Segre's work marked an important chapter in the passing of the torch of geometrical research from Germany to Italy in the 1880s.

Line geometry served here as a bridge between the older world of geometrical research and the newer style that soon led to the many more familiar achievements of Italian algebraic geometers. The world on the other side of that bridge had very little relevance for the mathematicians of the twentieth century, and even those with a serious interest in the early history of algebraic geometry were not usually inclined to grant work on line geometry much importance. Thus, when Jean Dieudonné wrote about Plücker coordinates, he merely remarked that "the analogy between the set of lines and the 'quadric' in projective [five–]space… is manifest" (Dieudonne 1985, 13). He further minimized the importance of this field by writing that the topic can be seen today from the viewpoint of "grassmannians", as first set forth in 1845 by Hermann Grassmann. Still, Dieudonné clearly recognized that

[16]His dissertation appeared as Klein (1884) and his note with Lie on the asymptotic curves on Kummer surfaces as Klein and Lie (1884).

geometers like Klein, even if they knew and valued Grassmann's work, were not inclined to remove geometry to an abstract realm of *n*-dimensions. He also knew that the mid 1880s represented an important turning point. Indeed, he quickly added the telling remark that it was not until around 1885 that the study of arbitrary varieties in general projective spaces really began, a development due to the pioneering work of Corrado Segre.

References

Aschieri, Ferdinando, Sopra un complesso di secondo grado, *Giornale di Matematiche* 8 (1870): 35–37; 229–234.

Battaglini, Giuseppe, Intorno ai sistemi di rette di secondo grado, *Atti della Reale Accademia delle scienze fisiche e matematiche di Napoli*, 5 (1866): 194–208.

Battaglini, Giuseppe, Intorno ai sistemi di rette di secondo grado, *Giornale di Matematiche* 6 (1868): 239–283.

Battaglini, Giuseppe, Intorno ai sistemi di rette di secondo grado, *Giornale di Matematiche* 7 (1869): 55–75.

Bottazzini, Umberto, Italy, in (Dauben and Scriba 2002), 61–96.

Cayley, Arthur, On Plücker's Models of certain Quartic Surfaces, *Proceedings of the London Mathematical Society*, 3 (1871): 281–285.

Clebsch, Alfred, Zum Gedächtnis an Julius Plücker, *Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen*, Band 16 (1871): 1–40.

Dauben, Joseph and Scriba, Christoph, eds., *Writing the History of Mathematics: its Historical Development*, (Science Networks, vol. 27), Basel: Birkhäuser, 2002.

Dieudonné, Jean, *History of Algebraic Geometry*. J. Sally, trans., Monterey, California: Wadsworth Inc. 1985.

Giacardi, Livia, Corrado Segre maestro a Torino. La nascita della scuola italiana di geometria algebrica, *Annali di storia delle università italiane*, 5 (2001): 139–163.

Hirst, Thomas Archer, On the complexes generated by two correlative planes, *Proceedings of the London Mathematical Society*, 10 (1879): 131–143; reprinted in *In memoriam Dominici Chelini. Collectanea mathematica*, L. Cremona and E. Beltrami, eds., Mediolani: Hoepli, 1881, 51–73.

Hudson, Ronald W. H. T., *Kummer's Quartic Surface*, Cambridge: Camdrige University Press, 1905; reprinted in 1990.

Fischer, Gerd, Hrsg., *Mathematische Modelle*, 2 Bde., Berlin: Akademie Verlag, 1986.

Klein, Felix, Zur Theorie der Liniencomplexe des ersten und zweiten Grades, *Mathematische Annalen* 2 (1870): 198–228.

Klein, Felix, Ueber Liniengeometrie und metrische Geometrie, *Mathematische Annalen* 5 (1872): 257–277.

Klein, Felix, Ueber die Plückersche Komplexfläche, *Mathematische Annalen* 7 (1874): 208–211.

Klein, Felix, Ueber die Transformation der allgemeinen Gleichung des zweiten Grades zwischen Linien-Coordinaten auf eine canonische Form, *Mathematische Annalen* 23 (1884): 579–586.

Klein, Felix and Lie, Sophus, Ueber die Haupttangentencurven der Kummer'schen Fläche vierten Grades mit 16 Knotenpunkten, *Mathematische Annalen* 23 (1884): 198–228.

Klein, Felix, *Gesammelte Mathematische Abhandlungen*, 3 Bde., Berlin: Julius Springer, 1921, 1922, 1923.

Luciano, Erika and Roero, Clara Silvia, From Turin to Göttingen: Dialogues and Correspondence (1879–1923), *Bollettino di Storia delle Scienze Matematiche* 32(1) (2012): 7–232.

Menghini, Marta, ed., *La corrispondenza di Luigi Cremona (1830–1903)*, Quaderni PRISTEM 9, Milano: Bocconi, 1996.

Painvin, Louis Félix, Étude d'un complexe du second ordre, *Nouvelles annales de mathématiques*, Série 2 (1872): 49–60.

Plücker, Julius, Neue Geometrie des Raumes gegründet auf die Betrachtung der geraden Linie als Raumelement, Erste Abteilung, Leipzig: Teubner, 1868.

Plücker, Julius, Neue Geometrie des Raumes gegründet auf die Betrachtung der geraden Linie als Raumelement, Zweite Abteilung, hrsg. F. Klein, Leipzig: Teubner, 1869.

Rohn, Karl, Transformation der hyperelliptischen Functionen $p = 2$ und ihre Bedeutung für die Kummer'sche Fläche, *Mathematische Annalen* 15 (1879): 315–354.

Schilling, Martin, Hrsg., *Katalog mathematischer Modelle für den höheren mathematischen Unterricht*, Leipzig: M. Schilling, 1911.

Schur, Friedrich, Ueber einen das System zweier Flächen 2. Grades betreffenden Satz und einen damit verbundenen Strahlenkomplex 2. Grades, *Mathematische Annalen* 21 (1883): 515–527.

Segre, Corrado, Loria, Gino, Sur les différentes espèces de complexes du 2e degré des droites qui coupent harmoniquement deux surfaces du second ordre, *Mathematische Annalen*, 23 (1883a): 213–234.

Segre, Corrado, Studio sulle quadriche in uno spazio lineare ad un numero qualunque di dimensioni, *Memorie della R. Accademia delle Scienze di Torino*, 2(36) (1883b): 3–86.

Segre, Corrado, Sulla geometria della retta e delle sue serie quadratiche, *Memorie della R. Accademia delle Scienze di Torino*, 2 (36) (1883c): 87–157.

Segre, Corrado. Note sur les complexes quadratiques dont la surface singulière est une surface du 2° dégré double, *Mathematische Annalen*, 23 (1884): 235–243.

Terracini, Alessandro, Corrado Segre (1863–1924), *Jahresbericht der Deutschen Mathematiker-Vereinigung* 35 (1926): 209–250.

Weiler, Adolf, Über die verscheidenen Gattungen der Complexe zweiten Grades, *Mathematische Annalen* 7 (1874): 145–207.

Ziegler, Renatus, *Die Geschichte der geometrischen Mechanik im 19. Jahrhundert: Eine historisch-systematische Untersuchung von Möbius und Plücker bis zu Klein und Lindemann*, Stuttgart: Franz Steiner Verlag, 1985.

Segre and the Foundations of Geometry: From Complex Projective Geometry to Dual Numbers

Aldo Brigaglia

Abstract

In 1886 Corrado Segre wrote to Felix Klein about his intention to study 'géométrie projective pure', completing and developing the work of von Staudt. He would continue this research project throughout the whole of his scientific life. In 1889, following a suggestion of Segre, Mario Pieri published his translation of the *Geometrie der Lage*, and from 1889 to 1890 Segre published four important papers, "Un nuovo campo di ricerche geometriche", in which he completely developed complex projective geometry, considering new mathematical objects such as antiprojectivities and studying the Hermitian forms from a geometrical point of view with the related 'hyperalgebraic varieties'. Segre developed the same ideas in 1892 in a new paper published in the Matematische Annalen, in which he also considers the so-called "bicomplex numbers" and provided the first example of a projective geometry on an algebra with zero-divisors. In 1891, during one of his celebrated courses in higher geometry, Segre asked his students to find a system of independent axioms for projective hyperspatial geometry. Fano, Enriques, Pieri and Amodeo, who wrote important papers, followed this proposal.

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1 Preliminaries: Von Staudt's Influence and of the Geometrical Introduction of Coordinates

Segre was charged with teaching the course in projective geometry at the University of Turin for the 1885−1886 academic year, two years after he received his degree. At the time, the young Piedmontese was acting as assistant to his thesis advisor, Enrico D'Ovidio, who had immediately discerned his student's extraordinary talent.

Segre developed his interests in the foundations of geometry—which resulted in the work (Segre 1886b)—thanks to this course: "It was precisely for the course in projective geometry that I was charged to teach this year at the University of Turin that I conceived this method".[1]

The course appears above all to have been inspired by von Staudt's concept of geometry, with a strong propensity towards the search for a rigorous framework that would eliminate the many vicious circles of the definitions most common at the time. Before examining this work, it is thus useful to give a brief overview of two aspects present in the mathematical milieu of Turin in which Segre moved.

The interest of Italian mathematicians in the work of von Staudt is quite precocious, and can be traced back to a review by Luigi Cremona of the German mathematician's *Beiträge* (Cremona 1858), which were not even complete at the time and had arrived at the second instalment, thus well before the appearance of the text by Reye (1866) which "began to reawaken interest in von Staudt" (Hartshorne 2008).

As is well known, von Staudt's framework was rigidly purist, in the sense that he refused to lay the basis of projective geometry in algebra, paving the way for the geometrical introduction of projective coordinates. The scheme was the following: introduce as a basic concepts harmonic quadruple and projectivity (the transformations that conserve the harmonic sets) to arrive at the fundamental theorem of projective geometry (there is one and only one projectivity between two forms of prime species that transform three given points into another three which are also given) to arrive at a kind of algebra of sets of four aligned points (*Würfe*) and to the isomorphism of that algebra with that of real numbers (projective coordinates).[2]

Further, in particular in the *Beiträge* (Staudt 1856–60), von Staudt had also attempted to free himself from algebraic notions with regard to the extension of projective geometry to complex numbers. His idea, which would be developed by Segre and about which we will say more below, is that of defining the pairs of conjugate complex points as elliptical involutions, that is, the involutions on a line without fixed points. These procedures are strictly related to the development of a modern concept of axiomatics.

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[1]"*Fu appunto pel corso di Geometria Proiettiva che ero incaricato di fare quest'anno nell'Università di Torino che imaginai questo metodo*" (Segre 1886b). In actual fact, Segre's interest in some aspects of the foundations of geometry might be traced back to his degree thesis. For example, the definition (albeit somewhat inexact) of vector space, the concept of isomorphism, the idea of the point as an undefined concept. I will not go into these aspects here. I refer the interested reader to Avellone et al. (2002).

[2]On von Staudt's work, see for example Freudenthal (1981) and Nabonnand (2008).

Cremona's interest, however, had no immediate visible effects on the teaching of projective geometry in Italy. As in the rest of the mathematics community, interest in the topic was reawakened by the publication of Klein's reflections (Klein 1873) on the foundations of geometry, in which, among other things, it was shown that the passage from the synthetic framework to the geometrical introduction of coordinates in the real line could not be justified in the absence of a geometric axiom on continuity, analogous to the arithmetical ones of Dedekind and Cantor, announced just a year earlier, in 1872. The one who developed the ideas of von Staudt, bearing in mind the debate that had arisen as a result of Klein's article, was a direct student of Cremona, Riccardo De Paolis, who published a significant work in 1881 (De Paolis 1880–81).[3] Thus we can see that by 1885, von Staudt's works were sufficiently well known in Italian scientific circles, even though they had little effect at the level of teaching.

However, Segre was not alone in moving in this direction[4]: in Turin it was precisely his *maestro* D'Ovidio who collaborated with Achille Sannia, his former professor and brother-in-law, who was engaged in those years in compiling his *Lezioni di Geometria Proiettiva* (Sannia 1891) which would adopt wholly the framework of von Staudt. At D'Ovidio's suggestion, Segre made direct contact with Sannia. Here is how he describes it in (Segre 1886b) (taking up the quotation that we cited earlier):

> It was precisely for the course in projective geometry that I was charged to teach this year at the University of Turin that I conceived this method. The most esteemed Prof. Sannia also introduced this theory in the Lessons of Projective Geometry that he was publishing … for that and for the correspondence that we had about it, and that was certainly of no little use in conceiving this work, I thank him here. It is precisely thanks to the publication of that treatise that here I save myself from giving my treatment an elementary character, I can, that is, omit going into too many small details.[5]

I believe that in this interaction, in the final analysis, Segre's influence was preponderant; in any case, it seems to me that Sannia's work in those years provided an important interpretative key, with respect as well to Segre's geometric

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[3]For more on this work, see Avellone et al. (2002).

[4]Perhaps it is worthwhile to note that an analogous process—that is, regarding a deepening of the topics related to the foundations of geometry born of the reflections that arose within an introductory course in geometry—would develop a few years later with the course in geometry given in Bologna by Federigo Enriques. For more about this course and its reflections in the articulation of his views relative to the foundations of geometry, see Ciliberto and Gario (1861). The close ties between didactic requirements and foundational necessities remained characteristic of Italian geometry for a long time.

[5]*Fu appunto pel corso di Geometria Proiettiva che ero incaricato di fare quest'anno nell'Università di Torino che imaginai questo metodo. Il chiar. prof. Sannia introduce pure questa teoria nelle Lezioni di Geometria Proiettiva che egli sta pubblicando … di ciò e della corrispondenza che intorno ad essa abbiamo avuto e che non mi fu certo inutile nel pensare questo lavoro gli faccio qui i miei ringraziamenti. Appunto grazie alla pubblicazione di quel trattato posso qui risparmiarmi di dare alla mia esposizione un carattere affatto elementare, posso cioè omettere di entrare in dettagli troppo minuti* (Segre *Opere*, v. 2, 211).

conception. I believe it is useful to quote what Sannia wrote in the preface to his *Lezioni*, which came out only in 1891 after much travail:

> … during printing, which began in October 1885, the desire, indeed, the need, to expound some of the fundamental theories in a new and more rigorous way, made the volume grow to more than 600 pages; which, together with other serious reasons, … delayed the publication at length. This forced me to put an end to the publication itself, without being able to carry out the entire programme that I had set for myself … I reserve for the aforementioned reprinting the preface, in which I count on going into details … about how much I owe to the correspondence with the most esteemed Prof. C. Segre.[6]

Thus the text incorporates not a few of Segre's suggestions,[7] which in my opinion shows that the very young mathematician (in 1885 he was 21 years old) played a role from the very beginning of this trend, among mathematicians not only older than he, but who also enjoyed much greater academic prestige. It is "the need" of a "new and more rigorous way" that would bring together the older mathematician from Molise and the very young one from Piedmont.

It is precisely in this milieu of the Torinese university, permeated with the search for new and rigorous ways, that Segre moved. Some years earlier, Peano had taken Genocchi's place in the course of analysis, while 1884 saw the publication of Genocchi (1884), in which Peano's additions contributed to the outline of a systematic programme of the rigorisation of analysis. Moreover, in that same period both Segre and Peano were engaged in the study of the foundations of mathematics, in particular, for geometry, of Pasch's *Vorlesungen* (Pasch 1882). In Segre (1886b), he shows a exigency for rigour and a detachment from the expositions of the texts most widely disseminated that is not much different from those of Peano.[8] The search for greater exactitude and greater rigour in the expression of the basic concepts of geometry, avoiding the frequent paralogisms, are found, for example, with respect to the problem relative to the introduction of points at infinity. Segre writes:

> The only rigorous way to introduce, for example, points at infinity as geometric entities, is had by being able to make use of reasoning, and of defining them not as the point of

[6]*… durante la stampa, incominciata nell'Ottobre 1885, il desiderio, anzi il bisogno di esporre alcune teorie fondamentali nel modo più nuovo e più rigoroso, facesse crescere il volume sino a più che 600 pagine; il che, insieme ad altre gravi cagioni, … menò a lungo la pubblicazione dell'opera. Ciò mi costringe a porre un termine alla pubblicazione stessa, senza aver potuto svolgere tutto intero il programma che mi ero proposto … riserbo all'accennata ristampa la prefazione, nella quale conto entrare in particolari … su quanto debbo alla corrispondenza col ch.º Prof. C. Segre* (Sannia 1891, XI–XII).

[7]Segre's contributions emerge amply in Sannia's correspondence with Federico Amodeo, Segre's student in Turin. These letters can be found in Palladino and Palladino (2006), which also contains a lengthy introduction that also deals with the development of Sannia's text.

[8]I think that the divergences between Segre's and Peano's mathematical ideas, at least before 1891, have been largely overstated. In my opinion, the controversy in 1891 cannot br reduced to a polemic of Peano against the alleged lack of rigour in Segre's works (as in the biography in the website of the Mac Tutor University).

intersections of parallel lines, that is, as lines that have no points of intersection (as do substantially all authors), but rather as synonymous of directions, or, if you will, of stars of parallel lines. The consideration that is usually made of the fact that when two lines in a plane tend to become parallel their point of intersection moves infinitely far away can only serve to justify the choice of the term point at infinity, but not to define it, as desired, as must be desired … so that it signifies a geometric entity. After all it can with much care be made the introduction to elements at infinity (an argument in which even von Staudt, a most precise author, leaves something to be desired) in the important *Vorlesungen über neuere Geometrie* by Pasch.[9]

This shows a need for rigour and precision not very different from that shown by Peano in his additions to Genocchi's *Lezioni*. Segre shows the same attitude with respect to the problem that interests him the most in the work cited, in which he criticises even the famous text by Reye:

It doesn't appear to me that the few pages dedicated by Reye … to the imaginary elements contain, as he says, the fundamental principles of the theory of von Staudt. In fact, he defines the imaginary elements precisely as the two fixed elements of a projectivity between real forms of the 1st species, which do not have (real) fixed elements. That definition, which other authors use as well, seems to me should be absolutely rejected, because it evidently contains something absurd in itself, and at the same time introduces the imaginary elements as a term that does not stand for any geometric entity. It is similar to the following definitions given by some to a pair of imaginary conjugate points in a line: the pair of points of intersection of this with a circle that does not intersect it. But Staudt did not do this. He defined … an imaginary element.. as an elliptical involution on a fundamental form of the 1st species together with one of the two directions of the form; and if you look carefully, it is the involution itself.. that he calls imaginary element and not its double element, since by hypothesis it does not have double elements. Thus the imaginary element constitutes, according to Staudt, a genuine geometric entity (real, although not of the same species of entities as the real element of the same name) about which it is legitimate to reason and that can thus be taken as the object of study.[10]

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[9]*Il solo modo rigoroso d'introdurre ad esempio i punti all'infinito come enti geometrici, sì da poterne far uso nei ragionamenti, è di definirli non già come punto d'intersezione di rette parallele, cioè di rette che non hanno punti d'intersezione (come fanno in sostanza quasi tutti gli autori), ma bensì come sinonimo di direzioni, oppure, se si vuole di stelle di rette parallele. La considerazione che si suol usare del fatto che quando due rette di un piano tendono a diventar parallele il loro punto d'intersezione s'allontana indefinitamente non può servire che per giustificare la scelta della locuzione punto all'infinito, ma non per definirla, se si vuole, come si deve volere ... che esso significhi un ente geometrico. Del resto si può con quanta cura vada fatta l'introduzione degli elementi all'infinito (argomento in cui persino lo Staudt, scrittore accuratissimo, lascia qualche cosa a desiderare) nelle importanti Vorlesungen über neuere Geometrie del Pasch* (Segre *Opere*, v. 2, 210).

[10]*Non mi pare che le poche pagine dedicate dal Reye ... agli elementi imaginari contengano, com'egli dice, i principii fondamentali della teoria di Staudt. In fatti egli definisce gli elementi imaginari appunto come i due elementi uniti di una proiettività di forme di 1^a specie sovrapposte reali, la quale non abbia elementi uniti (reali). Tale definizione, che anche altri autori usano, mi pare assolutamente da rigettare, perché contiene evidentemente in sé qualche cosa di assurdo, e nello stesso tempo introduce gli elementi imaginari come una locuzione che non sta per significare alcun ente geometrico. Essa rassomiglia alla seguente definizione che alcuni danno di una coppia di punti imaginari coniugati di una retta: la coppia dei punti d'intersezione di questa con un circolo che non lo incontri. Ma non così faceva Staudt. Questi definisce ... un elemento imaginario ... come una involuzione ellittica su una forma fondamentale di 1^a specie insieme con uno dei due*

I wanted to insert this long quotation because it so well expresses the central point (and the didactic validity) of Segre's work: to make his own the essential points of von Staudt's systematization, simplifying some points in order to make it transmissible at an elementary level. As I mentioned earlier, in this case, what is obtained by developing a projective geometry in which the constitutive elements are not the individual imaginary elements, but pairs of them, without the need to distinguish them by means of the complicated way devised by von Staudt. Segre succeeds superbly in this work, which constitutes his debut regarding problems of the foundations.

In these very first years of scientific activity there were two fixed lines of thinking that Segre would follow constantly: to give a rigorous foundation to the theory of hyperspace, and to follow von Staudt's line in defining complex (and later hypercomplex) geometric entities outside of algebra:

> I am in the course of writing a work in Italian in pure projective geometry on the theory of pairs of imaginary elements. The theory that Staudt gave in his Beiträge is complicated by the separation that he makes of conjugate imaginary elements; that complication is the fact that we have not yet introduced this theory in the treatises of projective geometry. But I notice that in him it is sufficient to introduce imaginary elements by pairs: and then I have managed to give the theory all of the desired simplification (renouncing the separation), and maintain the definition of Staudt but by means of considerations that appear to me to be new.[11]

In the years between 1886 and 1891 (during which, among other things, Peano published his masterful "trilogy" (Peano 1888, 1889a, b) on the foundations of mathematics) Segre devoted his attention deeply to the foundations, in the spirit of Staudt (1847). In keeping with the primarily didactic intentions behind these studies, the later cultural activity to which he dedicated himself was the publication of the Italian translation of von Staudt's *Geometrie der Lage*, on which he collaborated with Mario Pieri. As usual, among the first to be told of his intentions was Felix Klein:

(Footnote 10 continued)
versi della forma; e si badi bene, è l'involuzione stessa … che egli chiama elemento imaginario e non già un suo elemento doppio, poiché per ipotesi essa non ha elementi doppi. Quindi l'elemento imaginario costituisce secondo Staudt un vero ente geometrico (reale, benché non sia della stessa specie di enti che l'elemento reale omonimo) intorno a cui è lecito ragionare e che si può quindi prendere come oggetto di studio (Segre *Opere*, v. 2, 209).

[11]*Je suis en train d'écrire un travail en italien de géométrie projective pure sur la théorie des couples d'éléments imaginaires. La théorie que Staudt a donnée dans ses Beiträge est compliquée par la séparation qu'il a faite des éléments imaginaires conjugués; cette complication est ce qui fait qu'on n'a pas encore introduit cette théorie dans les traités de géométrie projective. Mais je remarque que dans ceux-ci il suffirait d'introduire les éléments imaginaires par couples: et alors je suis parvenu à donner à la théorie toute la simplicité désirable (en renonçant à la séparation), en conservant la définition de Staudt mais par des considérations qui me semblent nouvelles* (Luciano and Roero 2012).

> My admiration for the Geom[etrie] d[er] Lage (and Beiträge) of Staudt will have the satisfaction of seeing the appearance of an Italian version (for the present, only of the G.d. L.): which, since my eye malady does not permit me to carry it out, a young geometer who is here, Mr Mario Pieri, will do it.[12]

The introduction to the translation was written, as is known, by Segre himself, and in my opinion is a valuable critical-historic work. Expressed there are some of the guidelines of his scientific programme. Consider, for example, the following quotation:

> Treating the whole of geometry of position internally, without introducing metric concepts that are be foreign to it, constitutes a great step forward, since while it is true that various sciences must help each other mutually, and that many great discoveries are derived precisely from linking their apparently very disparate doctrines, it is no less true that results just as important have been seen … when … an attempt has been made to reduce the postulates, methods and instruments of research to the smallest possible number.[13]

It should also be noted that this translation, with the related annotations, completed in October 1888, when Peano had just published his first work of logic, constitutes the Pieri's first foray into the foundations of geometry.[14] Even at that time Pieri appears to have shared with Peano standards of rigour that were higher than those of Segre, as shown, for example, when he objects to the indiscriminate use that von Staudt makes of "hidden axioms", writing, "Many of the propositions contained in this section and in the following Sects. 2 and 3 are given without proof, because they are true postulates: on these the entire work is founded".[15]

To be sure, taking into account the publication that same year of Peano's principles of geometry, by then the path of the two Piedmontese mathematicians had clearly diverged: while Peano was involved in a programme of a rigorous resystemisation of existing mathematics, Segre envisioned a rigorisation and abstraction that would form the basis on which new results could be extended and constructed. What he wrote in the preface to Pieri's translation of von Staudt is sufficiently indicative:

> The importance of a new doctrine is greater the more capable it is of being extended and the more important the research is that derives from it … Thus in the research now underway in

[12] *Mon admiration pour la Geom. d. Lage (et Beiträge) de Staudt va avoir la satisfaction d'en voir paraître une version italienne (pour à-présent, seulement de la G. d. L.): ce que ma maladie aux yeux ne me permettait pas de faire, un jeune géomètre qui est ici, M. Mario Pieri, le fera* (Luciano and Roero 2012).

[13] *Trattare tutta la geometria di posizione da sé, senza introdurre concetti metrici che le sarebbero estranei, costituì un grande progresso, poiché se è vero che in varie scienze debbono prestarsi scambievoli aiuti e che molte grandi scoperte son derivate appunto dal collegare fra loro dottrine apparentemente molto disparate, non è men vero che risultati altrettanto importanti si son visti ... quando ... si è cercato di ridurre i postulati, i metodi e gli strumenti di ricerca al minor numero possibile* (Staudt 1889, XI).

[14] As is known, Pieri published his first work on the foundations five years later, in 1894.

[15] *Molte delle proposizioni contenute in questo paragrafo e nei successivi paragrafi 2 e 3 sono date senza dimostrazione, perché veri postulati: su di essi e fondata tutta quanta l'opera* (Staudt 1889, 1).

> the projective geometry of spaces of higher dimensions as in almost all of the work of S[taudt] it is possible, without altering the nature of the methods, to extend it to that science. Finally, we note that recently it has been possible to go further with that work, beginning to do for the synthetic treatment of plane curves of higher order (with their imaginary elements) what S[taudt] did for the curves and surfaces of 2nd order; and more, much more, we will shortly see done, again following in the footsteps of that great man.[16]

As we can see, this is a conception of the foundations that is very closely tied to the needs of the developments of advanced research.[17]

2 Segre's Antiprojective Mappings and Their Reception

It is worthwhile to repeat the last line of the previous quotation: "more, much more, we will shortly see done, again following in the footsteps of that great man". I believe that Segre is referring here to the in-depth study that he was doing on complex geometric entities, which would be brought out between 1889 and 1891. This is a work that clearly reveals (starting with its title: *Un nuovo campo di ricerche geometriche*, "A new field of geometric research") Segre's ambitions of its being foundational for geometry. It is perhaps necessary to say that, in my opinion, the title is not excessively presumptuous: Segre's work indeed appears completely original and foundational with respect to a new field a geometric research, giving a definitive meaning to the term "complex projective geometry", no longer seen as an "extension" of real geometry, but as a geometric object in its own right, to be studied for itself. I will briefly outline some of the features of this work.

The central point consists in the introduction and detailed study of anti-projectivities. I will recall briefly that von Staudt had characterised projectivity as the transformations that transform harmonic sets into harmonic sets, and had noted how, when we pass to complex geometry, that characterisation ceases to be valid in as much as there exist transformations that have that property, but are not projectivities. In general such transformations do not conserve the cross-ratio, but change it into its conjugate (in von Staudt's terminology, changing the direction of

[16] *L'importanza di una nuova dottrina è tanto maggiore quanto più capace è essa di venir estesa e quanto più importanti sono le ricerche che ne derivano. ... Così nelle ricerche che ora si vanno facendo nella geometria projettiva degli spazi superiori come quasi tutta l'opera di S[taudt] si possa, senza mutare la natura dei metodi, estendere a quella scienza. ... Infine osserveremo che recentemente si è riusciti a proseguire quell'opera cominciando a fare per la trattazione sintetica delle curve piane d'ordine superiore (coi loro elementi immaginari) ciò che S[taudt] aveva fatto per le curve e le superficie di 2° ordine; e di più, molto di più, si vedrà fatto tra breve, sempre seguendo le orme di quel grande* (Staudt 1889, XIII).

[17] An observation of Enriques has always struck me as being a keen interpretation that clearly distinguishes the interest of the algebraic geometers in the foundations from those of the logicians: *il concetto della geometria astratta ha ricevuto un grande sviluppo, divenendo poi (dopo Segre) un ordinario istrumento di lavoro nelle mani dei geometri italiani contemporanei* (the concept of abstract geometry has received a great development, becoming (after Segre) an ordinary tool for working in the hands of contemporary Italian geometers) (Enriques 1922, 139−40).

a *Wurf*), and thus preserves harmonicity. What Segre does is study those transformations, to which he gives the name anti-projectivities, describing them analytically in terms of linear algebra and determining their properties.

Segre begins from a continuous transformation in a complex line that conserves the harmonicity of four points and that transforms three points A, B, C into three points A', B', C', and he proves that these conditions determine exactly two transformations. One of these is linear; the other, by definition, is called anti-linear. In this second case the transformation is defined as an anti-projectivity, a definition and concept that is entirely new. The characterisation of these transformations in the sense closest to that of von Staudt, can become that of transforming chains into chains.[18] Segre immediately recognised that the projective and anti-projective transformations of the Riemann sphere constitute what Möbius called "circular affinities". This is how Segre justifies the definition that he gives for anti-projectivities:

> An anti-projectivity between two simple forms is … a one-to-one, continuous, and non-projective correspondence such that harmonic sets correspond to harmonic sets. … It is identified by 3 pairs of homologous elements. Two homologous tetrads in it have conjugate cross-ratios … The product of two anti-projective correspondences is a projectivity. The product of two correspondences of which one is projective and the other anti-projective, is an anti-projectivity.[19]

Segre's treatment is extremely detailed, and among other things he determines the analytical form of the anti-projectivity, which turns out to be obtained from the product of a projectivity by the conjugation, their relationship to the Hermitian forms, the characterisation of the chains by three elements, etc.

The extension of the complex projective line to complex spaces of dimension *r* gives rise to a very careful study. One result, for example, is that in projective spaces with an odd number of dimensions, such as the line, each anti-involution has either an infinite number of fixed points or none at all, while in spaces with an even number of dimensions, the number of fixed points is necessarily infinite. In the case of a complex projective line (Riemann's sphere) all that appertains to elementary geometry and was substantially known, for example, the anti-involutions as inversions (with a circumference of fixed points respect to real circles, and without

[18]It should be recalled that, viewed on Riemann's sphere, chains are constituted of circular circumferences (or by lines) and that among the anti-projectivities, the ones that are involuted are the well-known circular inversions (or reflections in the case of lines) of which the chains are the fixed points.

[19]*Un'antiproiettività fra due forme semplici è ... una corrispondenza univoca e continua non projettiva tale che a gruppi armonici corrispondono gruppi armonici ... Essa è individuata da 3 coppie di elementi omologhi. Due tetradi omologhe in essa hanno ... birapporti complessi coniugati ... Il prodotto di due corrispondenze antiprojettive è una projettività. Il prodotto di due corrispondenze, di cui una sia projettiva e l'altra antiprojettiva, è un'antiprojettività* (Segre *Opere*, v. 2, 250).

fixed points with respect to imaginary circles). However, completely new was the identification of Möbius transformations as the ones that conserve the harmonicity of the tetrads of aligned points, and above all the analogue of the fundamental theory of projective geometry for complex lines, mentioned earlier. But the most remarkable result in this direction was without a doubt the extension of the entire theory to any dimension whatsoever; in that case Segre proves that an anti-involution can always be given by means of a Hermitian form, and defines the hyperconics and hyperquadrics[20] as the set of (complex) isotropic vectors with respect to a given Hermitian form.[21] More generally there can be hyperalgebraic varieties, which Segre will in any case introduce in a later work.

I will not go into all the minute implications of this important work, and will limit myself to remarking some of the aspects destined to have a certain impact, at least potential, on future developments. In fact, Segre notes more than a few points of contact between his studies, in particular when translated in analytical terms, with other contemporary studies, as for example when he writes:

> Already in certain recent analytical research works there is a particular example of things that will be treated geometrically here. In the studies … on the functions of a complex variable the simple chains described by this, that is, the circles that represent them in the plane or on the sphere on which the variable is extended are frequently used: in particular they were used in the research on functions that remain unchanged for a group of linear transformations of the variable, and in particular in the brilliant and profound works of Mr Poincaré on Fuchsian and Kleinian functions. … And effectively in the research works on the functions of two complex variables carried out in recent years by Mr. Picard and Mr Poincaré, and especially those of the former on the first of the functions that he calls hyperfuchsian, the use of hyperconics is found, defined in a way that is different from ours … Mr Picard thus finds himself necessarily led to research on the reduction of the (3) [the Hermitian form] to a canonical form whose results in the geometric treatment will appear evident.[22]

[20]The term “hyperquadric” is introduced here with a meaning that is different from what today is commonly understood in the sense of a set of isotropic vectors of a (real) bilinear form of a hyperspace.

[21]Here I am giving Segre’s results in modern language; at the time that Segre was writing the term “Hermitian form” was not in use; however, the analytical expressions that he uses are exactly the ones relative to those forms. Instead, I will not give all of the geometric rationalisations from which Segre, in pure von Staudt style, derives the analytical forms; this would require a lengthy treatment that goes far beyond the scope of this present discussion.

[22]*Già in certe ricerche analitiche recenti si [ha] un esempio particolare di cose che qui si tratteranno geometricamente. Negli studi ... sulle funzioni di una variabile complessa le catene semplici descritte da questa, cioè i circoli che le rappresentano nel piano o nella sfera su cui la variabile vien distesa sono usati frequentemente: in particolare essi furono adoperati nelle ricerche delle funzioni che non mutano per un gruppo di trasformazioni lineari della variabile, e in particolare in quelle geniali e profonde del sig. Poincaré sulle funzioni fuchsiane e kleiniane ... Ed effettivamente nelle ricerche sulle funzioni di due variabili complesse fatte in questi ultimi anni dai sig.*[i] *Picard e Poincaré e specialmente in quelle del primo sulle funzioni che egli chiamò iperfuchsiane, si trovano usate le iperconiche, definite in modo diverso dal nostro ... Il sig. Picard si trova così condotto necessariamente a qualche ricerca sulla riduzione della (3) a forma canonica i cui risultati nella trattazione geometrica appariranno evidenti* (Segre *Opere*, v. 2, 246 −247).

He adds in a note: "I will add that the forms of type (3) [Hermitian]… with integer complex coefficients have also already been introduced in number theory thanks to Mr Hermite, Mr Picard and others".[23]

In the period in which Segre was writing, the theory of Hermitian forms underwent a significant development, particularly in the area of number theory and in that of functions of complex variables. He was, as the preceding quotation shows, well aware of this. In my opinion, it should be noted with regard to Segre's work that, while of great significance for the systematic framework that he gives to mathematical tools developed up to that time in diverse contexts, it is precisely the part that the Piedmontese mathematician probably had most at heart that did not achieve the prominence that it perhaps merited. In his commemoration of his friend, Castelnuovo commented:

> These research works … have not to the present received the widespread support that he perhaps believed they would. Innovations and generalisation penetrate slowly into science, unless they lead to an economy of thought in the study of those problems that, in a given period, attract the attention of researchers.[24]

Segre also developed his ideas in another work published shortly after (Segre 1891d), which thanks to its publication in an important international journal, but also for its slant that was more related to problems that were more usual at the time among mathematicians, had something of an echo. Although this work had many points of contact and many results analogous to those of the earlier one, the point of view is different. As Segre himself says in the introduction, while in Segre (1889–90, 1890–91) he was working directly in complex space and proceeded by analogy with the real case, here he preferred to reinterpret the same concepts in the real representation of complex numbers (thus in the Möbius plane or on the Riemann sphere in the one-dimensional case). As in the previous case, definitions are given of projectivity, anti-projectivity, chains, hyperconics and hyperquadrics, and significant results are obtained. To cite only one example, the representation analogous to that of the complex projective line on a real sphere leads to representing the complex plane on a four-dimensional variety Σ in real projective space of eight dimensions; that the images on Σ of the hyperconics of the complex plane are obtained as hyperplane sections of Σ, exactly as the chains of the complex projective line represented on the sphere, are its plane sections. This is followed by an accurate description of the hyperalgebraic variety, complete with an adequate elaboration of terminology: "fili" (complex points that correspond to a real curve), "tele" (those that correspond to a real surface), and so forth.

[23]*Aggiungerò che le forme del tipo (3)... a coefficienti complessi interi si sono pure già introdotti nella teoria dei numeri grazie ai sig.ⁱ Hermite, Picard ed altri* (Segre *Opere*, v. 2, 247).

[24]*Queste ricerche ... non hanno trovato sinora il largo appoggio sul quale egli forse faceva assegnamento. Le innovazioni o generalizzazioni penetrano lentamente nella scienza, a meno che esse non portino una economia di pensiero nello studio di quei problemi che, in un determinato periodo, attirano l'attenzione dei ricercatori* (Castelnuovo 1924).

Rather than going into the harvest of particular results obtained in this work, I will limit myself to a few considerations. In my opinion, Segre is well aware that this work is not so much aimed at the study of particular results as it is at furnishing a unifying scheme for the numerous areas of mathematics in which are found studies related to complex numbers with several variables. He thus writes:

> It is perhaps not inopportune to note that the topics mentioned in this [present] Note not only offer interests that are both geometric and analytical, and especially algebraic, in themselves, but can also provide numerous aids to a many mathematical theories. Wherever there are complex variables alongside which must be considered their conjugates, or, what amounts to the same things, wherever there occurs the necessity to consider, in complex variables or in their functions, the two real components separately: thus in general in the theory of functions of one or more complex variables (for example, of the automorphic ones); in questions that are strictly connected to that theory, of conform representations, of minimal surfaces, etc.; in certain modern studies in number theory (complex integers) and in particular groups of substitutions; ….[25]

In effect, it will be to number theorists that he ultimately addresses himself, given the substantial indifference of his principal interlocutors, including Castelnuovo. He would write to Hurwitz in this regard:

> I take this opportunity to recall your attention to my notes (which you have) entitled "Un nuovo campo di ricerche geom." and "Le rappresentaz.ⁱ reali delle forme complesse …" because if you, going forward with your arithmetic research, pass to the forms of Hermite, you will perhaps find some points of contact with my works. In fact, I study there, among other hyperalgebraic entities, those that I call hyperconics, etc., which are represented analytically by the forms of Hermite: as the conics, etc., give the geometric images of the forms of Dirichlet (ternaries, etc.). Neither Fricke nor Bianchi have yet profited from these works of mine, but I am persuaded that profit can be drawn from studying arithmetic questions with geometric aids.[26]

[25]*Non è forse inopportuno di rilevare che gli argomenti accennati i questa Nota non offrono solo interessi, sì geometrici che analitici e specialmente algebrici,* per se *stessi, ma possono fornire molteplici aiuti a parecchie teorie matematiche. Dovunque compaiono variabili complesse accanto a cui si debbano considerare le coniugate, o, ciò che fa lo stesso, dovunque accade di dover considerare, nelle variabili complesse o nelle loro funzioni, staccatamente le due componenti reali: quindi in generale nella teoria delle funzioni di una o più variabili complesse (ad esempio di quelle automorfe); nelle questioni strettamente connesse a quella teoria, delle rappresentazioni conformi, delle superfici minime, ecc.; in certe moderne ricerche sulla teoria dei numeri (interi complessi) e di particolari gruppi di sostituzioni …* (Segre *Opere*, v. 2, 340).

[26]*Colgo quest'occasione per richiamare la Sua attenzione anche sulle mie Note (che Ella ha) intitolate "Un nuovo campo di ricerche geom." e "Le rappresentaz.ⁱ reali delle forme complesse …" perché se Ella, proseguendo le sue ricerche aritmetiche passerà alle forme di Hermite, troverà forse qualche punto di contatto con quei miei lavori. Infatti io studio ivi, fra gli altri enti iperalgebrici, quelli che chiamo iperconiche, ecc. che sono rappresentati analiticamente da equazioni di Hermite; e che così danno l'equivalente geometrico delle forme di Hermite: come le coniche, ecc. danno l'immagine geometrica delle forme di Dirichlet (ternarie, ecc.). Né il Fricke né il Bianchi non hanno ancora approfittato di questi miei lavori, ma io sono persuaso che un profitto se ne possa trarre studiando le questioni aritmetiche con sussidi geometrici* (Letter from Segre to Hurwitz, 29 June 1894, in Luciano and Roero (2012).

3 Algebraic Structures, and in Particular Dual Numbers

The most interesting part of the work is, in my opinion, that in which he introduces a new hypercomplex algebra, that of bicomplex numbers.[27] I will look at the topic only from the algebraic point of view.

A bicomplex number is given by:

$$a + bi + cj + dk;\ a, b, c, d \in R;\ i^2 = j^2 = -1,\ k^2 = 1,\ ij = ji = k,\ jk = kj = i,\ ki = ik = i.$$

As can be easily seen, this is a commutative algebra, but with divisors of zero ("nullifici" to use Segre's terminology). As the Piedmontese mathematician himself recognised immediately, this algebra is isomorphic to a subalgebra of biquarternions, introduced by Hamilton in 1854.

One particularly important characteristic of the introduction of this algebra lies in the fact that Segre consciously intended to insert Italian geometric and algebraic research into the furrow of a line of inquiry that was among the most advanced (and destined to undergo extraordinary development in the course of the next century). He writes:

> These bodies were studied in general by Mr Weierstrass, whose results were published … only recently [1884] … This publication was quickly followed … by others … of Mr Schwarz (1884), Mr Dedekind (1885), Mr Holder (1886).[28]

Naturally, the presence of new properties, such as the existence of zero divisors, was carefully highlighted:

> The fundamental point at which general complex numbers of several units … detach themselves from ordinary complex numbers of a single imaginary unit is that while for ordinary complex numbers a product is null only when one of its factors is null, in the most general fields there exist particular non-null numbers, which when multiplied by suitable numbers that are likewise non-null give zero.[29]

[27]In this present work I will only mention the topic. A group of historians of mathematics in Palermo is currently carrying out a more detailed examination of Segre's contributions to those studies.

[28]*Tali corpi furono studiati in generale dal sig. Weierstrass, i cui risultati vennero pubblicati ... solo recentemente [1884] ... Questa pubblicazione fu tosto seguita ... da altre ... dei sig. Schwarz (1884), Dedekind (1885), Holder (1886)* (Segre *Opere*, v. 2, 385).

[29]*Il punto fondamentale in cui i numeri complessi generali a più unità ... si staccano dagli ordinari numeri complessi ad una sola unità imaginaria è quello che ..., mentre pei numeri complessi ordinari un prodotto s'annulla solo quando s'annulla uno dei suoi fattori, nei campi più generali esistono dei numeri particolari non nulli, i quali moltiplicati per convenienti numeri parimenti non nulli danno zero* (Segre *Opere*, v. 2, 386).

Also significant, to my mind, is the attention that he gives to the "structural" properties of this algebra, such as the fact that each bicomplex number can be decomposed into the sum of two divisors of zero. Further on I will come back to the fact that this work as well did not receive the echo that it merited.

1891 was a year dense in significant facts for Corrado Segre. In particular, it was the year in which he taught one of the most celebrated courses ever given by him, that on "Introduction to geometry of simply infinite algebraic entities" a course that was completed some years later, resulting in Segre (1894a), which in some way constituted the conclusive definition of the results and methods of the Italian School of geometry with respect to the geometry of algebraic curves.[30]

A very brief passage from Segre's notebook alerts us to the fact that in the context of this course he touched in a significant way on the theme of the foundations of geometry. He gave his listeners a precise assignment: "To find the axioms which define synthetically the space S_r".[31]

Further, that same year, in his advice to the students, he wrote:

> It has not yet been assigned and discussed (that I know of) a system of independent postulates that serve to characterise the linear space of n dimensions, so that it is possible to deduce the representation of the points of this with coordinates. It would be useful for some young person to occupy himself with this question (which does not seem difficult).[32]

Among those who attended the course were Federico Amodeo and Gino Fano, both of whom took up Segre's invitation, and very soon two works appeared (Amodeo 1891; Fano 1892) that were greatly important for the history of the foundations of geometry, above all that of Fano, which gave the first examples of finite geometries, in addition to earning, for its overall framework, a very complementary judgment on the part of Hans Freudenthal.[33] Once again Segre expresses mathematical ideas that would be developed by his students. In my opinion, even the work of a few years later by Enriques (1894) can be situated in this context. Further, its close link to didactic work, so similar to what happened in the case of Segre, which I mentioned earlier, begins to constitute a "style" of work that will be shared by many Italian algebraic geometers.

Still in 1891 Segre wrote another important work (Segre 1891a) in which, among the many observations,[34] he clearly expresses his point of view regarding certain aspects of the foundations of geometry. Here I wish to recall only his

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[30] About this course, see Conte (2002).

[31] *Trovare quali sono quei postulati che caratterizzano sinteticamente l'S_r* (Giacardi 2002, Quaderni. 3, 27).

[32] *Non è ancora stato assegnato e discusso (che io sappia) un sistema di postulati indipendenti che serva a caratterizzare lo spazio lineare ad n dimensioni, sì che se ne possa dedurre la rappresentazione dei punti di questo con coordinate. Sarebbe conveniente che qualche giovane si occupasse di tale questione (che non sembra difficile)* (Segre 1891a).

[33] I will not go more deeply into these works here; see Avellone et al. (2002).

[34] This work is particularly known for having led to the dispute with Peano. Here I will not discuss this polemic, but only with the references to problems relative to foundations.

presentation of pure geometric methods, which substantially regard the axiomatic method in geometry:

> These trends towards purity are truly of the greatest importance. It is in fact beyond doubt that the mathematician cannot be fully satisfied with the knowledge of a truth if he has not been able to deduce it with the utmost simplicity and naturalness of the smaller possible number of propositions known, of independent postulates, avoiding all hypothesis, all means of proof that do not appear necessary for the purpose. In so doing, didactic advantages are often achieved along with scientific advantages, in as much as from the scholar is required a smaller number of preliminary cognitions.[35]

For many years Segre's contributions to problems relative to the foundations of geometry did not show significant developments. However, I would like to examine some of the events between 1904 and 1905, when some of this ideas began to yield fruit, albeit indirectly.

The most important fact that I would like to highlight is the publication of Pieri (1905), which we can consider as the first result whose origins lay in the work of Segre (1889–90, 1890–91). Pieri, who up to then had been involved with numerative algebraic geometry, had begun to interest himself in the foundations of geometry in 1894 (with Pieri 1895), giving birth to an almost uninterrupted stream of works, always related to topics of projective geometry.[36]

These works were certainly the fruit of his encounter with Peano, his language, his search for an absolute rigour in the context of a mathematics that was deductive and axiomatic. However, they are also the fruit of his encounter with the work of von Staudt, tied to the translation of the Geometrie der Lage, towards which he had been steered, as we have seen, by Segre himself.

As already said, Pieri (1895) is the first work that reprises the concepts expressed in Segre (1889–90, 1890–91). In fact, we can even say that it is entirely based that work. It renders Segre's "new field of geometric research" rigorous from the point of view of axiomatics. Besides all else, its aim is to carry out, for complex geometry, the programme mentioned earlier that Segre had proposed in 1891: "Define space not by means of coordinates, but with a series of properties of which the representation with coordinates can be deduced as a consequence".

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[35] *Questi indirizzi puri sono veramente della massima importanza. È infatti fuor di dubbio che il matematico non può essere pienamente soddisfatto della conoscenza di una verità se non quando è riuscito a dedurla con la massima semplicità e naturalezza dal minor numero possibile di proposizioni note, di postulati indipendenti, evitando ogni ipotesi, ogni mezzo di dimostrazione che non appaia necessario per lo scopo. Così facendo si raggiungono spesso coi vantaggi scientifici anche vantaggi didattici, in quanto che dallo studioso si esigerà minor copia di cognizioni preliminari* (Segre *Opere,* v. 4, 396).

[36] On Mario Pieri and his mathematical work we can see various works of E. Marchisotto, in particular (Marchisotto and Smith 2007) and, with specific regard to the links between Pieri and the work of von Staudt, (Marchisotto 2006).

In keeping with this programme, Pieri wanted to follow a programme analogous to that followed by von Staudt in the *Geometrie der Lage*:

> For the complex linear variety there is yet no known system of intrinsic attributes and characters, aimed at qualifying it in a manner that from it derives without doubt the representability of its points by complex homogeneous points. The present essay proposes precisely to analyse the concept of complex linear variety: attempting to establish on new principles a complex projective geometry – or a geometric doctrine of imaginaries – not only exempt from any algebraic consideration whatsoever, but free from any deductive constraints of ordinary real projective geometry. And further: Staudt … proposed to establish projective geometry on its own foundations, excluding any dependence on elementary geometry. Likewise can be demanded a complex projective geometry independent of real projective geometry.[37]

As can be seen, this is the program of geometrical introduction of coordinates that was dear to Segre. Pieri bases himself on three primitive notions—complex point, complex line, chain—and on thirty postulates, obviously obtaining the representation of complex points on the real sphere, or by stereographic projections, on the Argand-Gauss plane. In that representation obviously the complex line is represented by the sphere itself, while the chains are represented by circumferences on the sphere. In this work are found again all of the principal results obtained by Segre in the preceding work, with the reintroduction of the anti-linear transformation, etc. It goes without saying that the "new field" is very frequently cited.

It should in any case be remarked that while for Segre (as for Fano and Enriques) complex projective geometry represents a (potential) instrument for the study of significant mathematical problems, for Pieri, as for Peano, the determination of a system of axioms is an object of research in itself. In other words, for Pieri the foundations of geometry was a new mathematical discipline; for Segre, no.

In the 1903−1904 academic year Segre taught a course in higher geometry on the "applications of Abelian integrals in geometry". His assistant in this course was the American mathematician Julian Coolidge, who came to be profoundly influenced by the teaching of the Piedmontese mathematician, in general on topics in algebraic geometry, but more in particular, precisely on topics relative to complex projective geometry. The following year Coolidge moved to Bonn to study with Eduard Study.

In effect, among European mathematicians Study was the one who came to closest to the kinds of study we have been talking about. Study and Segre were also in epistolary contact, and naturally knew and appreciated each other's works.

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[37] *Per la varietà lineare complessa non si conosce ancora un sistema di attributi e caratteri intrinseci, atti a qualificarla in maniera, che ne derivi senz'altro la rappresentabilità dei suoi punti per coordinate omogenee complesse. Il presente saggio si propone appunto l'analisi del concetto di varietà lineare complessa: cercando di istituire su nuovi principi una Geometria Projettiva complessa – o dottrina geometrica degli immaginari – esente non solo da qualsivoglia considerazione algebrica; ma sciolta eziandio da ogni vincolo deduttivo con l'ordinaria Geometria Projettiva reale*. E ancora: *Staudt ... si propose di stabilire la Geometria Projettiva su fondamenta proprie, esclusa ogni dipendenza dalla Geometria elementare. E così può domandarsi una Geometria Projettiva complessa indipendente dalla Geometria Projettiva reale* (Pieri 1905, 190–191).

Coolidge recalls, "Few geometers set a higher value on Segre's work than Study did" (Coolidge 1924, p. 133). It is in any case necessary to bear in mind that while the two mathematicians had the same interest and the same conceptual scheme regarding the use of complex and hypercomplex geometry, they conceived of applications that were very different: Segre, algebraic geometry; Study, to Lie groups.[38]

Perhaps we do well in any case to underline the fact that, to this aspect as well, Segre's students, engaged above all in the study of the subgroups of the group of birational transformations, made important contributions. In particular, it was Gino Fano who made important contributions to the topic, especially with Fano (1907); later we will mention the influence that this work had on Elie Cartan.

Study is rightly considered one of the leading European scholars of the theory of algebras (hypercomplex numbers) and his volume (Study 1903) exerted a profound influence. Under his advisement Coolidge wrote his doctoral thesis (Coolidge 1904), which, as the title indicates, regards the projective geometry on the algebra of dual numbers, a topic to which Segre himself would contribute some years later. Naturally I will not follow here the scientific career of the American mathematician, largely influenced by Segre's geometrical vision even on the subject of complex projective geometry; I will limit myself to quoting what Coolidge himself wrote in his book of 1924, which is among other things the first textbook on complex projective geometry ever published. Coolidge writes (as an aside, this is in the year of Segre's death):

> Every student in the complex domain will find that he is forced to refer continually to the work of two admirable contemporary geometers, Professor Corrado Segre of Turin, and Professor Eduard Study of Bonn. The names of both appear throughout this book; the author had the rare privilege to be the pupil of each of these masters. Geographical separation has cut him off from the one, the inexorable logic of history has impeded his communion with the other. But his sense of obligation has never wavered, and he begs to offer the present work as a small token of admiration and esteem (Coolidge 1924, 7).

I will come back more than once to the belated acknowledgment of his role in the founding of a fundamental branch of mathematics, one that is transversal to many of its branches.

In 1928 Coolidge returned to Italy for a scientific visit, publishing in that occasion (Coolidge 1928), which can be considered a genuine homage to his maestro, who had passed away a few years earlier.[39]

Again in 1904, Segre received an important international recognition, giving one of the plenary lectures at the third International Congress of Mathematicians in Heidelberg, thus assuming officially the role of leader of the Italian School of algebraic geometers—a role previously played by Cremona, who passed away the year before—, a role which he had in fact covered for some time.

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[38]On the work of Study relative to Lie groups, see Hawkins (2000).

[39]Coolidge never concealed his debt to and admiration of the Italian School of geometry. In Coolidge (1931, X) the dedication reads: "To the Italian geometers, dead and living".

In this important lecture (Segre 1905a) he confirmed his well-known points of view, with several significant statement regarding the topic of this present work. It seems to me useful to quote two passages from this lecture, because they represent a synthesis of his points of view on the questions dealt with. The first quotation refers to complex geometry and his interpretation of Hermitian space:

> In keeping with the tendency to broaden the field of geometry, the more general varieties can also be studied: obtained, that is, by considering separately, as independent variables, the two real components of each complex coordinate; and imposing links among the various real components. If these links are algebraic, we have the so-called hyperalgebraic varieties, about which I published several studies around 1890. Among these there are the geometric images of those quadratic forms of Hermite of conjugate complex variables, that have been seen so often in these years, in connection with the groups of linear substitutions and automorphic functions. Thus the forms of Hermite in R^4 represent correspondences between points and plains that are very similar to polarity with respect to a quadric.[40] … Among the hyperalgebraic varieties are also found those composed of the real points of an algebraic variety.[41]

This last observation will later receive an interesting confirmation in the works of Annibale Comessatti, particularly in Comessatti (1912), which makes full use of the frame of reference outlined by Segre: anti-projectivity (which Comessatti extends to anti-birational transformation), hyperconics, etc. That reference is explicit. In the opening section, entitled *Questioni generali su alcuni enti algebrici reali*, Comessatti writes: "for a treatment that is broader and more complete of these and many other similar questions, I refer the reader to the admirable memoirs of Segre"[42] which are obviously those we discussed earlier.[43] Yet again I would like to reiterate what I said earlier: what appears particularly significant to me in Segre's work is his having created a sufficiently coherent and solid general theoretical framework (complex projective geometry) in which develop different theories of diverse natures, not only geometrical, but also algebraic and analytical. Perhaps the natural place in which to situate Segre's results would have been a textbook,

[40]This obviously refers to the orthogonality with respect to a Hermitian form and its interpretation in real four-dimensional space.

[41]*Seguendo la tendenza ad ampliare il campo geometrico, si possono anche studiare delle varietà più generali: ottenute cioè considerando staccatamente, come variabili indipendenti, le due componenti reali di ogni coordinata complessa; e ponendo dei legami tra le varie componenti reali. Se questi legami sono algebrici, si hanno le così dette varietà iperalgebriche, intorno a cui ho pubblicato verso il 1890 alcune ricerche. Fra di esse vi sono le imagini geometriche di quelle forme quadratiche di Hermite a variabili complesse coniugate, che si sono presentate tanto spesso in questi anni, collegandosi ai gruppi di sostituzioni lineari ed alle funzioni automorfe. Così le forme di Hermite nel campo quaternario rappresentano delle corrispondenze fra punti e piani molto analoghe alla polarità rispetto a una quadrica. … Fra le varietà iperalgebriche si trovan pure quelle composte dai punti reali di una varietà algebrica* (Segre *Opere,* v. 4, 464).

[42]*Per una trattazione più diffusa e completa di tali questioni e di tantissime altre analoghe rimandiamo il lettore alle belle Memorie di Segre* (Comessatti 1912, 7).

[43]In my opinion it would be interesting to see a more complete recognition of the influence of Segre's points of view on Comessatti, the most important Italian algebraic geometer who dealt with the problem of reality. On his work see Ciliberto and Pedrini (1994).

something that was done many years later by Coolidge, whose volume I have already mentioned, and above all by Elie Cartan.

The second quotation from Segre's lecture refers directly to what I have dealt with here regarding projective geometry on an algebra:

> From complex geometry I passed to real geometry. I must however state that the abstraction, which I have repeatedly made evident as a characteristic of modern geometry, has also had the effect of multiplying, in a manner of speaking, the complex geometries. On one hand there might be the opportunity of considering certain geometric entities as points of a new nature, having for coordinates complex numbers of a higher species. Thus in the study of hyperalgebraic varieties, from the simplest problems that are born from the consideration of the real branches of an algebraic curve, there have arisen spontaneously bi-complex points. On the other hand, as instruments for research, it is well known, since the works of Grassmann and Hamilton, that various kinds of complex numbers can usefully serve in geometry. … In these past years, following an old mention of Clifford, consideration has been given in particular to the three systems of complex numbers of two units a + be, in which the square of the unit e is −1, +1, 0. These represent in a certain sense the three geometries, hyperboic, elliptical, parabolic. Those with $e^2 = 0$ had important applications, especially in line geometry. … Ample use of it is made in Study's *Geometrie der Dynamen*.[44]

Precisely on these topics, after a long silence about the subject, Segre returned to geometries on algebras, addressing, from this point of view, the topic of projective geometry on dual numbers in Segre (1911–12).

Dual numbers, as already presented by Segre in his lecture, are a generalisation of complex numbers in the sense of being second-degree associative algebras on real numbers, thus of the type a + be, with:

a. $e^2 = 0$ (the parabolic case, or of dual numbers proper)
b. $e^2 = -1$ (the hyperbolic case, of the usual complex numbers)
c. $e^2 = 1$ (the elliptical case, of numbers often called double or split complex numbers).

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[44]*Dalla geometria complessa ero passato a quella reale. Ma debbo però avvertire che l'astrazione, che ripetutamente ho messo in evidenza come un carattere della geometria moderna, ha avuto anche l'effetto di moltiplicare, per così dire, le geometrie complesse. Da un lato si può avere l'opportunità di considerare certi enti geometrici come punti di nuova natura, aventi per coordinate numeri complessi di specie superiore. Così nello studio delle varietà iperalgebriche, fin nei problemi più semplici che nascono dalla considerazione dei rami reali di una curva algebrica, si son presentati spontaneamente dei punti bicomplessi. D'altra parte, come strumenti di ricerca, si sa bene, fin dai lavori di Grassmann e di Hamilton, che varie sorte di numeri complessi possono servire utilmente in geometria. ... In questi ultimi anni, seguendo un antico accenno di Clifford, si sta considerando in particolare i tre sistemi di numeri complessi a due unità a + be, in cui il quadrato dell'unità e vale −1, +1, 0. Essi rappresentano in un certo senso le tre geometrie, iperbolica, ellittica, parabolica. Quelli con $e^2 = 0$ ebbero applicazioni importanti, specialmente nella geometria della retta ... Nella* Geometrie der Dynamen *dello Study ne è fatto ampio uso* (Segre *Opere* v. 4, 466).

In reality Segre examines the question in a way that is much more general, seeking, we might say, the most general associative algebra of dimension two, setting $e^2 = g + he$, and then deriving that, with an opportune change of base, these can always be reduced to the preceding form.

Moreover, he treats these numbers algebraically in a refined manner, studying them (as in the case of bi-complex numbers) the divisors of zero ("*nullifici*"). It is simple enough to find that the divisors of zero are, in the parabolic case, the numbers of the form as e, in the elliptical case, those of the form a (e + 1) or a (e − 1). Segre defines these numbers respectively as "of the first rank", and "of the second rank". This distinction is essentially in the definition of the dual projective point, in as much as it is defined as the quaternary of dual numbers not all null of the same rank (and naturally is less than a product by non-nullifying numbers).

I will say no more about this characterisation of projective geometries on algebra, which would require a treatment of its own, and which, even if not completely original,[45] appears to me to be of remarkable significance.[46]

As I have already said, in spite of his enormous prestige, the reception of Segre's ideas about geometries within the milieu of Italian geometers was really quite a cool one. I have already cited earlier the evaluation of Castelnuovo during his commemoration. It should be borne in mind that Segre had attempted, unsuccessfully, to arouse Castelnuovo's interest, writing him:

> For some weeks now I have taken up anew the ancient question about imaginary entities in the form of 1st, of 2nd, … species, on their representation in real forms (the p[oints] of the plane in the real p[oints] of S_4, etc.) and I have found and am finding results that I hope will interest you. At certain points you might say that I am continuing the Beiträge of Staudt (write me of the impression you get from this).[47]

Even a biographer who is attentive and certainly competent in the subject treated here gives similar indications; Beniamino Segre writes that these ideas of Segre "had directly or indirectly important reflections, at first perhaps though of an entity inferior to what the A[uthor] expected".[48] The lack of a dialogue on these topics can make it possible to better grasp the very nature of Segre's work, in many ways peculiar. One important exception to that absence of dialogue is in any case indicated in the work of Mario Pieri, of which I have spoken and who never failed to see in Segre on of the inspirers of the abstract vision of geometry. Thus in his

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[45] Segre himself says: queste premesse sono ben note (these premises are well known) and refers to several of the earlier works, in particular to Study and his school.

[46] For a modern treatment of the subject, not unlike that of Segre, see Yaglom (1966).

[47] *Ora da qualche settimana ho ripreso antiche questioni sugli enti imaginari nelle forme di 1ª, di 2ª, …specie, sulla loro rappresentaz.ᵉ in forme reali, (i p.ⁱ del piano nei p.ⁱ reali di S_4, ecc.) e ho trovato e trovo dei risultati che spero ti interesseranno. In certi punti si può dire che continuo i Beiträge di Staudt (scrivimi che impressione ricevi da questi)* (Letter from Segre to Castelnuovo, dated September, 6, 1889), the letters are edited by P. Gario in the website http://archivi-matematici.lincei.it/.

[48] *… hanno avuto in modo diretto od indiretto importanti riflessi, dapprima forse però d'entità inferiore a quanto l'A. non ne aspettasse* (B. Segre 1964, 15).

addition to the chapter of Zeuthen on numerative geometry in Klein's *Enzyklopädie* he signalled precisely the work of the Piedmontese mathematician being one of the founding ones of the abstract trend: "In the definition of C. Segre, the nature of the generating element or "point" becomes to all effect indeterminate".[49]

4 Ulterior Developments: Elie Cartan and Complex Projective Geometry

A genuine recognition of Segre's work on complex geometries arrived only much later, with the volume by Coolidge cited earlier, and above all that by Cartan. Later still (and we can say still in progress) is that regarding hyper-complex geometry. I would therefore like to conclude with a brief mention of the scientific relationships between Elie Cartan and Segre and his students.

A point of departure can be found in the studies of Fano regarding Lie groups and in his chapter (Fano 1907) in Klein's *Enzyklopädie*. A few years later, between 1912 and 1914, Cartan will be engaged in the study of this work, in which he will prepare a translation (Cartan 1915) that, because of the war, will not be published, but will ultimately appear only in his *Oeuvres*. In those same years, between 1913 and 1914, Cartan published what Hawkins (2000) calls his "trilogy" (Cartan 1913, 1914a, b), in which he makes broad use of Segre's results and language. It is worthwhile noting that in 1927 Beniamino Segre went to study with Cartan in Paris.

In any case, perhaps the most explicit acknowledgment of Segre's role as a founder of modern complex projective geometry came in the course of the 1928−1929 academic year, when Cartan taught the course in geometry at the Sorbonne. This course led to Cartan's important treatise *Leçons sur la Géométrie projective complexe* (1931), the most important text in which the material was treated at length and exhaustively, and the fundamental role of Segre was explicitly recognised:

> Complex projective geometry, considered as an autonomous discipline, … is primarily developed as a result of the work of Juel and especially C. Segre. This last geometer showed the importance of anti-projective transformations… alongside projective transformations, the only ones previously considered. Among anti-projectivities the anti-involutions and anti-polarities involve geometric objects whose role is in no way inferior to that played by quadrics and linear complex numbers, considered as basic elements of polarities properly speaking.[50]

[49]*Dans la définition de C. Segre, la nature de l'élément générateur ou "point" demeure tout à fait indéterminée* (Zeuthen and Pieri 1915, 327).

[50]*La géométrie projective complexe, considéré comme discipline autonome, ..., c'est principalement développée à la suite des travaux de Juel et surtout de C. Segre. Ce dernier géomètre a montré l'importance des transformations antiprojectives ... à côté des transformations projectives, qu'on avait seules considérées auparavant. Parmi les antiprojectivités, les antiinvolutions et les antipolarités font intervenir des êtres géométriques dont le rôle ne le cède en rien à celui joué par les quadriques et les complexes linéaires, envisagés comme éléments de base des polarités proprement dites* (Cartan 1931).

It seems to me that these words mark the conclusion of a pathway. Effectively what is dealt with is “a new field of geometric research”, as Segre wrote forty years earlier, but, as on the other hand he had intuited, the fields of application for these broad visions were numerous and, in the case of Cartan, fundamentally the study of Lie groups. The hopes of a broad use in the field of algebraic geometry were largely unrealised and perhaps confirm the impression that Segre’s wide-ranging and profound visions went forward above all thanks to the efforts of his students. In this field, what Segre lacked was an Enriques and a Castelnuovo.

Translated from the Italian by Kim Williams.

References

Amodeo, Federico, Quali possono essere i postulati fondamentali della geometria proiettiva in uno Sr, Atti della R. *Accademia delle Scienze di Torino,* 26 (1891): 741–770.

Avellone, Maurizio, Brigaglia, Aldo and Zappulla, Carmela, The Foundations of Projective Geometry in Italy from De Paolis to Pieri, *Archive for History of Exact Sciences,* 56 (2002): 363–425.

Cartan, Elie, *Leçons sur la Géométrie projective complexe*, Paris: Gauthier Villars, 1931.

Cartan, Elie, Les groupes projectifs qui ne laissent invariante aucune multiplicité plane, *Bulletin des Sciences mathématiques*, 41 (1913): 53–96.

Cartan, Elie, Les groupes projectifs qui ne laissent invariante aucune multiplicité plane, *Journal des mathématiques pures et appliquées*, 10 (1914a): 149–186.

Cartan, Elie, Les groupes réels simples finis et continus, *Annales scientifiques de l’École Normale Supérieur de Paris*, 10 (1914b): 263–355.

Cartan, Elie, La théorie des groupes finis et continus et la géométrie. Exposé d’après l’article allemand de G. Fano, *Encyclopédie des sciences mathématiques*, III, 1, (1915).

Castelnuovo, Guido, Commemorazione del socio Corrado Segre, *Atti della R. Accademia dei Lincei*, (5), 33 (1924): 353–359.

Ciliberto, Ciro and Gario, Paola, Federigo Enriques: the first years in Bologna, in Coen, Salvatore (ed.), *Mathematicians in Bologna 1861–1960*, Basel: Birkhauser, 2012, 105–142.

Ciliberto, Ciro and Pedrini, Claudio, Annibale Comessatti and real algebraic geometry, in Brigaglia Aldo, Ciliberto, Ciro and Sernesi, Edoardo, Algebra e Geometria (1860–1940): II contributo italiano, *Supplemento dei Rendiconti del Circolo Matematico di Palermo*, (2), 36 (1994): 71–102.

Comessatti, Annibale, Fondamenti per la geometria sopra le superficie razionali dal punto di vista reale, *Mathematische Annalen*, 73 (1912): 1–72.

Conte, Alberto, Introduzione al quaderno relativo al corso del 1890–91, in Giacardi, Livia (ed.), *I quaderni di Corrado Segre*, CD-ROM, Dipartimento di Matematica, Università di Torino, 2002.

Coolidge, Julian, *Die dual–projektive Geometrie im elliptischen und sphärischen Raume*. Dissertation. 1904.

Coolidge, Julian, *The Geometry of the Complex Domain*, Oxford: University Press, 1924.

Coolidge, Julian, Questioni di geometria nel campo complesso, *Rendiconti del Seminario Matematico di Roma*, (2), 5 (1928): 24–29.

Coolidge, Julian, *A Treatise on Algebraic Plane Curves*, Oxford: Clarendon Press, 1931.

Cremona, Luigi, Review of “Beiträge zur Geometrie der Lage”, *Annali di Matematica pura ed applicata*, (1), 1 (1858): 125–128.

De Paolis, Riccardo, Sui fondamenti della geometria projettiva, *Atti della R. Accademia dei Lincei*, (3), 9 (1880–81): 489–503.

Enriques, Federigo, *Per la storia della logica*, Bologna: Zanichelli, 1922.
Enriques, Federigo, Sui fondamenti della geometria proiettiva, *Rendiconti dell'Istituto Lombardo di Scienze, Lettere ed Arti*, (2), 27 (1894): 550–567
Fano, Gino, Sui postulati fondamentali della geometria proiettiva in uno spazio lineare a un numero qualunque di dimensioni, *Giornale di Matematiche*, 30 (1892): 106–132.
Fano, Gino, Kontinuierliche geometrische Gruppen. Die Gruppentheorie als geometrisches Einteilungsprinzip, *Enzyklopädie der mathematischen Wissenschaften*, III, 4a, 1907, 221–288.
Freudenthal, Hans, The impact of von Staudt's foundations of geometry, in Plaumann, Peter, Strambach, Karl, *Geometry–von Staudt's point of view*, Springer, 1981, 401–425.
Genocchi, Angelo, *Calcolo differenziale e principii del calcolo integrale, pubblicato con aggiunte dal Dr. Giuseppe Peano*, Torino: Bocca, 1884.
Giacardi, Livia (ed.), *I quaderni di Corrado Segre*, CD-ROM, Dipartimento di Matematica dell'Università di Torino, 2002.
Hartshorne, Robin, Publication History of von Staudt's Geometrie der Lage, *Archive for History of Exact Sciences*, 62 (2008): 297–299.
Hawkins, Thomas, *Emergence of the theory of Lie Groups*, New York: Springer, 2000.
Klein, Felix, Über die sogenannte Nicht–Euklidische Geometrie, *Mathematische Annalen*, 6 (1873): 112–145
Luciano, Erika and Roero, Clara Silvia, From Turin to Göttingen: Dialogues and Correspondence (1879–1923), *Bollettino di Storia delle Scienze Matematiche*, 32, 1 (2012): 7–232
Marchisotto, Elena, The projective geometry of Mario Pieri: a legacy of Georg Karl Christian von Staudt, *Historia Mathematica*, (3), 33 (2006): 277–314.
Marchisotto, Elena and Smith, James, *The Legacy of Mario Pieri in Geometry and Arithmetics*, Boston: Birkhäuser, 2007.
Nabonnand, Philip, La théorie des Würfe de von Staudt. Une irruption de l'algèbre dans la géométrie pure, *Archive for History of Exact Sciences*, 62 (2008): 201–242.
Palladino, Franco and Palladino, Nicla, *Dalla « moderna geometria » alla « nuova geometri-aitaliana »*, Olschki, 2006.
Pasch, Moritz, *Vorlesungen über neuer Geometrie*, Leipzig: Teubner, 1882.
Peano, Giuseppe, *Calcolo Geometrico secondo l'Ausdehnungslehre di H. Grassmann, preceduto dalle operazioni della logica deduttiva*, Torino: Fratelli Bocca Editori, 1888
Peano, Giuseppe, *Arithmetices Principia. Nova methodo exposita*, Torino: Fratelli Bocca Editori, 1889a.
Peano, Giuseppe, *I principii di geometria logicamente esposti*, Torino: Fratelli Bocca Editori, 1889b
Pieri, Mario, Sui principii che reggono la geometria di posizione. Nota I, *Atti della R. Accademia delle Scienze di Torino*, 30 (1895): 607–641.
Pieri, Mario, Nuovi principii di geometria proiettiva complessa, *Memorie della R. Accademia delle Scienze di Torino*, (2), 55 (1905): 189–235
Sannia, Achille, *Lezioni di Geometria proiettiva*, Napoli: Pellerano, 1891.
Segre, Beniamino, Nel primo centenario della nascita di Corrado Segre, *Rendiconti del Seminario Matematico dell'Università di Torino*, 23 (1964): 7–21.
Segre, Corrado, Le coppie di elementi imaginari nella geometria proiettiva sintetica, *Memorie della R. Accademia delle Scienze di Torino*, (2), 38 (1886b): 3−24.
Segre, Corrado, Un nuovo campo di ricerche geometriche, *Atti della R. Accademia delle Scienze di Torino*, 25 (1889–90): 180–205; 290–317; 376–396.
Segre, Corrado, Un nuovo campo di ricerche geometriche, *Atti della R. Accademia delle Scienze di Torino*, 26 (1890–91): 35–71
Segre, Corrado, Su alcuni indirizzi nelle investigazioni geometriche, *Rivista di Matematica*, 1, (1891a): 42−66.
Segre, Corrado, Le rappresentazioni reali delle forme complesse e gli enti iperalgebrici, *Mathematische Annalen*, 40 (1891d): 413–467.

Segre, Corrado, Introduzione alla geometria sugli enti algebrici semplicemente infiniti, *Annali di Matematica pura e applicata*, (2), 22, (1894a): 41−142.

Segre, Corrado, La geometria d'oggidì e i suoi legami con l'analisi, *Verhandlungen des 3en intern. Math-Kongress in Heidelberg*, 109–120; also *Rendiconti del Circolo Matematico di Palermo*, 19 (1905a): 81–93.

Segre, Corrado, Le geometrie proiettive nel campo dei numeri duali, *Atti della R. Accademia delle scienze di Torino*, 47 (1911–12): 114–133 and 164–185.

Segre, Corrado, *Opere*, Roma: Ed. Cremonese, 4 vols, 1953–63.

Staudt, Karl Georg Christian von, *Geometrie der Lage*, Nürnberg, 1847.

Staudt, Karl Georg Christian von, *Beiträge zur Geometrie der Lage*, Nürnberg, 1856, 1857, 1860.

Staudt, Karl Georg Christian von, *Geometria di Posizione*, translated by M. Pieri, Torino: Fratelli Bocca Editori, 1889.

Study, Eduard, *Geometrie der Dynamen*, Leipzig: Teubner, 1903.

Yaglom Isaac, *Les nombres complexes et leurs applications en géométrie*, Paris: Dunod, 1966.

Zeuthen Hieronymus Georg and Pieri Mario, Géométrie énumérative, in *Encyclopédie des Sciences Mathématiques*, III, 4, 1915, 60–331.

Segre, Castelnuovo, Enriques: Missing Links

Paola Gario

Abstract

At the end of the 1880s, Segre guided Castelnuovo's research towards the geometry of algebraic curves, introducing Castelnuovo, whose earlier studies had been focused on n-dimensional projective geometry, to birational geometry, which is the starting point of the Italian school of algebraic geometry. After graduating and attending one post-graduate year at the University of Pisa, Enriques got in touch with Segre, aiming to spend 1 year in Turin. He was attracted by the reputation of the young master and, perhaps, by the mathematical environment of Turin University, which was particularly lively in those years. Contrary to his expectations, Enriques was sent to Rome, where in the meantime Castelnuovo had moved. He began to study the birational geometry of algebraic surfaces under Castelnuovo's direct supervision. Segre followed Enriques's first results but he was committed to finding a rigorous proof of the theorem of resolution of singularities of algebraic surfaces. His article on singularities was concluded in December 1896 and in the following year his student Beppo Levi completed the proof of the resolution theorem. Meanwhile, Enriques had laid the foundations of the general theory of linear systems of curves on algebraic surfaces and Castelnuovo had proved his famous rationality criterion. The link between Segre, Castelnuovo and Enriques could have turned into a scientific partnership: at the end of 1896, the three geometers planned to collect their results in a general treatise on the theory of algebraic varieties. However, this treatise was never realised.

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1 Introduction

This paper concerns the relationship between Corrado Segre (1863–1922), Guido Castelnuovo (1865–1953) and Federigo Enriques (1872–1946). At the end of the 1880s, Segre guided Castelnuovo's research towards the geometry of algebraic curves, introducing Castelnuovo, whose earlier studies had been focused on n-dimensional projective geometry, to birational geometry, which is the starting point of the Italian school of algebraic geometry. After graduating and attending one post-graduate year at the University of Pisa, Enriques got in touch with Segre, aiming to spend 1 year in Turin. He was attracted by the reputation of the young master and, perhaps, by the mathematical environment of Turin University, which was particularly lively in those years. Contrary to his expectations, Enriques was sent to Rome, where in the meantime Castelnuovo had moved. He began to study the birational geometry of algebraic surfaces under Castelnuovo's direct supervision.

Segre followed Enriques's first results but he was committed to finding a rigorous proof of the theorem of resolution of singularities of algebraic surfaces. His article on singularities was concluded in December 1896 and in the following year his student Beppo Levi completed the proof of the resolution theorem. Meanwhile, Enriques had laid the foundations of the general theory of linear systems of curves on algebraic surfaces and Castelnuovo had proved his famous rationality criterion. The link between Segre, Castelnuovo and Enriques could have turned into a scientific partnership: at the end of 1896, the three geometers planned to collect their results in a general treatise on the theory of algebraic varieties. However, this treatise was never realised.

The three protagonists were closely linked both scientifically and personally throughout their lives and in this paper we refer to the period of their closest and best documented collaboration, that is, from 1887 to 1897.

The story told here refers to the period when they were young and newly graduated.[1] In particular we will describe the interests that drew them together, focusing on the events that may have influenced the development of their scientific relationship, without overlooking the interpersonal dynamics which can be reconstructed thanks to the correspondence that Enriques and Segre addressed to Castelnuovo.[2]

The reconstruction that I present of the events that deterred Segre from joining Castelnuovo and Enriques in more advanced research on birational geometry is partly based on the interpretation given by Castelnuovo in the obituary article for

[1]Segre had graduated in 1883 under the supervision of Enrico D'Ovidio (1843–1933). Castelnuovo, only 2 years younger, graduated with a thesis supervised by Giuseppe Veronese (1854–1917) in 1886 from the University of Padua. Enriques graduated in 1891 at Pisa, supervised by Riccardo de Paolis (1854–1892).

[2]The correspondence, which belonged to the personal archive of Guido Castelnuovo and was donated to the Accademia Nazionale dei Lincei by his daughter Emma, is available in Gario (2010). Enriques's correspondence was published in full in Bottazzini et al. (1996) with an appendix in Bottazzini et al. (2004).

the Accademia dei Lincei (Castelnuovo 1924). Very briefly, Castelnuovo wrote that although Segre had found "new paths of research for geometric investigation", contributing "towards spreading among us Italians" the study of invariant properties under birational transformations of algebraic curves, "he did not, however, proceed in this direction":

> Almost overcome with a sense of nostalgia, he quickly abandoned this field of study and returned to projective geometry.[3]

According to Castelnuovo, among the reasons why Segre did not go ahead with research on birational geometry was a question of style:

> The search for simplicity and elegance that makes his writings so attractive, the aversion to complicated arguments revealing the effort required, the arduous procedures that are sometimes necessary during the process of discovery – all of this is perhaps what deterred him from penetrating too far into the territory he had begun to explore.[4]

In his obituary, Castelnuovo did not mention Segre's painstaking work on the singularities of algebraic surfaces, which, as I will try to demonstrate, played a decisive role in this case.

2 Guido Castelnuovo. Towards the Birational Theory of Algebraic Curves

In July 1885, Segre, who had already embarked on an academic career as Assistant Professor of Descriptive and Projective Geometry and was reaching the end of his military service, received an article from Castelnuovo. The sending of this article is an indication of the recognition already achieved by Segre only 2 years after graduation, and it marked the first direct contact with Castelnuovo, then a student in the third year of the degree course in mathematics at the University of Padua:

> On my return from the [military] camp I found the paper[5] you kindly sent me. The word "camp" tells you what I am doing. Fortunately, I am nearing the end of my voluntary service. I didn't want to thank you before reading it but now I do thank you with all my heart because your work interests me and appears to me, with regard to the question of the angles of two spaces, to be more complete and refined than previous works on the same

[3]See Castelnuovo (1924) = (Castelnuovo 2002, II, 372): *Egli, però, quasi vinto da un senso nostalgico, abbandona ben presto questo campo e ritorna alla geometria proiettiva.*

[4]See Castelnuovo (1924) = (Castelnuovo 2002, II, 375): *La ricerca di semplicità ed eleganza che rende così attraenti i suoi scritti, l'avversione per i ragionamenti complicati ove si riveli lo sforzo, per i procedimenti arditi ai quali talora si è costretti a ricorrere nella fase della scoperta, lo hanno forse trattenuto dal troppo inoltrarsi nelle regioni che aveva cominciato ad esplorar*e.

[5]Segre is referring to Castelnuovo (1885–1886a).

> subject. However, among those works I would like to have seen a reference to D'Ovidio's paper[6] on the fundamental metric functions of any space ….[7]

A year later Castelnuovo sent a second article (Castelnuovo 1885–1886b) which was again carefully read by Segre:

> You are right; though it is widespread, the system of praising works without reading them is not a good one. I make a habit of reading all the articles that I receive and make my comments to the authors. I behave towards them as I hope others will behave towards me.[8]

During the 1886–1887 academic year, Castelnuovo held a post-graduate fellowship at the University of Rome, where he was able to attend the course of Luigi Cremona (1830–1903) in Higher Geometry, after which Segre informed him of the possibility of obtaining an assistantship at the University of Turin:

> I would like to take advantage of this opportunity to tell you about an idea that came to my mind concerning you. Would you agree to come here to Turin as Prof. D'Ovidio's assistant (for complementary algebra and analytical geometry) for the coming academic year? As I said, it's just an idea of mine.
>
> For many years that position was assigned by Prof. D'Ovidio to the best graduate of this university (I consider myself honoured to have held it).[9]

Segre had won acclaim for his research on hyperspatial projective geometry but in some of his writings, particularly in the notes on ruled surfaces of 1884 and 1886 (Segre 1883–84b, 1885–86c), there was evidence of a certain attention to invariant properties under birational transformations. His interest in this field of research emerges more clearly in the memoir "Recherches générales sur les courbes et les surfaces réglées algébriques" (Segre 1887b) of January 1887 and in a brief note (Segre 1887d) of the following April showing the relationship between certain properties of the linear systems of plane curves and the linear series of points cut out on the generic curve of the system by the curves of the system itself, in other words, what Castelnuovo later called the *characteristic series* of the system. Thus a

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[6]Segre is referring to D'Ovidio (1876–1877).

[7]Gario (2010), Segre to Castelnuovo, 31 July 1885: *Di ritorno dal campo trovo la memoria, che ella gentilmente volle inviarmi ... Non ho voluto ringraziarla prima d'averla letta; ed ora La ringrazio di cuore, perché il suo lavoro mi ha interessato e mi parve riguardo alla questione degli angoli di due spazi più completo ed elegante che i lavori precedenti in cui è trattato quell'argomento. Tra i quali lavori però avrei voluto veder citata la memoria del d'Ovidio sulle funzioni metriche fondamentali di uno spazio qualunque*, […].

[8]Gario (2010), Segre to Castelnuovo, 29 July 1886: *Ella ha ragione: è un brutto sistema quello pur così generale di lodare i lavori senza leggerli. Io ho invece l'abitudine di leggere tutte le memorie che ricevo e di farne la critica agli autori: faccio con gli altri ciò che vorrei si facesse con me*.

[9]Gario (2010), Segre to Castelnuovo, 6 October 1887 (letter written in a different hand: at the time, Segre was having trouble with his eyesight): *Profitto di questa occasione per parlarle di un'idea che mi è venuta in mente e che la riguarda. Accetterebbe Ella di essere qui a Torino assistente del professor D'Ovidio (per l'algebra complementare e la geometria analitica) durante l'anno scolastico che sta per cominciare? Si tratta, come le dissi, di una mia idea* [...] *Quel posto fu dato per molti anni da quel chiarissimo Professore, ogni anno al miglior laureato di quest'Università, (io considero come un mio titolo d'onore l'averlo coperto*).

relation was demonstrated between the algebraic theory of the linear series elucidated by Alexander von Brill (1842–1935) and Max Noether (1844–1921) in the fundamental memoir, "Ueber die algebraischen Functionen und ihre Anwendung in der Geometrie" (Brill and Noether 1874), and the theory of linear systems of plane curves.[10]

On his arrival in Turin, Castelnuovo was directed by Segre towards the study of geometry on a curve. With a work published at the end of 1887 (Castelnuovo 1887–1888) but dated July 1887, he brought to a close his earlier research. 1888 was a year of intensive study for him: in July the first results of his new field of study were published in the memoir entitled "Geometria sulle curve ellittiche" (Castelnuovo 1888–1889a). Here, drawing on Segre's research on elliptic ruled surfaces, he focuses on a property of the linear series on an elliptic curve, examining it through the n-dimensional variety to which the series can be made to correspond. We see here a first example of the so-called "hyperspatial method" which characterised the reinterpretation of Brill-Noether's theory of linear series. A few months later in the article "Ricerche di geometria sulle curve algebriche" (Castelnuovo 1888–1889b),[11] came the presentation of the new method in which, quoting Castelnuovo's words,

> we do not restrict ourselves to considering series of groups of points cut out on a plane curve by adjoint curves but rather curves in general, without limiting the dimension of the spaces that contain them, and we cut out the series by means of spaces of fundamental forms,[12]

[10]At the time when our story begins, the memoir (Brill and Noether 1874) was the fundamental reference for the algebro-geometric approach to the theory of algebraic curves. Brill-Noether's point of view was developed in more geometric terms by Segre and Castelnuovo. It should be recalled that Riemann's theory of abelian integrals $\int R(x,z)dz$, where R is a rational function, was the starting point of the general theory of algebraic curves. The genus p of a Riemann surface S could be described in an analytic way as the maximal number of linearly independent *Abelian integrals of the 1st kind* (integrals of holomorphic differential forms) on S. The explicit description of these differential forms made it possible to define, on the algebro-geometric side, the genus of plane irreducible algebraic curves C of $\mathbb{P}^2(\mathbb{C})$, of a given order n, by means of the *adjoint curves* of order $n-3$, defined by *suitable conditions* at the singular points of C (*adjoint conditions*): the genus of C is the maximal number of linearly independent homogeneous polynomials of degree $n-3$ describing adjoint curves. The genus of a smooth curve C is given by the formula $p=\frac{(n-1)(n-2)}{2}$. If the curve C has singularities, a correction term in the formula is necessary and this term depends on local properties of C at singular points. Noether, after having introduced the decomposition of a singular point into a finite number of *infinitely near multiple points* (see footnote (53)), gave the formula $p=\frac{(n-1)(n-2)}{2}-\sum\frac{\sigma_i(\sigma_i-1)}{2}$, where σ_i are the multiplicities of these points, generalizing the formula $p=\frac{(n-1)(n-2)}{2}-d$ for a curve having d nodal points only. These formulae are based on the property that the singular points impose independent conditions in the vector space of the adjoint curves. All these questions became crucial for developing the theory of algebraic surfaces from the algebro-geometric standpoint. As interesting reference, see Kleiman (2004).

[11]The article is dated February 1889.

[12]Castelnuovo (1888–1889b) = (Castelnuovo 2002, I, 204)... [non] *ci limitiamo a considerare serie di gruppi di punti segate su curve piane da curve aggiunte. Ma consideriamo le curve in generale senza limitare le dimensioni degli spazi che le contengono, e seghiamo le serie mediante spazi di forme fondamentali.*

in addition to significant applications such as the proof of the Riemann-Roch theorem independently from Noether's *Restsatz*. As Castelnuovo would later recall in the addendum to the edition of his *Memorie Scelte* (Castelnuovo 1937),[13] "the first step towards editing those works, the latter, in particular, sprang from the desire to explain geometry on a curve using hyperspatial language, which was already familiar to our school". In fact, "between 1880 and 1890, the projective geometry of hyperspaces was amply covered by Veronese, Segre and others".[14]

Between 1889 and 1891 Castelnuovo published seventeen articles. These publications are mainly about geometry on an algebraic curve and enumerative geometry, a fundamental tool for the new exposition of the Brill-Noether theory on linear series, following the hyperspatial method. Segre and Castelnuovo did not publish papers jointly[15] but unquestionably it was Segre who directed Castelnuovo towards the research that brought him international acclaim. Segre, with his work on ruled surfaces, where the use of the formulae of enumerative geometry recurs frequently, provided him with useful tools. In this regard, it is interesting to note Castelnuovo's observation after "a veil of several decades"[16] had been drawn over much of his work. While on one hand "the issue of language was secondary",[17] on the other hand the formulae of enumerative geometry suggested to him by the works of Segre were indispensable, as shown by later treatments of the theory by Enriques and his student Oscar Chisini (1889–1967) and by Francesco Severi (1879–1961).[18]

The combined work of a young master and his pupil, only 2 years his junior, was the starting point for what was soon to be recognised as the 'Italian school of Algebraic Geometry'. While Castelnuovo demonstrated the potential of the new method in his publications, Segre for his part was making a comparative reflection on the possible approaches to geometry on a curve through his course in Higher Geometry of 1890–1891,[19] also with a view to writing a general expository article.

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[13]The plan to publish a selection of the geometry memoirs was designed to mark Castelnuovo's scientific jubilee in 1935. The notes added by Castelnuovo were reproduced, maintaining the original location, in the latest edition of Castelnuovo's *Opere Matematiche* (Castelnuovo 2002). The *Introduction* and the *Index* of Castelnuovo (1937) were republished in the volume containing the writings of 1937.

[14]See Castelnuovo (1937, *Nota aggiunta*) = (Castelnuovo 2002, I, 229).

[15]It is worth noting Segre's additional observations to the article, "Sui sistemi lineari di curve piane algebriche di genere p" (Segre 1887d), reported in a footnote to Castelnuovo's article, "Sulle superficie algebriche le cui sezioni piane sono curve iperellittiche" (Castelnuovo 1890), and to the article, "Un'osservazione intorno alla riducibilità delle trasformazioni Cremoniane e dei sistemi lineari di curve piane per mezzo di trasformazioni quadratiche" (Segre 1900–01), reported in a footnote to Castelnuovo's article, "Le trasformazioni generatrici del gruppo cremoniano nel piano. Estratto di una lettera al prof. Corrado Segre" (Castelnuovo 1900–1901).

[16]See Castelnuovo (1937, *Introduzione*) = (Castelnuovo 2002, IV, 199).

[17]See Castelnuovo (1937, *Nota aggiunta*) = (Castelnuovo 2002, I, 229).

[18]Castelnuovo referred to Enriques and Chisini (1915, Vol. III, Chap. 1, Sects. 10–14) and Severi (1926, Chap. 7). Recall that Severi was a student of Segre's. He graduated in 1900.

[19]See the Part III in the present volume.

This came out in 1894 with the publication of "Introduzione alla geometria sopra un ente algebrico semplicemente infinito" (Segre 1894a).[20]

In the autumn of 1891, Castelnuovo moved to Rome, and Segre sent him words of esteem, affection and nostalgia that reveal feelings of profound friendship. This is Segre's farewell letter:

> Dear friend,
> I have received your affectionate letter and I thank you for it. I have missed you since Monday and I feel your absence strongly. You mention the benefits you may have experienced over the last four years from my company. If that is true, it is also true that these feelings are mutual. Your sharp intelligence and your generous heart have always made the many hours we spent together fruitful and pleasant. I had come to think of you as a younger brother, I mean in terms of years of experience (which justified my tendency to sometimes give you advice). Or perhaps at times, by exaggerating the difference in age and character, my feelings towards you were those of a father towards his son, and I was proud to hear you praised, proud to see you esteemed as you deserved […]. You have been good for me, I repeat, not only intellectually but also morally. And now that we are separated, I feel a void that no one can fill. May our friendship remain steadfast despite the distance, a friendship between two young men who place the ideals of goodness, honesty and the pursuit of science above philistine egoism.[21]

Castelnuovo had been appointed to the Chair of Analytical and Projective Geometry. It should be noted that, unlike Segre, who had been put in charge of the course in Higher Geometry, Castelnuovo took up this teaching appointment only after Cremona's death (starting in 1903).

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[20]Eugenio Bertini (1846–1933) wrote a reappraisal of Brill-Noether's memoir (Bertini 1894) along with the aforementioned paper by Segre. In this regard, Bertini wrote to Castelnuovo: "I have applied myself to the work that Segre will have spoken to you about in Turin and I wanted to finish it as quickly as possible …. Today I sent it to Segre. There is little that is new, but I think that it could be useful in order to spread the concept of geometry on a curve among us here". The paper served "to arouse the interest of the reader" in Castelnuovo's work (Castelnuovo 1888–1889b) and at the same time "to show the profusion of applications of geometry on a curve". See Gario (2010), Bertini to Castelnuovo, 19 September 1893.

As an interesting reference on Bertini's contribution, see Kleiman (1998).

[21]Gario (2010), Segre to Castelnuovo, 12 November 1891: *Caro Guido, ricevo la tua affettuosa lettera, e te ne ringrazio. Da Lunedì tu mi manchi ed io sento vivamente questa lacuna. Tu accenni a quel po' di giovamento che hai potuto avere in questi quattro anni dalla mia compagnia. Se ciò è vero, è pur vero che da te io ho avuto un completo ricambio, e che il tuo ingegno acuto, come la tua bontà di cuore m'han reso continuamente utili e piacevoli le tante ore che passavamo insieme. Io m'ero avvezzato a considerarti come un fratello minore: minore, intendo d'anni ed esperienza (il che giustificava la mia tendenza a consigliarti nelle varie occasioni). O forse, esagerandomi la differenza di età e di carattere, c'erano anzi talvolta in me al tuo riguardo i sentimenti di un padre verso un figlio: ed io mi sentivo fiero degli elogi che ti toccavano, fiero di vederti stimato come meritavi* […] *Tu m'hai fatto del bene, lo ripeto, non solo intellettualmente ma anche moralmente. Ed ora che tu mi manchi sento realmente un vuoto, che non sarà colmato da nessuno. Serbiamo almeno, anche a distanza inalterata la nostra amicizia: amicizia di due ragazzi che al di sopra dell'egoismo dei filistei ripongono i loro ideali di bontà, di onestà e di culto della scienza.*

3 Towards the Geometry on Algebraic Surfaces

In the article "Osservazioni intorno alla geometria sopra una superficie algebrica, Nota I" (Castelnuovo 1891), dated November 1890, Castelnuovo proved that "If a surface of geometric genus higher than 1 possesses an irrational pencil of plane cubics or of skew quartics, then the numerical genus of the surface is different from the geometric genus" and he showed that "such surfaces actually exist", providing the first example of an irregular unruled surface and drawing attention to the surfaces which contain an irrational pencil of curves.[22]

The example demonstrated the fallaciousness of the argument that by analogy, the properties of curves could be transferred to surfaces.[23] The article was explicitly connected to Noether's research and it revealed that interest was moving into the virtually unexplored territory of the theory of algebraic surfaces. Castelnuovo's "*Osservazioni*" attest that, at the end of 1890, the focus had already shifted away from the theory of algebraic curves and they reveal the direction that the research of the Italian geometers was taking.

In the autumn of 1891, after receiving his degree, Enriques was awarded a scholarship for new graduates at the Scuola Normale Superiore di Pisa.[24] Enriques first came into contact with Segre in the summer of 1892, when he was seeking a

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[22]See Castelnuovo (1891) = (Castelnuovo 2002, I, 316).

[23]The idea of a close analogy between curves and surfaces inspired the theory of algebraic surfaces at its beginnings (see footnote (10)). The algebro-geometric approach was developed by Alfred Clebsch (1833–1872) who in 1868 introduced the *double integrals of the 1st kind* (double integrals of algebraic differential forms $\int\int R(x,y,z)dxdy$, where R is a rational function, which are defined over a algebraic surface S and finite over each closed bounded real two-dimensional domain of integration in S) and the corresponding *adjoint surfaces*. If S, of equation $F(x,y,z)=0$ of degree n, has only *ordinary singularities* (see footnote (55)), $R(x,y,z)$ is of the form $\frac{Q(x,y,z)}{F'_z}$ where $Q(x,y,z)$ is a polynomial of degree $m \leq n-4$ vanishing along the nodal curve of S. The maximal number of linearly independent double integrals of the 1st kind linearly independent is finite. This number, later called by Noether *flächengeschlecht* (Noether 1875), corresponds to the *geometric genus* p_g and it is invariant under birational transformations of S. The algebraic surfaces of equations $Q(x,y,z)=0$ are the *adjoint surfaces*. An adjoint surface ψ_{n-4}, of order $n-4$, to a given surface S of $\mathbb{P}^3(\mathbb{C})$, of order n, has at the singular curves and isolated points of S a suitable behavior. The *adjunction conditions* depend on the nature of the singularities of S: for example (see Noether 1870), an ordinary s—fold curve for S is an $(s-1)$—fold curve for ψ_{n-4} and an ordinary (isolated) r—fold point of S is an $(r-2)$—fold point for ψ_{n-4}. Arthur Cayley (1821–1895) determined formulae for computing the genus by introducing the concept of *postulation* of a curve which is the number of conditions that a curve imposes on the surfaces of a given order, in order to be a singular curve of a specified type for those surfaces. Implicit in Cayley's *postulation formulae* is the independence of the conditions imposed on the surfaces ψ_{n-4}: the number they give was later called the *numerical* or *arithmetic genus* of the surface and denoted by p_n or p_a. For more references see Gario (1991). Castelnuovo's example showed that the two definitions of the genus could lead to different results, not only for ruled surfaces, and he therefore highlighted the distinction between the arithmetic genus and the geometric genus. For an interesting reference for the history of the concept of *irregularity* $q=p_g-p_a$, see Bardelli (1994).

[24]One of his teachers was Bertini.

position for 1892–1893 and he consulted Segre about the possibility of getting the post of assistant at Turin. Segre wrote about this to Castelnuovo as follows: "Enriques would like me to show him the way to obtaining a post in Turin, which doesn't seem at all easy to me …, but please remember this young man among your candidates".[25] In fact, between the last years of the 1880s and the first years of the 1890s, Turin was a centre of learning that attracted brilliant young people. In 1891, when he was only 28 years old, Segre wrote the article "Su alcuni indirizzi delle investigazioni geometriche" (Segre 1891a)[26] with the aim of offering guidance to young people who wanted to undertake mathematical research. He wrote and acted with the knowledge of a teacher whose goals transcended personal achievement and who sought to enlist new recruits. His scientific relationships were favoured by his youth, which helped him to engage in less formal ways of communicating. The conversations continued also outside the strictly academic context, in the cafés of Turin. The atmosphere was enlivened by controversy over the question of foundations and rigour in mathematics, in which Giuseppe Peano (1858–1932) was the key figure, and in which even Segre became involved, in spite of himself. In Turin Enriques might have found competent and attentive guidance and a milieu open to the philosophical issues for which he had developed a strong passion in his high school years and to which he contributed in the course of the twentieth century, taking his position in the international context of the philosophy of science with awards such as the appointment as *Correspondant* of the Académie des sciences morales et politiques of the Institut de France.

The letter we report below leaves no doubt as to Enriques's interest in Turin as the place for his specialisation. Segre wrote about it to Castelnuovo, proposing instead the opportunity of an appointment in Rome:

> Let's speak about Enriques …. He is going to apply for the post-graduate position; there's no doubt about that. My doubt, however, is whether coming to Turin will be advantageous to him, should he win the position. I do not deny that my company and my lectures may be useful to him. But in Rome, would he not benefit just as much or even more from your company and the lectures of Beltrami and Cremona?[27]

In fact, Enriques won the post-graduate position at the University of Rome. In the meantime he spent some time in Turin. Segre sent his impressions to Castelnuovo:

> Enriques is a likeable young man and every evening he comes to wait for me near here (where we meet up with D'Ovidio, Korsenberg, Foà, etc.) to discuss various issues. … He really needs to study more and above all he needs someone to make him reflect more

[25] Gario (2010), Segre to Castelnuovo, 15 July 1892.

[26] This paper had a significant influence on Enriques (1895). For more detail see Ciliberto and Gario (2012).

[27] Gario (2010), Segre to Castelnuovo, 18 July 1892: *"Parliamo dell'Enriques. ... Concorra pel posto di perfezionamento: su ciò non vi è alcun dubbio. Il dubbio l'ho invece, nel caso che vinca, se proprio debba profittare per venire a Torino. Non nego che la mia compagnia e le mie lezioni possano giovargli. Ma a Roma la tua compagnia e le lezioni di Beltrami e Cremona non gli gioverebbero altrettanto e più?*

carefully. He makes some really glaring mistakes through carelessness. He is slapdash both in his research and in his exposition so you will have a hard task with this young man.[28]

Almost half a century later, commemorating Enriques (Castelnuovo 1947), Castelnuovo recalled that at the outset “that young man had a broad vision of our science but had not yet established the goal of his research” and, looking back on the early months of 1893, he alluded to “those endless walks through the streets of Rome, during which our favourite topic of conversation was algebraic geometry”. Having rapidly assimilated the achievements of the Italian school in the theory of algebraic curves, “Enriques bravely turned his attention to the subject of geometry on an algebraic surface”, adding that he “kept me up to date with the progress of his research on a daily basis and I subjected it to harsh criticism”. Again in the words of Castelnuovo, “it is no exaggeration to say that during those conversations the theory of algebraic surfaces was constructed according to the Italian approach”.[29]

A letter dated 14 May reveals that Segre hoped to make his own contribution to the theory of algebraic surfaces. He had been given the opportunity with the course of Higher Geometry (1892–93), which focused on geometry on a curve. Segre aimed to extend the hyperspatial method devised for curves to the higher dimensional varieties, especially surfaces. Traces of this work can be found in his lecture notebooks and in the aforementioned “Introduzione alla geometria sopra un ente algebrico semplicemente infinito” (Segre 1894a) the publication of which he continued to postpone due to health problems and various commitments but also, and perhaps especially, because he intended to insert further developments, according to a program that he expounded to Castelnuovo in a letter written in May,[30] at the same time asking for information on how Enriques’s work was progressing:

I do not know if Enriques’s research on surfaces has points of contact with what I am doing. I will write to him in the next few days. Tell me what you think, but without specifying his results where they contain things that are doubtlessly not obvious.[31]

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[28]Gario (2010), Segre to Castelnuovo, 16 November 1892: *L’Enriques è un giovane simpatico, che ogni sera viene ad aspettarmi qui vicino (col quale poi mi trovo con d’Ovidio, Korsenberg, Foà ecc.) per discorrere di cose varie. ... Avrebbe gran bisogno di maggiori studi e sovratutto di chi insistesse per fargli acquistar maggior ponderatezza. Fa degli errori veramente grossolani, per effetto di leggerezza. Trascurato nella ricerca, come nell’esposizione avrai molto da fare intorno a questo giovane*.

[29]See Castelnuovo (1947) = (Castelnuovo 2002, Vol. IV, 218).

[30]*During my course I have spoken in several lectures about linear series on an M_k rather than on a curve. Many things are certainly developing. During the autumn holidays, for M_k, I plan to go ahead with a study on Restsatz, special series, Riemann-Roch theorem, etc., similar to the hyperspatial study done on curves 2 years ago. I am almost certain that it is possible.* Gario (2010), Segre to Castelnuovo, 14 May 1893.

[31]Gario (2010), Segre to Castelnuovo, 14 May 1893: *Non so se le ricerche di Enriques sulle superficie abbiano punti di contatto con quelle che io ora vado facendo. Gli scriverò in questi giorni. Tu dimmi che ne pensi, senza però enunciarmi i suoi risultati, dove contenessero cose non indovinabili senz’altro*.

Castelnuovo promptly replied to Segre, who with his customary frankness, did not hesitate to reveal to his friend a certain disappointment, but which did not in any case affect his readiness to present Enriques's paper (Enriques 1893), now nearing completion, to the Accademia delle Scienze di Torino:

> What I wrote to you recently made me want to wait a bit before hearing about his results. But you, who know me, will understand that I give little importance to personal considerations of this kind. On the contrary, I am curious to know if and how Enriques has done what I started to do.
>
> I would need to have Enriques's paper a few days before 11 June [...].
>
> I cannot assure you that Enriques's work will be published soon; it depends on how many papers are presented before his [...]. All else aside, in view of the favourable impression I have of Enriques and what you have already told me about his work, I willingly agree to present his paper if he wants me to.[32]

Thus, after 10 days or so, Segre received Enriques's manuscript:

> I have received Enriques's manuscript and I have read the first half carefully.
>
> Please advise him to take greater care with the exposition and especially the material form of his work. I strongly recommend rigour, rigour, rigour. I have already found some gratuitous statements in places [...]. It would be better to delay publication rather than diminish the importance of the paper with incomplete proofs or erroneous propositions.[33]

In June 1893, Segre presented Enriques's "Ricerche di geometria sulle superficie algebriche" (Enriques 1893) to the Accademia delle Scienze di Torino (D'Ovidio and Segre 1892–1893), underlining its value and originality, in spite of certain aspects which, in his opinion, would have to be refined in later studies. Castelnuovo, commemorating Enriques, used similar words: "it was natural that there were some imperfections in a work that had been formulated and hurriedly written down in just a few months, relating to an almost unexplored field, where the analogy with the theory of curves is often misleading".[34]

After the presentation to the Accademia, the revision of the article went ahead throughout the month of July. From the letters it is possible to discern a certain irritation on Segre's part over modifications that Enriques asked him about:

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[32]Gario (2010), Segre to Castelnuovo, 16 May 1893: *Quanto ti scrissi ultimamente mi faceva desiderare di tardare a conoscere i suoi risultati. Ma tu che mi conosci capirai che a considerazioni personali di questo genere io dò un peso assai piccolo. Anzi, vedrò con curiosità se e come l'Enriques abbia fatto ciò che io m'ero avviato a fare. Bisognerebbe che io avessi la memoria dell'Enriques alcuni giorni prima dell'11 (giugno) .... Non posso assicurare che il lavoro di Enriques sarebbe pubblicato presto: dipende dalla quantità di memorie che verranno presentate prima* [...]. *Del resto la simpatia che m'ispira l'Enriques e ciò che tu già mi dici sul suo lavoro fanno sì che io ben volentieri accetterò di presentarlo, se egli crede*.

[33]Gario (2010), Segre to Castelnuovo, 27 May 1893: *Ho ricevuto il manoscritto di Enriques e ne ho già letto attentamente la metà. Intanto ti prego di raccomandargli maggior cura nell'esposizione e specialmente nella forma materiale. Raccomando poi caldamente il rigore, il rigore, il rigore. Già ho trovato in qualche punto delle asserzioni gratuite* [...]. *Meglio ritardare la stampa del lavoro piuttosto che scemare l'importanza dei questo con dimostrazioni incomplete o proposizioni sbagliate*.

[34]See Castelnuovo (1947) = (Castelnuovo 2002, IV, 219).

> Enriques has made some modifications to his work and he wanted me to look at it again. But I refused! He should do something on his own, for goodness' sake!
>
> I certainly don't want him to come to Turin if he thinks he can keep on getting help with everything, at every step […]. I can fully understand why he might like to stay in Rome, where he has you, who are so good to him![35]

Enriques spent part of the autumn of 1893 in Turin while awaiting a post. His relationship became more relaxed with Segre, who appreciated his commitment, but Enriques remained somewhat in awe of him, as we can read in this letter to Castelnuovo:

> Was it really only a slight mistake?[36] […]. Certainly I am very sorry about it, not so much because I had assured you and Segre that I had taken every care with editing the work […] as because I realise that unfortunately, despite all my efforts (to which you can bear witness), my fundamental shortcoming remains […]. Up to now I have tried to persuade S[egre] (with some success) that the bad opinion he had of me regarding this defect is no longer justified. […]
>
> If there is anything that touches me and moves me to correct this defect of mine (in addition to feeling the necessity to do so), more than S[egre]'s "rigour" (however benevolent), it is your patience.
>
> You, more than any other witness to my mistakes, have never reproached me harshly, you have never shown impatience. And I am heartily grateful for this; you sympathise with me and correct me at the same time; your words always restore me, for otherwise I would often be overwhelmed with discouragement.[37]

In January 1894, Enriques moved to the University of Bologna to take up a teaching post. He stayed there until 1922.

[35]Gario (2010), Segre to Castelnuovo, 20 July 1893: *L'Enriques ha fatto varie modificazioni al suo lavoro e voleva ancora che lo rivedessi. Ma io mi son rifiutato! Faccia un po' da sé, che diamine! Non vorrei certo che venisse a Torino se pensa di continuare nel sistema di farsi aiutare in tutto, ad ogni passo* [...]. *Capisco bene che desideri continuare a Roma ove ha te, così buono!*".

[36]This refers to Enriques (1893–1894).

[37]Bottazzini et al. (1996), Enriques to Castelnuovo, 1 January 1894, 61: *L'errore era realmente lieve? ... Certo questo mi ha fatto un gran dispiacere, non tanto perché avevo assicurato il Segre e te di aver posto ogni cura nella redazione del lavoro ..., quanto perché mi accorgo disgraziatamente che nonostante tutti i miei sforzi (di cui tu sei stato testimone) il mio sostanziale difetto permane ancora.* [...]. *Io ho tentato fino ad ora di persuadere il S[egre] (e vi sono in parte riuscito) che la cattiva opinione che egli conservava di me su questo rapporto non è ora più giusta* [...]. *Se vi è una cosa che mi commuova e mi sproni a correggermi del mio difetto (oltre al sentimento della necessità) più che il rigore (pur tanto benevolo) del S[egre] è la tua longanimità. Tu più spesso d'ogni altro testimone dei miei errori non me ne hai mai rivolto un rimprovero acerbo, non mi hai mai mostrato impazienza. Ed io te ne sono grato di cuore; tu mi compatisci e mi correggi ad un tempo; la tua parola mi rianima sempre perché altrimenti io mi sentirei preso spesso da un grande scoraggiamento.*

4 Algebraic Surfaces: Teamwork, Synergy or Division of Labour?

Castelnuovo's work program had by now shifted towards issues of rationality. During the year 1893, he extended Lüroth's theorem to algebraic surfaces, demonstrating the rationality of every surface that is unirational. "Bravo, bravissimo! If you have managed to prove the great theorem, you deserve a medal!" These were the words of congratulation that Segre sent Castelnuovo in August that year.[38]

Fundamental in the article on the rationality of plane involutions (Castelnuovo 1894) was the finiteness of the *process of adjunction*, which consists in considering the subsequent adjoint systems to a linear system of curves on the given surface. This technique provided Castelnuovo with the means to deal successfully the following year with the problem of characterisation of rational surfaces, thanks in part to a happy intuition of Enriques. Castelnuovo's studies were based on the common conviction that the conditions $p_g = q = 0$ were not only necessary but also sufficient to characterise rational surfaces. In July 1894, Enriques gave him a "Bravo" for having "found the right way to decide the all-important question of the rationality of surfaces".[39] In the meantime, however, Castelnuovo had noticed that the conditions $p_g = q = 0$ did not exclude the possibility that the second adjoint system could contain the initial system (without this happening for the first adjoint system). If this were the case, the adjunction process would not be terminated. Of course, Castelnuovo immediately wrote to Enriques about this anomaly and the latter's reply gave him the famous example that invalidated the initial conjecture and suggested the correct formulation of the criterion of rationality that today is attributed to him ($p_g = q = 0$ and $P_2 = 1$).[40] The article of Castelnuovo, "Sulle superficie di genere zero" (Castelnuovo 1896b) is dated December 1894 but was only published in 1896.

In the meantime, Enriques had started to revise his "Ricerche" (Enriques 1893). Writing to Castelnuovo in April 1894, he said that "rather than explore new areas", he intended to review the foundations of the theory of linear systems of curves on a

[38]Gario (2010), Segre to Castelnuovo, 4 August 1893.

[39]Bottazzini et al. (1996), Enriques to Castelnuovo, 16 July 1894, 123. In this letter Enriques wrote: *I will send you the proof of the conservation of the numerical genus so that you can affirm that all the surfaces of geometric genus = numerical genus = 0 are rational.*

[40]*I am sorry about the new difficulties you are encountering with the question of rational surfaces. The existence of a surface of the genus 0 having ψ_{2n-8} and not ψ_{n-4} surfaces would indeed be strange; however, it wouldn't surprise me. Try with a surface of the 6th order having the 6 edges of a tetrahedron as double curve (if it exists).* Bottazzini et al. (1996), Enriques to Castelnuovo, 22 July 1894, 125.The surface described in this letter is of degree 6 and contains a nodal double curve consisting of the edges of the tetrahedron whose vertices are triple points for the surface. There are no quadrics passing through the edges of the tetrahedron, i.e., no adjoint surfaces ψ_{n-4} containing the nodal curve, and therefore $p_g = 0$. The existence of the reducible surface ψ_{2n-8} of degree 4, consisting of a sum of the four faces of tetrahedron ensures that the bigenus P_2 is not zero. In Enriques (1906) Enriques completed the birational classification of surfaces with $p_g = q = 0$ e $P_2 = 1$, known today as *Enriques surfaces*.

surface, in the hope that this work "will lay the foundations of my *Ricerche* on a firm basis". But there was "a lot to do" because "everywhere there are gaps filled by intuition". Fortunately, according to Enriques "everything fits, at least where I have checked, apart from the thing about complete systems". The revision work, which for Enriques had "a painful aspect, given that, at the end of the day, all this only serves to arrive at things that are already known",[41] occupied him in the months that followed, intensifying during the summer break, and leading to the publication of "Introduzione alla geometria sopra le superficie algebriche" (Enriques 1896) which came out together with Castelnuovo's articles on rational surfaces (Castelnuovo 1896b).

For his part, Segre did not give up the plan to contribute to the theory of algebraic surfaces even after the publication of Enriques's "*Ricerche*". The motivation came with the lectures for the Course of Higher Geometry (1893–1894), *Introduzione alla geometria delle trasformazioni birazionali del piano*.[42] Part of the course was devoted to rational surfaces studied through the corresponding linear systems of plane curves. On this topic, at the beginning of May, he wrote to Castelnuovo, who had "started on the extension of those things to linear systems on any surface".[43] The following declaration of his intentions left no doubt as to his study programme for the summer break:

> My (scientific) thoughts, now that the memoir[44] on the ∞ [simply infinite] algebraic entities has finally been published (have you read all of it?), will always be turned to surfaces; for these, there will always be analogous issues to deal with, in various directions. For some, similar methods can be used, but not all.[45]

As far as we can tell from Segre's next letter, Castelnuovo must have suggested to Segre a work of critical and comparative analysis of the results published in the existing literature but Segre was unenthusiastic about the proposal and his intentions remained unaltered.[46]

[41]Bottazzini et al. (1996, Enriques to Castelnuovo, 13 April 1894, 92).

[42]See Notebook 5 in Giacardi (2013).

[43]Gario (2010, Segre to Castelnuovo, 7 May 1894).

[44]The reference is to Segre (1894a).

[45]Gario (2010, Segre to Castelnuovo, 7 May 1894): *I miei pensieri (scientifici), ora che la memoria sugli enti* ∞ *[semplicemente infiniti] è finalmente pubblicata (l'hai letta tutta?), si rivolgeranno sempre alle superficie: per le quali vi sono da trattare questioni analoghe, in varie direzioni. Per alcune servono metodi simili: non per tutte.*

[46]"The issues you raise in your last letter are all interesting and worthy of my attention. What I fear is that time will go by, I will spend a few years only elaborating a little further what has been done so far on surfaces. Certain things may already be found in analytical works (Kronecher, Weierstrass and other scholars), but since their mode of exposition is completely different, it is difficult to make comparisons. Anyway, I very much appreciated your opinion of my work. Gario (2010, Segre to Castelnuovo, 17 May 1894). In a letter dated 11 July 1894, Segre wrote to Castelnuovo saying that he was working on extending to surfaces his memoir on the geometry on simply infinite algebraic entities and that he had already drafted a few sections.

So a busy summer of hard work lay ahead for our three protagonists, and, as we have already said, for Castelnuovo the result was highly satisfactory. The three remained in close touch, exchanging letters in turn, to stay up to date with each other's research.[47]

Segre was also thinking of devoting his course of the following year to the general theory of surfaces. "I would like to put together the course on surfaces. This will be my task for these holidays" he wrote at the beginning of August, saying: "as for adding something new, I have little hope".[48] But at the end of the month, having followed some of the updates on his friends' work, Segre had a change of plan:

> I have been thinking about the surfaces. What you have written to me on this subject is very interesting. But precisely for this reason I have doubts about whether or not a course on surfaces might be premature, because in a few months what can be done *now* will be made to look outdated by you (and Enriques). On the other hand the subject is so vast that I can only do so much in a year, and maybe I can choose some chapters that are not as imperfect as others. So it might be useful to study the singularities of surfaces.[49]

---

[47]In July, an opportunity for further correspondence between the three arose thanks to a letter from. Émile Picard (1856–1941), who had decided "to say something about the work done in Italy" (Gario 2010, Picard to Castelnuovo, 7 July 1894) and (Bottazzini et al. 1996, 659) in an article regarding recent developments in mathematics for the *Revue Générale des Sciences pures et appliquées* (Picard 1894). For this reason Picard had written to Castelnuovo asking him for "a short summary on the work done in Italy over the last 2 or 3 years on the theory of algebraic surfaces (Castelnuovo, Enriques, etc.)" so that he would not omit anything important. Naturally, Castelnuovo wrote to Segre about this, partly to ask him for suggestions as to what should be mentioned. "I am writing [...] to congratulate you about Picard's letter", wrote Segre, adding his first suggestions. See Gario (2010), Segre to Castelnuovo, 19 September 1894. Picard's contribution to the general theory of surfaces dates back to 1884–1885. For an account of Picard's work, see Houzel (1991). The letter quoted is one of the first examples testifying to Picard's interest in the new research on the theory of algebraic surfaces that had started in Italy. This interest is confirmed by numerous references found in Vol. I of the treatise by Picard and Simart (1897). The geometric approach to the theory of algebraic surfaces, whose concepts and results did not show a clear correspondence with the transcendent theory developed in France, is the cultural basis for the formative process that would lead to the identity of the Italian school of algebraic geometry. The conceptual links with the French mathematicians' results were clarified around the turn of the century and the chapter by Castelnuovo and Enriques at the end of Vol. II of the above-mentioned treatise (Castelnuovo and Enriques 1906) gives a brief elucidation of the issue; for details, see Gario (1997). Concerning the relationship between German and Italian school, see Gray (1994). For later developments of the Italian school of algebraic geometry, see Brigaglia and Ciliberto (1995).

[48]Gario (2010, Segre to Castelnuovo, 8 August 1894).

[49]Gario (2010, Segre to Castelnuovo, 13 September 1894): *Son passato a 'pensicchiare' alle superficie. Ciò che tu mi scrivi su questo argomento è assai interessante. Ma appunto perciò mi sarebbe venuto il dubbio che sarebbe prematuro ora un corso sulle superficie, perché tu (e l'Enriques) renderai in pochi mesi antiquato ciò che ora si potrebbe fare. D'altra parte l'argomento è così vasto che io potrò solo fare qualche cosa in un anno: e forse potrò scegliere qualche capitolo che non sia tanto imperfetto come altri. Così sarebbe forse utile un po' di studio delle singolarità delle superficie* [...] *non fosse che per vedere fino a qual punto si possono cacciare* via*. A ciò appunto penso da ieri*!.

Thus, Segre intended to restrict his field of interest to a few chapters that were "not so imperfect as others". On the basis of this conviction, which later turned out to be illusory, that Segre decided to devote himself to the study of the singularities of surfaces, "just to see how far it is possible to get rid of them."[50]

Segre set to work immediately, examining first of all the existing literature. This was the beginning of his involvement in a subject that turned out to be much more complex than expected and that perhaps contributed to his detachment from the research of Castelnuovo and Enriques.

5 Singularities of Algebraic Surfaces: Segre's Involvement

The problem presented or implicated by non-ordinary superior singularities with regard to the definition of the geometric genus[51] was highlighted in 1890 by Castelnuovo in his article "Sulle superficie algebriche le cui sezioni piane sono curve iperellittiche" (Castelnuovo 1890) and circumvented with the caveat that his results referred not only to surfaces with ordinary singularities but also to all those surfaces for which the "concept of the adjoint surface can be established" in order to satisfy suitable conditions.[52]

The question could not be avoided by Enriques; in his "Ricerche" (Enriques 1893), the paragraph devoted to the definition of the *canonical system* is weighted down by considerations aimed at giving a more general definition of adjoint surface than that of Noether, in other words, a definition that can also be applied to a surface with non-ordinary singularities. In his revision of the "Ricerche", Enriques would therefore have to overcome this initial difficulty.

In 1871, Noether had proved the possibility of decomposing the singular points of algebraic plane curves into *infinitely near multiple points* by the application of successive quadratic transformations[53] and had stated the theorem of resolution of

[50]Gario (2010, Segre to Castelnuovo, 13 September 1894).

[51]See footnote (23).

[52]These conditions, as clarified later, required that the given surface should be regular.

[53]See Noether (1871). In this article, which became a fundamental reference for successive studies on the analysis and resolution of singularities of algebraic curves and surfaces, the notion of *infinitely near multiple points* appeared for the first time. Let γ be an algebraic curve having at O a non-ordinary s-fold point and let $t_1, \ldots, t_l$, the distinct tangent lines at O to the curve of multiplicity $\tau_1, \ldots, \tau_l$, respectively. Noether applies a first quadratic transformation having as fundamental points the point O and two points A and B, not lying in γ and such that the lines OA and OB do not coincide with any of the lines $t_1, \ldots, t_l$. The transformed curve γ' of γ intersects the fundamental line, which corresponds to the point O, at distinct points $O'_1, \ldots, O'_l$ which correspond to the tangent lines $t_1, \ldots, t_l$. The multiplicity s'_i of the point O'_i for the curve γ' is less than or equal to τ_i. The curve γ is said to have in the first order neighbourhood of O the points O'_i of multiplicity s'_i infinitely near to O. Applying a second quadratic transformation to each of the points O'_i one gets the points O''_{ij} of multiplicity s''_{ij}, infinitely near to O in the second order neighbourhood of O, and so on. See also Noether (1876).

singularities by means of Cremona transformations of the plane. In the second part of his paper, Noether examined the possibility of extending the procedure designed for curves to surfaces, showing with an example how the application of suitable quadratic transformations could transform the given surface into one with only ordinary singularities. What he wrote could be taken to mean that he believed that this would always be the case, for arbitrary singularities of the given surface: it would be sufficient to adapt the method he had identified on a case by case basis. In fact, his belief in the possibility of the method led to a precise formulation of a first theorem[54] of resolution of singularities of algebraic surfaces by Cremona transformations of the space.

It was Segre who, in May 1894, with his recently-published paper (Segre 1894a, Sect. 3: "La geometria sopra una varietà algebrica") and a postcard, drew Enriques's attention to the possibility of a second theorem of resolution of singularities,[55] involving birational transformations of the surface. Enriques wrote to Castelnuovo about this in a letter dated 3 May 1894: "Segre (with his paper and a postcard) drew my attention to Kobb's paper. What it is substantially about (if I understand correctly) is the transformation of a surface into one with ordinary singularities (or without singularities in a hyperspace)". Gustav Kobb (1863–1934), in the memoir, "Sur la théorie des fonctions algébriques de deux variables" (Kobb 1892), gives an analytic representation of the neighbourhood of an isolated singular point of an algebraic surface, showing that it is possible to represent this neighbourhood by a finite number of developments in series of integer powers of two auxiliary variables. To obtain these developments, Kobb applied successively quadratic transformations of the space[56] and Enriques was of the opinion that from this process the theorem of the resolution of singularities by birational transformations of the surface would follow easily. His arguments, expressed in just a few

[54]Every algebraic surface of projective 3-space can be transformed by a Cremona transformation of the space into a surface with only ordinary multiples curves (i.e., with distinct tangent planes). The only isolated singularities, finite in number, on such curves are ordinary cuspidal points and *normal crossing points* common to two distinct multiple curves.

[55]Every algebraic surface of projective 3-space can be transformed birationally into a surface of projective 3-space with ordinary singularities only, i.e., into a surface having a nodal curve, on which there is a finite number of ordinary cuspidal points and of triple points for the curve with distinct tangent lines, which are also triple and triplanar points for the surface.

[56]Kobb claimed that these transformations produce, on the transformed surfaces, multiple curves whose points have singularities not higher than that of the starting point. He also affirmed that the quadratic transformations, for which the singularity of the points of the transformed curves of the given point does not decrease, are finite in number. Segre mentions Kobb's contribution in the paragraph concerning the first generalisations about geometry on a variety of any dimension: he says here that the singularities "must be considered as decomposed or resolved into elements" (Segre 1894a, Chap. 1, Sect. 3). He refers the reader to the article by Kobb without further comment. Segre goes into some details of the issue of singularities only in Chap. 2, which introduces the simply infinite algebraic entities, for the definition of the genus of a curve (Segre 1894a, Chap. 2, Sect. 9).

words, are obscure, but the letter is interesting because it anticipates some of the crucial aspects of the foundations of geometry on a surface.[57]

This, then, is the context surrounding Segre's commitment relating to the theory of singularities of algebraic surfaces; his contribution to the foundation of the theory of linear systems of curves on an algebraic surface was crucial. Thus, in October 1894 Segre's studies began, accompanied by the preparation of the course in Higher Geometry. The study and the review of existing literature proved disappointing and gave rise to his first criticisms:

> And now I come to the main reason for this letter.
>
> The singularities of surfaces. I must tell you right away that I have not concluded anything!
>
> Kobb's work gives developments in series obtained by applying successive quadratic transformations to singular points, but he does not say that these can produce higher singularities [*singolarità superiori*]. … But the same oversight by Del Pezzo[58] is enough (without speaking of the others) to annul all his so-called proof of the reducibility of higher singularities [*singolarità superiori*] through birational transformations of the space. … And so, after thinking it over for several days, this is where I find myself: I am doubtful as to whether or not it is possible to reduce surfaces to only ordinary singularities by means of birational spatial transformations.
>
> Now I would like to review the other, earlier work by Del Pezzo[59] on the reduction, by means of only birational correspondences between surfaces, to surfaces with double lines and triple points, in case I can get something out of it. But I am very much afraid I won't be able to.[60]

Shortly after this del Pezzo's article relating to the proof of the second theorem of resolution of singularities was also criticised by Segre: "I have looked at it again and I can see that there is nothing to be gotten out of it", he wrote in the following letter with the assurance, "however I am not going to stop thinking about the matter.

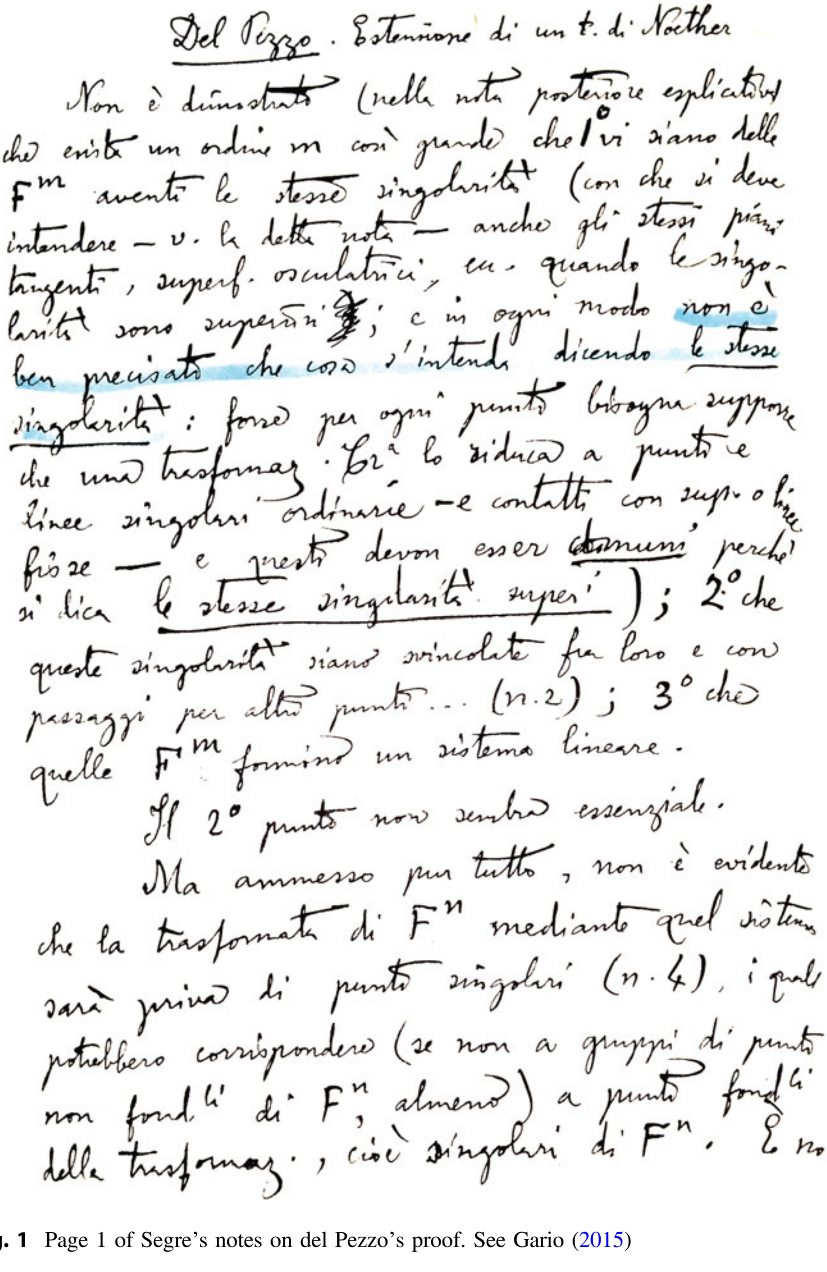
---

[57]"The same process seems to be applicable to multiple curves; but it is the multiple points that matter to me in particular. Dismissing these means having, on the surface, systems without fundamental curves, which disturb [orig.: "che danno noia"], systems for which the adjoint coincides with the pseudoadjoint [subadjoint]. So I don't despair of extending to all surfaces with the geometric genus 0 the theorem on the invariance of the numerical genus, as discussed in Chap. 3 of my Ricerche". Bottazzini et al. (1996, Enriques to Castelnuovo, 3 May 1894, 101–102).

[58]See del Pezzo (1892). The correct spelling is del Pezzo not Del Pezzo. The articulated preposition 'del' is typically found in the names of families of noble origin.

[59]See del Pezzo (1888).

[60]Gario (2010) Segre to Castelnuovo, 2 October 1894: *Ed ora vengo allo scopo principale di questa lettera. Le singolarità delle superficie. Debbo subito dichiararti che non ho concluso nulla! Il lavoro del Kobb dà degli sviluppi in serie ottenuti applicando a punti singolari staccati delle trasformazioni quadratiche successive: ma non rileva che queste possono produrre singolarità superiori. Forse da ciò non vengono infirmati i suoi risultati analitici. Ma la stessa inavvertenza fatta dal Del Pezzo basta da sé (senza parlare delle altre) ad annullare tutta la sua sedicente dimostrazione della riducibilità delle singolarità superiori mediante trasformazioni birazionali dello spazio.Ora voglio rivedere l'altro, più antico, lavoro di Del Pezzo sulla riduzione, mediante sole corrispondenze birazionali fra superficie, a superficie con linea doppia e punti tripli: caso mai qualcosa ne potessi cavare. Ma ho gran timore di non potere.*


Del Pezzo. Estensione di un t. di Noether

Non è dimostrato (nella nota posteriore esplicativa) che esista un ordine m così grande che vi siano delle F^m aventi le stesse singolarità (con che si deve intendere – v. la detta nota – anche gli stessi piani tangenti, superf. osculatrici, ecc. quando le singolarità sono superiori); e in ogni modo non è ben precisato che cosa s'intenda dicendo le stesse singolarità: forse per ogni punto bisogna supporre che una trasformaz. Cr.^a lo riduca a punti e linee singolari ordinarie – e contatti con sup. o linee fisse – e questi devon esser comuni perchè si dica le stesse singolarità super.); 2° che queste singolarità siano svincolate fra loro e coi passaggi per altri punti ... (n. 2); 3° che quelle F^m formino un sistema lineare.

Il 2° punto non sembra essenziale.

Ma ammesso pur tutto, non è evidente che la trasformata di F^n mediante quel sistema sarà priva di punti singolari (n. 4), i quali potrebbero corrispondere (se non a gruppi di punti non fond.li di F^n, almeno) a punti fond.li della trasformaz., cioè singolari di F^n. E no-

Fig. 1 Page 1 of Segre's notes on del Pezzo's proof. See Gario (2015)

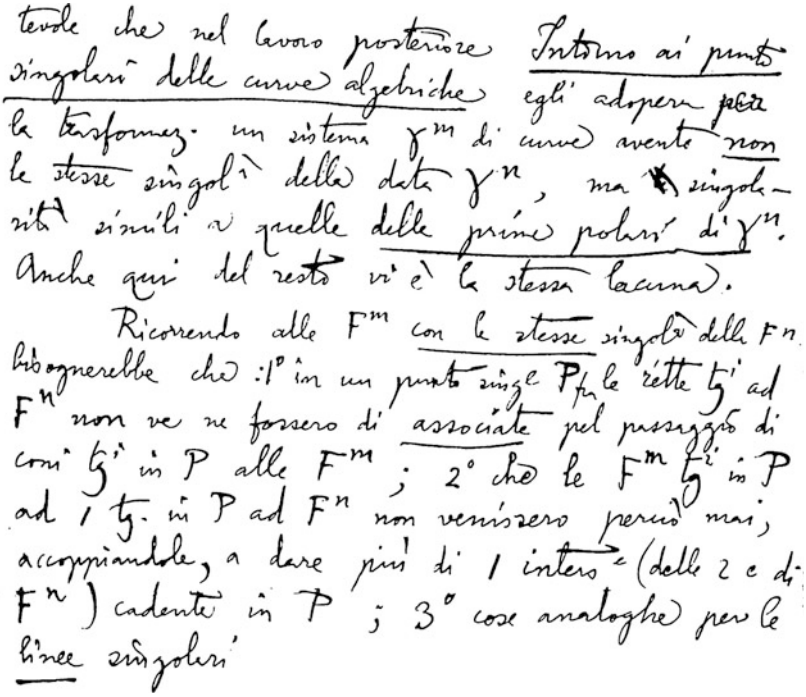

tende che nel lavoro posteriore Intorno ai punti
singolari delle curve algebriche egli adopera per
la trasformaz. un sistema γ^m di curve aventi non
le stesse singol.ᵗ della data γ^n, ma singola-
rità simili a quelle delle prime polari di γ^n.
Anche qui del resto vi è la stessa lacuna.
Ricorrendo alle F^m con le stesse singol.ᵗ della F^n
bisognerebbe che: 1° in un punto sing.e P fra le rette t_i ad
F^n non ve ne fossero di associate pel passaggio di
coni t_i in P alle F^m; 2° che le F^m t_i in P
ad i t_i in P ad F^n non venissero perciò mai,
accoppiandole, a dare più di 1 interse (delle 2 e di
F^n) cadente in P; 3° cose analoghe per le
linee singolari

Fig. 2 Page 2 of Segre's notes on del Pezzo's proof. See Gario (2015)

You think about it, too!"[61] The two sheets of notes written by Segre on del Pezzo's proof of 1888 are shown in Figs. 1 and 2.[62]

[61]Gario (2010, Segre to Castelnuovo, 10 October 1894).

[62]Figures 1 and 2 show a copy of the two sheets of notes written by Segre on del Pezzo's proof of 1888. This document, which is part of a copious set of notes relating to his reading, was discovered in a private archive at the villa known as "Il Pinocchio" in Ancona, which belonged to the family of Segre's wife, Olga Michelli. As the document is not dated, it is impossible to determine whether it refers to the study mentioned in the letter. For a description of this archive, see Gario (1989, 2015). The document touches on one of the essential points of del Pezzo's proof. Given the surface F in the ordinary projective space $\mathbb{P}^3(\mathbb{C})$, del Pezzo applied a birational transformation to F, transforming F into a surface F' without singularities of a suitable $\mathbb{P}^r(\mathbb{C})$. In order to define the rational map from F to F', he used the linear system cut out on F by the surfaces of a given order (sufficiently high) passing through the singular points and the singular curves of F and having there the "same singularity" as F. Segre observed that "it is not clear what is intended by the same singularities" (see the sentence highlighted in Fig. 1). As a result of this observation Segre felt that it was necessary to develop an instrument of analysis of singularities of algebraic surfaces that would precisely define the equisingularity introduced by del Pezzo (here I take the liberty of using the term *equisingularity*, see Zariski (1979). In his paper "*Alcune proprietà fondamentali dei sistemi lineari di curve tracciate sopra una superficie*" (Castelnuovo 1897), Castelnuovo specified the meaning of equisingularity for the case of plane curves to introduce the concept of "linear system of plane curves defined by the base points", bringing into play the infinitely near multiple

The month of October went by without results and from the correspondence it is clear that Castelnuovo was getting impatient. Examinations, the preparation of his course and paternal duties—a pleasure he was unwilling to renounce[63]—all these prevented him from devoting himself full time to "the work you desire", he wrote to Castelnuovo at the end of the month. The work would be ready "perhaps by the end of the year".[64] In fact, during the end-of-year holidays he would have the opportunity "to reflect a lot on the minutest details of the proof that the surface can be reduced to one of S^3 with only one double curve, etc.", and he reassured his friends: "I hope to satisfy you somehow". The work seemed to be coming to fruition. In early February, Segre was able to write to Castelnuovo that "I will also explain the theorem that any surface can be referred to a hyperspatial surface without multiple points" and he urged him to inform Enriques and, above all, to tell the journal in which their works were about to be published that "some space should be reserved" for him (12–20 pages) because, he concluded:

> I would really like to publish in your company. In this way our whole school would be represented![65]

But the optimism shown by Segre was once again thwarted by serious, seemingly endless difficulties.[66] The words of Enriques "the way to surfaces, like the way to heaven, is strewn with thorns"[67] had proved prophetic. Segre continued to work on the problem throughout 1895 and 1896, obtaining significant results but without managing to complete the proof of the resolution theorems. Segre dealt with the singularities of algebraic surfaces also in his course of 1896–1897.[68] In the autumn of 1896 he decided in any case to publish the results he had achieved so far.

> The work plan is now completely different from the old one, and the problem you are interested in is no longer in first place. I think I have come very close to the solution even

(Footnote 62 continued)
points, thus delineating a classic theory of equisingularity for plane curves: for further details, see Gario (1990). In the same memoir, Castelnuovo introduced an analogous notion for the linear systems of algebraic surfaces. This is a demonstration of the great significance of Segre's work on singularities.

[63]"When I arrive home I find an adorable little girl who holds out her arms to me and cries if I don't pick her up and start playing with her" (Gario 2010, Segre to Castelnuovo, 29 October 1894).

[64]Gario (2010), Segre to Castelnuovo, 24 December 1894.

[65]Gario (2010), Segre to Castelnuovo, 9 February 1895: […] *avrei molto piacere di uscire in vostra compagnia. Così appunto sarebbe tutta la nostra scuola rappresentata!* Segre referred to the publication of the papers (Castelnuovo 1896a, b; Enriques 1896).

[66]The program for the course of 1894–1895 was also affected by those difficulties, as can be seen from the notebook (Gario 2002a) which, despite the title on the cover, "*Teoria delle singolarità delle curve e delle superficie algebriche*", is actually about the singularities of plane curves.

[67]Bottazzini et al. (1996, Enriques to Castelnuovo, 20 October 1894, 140–141): *La* via *delle superfici come quella dei cieli, è seminata di spine.*

[68]See the notebook "Lezioni sulla teoria delle singolarità delle curve e superficie algebriche" (Gario 2002b).

> though there are still some gaps, but I have had to decide to publish at all costs, otherwise it would mean at least another year's delay![69]

At the end of the year, the memoir "Sulla scomposizione dei punti singolari delle superficie algebriche" (Segre 1897a) was ready to go to press. The aim of Segre's paper was to analyze the singularities of algebraic surfaces using the techniques of *infinitely near multiple points* and to use this notion to deal with the problem of the resolution of singularities of algebraic surfaces by birational transformations (not Cremona transformations). In this paper Segre also examines del Pezzo's method in "Estensione di un teorema di Noether" (del Pezzo 1888) making some critical remarks and sparking a sharp dispute which lasted several months.[70] The finiteness of the process of decomposition of singularities introduced by Segre was the subject of the controversy with del Pezzo. This question was rigorously treated by Segre's student Beppo Levi (1875–1961), who had helped to revise the draft and who had been given the task of "completing a certain line of reasoning,"[71] in the article "Sulla riduzione delle singolarità puntuali delle superficie algebriche dello spazio ordinario per trasformazioni quadratiche" (Levi 1897). Levi also proposed a proof of the resolution theorem by birational transformations of the surface "Risoluzione delle singolarità puntuali delle superficie algebriche" (Levi 1897–1898). Levi's proof was considered to be exhaustive until the critical revision of the various proofs of resolution theorems that Oscar Zariski (1899–1986) published as the introduction to his *Algebraic Surfaces,* in 1935.[72]

Segre's paper, therefore, did not contain any proof of the resolution of singularities, and this may have disappointed his two friends, but his technique for the analysis of the singular points of algebraic surfaces still stands in the history of mathematics for its rigour and elegance:

> The foundations of a geometric theory of singularities of algebraic surfaces have been laid down by C. Segre in his important paper. Using the method of successive quadratic transformations of the 1st kind he arrives at a theoretically satisfactory definition of infinitely near multiple points on a surface.
>
> […] It is important, however, to bear in mind that in the theory of singularities the details of the proofs acquire a special importance and make all the difference between theorems which are rigorously proved and those which are only rendered highly plausible.[73]

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[69]Gario (2010), Segre to Castelnuovo, 30 October 1896: *Il piano del lavoro è ora completamente diverso dall'antico: ed il problema che a te interessa non occupa il posto principale. Credo di averlo portato assai vicino alla soluzione, sebbene qualche lacuna vi sia ancora: ma ho dovuto decidermi a pubblicare senz'altro se no era di nuovo un anno almeno di ritardo*!.

[70]The controversy emerged with the publication of the articles (Segre 1896–97, 1897–98; del Pezzo 1897a, b, c). Del Pezzo had the last word, however. For details see Gario (1989a).

[71]Gario (2010), Segre to Enriques and Castelnuovo, 30 December 1896. The letter is addressed to Enriques and Castelnuovo, in that order. It would seem that Segre sent it to Enriques who then forwarded it to Castelnuovo the following day, with a short accompanying letter.

[72]See Zariski (1935, *Theory and Reduction of Singularities*, Chap. 1, 1–23).

[73]The two quotations are from the second (1971) supplemented edition: Zariski (1935), p. 13 and p. 18, respectively.

We note that Segre used similar words to comment on his work on singularities in the letter to Enriques and Castelnuovo at the end of December 1896:

> On this issue, my friends, it is necessary to proceed with great caution. The merit often lies in the details, not in the general concepts, which are easily discerned.[74]

Segre often lent out his notebooks. The appendix on the singularities of algebraic curves in Bertini's treatise "Introduzione alla geometria proiettiva degli iperspazi" (Bertini 1907) clearly indicates Segre's treatment of the topic and this is also shown by the succession of subjects. His notebooks on singularities are also cited in *Lezioni sulla teoria geometrica delle equazioni e delle funzioni algebriche* (Enriques and Chisini 1915). Segre's treatment was also used by Severi for drafting his *Trattato di Geometria algebrica* (Severi 1926).

6 A Joint Project

Segre's involvement in original research on singularities of algebraic surfaces, which it has been possible to reconstruct in its most dramatic aspects, drew to a close at the end of 1896. Recognition, also at international level,[75] of the value of the results on curves and algebraic surfaces led our three protagonists to consider producing a "great treatise on Higher Geometry":

Let us now speak about the great treatise of Higher Geometry. Castelnuovo will remember that here in Turin we sometimes spoke about jointly producing a work later on. Now that a new, energetic element has been introduced, the prospect seems much easier.

I would also like to point out that for Enriques it might be better to delay the project for a few years because, for better or for worse, the rules demand that he should prepare some publications in order to be promoted to full professor; so it would be better if, for a while, he focused his scientific activity only on original research. There is also the usual problem of the publisher: if possible, Gauthier-Villars or Teubner. They would be fine! I fully approve of the idea and I believe that the three of us could produce a good, harmonious and original treatise.

Whatever type of collaboration is adopted, I already know that I will be part of it: that is, with the notes (written by me) for some of my courses, namely, general courses on algebraic curves, surfaces, etc., and specialised courses on superior

[74]Gario (2010, Segre to Enriques and Castelnuovo, 30 December 1896): *In questo argomento, amici miei, bisogna andare molto guardinghi: il merito spesso sta nelle minuzie delle cose, non nei concetti generali che facilmente s'intuiscono.*

[75]One of the things that contributed towards the international diffusion of the results in the theory of algebraic surfaces was the paper "Sur quelques récents résultats dans la théorie des surfaces algébriques" (Castelnuovo and Enriques 1896) that Noether had agreed to publish in the *Mathematische Annalen* in place of the original articles (Castelnuovo 1896a, b; Enriques 1896), which were published instead in an Italian journal, partly on Segre's insistence. The article did not go into any detail on more delicate questions such as the construction of the adjoint system.

singularities, on geometry on a curve, on rational surfaces and linear systems of plane curves, etc.

If the project is postponed for a few years, some of my new courses might be of use. In addition I have to prepare the article on hyperspaces (algebraic varieties) for the German Encyclopaedia, and that could be useful for the corresponding chapters of the treatise (the same goes for the article on algebraic surfaces that Guido promised for the Encyclopaedia).[76]

Segre made it clear that in any case:

> I am completely in favour of the treatise being written by the three of us and no one else, because we share similar ideas and other people might upset the harmony of the work[77]

and, signing off, he added, "I'll be pleased to hear more from you on this subject." In reply, in the note that Enriques sent Castelnuovo along with the letter from Segre,[78] Enriques rejected Segre's concerns and scruples regarding his career and confirmed his commitment to the project:

> Predictably, Segre has agreed to take up the idea of the treatise again. I am not worried about his doubts as to whether it is expedient for me to start working on it (after all, it gives me no cause to set aside my original research, especially as there's no rush), so I am seriously thinking of transforming the idea into concrete reality.[79]

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[76]Gario (2010), Segre to Enriques and Castelnuovo, 30 December 1896: *Parliamo ora del grande trattato di Geometria Superiore. Castelnuovo ricorderà che qui a Torino ne abbiam parlato insieme qualche volta come di un lavoro che avremmo fatto insieme più tardi. Ora che un nuovo elemento, energico, s'introduce, la cosa diventa molto più facile. Per altro fo notare che appunto per Enriques converrebbe differire ancora di qualche anno l'impresa: poiché, buono o cattivo che sia, il regolamento vuole che egli si prepari titoli per la sua promozione ad ordinario; sicché per qualche tempo è meglio che egli dedichi la sua operosità scientifica solo a ricerche originali. Vi è poi la solita difficoltà dell'editore: se fosse possibile ricorrere a Gauthier-Villars oppure a Teubner, andrebbe bene! Del resto io approvo completamente l'idea, e credo che fra noi tre si potrebbe fare un buon trattato, armonico, originale. Qualunque sia la forma di collaborazione da adottare, io ho già un modo sicuro di prender parte all'opera: e cioè coi sunti (da me fatti) di vari miei corsi: cioè di qualche corso generale sulle curve, superficie ecc. algebriche, e di corsi speciali sulle singolarità superiori, sulla geometria sopra una curva, sulle superficie razionali e sistemi lineari di curve piane, ecc. Se l'impresa verrà ritardata di qualche anno, qualche nuovo corso farò che potrà pur servire. Inoltre io debbo preparare per l'Enciclopedia tedesca l'articolo sugli iperspazi (varietà algebriche), e ciò potrà pur servire pei capitoli corrispondenti del trattato (analogamente per l'articolo sulle superficie algebriche promesso da Guido all'Enciclopedia).*

[77]Gario (2010), Segre to Enriques and Castelnuovo, 30 December 1896: *Approvo poi pienamente che il trattato sia di noi tre e non d'altri: perché noi tre siamo in piena uniformità d'idee; ed altri forse turberebbe l'armonia dell'opera.*

[78]This is the letter cited earlier (Gario 2010, Segre to Enriques and Castelnuovo, 30 December 1896) received first by Enriques, who forwarded it to Castelnuovo with a note of accompaniment of his own; see (footnote (71)).

[79]Bottazzini et al. (1996), Enriques to Castelnuovo, 31 December 1896, 303–304: *Il Segre accetta, come si prevedeva, di riprendere l'idea del trattato. Siccome per parte mia non mi preoccupo dei dubbi che egli muove circa la convenienza per me di cominciarlo (non sarà una ragione per lasciare da parte le ricerche originali, tanto più che non importa andare con fretta), così si può pensare sul serio a tradurre l'idea nel campo dei fatti.*

The idea of a joint project, which had been mooted during Castelnuovo's Turin years before Enriques arrived on the scene, was now reproposed; the initiative appears to have come from Castelnuovo and Enriques, setting the seal on 10 years of work. The publishers mentioned, Gauthier-Villars and Teubner, would have guaranteed wide distribution and international prestige for the treatise. All that remained was to define what form the collaboration between the three authors was to take. The list of topics on which Segre thought he could make a contribution would mean dividing the work according to areas of competence: we are led to imagine a joint work, written in collaboration with Castelnuovo, though less probably with Enriques. The treatise never appeared. Segre's concerns about Enriques's career should not have affected the matter, however, considering the determination expressed by Enriques in the extract from his letter above.

In the 1897–1898 academic year, the opportunity of teaching a course in Higher Geometry was offered to Enriques. The course consisted in an introduction to algebraic geometry. Could it be considered as a step towards the "great treatise" project? Was his program an outline of the reflections shared with Castelnuovo and Segre during the preceding months?[80]

7 Lost Opportunities

Immediately following the above correspondence there was no further mention of the "great treatise" project. In the meantime, the project for the *Encyklopädie der mathematischen Wissenschaften* was already underway; as is well known, this would involve all three protagonists.[81] Segre wrote to Castelnuovo that the work for the German encyclopaedia might have been useful "for the corresponding chapters of the treatise".[82] On the other hand, it is not beyond the bounds of possibility that it was precisely the proposed collaboration with the German encyclopaedia that

[80]These questions are treated in more details in the article (Ciliberto and Gario 2012).

[81]The initial plan for the *Encyklopädie* was modified during its realisation, also because the publication of the various volumes was spread over several years. From the correspondence addressed to Castelnuovo and from the publishing plans, it is possible to reconstruct the development of the project. Castelnuovo and Enriques were also to have written (see Bottazzini et al. 1996, Enriques to Castelnuovo, 26 February 1897, 324) the article "Transformazionen und Korrespondenzen", later written by Luigi Berzolari (1863–1949) and published in Berzolari (1933), who also wrote Berzolari (1906). The article "Prinzipien der Geometrie", initially assigned to Heinrich Burkhardt (1861–1914), was reassigned (see Bottazzini et al. 1996, Enriques to Castelnuovo, 16 March 1897, 325) to Enriques (see Enriques 1907). Segre contributed with the article "Mehrdimensionale Räume" (Segre 1921c). Together, Castelnuovo and Enriques contributed with the articles "Grundeigenschaften der algebraischen Flächen" (Castenuovo and Enriques 1908) and "Die algebraischen Flächen vom Gesichtspunkte der birationalen Transformationen aus" (Castenuovo and Enriques 1915).

[82]Gario (2010, Segre to Enriques and Castelnuovo, 30 December 1896).

motivated Castelnuovo and Enriques to take up the idea of a joint project with Segre again.[83]

The contribution of the three geometers to the *Encyklopädie* is mentioned again in a few letters written in the early months of 1897, followed by a period of silence. When Segre takes up the subject again, in a letter written in August 1898, he is clearly in favour of the idea that had matured in the meantime, of dividing the work into areas of competence with regard to the articles on the theory of algebraic surfaces. Segre proposed that he should deal exclusively with the projective properties of the surfaces:

> It is also necessary that we should come to an agreement about the surfaces, to define properly what I am going to deal with, and what you and Enriques are going to write in your article concerning surfaces. For the sake of uniformity I think it would be best to leave to you and Enriques everything to do with the properties of geometry on a surface, even if they are obtained using hyperspaces; in general, for what concerns the surfaces of hyperspaces, I would be responsible only for the projective properties. So I will reproduce your results on surfaces with elliptic, hyperelliptic sections, of genus 3, etc., not those on canonical curves, on bigenus (on the surfaces considered in your paper on the rationality of plane involutions); or, if I find it necessary, I will only mention them. Tell me if you approve and if Enriques does.[84]

A year later, in August 1899, Segre wrote to Castelnuovo about the content and financial arrangements with the publisher Teubner concerning the publication of a book tentatively entitled *Vorlesungen über algebraischen Curven und höhere Räumen*, referring to it explicitly as "my book", while he asked Castelnuovo to indicate to him any other topics that he should have inserted in order to "give some help to your book" adding:

> Please do me the favour of sending this letter to Enriques, to whom it is also directed. I expect him, too, to express all his demands with regard to my book, as a separate work in relation to yours, as well as all the advice that comes into his mind.[85]

[83]"I am very keen to collaborate with you for Meyer's Encyclopedia, whether you want to write each article together, or whether you prefer to split the job between us", Enriques wrote to Castelnuovo on 9 June 1896, adding, "However, I would advise you to accept the topic of *plane algebraic curves* because the topic *algebraic curves*, at least for what concerns geometry on a curve, is a unitary theme". Instead the theme of the algebraic curves was dealt with in Berzolari (1906).

[84]Gario (2010), Segre to Castelnuovo, 28 August 1898: *Occorre anche che ci accordiamo riguardo alle superficie, per separare bene quel che spetta a me e quel che spetta all'articolo tuo e di Enriques sulle superficie. Per ragioni di uniformità mi sembra conveniente di lasciare a voi tutte quante le proprietà di geometria sulla superficie, anche se ottenute per mezzo degli iperspazi; io mi limiterei in generale, quanto alle superficie degli iperspazi, alle proprietà projettive. Così io riprodurrò i risultati tuoi sulle superficie a sezioni ellittiche, iperellittiche, di genere 3, ecc.: non quelli sulle curve canoniche, sul bigenere (sulle superficie considerate nel lavoro sulla razionalità delle involuzioni piane); od, occorrendomi, li citerò soltanto. Dimmi se approvi tu e se approva l'Enriques*.

[85]Gario (2010), Segre to Castelnuovo, 9 August 1899: *Poi fammi il favore di inviare questa mia lettera ad Enriques, al quale è pure diretta. Anche da lui aspetto che mi esprima tutti i suoi desideri riguardo alla mia opera, in relazione colla vostra, come considerata da sé, e tutti i consigli che gli posson venire in mente.*

Segre wanted to “produce a work, wide-ranging, yes, but not too wide; a harmonious work, not an encyclopaedia”. Its accomplishment had to “be carried out according to the most modern points of view”. His book—because this is what we are talking about, not the article for the German encyclopaedia—“along with the classic treatises on Analytic Geometry of Salmon and Clebsch”,[86] and with that “on algebraic surfaces” that Castelnuovo and Enriques were to write, as stated in the same letter,[87] would have contributed to “giving a fairly comprehensive idea of the [then] current state of algebraic geometry”. The project of the “great treatise on Higher Geometry”, which was to have been written by all three and which would have given Segre the opportunity to work together with his friends, had effectively been abandoned. In Segre’s correspondence with Castelnuovo there is just one more mention of his “book still to come”[88] in February 1900, after which no further trace of it is to be found.

The Italian treatises on algebraic geometry came out in Italy much later, even well into the twentieth century and, of the three, only Enriques appeared as author or co-author, along with other mathematicians who had arrived on the scene in the meantime.

8 Conclusions

> On re-reading my work done during the Turin period (1887–1891), I see before me the sweet and thoughtful figure of Corrado Segre, who, during the preparation of those writings was generous with his advice and encouragement. With gratitude and sadness I pay homage to this dear friend, incomparable master, who with his example taught me devotion to science and research. I also extend my gratitude to all my other masters.[89]

The story told here is that of three brilliant young researchers, whose relationships were not confined to the sphere of science but were also personal and emotional. Segre, the eldest, had graduated only 3 years before coming into contact with Castelnuovo. He used the tone of a master, he expressed himself with serious and severe words that perhaps reflect the tragic events which had afflicted his family and which could have been responsible for certain character traits. The contact with

[86]Segre referred to Clebsch (1875–1876) and Salmon (1879).

[87]Gario (2010), Segre to Castelnuovo, 9 August 1899.

[88]These were the topics that Segre was thinking of dealing with in the book: *Hyperspaces. The most important algebraic varieties present in hyperspaces. Geometry on a curve (linear series of groups of points,* etc.*) and its applications to spatial and hyperspatial curves. Rational surfaces lying in spaces of various dimensions, in relation to linear systems of plane curves, reduction of these linear systems to types,* etc. Gario (2010, Segre to Castelnuovo, 9 August 1899).

[89]See Castelnuovo (1937, *Introduzione*) = (Castelnuovo 2002, IV, 200): *Nel rileggere i miei lavori del periodo torinese (1887–1891) risorge innanzi a me la figura dolce e pensosa di Corrado Segre che durante la redazione di quegli scritti mi fu largo di consigli ed incoraggiamenti. Alla memoria dell’amico carissimo, del maestro incomparabile, che con l’esempio mi insegnò la devozione alla scienza ed alla scuola, invio un mesto e riconoscente saluto. Un pensiero di gratitudine rivolgo pure agli altri miei maestri.*

Castelnuovo was beneficial for him, by his own admission: “you have been good for me, I repeat, not only intellectually but also morally”. In his relationship with Enriques as well, despite some signs of impatience at first, due to the youthful errors of Enriques’s exuberant personality, he later developed feelings of friendliness and affection for him. In Segre’s article “Su alcuni indirizzi delle investigazioni geometriche” (Segre 1891a), we can see the teacher’s passionate commitment to young people, who he wants to direct towards the highest topics of mathematical research and towards a style of working open to the different areas of the discipline, in contrast to mathematical purism which is an obstacle to research. Enriques was attracted to him and influenced by him. Segre’s words are echoed by Enriques in the lecture notes for his *Conferenze di geometria: fondamenti di una geometria iperspaziale* (Enriques 1895), lectures that he gave during his second year at Bologna. Enriques’s *Conferenze* clearly show the influence of Segre’s vision of the evolution of mathematics as well as illustrating how the pupil imitates his master when he himself becomes a master.[90]

The story is, however, also that of connections that failed to evolve and develop scientifically and that fulfilled their early promise only in part. The relationship between Segre and Castelnuovo, which matured on the scientific level with the creation of geometry on a curve, and the relationship between Castelnuovo and Enriques, which matured on the scientific level with the creation of geometry on a surface, along with the work on classification of surfaces which they initiated, was never consolidated as a scientific partnership that involved these mathematicians in any truly common pursuit. The scientific association between Segre and Castelnuovo seemed to be interrupted when Enriques arrived on the scene. At the same time Segre was becoming progressively distant from the core research of his two friends. However, there is sufficient evidence to confirm that the reasons for this were not of a personal nature. The ‘accident’ of the singularities marks the beginning of his isolation and partly justifies it. When Segre finished writing the article on the decomposition of singular points of algebraic surfaces at the end of 1896, Castelnuovo and Enriques proposed the project of the “great treatise” of algebraic geometry to him. By entrusting his student Beppo Levi with the task of completing the remaining aspects of the theory of singularities, Segre would have had the time and energy to devote to this new and attractive project. The treatise could have been for him a bridge to new research on the theory of algebraic surfaces. Subsequent developments show, however, an evolution of the project which did not require Segre’s participation in the research of Castelnuovo and Enriques and which did not entail the possibility of closer collaboration.

[90]“For what concerns the opinions expressed here, we refer to Segre’s paper; here we can find the exposition of broad views on geometric trends which could turn out to be useful to young people in their research”. The citation is from the article (Ciliberto and Gario 2012) which makes a comparative analysis of the writings (Segre 1891a) and (Enriques 1895) and, more generally, examines the influence of the Turin milieu frequented by Enriques periodically before he moved definitively to Bologna.

The singularities of algebraic surfaces proved to be a thorny issue also for Beppo Levi, whose research Segre continued to follow, as stated in a letter addressed to Castelnuovo in the late spring of 1897:

> Anyone that has not explored them cannot imagine how many difficulties one encounters in the analysis of higher singularities. Levi knows it, after being engaged for so many months, first in completing my proof of the reducibility of singularities for birational transformations, then with that thing you needed about generic points of multiple lines and then with the theorem that the number of the characters s, s_i, s_{ik},… is finite; and every so often he thinks he's solved a problem but instead new difficulties arise. Now he seems to have proved this last theorem properly (according to Levi, Kobb's proof is wrong!).[91]

In the long introduction to the article "Introduzione alla geometria sopra le superficie algebriche" (Enriques 1896) Enriques dwelt on the problem of singularities, distinguishing between two kinds of difficulty: "the projective question of the singularities of a surface, as opposed to the question of their reducibility, i.e. of the transformation of a surface into one having only ordinary singularities". Relying on the positive outcome of Levi's studies, he wrote that, regarding the second question, "although not definitively resolved in a rigorous way," a "positive answer" was expected. Thus, he assumed the possibility that every surface can be birationally transformed into a surface with ordinary singularities only, specifying that this assumption could have been "limiting" for the surfaces considered, i.e., if it is not true that every surface can be transformed into one with only ordinary singularities, his results could be applied only to the surfaces that satisfy this hypothesis. But the "difficulty of their projective study", in particular of the singular curves that Segre had not dealt with in his paper, could have led to admitting "a restriction no longer relative to surfaces but to the systems [linear systems of curves] on it". Therefore, it was necessary to find a means to "remove this restriction," or rather, to find a strategy to get round the problem:

> Instead of addressing the question directly, I will say how, in a different way, I succeeded in applying the very general definition of adjoint curves to a linear system (and therefore of adjoint surfaces to a given [surface] S_3) thus indirectly overcoming the difficulty of the projective study of singularities of surfaces.[92]

[91]Gario (2010), Segre to Castelnuovo, 11 May 1897: *Non può immaginare chi non le ha approfondite quante difficoltà s'incontrano nell'analisi delle singolarità superiori: lo sa quel Levi che da tanti mesi si occupa, ora di completare la mia dimostrazione della riducibilità delle singolarità per trasformazioni birazionali, ora di quella tal cosa che a te sarebbe occorsa sui punti generici delle linee multiple, ed ora del teorema che i caratteri s_i, s_{ij}, s_{ijk},... sono in numero finito; e crede di tanto in tanto di aver risolta una questione e invece vengon fuori nuove difficoltà. Ora pare che abbia dimostrato bene quest'ultimo teorema (la dimostrazione del Kobb secondo Levi è sbagliata!).*

[92]*In luogo di abbordare la questione diretta, dirò come sono riuscito per altra* via *a porre la definizione assolutamente generale delle curve aggiunte ad un sistema lineare (e quindi delle superficie aggiunte ad una data S_3) superando quindi indirettamente la difficoltà dello studio proiettivo delle singolarità delle superficie.* The citations are from Enriques (1896) = (Enriques 1956, Vol. I, 216–217). In 1901 (see Enriques 1901–1902), he provided another construction of the adjoint system using the Jacobian system. In this article, however, the viewpoint is the modern

After the course on Higher Geometry of 1896–1897, Segre did not teach any other specific courses devoted to the theory of singularities of algebraic surfaces. He had occasion to return to the subject in the course of 1900–1901 on the theory of rational surfaces and of linear systems of plane curves[93] and in the course of the following year "on the geometry over an algebraic surface".[94]

In order to fully understand all the implications of the events we are focusing on, we should consider them in the context of Segre's activities as a whole. His commitment to teaching, never separated from his scientific work, was aimed at creating a school of mathematics in Italy comparable to the other European schools, open to all areas in which the discipline can be structured. In order to achieve this objective he probably sacrificed personal ambitions and achievements. As we can read in the article on some trends in geometric investigations (Segre 1891a) nothing is further from Segre's mind than a school that is closed in on itself; here Segre sees himself as a promoter of mathematical culture. The pride in his own achievements and those of his friends, which, at the end of the nineteenth century, led to the creation of a school of algebraic geometry, merges with the desire shared by many scientists of post-unification Italy: to create an identity for their country and a future for the younger generation.

Translated from the Italian by Helen Downes.

References

Bardelli, Fabio, On the Origins of the Concept of Irregularity of an Algebraic Surface, *Studies in the History of Modern Mathematics. Rendiconti del Circolo matematico di Palermo,* (2) 36 (1994): 11–26.

Bertini, Eugenio, La geometria delle serie lineari sopra una curva piana secondo il metodo algebrico, *Annali di Matematica pura ed applicata*, (2) 22 (1894): 1–40.

Bertini, Eugenio, *Introduzione alla geometria proiettiva degli iperspazi con appendice sulle curve algebriche e loro singolarità,* Pisa: E. Spoerri, 1907.

Berzolari, Luigi, Allgemeine Theorie der hören ebenen algebraischen Kurven, *Encyklopädie der mathematischen Wissenschaften*, III C4 (1906): 314–455.

(Footnote 92 continued)
one: the surface to be considered is a non-singular surface of a projective space of opportune dimension.

[93]See notebook 14, *Lezioni sulla teoria delle superficie razionali e dei sistemi lineari di curve piane,* in Giacardi (2013).

[94]See notebook 15, *Introduzione alla geometria sopra una superficie algebrica*, in Giacardi (2013). This was Segre's first course on geometry on a surface. In the meantime the theory of linear systems of curves on algebraic surfaces had been placed on a firmer basis thanks to Castelnuovo and Enriques, in particular for what concerns the construction of the adjoint system, as already mentioned, and with regard to the question of the *exceptional curves*, with the introduction of the *minimal model* (see the aforementioned article (Enriques 1901–1902) and the article (Castelnuovo and Enriques 1901). Segre took this into account in his course, as can be seen from his notebook. References to these questions can also be found in Enriques's letters to Castelnuovo. See Gario (1994).

Berzolari, Luigi, Algebraische Transormationen und Korrespondenzen, *Encyklopädie der mathematischen Wissenschaften*, III C11 (1933): 1781–2218.

Bottazzini, Umberto, Alberto Conte, Paola Gario, Riposte armonie. Lettere di Federigo Enriques a Guido Castelnuovo, Torino: Bollati Boringhieri, 1996.

Bottazzini, Umberto, Alberto Conte, Paola Gario, Lettere di Enriques a Castelnuovo: 1895–1905, *Studies in the History of Modern Mathematics. Rendiconti del Circolo matematico di Palermo,* (2) 74 (2004): 31–70.

Brigaglia, Aldo, Ciro Ciliberto, Italian Algebraic Geometry between the Two World Wars, *Queen's papers in Pure and Applied Mathematics,* 100 (1995), Kingston: Queen's University.

Brill, Alexander and Max Noether, Ueber die Algebraische Functionen und ihre Anwendungen in der Geometrie, *Mathematische Annalen,* 7 (1874): 269–310.

Castelnuovo, Guido, Angoli di due spazi contenuti nello spazio a *n* dimensioni, *Atti del R. Istituto veneto di Scienze, Lettere ed Arti,* (6) 4 (1885–1886a): 1167–1199; also in: [Castelnuovo 2002, I: 7–18].

Castelnuovo, Guido, Studio dell'involuzione generale sulle curve razionali mediante la loro curva normale dello spazio a *n* dimensioni, *Atti del R. Istituto veneto di Scienze, Lettere ed Arti,* (6) 4 (1885–1886b): 1167–1199; also in: [Castelnuovo 2002, I: 18–40].

Castelnuovo, Guido, Sulle congruenze del terzo ordine dello spazio a quattro dimensioni, *Atti del R. Istituto veneto di Scienze, Lettere ed Arti,* (6) 6 (1887–1888): 525–579; also in: [Castelnuovo 2002, I: 139–176].

Castelnuovo, Guido, Geometria sulle curve ellittiche, *Atti della R. Accademia delle Scienze di Torino. Classe di Scienze fisiche, matematiche e naturali,* 24 (1888–1889a): 4–22; also in: [Castelnuovo 1937: 1–17] and [Castelnuovo 2002, I: 177–194].

Castelnuovo, Guido, Ricerche di geometria sulle curve algebriche, *Atti della R. Accademia delle Scienze di Torino. Classe di Scienze fisiche, matematiche e naturali*, 24 (1888–1889b): 346–73; also in: [Castelnuovo 1937: 19–44] and [Castelnuovo 2002, I: 204–231].

Castelnuovo, Guido, Sulle superficie algebriche le cui sezioni piane sono curve iperellittiche, *Rendiconti del Circolo matematico di Palermo,* 4 (1890): 73–88; also in: [Castelnuovo 1937: 189–202] and [Castelnuovo 2002, I: 263–277].

Castelnuovo, Guido, Osservazioni intorno alla geometria sopra una superficie algebrica. Nota I, *Rendiconti dell'Istituto lombardo di Scienze e Lettere,* (2) 24 (1891): 127–137; also in: [Castelnuovo 1937: 245–25] and [Castelnuovo 2002, I: 309–318].

Castelnuovo, Guido, Sulla razionalità delle involuzioni piane, *Mathematische Annalen*, 44 (1894): 125–155; also in: [Castelnuovo 1937: 273–306] and [Castelnuovo 2002, II: 15–48].

Castelnuovo, Guido, Alcuni risultati sui sistemi lineari di curve appartenenti a una superficie algebrica, *Memorie di Matematica e Fisica della Società italiana delle Scienze, detta dei XL,* (3) 10 (1896a): 82–102; also in: [Castelnuovo 1937: 335–360] and [Castelnuovo 2002, II: 64–89].

Castelnuovo, Guido, Sulle superficie di genere zero, *Memorie di Matematica e Fisica della Società italiana delle Scienze, detta dei XL*, (3) 10 (1896b): 103–123; also in: [Castelnuovo 1937: 307–334] and [Castelnuovo 2002, II: 90–117].

Castelnuovo, Guido, Alcune proprietà fondamentali dei sistemi lineari di curve tracciati sopra una superficie algebrica, *Annali di matematica pura ed applicata*, (2) 25 (1897): 235–318; also in: [Castelnuovo 1937: 361–442] and [Castelnuovo 2002, II: 200–280].

Castelnuovo, Guido, Le trasformazioni generatrici del gruppo cremoniano nel piano. Estratto di una lettera al prof. Corrado Segre, *Atti della R. Accademia delle Scienze di Torino. Classe di Scienze fisiche, matematiche e naturali*, 36 (1900–1901): 861–874; also in: [Castelnuovo 1937: 459–471] and [Castelnuovo 2002, II: 339–352].

Castelnuovo, Guido, Commemorazione del Socio nazionale Corrado Segre, *Atti della R. Accademia dei Lincei. Classe di Scienze fisiche, matematiche e naturali. Rendiconti*, (5) 33 (2° semestre 1924), 353–359; also in [Castelnuovo 2002, III: 370–376].

Castelnuovo, Guido, *Memorie scelte*. Bologna: Zanichelli, 1937.

Castelnuovo, Guido, Commemorazione del Socio Federigo Enriques, *Atti dell'Accademia N. dei Lincei. Classe di Scienze fisiche, matematiche e naturali. Rendiconti*, (8) 2 (1° semestre 1947): 3–21; also in: [Castelnuovo 2002, IV: 218–242].

Castelnuovo, Guido, *Opere matematiche. Memorie e note, Pubblicate a cura dell'Accademia Nazionale dei Lincei,* Roma: Accademia Nazionale dei Lincei, I (2002), II (2003), III (2004), IV (2007).

Castelnuovo, Guido, Federigo Enriques, Sur quelques récents résultats dans la théorie des surfaces algébriques, *Mathematische Annalen*, 48 (1896): 241–316; also in: [Enriques 1956, I: 355–433] and [Castelnuovo 2002, II: 121–199].

Castelnuovo, Guido, Federigo Enriques, Sopra alcune questioni fondamentali nella teoria delle superficie algebriche, *Annali di matematica pura ed applicata*, 6 (1901): 165–225; also in: [Enriques 1956, II, 85–144.] and [Castelnuovo 2002, II: 353–410].

Castelnuovo, Guido, Federigo Enriques, Sur quelques résultats nouveaux dans la théorie des surfaces algébriques. In [Picard and Simart 1897, II: 485–522]; also in: [Castelnuovo 2002, II: 461–499].

Castelnuovo, Guido, Federigo Enriques, Grundeigenschaften der algebrischen Flächen (Abgeschlossen im jahre 1908), *Encyklopädie der mathematischen Wissenschaften,* III 2 1C (1915): 635–673; also in: [Castelnuovo 2002, III: 100–152].

Castelnuovo, Guido, Federigo Enriques, Die algebraischen Flächen vom Gesichtpunkte der birationalen Transformationen aus. *Encyklopädie der mathematischen Wissenschaften,* III 2 1C (1915): 674–768; also in: [Castelnuovo 2002, III: 153–266].

Ciliberto, Ciro, Paola Gario, Federigo Enriques: The First Years in Bologna, in, *Mathematicians in Bologna 1861–1960*, Salvatore Coen, ed., Basel: Birkhäuser, 2012, 105–142.

Clebsch, Alfred, Vorlesungen über Geometrie. Bearbeitet und herausgegeben von dr. Ferdinand Lindemann, I-III, Leipzig: Teubner, 1875–1876.

D'Ovidio, Enrico, Le funzioni metriche fondamentali negli spazi di quante si vogliono dimensioni e di curvatura costante, *Atti dell'Accademia R. dei Lincei. Memorie della Classe di scienze matematiche fisiche e naturali*, (3) 1 (1876–1877): 929–968.

D'Ovidio, Enrico, Corrado Segre, Relazione intorno alla memoria intitolata "Ricerche di geometria sulle superficie algebriche" del Dott. F. Enriques. Seduta del 25 giugno 1893, *Rendiconti della R. Accademia delle Scienze di Torino. Classe di Scienze fisiche, matematiche e naturali*, 28 (1892–1893): 867–868.

del Pezzo, Pasquale, Estensione di un teorema di Noether, *Rendiconti del Circolo matematico di Palermo*, 2 (1888): 139–144.

del Pezzo, Pasquale, Intorno ai punti singolari delle superficie algebriche, *Rendiconti del Circolo matematico di Palermo*, 6 (1892): 139–154.

del Pezzo, Pasquale, Osservazioni su una memoria del Prof. Corrado Segre e risposta ad alcuni suoi appunti, *Atti dell'Accademia Pontaniana,* 27 (1897): 1–13.

del Pezzo, Pasquale, Replica ad una nota del Prof. Corrado Segre in risposta ad alcune mie osservazioni, *Atti dell'Accademia Pontaniana*, 27 (1897): 1–7.

del Pezzo, Pasquale, *Contra Segrem*, *6 luglio 1897*, Napoli: Biblioteca Nazionale, Miscellanea Amodeo IX-64, 1897.

Enriques, Federigo, Ricerche di geometria sulle superficie algebriche, Torino: Clausen, 1893, and in, *Memorie dell'Accademia delle Scienze di Torino. Classe di Scienze fisiche, matematiche e naturali*, (2) 44 (1894): 171–232; also in: [Enriques 1956, I: 31–106].

Enriques, Federigo, Sulla massima dimensione dei sistemi lineari di curve di dato genere appartenenti ad una superficie algebrica, *Atti dell'Accademia delle Scienze di Torino. Classe di Scienze fisiche, matematiche e naturali*, 29 (1893–1894): 275–296; also in: [Enriques 1956, I: 107–124].

Enriques, Federigo, Conferenze di geometria: fondamenti di una geometria iperspaziale, Bologna: lithogr., 1895.

Enriques, Federigo, Introduzione alla geometria sopra le superficie algebriche, *Memorie di Matematica e Fisica della Società italiana delle Scienze, detta dei XL*, (3) 10 (1896): 1–81; also in: [Enriques 1956, I: 21–312].

Enriques, Federigo, Intorno ai fondamenti della geometria sopra le superfici algebriche, *Atti dell'Accademia delle Scienze di Torino. Classe di Scienze fisiche, matematiche e naturali*, 37 (1901–1902): 19–40; also in: [Enriques 1956, II: 65–83].

Enriques, Federigo, Sopra le superficie algebriche di bigenere uno, *Memorie della Società Italiana delle Scienze, detta dei XL*, (3) 14 (1906): 327–352; also in: [Enriques 1956, II: 241–272].

Enriques, Federigo, Prinzipien der Geometrie. *Encyklopädie der mathematischen Wissenschaften*, III 1 1 (1907): 1–129.

Enriques, Federigo, *Memorie Scelte di Geometria, Pubblicate a cura dell'Accademia Nazionale dei Lincei*, Bologna: Zanichelli, I (1956), II (1959), III (1966).

Enriques, Federigo, Oscar Chisini, Oscar, *Lezioni sulla teoria geometrica delle equazioni e delle funzioni algebriche*, Bologna: Zanichelli, I (1915), II (1918), III (1924), IV (1934).

Gario, Paola, Resolution of Singularities of Surfaces by P. del Pezzo. A Mathematical Controversy with Segre, *Archive for History of Exact Sciences*, 40 (1989a): 247–274.

Gario, Paola, Su alcune carte di Corrado Segre recentemente rinvenute, *Atti della Accademia delle Scienze di Torino. Classe di Scienze fisiche, matematiche e naturali*, 123 (1989b): 187–197.

Gario, Paola, La teoria classica dell'equisingolarità per le curve algebriche piane, *Bollettino di Storia delle Scienze matematiche*, 10 (1990), 77–97.

Gario, Paola, Singolarità e geometria sopra una superficie nella corrispondenza di C. Segre con G. Castelnuovo, *Archive for History of Exact Sciences*, 43 (1991): 145–188.

Gario, Paola, Singolarità e fondamenti della geometria sopra una superficie nelle lettere a Castelnuovo, *Studies in the History of Modern Mathematics. Rendiconti del Circolo matematico di Palermo,* (2) 36 (1994): 117–150.

Gario, Paola, Enriques e le comunità matematiche europee, in, *Seminari di geometria 1996–1997*, Bologna: Dipartimento di Matematica-Università di Bologna, 1997, 279–306.

Gario, Paola, *Teoria delle singolarità delle curve e delle superficie algebriche*, Quaderni di Corrado Segre 1894–95, (2002a): in [Giacardi 2013].

Gario, Paola, *Lezioni sulle singolarità delle curve e superficie algebriche*, Quaderni di Corrado Segre 1896–97, (2002b): in [Giacardi 2013].

Gario, Paola, Lettere e Quaderni dell'Archivio di Guido Castelnuovo, http://archivi-matematici.lincei.it/Castelnuovo/Lezioni_E_Quaderni/menu.htm. Roma: Accademia N. dei Lincei, 2010.

Gario, Paola, *Corrado Segre: Archivio di Ancona*, Dipartimento di Matematica - Università degli Studi di Milano, (23 febbraio 2015), http://users.mat.unimi.it/users/gario/Elenco-Segre.html

Giacardi, Livia (ed.), *Corrado Segre e la Scuola Italiana di Geometria Algebrica*: http://www.corradosegre.unito.it/. Torino 2013.

Gray, Jeremy, German and Italian Algebraic Geometry, *Studies in the History of Modern Mathematics. Rendiconti del Circolo matematico di Palermo,* (2) 36 (1994): 151–184.

Houzel, Christian, Aux origines de la Géométrie algébrique: les traveaux de Picard sur les surfaces (1884–1905), in, La France Mathématique, *Cahiers d'Histoire et de Philosophie des Sciences, Société Mathématique de France*, 34 (1991): 243–276 (also in: C. Houzel, *La géométrie algébrique. Recherches historiques*, 203–234. Paris: A. Blanchard, 2002).

Kleiman, Steven, Bertini and his Two fundamental Theorems, *Studies in the History of Modern Mathematics. Rendiconti del Circolo matematico di Palermo,* (2) 55 (1998): 9–37.

Kleiman, Steven, What is Abel's Theorem Anyway?, in, Laudal, Olav A. and Piene, Ragni, eds., The Legacy of Niels Henrik Abel: The Abel Bicentennial, Oslo 2002, Berlin-Heidelberg: Springer, 2004.

Kobb, Gustav, Sur la théorie des fonctions algébriques de deux variables, *Journal de Mathématiques pures et appliquées*, (4) 8 (1892): 385–419.

Levi, Beppo, Sulla riduzione delle singolarità puntuali delle superficie algebriche dello spazio ordinario per trasformazioni quadratiche, *Annali di Matematica pura ed applicata* (2) 26 (1897): 219–253.

Levi, Beppo, Risoluzione delle singolarità puntuali delle superficie algebriche, *Atti della R. Accademia delle Scienze di Torino. Classe di Scienze fisiche, matematiche e naturali,* 33 (1897–1898): 66–86.

Noether, Max, Zur Theorie des eindeutigen Entsprechens algebraischer Gebilde, *Mathematische Annalen*, 2 (1870): 293–316.

Noether, Max, Ueber die algebraischen Functionen einer und zweier Variabeln. Note 2, *Nachrichten von der K. Gesellschaft der Wissenschaften und der G.A.Universität zu Göttingen*, 9 (1871): 267–278.

Noether, Max, Zur Theorie des eindeutigen Entsprechens algebraischer Gebilde. Zweiter Aufsatz, *Mathematische Annalen*, 8 (1875): 495–533.

Noether, Max, Ueber die singulären Werthsysteme einer algebraischen Function und die singulären Punkte einer algebraischen Curve, *Mathematische Annalen*, 9 (1876), 166–182.

Picard, Émile, Sur la théorie des surfaces algébriques, *Revue générale des Sciences pures et appliquées,* 5 (1894): 945–949.

Picard, Émile, Georges Simart, *Théorie des fonctions algébriques de deux variables indépendantes*, Paris: Gauthier-Villars, I (1897), II (1906).

Salmon, George, *Analytische Geometrie des Raumes, bearbeitet von W. Fiedler*, Leipzig: Teubner, I (1879), II (1880).

Segre, Corrado, Sulle rigate razionali in uno spazio lineare qualunque, *Atti della R. Accademia delle Scienze di Torino. Classe di Scienze fisiche, matematiche e naturali*, 19 (1883–1884b): 265–282; also in: (Opere, 1: 1–16).

Segre, Corrado, Ricerche sulle rigate ellittiche di qualunque ordine, *Atti della R. Accademia delle Scienze di Torino. Classe di Scienze fisiche, matematiche e naturali,* 21 (1885–1886c): 628–651; also in: (*Opere*, 1: 56–77).

Segre, Corrado, Recherches générales sur les courbes et les surfaces réglées algébriques, *Mathematische Annalen*, 30 (1887b): 203–226; also in: (*Opere*, 1: 80–104).

Segre, Corrado, Sui sistemi lineari di curve piane algebriche di genere *p*. Estratto di una lettera sl dott. G. B. Guccia, *Rendiconti del Circolo matematico di Palermo*, 1 (1887d): 217–221; also in: (*Opere*, 1: 105–109).

Segre, Corrado, Su alcuni indirizzi nelle investigazioni geometriche, *Rivista di Matematica*, 1 (1891a): 42–66; also in: (*Opere*, 4: 387–412).

Segre, Corrado, Introduzione alla geometria sopra un ente algebrico semplicemente infinito, *Annali di Matematica pura ed applicata*, (2) 22 (1894a): 41–142; also in: (*Opere*, 1: 198–304).

Segre, Corrado, Sulla scomposizione dei punti singolari delle superficie algebriche, *Annali di Matematica pura ed applicata*, (2) 25 (1897a): 1–54; also in: (*Opere*, 1: 327–379).

Segre, Corrado, Intorno ad una mia memoria "Sulla scomposizione dei punti singolari delle superficie algebriche", *Atti della R. Accademia delle Scienze di Torino. Classe di Scienze fisiche, matematiche e naturali*, 32 (1896–1897): 781–789.

Segre, Corrado, Su un problema relativo alle intersezioni di curve e superficie, *Atti della R. Accademia delle Scienze di Torino. Classe di Scienze fisiche, matematiche e naturali*, 33 (1897–1898): 19–23.

Segre, Corrado, Un'osservazione intorno alla riducibilità delle trasformazioni Cremoniane e dei sistemi lineari di curve piane per mezzo di trasformazioni quadratiche, *Atti della R. Accademia delle Scienze di Torino. Classe di Scienze fisiche, matematiche e naturali*, 36 (1900–1901): 645–651;also in: (*Opere*, 1: 433–439).

Segre, Corrado, Mehrdimensionale Räume, *Encyklopädie der mathematischen Wissenschaften*, III C 7 (1921c): 769–972.

Severi, Francesco, *Trattato di geometria algebrica*. Vol. I, Parte I. Bologna: Zanichelli, 1926.

Zariski, Oscar, Algebraic surfaces, *Ergebnisse der Mathematik und ihrer Grenzgebiete*, III 5 (1935) and Berlin: Springer, 1937. Second Supplemented Edition with Appendices by Abhyankar, Shreeram, Lipman, Joseph and David Munford, New York-Heidelberg-Berlin: Springer, 1971.

Zariski, Oscar, *Collected Papers.* Vol. IV: *Equisingularity on Algebraic Varieties*. Cambridge Mass.: MIT Press, 1979.

Corrado Segre: Biographical Timeline

Livia Giacardi

20 August 1863

Corrado is born in Saluzzo to a well-to-do Jewish family. In 1870 the family moves to Turin. His father, Abramo Segre, was an industrialist in silk production, and his mother, Estella De Benedetti, came from a cultivated family of the upper-middle class. Corrado had two brothers, Mario e Arturo,[1] and a sister, Palmira.

1876/1877–1878/1879

Corrado finishes his secondary schooling at the Istituto Tecnico Sommeiller in Turin, where his mathematics teacher was Giuseppe Bruno, who at the time also taught a course in descriptive geometry as a non-tenured professor at the University of Turin. He finishes his secondary studies before he is 16 years old:

> When not yet 16 years old, Corrado received his diploma from the Technical Institute, first in his class with a prize of 300 lire awarded by the Chamber of Commerce. And I recall that he used a good part of the prize to acquire the works of Lagrange, so that we jokingly called him *Lagrange* (1879).[2]

October 1879–July 1883

Although his father wishes him to study engineering, Corrado prefers to study mathematics and enrols in the degree program for mathematical sciences in the

[1]Arturo Segre (1873–1928) was an esteemed historian.

[2]*Non ancora di 16 anni Corrado ebbe la licenza dell'Istituto 1° del suo corso col premio di £ 300 assegnato dalla Camera di Commercio. E rammento ch'egli impiegò buona parte del premio nell'acquisto delle opere di Lagrange, tantoché noi scherzosamente lo chiamavamo* Lagrange (*1879*) (A. Segre to G. Fano, Turin, 29 June 1924, see Conte et al. (2013, 100)).

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Faculty of Mathematical, Natural and Physical Sciences. Here he has teachers of the calibre of Enrico D'Ovidio for geometry, Angelo Genocchi and Francesco Faà di Bruno for analysis, and Francesco Siacci for mechanics.

Corrado Segre student

Source: Courtesy of the Department of Mathematics, University of Turin.

1880/1881 Corrado adds to his study program the course in mineralogy taught by Giorgio Spezia and those in political economics and criticism of socialist doctrines taught by Salvatore Cognetti De Martiis.

1881/1882 As the subject of his course in higher geometry D'Ovidio proposes the line geometry, and Segre, then just 18 years old, takes it as the starting point for carrying out his personal re-elaboration of Giuseppe Battaglini's theory of complexes, enriching it with new properties and giving a lecture on it at the *Scuola di Magistero*, the Teachers College of the University of Turin.

1882/1883 In his fourth and final year of study, in addition to the compulsory courses in higher mechanics, astronomy and mathematical physics, Segre again attends D'Ovidio's course in higher geometry and that in higher analysis taught by Faà di Bruno. This year is blighted by a family tragedy: a crisis in his father's business leads to his suicide. His brother Arturo recalls:

> […] especially serious for him was the 4th year of university (1882-1883), a year that was extremely painful for us, in which my family suffered financial ruin and the sad epilogue of my poor father. He wrote his thesis in those terrible straits and took his degree in July 1883 with honours.[3]

1 July 1883

Not yet 20 years old, Segre earns his degree with the maximum score of 70/70 with honours, with a dissertation assigned to him by D'Ovidio entitled "Studio sulle quadriche in uno spazio lineare ad *n* dimensioni ed applicazioni alla geometria della retta e specialmente delle sue serie quadratiche", published that same year in two memoirs in the *Memorie della R. Accademia delle Scienze di Torino*,[4] which reveal, as Castelnuovo would later note, such "confidence and a vastness of vision and means" that it appears to be the work "not of a beginner, but of a experienced mathematician".[5]

16 August 1883

The first letter of the important correspondence with Felix Klein, extremely frequent during 1883 and 1884 with an average of two letters a month, and destined to last until 1923.

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[3][…]*grave soprattutto fu per lui il 4° anno d'università (1882–83), anno per noi dolorosissimo, nel quale la mia famiglia ebbe il crollo economico e l'epilogo triste di mio povero padre. Egli compose la tesi in quei terribili frangenti e prese la laurea nel luglio 1883 colla lode* (A. Segre to G. Fano, Turin, 29 June 1924, see Conte et al. (2013, 100)).

[4]The dissertation was published in two memoirs: Segre (1883b, c). The manuscript of the dissertation is conserved in BMP, *Fondo Segre*, SCRITTI 1.

[5]… *sicurezza e vastità di vedute e di mezzi* […] *non già di un principiante, ma di un matematico provetto* (Castelnuovo (1924, 353)).

1883/1884

Having obtained a postponement of a year of military service, Segre assumes the position as assistant to D'Ovidio. This is a very intense year for him, scientifically speaking. He publishes no fewer than 16 articles.

1884

Segre is awarded the Mathematics Prize of the Società dei XL (now Accademia Nazionale delle Scienze, known as Accademia dei XL). The jury, composed of Giuseppe Battaglini, Eugenio Beltrami and Enrico Betti, writes:

> For the generality of the research contained in this work of Segre's, for the refinements that it makes to various theories treated previously by diverse eminent geometers, for the many and important new results obtained, and finally for the very broad field of research to which he paves the way, we believe that the two aforesaid Memoirs by Segre are most deserving of the Mathematics Prize that the Italian Society of Sciences must confer for the year 1884.[6]

1884/1885

Corrado does, to his great regret, his military service, which does not, however, prevent him from staying up-to-date with what is being published in his area of research. He writes to Felix Klein:

> I spend all day doing boring material jobs one after another. But in the evening when I have some free time I take the opportunity to study and work; so, while I was afraid of remaining, this year, completely behind in the scientific movement, I manage to follow well enough what is published in scientific collections and learn something new. I am even able to work a little for myself and publish some works […] Of course all this is far from sufficient and I eagerly await the time when I cease to be in the military.[7]

During his military service, he is awarded a scholarship from the government for 1,200 lire for a period of research in higher geometry at the University of Pavia (ministerial decree of 7 November 1884). He is then made an officer in the reserves for three months, but he was always against military life and avoided ever being called again (A. Segre to G. Fano, Turin, 29 June 1924, in Conte et al. (2013, 101)).

[6] *Per la generalità delle ricerche contenute in questo lavoro del Segre, pel perfezionamento che esso apporta a varie teorie trattate precedentemente da diversi valenti Geometri, per i molti ed importanti risultati nuovi ottenuti, ed infine per il campo vastissimo di ricerche cui esso apre la via, crediamo che le indicate due Memorie del Segre siano ben meritevoli del premio delle Matematiche, che la Società italiana delle Scienze deve conferire per l'anno 1884* (RAPPORTO relativo al conferimento del premio nelle Matematiche, dalla Società italiana delle Scienze, per l'anno 1884, *Memorie di Matematica e Fisica della Società italiana delle Scienze* (3), VII (1890): XXXIV–XXXVI, on p. XXXVI).

[7] *Je passe toute la journée parmi des travaux matériels et ennuyeux qui se succèdent les uns aux autres. Mais le soir j'ai quelques heures de liberté dont je profite pour étudier et travailler; de sorte que, tandis que je craignais de rester tout-à-fait en arrière, pendant cette année, du mouvement scientifique, je parviens à suivre suffisamment ce que l'on publie dans les recueils scientifiques et à apprendre quelque chose de nouveaux. Je parviens même à travailler quelque peu pour mon compte et à publier quelques travaux* […] *Naturellement tout cela est loin de me suffire et j'attende avec impatience le moment où je cesserai d'être militaire* (C. Segre to F. Klein, Turin, 20 May 1885, in Luciano and Roero (2012, 138)).

31 October 1885

By ministerial decree Segre is certified to teach higher geometry at the University of Turin. To his application were attached 18 publications, about which D'Ovidio expressed a very flattering assessment:

> Without going into detail about the individual works of Dr Segre, I will limit myself to observing that they all refer to the theory of spaces of *n* dimensions and its applications, and constitute a very noteworthy set of research that is mostly new, and partly aimed at bestowing unity and elegance on previous research by other authors. […] A separate examination of the many elaborate works by Segre would only, in my opinion, better demonstrate that he is gifted with a singularly acute mind, most active, most accurate, and suitable for successfully treating the most arduous and comprehensive questions of geometry with lucidity and elegance. He already enjoys a fine reputation among Italian and foreign mathematicians and will certainly proceed with sure steps along the path that he has started down so well. As a teacher he has given good proof of himself during the year in which he was my assistant, turning out to be clear, exact and effective.[8]

1885/1886

Segre is assigned to teach the course of projective geometry with drawing, aimed at first-year students in the program. In that same year he also teaches a *corso libero* in higher geometry.

23 April 1886

Opening of the competitive examination for the chair of higher geometry at the University of Catania. In the autumn Segre is certified as eligible for the position of non-tenured professor (*professore straordinario*), with a score of 49/50. The commission of jurors writes:

> The preceding works, as a whole, are of exceptional merit, for the importance and the difficulty of the subjects treated, for the rigour and for the lucidity with which they are carried out, for the originality and the interest of the results. The first two, constituting the dissertation presented by Segre for the degree, show the precocious maturity of his mind,

[8] *Senza entrare nell'esame particolareggiato dei singoli lavori del dottor Segre, mi limiterò ad osservare che essi si riferiscono tutti alla teoria degli spazi di n dimensioni e alle sue applicazioni, e costituiscono un insieme molto notevole di ricerche in gran parte nuove, e in parte atte a recare unità ed eleganza in ricerche precedenti di altri autori.* […] *Un esame separato dei molti elaborati lavori del Segre, non potrebbe a mio avviso, che dimostrare sempre meglio come egli sia dotato di un ingegno singolarmente acuto, operosissimo, accuratissimo, atto a trattare con successo le questioni geometriche più ardue e più comprensive e ad esporle con lucidità ed eleganza. Egli gode già una bella riputazione fra i matematici italiani e stranieri e certo progredirà con passo sicuro nella via in cui si è messo così bene. Come docente egli ha già dato buone prove durante l'anno in che fu mio assistente, riuscendo chiaro, esatto ed efficace* (ASUT, *Verbale dell'adunanza degli insegnanti della Facoltà di Scienze Matematiche Fisiche e Naturali del 16 luglio 1885*, VII 79, n. 17).

> and together with the others prove his admirable industriousness; these gifts have already allowed him to acquire, at the age of 23, the esteem of learned men and a conspicuous place among geometers. An uncommon teaching ability is associated to these.[9]

27 December 1886

Because he held the chair of projective geometry and descriptive geometry as well as being dean of the Faculty of Sciences, Bruno declares himself 'quite willingly' disposed to give up teaching projective geometry in favour of Segre. The rector of the University of Turin, Giorgio Anselmi, referring to the outcome of the examination in Catania, proposes to the Minister that Segre be appointed non-tenured professor of projective geometry, as a course separate from that of descriptive geometry. He writes:

> Some professors of mathematical, physical and natural sciences of this University have very warmly described to me the excellent teaching given by Dr Corrado Segre, currently assigned to teach projective geometry with drawing, and the serious damage that would be done to these studies should he become a candidate for a position in another university to improve his situation. […] I permit myself to draw your attention to this, faithful that you might possibly find a suitable position for a young teacher who was able to acquire, in the space of just a few years, a reputation among scientists and affection among his students, who have profited so much from his lessons.[10]

The Minister does not give his consent, because "it is an unbroken maxim of the Ministry to not avail itself of the results of university examinations for positions other than those for which they were held".[11]

[9]*I precedenti lavori, nel loro complesso, sono di un merito eccezionale, per la importanza e la difficoltà degli argomenti trattati, pel rigore e per la lucidità dello svolgimento, per la novità e l'interesse dei risultati. I due primi, costituenti la dissertazione presentata dal Segre per la laurea, mostrano la precoce maturità del suo ingegno, e insieme agli altri provano la sua mirabile operosità; le quali doti gli han fatto già acquistare a 23 anni la stima dei dotti e un posto cospicuo fra' geometri. Ad esse si associa una non comune abilità didattica* (Relazione della Commissione pel concorso alla detta cattedra di professore straordinario, *Bollettino Ufficiale dell'Istruzione. Atti e documenti scolastici*, XIII, May, 1887, p. 342).

[10]*Alcuni professori di Scienze matematiche, fisiche e naturali di questa Università mi hanno con molto calore riferito sull'ottimo insegnamento che dà il dottor Corrado Segre, attuale incaricato della Geometria proiettiva con disegno, sul danno grave che deriverebbe a questi studi qualora il predetto, per migliorare la sua condizione, concorresse per una nomina in altra Università* […] *io mi permetto chiamare su di essa* [proposta] *l'attenzione dell'E[ccellenza] V[ostra] nella fiducia che vorrà possibilmente fare una posizione conveniente ad un giovine insegnante che ha saputo acquistarsi, nel volgere di pochi anni, riputazione fra gli scienziati e affetto presso la scolaresca, che tanto profitto ritrae dalle sue lezion*(G. Anselmi, Turin, 27 December 1886, see Conte et al. (2013, 28)).

[11]*È massima costante del Ministero di non valersi dei risultati dei concorsi per Università diversa da quella per cui furono banditi.* ASUT, *Corrispondenza, Carteggio classificato, 1886–87, fasc. III.2 Disposizioni relative al personale insegnante,* 17 Gennaio 1887. The Faculty of Science did not relent, and made further pressure in Segre's favour with the Ministry; see Conte et al. (2013, 27–29).

1887

Thanks to Segre's intervention, Guido Castelnuovo (1865–1952) arrives in Turin to be assistant to D'Ovidio. Thus is born a fruitful scientific collaboration that leads to the creation of the Italian line of research on the geometry of algebraic curves. Castelnuovo will remain in Turin until 1891, when he is appointed professor of analytic and projective geometry at the University of Rome.

1887/1888–1890/1891

Segre gives lectures in geometry at the *Scuola di Magistero*, whose mathematics section was created in 1876/1877.

1888

Opening, in May, of the competition for a position of non-tenured professor of higher geometry at the University of Turin. This is made possible by D'Ovidio's move from teaching higher geometry to teaching higher analysis, a chair left vacant by the death of Francesco Faà di Bruno.

25 October 1888

Segre is the only candidate, and the jury commission, composed, among others, of D'Ovidio, declares him eligible with the maximum score of 50/50, proposing that he be named non-tenured professor of higher geometry at the University of Turin:

> For the importance and difficulty of the arguments that he treats in the works cited, for the elegance and rigour that he shows in carrying them out and displaying them, for the originality and interest obtained and for his admirable industriousness, Segre, though still at a very young age, has already been able to acquire a conspicuous place among geometers, possesses power and maturity of mind, and has already given proof of noteworthy ability for teaching, carrying out, over three consecutive years, at the Royal University of Turin, a course in projective geometry, in which, as the Commission has verified, he has also introduced several innovations.[12]

[12]*Per la importanza e difficoltà degli argomenti che tratta nei citati lavori, per la eleganza e il rigore che pone nello svolgerli ed esporli, per la novità ed interesse ottenuti e per la sua mirabile operosità, il Segre, benché in età giovanissima, ha saputo già conquistare un posto cospicuo tra i geometri, possiede potenza e maturità di ingegno, e ha dato già prove di notevole attitudine all'insegnamento, svolgendo per tre anni consecutivi, presso la R. Università di Torino, un corso di Geometria proiettiva, nel quale corso, come consta alla Commissione, ha anche introdotto alcune innovazioni,* […]. (Concorso alla cattedra di professore straordinario di Geometria superiore, vacante presso la R. Università di Torino. Relazione della Commissione. Roma, 25 ottobre 1888, *Bollettino ufficiale dell'Istruzione,* 1888, 636–638).

1888/1889

Segre begins teaching higher geometry and the University of Turin, and will continue to do so for 36 years, until his death.

1888–1893

Segre is one of the editors of the journal *Rendiconti del Circolo Matematico di Palermo*. In 1893 he is excluded from the Editorial Board because of disagreements with the director Giovan Battista Guccia,[13] and will be reinstated only in 1909.

1889

The Italian translation of Karl G. von Staudt's *Geometrie der Lage* is published by Bocca in Turin. The translation was carried out by Mario Pieri, at Segre's invitation, and is prefaced by a valuable biographical-bibliographical study about von Staudt written by Segre himself.

10 February 1889

Segre is directly appointed a National Member of the Accademia delle Scienze in Turin at the age of only 26. He will be named director of the Class of Physical, Mathematical and Natural Sciences on 11 April 1920, and once again shortly before his death, on 3 February 1924.

[13]See, for example, the letter of C. Segre to G. Castelnuovo, Turin, 6 January 1894, in Gario (2010).

Corrado Segre in 1889

Source: Courtesy of the Academy of Science of Turin and of the Department of Mathematics, University of Turin.

1889/1890–1890/1891

Segre publishes the four papers entitled "Un nuovo campo di ricerche geometriche" in the *Atti della Reale Accademia delle Scienze di Torino* Segre (1889–90) and (1890–91). These will pave the way for a new field of research, that of hyperalgebraic entities.

1890

The Italian translation of Klein's Erlangen Program is published, prepared by Gino Fano (1871–1952) at Segre's invitation (Klein 1890). Segre writes:

> This work is not, in my opinion, well enough known to *young Italian geometers*; and it is especially for them that I desired this reprint to be published.[14]

1890/1891

Segre teaches the course entitled "Introduzione alla geometria sugli enti algebrici semplicemente infiniti" (Introduction to the geometry on simply infinite algebraic entities), very important both scientifically, due to the innovative nature of Segre's approach to the study of the geometry of an algebraic curve, and for the formation and consolidation around him of a group of researchers animated by a common project.

The course was attended by the young Neapolitan Federico Amodeo (1859–1946), who writes:

> In the 1890–1891 academic year Segre replicated with D'Ovidio in Turin the excellent experiment done by Brioschi, Casorati and Cremona in 1869 in Milan. While D'Ovidio gave a series of lectures on *Functions of complex variables and Abelian integrals*, he [Segre] taught *Geometry on a simply infinite algebraic variety* under the triad of *hyper-spatial, algebraic and functional* aspects.[15]

A kind of scientific community baptised the *Pitareide*,[16] forms around Segre and Peano and meets in the American Bar in the Galleria Nazionale in Turin.

1891

Segre publishes in the *Rivista di matematica* the article "Su alcuni indirizzi nelle investigazioni geometriche. Osservazioni dirette ai miei studenti" (Segre 1891a),[17] which becomes the starting point of a well-known dispute with the director of the journal, Giuseppe Peano, regarding the way scientific research is conceived.

In the note "Sulle varietà che rappresentano le coppie di punti di due piani o spazi" (Segre 1891c), Segre defines for the first time the product of projective spaces, now called "Segre variety".

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[14]*Questo lavoro non è, a mio avviso, abbastanza noto ai* giovani geometri italiani; *ed è specialmente per essi che ho desiderato si facesse questa ristampa* (Klein 1890, 307–308).

[15]*Nell'anno scolastico 1890–91 Segre ripetette con D'Ovidio a Torino la eccellente prova fatta da Brioschi, Casorati e Cremona nel 1869 a Milano. Mentre D'Ovidio faceva un corso di lezioni sulle* Funzioni di variabile complessa e sugli integrali abeliani, *egli* [Segre] *esponeva la* Geometria su di una varietà algebrica semplicemente infinita *sotto il triplice aspetto* iperspaziale, algebrico e funzionale (Amodeo 1945, 245).

[16]See, for example, C. Segre to F. Amodeo, 24 November 1891 in Palladino F. & N. (2006, 185–187) and C. Segre to G. Castelnuovo, Turin, 28 November 1891, ANL-Castelnuovo in Gario (2010).

[17]The English translation (Segre 1904) was done by John Wesley Young and revised by Segre himself, who added some notes.

18 July 1891

Segre becomes a Corresponding Member of the Accademia Nazionale dei Lincei. He will become a National Member of that Academy on 28 August 1901.

Summer 1891

Segre makes a trip to Germany with the aim of visiting the principal institutes and libraries and making contact with those who had influenced his research. He visits Frankfurt a. M., Göttingen, Berlin, Leipzig, Dresden, Nuremberg, and Munich, and has the opportunity to be in contact, among others, with Leopold Kronecker, Max Noether, Theodor Reye, Rudolf Sturm and Moritz Cantor, as well as Felix Klein, with whom up to that time he had only maintained epistolary relations.

22 June 1892

Gino Fano receives his degree with a maximum score of 90/90 and honours, with a thesis on hyperspatial geometry written under Segre's supervision.

November 1892

Federigo Enriques (1871–1946) comes to Turin to make contact with Segre.

25 November 1892

Segre is promoted to tenured professor.

25 March 1893

Segre marries (civil wedding)[18] Olga Michelli, "an amiable young lady, of most remarkable mind and artistic abilities, modest, affectionate, family-oriented".[19] She is also Jewish, and comes from a well-to-do family from Ancona. Her father, Giuseppe, manages an oil importation company, and her mother, Clementina Penso, comes from a Jewish family from Trieste.

[18]Religious wedding took place on 26 March 1893.

[19]C. Segre to G. Castelnuovo, Ancona, 29 October 1892, ANL-Castelnuovo in Gario (2010): *signorina simpatica, d'ingegno ed attitudini artistiche spiccatissime, modesta, affettuosa, famigliare.*

Corrado Segre and his wife Olga Michelli

Source: Courtesy of the Department of Mathematics, University of Turin.

November 1893–January 1894

Federigo Enriques is in Turin to work with Segre.

1894

Segre publishes the important memoir "Introduzione alla geometria sopra un ente algebrico semplicemente infinito".[20] Alessandro Terracini had this to say about it:

> His "Introduction to the geometry on a simply infinite entity" published in 1894 in the *Annali di Matematica* […] was like the Magna Carta that became a point of reference for the geometry of an algebraic curve according to Segre's ideas. That Introduction was the fruit of a course given by Segre here in Turin in the 1890–91 academic year, in which – it was important to Segre to say this – he expounded not only the geometric method, due to him and to Castelnuovo, but also those pre-existing: chiefly the algebraic method of Brill and Noether, and the transcendent one of Riemann.[21]

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[20] *Annali di Matematica pura ed applicata*, s. 2, 22, (1894a): 41–142 (*Opere*, 1, 198–304).

[21] *La sua "Introduzione alla geometria sopra un ente algebrico semplicemente infinito" pubblicata nel 1894 sugli Annali di Matematica* […] *è stata come la magna charta che ha fatto testo per la geometria sulla curva secondo le idee di Segre. Quell'Introduzione è il frutto di un corso tenuto da Segre qua a Torino nell'anno accademico 1890–91, nel quale—Segre ci teneva a dirlo—egli aveva esposto non solo il metodo geometrico, dovuto a lui e a Castelnuovo, ma anche quelli preesistenti: segnatamente il metodo algebrico di Brill e Noether e quello trascendente di Riemann* (Terracini 1962, 12).

14 March 1894

Segre's daughter Elena is born.[22]

1895/1896

In the article "Intorno ad un carattere delle superficie e delle varietà superiori algebriche" (Segre 1895–96), Segre introduces one of the most important topological invariants of an algebraic surface, today known as the *Zeuthen-Segre invariant*.

1895–96 and 1896–97

Segre is assigned to teach the course in mathematical physics.

6 July 1896

Beppo Levi (1875–1961) receives is degree with a maximum score of 70/70 with honours, with a thesis on the "Singolarità delle curve algebriche sghembe (iperspaziali)", written under Segre's supervision.

1897

Segre is invited to be vice-president of the geometry section of the International Congress of Mathematicians in Zurich (9–11 August 1897).

He publishes a memoir entitled "Sulla scomposizione dei punti singolari delle superficie algebriche" (Segre 1897a), in which, extending one of Noether's results, he gives a general and rigorous definition for the notion of "infinitely near multiple points" of a surface and uses it to investigate the problem of the resolution of singularities of algebraic surfaces. Here Segre also presents several critical observations on the proof given by Pasquale del Pezzo in 1888, observations that give rise to a heated polemic between the two.

28 October 1897

Segre's daughter Adriana is born.[23]

1898

Segre and Vito Volterra share the prestigious Premio Reale for mathematics awarded by the Accademia dei Lincei. The jury's statement reads:

[22]Elena would grow up to marry Riccardo Fuà, a pediatrician in Ancona, and would have two children: Corrado, a noted physician, and Giorgio, a well-known economist; see Rosenthal Fuà (2004, 95–99).

[23]Adriana will grow up to marry Guglielmo Morpurgo, a Milan businessman, and will have three daughters: Elena, Matilde and Clara; see (Rosenthal Fuà 2004, 99).

> Segre's scientific work is among the most admirable. He has left a mark of his strong intelligence and his great and ceaseless industriousness in a vast range of fields, part of which are yet unexplored. […] Nor is another, fundamental merit of Segre's to be neglected: that of having opened the way for the present line of Italian research in studies of geometry of curves and surfaces, effectively making contributions of his own there.[24]

1898/1899

The British couple Grace Chisholm Young (1868–1944) and William Young (1863–1942) is in Turin to attend Segre's courses and to "live a mathematical life". Segre places himself at their disposal:

> If you and your husband would like me to come speak to you on a geometric topic of interest to you, please write to me, indicating the topics.[25]

April 1899

Klein arrives in Turin and celebrates his fiftieth birthday with the Turinese mathematicians. William Young recalls that day:

> He arrived, on his 50th birthday, at Turin, and was fêted by the mathematicians of that city, where the present writer and his wife […] were then studying (Young 1928, xiii).

8 July 1899

Alberto Tanturri (1877–1924) receives his degree with a maximum score of 80/80 with honours, defending a thesis on enumerative geometry written under Segre's supervision.

[24] *L'opera scientifica del Segre è delle più ammirevoli. Egli ha lasciato tracce del suo forte ingegno e della sua grande e continua operosità in vasti campi, in parte ancora inesplorati.* […] *Né è da tacersi un altro e principale merito del Segre: di avere, cioè, avviato il presente indirizzo italiano degli studi di Geometria sopra una curva ed una superficie, contribuendovi egli stesso efficacemente* (Relazione sul concorso al premio reale per la Matematica, pel 1895, *Atti della R. Accademia dei Lincei, Rendiconti delle sedute solenni*, 1 (1898): 354–374, quotation on p. 367).

[25] *Se lei e Suo marito desiderano che uno di questi giorni io venga a parlar Loro su qualche argomento geometrico di Loro interesse, favorisca scrivermi, indicandomi gli argomenti* (C. Segre to G. Chisholm, 11 March [1899]). See the essay by Alberto Conte and Livia Giacardi, Appendix 1.

Corrado Segre in 1899
Source: Courtesy of the Department of Mathematics, University of Turin.

August 1899

Together with the German publisher Teubner, Segre defines the title of the treatise on algebraic geometry that he intends to write together with Castelnuovo and Enriques: *Vorlesungen über höhere algebraische Geometrie, mit besonderer Berücksichtigung der mehrdimensionalen Räume* (*Lectures on higher algebraic geometry, with special emphasis on multidimensional spaces*), and the topics to be included:

> The treatment must proceed according to the most modern points of view, and in such a way that my book, together with the classic treatises on analytic geometry by Salmon and Clebsch, and with those which will be written by Castelnuovo and Enriques on algebraic surfaces, will contribute to providing a rather complete idea of the current state of algebraic geometry.[26]

The treatise will never appear.

[26] *Lo svolgimento dovrebbe farsi secondo i punti di vista più moderni, ed in modo che il mio libro, insieme con i classici trattati di Geometria analitica di Salmon e Clebsch, e con quello che scriveranno Castelnuovo ed Enriques sulle superficie algebriche contribuisca a dare un'idea abbastanza completa dello stato attuale della geometria algebrica* (C. Segre to G. Castelnuovo, Ancona, 9 August 1899, ANL-Castelnuovo, in Gario (2010)).

30 June 1900

Francesco Severi (1879–1961) receives his degree with a maximum score of 80/80 with honours, with a thesis on the singularities of curves in a hyperspace, written under Segre's supervision.

4 November 1901

Giovanni Zeno Giambelli (1879–1953) receives his degree with a maximum score of 80/80 with honours, with a thesis on enumerative geometry, written under Segre's supervision.

28 August 1901

Segre becomes a National Member of the Accademia dei Lincei.

1903/1904

Julian Coolidge (1873–1954) comes to Turin to do post-graduate studies with Segre. In his recollections of Segre, he places him among the greatest figures of the "geometric *risorgimento* in Italy" (Coolidge 1927, 352). In addition to acknowledging his scientific debt to his Italian *maestro*, he praises his devotion to his students:

> There was no limit to the amount of care and patience which he would bestow on one of his pupils (Coolidge 1927, 357).

1904

Segre is invited to the International Congress of Mathematicians in Heidelberg (8–13 August 1904), where he gives a plenary lecture entitled "La geometria d'oggidì e i suoi legami con l'analisi", in which he duly illustrates with examples how "the two sister sciences each render immense services to the other!"[27] Here he meets, among others, Eduard Study, Ernest Wilczynski and Samuel Dickstein, who immediately translates into Polish his lecture (Segre 1905b).

1904–1924

Segre is one of the editors of one of the most important scientific journals of the day, the *Annali di Matematica pura ed applicata*. He is involved not only with scientific aspects but also with financial aspects, as shown by his correspondence, in particular with Tullio Levi-Civita, Giuseppe Jung, Salvatore Pincherle, Virgil Snyder of Cornell University, as well as with Emilio De Benedetti and Gino Olivetti. A year before his death he writes to Levi-Civita:

[27] […] *le due scienze sorelle rendono l'una all'altra servizi immensi!*, Segre (1905a, 110).

> Today I am writing to you about the *Annali di Mat*[*ematica*]. Perhaps you have already heard that the crisis that these [*Annali*] have gone through is by now over. […] If you have works by *others* to submit, do so: but with a certain *severity*, as these are not the times in which we can be excessive in printing; on the other hand, the important thing is that the *Annali* maintain, or even augment further, its fame![28]

1907–1924

Segre is the director of the "Biblioteca speciale di matematica", the mathematics library, today named for Giuseppe Peano.

1907/1908–1920/1921

On 23 November 1907, following D'Ovidio's resignation, the Faculty of Sciences of the University of Turin selects Segre as professor for the lectures in mathematics in the *Scuola di Magistero* for the three-year period 1907–1910. He continues in that role until 1920/21, when the teacher training schools were suppressed by the Minister for Education Benedetto Croce.

1908

Segre is named a member of the awards commission for the Medaglia Guccia,[29] together with Max Noether and Henri Poincaré. The first medal was awarded to Francesco Severi during the International Congress of Mathematicians held in Rome, 6–11 April 1908 (Segre, Noether, Poincaré 1909).

March 1908

Clarence Lemuel E. Moore (1876–1931), from the Cornell University, attends Segre's lectures.[30]

1908/1909

Charles Herschel Sisam (1879–1964), from the Cornell University, attends Segre's lectures. He wrote:

> I was student at the University of Turin during the year 1908-9. I attended Segre's lectures, which will ever stand out in my mind as models of clearness, force and value.[31]

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[28]*Oggi ti scrivo per gli Annali di Mat*[a]*. Forse avrai già sentito dire che la crisi che questi hanno attraversato è ormai superata.* […] *Se avete lavori di altri da presentare fatelo: ma con una certa severità, ché non siamo in tempi in cui si possa esser larghi nello stampare; e d'altra parte c'importa che gli Annali mantengano, od anche elevino ulteriormente, la loro fama!* (C. Segre to T. Levi-Civita, Turin, 9 April 1923, ANL-Levi-Civita).

[29]The Medaglia Guccia was instituted in 1908 by Giovanni Battista Guccia, founder of the Circolo Matematico di Palermo. The 3,000 gold francs prize (see G. Guccia to M. Noether, 20 July 1904, in Brigaglia and Masotto (1982, 264) was to be awarded for a memoir containing a significant improvement on the theory of algebraic curves.

[30]See the essay by Luciano and Roero in this present volume.

[31]See C. Sisam to Olga Michelli Segre, Colorado College,14 July 1924, UTo-ACS, II.

1909/1910–1915/1916

On 4 March 1909 Segre is officially named dean of the Faculty of Sciences of the University of Turin for the remainder of the 3 year term 1907–1910. In 1910 he is reconfirmed for a second 3 year term, and is once again confirmed in 1913.

1910

Segre publishes the memoir entitled "Preliminari di una teoria delle varietà luoghi di spazi, (Segre 1910a) which is one of the cornerstones of that field of research known as "projective differential geometry". Great impetus was given to this line of research inaugurated by Segre by his student Alessandro Terracini, and by Enrico Bompiani, student of Guido Castelnuovo.

17 March 1910

In faculty meeting, Segre, as dean of the Faculty of Sciences, address the problem of the teaching of higher analysis that had been carried out by Peano since 1908 in a way that was not, in his opinion, conform with the primary aim of a higher-level course, that is, to direct young people towards research.

The conflict ends with Peano's removal from teaching that course.

5 July 1911

Alessandro Terracini (1889–1968) receives his degree with a maximum score of 100/100 and honours, with a thesis on projective differential geometry, written under Segre's supervision.

3 July 1912

Eugenio Togliatti (1889–1977) receives his degree with a maximum score of 90/90 and honours, with a thesis concerning algebraic surfaces, written under Segre's supervision.

24 May 1915

Before Italy's entry into World War I, Segre is among those in favour of neutrality, but after he shows genuine patriotic sentiments. For example, on 27 May 1915 he writes to Volterra "Long live Italy".[32] In another letter to Volterra, he denounces the obstructionism of the military command in Turin against Mauro Picone, who was charged with compiling new firing tables for the artillery, writing: "They don't want help from those who might be able to give it. And at stake is our Italy!"[33]

[32] *Viva l'Italia*, C. Segre to V. Volterra, Turin, 27 May 1915, ANL-Volterra.

[33] *Non vogliono aiuti da chi potrebbe darli! E si tratta della nostra Italia*, C. Segre to V. Volterra, Turin, 13 January 1917, ANL-Volterra. About this see Nastasi and Tazzioli (2014) and the essay by Luciano and Roero in this present volume.

1916

Segre is appointed director of the *Scuola di Magistero*.

Corrado Segre in the last years of his life

Source: Courtesy of the Department of Mathematics, University of Turin.

1918/1919

Segre gives the inaugural lecture of the academic year, the first following the end of the Great War, dedicating it to predictions. After having discussed that topic with due reference to physics, chemistry, astronomy and meteorology, Segre concludes his talk by addressing the students:

> Now the day has arrived in which humanity, freed from the overbearing powers, can take up anew, with greater confidence than ever before, the works of peace. And all of us, together, can, in these halls, without that shadow of remorse we seemed to feel during the war, once again take up and cultivate Science: not only that which is applied to procure material wellbeing for mankind, but also that other whose sole aim is the satisfaction of our spirit. And Science will give you – permit me, in closing, to make this prediction – the highest, purest satisfactions: Science, whose supreme aim, as was so well said, is the *Honour of human intellect!*[34]

[34]*Ecco, è giunto il giorno, in cui l'umanità, liberata dai prepotenti, potrà riprendere con maggior sicurezza di quanta non abbia mai avuto, le opere di pace. E noi potremo in queste aule, senza quell'ombra di rimorso che durante la guerra pareva di sentire, tutti insieme riprendere a*

1921

Segre's long article "Mehrdimensionale Räume" comes out in the *Encyklopädie der mathematischen Wissenschaften* (Segre 1921c), dedicated to higher-dimensional spaces, and, as Henry F. Baker wrote, it will "remain for many years a monument of the comprehensiveness of the man" (Baker 1926, 269).

1921–23

Segre gives the lectures in complementary mathematics for the combined degrees in mathematics and physics, which had just been established by the Minister of Education Orso Mario Corbino.

19 May 1922

Segre writes to Salvatore Pincherle recommending that the newly constituted Unione Matematica Italiana (Italian Mathematical Union, UMI) be given an international character:

> I will absolutely not approve of the Italian Union becoming part of an International Union from which the nations who were former enemies are excluded! I firmly disapprove of the anti-German trend that is still being nurtured by a certain group of scientists, so long after peace has been achieved. I want the victors to extend a hand to the vanquished. If, once our Union is constituted, I see that this concept is not supported in the International Union, and that the word "International" is used […] in the same sense as in the "International Congress of Strasbourg" (!), I believe I will submit my resignation from the Italian Union.[35]

(Footnote 34 continued)
coltivare la Scienza: non solo quella che si applica a procurare agli uomini il benessere materiale, ma ancora quell'altra che ha per unica mira il compiacimento del nostro spirito. E la Scienza vi darà—consentitemi, nel finire, questa previsione—le più alte, le più pure soddisfazioni: la Scienza, il cui scopo supremo, come ben fu detto, è l'onore dell'intelletto umano! (Segre 1918–19, 24).

[35]*Non approverei assolutamente che l'Unione italiana entrasse a far parte di un'Unione Internaz^{le} dalla quale fossero escluse le Nazioni ex-nemiche! Io disapprovo recisamente le tendenze antitedesche che si coltivano ancora da certi gruppi di scienziati, dopo tanto tempo che s'è conclusa la pace. Voglio che i vincitori stendan la mano ai vinti. Se, costituita la nostra Unione, io vedrò che nell'Unione Internaz.^{le} essa non sostenga questi concetti, e si adatti a che la parola "Internaz.^{le}" significhi* […] *quel che ha significato "congr.^{o} Internaz.^{le} di Strasburgo" (!), io credo che darò le mie dimissioni dall'Unione Italiana* (C. Segre to S. Pincherle, Turin, 19 May 1922, UMI-Archivio). Segre is referring here to the 1920 International Congress of Mathematicians held in Strasbourg, from which mathematicians from the countries of the former Central Powers were excluded.

14 July 1923

Beniamino Segre (1903–1977) receives his degree with a maximum score of 90/90 with honours, with a thesis on algebraic geometry written under Segre supervision.

18 November 1923

The engineer Guido Ghersina, as a sign of his esteem for Segre, gives an endowment to the University of Turin to institute a prize carrying his name. The first two winners are Beniamino Segre in 1926 and Maria Cibrario in 1929.

18 May 1924

Segre dies in Turin. He is quoted as having said:

> I will go calmly and serenely, because the father must make way for his children, because it is necessary that the past die each day to make way for the other mornings to rise triumphantly.[36]

Expressions of sorrow and condolence arrived to the rector of the University of Turin and the Faculty of Sciences from all over Italy and abroad.

March 1926

Segre's widow, Olga Michelli, stipulates an agreement with the director of the mathematics library, Gino Fano, for "temporary deposit of manuscripts".[37] Segre's notebooks are still today conserved in the *Fondo Segre* and can be accessed at the website (Giacardi 2013).

18 May 1928

On the fourth anniversary of his death, a solemn commemoration was held for Segre in the Aula Magna of the University of Turin, in the presence of his wife Olga Michelli and his daughters Elena Fuà Segre and Adriana Morpurgo Segre, followed by the inauguration of a marble plaque placed in the Faculty of Sciences, with this inscription:

[36] *Io me ne andrò calmo e sereno, perché il padre deve far posto ai figli, perché bisogna che ogni giorno il passato muoia affinché altri mattini si alzino trionfanti* (Discorso del Rettore della R. Università di Torino, Prof. Comm. Alfredo Pochettino, in "*Corrado Segre* (*20 agosto 1863–18 maggio 1924*)", *Supplemento ai Rendiconti del Circolo matematico di Palerm*o, XV, Anno 1926–1928: 42).

[37] ASUT, *Verbale di deposito temporaneo dei quaderni manoscritti di Corrado Segre. Torino-Ancona, 1° marzo 1926*, in *Corrispondenza, Carteggio classificato, 1926, fasc. I.4 Biblioteche.*

In queste aule
che lo videro allievo
CORRADO SEGRE
con elevata parola
per XXXVI anni
promuoveva il culto – ispirava l'amore
delle discipline geometriche.
Sulla cattedra – cogli scritti
altamente onorò la scienza italiana.
Alla Biblioteca Matematica
diede efficace duraturo incremento.
Colleghi e discepoli
Affettuosamente riverenti
P.P.
MCMXXVI
(In these halls that saw him a student
CORRADO SEGRE
with lofty speech
for XXXVI years
promoted knowledge – inspired love
of geometric disciplines.
In the chair – with his writings
he highly honoured Italian science.
To the Mathematics Library
he brought effective lasting increase.
Colleagues and disciples
affectionately reverent
P[ublicly] P[osed]
MCMXXVI)

Various of Segre's students spoke and many illustrious Italian and foreign mathematicians sent messages.[38]

[38]Participating in the ceremony were, among others, Guido Castelnuovo, Federigo Enriques, Gino Loria, Eugenio Togliatti, Luigi Berzolari, Beppo Levi and Ugo Cassina. Professor Gino Fano gave the commemorative address, and Carlo Somigliana, dean of the Faculty of Sciences, also spoke. Messages of participation were sent by, among others, Luigi Bianchi, Michele De Franchis, Eugenio Bertini, Vito Volterra, Tullio Levi-Civita, Francesco Severi, Giuseppe Armellini, Giulio Pittarelli, Enrico Bompiani, Giulio Vivanti, Oscar Chisini, Salvatore Pincherle, Leonida Tonelli, Ettore Bortolotti, Giuseppe Vitali, Annibale Commessatti, Ernesto Laura, Francesco Gerbaldi, Francesco Sbrana, Ermenegildo Daniele, Giovanni Sansone, Enrico Persico, Ernesto Pascal, Mauro Picone, Gaetano Scorza, Gustavo Sannia, Domenico Montesano, Antonio Signorini, Luigi Brusotti, Margherita Piazzolla Beloch, Carlo Bonferroni, Mineo Chini, Angelo Ramorino, Émile Picard, Jacques Hadamard, Élie Cartan, Alexander Brill, Ferdinand von Lindemann, Friedrich Schur, Friedrich Schilling, Lucien Godeaux, Alfred Rosenblatt. Besides the main Italian Faculties of Sciences, the following scientific societies also took part in the ceremony: Accademia delle scienze di Torino, Istituto Lombardo di scienze e lettere, Deutsche Mathematiker Vereinigung, American Mathematical Society, Jednota Ceskoslovenskych Matematicu a Fysiku v Praze. All of the speeches and messages are published in *Rendiconti del Circolo matematico di Palermo. Supplemento*, XV (1926–1928): 40–73. The marble plaque mentioned above was placed in the boardroom of the Faculty of Sciences in the main building of the university in Via Po, but was destroyed during the air raids of 1943. One similar was placed in the Institute of Geometry in December 1963, when that Institute was named for Corrado Segre (Terracini 1968, 14–15).

Translated from the Italian by Kim Williams

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References

Amodeo, Federico. *Sintesi storico-critica della geometria delle curve algebriche*, Napoli: Conte, 1945.

Baker, Henry F, Corrado Segre, *Journal London Mathematical Society,* 1 (1926): 263–271, It. trans. in *Bollettino della Unione Matematica Italiana,* 6 (1927): 276–284.

Brigaglia, Aldo and Masotto, Guido, *Il Circolo Matematico di Palermo*, Bari: Edizioni Dedalo, 1982.

Conte, Alberto, Livia Giacardi and Paola Novaria, *Corrado Segre (1863–1924). A 150 anni dalla nascita*. Catalogo delle Mostre documentarie – Novembre 2013. Archivio Storico dell'Università di Torino, Accademia delle Scienze, Torino: Kim Williams Books, 2013.

Castelnuovo, Guido, [Notizia della morte], *Atti della R. Accademia dei Lincei, Rendiconti*, s. 5, 331 (1924): 460.

Coolidge, Julian. Corrado Segre, *Bulletin of the American Math. Society*, 33 (1927): 352–357.

Klein, Felix (translated by Gino Fano) Considerazioni comparative intorno a ricerche geometriche recenti, *Annali di matematica pura ed applicata*, s. 2, 17 (1890): 307–343.

Gario, Paola (ed.), Lettere e Quaderni dell'Archivio di Guido Castelnuovo, Roma: Accademia N. dei Lincei, 2010: *http://operedigitali.lincei.it/Castelnuovo/Lettere_E_Quaderni/menu.htm*.

Giacardi, Livia, Corrado Segre maestro a Torino. La nascita della scuola italiana di geometria algebrica, *Annali di storia delle università italiane*, 5 (2001): 139–163.

Luciano, Erika and C. Silvia Roero, From Turin to Göttingen: Dialogues and Correspondence, *Bollettino di Storia delle Scienze Matematiche*, XXXII, 1 (2012): 9–232.

Nastasi, Pietro and Rossana, Tazzioli, Italian Mathematicians and the First World War: Intellectual Debates and Institutional Innovations. In David Aubin, Catherine Goldstein (Eds.), *The war of guns and mathematics, Mathematical Practices and Communities in France and Its Western Allies around World War I*, AMS/LMS, 2014, 181–228.

Palladino, Franco and Nicla Palladino. *Dalla 'moderna geometria' alla 'nuova geometria italiana'. Viaggiando per Napoli, Torino e dintorni*, Firenze: Olschki, 2006.

Rosenthal Fuà, Erika, *Fuga a due*, Bologna: Il Mulino, 2004.

Segre, Corrado, Noether, Max and Poincaré Henri, Relazione del concorso internazionale per la Medaglia Guccia, in G. Castelnuovo (ed.) *Atti del IV Congresso Internazionale dei Matematici, Roma, 6–11, Aprile, 1908,* 3 vols. Roma: Accademia Nazionale dei Lincei, vol. I., 209–216.

Segre, Corrado, Studio sulle quadriche in uno spazio lineare ad un numero qualunque di dimensioni, *Memorie della R. Accademia delle Scienze di Torino*, 2, 36 (1883b): 3–86. (*Opere*, 3, 25–126).

Segre, Corrado, Sulla geometria della retta e delle sue serie quadratiche, *Memorie della R. Accademia delle Scienze di Torino*, 2, 36 (1883c): 87–157. (*Opere*, 3, 127–217).

Segre, Corrado, Un nuovo campo di ricerche geometriche, Note I-III, *Atti della R. Accademia delle Scienze di Torino*, 25 (1889–90): 180–205; 290–317; 376–396. (*Opere*, 2, 237–303).

Segre, Corrado, Un nuovo campo di ricerche geometriche, Nota IV, *Atti della R. Accademia delle Scienze di Torino*, 26 (1890–91): 35–71. (*Opere*, 2, 303–337).

Segre, Corrado, Su alcuni indirizzi nelle investigazioni geometriche. *Osservazioni dirette ai miei studenti, Rivista di Matematica*, 1 (1891a): 42–66. (*Opere*, 4, 387–412).

Segre, Corrado, Sulle varietà che rappresentano le coppie di punti di due piani o spazi, *Rendiconti del Circolo Matematico di Palermo*, 5 (1891c): 192–204. (*Opere*, 1, 173–184).

Segre, Corrado, Intorno ad un carattere delle superficie e delle varietà superiori algebriche, *Atti della R. Accademia delle Scienze di Torino*, 31 (1895–96): 485–501. (*Opere*, 1, 312–326).

Segre, Corrado, Sulla scomposizione dei punti singolari delle superficie algebriche, *Annali di Matematica pura ed applicata*, 2, 25 (1897a): 2–54. (*Opere*, 1, 327–379).

Segre, Corrado, On some tendencies in geometric investigations, *Bulletin of the American Mathematical Society*, 2, 10 (1904): 442–468. English translation by John Wesley Young of Segre (1891a).

Segre, Corrado, La Geometria d'oggidì e i suoi legami coll'analisi, in *Verhandlungen des dritten internationalen Mathematiker-Kongresses in Heidelberg vom 8 bis 13 August 1904,* 1905a, 109–120; also in *Rendiconti del Circolo Matematico di Palermo*, 19 (1905a): 81–93. (*Opere*, 4, 456–468).

Segre, Corrado, Geometrya dzisiejsza i jej związki z Analizą, *Wiadomości Matematyczne*, Warszawa, 9 (1905b): 7–21. Polish translation by Samuel Dickstein of Segre (1905a).

Segre, Corrado, Preliminari di una teoria delle varietà luoghi di spazi, *Rendiconti del Circolo Matematico di Palermo*, 30 (1910a): 87–121. (*Opere*, 2, 71–114).

Segre, Corrado, Discorso inaugurale. Le previsioni, *R. Università di Torino* 1918–1919. *Inaugurazione dell'anno accademico. Discorsi e dati statistici*, Torino: Stamperia Reale, 1918–19, 11–25.

Segre, Corrado, Mehrdimensionale Räume, in *Encyklopädie der mathematischen Wissenschaften*, III.2 7, 1921c , 769–972.

Terracini, Alessandro, Parole del Prof. Terracini, in *Atti del Convegno internazionale di geometria algebrica*, Torino, 24–27.5.1961, Torino: Rattero, 9–14.

Young, William, Christian Felix Klein—1949–1925, *Proceedings of the London Royal Society*, Series A 121 (1928): 1–xix.

Part II
Perspectives on Segre's Legacy

Algebraic Elements of the Cremona Groups

Jérémy Blanc

Abstract
This article studies algebraic elements of the Cremona group. In particular, we show that the set of all these elements is a countable union of closed subsets but it is not closed.

1 Introduction

In the sequel, the ground field k will be a fixed algebraically closed field. The Cremona group of rank n is the group $\mathrm{Bir}(\mathbb{P}^n)$ of birational transformations of the projective space $\mathbb{P}^n$. There is a natural topology on it, called the *Zariski topology* (see Sect. 2 for a precise definition).

An element $\varphi \in \mathrm{Bir}(\mathbb{P}^n)$ is said to be *algebraic* if it is contained in an algebraic subgroup G of $\mathrm{Bir}(\mathbb{P}^n)$. This is equivalent to the fact that the sequence $\{\deg(\varphi^m)\}_{m\in\mathbb{N}}$ is bounded (Corollary 2.9). We can also observe that G is in this case an affine algebraic group (Blanc and Furter 2013, Remark 2.21), so there exists a Jordan decomposition $\varphi = \varphi_s\varphi_u$, where $\varphi_s, \varphi_u \in G$ are semi-simple and unipotent respectively. As observed in Popov (2013, Sect. 9.1), this decomposition does not depend on the choice of G, so there is a natural notion of semi-simple and unipotent elements of $\mathrm{Bir}(\mathbb{P}^n)$. In fact, the group G could even be chosen to be the commutative algebraic subgroup $\overline{\{\varphi^i | i \in \mathbb{Z}\}}$ of $\mathrm{Bir}(\mathbb{P}^n)$ (Proposition 2.8).

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In Popov (2013), V.L. Popov asks whether the set of unipotent elements of $\mathrm{Bir}(\mathbb{P}^n)$ is closed, as it is the case in all linear algebraic groups. This also raises the question of knowing if the set of algebraic elements is in fact closed.

After giving some properties of the Zariski topology of $\mathrm{Bir}(\mathbb{P}^n)$ in Sect. 2, we describe in Sect. 3 two families of birational maps that give the following result:

Theorem 1 *For each $n \geq 2$, there are two closed subsets $U, S \subset \mathrm{Bir}(\mathbb{P}^n)$, canonically homeomorphic to $\mathbb{A}^1$ and $\mathbb{A}^1 \times (\mathbb{A}^1 \backslash \{0\})$ respectively, such that the following holds:*

(1) *The set of algebraic elements of U is equal to the set of unipotent elements of U, and corresponds to the elements $t \in \mathbb{A}^1$ that belong to the subgroup of* (k, +) *generated by* 1.
(2) *The set of algebraic elements of S is equal to the set of semi-simple elements of S, and corresponds to the elements $(a, \xi) \in \mathbb{A}^1 \times (\mathbb{A}^1 \backslash \{0\})$ such that $a = \xi^k$ for some $k \in \mathbb{Z}$.*

In particular, the set $\mathrm{Bir}(\mathbb{P}^n)_{\mathrm{alg}}$ of algebraic elements of $\mathrm{Bir}(\mathbb{P}^n)$ is not closed in $\mathrm{Bir}(\mathbb{P}^n)$. Moreover, if $\mathrm{char}(\mathrm{k}) = 0$, the set of unipotent elements of $\mathrm{Bir}(\mathbb{P}^n)$ is not closed.

Let us finish this introduction with some remarks:

(1) The set $\mathrm{Bir}(\mathbb{P}^n)_{\mathrm{alg}}$ is a countable union of closed sets of $\mathrm{Bir}(\mathbb{P}^n)$ (Proposition 2.11).
(2) We do not know if the set of unipotent elements of $\mathrm{Bir}(\mathbb{P}^n)$ is closed in $\mathrm{Bir}(\mathbb{P}^n)_{\mathrm{alg}}$ (although it is not closed in $\mathrm{Bir}(\mathbb{P}^n)$).
(3) One can restrict ourselves to the subgroup $\mathrm{Aut}(\mathbb{A}^n) \subset Bir(\mathbb{A}^n) \simeq Bir(\mathbb{P}^n)$. Over $\mathrm{k} = \mathbb{C}$, it follows from Furter (1999) that the set of algebraic elements of $\mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}})$ is closed in $\mathrm{Aut}(\mathbb{A}^2)$. The question is however open for $\mathbb{A}^n$, $n \geq 3$.

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2 A Few Properties of the Zariski Topology of $\mathrm{Bir}(\mathbb{P}^n)$

2.1 Families of Birational Maps and the Zariski Topology Induced

We recall the notion of families of birational maps, introduced by Demazure (1970) (see also Serre 2010; Blanc and Furter 2013).

Definition 2.1 Let A, X be irreducible algebraic varieties, and let f be a A-birational map of the A-variety $A \times X$, inducing an isomorphism $U \to V$, where U, V are open subsets of $A \times X$, whose projections on A are surjective.

The rational map f is given by $(a, x) \dashrightarrow (a, p_2(f(a, x)))$, where p_2 is the second projection, and for each k-point $a \in A$, the birational map $x \dashrightarrow p_2(f(a, x))$ corresponds to an element $f_a \in \mathrm{Bir}(X)$. The map $a \mapsto f_a$ represents a map from A (more precisely from the $A(k)$-points of A) to $\mathrm{Bir}(X)$, and will be called a *morphism* from A to $\mathrm{Bir}(X)$.

These notions yield the natural Zariski topology on $\mathrm{Bir}(X)$, introduced by Demazure (1970) and Serre (2010):

Definition 2.2 A subset $F \subseteq \mathrm{Bir}(X)$ is closed in the Zariski topology if for any algebraic variety A and any morphism $A \to \mathrm{Bir}(X)$ the preimage of F is closed.

We can make the following simple observations:

Lemma 2.3 *Let X, Y be irreducible algebraic varieties, let $\mu : XY$ and $\psi : XX$ be birational maps and let $m \in \mathbb{Z}$ be some integer. The following maps are continuous*

$$\begin{array}{ll} (1)\ \mathrm{Bir}(X) \to \mathrm{Bir}(X), & (2)\ \mathrm{Bir}(X) \to \mathrm{Bir}(X), \\ \quad \varphi \mapsto \psi\varphi & \quad \varphi \mapsto \varphi\psi \\ (3)\ \mathrm{Bir}(X) \to \mathrm{Bir}(X), & (4)\ \mathrm{Bir}(X) \to \mathrm{Bir}(X) \\ \quad \varphi \mapsto \varphi^m & \quad \varphi \mapsto \varphi\psi\varphi^{-1}, \\ (5)\ \mathrm{Bir}(X) \to \mathrm{Bir}(Y) & \\ \quad \varphi \mapsto \varphi\psi\varphi^{-1}, & \end{array}$$

Proof Let A be an irreducible algebraic variety. If f, g are two A-birational maps $f, g : A \times XA \times X$ inducing morphisms $A \to \mathrm{Bir}(X)$, then $f \circ g$ and f^{-1} are again A-birational maps that induce morphisms $A \to \mathrm{Bir}(X)$. This shows that the map $\mathrm{Bir}(X) \to \mathrm{Bir}(X)$ given by $\varphi \mapsto \varphi^m$ is continuous. Similarly, $(\mathrm{id} \times \psi) \circ f$, $f \circ (\mathrm{id} \times \psi)$ and $f \circ (\mathrm{id} \times \psi) \circ f^{-1}$ are A-birational maps that induce morphisms $A \to \mathrm{Bir}(X)$, so the maps $\mathrm{Bir}(X) \to \mathrm{Bir}(X)$ given by $\varphi \mapsto \varphi\psi$, $\varphi \mapsto \psi\varphi$ and $\varphi\psi\varphi^{-1}$ are continuous. The continuity of the last map is given in a similar way, by observing that $(\mathrm{id} \times \mu^{-1}) \circ f \circ (\mathrm{id} \times \mu^{-1})$ also yields a A-birational map that induces a morphism $A \to \mathrm{Bir}(X)$. □

Corollary 2.4 *Let $\varphi \in \mathrm{Bir}(X)$. Denote by F the closure of $\{\varphi^i | i \in \mathbb{Z}\}$ in $\mathrm{Bir}(X)$. Then, F is a closed abelian subgroup of $\mathrm{Bir}(X)$.*

Proof The argument is the same as for algebraic groups or topological groups, and follows from Lemma 2.3, which gives the properties needed for the proof. Let us recall how it works.

(1) For each $j \in \mathbb{Z}$, the set $\varphi^j F$ is a closed subset of $\mathrm{Bir}(X)$ which contains $\{\varphi^i | i \in \mathbb{Z}\}$, and contains thus F. This implies that $\varphi^j F = F$ for each $j \in \mathbb{Z}$.

(2) Let us write $M = \{\psi \in Bir(X) | \psi F \subset F\} = \bigcap_{f \in F} Ff^{-1}$. Since M is closed and contains $\{\varphi^i | i \in \mathbb{Z}\}$, M contains F. This shows that F is closed under composition.
(3) Similarly, the set $I = \{\psi^{-1} | \psi \in F\}$ is closed in $\mathrm{Bir}(X)$ and contains $\{\varphi^i | i \in \mathbb{Z}\}$; hence it contains F. The set F is then a subgroup of $\mathrm{Bir}(X)$.
(4) It remains to see that F is abelian.

We denote by $C(\mu) = \{\psi \in \mathrm{Bir}(X) | \psi\mu = \mu\psi\}$ the centraliser of an element $\mu \in \mathrm{Bir}(X)$. Note that $C(\mu)$ is the preimage of the identity by the continuous map $\mathrm{Bir}(X) \to \mathrm{Bir}(X)$ which sends ψ onto $\psi\mu\psi^{-1}\mu^{-1}$. A point of $\mathrm{Bir}(X)$ being closed by definition of the topology, this shows that $C(\mu)$ is closed.

Because $C(\varphi)$ is a closed subgroup of $\mathrm{Bir}(X)$ which contains $\{\varphi^i | i \in \mathbb{Z}\}$, it contains F, so each element of F commutes with φ.

Finally, we write $S = \{\psi \in \mathrm{Bir}(X) | \psi f = f\psi \text{ for each } f \in F\} = \bigcap_{f \in F} C(f)$, which is again closed, contains $\{\varphi^i | i \in \mathbb{Z}\}$, and thus contains F. This shows that F is abelian. □

2.2 Reminders of Results of Blanc and Furter (2013)

Recall the following natural construction associated to $\mathrm{Bir}(\mathbb{P}^n)$ (which is Blanc and Furter 2013, Definition 2.3):

Definition 2.5 Let d be a positive integer.

(1) We define W_d to be the set of equivalence classes of non-zero $(n+1)$-uples $(h_0, \ldots, h_n)$ of homogeneous polynomials $h_i \in \mathrm{k}[x_0, \ldots, x_n]$ of degree d, where $(h_0, \ldots, h_n)$ is equivalent to $(\lambda h_0, \ldots, \lambda h_n)$ for any $\lambda \in \mathrm{k}^*$. The equivalence class of $(h_0, \ldots, h_n)$ will be denoted by $(h_0 : \ldots : h_n)$.
(2) We define $H_d \subseteq W_d$ to be the set of elements $h = (h_0 : \ldots : h_n) \in W_d$ such that the rational map $\psi_h : \mathbb{P}^n \mathbb{P}^n$ given by $(x_0 : \ldots : x_n)(h_0(x_0, \ldots, x_n) : \ldots : h_n(x_0, \ldots, x_n))$ is birational. We denote by π_d the map $H_d \to Bir(\mathbb{P}^n_{\mathrm{k}})$ which sends h onto ψ_h.

It follows from the construction that W_d is a projective space and that $\pi_d(H_d) = \mathrm{Bir}(\mathbb{P}^n)_{\leq d}$. Moreover, we have the following properties:

Proposition 2.6 (Blanc and Furter 2013, Lemma 2.4, Corollaries 2.7 and 2.9)

(1) *The set $H_d \subset W_d$ is locally closed, and is thus an algebraic variety.*
(2) *The map $\pi_d : H_d \to \mathrm{Bir}(\mathbb{P}^n)$ is a morphism. It yields a map $H_d \to \mathrm{Bir}(\mathbb{P}^n)_{\leq d}$ which is surjective, closed and continuous. In particular, it is a topological quotient map.*

(3) *A subset* $F \subset \mathrm{Bir}(\mathbb{P}^n)$ *is closed if and only if* $(\pi_d)^{-1}(F)$ *is closed in* H_d *for each* d.

We also have the following description of algebraic subgroups of $\mathrm{Bir}(\mathbb{P}^n)$:

Proposition 2.7 (Blanc and Furter 2013, Corollary 2.18 and Lemma 2.19)
A subgroup of $\mathrm{Bir}(\mathbb{P}^n)$ *is an algebraic subgroup if and only if it is closed and of bounded degree.*

2.3 Algebraicity and Boundedness of the Degree Sequence

Proposition 2.8 Let $\varphi \in \mathrm{Bir}(\mathbb{P}^n)$.

(1) *If the sequence* $\{\deg(\varphi^m)\}_{m\in\mathbb{N}}$ *is bounded, then* $\overline{\{\varphi^i | i \in \mathbb{Z}\}}$ *is a commutative algebraic subgroup of* $\mathrm{Bir}(\mathbb{P}^n)$.
(2) *If the sequence* $\{\deg(\varphi^m)\}_{m\in\mathbb{N}}$ *is unbounded, then* φ *is not contained in any algebraic subgroup of* $\mathrm{Bir}(\mathbb{P}^n)$.

Proof Proposition 2.7 directly yields (2). Let us prove (1).

We suppose then that $\{\deg(\varphi^m)\}_{m\in\mathbb{N}}$ is bounded. Because $\deg(\varphi^{-m}) \leq (\deg(\varphi^m))^{n-1}$ for each m (Bass et al. 1982, Theorem 1.5, p. 292), the set $\{\varphi^i\}_{i\in\mathbb{Z}}$ is contained in $\mathrm{Bir}(\mathbb{P}^n)_{\leq d}$ for some d. The closure F of $\{\varphi^i | i \in \mathbb{Z}\}$ in $\mathrm{Bir}(\mathbb{P}^n)$ is then again contained in $\mathrm{Bir}(\mathbb{P}^n)_{\leq d}$. By Corollary 2.4, F is a commutative subgroup of $\mathrm{Bir}(\mathbb{P}^n)$ and is then a commutative algebraic subgroup of $\mathrm{Bir}(\mathbb{P}^n)$ (Proposition 2.7). □

Corollary 2.9 *Let* $\varphi \in \mathrm{Bir}(\mathbb{P}^n)$. *The following are equivalent:*

(1) *The element* φ *is algebraic.*
(2) *The sequence* $\{\deg(\varphi^m)\}_{m\in\mathbb{N}}$ *is bounded.*

Proof Directly follows from Proposition 2.8. □

Lemma 2.10 *The Zariski topology of* $\mathrm{Bir}(\mathbb{P}^n)_{\leq d}$ *is noetherian, i.e. every decreasing sequence of closed subsets is eventually stationary.*

This is not the case for $\mathrm{Bir}(\mathbb{P}^n)$.

Proof By Proposition 2.6, we have a map $\pi_d : H_d \to \mathrm{Bir}(\mathbb{P}^n)_{\leq d}$, which is surjective, continuous and closed. The topology of H_d being noetherian (it is an algebraic variety), the same holds for $\mathrm{Bir}(\mathbb{P}^n)_{\leq d}$.

The fact that the topology of $\mathrm{Bir}(\mathbb{P}^n)$ is not noetherian has already being observed in Pan and Rittatore (2016). It can be shown by taking a sequence $\{\varphi_i\}_{i\in\mathbb{N}}$ of maps φ_i of degree i. Then $F_i = \{\varphi_j | j \geq i\}$ is closed in $\mathrm{Bir}(\mathbb{P}^n)$ for each i (follows from Proposition 2.6) but the sequence $F_1 \supset F_2 \supset F_3 \supset \ldots$ is not stationary. □

Proposition 2.11 *For each integers $k, d \in \mathbb{N}$ let us write*

$$Bir(\mathbb{P}^n)_{k,d} = \{f \in Bir(\mathbb{P}^n) | \deg(f^k) \leq d\}$$
$$Bir(\mathbb{P}^n)_{\infty,d} = \{f \in Bir(\mathbb{P}^n) | \deg(f^i) \leq d \text{ for all } i \in \mathbb{N}\}.$$

Then, the following hold:

(1) *The set* $\mathrm{Bir}(\mathbb{P}^n)_{k,d}$ *is closed in* $\mathrm{Bir}(\mathbb{P}^n)$.
(2) *The set* $\mathrm{Bir}(\mathbb{P}^n)_{\infty,d} = \bigcap_{i \in \mathbb{N}} \mathrm{Bir}(\mathbb{P}^n)_{i,d}$ *is closed in* $\mathrm{Bir}(\mathbb{P}^n)$.
(3) *The set* $\mathrm{Bir}(\mathbb{P}^n)_{alg}$ *of all algebraic elements of* $\mathrm{Bir}(\mathbb{P}^n)$ *is equal to the union of all* $\mathrm{Bir}(\mathbb{P}^n)_{\infty,d}$.

Proof By Proposition 2.6, the set $\mathrm{Bir}(\mathbb{P}^n)_{\leq d}$ is closed in $\mathrm{Bir}(\mathbb{P}^n)$ for each d. The map $\mathrm{Bir}(\mathbb{P}^n) \to \mathrm{Bir}(\mathbb{P}^n)$ which sends φ onto φ^k being continuous (Lemma 2.3), this directly shows that $\mathrm{Bir}(\mathbb{P}^n)_{k,d}$ is closed in $\mathrm{Bir}(\mathbb{P}^n)$. This yields (1), which implies (2).

Corollary 2.9 yields the equality $\mathrm{Bir}(\mathbb{P}^n)_{alg} = \bigcup_{d \in \mathbb{N}} \mathrm{Bir}(\mathbb{P}^n)_{\infty,d}$, which corresponds to (3). ☐

3 Two Explicit Families

3.1 A Unipotent Example

Example 3.1 For $n \geq 2$, let $\rho : \mathbb{A}^1 \to Bir(\mathbb{P}^n)$ be the morphism given by

$$\begin{array}{ccc} \mathbb{A}^1 \times \mathbb{P}^n & \dashrightarrow & \mathbb{A}^1 \times \mathbb{P}^n \\ (a, [x_0 : \cdots : x_n]) & \mapsto & (a, [x_0 x_1 : x_1(x_1 + x_0) : x_2(x_1 + a x_0) : x_3 x_1 : \cdots : x_n x_1]) \end{array}$$

which corresponds on the affine open subset where $x_0 = 1$ to the family of birational maps given by

$$(x_1, \ldots, x_n) \to (x_1 + 1, x_2 \cdot \frac{x_1 + a}{x_1}, x_3, \ldots, x_n).$$

Lemma 3.2 *The map* $\rho : \mathbb{A}^1 \to \mathrm{Bir}(\mathbb{P}^n)$ *of Example 3.1 is a topological embedding.*

Proof The fact that ρ is injective can be directly checked on the formula given above. We then consider the closed embedding $\hat{\rho} : \mathbb{P}^1 \to W_2$ that sends $[\mu : \lambda] \in \mathbb{P}^1$ to

$$[\mu x_0 x_1 : \mu x_1(x_1 + x_0) : \mu x_2 x_1 + \lambda x_2 x_0 : \mu x_3 x_1 : \ldots : \mu x_n x_1].$$

When $\mu = 0$, this does not give a birational map of $\mathbb{P}^n$, so $\hat{\rho}([0:1]) \notin H_2$. However, we have $\pi_2(\hat{\rho}([1:t])) = \rho(t)$ for each $t \in \mathbb{A}^1$, so the restriction to $\mathbb{A}^1$ yields a closed embedding $\mathbb{A}^1 \to H_2$. It remains to show that the restriction of π_2 to $\hat{\rho}(\mathbb{P}^1 \backslash [0:1])$ is an homeomorphism, which is given by Proposition 2.6. □

Proposition 3.3 *The morphism $\rho : \mathbb{A}^1 \to Bir(\mathbb{P}^n)$ of Example 3.1 has the following properties:*

(1) *For $t \in$ k, the following conditions are equivalent:*

  (a) *$\rho(t)$ is algebraic;*
  (b) *$\rho(t)$ is unipotent;*
  (c) *$\rho(t)$ is conjugate to $\rho(0) : (x_1, \ldots, x_n) \mapsto (x_1 + 1, x_2, \ldots, x_n)$;*
  (d) *t belongs to the subgroup of* (k, +) *generated by 1.*

(2) *The pull-back by ρ of the set of algebraic elements is not closed if* char(k) = 0.

Proof (1) Proceeding by induction, the iterates of $\rho(a)$ send $(x_1, \ldots, x_n)$ onto:

$$\begin{array}{ll} \rho(a): & (x_1 + 1, x_2 \cdot \frac{x_1 + a}{x_1}, x_3, \ldots, x_n), \\ \rho(a)^2: & (x_1 + 2, x_2 \cdot \frac{(x_1 + a)(x_1 + a + 1)}{x_1(x_1 + 1)}, x_3, \ldots, x_n), \\ \rho(a)^m: & (x_1 + m, x_2 \cdot \frac{(x_1 + a)(x_1 + a + 1) \cdots (x_1 + a + m - 1)}{x_1(x_1 + 1) \cdots (x_1 + m - 1)}, x_3, \ldots, x_n). \end{array}$$

Then, the second coordinate of $\rho(a)^m(x_1, \ldots, x_n)$ is

$$\frac{\prod_{i=0}^{m-1} (x_1 + a + i)}{\prod_{i=0}^{m-1} (x_1 + i)}.$$

If a does not belong to the subgroup of (k, +) generated by 1, then the denominator and numerators have no common factor, for each $m \in \mathbb{N}$, so the degree growth of $\rho(a)^m$ is linear, which implies that $\rho(a)$ is not algebraic.

If a belongs to the subgroup of (k, +) generated by 1, it is equal to $k \in \mathbb{Z}$, and the degree of $\{\rho(a)^m\}_{m \in \mathbb{N}}$ is bounded by $|k| + 1$, so $\rho(a)$ is algebraic. We can moreover see that $\rho(a)$ is unipotent. Indeed, $\rho(a)$ is conjugate to

$$\rho(0) = (x_1, \ldots, x_n) \to (x_1 + 1, x_2, x_3, \ldots, x_n)$$

by

$$(x_1, \ldots, x_n) \to (x_1, \frac{x_2}{x_1(x_1+1)\ldots(x_1+a-1)}, x_3, \ldots, x_n)$$

if $a = k > 0$ or by

$$(x_1, \ldots, x_n) \to (x_1, x_2 \cdot x_1(x_1-1)\ldots(x_1+a), x_3, \ldots, x_n)$$

if $a = k < 0$.

Assertion (2) follows directly from (1) and the fact that the subgroup of (k, +) generated by 1 is closed if and only if char(k) $\neq 0$. □

3.2 A Semi-simple Example

Example 3.4 For $n \geq 2$, let $\rho : \mathbb{A}^1 \times (\mathbb{A}^1 \setminus \{0\}) \to \mathrm{Bir}(\mathbb{P}^n)$ be the morphism given by

$$\begin{aligned}\rho(a,\xi)([x_0 : x_1 : \ldots : x_n]) = [&x_0(x_1+x_0) : \xi x_1(x_1+x_0) \\ &: x_2(x_1+ax_0) : x_3(x_1+x_0) : \ldots : x_n(x_1+x_0)]\end{aligned}$$

which corresponds on the affine open subset where $x_0 = 1$ to the family of birational maps

$$(x_1, \ldots, x_n) \dashrightarrow (x_1+1, x_2 \cdot \frac{x_1+a}{x_1}, x_3, \ldots, x_n).$$

Lemma 3.5 *The map* $\rho : \mathbb{A}^1 \to \mathrm{Bir}(\mathbb{P}^n)$ *of Example* 3.4 *is a topological embedding.*

Proof The proof is similar to the one of Lemma 3.2. The fact that ρ is injective can be directly checked on the formula given above. We then consider the closed embedding $\hat{\rho} : \mathbb{P}^2 \to W_2$ that sends $[\mu : \eta : \lambda] \in \mathbb{P}^2$ to

$$[\mu x_0(x_1+x_0) : \lambda x_1(x_1+x_0) : x_2(\mu x_1 + \eta x_0) : \mu x_3(x_1+x_0) : \ldots : \mu x_n(x_1+x_0)].$$

These elements correspond to birational maps if and only if $\mu\lambda \neq 0$. Hence, we have a closed embedding $\mathbb{A}^1 \times (\mathbb{A}^1 \setminus \{0\}) \to H_2$ that sends (a, ξ) onto $\hat{\rho}([1 : a : \xi])$. Moreover, we have $\pi_2(\hat{\rho}([1 : a : \xi])) = \rho(a, \xi)$. The fact that the restriction of π_2 to the image is a homeomorphism is then given by Proposition 2.6. □

Proposition 3.6 *The morphism* $\rho : \mathbb{A}^1 \to Bir(\mathbb{P}^n)$ *of Example* 3.4 *has the following properties:*

(1) *For* $(a, \xi) \in \mathbb{A}^1 \times (\mathbb{A}^1 \backslash \{0\})$, *the following conditions are equivalent:*

(a) $\rho(a, \xi)$ *is algebraic;*
(b) $\rho(a, \xi)$ *is semi-simple;*
(c) $\rho(a, \xi)$ *is conjugate to* $\rho(1, \xi) : (x_1, \ldots, x_n) \mapsto (\xi x_1, x_2, \ldots, x_n)$;
(d) *There exists* $k \in \mathbb{Z}$ *such that* $a = \xi^k$.

(2) *The pull-back by* ρ *of the set of algebraic elements is not closed.*

Proof (1) Proceeding by induction, the iterates of $\rho(a, \xi)$ send $(x_1, \ldots, x_n)$ onto:

$$\begin{array}{ll}
\rho(a,\xi): & \left(\xi x_1, x_2 \cdot \frac{x_1 + a}{x_1 + 1}, x_3, \ldots, x_n\right), 0.2cm \\
\rho(a,\xi)^2: & \left(\xi^2 x_1, x_2 \cdot \frac{(x_1 + a)(\xi x_1 + a)}{(x_1 + 1)(\xi x_1 + 1)}, x_3, \ldots, x_n\right), 0.2cm \\
\rho(a,\xi)^m: & \left(\xi^m x_1, x_2 \cdot \frac{(x_1 + a)(\xi x_1 + a)\cdots(x_1 \xi^{m-1} + a)}{(x_1 + 1)(x_1 \xi + 1)\cdots(x_1 \xi^{m-1} + 1)}, x_3, \ldots, x_n\right).
\end{array}$$

Then, the second coordinate of $\rho(a, \xi)^m(x_1, \ldots, x_n)$ is

$$\frac{\prod_{i=0}^{m-1} (\xi^i x_1 + a)}{\prod_{i=0}^{m-1} (\xi^i x_1 + 1)}.$$

If a does not belong to the subgroup of $(\mathrm{k}, \cdot)$ generated by ξ, then the denominator and numerators have no common factor, for each $m \in \mathbb{N}$, so the degree growth of $\rho(a, \xi)^m$ is linear, which implies that $\rho(a, \xi)$ is not algebraic.

If a belongs to the subgroup of $(\mathrm{k}, \cdot)$ generated by ξ, it is equal to $a = \xi^k$ for some $k \in \mathbb{Z}$, and the degree of $\{\rho(a, \xi)^m\}_{m \in \mathbb{N}}$ is bounded by $|k| + 1$, so $\rho(a, \xi)$ is algebraic. We can moreover see that $\rho(a, \xi)$ is semi-simple. Indeed, $\rho(a, \xi)$ is conjugate to
$\rho(0) = (x_1, \ldots, x_n) \dashrightarrow (x_1 + 1, x_2, x_3, \ldots, x_n)$
by
$$(x_1, \ldots, x_n) \dashrightarrow (x_1, \frac{x_2}{x_1(x_1+1)\ldots(x_1+a-1)}, x_3, \ldots, x_n)$$
if $k > 0$ or by
$$(x_1, \ldots, x_n) \dashrightarrow (x_1, x_2 \cdot x_1(x_1-1)\ldots(x_1+a), x_3, \ldots, x_n)$$
if $k < 0$.

Assertion (2) follows from (1) and the fact that the subset of $\mathbb{A}^1 \times (\mathbb{A}^1 \backslash \{0\})$ that consists of elements (a, ξ) such that $a = \xi^k$ for some $k \in \mathbb{Z}$ is not closed. □

References

Hyman Bass, Edwin H. Connell and David Wright, *The Jacobian conjecture: reduction of degree and formal expansion of the inverse,* Bull. of the A.M.S. **7** (1982), 287–330.

Ivan Pan and Alvaro Rittatore, *Some remarks about the Zariski topology of the Cremona group.* http://arxiv.org/abs/1212.5698.

Jean-Philippe Furter, *On the degree of iterates of automorphisms of the affine plane,* Manuscripta Math. **98** (1999), no. 2, 183–193.

Jean-Pierre Serre, *Le groupe de Cremona et ses sous-groupes finis.* Séminaire Bourbaki. Volume 2008/2009. Astérisque No. **332** (2010), Exp. No. 1000, vii, 75–100.

Jérémy Blanc and Jean-Philippe Furter, *Topologies and structures of the Cremona groups*, Ann. of Math. **178** (2013), no. 3, 1173–1198.

Michel Demazure, *Sous-groupes algébriques de rang maximum du groupe de Cremona,* Ann. Sci. École Norm. Sup. (4) **3** (1970), 507–588.

Vladimir L. Popov, *Tori in the Cremona groups.* Izv. Math. **77** (2013), no. 4, 742–771.

Segre Functions in Multiprojective Spaces and Tensor Analysis

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Abstract

We introduce the notion of Segre function s_X for a variety X embedded in a product of projective spaces and determine some initial property of s_X, when X is a finite subset. We show in the last section how these properties can be used to derive results on the identifiability of specific tensors.

1 Introduction

The aim of this note is to raise attention to some aspects of the geometry of subvarieties of Segre varieties (i.e. products of projective space, with the natural polarization), that are relevant for applications in other fields of Mathematics.

The embedding of a variety X in a projective space $\mathbb{P}^N$ is determined by one (not necessarily complete) linear series H and many properties of the geometry of X, as well as many invariants of the embedding, can be read in the *Hilbert function* h_X, which sends an integer r to the dimension of the multiple rH (since we do not care about completeness, rH means the minimal sum $H + \cdots + H$, r times). This is true even when X is a finite set of points, if we identify H with a subspace of the vector space of sections of the structure sheaf (after fixing coordinates on $\mathbb{P}^n$).

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We do not want even to try here a survey of the enormous literature on the Hilbert functions of projective varieties. Let us notice that Hilbert functions allow us to prove some results on the structure of $X \subset \mathbb{P}^n$, by arguing inductively on the degree r.

The aspect of the geometry of varieties that we consider is related with the study of secant varieties. The theory of secant varieties lives now an outburst of interest, since it is strictly connected with the notion of *rank* of a multilinear object, thus also with the complexity of ubiquitous mathematical tools (polynomials, tensors, etc.). It turns out that algebraic-geometric methods allowed some recent progress in our knowledge of spaces of tensors of given rank (see e.g. Alexander and Hirschowitz 1995; Chiantini and Ciliberto 2002; Ballico 2006; Abo et al. 2009; Landsberg and Ottaviani 2013), on the identifiability of general or even specific tensors (see e.g. Strassen 1983; Chiantini and Ciliberto 2006; Bocci et al. 2014; Chiantini et al. 2014) and on the decomposition of tensors by means of elementary objects (see e.g. Ballico and Bernardi 2013a; Carlini et al. 2014).

One important feature in the study of secant varieties is the fact that pathological behavior is associated with the existence of degenerate subvarieties. This happens when one studies the dimension of secant varieties (see Theorem 1.1 of Chiantini and Ciliberto 2002) or the identifiability of points (see Theorem 2.5 of Chiantini and Ciliberto 2006).

Secant varieties are mostly important for applications when X is a Segre embedding of a product of projective spaces, or a suitable projection or a linear section (thus including Veronese embeddings of a single space). From this point of view, a description of degenerate subvarieties of Veronese or Segre varieties could provide a fundamental step towards progress on the study of the complexity of multilinear objects. As an example, we refer to Proposition 6.7 of Chiantini and Ciliberto (2005), where it turns out that the classification of degenerate subvarieties of some Segre embeddings determines the final step in the classification of defective threefolds. Also notice how in Chiantini and Ottaviani (2012) Theorem 1.3, the existence of degenerate elliptic curves through 6 general points of $\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3$ is used to prove the non-identifiability of generic tensors of type $4 \times 4 \times 4$ and rank 6.

There are aspects for which results on the Hilbert function of varieties determine results for the complexity of *symmetric* tensors. The connection is evident from the fact that degenerate subvarieties of the Veronese embedding X of degree d of $\mathbb{P}^n$ correspond to subvarieties $Y \subset \mathbb{P}^n$ for which the Hilbert function in degree d is low. Thus, knowing the minimal Hilbert function of projective varieties, one has tools for understanding the behavior of defective Veronese varieties.

Another interesting application of Hilbert functions concerns the study of the identifiability of symmetric tensors. If one can find two different decompositions Z, Z' of the same symmetric tensor of degree d, then the union $Z \cup Z'$ is a linearly dependent subset of the dth Veronese embedding of $\mathbb{P}^n$. In turn, this determines a subset W of $\mathbb{P}^n$ whose Hilbert function in degree d is strictly positive. When the number of points (i.e. the rank) is small enough with respect to d, this can be excluded (maybe by means of some extra condition on the set Z). This is the idea behind a series of results on identifiability (Buczy'nski et al. 2016, Proposition 1.5.1; Ballico and Bernardi 2012; Ballico and Chiantini 2013b; Carlini et al. 2014).

Trying to extend the previous analysis to non-symmetric tensors, we must substitute the Veronese embedding of a single projective space with the Segre embedding of a *product of* projective spaces. Then, a first problem that one faces is the introduction of an object analogous to the Hilbert function, having similar properties, which can determine the dimension of the span of subvarieties of a product of projective spaces.

The aim of this note is exactly the definition of an analogue of the Hilbert function (the *Segre function* s_Y) for the study of subvarieties Y of multiprojective spaces (= products of projective spaces). Roughly speaking, if $Y \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_q}$ and H_i is the linear series determining the projection $X \to \mathbb{P}^{n_i}$, we define $s_Y(d)$ as the dimension of the (minimal) sum $H_1 + \cdots + H_d$. In other words, $s_Y(d)$ is the affine dimension of the linear span of the projection of X to the Segre embedding of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_d}$. This last characterization is used to define s_Y for a finite set Y.

We invite comparison of the definition of $s_Y(d)$ with the definition of the Hilbert function $h_Y(d)$, which corresponds to the dimension of the sum $H + \cdots + H$, d times. Let us also notice that the Segre function has not much to do with the *normal* Hilbert function of the Segre embedding of $Y \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_q}$ in the corresponding space $\mathbb{P}^N$, $(N + 1 = \Pi(n_i + 1))$ which associates to d the dimension of the series $d(H_1 + \cdots + H_q)$. Technically, the Segre function that we define is the initial block of the *multigraded* Hilbert function.

From this point of view, the classical Castelnuovo's theorem on the sum of linear series (Castelnuovo 1889) as well as the results of Accola on curves (Accola 1979) can be considered as steps towards the understanding of Segre functions of positive dimensional varieties.

In this note, we will study the Segre function mainly for finite subsets of a multiprojective space, and determine some of their initial properties. In the final section, we will give a hint of how Segre functions can be used to determine results on the identifiability of *specific* tensors of type $2 \times \cdots \times 2$. The result we prove is somehow analogous to Proposition 1.5.1 of Buczy'nski et al. (2016), valid for symmetric tensors.

More than the results themselves, which probably can be obtained even with other tools, our scope is to stimulate a systematic research on Segre functions, which, in our opinion, can produce advances in our knowledge of the complexity of general tensors.

We also believe that the best possible collocation for this note is the volume in honor of Corrado Segre, whose pioneering studies on multiprojective spaces (see e.g. Segre 1920) marked one of the most interesting (and multidisciplinary) objects for contemporary mathematics.

2 Notation

We work over the complex field $\mathbb{C}$.

We will indicate with $[P]$ a representative for the homogeneous coordinates of $P \in \mathbb{P}^n$.

By abuse, we will often indicate with the same letter a multihomogeneous form f and the locus defined by f = 0 in a suitable product of projective spaces.

Consequently, we will use locutions like *f misses a point P* or *f contains a point Q* to indicate respectively that $f(P) \neq 0$ and $f(Q) = 0$.

The Segre embedding v of a product of projective spaces (=*multiprojective space*) $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_q}$ is defined by choosing coordinates on each $\mathbb{P}^{n_i}$ and sending a point

$$P = ([a_{10}, \ldots, a_{1n_1}], \ldots, [a_{q0}, \ldots, a_{qn_q}])$$

to the point $v(P) = [M_0, \ldots, M_N] \in \mathbb{P}^N$, where $N + 1 = \Pi(n_i + 1)$ and the M_k's are the linear multihomogeneous monomials in the coordinates a_{ij}'s.

It follows that the image of the Segre map is defined only up to a projective change of coordinates in $\mathbb{P}^N$.

For a given product of projective spaces $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_q}$ we define the *i*th projection π_i as the surjective rational map

$$\pi_i : \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_q} \to \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_i}$$

which sends a point $(P_1, \ldots, P_q)$ to $(P_1, \ldots, P_i)$.

Remark 2.1 There is a way to reproduce the projection π_i in the Segre embedding of the product. Namely it corresponds to the map

$$v(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_q}) \to v(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_i})$$

obtained as a projection from the span of all the points $v(P_1, \ldots, P_i, Q_{i+1}, \ldots, Q_q)$ where $Q_j = [1, 0, \ldots, 0] \in \mathbb{P}^{n_j}$ for $j = i + 1, \ldots, q$.

From this point of view, we can identify the set where π_i is not defined as the span of

$$v(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_i} \times \{Q_{i+1}\} \times \cdots \times \{Q_q\}),$$

hence it is isomorphic to $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_i}$.

Similarly, the (closure of) the fibers of π_i are isomorphic to $\mathbb{P}^{n_{i+1}} \times \cdots \times \mathbb{P}^{n_q}$.

In particular, when $i = q - 1$, the fibers of π_{q-1} are projective spaces.

Let Z be a subscheme of the (ordered) product of projective spaces $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_q}$.

For each i the sub-product $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_i}$ is naturally embedded by the Segre map in a projective space $\mathbb{P}^{N_i}$, with $N_i + 1 = (n_1 + 1) \cdots (n_i + 1)$. Let I_i be the homogeneous ideal of the projection $\pi_i(Z)$ in $\mathbb{P}^{N_i}$.

Definition 2.2 We define the *Segre function* of the subvariety Z of the product $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_q}$ as follows:

- For non-positive values we set:

$$s_Z(i) = 0 \text{ for } i < 0 \quad \text{and} \quad s_Z(0) = 1.$$

- For $q \geq i > 0$ we set

$$s_Z(i) = N_i + 1 - \dim(I_i)_1.$$

- For $i > q$ we set $s_Z(i) = s_Z(q)$.

It is clear that $s_Z(i)$ is well defined for any choice of coordinates, and the only relevant part of $s_Z(i)$ sits between 0 and q.

Example 2.3 Consider a set Z of four distinct points in $\mathbb{P}^1 \times \mathbb{P}^1$, which is embedded by the Segre map as a quadric in $\mathbb{P}^3$.

If the projection to the first factor π_1 is a single point, then Z lies in a line of $\mathbb{P}^3$ and its Segre function is

i	-1	0	1	2	3	...
$s_Z(i)$	0	1	1	2	2	...

If $\pi_1(Z)$ has at least two points, but the image of Z lies in a plane of $\mathbb{P}^3$, then the Segre function is

i	-1	0	1	2	3	...
$s_Z(i)$	0	1	2	3	3	...

If Z spans $\mathbb{P}^3$ then the Segre function is

i	-1	0	1	2	3	...
$s_Z(i)$	0	1	2	4	4	...

Example 2.4 If a finite set $Z \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_q}$, of cardinality $z > N$, is not contained in any hyperplane of $\mathbb{P}^N$, $N = -1 + (n_1 + 1)(n_2 + 1) \cdots (n_s + 1)$, then any projection $\pi_i(Z)$ must span the whole space spanned by the Segre embedding of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_q}$, thus the Segre function of Z is:

i	-1	0	1	2	$\ldots$	q	$q+1$	$\ldots$
$s_Z(i)$	0	1	a_1+1	$(a_1+1)(a_2+1)$	$\ldots$	$N+1$	$N+1$	$\ldots$

This is a particular case of *linear general position*, which will be defined at the beginning of Sect. 4.

Remark 2.5 The number $s_Z(i)$ can be seen also as the *affine* dimension of the span of the set $\pi_i(Z)$ in the Segre embedding of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_i}$.

Let $Z = (P_1, \ldots, P_z)$ be an ordered finite set of z points in $\mathbb{P}^n$. Fix a linear form f_0 in $\mathbb{P}^n$ which does not vanish at any point P_i.

We define the f_0-*residue* at Z to be the subspace Res_{f_0} of $\mathbb{C}^z$ generated by the vectors

$$\{(f/f_0(P_1), \ldots, f/f_0(P_z)) : f = \text{ linear form in } \mathbb{P}^n\}.$$

Notice that replacing f_0 by another linear form which does not vanish at any point P_i, the residue is isomorphic to Res_{f_0}. Indeed it changes, in any component of $\mathbb{C}^z$, by multiplication by a non-zero scalar.

The notion can be extended to ordered finite subsets of *products* of projective spaces. Namely, if $Z \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_q}$, then one chooses a multilinear form f_0 in $\mathbb{P}^n$, vanishing at no point P_i, and defines the f_0-*residue* at Z as above.

We emphasize that the dimension of the f_0-residue is independent of the choice of f_0.

This procedure is clearly equivalent to determine the g_0-residue of the image of Z in the Segre embedding, where g_0 is the linear form in $\mathbb{P}^N$ $(N + 1 = (n_1 + 1) \cdots (n_q + 1)$ associated to f_0.

Remark 2.6 One obtains an equivalent definition of the Segre function by setting, for $0 < i \leq q$, $s_Z(i)$ equal to the dimension of the residue at $\pi_i(Z)$ (with respect to any multilinear form).

Indeed, since the Segre map is defined by multilinear polynomials, the dimension of the residue is equal to the dimension of the cokernel of the embedding $0 \to (I_i)_1 \to (R_i)_1$, where R_i is the polynomial ring in $(n_1 + 1) \cdots (n_i + 1)$ variables.

Remark 2.7 It is trivial that if $Z' \subset Z$, then $s_Z(i) \geq s_{Z'}(i)$ for all i.

Remark 2.8 If we consider a permutation of the indexes σ, then we have an obvious isomorphism

$$\sigma' : \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_q} \longrightarrow \mathbb{P}^{n_{\sigma(1)}} \times \cdots \times \mathbb{P}^{n_{\sigma(q)}}$$

which can be realized from an automorphism of the ambient space of the two Segre embeddings.

Let us notice that, even when the indexes are all equal, by no means can we guarantee that the Segre functions of Z and $\sigma'(Z)$ are equal.

As the Segre embedding is related with multilinear forms, the linear geometry of points in projective spaces is deeply involved. We introduce a notion which will be useful later.

Definition 2.9 Let $v_1, \ldots, v_n$ be elements of a linear space. We say that $v_1, \ldots, v_n$ are *minimally dependent*, or that they form a *circuit*, if they are linearly dependent but any proper subset is independent.

We can extend this notion, which is invariant under multiplication by non-zero scalars, to sets of points in a projective space.

The name of *circuit* for a system of points as above was introduced by Gelfand et al. (1994), Chap. 7, Sect. 1.

There are plenty of equivalent definitions. All of them will be used freely in the sequel.

Remark 2.10 The set W is minimally dependent if and only if it is dependent and either:

- the null vector can be expressed uniquely, up to scalar multiplication, as a non-trivial linear combination of the v_i's, and in this combination all the scalars are non-zero; or
- all non-trivial combinations of the v_i's which give the null vector have no zero scalars; or
- each v_i can be expressed uniquely as a linear combination of the remaining elements of W.

If we take an independent set W' and add a vector which sits in the span of W', but not in the span of any proper subset, then we get a minimally dependent set. Conversely, any minimally dependent set arises in this way.

The definition of *minimally dependent* follows the same circle of ideas of the definition of *Kruskal rank* of a matrix (see Kruskal 1977), so it sounds natural that it has applications in the study of tensors.

3 Some Results on the Segre Function of Finite Sets

In this section, we provide some results for the Segre functions of finite sets of points, which are analogue to similar results for the Hilbert functions.

Proposition 3.1 *For any finite subset $Z \subset X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_q}$ of z distinct points, and for all i, the (oriented) Segre function s_Z of Z satisfies*

$$0 \leq s_Z(i-1) \leq s_Z(i) \leq z.$$

Proof The only non trivial part is for $0 < i \leq q$. Moreover, by taking the projection π_i, we may always assume $i = q$.

Call g_0 a linear form in $\mathbb{P}^{n_q}$ that misses all the points of the projection of Z to $\mathbb{P}^{n_q}$. Fix a multilinear form f_0 in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_{q-1}}$ that misses all points of $\pi_q(Z)$. Then we can define a map $Res_{f_0} \to Res_{f_0 g_0}$ by sending the evaluation of f/f_0 to the evaluation of $fg/f_0 g$ and the map is injective. The claim follows. □

Definition 3.2 A subset $Z \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_q}$ is *i-degenerate* for some index i if there exists a point $P_i \subset \mathbb{P}^{n_i}$ such that

$$Z \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_{i-1}} \times \{P_i\} \times \mathbb{P}^{n_{i+1}} \times \cdots \times \mathbb{P}^{n_q}.$$

Equivalently, Z is i-degenerate if the projection of Z to the factor $\mathbb{P}^{n_i}$ is constant.

Proposition 3.3 *Let $Z \subset X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_q}$ be a finite set of z distinct points and call s_Z the (oriented) Segre function of Z. Fix an index i with $0 < i \leq q$.*
If Z is i-degenerate then $s_Z(i-1) = s_Z(i)$.

Proof By taking the projection π_i, we may always assume $i = q$.

Assume that Z is q-degenerate. After a change of coordinates we may assume that $Z \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_{q-1}} \times \{P\}$ with $P = (0, \ldots, 0, 1)$. For any multilinear form in X, call f^P the form obtained by partially evaluating f at P (i.e. with respect to the last coordinates). Clearly f^P is a multilinear form in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_{q-1}}$ and the evaluation of f/f_0 at Z equals the evaluation of f^P/f_0^P at $\pi_{q-1}(Z)$. The claim follows. □

We can refine the previous fact and invert it as follows.

Remark 3.4 If $W = \{P_1, P_2\}$ is a couple of points such that $\pi(P_1) = \pi(P_2)$ in some projection π, we will say (by abuse) that the points of $\pi(W)$ are minimally dependent.

Indeed, there are choices of coordinates for $\pi(P_1), \pi(P_2)$ such that $[\pi(P_1)] - [\pi(P_2)] = 0$, while clearly each $[\pi(P_i)]$ is non-zero.

Proposition 3.5 *Let $Z \subset X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_q}$ be a finite set of z distinct points and call s_Z the (oriented) Segre function of Z. Fix an index i with $1 \le i \le q$ and assume $s_Z(i-1) < z$.*

Then $s_Z(i-1) = s_Z(i)$ if and only if for any subset $Z' \subset Z$ such that the points $\pi_{i-1}(Q)$, $Q \in Z'$, are minimally dependent, then Z' is i-degenerate.

Proof By taking the projection π_i, as above we may always assume $i = q$.

Assume that for any subset $Z' \subset Z$ such that $\pi_{q-1}(Z')$ is minimally dependent, then Z' is q-degenerate. As established in the previous remark, the condition implies that if $\pi_{q-1}(P_1) = \pi_{q-1}(P_2)$ for a couple of points of Z, then also the projections of P_1, P_2 to $\mathbb{P}^{n_q}$ must coincide, so that $P_1 = P_2$. In other words, the condition implies, in particular, that π_{q-1} is injective on Z. Let $W \subset Z$ be a maximal subset such that $\pi_{q-1}(W)$ is linearly independent. Then by definition $s_Z(q-1)$ is the cardinality of W, so we get $s_Z(q-1) = s_Z(q)$ if we can prove that $\langle Z \rangle = \langle W \rangle$. Fix $P \in Z \backslash W$ and let W' be a subset of W such that $\pi_{q-1}(W' \cup \{P\})$ is minimally dependent. Then, by assumption, the projection of P to $\mathbb{P}^{n_q}$ is equal to the projection of all the other points of W'. Hence both the projections of P to $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_{q-1}}$ and to $\mathbb{P}^{n_q}$ are spanned by the projections of W. As in the proof of Proposition 3.3, this implies that P belongs to the span of W. Hence one implication of the claim follows.

Assume $s = s_Z(q-1) = s_Z(q) < z$. Then the points of $\pi_{q-1}(Z)$ are dependent. Take a subset Z_0 of s points of Z such that $\pi_{q-1}(Z_0)$ (thus also Z_0) is independent. Take $P_0 \in Z - Z_0$ and take a subset $Z' \subset Z_0$ such that $\{P_0\} \cup Z'$ is minimally dependent. We claim that $\pi_{q-1}(\{P_0\} \cup Z')$ is also minimally dependent. Indeed the unique representation of $\pi_{q-1}(P_0)$ in terms of the $\pi_{q-1}(P_j)$'s is obtained from the unique representation of P_0 in terms of the P_j's. Thus any non-trivial representation of zero as a combination of coordinates of the $\pi_{q-1}(P_i)$'s has non-zero scalars. Notice that any minimally dependent subset of $\pi_{q-1}(Z)$ arises in this way.

Write $Z' = \{P_1, \ldots, P_w\}$ for some $w \le s < z$ and write $P_i = (\pi_{q-1}(P_i), Q_i)$ for $i = 0, 1, \ldots, w$, where $Q_j \in \mathbb{P}^{n_q}$. After possibly a change of coordinates in $\mathbb{P}^{n_q}$, we may assume that any point Q_j has coordinates of type $[Q_j] = (1, q_j^1, \ldots, q_j^{n_q})$ for some numbers q_j^k.

We have a linear relation

$$[\pi_{q-1}(P_0)] = a_1[\pi_{q-1}(P_1)] + \cdots + a_w[\pi_{q-1}(P_w)]$$

where the a_j's are non-zero scalars. Thus we also have

$$[P_0] = a_1[P_1] + \cdots + a_w[P_w]$$

which implies in the Segre embedding, for any $k = 1, \ldots, n_q$,

$$\begin{aligned} q_0^k[\pi_{q-1}(P_0)] &= q_0^k a_1[\pi_{q-1}(P_1)] + \cdots + q_0^k a_w[\pi_{q-1}(P_w)] \\ &= q_1^k a_1[\pi_{q-1}(P_1)] + \cdots + q_w^k a_w[\pi_{q-1}(P_w)] \end{aligned}$$

and since the points $\pi_{q-1}(P_i)$'s are independent and all the a_j's are non-zero, we get $q_0^k = q_j^k$ for $j = 1, \ldots, w$, i.e.

$$[Q_j] = [Q_0] \qquad \text{for all } j.$$

The claim follows. □

In particular, it follows from the previous proposition that when the points $\pi_{i-1}(Q)$, $Q \in Z$, are minimally dependent, then $s_Z(i-1) = s_Z(i)$ if and only if Z is i-degenerate.

Example 3.6 Consider in $\mathbb{P}^3$ a set W of five points $P_1, \ldots, P_5$, such that P_1, P_2, P_3 sit in a line which misses the line P_4, P_5.

W is not minimally dependent, but contains the minimally dependent subset $W' = \{P_1, P_2, P_3\}$.

If $Z = \{Q_1, \ldots, Q_5\}$ is a subset of (the Segre embedding of) $\mathbb{P}^3 \times \mathbb{P}^1$ such that $i_1(Q_i) = P_i$ for all i, and the span of Z has dimension 3, then Q_1 must sit in the span of $Q_2, \ldots, Q_5$, and these 5 points are independent (as their projections are). Since there is a unique expression of P_1 in terms of the other points, and it is obtained by P_2, P_3 alone, then Q_1 must sit in the line Q_2, Q_3. Since the Segre embedding of $\mathbb{P}^3 \times \mathbb{P}^1$ is cut by quadrics, this is possible only if the line Q_1, Q_2, Q_3 sits in $\mathbb{P}^3 \times \mathbb{P}^1$, which implies that Q_1, Q_2, Q_3 map to the same point in $\mathbb{P}^1$.

Corollary 3.7 *In the previous setting, assume that* $s_Z(i-1) = s_Z(i)$. *For any subset* $Z' \subset Z$ *such that* $\pi_{i-1}(Z')$ *is minimally dependent, then also* $\pi_i(Z')$ *is minimally dependent.*

Proof When $\pi_i(Z')$ is i-degenerate, then the projection on the first $i-1$ factors determine an isomorphism of the span of Z' to the span of $\pi_i(Z')$. □

If we add some conditions on the sets of points, then we can refine the previous statement.

Definition 3.8 We say that a set Z of points in a product of projective spaces is in *Segre uniform position* if the Segre functions of all subsets of the same cardinality are equal.

Corollary 3.9 *Let $Z \subset X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_q}$ be a finite set of z distinct points, in Segre uniform position, and call s_Z the (oriented) Segre function of Z. Fix an index i with $0 < i \leq q$ and assume $s_Z(i-1) < z$.*

Then $s_Z(i-1) = s_Z(i)$ if and only if Z is i-degenerate.

Proof Segre uniform position implies that, with the notation of Proposition 3.5, there exists a subset Z' such that for *all* $P_0 \in Z - Z_0$ the set $\{P_0\} \cup Z'$ is minimally dependent. □

Going one step further, we analyzes what happens if $s_Z(i) = s_Z(i-1) + 1$.

We do that in the case of a product $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_{q-1}} \times \mathbb{P}^1$, i.e. $n_q = 1$. Notice that in this case the product is contained in $\mathbb{P}^M \times \mathbb{P}^1$, where $M + 1 = (n_1 + 1)\ldots(n_{q-1} + 1)$.

Remark 3.10 The Segre embedding Y of $\mathbb{P}^M \times \mathbb{P}^1$ in $\mathbb{P}^{2M+1}$ is a smooth irreducible variety of dimension $M + 1$ and degree $M + 1$.

Fix a fiber $H = \mathbb{P}^M \times \{Q\}$. Let L be a linear space in $\mathbb{P}^{2M+1}$ of dimension $M + 1$ which meets H in one point. Then any irreducible component W of $L \cap Y$ is either a rational normal curve of degree $\leq M + 1$, which maps birationally to the last factor $\mathbb{P}^1$, or it is contained in a fiber $\mathbb{P}^M \times \{Q'\}$.

Indeed If L is general, then $L \cap Y$ is an irreducible rational normal curve of degree $M + 1$. Thus if $\dim(W) = 1$, then W is a component of a specialization of a rational normal curve, hence it is a rational normal curve itself. If $\dim(W) > 1$ and the map to last factor $W \to \mathbb{P}^1$ surjects, then the dimension of any fiber, hence also of the fiber over Q, is at least 1, contradicting the assumption.

Proposition 3.11 *Let $Z \subset X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_{q-1}} \times \mathbb{P}^1$ be a finite set of z distinct points and call s_Z the Segre function of Z. Set $M + 1 = (n_1 + 1) \cdots (n_{q-1} + 1)$ and assume $s_Z(q-1) = M + 1$, i.e. assume that $\pi_{q-1}(Z)$ spans the whole space of the Segre embedding of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_{q-1}}$.*

If $s_Z(q) = s_Z(q-1) + 1 (= M + 2)$, then we can decompose Z in a union

$$Z = Z_0 \cup Z_1 \cup \cdots \cup Z_k,$$

where Z_0 is contained in a rational curve of degree $\leq M + 1$ and the image of each Z_i, $i > 0$, in the projection to the last factor $\mathbb{P}^1$, is a singleton. Moreover if u_i is the dimension of the span of Z_i, then $u_1 + \cdots + u_k + k \leq M + 1$.

Proof Call L the span of Z. The first part of the statement follows immediately by Remark 3.10. For the last equality, notice that the spans of the Z_i's belong to different fibers of the projection of X to $\mathbb{P}^1$, thus the span of Z has dimension at least $u_1 + \cdots + u_k + k - 1$. □

4 Consequences on the Decomposition of Tensors

Definition 4.1 We say that a finite set $Z \subset \mathbb{P}^N$ is in *linear general position* (LGP) if the affine dimension of the span of any subset $Z' \subset Z$ is the minimum between $N+1$ and the cardinality of Z'.

We say that a finite set $Z \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_q}$ is in *linear general position* (LGP) if for any subset $Z' \subset Z$ and for any choice of indexes $p_1, \ldots, p_b \in \{1, \ldots, q\}$, the projection of Z to the Segre embedding of $\mathbb{P}^{n_{p_1}} \times \cdots \times \mathbb{P}^{n_{p_b}}$ is in a linear general position. With this we mean, in particular, that no pairs of points in Z can glue together in any such projection.

It is clear that any subset of a set in LGP is itself in LGP.

Moreover, if $Z \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_q}$ is in LGP, then for any permutation σ, Z is also in LGP as a subset of $\mathbb{P}^{n_{\sigma(1)}} \times \cdots \times \mathbb{P}^{n_{\sigma(q)}}$.

Notice that, for a given set Z, one can check with (maybe long but) trivial computations whether or not Z is in LGP.

Remark 4.2 If the finite set $Z \subset \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_q}$, of cardinality z is in LGP, then the Segre function of Z has maximal rank, in the sense that, setting $M_i + 1 = (n_1+1)\ldots(n_i+1)$, then for all i

$$s_Z(i) = \min\{M_i + 1, z\}.$$

Theorem 4.3 *Let X be the image of the Segre embedding of $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ (k times) in $\mathbb{P}^N$, $N = 2^k - 1$. Let T be a point of $\mathbb{P}^N$ which sits in the span of r points $T_1, \ldots, T_r$ of X and not in any span of $r-1$ points. Assume that $W = \{T_1, \ldots, T_r\}$ is in linear general position.*

If $r + \log_2(r) < k$, then $T_1, \ldots, T_r$ is the unique set of r points of X whose span contains T.

Proof Make induction on r, the case $r = 1$ being obvious.

Assume there exists another set of points $W' = \{T'_1, \ldots, T'_r\}$ whose span contains T. If $T_i = T'_j$ for some indexes, then we can replace T with $T - T_i = T - T'_j$ and get a contradiction by induction. So assume that no T'_j belongs to W.

The set

$$Z = \{T_1, \ldots, T_r, T'_1, \ldots, T'_r\} \subset X,$$

of cardinality $2r$, is linearly dependent, because there is a point in the span of two proper subsets. If we consider the Segre function of Z, then $s_Z(i) \geq s_W(i)$ (by Remark 2.7 i, and Remark 4.2) implies that

$$s_Z(i) \geq \begin{cases} 2^i & \text{for } i \leq \log_2(r) \\ s_Z(i-1) & \text{for } i > \log_2(r). \end{cases}$$

Since $r + \log_2(r) < k$, one cannot have $s_Z(i) > s_Z(i-1)$ for all indexes i with $\log_2(r) < i < k$. Fix an index $b > \log_2(r)$ for which $s_Z(b) = s_Z(b-1)$.

If π is the projection to the first $b-1$ copies of $\mathbb{P}^1$, we know by Proposition 3.5 that any minimally dependent subset of $\pi(Z)$ maps to the same point of the bth copy of $\mathbb{P}^1$. Since W is in LGP, then any minimally dependent subset $\pi(Z')$ of $\pi(Z)$ can contain at most one point of $\pi(W)$.

Assume that for some $j > b$ the projection of Z' to the jth copy of $\mathbb{P}^1$ is not constant. Then after switching the bth and the jth copy of $\mathbb{P}^1$, we get $s_Z(b) > s_Z(b-1)$ (by Proposition 3.5 again).

Thus, after rearranging the copies of $\mathbb{P}^1$, eventually we arrive to a situation in which the minimally dependent set $\pi(Z')$ sits in one fiber of the projection to the remaining copies, from b to k, of $\mathbb{P}^1$'s.

Thus Z' itself is minimally dependent, by Corollary 3.7. After renumbering we may assume $Z' = \{T_1, T'_1, \ldots, T'_j\}$ for some j, and each point of Z' sits in the span of the remaining points. Thus we may replace T'_1 by T_1 in W' and still get that T belongs to the span of W'. Now W and W' have a common point and then, as at the beginning of the proof, we can conclude by induction. □

Corollary 4.4 *Let T be a tensor of type $2 \times 2 \times \cdots \times 2$ (k times) and of rank r. Assume*

$$r + \log_2(r) < k$$

and assume that we have a decomposition $T = T_1 + \cdots + T_r$ with tensors T_i of rank 1, such that the set $\{T_1, \ldots, T_r\}$ is in linear general position, as a subset of $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$.

Then the decomposition is unique.

References

H. Abo, G. Ottaviani and C. Peterson, *Induction for secant varieties of Segre varieties*. Trans. Amer. Math. Soc. 361 (2009), 767–792.

R. Accola, *On Castelnuovo's inequalities for algebraic curves I.* Trans. Amer. Math. Soc. 251 (1979), 357–373.

J. Alexander and A. Hirschowitz, *Polynomial interpolation in several variables.* J. Alg. Geom. 4 (1995), 201–222.

E. Ballico, *On the non-defectivity and non weak defectivity of Segre-Veronese embeddings of products of projective spaces.* Portug. Math. 63 (2006), 101–111.

E. Ballico and A. Bernardi, *Decomposition of homogeneous polynomials with low rank.* Math. Zeit. 271 (2012), 1141–1149.

E. Ballico and A. Bernardi, *Unique decomposition for a polynomial of low rank.* Ann. Polon. Math. 108 (2013a), 219–224.

E. Ballico and L. Chiantini, *A criterion for detecting the identifiability of symmetric tensors of size three*. Diff. Geom. Appl. 30 (2013b), 233–237.

C. Bocci, L. Chiantini and G. Ottaviani, *Refined methods for the identifiability of tensors*. Ann. Mat. Pura Appl. 193 (2014), 1691–1702.

J. Buczyǹski, A. Ginenski and J. M. Landsberg, *Determinantal equations for secant varieties and the Eisenbud-Koh-Stillman conjecture*. J. London Math. Soc. to appear. doi:10.1112/jlms/jds073.

E. Carlini, M. V. Catalisano and L. Chiantini, *Progress on the symmetric Strassen conjecture*. J. Pure Appl. Alg. to appear. doi:10.1016/j.jpaa.2014.10.006.

G. Castelnuovo, *Ricerche di geometra sulle curve algebriche*. Atti R. Accad. Sci. Torino 24 (1889), 346–373.

L. Chiantini and C. Ciliberto, *Weakly defective varieties*. Trans. Amer. Math. Soc. 354 (2002), 151–178.

L. Chiantini and C. Ciliberto, *On the concept of k-secant order of a variety*. J. London Math. Soc. 73 (2006), 436–454.

L. Chiantini and C. Ciliberto, *On the classification of defective threefolds*. Projective Varieties with Unexpected Properties, proceedings of the Siena Conference, C. Ciliberto, A. V. Geramita, B. Harbourne, R. M. Mirò–Roig, K. Ranestad ed. W. de Gruyter (2005), 131–176.

L. Chiantini, G. Ottaviani and N. Vannieuwenhoven, *An algorithm for generic and low-rank specific identifiability of complex tensors*. SIAM J. Matrix Anal. Appl. 35 (2014), 1265–1287.

L. Chiantini, G. Ottaviani (2012), *On generic identifiability of 3-tensors of small rank*. SIAM J. Matrix Anal. Appl vol. 33(0) p 1018–1037.

M. V. Catalisano, A. V. Geramita and A. Gimigliano (2011), *Secant varieties of* $\mathrm{P}^1 \times \cdots \times \mathrm{P}^1$ *(n-times) are NOT Defective for* $n \geq 5$ J. Alg. Geom vol. 20(0) p 295–327.

I. M. Gelfand, M. Kapranov and A. Zelevinsky, *Discriminants, Resultants, and Multidimensional Determinants*. Birkäuser, Zurich (1994).

J. B. Kruskal (1977), *Three-way arrays: rank and uniqueness of trilinear decompositions with applications to arithmetic complexity and statistics* Lin. Alg. Applic. vol.18(0) p 95–138.

J. M. Landsberg and G. Ottaviani (2013), *Equations for secant varieties of Veronese and other varieties* Ann. Mat. Pura. Appl. vol.192(0) p 569–606.

C. Segre (1920), *Sulle corrispondenze quadrilineari fra forme di* 1 *a specie e su alcune loro rappresentazioni*. Ann. Mat. Pura. Appl. vol.29(0) p 105–140.

V. Strassen (1983), *Rank and optimal computation of generic tensors* Lin. Alg. Applic. vol. 52(0) p 645–685.

Cremona Linearizations of Some Classical Varieties

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Abstract

In this paper we present an effective method for linearizing rational varieties of codimension at least two under Cremona transformations, starting from a given parametrization. Using these linearizing Cremonas, we simplify the equations of secant and tangential varieties of some classical examples, including Veronese, Segre and Grassmann varieties. We end the paper by treating the special case of the Segre embedding of the n-fold product of projective spaces, where cumulant Cremonas, arising from algebraic statistics, appear as specific cases of our general construction.

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1 Introduction

Computations in Algebraic Geometry may be very sensitive to the choice of coordinates. Often, by picking the appropriate coordinate system, calculations or expressions can be greatly simplified. Changes of variables with rational functions are classically known as *Cremona transformations* and give a huge flexibility when dealing with systems of polynomial equations. In this paper, we focus on varieties and maps defined over $\mathbb{C}$.

Cremona transformations, one of the most venerable topics in algebraic geometry, were widely studied in the second half of XIX and the first of XX century. They became again fashionable in recent times, after the spectacular developments of birational geometry due to Mori, Kawamata, Kollár, et al. and even more recently by Birkar, Cascini, Hacon, McKernan (see Druel 2007 and references therein). However, despite this great progress, Cremona transformations still reserve a great deal of surprises. The aim of the present, mainly expository, paper is to show how useful they can be in studying some classical geometric invariants of complex projective varieties, linking previous independent work of the last three authors (see Goodman et al. 2006; Mella and Polastri 2009; Zwiernik 2012). For a recent survey on the properties of the group of Cremona transformations, we refer the reader to Cantat (2013).

The work of Mella and Polastri (2009) shows that any rational variety X of codimension at least two in $\mathbb{P}^r$ can be *linearized* by a Cremona transformation: a *linearizing Cremona map* is a transformation that maps X birationally to a linear subspace of $\mathbb{P}^r$. In Sect. 3 we provide a proof of the aforementioned result, close to the original one in spirit, but best suited for effective computations.

If X in $\mathbb{P}^r$ admits a birational linear projection to a linear subspace, a Cremona linearizing map can be directly constructed as a *triangular Cremona transformation* (see Sect. 2.4). In fact, in Sect. 2 we present a systematic approach to Cremona transformations linearizing a rational variety. They turn out to be building blocks for the *cumulant Cremona transformations*, which we present in Sect. 6. In Sect. 4 we discuss the effect of linearizing Cremona transformations on tangential and secant varieties. We devote Sect. 5 to the study of a number of classical examples, including Veronese, Segre and Grassmann varieties, and in these cases we observe an interesting feature of linearizing Cremonas: they tend to simplify also tangential and secant varieties.

The final Sect. 6 is, as we said, devoted to cumulant Cremona transformations which appear already, in a simple case, in Sect. 5. Cumulants arise from algebraic statistics (see Zwiernik 2012) and can be viewed as the choice of preferable coordinates in which varieties coming from algebraic statistics simplify. For instance, Sturmfels and Zwiernik (2013) used cumulants to simplify the equations of the Segre embedding of the n-fold product of $\mathbb{P}^1$ and of its tangent variety. More recent results in the same direction are contained in Manivel and Michałek (2014), Michałek et al. (2014). Cumulants are particular instances of linearizing Cremona transformations. We conclude by indicating how to generalize some combinatorial formulas in Michałek et al. (2014), Sturmfels and Zwiernik (2013), and Zwiernik (2012).

2 Construction of Some Cremona Transformations

In this section we present some recipes for constructing Cremona transformations. We focus on two specific closely-related types: *monoidal extensions* and *triangular Cremona transformations*. All constructions in this paper may be seen as iterated applications of monoidal extensions as in Sect. 2.3 (see also Costa and Simis 2013).

2.1 Basics on Cremona Transformations

A *Cremona transformation* is a birational map

$$\varphi : \mathbb{P}^r \dashrightarrow \mathbb{P}^r, \qquad [x_0, \ldots, x_r] \mapsto [F_0(x_0, \ldots, x_r), \ldots, F_r(x_0, \ldots, x_r)], \tag{1}$$

where $F_i(x_0, \ldots, x_r)$ are coprime homogeneous polynomials of the same degree $\delta > 0$, for $0 \leq i \leq r$. The inverse map is also a Cremona transformation, and it is defined by coprime homogeneous polynomials $G_i(x_0, \ldots, x_r)$ of degree $\delta' > 0$, for $0 \leq i \leq r$. In this case, we say that φ is a (δ, δ')-*Cremona transformation*. The subscheme $\mathrm{Ind}(\varphi) := \{F_i(x_0, \ldots, x_r) = 0\}_{0 \leq i \leq r}$ is the *indeterminacy locus* of φ. Since the composition of φ and its inverse is the identity, we have

$$G_i(F_0(x_0, \ldots, x_r), \ldots, F_r(x_0, \ldots, x_r)) = \Phi \cdot x_i, \quad \text{for} \quad 0 \leq i \leq r$$

where Φ is a homogeneous polynomial of degree $\delta \cdot \delta' - 1$. The hypersurface $\mathrm{Fund}(\varphi) := \{\Phi = 0\}$ is the *fundamental locus* of φ and its support is the *reduced fundamental locus* $\mathrm{Fund}_{\mathrm{red}}(\varphi)$. In general one cannot reconstruct $\mathrm{Fund}(\varphi)$ from $\mathrm{Fund}_{\mathrm{red}}(\varphi)$, except when $\mathrm{Fund}_{\mathrm{red}}(\varphi)$ is irreducible. Indeed, in this case, we recover $\mathrm{Fund}(\varphi)$ from its multiplicity value $(\delta \cdot \delta' - 1)/e$, where e is the degree of $\mathrm{Fund}_{\mathrm{red}}(\varphi)$. By construction, $\mathrm{Ind}(\varphi) \subset \mathrm{Fund}(\varphi)$ and ϕ is one-to-one on the complement of $\mathrm{Fund}_{\mathrm{red}}(\varphi)$.

Often times in this paper, the loci $\mathrm{Fund}_{\mathrm{red}}(\varphi)$ and $\mathrm{Fund}_{\mathrm{red}}(\varphi^{-1})$ induced by the Cremona transformation φ are the same hyperplane. In those cases, we can see φ as a polynomial automorphism $\varphi_a : \mathbb{A}^r \to \mathbb{A}^r$ (often denoted by φ) whose extension to $\mathbb{P}^r$ contains the hyperplane at infinity as its fundamental locus.

2.2 Monoids

Let $X \subset \mathbb{P}^r$ be a hypersurface of degree d where $r \geq 2$. We say that X is a *monoid* with *vertex* $p \in \mathbb{P}^r$ if X is irreducible and p is a point in X of multiplicity exactly $d-1$. Note that a monoid can have more than one vertex. If we choose projective coordinates in such a way that $p = [0, \ldots, 0, 1]$, then the defining equation of X is

$$f(x_0, \ldots, x_r) = f_{d-1}(x_0, \ldots, x_{r-1})x_r + f_d(x_0, \ldots, x_{r-1}) = 0,$$

where f_{d-1} and f_d are homogeneous polynomials of degree $d-1$ and d respectively and f_{d-1} is nonzero. The hypersurface X is irreducible if and only if f_{d-1} and f_d are coprime.

A monoid X is rational. Indeed, the projection of X from its vertex p onto a hyperplane H not passing through p is a birational map $\pi : X \dashrightarrow H \cong \mathbb{P}^{r-1}$. If H has equation $x_r = 0$, then the inverse map $\pi^{-1} : \mathbb{P}^{r-1} \dashrightarrow X$ is given by

$$[x_0, \ldots, x_{r-1}] \mapsto [f_{d-1}(x_0, \ldots, x_{r-1})x_0, \ldots, f_{d-1}(x_0, \ldots, x_{r-1})x_{r-1}, -f_d(x_0, \ldots, x_{r-1})].$$

The map π is called the *stereographic projection* of X from p. Its indeterminacy locus is p. Each line through p contained in X gets contracted to a point under π. The set of all such lines is defined by the equations $\{f_d = f_{d-1} = 0\}$. This is the indeterminacy locus of π^{-1}, whereas the hypersurface of H with equation $\{x_r = f_{d-1} = 0\}$ is contracted to p by the map π^{-1}.

2.3 Monoidal Extensions of Rational Maps

Let $\omega : \mathbb{P}^r \dashrightarrow \mathbb{P}^r$ be a dominant rational map defined, in homogeneous coordinates, as in (1). The homogeneous polynomials $F_0, \ldots, F_r$ have the same degree $\delta > 0$ and are coprime. We construct a *monoidal extension* Ω of ω as follows. First, we embed $\mathbb{P}^r$ in $\mathbb{P}^{r+1}$ as the hyperplane $H = \{x_{r+1} = 0\}$ and we consider the point $p = [0, \ldots, 0, 1] \in \mathbb{P}^{r+1}$. Fix an integer $d \geq \delta$, a nonzero homogeneous polynomial $h(x_0, \ldots, x_r)$ of degree $d - \delta$ and an irreducible monoid of degree d with vertex at p defined by

$$f(x_0, \ldots, x_{r+1}) = f_{d-1}(x_0, \ldots, x_r)x_{r+1} + f_d(x_0, \ldots, x_r) = 0.$$

Then we let $\Omega : \mathbb{P}^{r+1} \dashrightarrow \mathbb{P}^{r+1}$ be defined by

$$[x_0, \ldots, x_{r+1}] \mapsto [h(x_0, \ldots, x_r)F_0(x_0, \ldots, x_r), \ldots, h(x_0, \ldots, x_r)F_r(x_0, \ldots, x_r), f(x_0, \ldots, x_{r+1})].$$

Note that p is an indeterminacy point of Ω. If $\pi : \mathbb{P}^{r+1} \dashrightarrow \mathbb{P}^r$ is the projection from p to H, we have

$$\pi \circ \Omega = \omega \circ \pi. \tag{2}$$

Lemma 2.1 *The map* $\Omega : \mathbb{P}^{r+1} \dashrightarrow \mathbb{P}^{r+1}$ *is dominant and has the same degree as* ω. *Hence* Ω *is a Cremona transformation if and only if* ω *is.*

Proof By definition, the degree of the map Ω coincides with the degree of the induced field extension. Let $y = [y_0, \ldots, y_{r+1}] \in \mathbb{P}^{r+1}$ be a general point and let $y' = [y_0, \ldots, y_r] = \pi(y)$. The rational map Ω may be written as $\Omega(y) = [F_0(y'), \ldots, F_r(y'), (f_{d-1}(y')y_{r+1} + f_d(y'))/h(y')]$. In particular, $\Omega(\mathbb{C}(y_0, \ldots, y_r)) = \omega(\mathbb{C}(y_0, \ldots, y_r)) \subset \mathbb{C}(y_0, \ldots, y_r)$, while $\Omega(y)_{r+1}$ is linear in y_{r+1} over $\Omega(\mathbb{C}(y_0, \ldots, y_r))$. The field extension has degree $[\mathbb{C}(y_0, \ldots, y_{r+1}) : \Omega(\mathbb{C}(y_0, \ldots, y_r))] = [\mathbb{C}(y_0, \ldots, y_r) : \omega(\mathbb{C}(y_0, \ldots, y_r))]$

$$[\mathbb{C}(y_0, \ldots, y_{r+1}) : \Omega(\mathbb{C}(y_0, \ldots, y_r))] = [\mathbb{C}(y_0, \ldots, y_r) : \omega(\mathbb{C}(y_0, \ldots, y_r))]$$

and the lemma follows. □

The indeterminacy locus of Ω (as a scheme) is the union of the cone over the locus of indeterminacy of ω with vertex p intersected with the monoid $\{f = 0\}$ and the codimension two subvariety $\{h = f = 0\}$. The reduced fundamental locus of Ω is the union of the hypersurface $\{h = 0\}$ and the cone over the fundamental locus of ω with vertex p.

2.4 Triangular Cremona Transformations

Triangular Cremona transformations are obtained as iterated applications of monoidal extensions, as we now explain. Consider a rational map $\tau : \mathbb{P}^r \dashrightarrow \mathbb{P}^r$ defined, in affine coordinates over $\{x_0 \neq 0\}$, by formulas of the type

$$(x_1, \ldots, x_r) \to (f_1(x_1), \ldots, f_i(x_1, \ldots, x_i), \ldots f_r(x_1, \ldots, x_r))$$

where each $f_i(x_1, \ldots, x_i) \in K(x_1, \ldots, x_{i-1})[x_i]$ is nonconstant and linear in x_i, for $1 \leq i \leq r$. If $f_i(x_1, \ldots, x_i) \in K[x_1, \ldots, x_i]$ for all $1 \leq i \leq r$, the indeterminacy locus of τ is contained in the *hyperplane at infinity* $\{x_0 = 0\}$. Any such map τ is birational, with inverse of the same type. To find τ^{-1}, we have to solve the system

$$y_i = f_i(x_1, \ldots, x_i), \quad 1 \leq i \leq r$$

in the indeterminates $x_1, \ldots, x_r$. This can be done stepwise as follows. From the linear expression $y_1 = f_1(x_1)$ we find a linear polynomial g_1 such that $x_1 = g_1(y_1)$. Given $i > 1$, assume we know that

$$x_h = g_h(y_1, \ldots, y_h), \quad \text{for} \quad 1 \leq h < i \leq n \tag{3}$$

where $g_h(y_1, \ldots, y_h) \in K(y_1, \ldots, y_{h-1})[y_h]$ is linear in y_h. From $y_i = f_i(x_1, \ldots, x_i)$ we obtain the expression $x_i = \xi(x_1, \ldots, x_{i-1}, y_i)$, where $\xi(x_1, \ldots, x_{i-1}, y_i) \in K(x_1, \ldots, x_{i-1})[y_i]$ is linear in y_i. Substituting x_h from (3), we conclude that $x_i = g_i(y_1, \ldots, y_i)$ with $g_i(y_1, \ldots, y_i) \in K(y_1, \ldots, y_{i-1})[y_i]$ of degree 1 in y_i.

Example 2.2 Fix an integer $n \geq 2$ and use the same notation as above. The following Cremona quadratic transformation ω_n of $\mathbb{P}^{\binom{n+1}{2}}$ is triangular

$$[x_0, \ldots, x_i, \ldots, x_{ij}, \ldots] \to [x_0^2, \ldots, x_0 x_i, \ldots, x_0 x_{ij} - x_i x_j, \ldots], \quad \text{where} \quad 1 \leq i < j \leq n.$$

The inverse is

$$[y_0, \ldots, y_i, \ldots, y_{ij}, \ldots] \to [y_0^2, \ldots, y_0 y_i, \ldots, y_0 y_{ij} + y_i y_j, \ldots]$$

From a geometric viewpoint, ω_n is defined by a linear system $\mathcal{L}$ of quadrics as follows. Consider the coordinate hyperplane $\Pi = \{x_0 = 0\}$. Let S be the linear subspace of Π with equations $\{x_0 = x_1 = \cdots = x_n = 0\}$ and let S' be the complementary subspace in Π with equations $\{x_0 = x_{12} = \cdots = x_{ij} = \cdots = x_{(n-1)n} = 0\}$, where $1 \leq i < j \leq n$. Then $\mathcal{L}$ cuts out the complete linear system of quadric cones on Π that are singular along S and that pass through the n independent points $p_i \in S'$, where p_i is the torus-fixed point with all coordinates 0 but x_i, with $1 \leq i \leq n$. After splitting off Π from $\mathcal{L}$, the residual system consists of all hyperplanes containing S. $\diamondsuit$

3 Cremona Equivalence

An irreducible variety $X \subset \mathbb{P}^r$ is *Cremona linearizable* (CL) if there is a *linearizing Cremona transformation* of $\mathbb{P}^r$ which maps X birationally onto a linear subspace. It is a consequence of Theorem 3.5 (presented in Mella and Polastri 2009) that, if X is rational of dimension $n \leq r - 2$ in $\mathbb{P}^r$, then X is CL. In this section we recall this theorem and present a slightly different proof.

3.1 Monoids and Cremona Transformations

Let $X \subset \mathbb{P}^r$ be a monoid of degree d. Let $p_1, p_2 \in X$ be two vertices, let H_1, H_2 be hyperplanes with $p_i \notin H_i$, and consider the stereographic projections of X from p_i, which is the restriction of the projection $\pi_i : \mathbb{P}^r \dashrightarrow H_i$ from p_i, with $i = 1, 2$. The map

$$\pi_{X,p_1,p_2} := \pi_2 \circ \pi_1^{-1} : H_1 \dashrightarrow H_2$$

is a Cremona transformation. If $p_1 = p_2 = p$, then $\pi_{X,p} := \pi_{X,p,p}$ does not depend on X, being the (linear) *perspective* of H_1 to H_2 with center p. From now on, we restrict to the case when $p_1 \neq p_2$. In this setting, the map π_{X,p_1,p_2} depends on X and is in general nonlinear. In the following we assume that H_1 and H_2 have equations $x_r = 0$ and $x_{r-1} = 0$ respectively, and $p_1 = [0, \ldots, 0, 1], p_2 = [0, \ldots, 0, 1, 0]$. The defining equation of X has the form

$$f_d(x_0, \ldots, x_{r-2}) + x_{r-1}g_{d-1}(x_0, \ldots, x_{r-2}) + x_r h_{d-1}(x_0, \ldots, x_{r-2}) + x_r x_{r-1} f_{d-2}(x_0, \ldots, x_{r-2}) = 0.$$

Then

$$\pi_{X,p_1,p_2}([x_0, \ldots, x_{r-1}]) = [(f_{d-2}x_{r-1} + h_{d-1})x_0, \ldots, (f_{d-2}x_{r-1} + h_{d-1})x_{r-2}, -f_d - x_{r-1}g_{d-1}].$$

Lemma 3.1 *Let $Z \subset \mathbb{P}^r$, with $r \geq 3$, be an irreducible variety of dimension $r - 2$ and let $p \in \mathbb{P}^r$ be such that the projection of Z from p is birational to its image. For $d \gg 0$ there is a monoid of degree d with vertex p, containing Z but not containing the cone $C_p(Z)$ over Z with vertex p.*

Proof Let $V \to \mathbb{P}^r$ be the blow-up of $\mathbb{P}^r$ at p. We denote by E the exceptional divisor and by H the proper transform of a general hyperplane of $\mathbb{P}^r$ and by Z' the proper transform of Z.

Consider $M_d = |dH - (d-1)E| = |(d-1)(H-E) + H|$, i.e. the proper transform on V of the linear system of monoids we are interested in. We have

$$\dim(M_d) = \frac{2d^{r-1}}{(r-1)!} + \frac{(r-1)d^{r-2}}{(r-2)!} + O(d^{r-3}). \tag{4}$$

Since the projection of Z from p is birational, the line bundle $\mathcal{O}_{Z'}(H-E)$ is big and nef. Then by Lazarsfeld (2004, Theorem 1.4.40, vol. I, p. 69) it follows that

$$h^0(Z', \mathcal{O}_{Z'}(d(H-E))) = \frac{\delta}{(r-2)!}d^{r-2} + O(d^{r-3}), \quad \text{for} \quad d \gg 0$$

where $\delta = (H-E)^{r-2}\cdot Z' > 0$ is the degree of the variety obtained under the projection of Z from p, i.e. the degree of the cone $C_p(Z)$. Thus, if M'_d is the sublinear system of M_d of the divisors containing Z', then

$$\begin{aligned}\dim(M'_d) \geq\ & \dim(M_d) - h^0(Z', \mathcal{O}_{Z'}(d(H-E)))\\ = & \frac{2d^{r-1}}{(r-1)!} + \frac{(r-1-\delta)d^{r-2}}{(r-2)!} + O(d^{r-3}).\end{aligned}$$

We let M''_d be the sublinear system of M'_d of the divisors containing the proper transform Y of the cone $C_p(Z)$, which is a hypersurface of degree δ, i.e. $Y \in |\delta(H-E)|$. Hence, $M''_d \subseteq |(d-\delta-1)(H-E)+H|$ so by (4) we have

$$\begin{aligned}\dim(M''_d) \leq & \frac{2(d-\delta)^{r-1}}{(r-1)!} + \frac{(r-1)(d-\delta)^{r-2}}{(r-2)!} + O(d^{r-3})\\ = & \frac{2d^{r-1}}{(r-1)!} + \frac{(r-1-2\delta)d^{r-2}}{(r-2)!} + O(d^{r-3}).\end{aligned}$$

Hence

$$\dim(M'_d) - \dim(M''_d) = \frac{\delta d^{r-2}}{(r-2)!} + O(d^{r-3}) > 0, \quad \text{for} \quad d \gg 0,$$

as we wanted to show. □

Lemma 3.2 *Let $Z \subset \mathbb{P}^r$ be an irreducible variety of positive dimension $n \leq r-3$ and let $p_1, p_2 \in \mathbb{P}^r$ be distinct points such that the projection of Z from the line ℓ joining p_1 and p_2 is birational to its image. For $d \gg 0$ there is a monoid of degree d with vertices p_1 and p_2, containing Z but not containing any of the cones $C_i(Z)$ over Z with vertices p_i, for $i = 1, 2$.*

Proof We start with the following

Claim *It suffices to prove the assertion for $n = r-3$.*

Proof of the Claim Consider the projection of $\mathbb{P}^r$ to $\mathbb{P}^{n+3}$ from a general linear subspace Π of dimension $r-n-4$ and call Z', p'_1, p'_2, ℓ' the projections of Z, p_1, p_2, ℓ respectively. Then Z' is birational to Z and it is still true that the projection of Z' form ℓ' is birational to its image. The dimension of Z' is $n-3$.

Assume the assertion holds for Z', p'_1, p'_2 and let $F' \subset \mathbb{P}^{n+3}$ be a monoid of degree $d \gg 0$ with vertices p'_1, p'_2 containing Z' but not $C_i(Z')$, for $i = 1, 2$. Let $F \subset \mathbb{P}^r$ be the cone over F' with vertex Π. Then F is a monoid with vertices p_1, p_2 containing Z. It does not contain either one of $C_i(Z)$, for $i = 1, 2$, otherwise F' would contain one of the cones $C_i(Z')$, for $i = 1, 2$, contradicting our hypothesis on F'. □

We can thus assume from now on that $n = r-3$. Fix two hyperplanes H_1 and H_2, where $p_1 \notin H_1$ and $p_2 \notin H_2$. Let Z_1 and Z_2 be the projections from p_1 and p_2 to

H_1 and H_2, respectively. We set $p'_{3-i} := \pi_i(p_{3-i})$, for $i = 1, 2$. Our result follows by Lemma 3.1 and the following claim:

Claim *It suffices to prove that for $d \gg 0$, there is a monoid of degree d in H_i with vertex p'_i containing Z_i but not containing the cone $C(Z_i)$ over Z_i with vertex p'_i, for $i = 1, 2$.*

Proof of the Claim Let $F'_i \subset H_i$ be such a monoid, and let F_i be the cone over F'_i with vertex p_{3-i}, for $i = 1, 2$. Then F_i is a monoid with vertex p_i (by construction, we can take any point in the line joining p_i and p_{3-i} as its vertex). In addition, F_i contains Z but does not contain $C_i(Z)$ (same argument as in the proof of the previous Claim). Then the assertion of Lemma 3.1 holds for a general linear combination F of F_1 and F_2. □

Let Z, p_1, p_2 be as in Lemma 3.2. Fix a general monoid $X \supset Z$ with vertices in p_1 and p_2; by Lemma 3.2, X does not contain the cones $C_i(Z)$. Let H_1 and H_2 be hyperplanes such that $p_i \notin H_i$, for $i = 1, 2$. Let Z_i be the projection of Z from p_i to H_i, for $i = 1, 2$. Then the map $\varphi := \pi_{X,p_1,p_2} : H_1 \dashrightarrow H_2$ is birational and Z_1 is not contained in the indeterminacy locus of both φ and φ^{-1}. Thus φ induces a birational transformation $\phi : Z_1 \dashrightarrow Z_2$. For future reference we summarize this construction in the following Proposition.

Proposition 3.3 *In the above setting, the double projection $\varphi : H_1 \dashrightarrow H_2$, (resp. φ^{-1}) is defined at the general point of Z_1 (resp. of Z_2), hence it defines a birational map $\phi : Z_1 \dashrightarrow Z_2$ (resp. $\phi^{-1} : Z_2 \dashrightarrow Z_1$).*

3.2 Cremona Equivalence

Let $X, Y \subset \mathbb{P}^r$ be irreducible, projective varieties. We say that X and Y are *Cremona equivalent* (CE) if there is a Cremona transformation $\omega : \mathbb{P}^r \dashrightarrow \mathbb{P}^r$ such that ω [resp. ω^{-1}] is defined at the general point of X [resp. of Y] and such that ω maps X to Y [accordingly ω^{-1} maps Y to X]. This is an equivalence relation among all irreducible subvarieties of $\mathbb{P}^r$.

The following result is due to Mella and Polastri (2009, Theorem 1). We present an alternative proof, close to the original ideas, but more computational in spirit.

Remark 3.4 We take the opportunity of correcting a mistake in the proof of Mella and Polastri (2009, Theorem 1). In the notation of Mella and Polastri (2009, Theorem 1), let $T_i = \varphi_{\mathcal{H}_i}(X)$, $Y_i = T_i \cap (x_{n+1} = 0)$ and Z the cone over T_i with vertex q_1. Let S be the monoid containing Y_i and W_i the projection of T_i onto S. Then $X_{i+1} := \pi_{q_2}(W_i)$ and not $\pi_{q_2}(Y_i)$ as written in page 92 of the published paper.

Theorem 3.5 *Let $X, Y \subset \mathbb{P}^r$, with $r \geq 3$, be two irreducible varieties of positive dimension $n < r - 1$. Then X, Y are CE if and only if they are birationally equivalent.*

Proof We prove the nontrivial implication.

Claim *We may assume that the projection of Y from any coordinate subspace of dimension m is birational to its image if $r > n+m+1$ and dominant to $\mathbb{P}^{r-m-1}$ if $r \leq n+m+1$.*

Proof of the Claim If we choose the $r+1$ torus-fixed points of $\mathbb{P}^r$ to be generic (which we can do after a generic change of coordinates), then each one of the coordinate subspaces of a given dimension m (spanned by $m+1$ coordinate points) is generic in the corresponding Grassmannian, hence the assertion follows. □

Let Z be a smooth variety and let $\phi : Z \dashrightarrow X$ and $\psi : Z \dashrightarrow Y$ be birational maps. Passing to affine coordinates, we may assume that ϕ and ψ are given by equations

$$x_j = \phi_j(t), \text{and } y_j = \psi_j(t), \quad \text{for} \quad 1 \leq j \leq r,$$

where ϕ_j, ψ_j are rational functions on Z and t varies in a suitable dense open subset of Z.

We prove the theorem by constructing a sequence of birational maps $\varphi_i : Z \dashrightarrow X_i \subset \mathbb{P}^r$, with X_i projective varieties, for $0 \leq i \leq r$, such that:

(a) $\varphi_0 = \phi$ and $\varphi_r = \psi$, thus $X_0 = X$ and $X_r = Y$;
(b) for $0 \leq i \leq r-1$, there is a Cremona transformation $\omega_i : \mathbb{P}^r \dashrightarrow \mathbb{P}^r$, such that ω_i (resp. ω_i^{-1}) is defined at the general point of X_i (resp. of X_{i+1}), and it satisfies $\omega_i(X_i) = X_{i+1}$ (accordingly, $\omega_i^{-1}(X_{i+1}) = X_i$) and $\varphi_{i+1} = \omega_i \circ \varphi_i$.

The construction is done recursively. We assume φ_i is of the form

$$\varphi_i(t) = (\tilde{\phi}_{i,0}(t), \ldots, \tilde{\phi}_{i,r-i}(t), \psi_{r-i+1}(t), \ldots, \psi_r(t)), \quad \text{for} \quad t \in Z \quad \text{and} \quad 0 \leq i \leq r-1,$$

where the $\tilde{\phi}_{i,j}$'s are suitable rational functions on Z. For $i = 0$, the starting case, we fix $\tilde{\phi}_{0,j} = \phi_j$ for all $0 \leq j \leq r$. Thus, requirement (a) is satisfied.

Assume $0 \leq i \leq r-1$. In order to perform the step from i to $i+1$, we consider the map

$$g_i : Z \dashrightarrow \mathbb{A}^{r+1}, \qquad g_i(t) = (\tilde{\phi}_{i,0}(t), \ldots, \tilde{\phi}_{i,r-i}(t), \psi_{r-i}(t), \psi_{r-i+1}(t), \ldots, \psi_r(t)).$$

Let Z_i be the closure of the image of g_i, and $\pi_j : \mathbb{A}^{r+1} \dashrightarrow \mathbb{A}^r$ the projection to the coordinate hyperplane $\{x_j = 0\}$ from the point at infinity of the axis x_j, for all $1 \leq j \leq r+1$. We have $\varphi_i = \pi_{r-i+1} \circ g_i$, and since φ_i is birational onto its image, the same holds for g_i.

Claim *The projection of Z_i from a general point of the space at infinity of the affine linear space $\Pi_i := \{x_{r-i+1} = \cdots = x_{r+1} = 0\}$ is birational to its image.*

Proof of the Claim The variety Z_i is not a hypersurface. Then, by Calabri and Ciliberto (2001, Theorem 1), the locus of points from which the projection is not birational has dimension strictly bounded by the dimension of Z_i. We may therefore assume that $\dim(\Pi_i) = r - i - 1 < n$. On the other hand the map ψ is birational, therefore we may as well assume that $i < r - 1$. So it remains to prove the result in the range $0 < r - i - 1 < n$.

The projection of Z_i from Π_i is the closure of the image of the map

$$h_i : Z \dashrightarrow \mathbb{A}^{i+1}, \qquad h_i(t) = (\psi_{r-i}(t), \psi_{r-i+1}(t), \ldots, \psi_r(t)).$$

By the previous claim applied to Z, either h_i is birational to its image (if $i \geq n$) or h_i is dominant. In the former case the projection from a general point of Π_i is also birational, so the assertion follows. In the latter case, the cone over Z_i with vertex Π_i is the whole $\mathbb{P}^r$, and the assertion follows from Calabri and Ciliberto (2001, Theorem 1).

By the Claim, we can make a general change of the first $r - i$ coordinates of g_i so that $\varphi_{i+1} := \pi_{r-i+i} \circ g_i$ is birational to its image X_{i+1}. Finally, iterated applications of Proposition 3.3 show that also requirement (b) is satisfied, thus ending the proof. □

3.3 Linearizing Cremona

Theorem 3.5 ensures that any variety X of dimension $n \leq r - 2$ in $\mathbb{P}^r$ is CE to a hypersurface in a $\mathbb{P}^{n+1} \subset \mathbb{P}^r$. If, in addition, X is rational, then it is CL, e.g. it is CE to the subspace $\{x_{n+1} = \cdots = x_r = 0\}$. In particular, suppose that there is a linear subspace Π of dimension $r - n - 1$ of $\mathbb{P}^r$ such that the projection from Π induces a birational map $\pi : X \dashrightarrow \mathbb{P}^n$. Equivalently, X admits an affine parametrization of the form

$$x_i = t_i, \quad \text{for} \quad 1 \leq i \leq n, \qquad x_j = f_j(t_1, \ldots, t_n), \quad \text{for} \quad n+1 \leq j \leq r, \tag{5}$$

where the f_i 's are rational functions of $t_1, \ldots, t_n$. For instance, smooth toric varieties and Grassmannians enjoy this property (see Sect. 5).

Using (5), we define a Cremona map $\phi : \mathbb{P}^r \dashrightarrow \mathbb{P}^r$ in affine coordinates as

$$\phi(x_1, \ldots, x_r) = (x_1, \ldots, x_n, x_{n+1} - f_{n+1}(x_1, \ldots, x_n), \ldots, x_r - f_r(x_1, \ldots, x_n)). \tag{6}$$

The map ϕ gives a birational equivalence between X and the subspace $\{x_{n+1} = \cdots = x_r = 0\}$, hence ϕ linearizes X. The above construction can be slightly modified to make it more general. Fix a collection of rational functions $g_i(x_1, \ldots, x_{i-1})$ and $h_i(x_1, \ldots, x_{i-1})$, with $n+1 \leq i \leq r$, with all the $h_i(x_1, \ldots, x_{i-1})$'s nonzero. We replace the i- th coordinate of ϕ with the expression

$$\phi_i(x_1, \ldots, x_r) = h_i(x_1, \ldots, x_{i-1})(x_i - f_i(x_1, \ldots, x_n)) + g_i(x_1, \ldots, x_{i-1}),$$
$$i = n+1, \ldots, r.$$

The following is clear:

Lemma 3.6 *The image $\phi(X)$ is the linear subspace $\{x_{n+1} = \cdots = x_r = 0\}$ if and only if the functions g_i vanish on X for $n+1 \leq i \leq r$.*

4 Secant and Tangential Varieties

In this section, we focus on Cremona transformations of secant and tangential varieties. Similar techniques can be applied to osculating varieties, although we will not do this here.

Definition 4.1 Let $X \subset \mathbb{P}^r$ be a variety of dimension n. The *k-secant variety* $\mathrm{Sec}_k(X)$ of X (simply $\mathrm{Sec}(X)$ if $k = 1$) is the Zariski closure of the union of all $(k+1)$-secant linear spaces of dimension k to X, i.e. those containing $k+1$ linearly independent points of X. The *k-defect* of X is $\min\{r, n(k+1)+k\} - \dim(\mathrm{Sec}_k(X))$ (which is always nonnegative), and X is *k-defective* if the k-defect is positive.

The k-secant variety of X has expected dimension $nk+n+k$. It is parametrized as

$$\psi : \mathrm{Sym}^{k+1}(X) \times \mathbb{P}^k \dashrightarrow \mathrm{Sec}_k(X) \subset \mathbb{P}^r, \quad ([p^{(0)}, \ldots, p^{(k)}], [s_0, \ldots, s_k]) \mapsto \sum_{j=0}^{k} s_j p^{(j)}. \tag{7}$$

Assume that there is a codimension $n+1$ linear subspace Π such that the projection from Π induces a birational map $\pi : X \dashrightarrow \mathbb{P}^n$. From Sect. 3, we know that X can be parametrized as in (5). Then, we can combine the maps ψ and π to simplify the parametrization of $\mathrm{Sec}_k(X)$, as we now show.

Pick affine variables $s_1, \ldots, s_k$ and set $s_0 := 1 - \sum_{j=1}^{k} s_j$ in (7). Consider $k+1$ vectors of unknowns

$$\mathbf{t}_i = (t_{i1}, \ldots, t_{in}) \quad \text{for} \quad 0 \leq i \leq k.$$

Then, $\mathrm{Sec}_k(X)$ is parametrized as follows

$$x_i = \begin{cases} s_0 t_{0i} + s_1 t_{1i} + \cdots + s_k t_{ki} & \text{for} \quad 1 \leq i \leq n, \\ s_0 f_i(\mathbf{t}_0) + s_1 f_i(\mathbf{t}_1) + \cdots + s_k f_i(\mathbf{t}_k) & \text{for} \quad n+1 \leq i \leq r. \end{cases}$$

We let ϕ be the Cremona transformation from (6), that linearizes X. Applying ϕ to $\mathrm{Sec}_k(X)$ gives

$$x_i = \begin{cases} s_0t_{0i} + s_1t_{1i} + \cdots + s_kt_{ki} & \text{for} \quad 1 \le i \le n, \\ s_0f_i(\mathbf{t}_0) + s_1f_i(\mathbf{t}_1) + \cdots + s_kf_i(\mathbf{t}_k) - f_i(s_0\mathbf{t}_0 + s_1\mathbf{t}_1 + \cdots + s_k\mathbf{t}_k) & \text{for} \quad n+1 \le i \le r. \end{cases} \tag{8}$$

This change of coordinates can be useful for computing geometric invariants of X, such as its k-defect. The next example, illustrates this situation.

Example 4.2 Suppose that the f_i in (5) are quadratic polynomials. In this case, X is a projection of the Veronese variety, hence it is 1-defective. We write the homogeneous decomposition of f_i

$$f_i = f_{i0} + f_{i1} + f_{i2}, \qquad \text{for} \quad n+1 \le i \le r,$$

where f_{ij} is the homogeneous component of f_i of degree j. Let Φ_i be the bilinear form associated to f_{i2}. Then, the parametrization (8) yields the expression

$$x_i = \begin{cases} s_0t_{0i} + s_1t_{1i} + \cdots + s_kt_{ki}, & \text{for} \quad 1 \le i \le n, \\ \sum_{j=0}^{k} s_j(1-s_j)f_{i2}(\mathbf{t}_j) - 2 \sum_{0 \le u < v \le k} s_us_v\Phi_i(\mathbf{t}_u, \mathbf{t}_v), & \text{for} \quad n+1 \le i \le r. \end{cases}$$

Suppose that $k = 1$. Then

$$x_i = s_0(1-s_0)f_{i2}(\mathbf{t}_0) + s_1(1-s_1)f_{i2}(\mathbf{t}_1) - 2s_0s_1\Phi_i(\mathbf{t}_0, \mathbf{t}_1), \quad \text{for} \quad n+1 \le i \le r.$$

Since $s_0 = 1 - s_1$, then $s_0(1-s_0) = s_1(1-s_1) = s_0s_1$ and we obtain

$$x_i = s_0s_1f_{i2}(\mathbf{t}_1 - \mathbf{t}_0) \qquad \text{for} \quad n+1 \le i \le r.$$

Replacing $\mathbf{t}_0 - \mathbf{t}_1$ with $\mathbf{u} := (u_1, \ldots, u_n)$ and setting $\mathbf{t}_0 =: \mathbf{t} = (t_1, \ldots, t_n)$ and $s_1 =: s$ yields

$$x_i = \begin{pmatrix} t_i + su_i, & \text{for} \quad 1 \le i \le n, \\ s(1-s)f_{i2}(\mathbf{u}), & \text{for} \quad n+1 \le i \le r. \end{pmatrix}$$

The image of a general secant line is a conic with two points in the linear image of the variety X. The dimension of the secant variety can be deduced from the rank of the Jacobian of this parametrization. □

Next we discuss the interplay between tangential varieties and Cremona transformations.

Definition 4.3 Let $X \subset \mathbb{P}^r$ be a variety. The *tangential variety* $T(X)$ of X is the Zariski closure of the union of all tangent spaces to X at smooth points of X.

Assume that X has dimension n. The tangential variety has expected dimension $2n$. If X is (locally) parametrized by a map

$$\mathbf{t} = (t_1, \ldots, t_n) \in U \mapsto [x_0(\mathbf{t}), \ldots, x_r(\mathbf{t})] \in X,$$

where $U \subset \mathbb{C}^n$ is a suitable nonempty open subset, then $T(X)$ is represented by

$$\tau : U \times \mathbb{C}^n \dashrightarrow T(X),$$
$$(\mathbf{t}, \mathbf{s}) = (t_1, \ldots, t_n, s_1, \ldots, s_n) \mapsto [x_0(\mathbf{t}) + \sum_{j=1}^{n} s_j \frac{\partial x_0}{\partial t_j}(\mathbf{t}), \ldots, x_r(\mathbf{t}) + \sum_{j=1}^{n} s_j \frac{\partial x_r}{\partial t_j}(\mathbf{t})].$$

Assume again that X is described as in (5). Then, the parametric equations of $T(X)$ have a simplified expression

$$x_i = \begin{cases} t_i + s_i, & \text{for} \quad 1 \le i \le n, \\ f_i(\mathbf{t}) + \sum_{j=1}^{n} s_j \frac{\partial f_i}{\partial t_j}(\mathbf{t}), & \text{for} \quad n+1 \le i \le r. \end{cases}$$

Under the linearizing Cremona transformation ϕ from Sect. 3, the variety $T(X)$ has image

$$x_i = \begin{cases} t_i + s_i, & \text{for} \quad 1 \le i \le n, \\ f_i(\mathbf{t}) - f_i(\mathbf{t} + \mathbf{s}) + \sum_{j=1}^{n} s_j \frac{\partial f_i}{\partial t_j}(\mathbf{t}), & \text{for} \quad n+1 \le i \le r. \end{cases} \tag{9}$$

Example 4.4 Assume the f_i's in (5) are homogeneous quadratic polynomials. Then (9) becomes

$$x_i = \begin{cases} t_i + s_i, & \text{for} \quad 1 \le i \le n, \\ -f_i(\mathbf{s}), & \text{for} \quad n+1 \le i \le r. \end{cases} \tag{10}$$

Formula (10) describes a cone with vertex the space at infinity of the n-dimensional linear space $\{x_{n+1} = \cdots = x_r = 0\}$, over the variety parametrically represented by the last $n - r$ coordinates of (10)

$$x_i = -f_i(\mathbf{s}), \qquad \text{for} \quad n+1 \le i \le r.$$

□

In Sect. 5 we will see how the equations of secant and tangential varieties simplify in classical defective cases, as predicted by the above example. If the parametrization involves forms of degree higher than 2, the tangent variety is in general no longer transformed to a cone. In Sect. 6 we will see alternative linearizing Cremonas that work better for certain varieties. For instance, for Segre varieties cumulant Cremonas enable us to write the tangential variety in the form (10) even though the parametrizing polynomials are not quadratic.

5 Cremona Linearization of Some Classical Varieties

Segre, Veronese and Grassmannian varieties and their secants play a key role in the study of determinantal varieties. Here we describe some triangular Cremona transformations that linearize these varieties, and we compute the image of their secant varieties under these transformation. Similar considerations can be applied to *Spinor varieties* (see Angelini 2011 for a parametrization of these varieties), and to *Lagrangian Grassmannians* $LG(n, 2n)$, etc., on which we do not dwell here.

5.1 Segre Varieties

The *Segre variety* $Seg(r_1, \ldots, r_k)$ is the image of $\mathbb{P}^{r_1} \times \cdots \times \mathbb{P}^{r_k}$ under the Segre embedding in $\mathbb{P}^r$, with $r + 1 = \prod_{i=1}^{k}(r_i + 1)$ (we may assume $r_1 \geq r_2 \geq \cdots \geq r_k \geq 1$). Sometimes we may use the exponential notation $\mathrm{Seg}(m_1^{h_1}, \ldots, m_k^{h_k})$ if m_i is repeated h_i times, for $1 \leq i \leq k$.

In this section, we find Cremona linearizations for $Seg(m, n)$ and we show how they simplify the equations for their secant varieties. In Sect. 6 we will extend this to higher Segre varieties.

We interpret $\mathbb{P}^{mn+m+n}$ as the space of nonzero $(m+1) \times (n+1)$ matrices modulo multiplication by a nonzero scalar, so we have coordinates $[x_{ij}]_{0 \leq i \leq n, 0 \leq j \leq m}$ in $\mathbb{P}^{mn+m+n}$. Then, $\mathrm{Seg}(m, n)$ is defined by the rank condition

$$\mathrm{rk}(x_{ij})_{0 \leq i \leq n, 0 \leq j \leq m} = 1.$$

This condition amounts to equate to zero all 2×2 minors of the matrix $x = (x_{ij})_{0 \leq i \leq n, 0 \leq j \leq m}$. We pass to affine coordinates by setting $x_{00} = 1$, and we let

$$x = \begin{pmatrix} 1 & x_{01} & x_{02} & \cdots & x_{0n} \\ x_{10} & x_{11} & x_{12} & \cdots & x_{1n} \\ \vdots & & & & \vdots \\ x_{m0} & x_{m1} & x_{m2} & \cdots & x_{mn} \end{pmatrix}$$

be the corresponding matrix. Then the affine equations of $\mathrm{Seg}(m, n)$ are $\{x_{ij} - x_{i0}x_{0j} = 0\}_{1 \leq i \leq n, 1 \leq j \leq m}$. This shows that $\mathrm{Seg}(m, n)$ has parametric equations of type (5) with parameters x_{i0}, x_{0j}, for $1 \leq i \leq n, 1 \leq j \leq m$.

As in Sect. 3.3 a linearizing affine Cremona has equations (in vector form)

$$(y_{ij})_{0 \leq i \leq m, 0 \leq j \leq n, (i,j) \neq (0,0)} = (x_{i0}, x_{0j}, x_{ij} - x_{i0}x_{0j})_{1 \leq i \leq n, 1 \leq j \leq m}, \tag{11}$$

which is of type (2,2) and in homogeneous coordinates reads

$$[\mathbf{y}] = [y_{ij}]_{0 \leq i \leq m, 0 \leq j \leq n)} = [x_{00}^2, x_{00}x_{i0}, x_{00}x_{0j}, x_{00}x_{ij} - x_{i0}x_{0j}]_{1 \leq i \leq n, 1 \leq j \leq m}.$$

The indeterminacy locus has equations $\{x_{00} = x_{i0}x_{0j} = 0\}_{1 \le i \le n, 1 \le j \le m}$ and the reduced fundamental locus is $\{x_{00} = 0\}$.

To see the image of the secant varieties, we perform column operations on x and use (11) to see that

$$\operatorname{rank}(x) = \operatorname{rank}\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ y_{10} & y_{11} & y_{12} & \cdots & y_{1n} \\ \vdots & & & & \vdots \\ y_{m0} & y_{m1} & y_{m2} & \cdots & y_{mn} \end{pmatrix} = 1 + \operatorname{rank}\begin{pmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ \vdots & & & \vdots \\ y_{m1} & y_{m2} & \cdots & y_{mn} \end{pmatrix}.$$

Therefore the k -secant variety to $\mathrm{Seg}(m,n)$ is mapped to the cone over the $(k-1)$-secant variety of $\mathrm{Seg}(m-1,n-1)$ with vertex along the linear image of $\mathrm{Seg}(m,n)$.

Example 5.1 The (first) secant and tangent variety to $\mathrm{Seg}(2,2)$ is the cubic hypersurface defined by the 3×3-determinant

$$\det(\mathbf{x}) = x_{00}(x_{11}x_{22} - x_{12}x_{21}) - x_{01}(x_{10}x_{22} - x_{20}x_{12}) + x_{02}(x_{10}x_{21} - x_{20}x_{11}) = 0.$$

In the new coordinates this hypersurface has the simpler binomial equation $y_{11}y_{22} - y_{12}y_{21} = 0$. ◊

5.2 Projectivized Tangent Bundles

The projectivized tangent bundle TP^n over $\mathbb{P}^n$ is embedded in $\mathrm{Seg}(n,n)$ as the traceless nonzero $(n+1) \times (n+1)$-matrices modulo multiplication by nonzero scalar, i.e. as the hyperplane section $\mathrm{tr}(\mathbf{x}) = 0$ of $\mathrm{Seg}(n,n)$ in $\mathbb{P}^{n^2+2n}$. On the affine chart $x_{00} \ne 0$, we view TP^n as the set of rank 1 matrices of the form

$$x = \begin{pmatrix} 1 & x_{01} & x_{02} & \cdots & x_{0n} \\ x_{10} & x_{11} & x_{12} & \cdots & x_{1n} \\ \vdots & & & & \vdots \\ x_{n0} & x_{n1} & x_{n2} & \cdots & -x_{11} - \cdots - x_{n-1,n-1} - 1 \end{pmatrix}.$$

We parametrize TP^n with the $2n-1$ coordinates $x_{0i} \ne 0$, with $1 \le i \le n$, and x_{ii}, with $1 \le i \le n-1$. The parametric equations for the remaining coordinates are

$$\begin{cases} x_{i0} = \frac{x_{ii}}{x_{0i}} & \text{for } 1 \le i \le n-1, \\ x_{n0} = -\frac{1 + x_{11} + \cdots + x_{n-1,n-1}}{x_{0n}}, & \\ x_{ij} = \frac{x_{ii}x_{0j}}{x_{0i}} = x_{i0}x_{0j} & \text{for } 1 \le i < j \le n. \end{cases}$$

According to Sect. 3.3 we have a linearizing Cremona map $\phi : \mathbb{P}^{n^2+2n-1} \dashrightarrow \mathbb{P}^{n^2+2n-1}$ given in affine coordinates by

$$\begin{array}{lll} y_{0i} = x_{0i} & \text{if} & 0 \leq i \leq n \\ y_{ii} = x_{ii} & \text{if} & 1 \leq i \leq n-1 \\ y_{i0} = x_{ii} - x_{i0}x_{0i} & \text{for} & 1 \leq i \leq n-1 \\ y_{n0} = -(1 + x_{11} + \cdots + x_{n-1,n-1} + x_{n0}x_{0n}) & & \\ y_{ij} = x_{ij} - x_{i0}x_{0j} & \text{for} & 1 \leq i < j \leq n. \end{array} \tag{12}$$

Performing row operations on x and using (12), we see that x has rank k if and only if

$$y' = \begin{pmatrix} y_{10} & y_{12} & y_{13} & \cdots & y_{1n} \\ y_{21} & y_{20} & y_{23} & \cdots & y_{2n} \\ \vdots & & & & \vdots \\ y_{n1} & y_{n2} & y_{n3} & \cdots & y_{n0} \end{pmatrix}$$

has rank $k-1$. This shows that the k-th secant variety of TP^n is mapped to a cone over the $(k-1)$-st secant variety of $\mathrm{Seg}(n-1, n-1)$ with vertex along the linear image of TP^n.

Example 5.2 The first secant variety of TP^2 coincides with the tangent variety and it is the cubic hypersurface defined in $\mathbb{P}^7$, with coordinates $[x_{ij}]_{0 \leq i \leq j \leq 2,(i,j) \neq (2,2)}$, by the equation

$$\begin{aligned} \det(x) = x_{00}^2 x_{11} + x_{00}(x_{11}^2 + x_{12}x_{21} - x_{01}x_{10}) - x_{01}(x_{10}x_{11} + x_{20}x_{12}) \\ - x_{02}(x_{10}x_{21} - x_{20}x_{11}) = 0. \end{aligned}$$

In the new coordinates it has the simpler equation $y_{12}y_{21} = y_{10}y_{20}$, which defines the cone over $\mathrm{Seg}(1,1)$ with vertex along the subspace $\{y_{12} = y_{21} = y_{10} = y_{20} = 0\}$, the linear image of TP^2. ◊

5.3 Veronese Varieties

Consider the 2-*Veronese variety* $V_{2,n}$ of quadrics in $\mathbb{P}^n$ embedded in $\mathbb{P}^{\frac{n(n+3)}{2}}$ with coordinates $[x_{ij}]_{0 \leq i \leq j \leq n}$. The following map ϕ is a linearizing affine (2, 2)-Cremona transformation for $V_{2,n}$ defined on $\{x_{00} \neq 0\}$

$$(x_{ij})_{0 \leq i \leq j \leq n} \mapsto (y_{ij})_{0 \leq i \leq j \leq n} = (x_{01}, \ldots, x_{0n}, x_{ij} - x_{0i}x_{0j})_{1 \leq i \leq j \leq n}.$$

Its reduced fundamental locus is $\{x_{00} = 0\}$ and the indeterminacy locus is $\{x_{00} = \cdots = x_{0n} = 0\}$.

We interpret $V_{2,n}$ as the set of rank 1 symmetric matrices $x = (x_{ij})_{0 \le i,j \le n}$ with $x_{ji} = x_{ij}$ if $j < i$. The $(k-1)$-secant variety to $V_{2,n}$ is defined by the $k \times k$-minors of x. On $\{x_{00} \neq 0\}$ the rank of x coincides with the rank of

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ x_{01} & x_{11} - x_{01}^2 & x_{12} - x_{01}x_{02} & \dots & x_{1n} - x_{01}x_{0n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{0n} & x_{1n} - x_{01}x_{0n} & x_{2n} - x_{02}x_{0n} & \dots & x_{nn} - x_{0n}^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ x_{01} & y_{11} & y_{12} & \dots & y_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{0n} & y_{1n} & y_{2n} & \dots & y_{nn} \end{pmatrix},$$

So the $(k-1)$-secant variety to $V_{2,n}$ is mapped by ϕ to the cone over the $(k-2)$-secant variety of $V_{2,n-1}$ with vertex along the linear image of $V_{2,n}$ in $\mathbb{P}^{\frac{n(n+3)}{2}}$.

Example 5.3 The secant variety $\mathrm{Sec}(V_{2,2})$ of the Veronese surface in $\mathbb{P}^5$ is mapped to the cone over the conic $V_{2,1} = \{y_{11}y_{22} - y_{12}^2 = 0\} \subset \mathbb{P}^2 = \{y_{00} = y_{10} = y_{20} = 0\}$ with vertex $\mathbb{P}^2 = \{y_{11} = y_{12} = y_{22} = 0\}$. ♢

In general, the *d-Veronese variety* $V_{d,n}$ of $\mathbb{P}^n$ is embedded in $\mathbb{P}^{\binom{n+d}{n}-1}$ with coordinates $[x_{i_0 \dots i_n}]_{i_0 + \dots + i_n = d}$, with $i_j \ge 0$ for $0 \le j \le n$. Its projection from the linear space $\{x_{i_0 \dots i_n} = 0\}_{i_0 \ge d-1}$ to the n-space $\{x_{i_0 \dots i_n} = 0\}_{i_0 < d-1}$, is birational. Accordingly, we can find a Cremona linearizing map. We will treat the curve case in Sect. 5.6 but we will not dwell on the higher dimensional and higher degree cases.

5.4 Grassmannians of Lines

In this section we present Cremona linearizations of Grassmannians of lines. Analogous linearizations exist for higher Grassmannians, an example of which we treat in Sect. 5.7.

Let V be a complex vector space of dimension n. We can identify V with $\mathbb{C}^n$, once we fix a basis $(\mathbf{e}_0, \dots, \mathbf{e}_{n-1})$ of V. The *Plücker embedding* maps the Grassmannian $G(2,n)$ of 2-dimensional vector subspaces (i.e. 2-*planes*) of V into $\mathbb{P}^{\frac{n(n-1)}{2}-1} = \mathbb{P}(\wedge^2 V)$, which we identify with the projective space associated to the vector space of antisymmetric matrices of order n, thus the coordinates are $[x_{ij}]_{0 \le i < j \le n-1}$.

Two vectors

$$\xi_0 = (\xi_{00}, \ldots, \xi_{0,n-1}),\ \ \xi_1 = (\xi_{10}, \ldots, \xi_{1,n-1})$$

in V that span a 2-plane W form the rows of a $2 \times n$-matrix x, whose minors are independent on the chosen points, up to a nonzero common factor. The *Plücker point* associated to W is $[x_{ij}]_{0 \le i < j \le n-1}$, where x_{ij} denotes the minor of X obtained by choosing the ith and jth columns.

The *Plücker ideal* $I_{2,n}$ is the homogeneous ideal of $G(2,n)$ in its Plücker embedding. This ideal is prime and it is generated by quadrics. More precisely, $I_{2,n}$ is generated by the $\binom{n}{4}$ *three terms Pücker relations*

$$x_{ij}x_{kl} - x_{ik}x_{jl} + x_{il}x_{jk} \qquad \text{for } 0 \le i<j<k<l \le n-1. \tag{13}$$

Using (13), in the open affine $\{x_{01} \neq 0\}$, we have parametric equations for $G(2,n)$: the parameters are the $2n-4$ coordinates x_{ij} with $i = 0, 1$, and the equations for the remaining coordinates are

$$x_{ij} = x_{0i}x_{1j} - x_{0j}x_{1i}, \quad \text{for} \quad 2 \le i<j \le n-1.$$

Hence $G(2,n)$ is rational, and a birational map $G(2,n) \dashrightarrow \mathbb{P}^{2n-2}$ is given by projecting $G(2,n)$ from the linear span $\mathbb{P}^{\frac{n(n-5)}{2}+2}$ of $G(2,n-2)$ viewed inside $G(2,n)$ as the Grassmannian of 2-planes in $V' = \langle \mathbf{e}_2, \ldots, \mathbf{e}_{n-1} \rangle \subset V$.

According to Sect. 3.3, we have a triangular (2, 2)-Cremona linearization $\varphi : \mathbb{P}^{\frac{n(n+1)}{2}-1} \dashrightarrow \mathbb{P}^{\frac{n(n+1)}{2}-1}$ of $G(2,n)$, given in affine coordinates by

$$y_{ij} = \begin{cases} x_{ij} & \text{if} \quad i = 0, 1,\ 2 \le j \le n-1, \\ x_{ij} - x_{0i}x_{1j} + x_{0j}x_{1i} & \text{if} \quad 2 \le i<j \le n-1. \end{cases}$$

The reduced fundamental locus is $\{x_{01} = 0\}$, and the indeterminacy locus is the union of the two linear spaces $\{x_{01} = x_{02} = \cdots = x_{0(n-1)} = 0\}$ and $\{x_{01} = x_{12} = \cdots = x_{1(n-1)} = 0\}$.

On the complement of $\{x_{01} = 0\}$, the Grassmannian $G(2,n)$ is the set of rank 2 matrices of the form

$$x = \begin{pmatrix} 0 & 1 & x_{02} & x_{03} & \cdots & x_{0n} \\ -1 & 0 & x_{12} & x_{13} & \cdots & x_{1n} \\ -x_{02} & -x_{12} & 0 & \ddots & \vdots & \vdots \\ -x_{03} & -x_{13} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & x_{n-2,n-1} \\ -x_{0,n-1} & -x_{1,n-1} & -x_{2,n-1} & \vdots & -x_{n-2,n-1} & 0 \end{pmatrix}.$$

Performing suitable column operations on x and using the y-coordinates, we see that the rank of x is 2 plus the rank of the matrix

$$\begin{pmatrix} 0 & y_{23} & \cdots & \cdots & y_{2,n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & 0 & y_{n-2,n-1} \\ -y_{2,n-1} & \cdots & \cdots & -y_{n-2,n-1} & 0 \end{pmatrix}.$$

Since $\mathrm{Sec}_k(G(2,n))$ is the set of antisymmetric matrices of rank $2k+2$, we see that $\mathrm{Sec}_k(G(2,n))$ is mapped by φ to the cone over $\mathrm{Sec}_{k-1}(G(2,n-2))$ with vertex along the linear image of $G(2,n)$.

Example 5.4 The first secant and tangent variety of $G(2,6)$ coincide and are defined by the Pfaffian cubic polynomial

$$\begin{aligned} &x_{01}(x_{23}x_{45}-x_{24}x_{35}+x_{25}x_{34})-x_{02}(x_{13}x_{45}-x_{14}x_{35}+x_{15}x_{34})+x_{03}(x_{12}x_{45}-x_{14}x_{25}+x_{15}x_{24}) \\ &-\ x_{04}(x_{12}x_{35}-x_{13}x_{25}+x_{15}x_{23})+x_{05}(x_{14}x_{23}-x_{13}x_{24}+x_{12}x_{34})=0. \end{aligned}$$

In the y-coordinates, this hypersurface has a much simpler equation, namely the Plücker equation of $G(2,4)$

$$y_{23}y_{45}-y_{24}y_{35}+y_{25}y_{34}=0.$$

◊

Example 5.5 A different Cremona transformation that linearizes $G(2,n)$ was considered in Goodman et al. (2006), namely

$$y_{0i}\mapsto\frac{1}{x_{0i}}\quad i=1,\ldots,n-1,\qquad y_{ij}\mapsto\frac{x_{ij}}{x_{0i}x_{0j}}\quad 1\le i<j\le n-1.$$

It maps $G(2,n)$ to the linear space defined by

$$y_{ij}-y_{ik}+y_{jk}=0\qquad 1\le i<j<k\le n-1.$$

This transformation is studied in Goodman et al. (2006) to compare various notions of convexity for lines. ◊

5.5 Severi Varieties

The Veronese surface $V_{2,2}$, the Segre variety $\mathrm{Seg}(2,2)$ and the Grassmannian $G(2,6)$ mentioned in Examples 5.1, 5.3 and 5.4 are *Severi varieties*, i.e. smooth 1-defective varieties of dimension n in $\mathbb{P}^{\frac{3}{2}n+2}$ (see Zak 1993). There is one more Severi variety: the so-called *Cartan variety* of dimension 16 embedded in $\mathbb{P}^{26}$.

Let X be a Severi variety. It is known that X is swept out by a n-dimensional family $\mathcal{Q}$ of $\frac{n}{2}$-dimensional smooth quadrics, such that, given two distinct points $x, y \in X$, there is a unique quadric of $\mathcal{Q}$ containing x, y. If $Q \in \mathcal{Q}$, the projection of X from the linear space $\langle Q \rangle$ of dimension $\frac{n}{2}+1$ to $\mathbb{P}^n$ is birational and, as usual by now, we get a Cremona linearization ϕ of X^2.

Being X defective, its tangent and first secant varieties coincide. By Example 4.4 we see that ϕ maps $T(X) = \mathrm{Sec}(X)$ to the cone over $\mathcal{Q}$ with vertex the n-dimensional linear image of X. This agrees with the contents of the previous examples and applies to the Cartan variety as well.

5.6 Rational Normal Curves

Let $V_n := V_{1,n}$ be the rational normal curve of degree n in $\mathbb{P}^n$

$$V_n = \{[t^n, st^{n-1}, \ldots, s^{n-1}t, s^n] : [s,t] \in \mathbb{P}^1\} \subset \mathbb{P}^n.$$

Let $[x_0, \ldots, x_n]$ be the coordinates of $\mathbb{P}^n$. Then, V_n is the determinantal variety

$$\mathrm{rk}\begin{pmatrix} x_0 & x_1 & \ldots & x_{n-1} \\ x_1 & x_2 & \ldots & x_n \end{pmatrix} = 1.$$

Assume $n = 2k$ is even (similar considerations can be made in the odd case). Then, we can linearize V_n via the following affine triangular $(2,2)$-Cremona map ϕ on $\{x_0 \neq 0\}$

$$y_i = \begin{cases} x_1 & \text{if } i = 1, \\ x_i - x_{i-1}x_1 & \text{if } i > 1 \text{ and } i \text{ is odd}, \\ x_i - x_{\frac{i}{2}}^2 & \text{otherwise}. \end{cases}$$

The ϕ-image of V_n is the linear space $\{y_2 = y_3 = \cdots = y_n = 0\}$. The reduced fundamental locus is $\{x_0 = 0\}$ and the indeterminacy locus is $\{x_0 = x_1 = \cdots = x_k = 0\}$.

The secant variety $\mathrm{Sec}(V_n)$ is defined by the 3×3-minors of the $3 \times (n-1)$ *catalecticant* matrix

$$\begin{pmatrix} x_0 & x_1 & \dots & x_{n-2} \\ x_1 & x_2 & \dots & x_{n-1} \\ x_2 & x_3 & \dots & x_n \end{pmatrix},$$

(see Dolgachev 2012), where we as usually set $x_0 = 1$. Using column operations, this matrix can be transformed into the following one expressed in terms of the y-coordinates

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ y_1 & y_2 & \dots & y_{n-1} \\ y_2 + y_1^2 & y_3 & \dots & y_n \end{pmatrix}.$$

This shows that $\mathrm{Sec}(V_n)$ is mapped by ϕ to the cone over a V_{n-2} with vertex the line $\{y_2 = \cdots = y_n = 0\}$ which is the ϕ-image of V_n. A similar situation occurs for all higher secant varieties of V_n. For instance, $\mathrm{Sec}_{k-1}(V_n)$ is the hypersurface defined by the catalecticant determinantal equation of degree $k + 1 = \frac{n}{2} + 1$

$$\det\begin{pmatrix} x_0 & x_1 & \dots & x_k \\ x_1 & x_2 & \dots & x_{k+1} \\ \dots & \dots & \dots & \dots \\ x_k & x_{k+1} & \dots & x_n \end{pmatrix} = 0.$$

Hence, ϕ maps $\mathrm{Sec}_{k-1}(V_n)$ to the cone over $\mathrm{Sec}_{k-2}(V_{n-2})$ with vertex the $\mathbb{P}^1$ obtained as the ϕ-image of V_n.

5.7 The Grassmannians $G(3, 6)$

Let $X = (x_{ij})_{1 \le i,j \le 3}$ and $Y = (y_{ij})_{1 \le i,j \le 3}$ be 3×3-matrices, and let

$$[x_0, X, Y, y_0]$$

be coordinates in $\mathbb{P}^{19}$.

Let $A = (a_{ij})_{1 \le i,j \le 3}$ be a 3×3 matrix. We denote by A_{ij} the minor of A obtained by deleting row i and column j, so that

$$\wedge^2 A = (A_{ij})_{1 \le i,j \le 3}, \quad \text{and} \quad \wedge^3 A = \det(A).$$

We parametrize $G(3, 6)$ as follows:

$$(I_3|A) \in \mathbb{C}^9 \mapsto (1, A, \wedge^2 A, \wedge^3 A) \in \{x_0 \neq 0\} \subset \mathbb{P}^{19}.$$

This parametrization is precisely the inverse of the birational projection of $G(3,6)$ from its tangent space at the point $[0,0,0,1]$.

By our discussion in Sect. 3.3, this gives rise to a family of triangular Cremona transformations linearizing $G(3,6)$

$$\phi : [x_0, X, Y, y_0] \in \mathbb{P}^{19} \dashrightarrow [z_0, Z, W, w_0] \in \mathbb{P}^{19}$$

where $Z = (z_{ij})_{1 \le i,j \le 3}$ and $W = (w_{ij})_{1 \le i,j \le 3}$.

We can view the determinant of A in two ways: as a cubic polynomial in the entries of A and as a bilinear quadric form in the variables (a_{ij}, A_{ij}). This yields different Cremona transformations, one defined by quadrics whose inverse transformation is defined by cubics (a *quadro–cubic* transformation), the other defined by cubics with the inverse also defined by cubics (a *cubo–cubic* transformation).

Let us start with the quadro-cubic transformation ϕ. On $\{x_0 \neq 0\}$ it is defined by

$$z_{ij} = x_{ij}, \quad w_{ij} = y_{ij} - X_{ij}, \quad w_0 = y_0 - \sum_{i=1}^{3} (-1)^{i+1} x_{1i} y_{1i}.$$

The reduced fundamental locus is $\{x_0 = 0\}$ and the indeterminacy locus is $\{x_0 = x_{ij} = 0\}$. The inverse of ϕ, on $\{z_0 \neq 0\}$, is given by

$$x_{ij} = z_{ij}, \quad y_{ij} = w_{ij} + Z_{ij}, \quad y_0 = w_0 + \sum_{i=1}^{3} (-1)^{i+1} z_{1i}(w_{1i} + Z_{1i}). \tag{14}$$

The cubo–cubic Cremona transformation ψ is given on the affine set $\{x_0 \neq 0\}$ by the following expressions

$$z_{ij} = x_{ij}, \quad w_{ij} = y_{ij} - X_{ij}, \quad w_0 = y_0 - \det(X).$$

Its reduced fundamental locus is $\{x_0 = 0\}$, and its indeterminacy locus is $\{x_0 = x_{ij} = 0\}$. The inverse Cremona transformation restricted to $\{z_0 \neq 0\}$ is defined by

$$x_{ij} = z_{ij}, \quad y_{ij} = w_{ij} + Z_{ij}, \quad y_0 = w_0 + \det(Z).$$

The image of $G(3,6)$ under both ϕ and ψ is the linear space defined by $\{W = 0, w_0 = 0\}$.

It is known that $\mathrm{Sec}(G(3,6)) = \mathbb{P}^{19}$ (see Donagi 1977), while $T(G(3,6))$ is the quartic hypersurface defined by

$$P = (x_0 y_0 - \mathrm{tr}(XY))^2 + 4x_0 \det(Y) + 4y_0 \det(X) - 4 \sum_{i,j} \det(X_{ij}) \det(Y_{ji}) = 0,$$

(see Sato and Kimura 1977, p. 83). We find the equation of $\phi(T(G(3,6)))$ by substituting (14) in P (where $x_0 = 1$). We obtain a degree 6 equation

$$z_{13}^4 z_{22}^2 - 2z_{12}z_{13}^3 z_{22}z_{23} + \text{appr. } 600 \text{ terms} = 0.$$

Analogously, for $\psi(T(G(3,6)))$ we obtain a degree 6 equation containing the same two highlighted terms.

Since $T(G(3,6))$ is singular along $G(3,6)$, the same happens for the above two sextics along $\{W = 0, w_0 = 0\}$. In any event, none of these two linearizing Cremonas simplify the equation of $T(G(3,6))$, which actually becomes more complicated.

Similar considerations can be done for Grassmannians $G(n,2n)$ with $n \geq 4$.

6 Cumulant Cremonas

As we saw in Sect. 5, there are several examples in which a Cremona linearization of a rational variety simplifies the equations of its secant varieties. Here is another instance of this behavior.

Example 6.1 Consider the Segre embedding Σ_n of $(\mathbb{P}^1)^n$ in $\mathbb{P}^{2^n-1}$. In particular take the case $n = 3$. Then, Σ_3 is parametrically given by the equations

$$x_1 = t_1, \quad x_2 = t_2, \quad x_3 = t_3, \quad x_4 = t_1t_2, \quad x_5 = t_1t_3, \quad x_6 = t_2t_3, \quad x_7 = t_1t_2t_3.$$

We have $\text{Sec}(\Sigma_3) = \mathbb{P}^7$, whereas $T(\Sigma_3)$ is a hypersurface of degree four in $\mathbb{P}^7$. Its defining equation is the so called *hyperdeterminant* (see Gelfand et al. 1994).

The linearizing Cremona transformation ϕ defined in (6) maps Σ_3 to the linear space $x_4 = \cdots = x_7 = 0$. Following (9), the variety $T(\Sigma_n)$ is mapped by ϕ to a (symmetric) degree four hypersurface with defining equation

$$\begin{aligned}&x_3^2x_4^2 + x_2^2x_5^2 + x_1^2x_6^2 + 2(x_1x_2x_5x_6 + x_1x_3x_4x_6 + x_2x_3x_4x_5) + 4x_4x_5x_6\\ &\quad - 2x_7(x_1x_6 + x_3x_4 + x_2x_5) + x_7^2\\ &\quad = 0.\end{aligned}$$

◊

In this case, the linearization process simplifies the equation of $T(\Sigma_3)$, but the degree remains the same. The question is: can we find a linearizing Cremona for Σ_3 that lowers the degree of $T(\Sigma_3)$? An affirmative answer to this question is given by *cumulant Cremonas* arising from algebraic statistics. Indeed, this family of Cremonas gives the following very simple equation for the image of $T(\Sigma_3)$ (see Sturmfels and Zwiernik 2013, (2.1))

$$x_7^2 + 4x_4x_5x_6 = 0.$$

6.1 Binary Cumulants

We recall the setting of binary cumulants from Sturmfels and Zwiernik (2013). Let $\Pi(I)$ denote the set of all nonempty set partitions of $I \subseteq [n] := \{1, \ldots, n\}$. We write $\pi = B_1 | \cdots | B_r$ for a typical element of $\Pi(I)$, where all $\emptyset \neq B_i \subset I$ are the unordered disjoint *blocks* of π, and $I = \cup_{i=1}^r B_i$. For example, if $n = 3$, then

$$\Pi([3]) = \{123,\ 1|23,\ 2|13,\ 3|12,\ 1|2|3\}.$$

We denote by $|\pi|$ the number of blocks of $\pi \in \Pi(I)$.

Consider two copies of $\mathbb{P}^{2^n-1} = \mathbb{P}(\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2)$ with coordinates $[x_I]_{I \subseteq [n]}$ and $[y_I]_{I \subseteq [n]}$. Following Sturmfels and Zwiernik (2013), we define the (*binary*) *cumulant Cremona transformation* or the (*binary*) *cumulant change of coordinates*

$$\psi : [x_I]_{I \subseteq [n]} \in \mathbb{P}^{2^n-1} \dashrightarrow [y_I]_{I \subseteq [n]} \in \mathbb{P}^{2^n-1}$$

via the formula

$$y_\emptyset = x_\emptyset^n, \quad \text{and} \quad y_I = \sum_{\pi \in \Pi(I)} (-1)^{|\pi|-1}(|\pi| - 1)!\, x_\emptyset^{n-|\pi|} \prod_{B \in \pi} x_B \qquad \text{for } \emptyset \neq I \subseteq [n], \tag{15}$$

where the product in (15) is taken over all blocks B of π. We will call $[y_I]_{I \subseteq [n]}$ the *cumulant coordinates*. Note that I is the maximal element in the poset $\Pi(I)$, hence ψ is a triangular Cremona transformation. It linearizes Σ_n, which is mapped to the linear space $\{y_I = 0\}_{|I| \geq 2}$ (see Sturmfels and Zwiernik 2013, Remark 3.4), and $T(\Sigma_n)$ is toric in the cumulants coordinates (see Sturmfels and Zwiernik 2013, Theorem 4.1).

The inverse map of ψ is given by the standard Möbius inversion formula for the partition lattice $\Pi([n])$ (see Stanley 2002, Proposition 3.7.1)

$$x_\emptyset = y_\emptyset^n, \quad \text{and} \quad x_I = \sum_{\pi \in \Pi(I)} y_\emptyset^{n-|\pi|} \prod_{B \in \pi} y_B \qquad \text{for } I \subseteq [n].$$

Both maps are morphisms on the open affine subsets $\{x_\emptyset \neq 0\}$ and $\{y_\emptyset \neq 0\}$, respectively.

Example 6.2 Fix $n = 2$. Then

$$y_\emptyset = x_\emptyset^2, \quad y_1 = x_\emptyset x_1, \quad y_2 = x_\emptyset x_2 \quad y_{12} = x_\emptyset x_{12} - x_1 x_2,$$

which coincides with (11) in this case. The inverse is given by

$$x_\emptyset = y_\emptyset^2,\ x_1 = y_\emptyset y_1, x_2 = y_\emptyset y_2,\ x_{12} = y_\emptyset y_{12} + y_1 y_2.$$

The fundamental locus is $\{x_\emptyset^3 = 0\}$.

Let $n = 3$. Then

$$y_\emptyset = x_\emptyset^3,\ y_i = x_\emptyset^2 x_i, \quad \text{for} \quad 1 \le i \le 3,\ y_{ij} = x_\emptyset^2 x_{ij} - x_\emptyset x_i x_j, \quad \text{for} \quad 1 \le i < j \le 3$$
$$y_{123} = x_\emptyset^2 x_{123} - x_\emptyset x_1 x_{23} - x_\emptyset x_2 x_{13} - x_\emptyset x_3 x_{12} + 2x_1 x_2 x_3.$$

The inverse map is

$$x_\emptyset = y_\emptyset^3, x_i = y_\emptyset^2 y_i, \quad \text{for} \quad 1 \le i \le 3, x_{ij} = y_\emptyset(y_\emptyset y_{ij} + y_i y_j), \quad \text{for} \quad 1 \le i < j \le 3$$
$$x_{123} = y_\emptyset^2 y_{123} + y_\emptyset(y_{12} y_3 + y_{13} y_2 + y_{23} y_1) + y_1 y_2 y_3.$$

The fundamental locus is now $\{x_\emptyset^8 = 0\}$. ♢

6.2 Linearizing Higher Segre Varieties

The above construction can be generalized to $\mathrm{Seg}(r_1, \ldots, r_k) \subset \mathbb{P}^r$ with $r + 1 = \prod_{i=1}^3 (r_i + 1)$, for any $k \ge 2$ and $r_1 \ge \cdots \ge r_k \ge 1$. The case $k = 2$ was treated in Sect. 5.1. If $k = 3$, let $[x_{ijk}]_{0 \le i \le r_1, 0 \le j \le r_2, 0 \le k \le r_3}$ be the coordinates on $\mathbb{P}^r$. Define a Cremona transformation by

$$y_{000} = x_{000}^3, \quad y_{i00} = x_{000}^2 x_{i00}, \quad y_{0j0} = x_{000}^2 x_{0j0}, \quad y_{00k} = x_{000}^2 x_{00k},$$
$$y_{ij0} = x_{000}(x_{000} x_{ij0} - x_{i00} x_{0j0}), \quad y_{i0k} = x_{000}(x_{000} x_{i0k} - x_{i00} x_{00k}), \quad y_{0jk} = x_{000}(x_{000} x_{0jk} - x_{0j0} x_{00k}),$$
$$y_{ijk} = x_{000}^2 x_{ijk} - x_{000} x_{i00} x_{0jk} - x_{000} x_{0j0} x_{i0k} - x_{000} x_{00k} x_{ij0} + 2 x_{i00} x_{0j0} x_{00k},$$

where $i, j, k \ge 1$. This linearizes $\mathrm{Seg}(r_1, r_2, r_3)$ by mapping it to the linear space $\{y_{ijk} = 0\}$ for all triples $(i, j, k) \in \prod_{i=1}^3 \{0, \ldots, r_i\}$ with at least two nonzero coordinates.

This generalizes to any k as follows (see Michałek et al. 2014, Sects. 7 and 8). Let $S(\mathbf{i}) \subseteq [n]$ be the support of $\mathbf{i} = (i_1, \ldots, i_k) \in \prod_{i=1}^k \{0, \ldots, r_i\}$, i.e. the set of coordinates of nonzero entries in $\mathbf{i}$. For every $B \subseteq [k]$, we define the k-tuple $\mathbf{i}(B)$ that agrees with $\mathbf{i}$ on those indices in B and is zero otherwise. We define the Cremona transformation $\psi : \mathbb{P}^r \dashrightarrow \mathbb{P}^r$ by the formulas

$$y_{\mathbf{i}} = \sum_{\pi \in \Pi(S(\mathbf{i}))} (-1)^{|\pi|-1}(|\pi|-1)!\, x_{0\cdots 0}^{n-|\pi|} \prod_{B \in \pi} x_{\mathbf{i}(B)}, \quad \text{for all} \quad \mathbf{i} \in \prod_{i=1}^{k} \{0, \ldots, k_i\}.$$

The image of $\mathrm{Seg}(r_1, \ldots, r_k)$ lies in the subspace $\{y_{\mathbf{i}} = 0\}_{|S(\mathbf{i})| \geq 2}$. This can be shown by mimicking the proof of Theorem 6.8, so we leave the proof to the reader.

6.3 *L*-Cumulant Cremonas

One of the advantages of working with cumulants is that the change of coordinates is conveniently encoded by the cumulant generating function (Sturmfels and Zwiernik 2013). However, to fully exploit the involved combinatorics, we will generalize cumulants to situations when such a generating function is not known. As we will see, $\mathcal{L}$-cumulants, introduced in Zwiernik (2012), enjoy this property.

First we show how the homogeneous binary cumulant change of coordinates generalizes. We replace the partition lattice $\Pi(I)$ by a *partial order set* (*poset*) $(P, <_P)$ (or simply $(P, <)$ if there is no danger of confusion) with its associated *Möbius function* μ_P. The function $\mu_P : P \times P \to \mathbb{Z}$ (or simply μ) is recursively defined by $\mu(\pi, \pi) = 1$ for all $\pi \in P$, $\mu(\pi, \nu) = 0$ if $\pi \not< \nu$, and

$$\mu(\pi, \nu) = -\sum_{\pi \leq \delta < \nu} \mu(\pi, \delta), \quad \text{for all } \pi < \nu \text{ in } P.$$

The two main features of this function that we will use in the rest of this section are the *Möbius inversion formula* and the *product theorem*, which we now recall. Even though they hold in a more general setting, we state them for finite posets, since this will suffice for our purposes.

Proposition 6.3 (Möbius inversion formula, Stanley 2002, Proposition 3.7.1). *Let $(P, <)$ be a finite poset and $f, g : P \to \mathbb{C}$. Then*

$$g(x) = \sum_{y \leq x} f(y) \quad \textit{for all } x \in P \quad \textit{if and only if} \quad f(x) = \sum_{y \leq x} g(y)\mu(y, x)$$
for all $x \in P$.

Theorem 6.4 (Product theorem, Stanley 2002, Proposition 3.8.2) *Let $(P, <_P)$ and $(Q, <_Q)$ be finite posets and let $(P \times Q, <)$ be their product, with order given coordinatewise, i.e. $(p, q) \leq (p', q')$ if and only if $p \leq_P p'$ and $q \leq_Q q'$. If $(p, q) \leq (p', q')$ in $P \times Q$, then*

$$\mu_{P\times Q}((p,q),(p',q')) = \mu_P(p,p')\,\mu_Q(q,q').$$

For further basic results concerning Möbius functions we refer the reader to Stanley (2002, Chap. 3). Some of them will be recalled later on in this section.

The set partitions of a given nonempty set form a poset, where the order < corresponds to refinement, that is, $\pi < \nu$ if π refines ν. To generalize cumulant Cremonas, we replace $\Pi([n])$ by a subposet $\mathcal{L}$ containing the maximal and minimal elements of $\Pi([n])$, i.e., the partitions

$$\hat{0} = 1|\cdots|n \quad \text{and} \quad \hat{1} = [n].$$

These two elements coincide if and only if $n = 1$.

Let us fix such an $I \subseteq [n]$. For each ϕ, we construct a subposet $\mathcal{L}(I)$ of $\Pi(I)$ by restricting each partition in $\mathcal{L}$ to I. In particular, $\mathcal{L}([n]) = \mathcal{L}$. Each poset $\mathcal{L}(I)$ has an associated Möbius function. To simplify notation, we denote all of them by μ. Similarly, we denote the maximal and the minimal element of each poset by $\hat{0}$ and $\hat{1}$, so $\hat{1} = I$ in $\mathcal{L}(I)$. The identification will be clear from the context.

Given $\mathcal{L}$, we define a map $\psi_{\mathcal{L}} : \mathbb{P}^{2^n-1} \dashrightarrow \mathbb{P}^{2^n-1}$ as

$$y_I = \begin{cases} \sum_{\pi\in\mathcal{L}(I)} \mu(\pi,\hat{1})\, x_{\emptyset}^{n-|\pi|} \prod_{B\in\pi} x_B & \text{if } I \neq \emptyset, \\ x_{\emptyset}^{n} & \text{otherwise.} \end{cases} \tag{16}$$

Here, $B \in \pi$ if it is a block of the partition. Note that

$$y_i = x_{\emptyset}^{n-1} x_i, \quad \text{for} \quad \text{all} \quad i \in [n], \quad \text{and} \quad y_{ij} = x_{\emptyset}^{n-2}(x_{\emptyset}x_{ij} - x_i x_j)$$

do not depend on $\mathcal{L}$. Since $\hat{1} \in \mathcal{L}$, we know that $\mathcal{I}$ is the maximal element of $\mathcal{L}(I)$ for every $I \subset [n]$. This implies that $\psi_{\mathcal{L}}$ is a triangular Cremona transformation. It is defined over the open set $\{x_{\emptyset} \neq 0\}$. We call such a map an *$\mathcal{L}$-cumulant Cremona*. Its fundamental locus is $\{x_{\emptyset}^{n^2-1} = 0\}$.

Example 6.5 If $\mathcal{L} = \Pi([n])$, the Möbius function satisfies $\mu(\pi,\hat{1}) = (-1)^{|\pi|-1}(|\pi|-1)!$, so we recover the cumulant change of coordinates in (15).

To the other extreme, if $n > 1$ and $\mathcal{L} = \{\hat{0},\hat{1}\}$, then (16) becomes

$$y_I = \begin{cases} x_{\emptyset}^{n-1} x_I - x_{\emptyset}^{n-|I|} \prod_{i\in I} x_i & \text{if } I \neq \emptyset, \\ x_{\emptyset}^{n} & \text{otherwise,} \end{cases}$$

which is the linearizing Cremona of Σ_n arising, as in (6), from the affine parametrization of Σ_n given by

$$x_I = \prod_{i \in I} t_i, \quad \text{for} \quad \emptyset \neq I \subseteq [n] \quad \text{and} \quad (t_1, \ldots, t_n) \in \mathbb{C}^n.$$

Example 6.6 (Interval partitions of $[n]$) Fix a positive integer n and let $\mathcal{L}$ be the set of *interval partitions* of $[n]$, ordered by refinement. An interval partition of $[n]$ is obtained by cutting the sequence $1, 2, \ldots, n$ into subsequences. For example, there are four interval partitions on [3], i.e., 123, 1|23, 12|3 and 1|2|3.

The interval partitions form a poset isomorphic to the Boolean lattice of a set of $n - 1$ elements. In particular, its Möbius function satisfies $\mu(\pi, \hat{1}) = (-1)^{|\pi|-1}$ (c.f. Stanley 2002, Example 3.8.3). If $n = 3$, this gives the following formulas for the map $\psi_{\mathcal{L}}$ from (16)

$$y_\emptyset = x_\emptyset^3, \quad y_i = x_\emptyset^2 x_i \ (i = 1, 2, 3), \quad y_{12} = x_\emptyset^2 x_{12} - x_\emptyset x_1 x_2, \quad y_{13} = x_\emptyset^2 x_{13} - x_\emptyset x_1 x_3,$$
$$y_{23} = x_\emptyset^2 x_{23} - x_\emptyset x_2 x_3, \quad y_{123} = x_\emptyset^2 x_{123} - x_\emptyset x_1 x_{23} - x_\emptyset x_3 x_{12} + x_1 x_2 x_3.$$

The formula for the inverse of $\psi_{\mathcal{L}}$ over $\{y_\emptyset \neq 0\}$ follows by the standard Möbius inversion formula on each poset $\mathcal{L}(I)$. Let us show how to do this. For every $\pi \in \Pi(I)$ we set

$$x_\pi := \prod_{B \in \pi} x_B. \tag{17}$$

Given $I \subseteq [n]$ and $\nu \in \mathcal{L}(I)$, we define

$$y_\nu := \sum_{\pi \leq \nu \ \pi \in \mathcal{L}(I)} \mu(\pi, \nu) x_\pi. \tag{18}$$

By the Möbius inversion formula on $\mathcal{L}(I)$, we conclude that

$$x_\nu = \sum_{\pi \leq \nu \pi \in \mathcal{L}(I)} y_\pi \quad \text{for all } I \subset [n] \quad \text{and} \quad \nu \in \mathcal{L}(I). \tag{19}$$

In particular

$$x_I = \sum_{\pi \in \mathcal{L}(I)} y_\pi, \quad \text{for all } I \subset [n]. \tag{20}$$

The following lemma ensures that, for each $v \in \mathcal{L}(I)$, y_v is a polynomial in the variables y_J's with $J \subseteq I$. It also proves (20) and yields an explicit formula for $\psi_{\mathcal{L}}^{-1}$.

Lemma 6.7 *For each $I \subset [n]$ and each $v \in \mathcal{L}(I)$, the variable y_v is a polynomial in y_J's where J runs over all subsets of each one of the blocks of v.*

Proof We prove the result by induction on the subsets of $[n]$. If $I = \emptyset$, there is nothing to prove since $y_\emptyset = 1$. Suppose that $I \supsetneq \emptyset$ and that the result holds for all $J \subsetneq I$. If v is a one block partition, there is nothing to prove. Assume that v contains more than one block. By (17)–(19) we obtain

$$y_v = \sum_{\pi \leq v} \mu(\pi, v) \prod_{B \in \pi} \Big(\sum_{\tau \in \mathcal{L}(B)} y_\tau \Big).$$

Since all B's on the right-hand side are strictly included in I, the result follows by induction. □

As it happens with the homogeneous cumulant change of coordinates (15), the map $\psi_{\mathcal{L}}$ linearizes Σ_n:

Theorem 6.8 *For any choice of $\mathcal{L}$, the map $\psi_{\mathcal{L}}$ from (16) linearizes Σ_n. Its image is the linear space $\Pi := \{y_I = 0\}_{I \subseteq [n], |I| \geq 2}$.*

Proof Denote by $a_i = [a_{i0}, a_{i1}]$ the coordinates of the ith copy of $\mathbb{P}^1$ in $(\mathbb{P}^1)^n$. The Segre embedding $\sigma_n : (\mathbb{P}^1)^n \to \mathbb{P}^{2^n - 1}$, maps $a = (a_1, \ldots, a_n)$ to the point in $\mathbb{P}^{2^n-1}$ whose I-th coordinate is

$$a_I = \prod_{i \in I} a_{i1} \prod_{i \notin I} a_{i0}, \quad \text{for every} \quad I \subseteq [n].$$

We compute $\psi_{\mathcal{L}} \circ \sigma_n$ using (16). For every $I \subseteq [n]$ and every partition $\pi \in \mathcal{L}(I)$ we have

$$a_\emptyset^{n-|\pi|} \prod_{B \in \pi} a_B = \prod_{i \in I} (a_{i0}^{n-1} a_{i1}) \prod_{i \notin I} a_{i0}^n$$

which does not depend on π. Therefore, the Ith coordinate of $\psi_{\mathcal{L}}(\sigma_n(a))$ is

$$b_I = \Big(\prod_{i \notin I} a_{i0}^n \prod_{i \in I} (a_{i0}^{n-1} a_{i1})\Big) \sum_{\pi \in \mathcal{L}(I)} \mu(\pi, \hat{1}). \tag{21}$$

If $|\mathcal{L}(I)| \geq 2$, Lemma 6.9, applied to $P = \mathcal{L}(I)$, yields $\sum_{\pi \in \mathcal{L}(I)} \mu(\pi, \hat{1}) = 0$. Combining this fact with (21), we conclude that the image of Σ_n by $\psi_{\mathcal{L}}$ is contained in the linear space $\{y_I = 0\}_{|\mathcal{L}(I)| \geq 2}$. Note that since $\hat{1}$ and $\hat{0}$ lie in $\mathcal{L}$, the condition $|\mathcal{L}(I)| \geq 2$ is equivalent to $|I| \geq 2$. So this linear space is Π, and it has dimension

n. Moreover Σ_n is not contained in the fundamental locus of $\psi_{\mathcal{L}}$, so the induced map $\psi_{\mathcal{L}|\Sigma_n} : \Sigma_n \dashrightarrow \Pi$ is birational. □

Lemma 6.9 *Let $(P, \leq)$ be a finite poset of size at least two with unique maximal and minimal elements $\hat{1}, \hat{0}$. Let μ be its Möbius function. Then,*

$$\sum_{x \in P} \mu(x, \hat{1}) = 0.$$

Proof Consider the dual poset $(P^*, \leq^*)$ obtained by reversing the order in $(P, \leq)$. In particular, the roles of the minimal and maximal elements are exchanged, namely $\hat{0}^* = \hat{1}$ and $\hat{1}^* = \hat{0}$. The Möbius function μ^* of P^* satisfies $\mu^*(x, y) = \mu(y, x)$ for all $(x, y) \in P \times P$ (see Stanley 2002, p. 120). Therefore

$$\sum_{x \in P} \mu(x, \hat{1}) = \mu(\hat{0}, \hat{1}) + \sum_{\hat{0} < x \leq \hat{1}} \mu(x, \hat{1}) = \mu^*(\hat{0}^*, \hat{1}^*) + \sum_{\hat{0}^* \leq^* x <^* \hat{1}^*} \mu^*(\hat{0}^*, x) = 0,$$

where the last equality follows from the recursive definition of μ^*. □

6.4 Secant Cumulants

As we mentioned earlier, one of the useful features of binary cumulants is that the tangential variety of Σ_n, expressed in cumulants, becomes toric. This is not the case, in general, for $\mathcal{L}$-cumulant Cremonas. However, with a careful choice of the defining poset $\mathcal{L}$ one may obtain other desired properties. For example, if $\mathcal{L}$ is the poset of interval partitions of $[n]$ defined in Example 6.6, the Cremona transformation $\psi_{\mathcal{L}}$ is an involution.

The next example is related to $\mathrm{Sec}(\Sigma_n)$ (see Michałek et al. 2014; Zwiernik 2012, Sect. 3.3).

Example 6.10 In what follows, we parametrize the secant variety $\mathrm{Sec}(\Sigma_n)$ inside $\mathbb{P}^{2^n-1}$ starting from the parametrization of Σ_n:

$$p_I^{(0)} = \prod_{i \in [n] \setminus I} a_{i0} \prod_{i \in I} a_{i1}, \quad p_I^{(1)} = \prod_{i \in [n] \setminus I} b_{i0} \prod_{i \in I} b_{i1} \qquad \text{for all } I \subseteq [n].$$

Denote by $\mathbf{A}$ the affine subspace given by $x_\emptyset = 1$. The affine variety $\mathrm{Sec}(\Sigma_n) \cap \mathbf{A}$ is parametrized by

$$x_I = (1 - s_1) \prod_{i \in I} a_{i1} + s_1 \prod_{i \in I} b_{i1}.$$

Consider a sequence of two $\mathcal{L}$-cumulant transformations. The first one corresponds to the lattice $\mathcal{L}_1$ of all *one-cluster partitions* of $[n]$, i.e. partitions with at most one block of size greater than one. The second one comes from the lattice $\mathcal{L}_2$ of interval partitions of $[n]$. The first map $\psi_1 : \mathbf{A} \to \mathbf{A}$ is defined by

$$y_I = \sum_{A \subseteq I} (-1)^{|I \setminus A|} x_A \prod_{i \in I \setminus A} x_i, \quad \text{for } I \subseteq [n].$$

The second map $\psi_2 : \mathbf{A} \to \mathbf{A}$ is given by 16, i.e.

$$z_I = \sum_{\pi \in \mathcal{I}(I)} (-1)^{|\pi|-1} \prod_{B \in \pi} y_B, \quad \text{for } I \subseteq [n].$$

To see how this sequence of maps can be written as a single $\mathcal{L}$-cumulant transformation we refer to Zwiernik (2012). By Michałek et al. (2014, Lemma 3.1), for every $I \subseteq [n]$ such that $|I| \geq 2$, the result of $\psi_2 \circ \psi_1$ applied to $\mathrm{Sec}(\Sigma_n)$ is

$$z_I \;=\; s_1(1-s_1)(1-2s_1)^{|I|-2} \prod_{i \in I} (b_{i1} - a_{i1}). \tag{22}$$

Taking $d_i = (1-2s_1)(b_{i1} - a_{i1})$ for $i \in [n]$, and $t = s_1(1-s_1)(1-2s_1)^{-2}$ in Eq. (22) we conclude that that secant variety, when expressed in cumulants, becomes locally toric with $z_I = t \prod_{i \in I} d_i$ for $|I| \geq 2$.◇

This simple local description of $\mathrm{Sec}(\Sigma_n)$ can be generalized to the secant variety of the Segre product of projective spaces of arbitrary dimensions. This gives the following result:

Theorem 6.11 (see Michałek et al. 2014) *The secant variety* $\mathrm{Sec}(\mathrm{Seg}(r_1, \ldots, r_k))$ *is covered by normal affine toric varieties. In particular, it has rational singularities.*

It turns out that similar techniques can be applied to study the tangential variety $T(\mathrm{Seg}(r_1, \ldots, r_k))$ (see Michałek et al. 2014 for details).

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References

E. Angelini, *Higher secants of Spinor varieties*. Boll. Unione Mat. Ital. (9) **4**, no. 2, 213–235, 2011.

A. Calabri and C. Ciliberto, *On special projections of varieties: epitome to a theorem of Beniamino Segre*, Adv. Geom. **1**, 97–106, 2001.

S. Cantat, *The Cremona group in two variables*, *Proceedings of the sixth European Congress of Mathematics*, 211–225, Europ. Math. Soc., 2013.

B. Costa and A. Simis, *New Constructions of Cremona Maps*, Math. Res. Lett. **20**(4), 629–645, 2013.

I. Dolgachev, *Classical Algebraic Geometry: A Modern View*, Cambridge Univ. Press, 2012.

R. Donagi, *On the geometry of Grassmannians*. Duke Math. Journal **44**, 795–837, 1977.

S. Druel , *Existence des modèles minimaux pour les variétés de type général*, Séminaire BOURBAKI, Novembre 2007, 2007–2008, no. 982.

I. M. Gelfand, M. M. Kapranov and A. V. Zelevinsky, *Discriminants, resultants, and multidimensional determinants*. Mathematics: Theory & Applications. Birkhäuser Boston Inc., Boston, MA, 1994.

J. E. Goodman, A. Holmsen, R. Pollack, K. Ranestad and F. Sottile, *Cremona Convexity, Frame Convexity, and a Theorem of Santaló*. Adv. Geom. **6**, 301–321, 2006.

R. Lazarsfeld, *Positivity in Algebraic geometry, vol. I and II*, Springer Verlag, 2004.

L. Manivel and M. Michałek, *Secants of minuscule and cominuscule minimal orbits*. Linear Algebra Appl. 481, 288–312, 2015.

M. Mella and E. Polastri, *Equivalent birational embeddings.* Bull. Lond. Math. Soc. **41**(1), 89–93, 2009.

M. Michałek, L. Oeding and P. Zwiernik, *Secant cumulants and toric geometry*. Int. Math. Res. Not. **12**, 4019–4063, 2015

M. Sato and T. Kimura, *A classification of irreducible prehomogeneous vector spaces and their relative invariants*. Nagoya Math. J. **65**, 1–155, 1977.

R. P. Stanley, *Enumerative combinatorics. Volume I.* Cambridge Studies in Advanced Mathematics **49**, Cambridge University Press, 2002.

B. Sturmfels and P. Zwiernik, *Binary cumulant varieties*. Ann. Comb. **17**, no. 1, 229–250 (2013).

F. L. Zak, *Tangents and secants of algebraic varieties*, Translations of Mathematical Monographs 127, Amer. Math. Soc., Providence, RI, 1993.

P. Zwiernik, *L-cumulants, L-cumulant embeddings and algebraic statistics*. J. Alg. Stat. **3**, 11–43, 2012.

An Account of Instanton Bundles on Hyperquadrics

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Abstract
We study instanton bundles on three-dimensional quadrics, paying special attention to the family of 't Hooft bundles. We give explicit families of instanton bundles which are not 't Hooft. In the last section we propose a generalization of an instanton bundle on odd dimensional hyperquadrics $Q_{2n+1} \subset \mathbb{P}^{2n+2}$ for arbitrary n and we state several open questions.

1 Introduction

Among many other contributions to the early development of Algebraic Geometry, Corrado Segre's work on line geometry deserves special attention, i.e., the study of the geometrical properties of the Grassmannian variety $G := Gr(2, V_{n+1})$ of lines of an n-dimensional projective space $\mathbb{P}^n := \mathbb{P}(V_{n+1})$. In particular, he devoted several papers (cf. Loria and Segre 1884; Segre 1884) to the analysis of complexes

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of lines, which are defined as hypersurfaces of G or, equivalently, elements of the linear series $|\mathcal{O}_G(d)|, d \geq 1$.

In the particular case $n = 5$, it turns out that G corresponds to a hyperquadric on $\mathbb{P}^5 := \mathbb{P}(\wedge^2 V)$ under the Plücker embedding. Under this incarnation, linear line complexes correspond to 3-dimensional quadric hypersurfaces Q_ω in $\mathbb{P}^4 := \mathbb{P}^5 \cap H_\omega$ of isotropic lines for a skew-symmetric form $\omega \in \wedge^2 V^\vee$. Under this correspondence, the quadric Q_ω will be smooth if and only if the form ω is non-degenerate.

On the other hand, a central role in modern Algebraic Geometry is played by vector bundles. However, a strong link still exists (at least in small codimension) between subvarieties of a given projective variety and global sections of algebraic vector bundles. As a paradigmatic example of this philosophy we can mention Serre's correspondence. So, in particular it makes sense to approach the study of certain configurations of lines on $\mathbb{P}^n$ through vector bundles defined on G (see Example 3.11 and Proposition 3.12).

One of the families of vector bundles that has received a central attention in the last decades is the family of instanton bundles. The notion of instanton comes from Physics. The formal mathematical definition is that of a self-dual connection on a principal bundle on a four-dimensional Riemannian manifold. The Penrose-Ward correspondence identifies instantons on the four-dimensional sphere with certain holomorphic bundles on the three-dimensional complex projective space $\mathbb{P}^3$. In fact, understanding of the geometric properties of the moduli spaces of stable instanton bundles revealed itself as one of the major breakthroughs in Algebraic Geometry in recent years.

Recently, the definition of instanton bundle has been extended to other ambient varieties than $\mathbb{P}^3$. Basically, two possibilities have been taken into account: on one hand, they have been defined as certain $2n$-vector bundles on $\mathbb{P}^{2n+1}$. In this setting, many results have been obtained; on the other hand, due to some analogies with the projective space, a notion of instanton bundle on Fano threefolds X of Picard index one has been proposed in Faenzi (2014) and Kuznetsov (2012). They are defined as the normalization of rank two stable bundles $\mathcal{F}$ satisfying

$$\mathcal{F} \cong \mathcal{F}^* \otimes \omega_X, H^1(X, \mathcal{F}) = 0.$$

Notice that after $\mathbb{P}^3$, the three-dimensional quadric Q is one of the most representative Fano threefolds of Picard index one. Therefore the main goal of this paper is to study instanton bundles on Q leaning on the aforementioned presentation of Q as a linear line complex of the Grassmannian $Gr(2, V_5)$, trying to underline how classical techniques à la Segre and contemporary tools can be placed together and stressing the major role of line complexes for whose understanding all of us are deeply indebted to Corrado Segre.

The structure of the paper is as follows: in the next section we recall a presentation of the three-dimensional quadric Q as the variety of isotropic lines of $\mathbb{P}^3$ with respect to a non-degenerate skew-symmetric form; we also define the spinor variety parameterizing lines contained in Q as well as its associated spinor bundle. In section three we study instanton bundles E on Q, particularly focusing on sections of $E(1)$ and on their restrictions to lines. We give an explicit construction of instanton bundles whose first twist has no global sections. Finally, in the last section we propose a definition of instanton bundle on higher dimensional hyperquadrics and deal with its existence. We end the paper stating several open questions.

Notation: Given an n-dimensional vector space V and $k \in \mathbb{Z}$, we will denote by $Gr(k, V)$ the Grassmann variety of k-dimensional subspaces of V and, given any non-zero vector $p \in V$, we will denote by $[p] \in \mathbb{P}(V)$ the corresponding point on the associated projective space.

2 Lines on the Quadric Threefold in $\mathbb{P}^4$

Throughout the paper the ambient variety will be a smooth 3-dimensional quadric hypersurface in $\mathbb{P}^4$ that will be denoted by Q. We will identify $Q \subset \mathbb{P}^4$ with the smooth hyperplane section $\mathbb{P}^4 \cap Gr(2,4)$ of the Plücker embedding into $\mathbb{P}^5$ of the Grassmannian of lines in $\mathbb{P}^3$, $Gr(2,4)$. Indeed, we have the following description.

Let U be a 4-dimensional vector space over the complex numbers $\mathbb{C}$ and let us consider the Grassmannian $Gr(2,4)$ of lines in $\mathbb{P}^4$. It is well-known that its Plücker embedding in $\mathbb{P}^5 := \mathbb{P}(\wedge^2 U)$ defines a four-dimensional quadric

$$Gr(2, U) \cong Q_4 \subset \mathbb{P}^5$$

given by the quadric equation $w \wedge w = 0$ for $w \in \wedge^2 U$. Any linear form on $\mathbb{P}(\wedge^2 U)$ can be interpreted as the skew-symmetric bilinear pairing

$$B : U \times U \to \mathbb{C}.$$

Associated to B we have the induced isomorphism

$$B^* : U \to U^*$$

where for any $p \in U, B^*(p) := B(p, \cdot)$. B will be non-degenerated if and only if the associated hyperplane intersects Q_4 transversely. Suppose that this is the case and consider the three-dimensional smooth quadric

$$Q = Q_B := \{\langle u, v\rangle \in Gr(2, U) | B(u, v) = 0\} \subset Gr(2, U) \subset \mathbb{P}^5 = \mathbb{P}(\overset{2}{\wedge} U)$$

parameterizing 2-dimensional isotropic linear subspaces of U. Recall that Q is a Fano variety of index 3 with the canonical line bundle given by $K_Q = \mathcal{O}_Q(-3)$.

Lemma 2.1 *For any non-zero point $p \in U$, we define*

$$l_{[p]} := \{\langle u, p\rangle \in Gr(2, U) | B(u, p) = 0\} \subset Q.$$

For any $0 \neq p \in U, l_{[p]}$ is a line in Q and conversely any line in Q is of this form for a unique $p \in U$. In other words, $\mathbb{P}(U) \cong \mathbb{P}^3$ is isomorphic to the Fano variety $F(Q)$ parameterizing lines in Q.

Proof Let us first see that for any $p \in \mathbb{P}(U), l_{[p]}$ is actually a line on $Q \subset \mathbb{P}^4$. Notice that a point $x = [u \wedge v] \in Q$ belongs to $l_{[p]}$ if and only if $u \wedge v \wedge p = 0 \in \wedge^3 U$. This last statement can be expressed by a set of three independent linear conditions on the Plücker coordinates and therefore they define a line.

In the other direction, in order to see that any line on Q is of this form, notice that for any point $x \in Q$, any line l passing through x and contained in Q will lie on $Q \cap T_x Q$, which is a quadric cone in $T_x Q \cong \mathbb{P}^3$. Hence the set of lines contained in Q passing through a given point $x = [u \wedge v]$ are parameterized by a projective line $\mathbb{P}^1$. Since for any $w \in \langle u, v\rangle$, the line $l_{[w]}$ passes through x, we are done. □

Remark 2.2 The previous Lemma can be also seen as follows: $\mathbb{P}(U)$ parameterizes 2-dimensional planes in Q_4, namely the point $u \in U$ determines the plane $\mathbb{P}(u \wedge U) \subset Q_4$ of lines passing through u. These are the so-called α-planes of Q_4. Since any line in $Q = Q_4 \cap B$ is contained in a single α-plane of Q_4 the result follows.

Remark 2.3 Given two lines $l_{[p]}$ and $l_{[q]}$ in Q corresponding to non-zero vectors $p, q \in U$, they intersect in a point of Q if and only if $B(p, q) = 0$. Indeed, since B is skew-symmetric, the subspaces of $Gr(2, U)$,

$$l_{[p]} = \{\langle u, p\rangle \in Gr(2, U) | B(u, p) = 0\}$$
$$l_{[q]} = \{\langle u, q\rangle \in Gr(2, U) | B(u, q) = 0\},$$

intersect on $\langle q, p\rangle \in Gr(2, U)$. Hence, they intersect in a point of Q if and only if $\langle q, p\rangle \in Q$, that is, if and only if $B(p, q) = 0$.

Now we are going to introduce the Spinor bundle S on Q_3 (see also Ottaviani 1989). Recall that on the Grassmannian $Gr(2, 4) \cong Q_4 \subset \mathbb{P}(\wedge^2 U)$ we have the exact sequence of vector bundles

$$0 \to R \to U \otimes \mathcal{O}_{Q_4} \to P \to 0, \tag{2.1}$$

where R and P are the universal subbundle and the universal quotient bundle of $Gr(2, 4)$, respectively. Moreover, we have $U^* = Hom(R, \mathcal{O}_{Q_4}) = H^0(R^\vee)$. Then we define the Spinor bundle S on Q as the restriction of the universal bundle R to Q, $S := R_{|Q}$. It turns out that S is a rank two stable vector bundle on Q with $S^\vee \cong S(1)$. The exact sequence (2.1) restricts on Q to

$$0 \to S \to U \otimes \mathcal{O}_Q \to S(1) \to 0. \tag{2.2}$$

Furthermore, we have

$$c_1(S) = -1, \quad c_2(S) = 1.$$

The Spinor bundle can also be defined through the incidence variety

$$\mathcal{F} := \{(x, l) \in Q \times \mathbb{P}(U) | x \in l\}$$

together with the natural projections $\pi_1 : \mathcal{F} \to Q$ and $\pi_2 : \mathcal{F} \to \mathbb{P}(U)$. Then, the Spinor bundle S on Q is defined by

$$S(1) \cong \pi_{1*}\pi_2^*\mathcal{O}_{\mathbb{P}(U)}(1).$$

Moreover we have, for any line $l \subset Q$,

$$S_{|l} \cong \mathcal{O}_l(-1) \oplus \mathcal{O}_l \tag{2.3}$$

and $H^1(S(t)) = H^2(S(t)) = 0$ for $t \in \mathbb{Z}$ and $H^0(S(t)) = 0$ for $t \leq 0$ (that is, S is an ACM bundle on Q). Finally, in order to understand the zero locus of global sections of $S(1)$, notice that from the exact sequence

$$0 \to \mathcal{O}_{Q_4}(-1) \xrightarrow{B} \mathcal{O}_{Q_4} \to \mathcal{O}_Q \to 0$$

tensored with $R^\vee$ we obtain

$$0 \to R \xrightarrow{B} R^\vee \to S^\vee \to 0$$

and thus $H^0(S^\vee) \cong H^0(R^\vee) \cong Hom(R, \mathcal{O}_{Q_4}) = U^*$. Moreover, an element $u^* \in U^*$ corresponds to a map:

$$u^* : R \to \mathcal{O}_{Q_4}$$

whose image is the ideal sheaf of the β-plane $\mathbb{P}(\bigwedge^2 \ker u^*) \subset Q_4$. Therefore its restriction to $Q = Q_4 \cap B$ is the ideal sheaf of the line $l := \mathbb{P}(\bigwedge^2 \ker u^*) \cap B$. Summing up, we have just seen that the zero locus of a global section $s \in H^0(S(1)))$ is a line $l \subset Q$.

3 Instanton Bundles

Definition 3.1 A k-instanton bundle on Q is a stable rank 2 vector bundle E on Q such that

$$c_1(E) = -1, \quad c_2(E) = k \quad \text{and} \quad H^1(E(-1)) = 0.$$

As in the case of instanton bundles on $\mathbb{P}^3$, instanton bundles on Q can be described in terms of the cohomology bundles of a very particular monad. Indeed, we have:

Lemma 3.2 *Fix an integer $k \geq 2$ and let E be a k-instanton bundle on Q. Then E is the cohomology of a monad of the following type:*

$$M_\bullet : I^* \otimes \mathcal{O}_Q(-1) \overset{D\circ\alpha^t}{\rightarrow} W^* \otimes S \overset{\alpha}{\rightarrow} I \otimes \mathcal{O}_Q,$$

where $I \cong \mathbb{C}^{k-1}, W \cong \mathbb{C}^k, D : W \rightarrow W^*$ is a symmetric duality and α is surjective. Conversely, the cohomology of a monad of this form is a k-instanton bundle on Q.

Proof (Faenzi 2014, Lemma 2.2). □

Associated to this monad $M_\bullet$ there are two exact sequences called the display of the monad: namely, if we denote by $K = \ker(\alpha)$, we have

$$0 \rightarrow K \rightarrow W^* \otimes S \overset{\alpha}{\rightarrow} I \otimes \mathcal{O}_Q \rightarrow 0, \tag{3.1}$$

$$0 \rightarrow I^* \otimes \mathcal{O}_Q(-1) \overset{D\circ\alpha^t}{\rightarrow} K \overset{\pi}{\rightarrow} E \rightarrow 0. \tag{3.2}$$

Notation 3.3 Let $l_{[p]}$ be a line in Q associated to a non-zero vector $p \in U$ and consider the monad

$$I^* \otimes \mathcal{O}_Q(-1) \overset{D\circ\alpha^t}{\rightarrow} W^* \otimes S \overset{\alpha}{\rightarrow} I \otimes \mathcal{O}_Q$$

associated to a k-instanton bundle on Q. Observe that the morphism α corresponds to the operator

$$A : W^* \rightarrow I \otimes U^*$$

which induces the morphism α via

$$\begin{array}{ccc} W^* \otimes S & \overset{\alpha}{\rightarrow} & I \otimes \mathcal{O}_Q \\ A \otimes Id_S \searrow & & \nearrow \\ & I \otimes U^* \otimes S. & \end{array}$$

We will denote by $\alpha(p) : W^* \to I$ the result of composing A with $ev_{l_{[p]}} : I \otimes U^* \to I$ defined by $ev_{l_{[p]}}(v \otimes u^*) := u^*(p)v$. In other words, since α is given by a matrix with linear entries on U, $\alpha(p)$ is given by the matrix obtained by evaluating α on $p \in U$.

According to Grothendieck's theorem, given any k-instanton bundle E on Q and any line $l_{[p]} \subset Q, E|_{l_{[p]}} \cong \mathcal{O}_{l_{[p]}}(a) \oplus \mathcal{O}_{l_{[p]}}(-a-1)$ for some integer $a \geq 0$. This drives us to the following definition:

Definition 3.4 Let E be a k-instanton bundle on Q and $l_{[p]} \subset Q$ a line. We say that $l_{[p]}$ is a jumping line of order i (an i-jumping line) of E if

$$E|_{l_{[p]}} \cong \mathcal{O}_{l_{[p]}}(i) \oplus \mathcal{O}_{l_{[p]}}(-i-1), \quad i > 0.$$

We say that E has the trivial splitting type on $l_{[p]}$ (or equivalently, $l_{[p]}$ is a 0-jumping line) if $E|_{l_{[p]}} \cong \mathcal{O}_{l_{[p]}} \oplus \mathcal{O}_{l_{[p]}}(-1)$.

The following well known result due to Ein-Sols in particular states that any k-instanton bundle on Q has trivial splitting type on a general line of Q.

Proposition 3.5 *Let E be a semistable reflexive sheaf of rank r on Q. Suppose that the restriction of E to a general line l is isomorphic to $\oplus_{i=1}^{r}\mathcal{O}_l(a_i)$ with $a_1 \leq a_2 \leq \cdots \leq a_r$. Then, $a_{i+1} - a_i \leq 1$ for $1 \leq i \leq r-1$.*

Proof See Ein and Sols (1984, Proposition 1.3). □

Next we are going to characterize the splitting type of a k-instanton bundle E on any line $l_{[p]} \subset Q$ in terms of the matrix $\alpha(p)$ associated to the monad defining E.

Lemma 3.6 *Let E be a k-instanton bundle on Q given as the cohomology of the monad*

$$M : \quad I^* \otimes \mathcal{O}_Q(-1) \stackrel{D\circ\alpha^t}{\to} W^* \otimes S \stackrel{\alpha}{\to} I \otimes \mathcal{O}_Q,$$

and $l_{[p]} \subset Q$ a line associated to $p \in U$. Then,

$$E|_{l_{[p]}} \cong \mathcal{O}_{l_{[p]}}(i) \oplus \mathcal{O}_{l_{[p]}}(-i-1)$$

if and only if $\operatorname{rk}\alpha(p) = k-(i+1)$.

Proof Restricting both exact sequences (3.1) and (3.2) from the display of the monad to the line $l_{[p]} \subset Q$ and taking cohomology we get

$$\mathrm{h}^0(E|_{l_{[p]}}) = \mathrm{h}^0(K|_{l_{[p]}}) \tag{3.3}$$

and the long exact sequence

$$\begin{array}{ccccccc} 0 \to & \mathrm{H}^0(K|_{l_{[p]}}) & \to & W^* \otimes \mathrm{H}^0(S_{|l_{[p]}}) & \to & I \otimes \mathrm{H}^0(\mathcal{O}_{l_{[p]}}) \\ & & & \parallel & & \parallel \\ & & & W^* & \overset{\alpha(p)}{\to} & I \end{array} \tag{3.4}$$

where the isomorphism follows from the fact that $S_{|l_{[p]}} \cong \mathcal{O}_{l_{[p]}}(-1) \oplus \mathcal{O}_{l_{[p]}}$ (see also Notation 3.3). Hence, $E|_{l_{[p]}} \cong \mathcal{O}_{l_{[p]}}(i) \oplus \mathcal{O}_{l_{[p]}}(-i-1)$, for some $i \geq 0$, if and only if $\mathrm{h}^0(E|_{l_{[p]}}) = i+1$ which by (3.3) is equivalent to $\mathrm{H}^0(K|_{l_{[p]}}) = i+1$. On the other hand, by (3.4)

$$\mathrm{H}^0(K|_{l_{[p]}}) = k - rk(\alpha(p)).$$

Therefore, $E|_{l_{[p]}} \cong \mathcal{O}_{l_{[p]}}(i) \oplus \mathcal{O}_{l_{[p]}}(-i-1)$ if and only if $\mathrm{rk}\,(\alpha(p)) = k - (i+1)$. □

As an immediate consequence, we have an upper bound for the order of jumping of any line:

Corollary 3.7 *For any k-instanton bundle E on Q and any line $l_{[p]} \subset Q$, the order of jumping of $l_{[p]}$ on E is less or equal than $k-1$.*

The same bound was obtained by Coanda and Faenzi (2014; Corollary 5.6). Later on, we will see that the bound is sharp and that, in addition, all the possible orders of jumping are realized. To this end, we need to introduce the definition of the k-instanton bundles on Q called 't Hooft bundles. But first, let us study the zero locus of global sections of $E(1)$. We obtain a similar result as in Böhmer and Trautmann (1987):

Proposition 3.8 *Let E be a k-instanton bundle on Q given as the cohomology of the monad*

$$M_\bullet: \quad I^* \otimes \mathcal{O}_Q(-1) \overset{D\circ\alpha^t}{\to} W^* \otimes S \overset{\alpha}{\to} I \otimes \mathcal{O}_Q,$$

and denote by $K = \ker(\alpha)$.

(i) *The zero locus of a nonzero global section of $K(1)$ is either empty, a point or a line.*
(ii) *If it exists, the zero locus of a nonzero global section of $E(1)$ is a disjoint union of (not necessary reduced) lines.*

Proof The proof is analogous to the one given in Böhmer and Trautmann (1987, Lemma 1.3), but we reproduce it here for the sake of completeness. From (3.1) it turns out that a global section t of $K(1)$ corresponds to k global sections $u = (u_1, \ldots, u_k) \in H^0(S(1))^k \cong (U^*)^k$ such that $H^0(\alpha)(u) = 0$. Since, as it was explained, the zero locus of a single nonzero global section of $S(1)$ is a line, the zero locus of t will be either empty, a point or a line.

Let us consider a nonzero section s of $S(1)$ and suppose that $(s)_0$ is non-empty. Then, since $H^0(S) = 0, (s)_0$ is pure two codimensional. Let X be a connected component of the zero locus and let us consider its reduction X_{red}. By the exact sequence (3.2) there exists a global section $t \in H^0(K(1))$ such that $\pi(t) = s$ and therefore $\pi(t_{|X_{red}}) = 0$. Restricting the exact sequence to X_{red} and taking into account that $H^0(\mathcal{O}_{X_{red}}) = H^0(\mathcal{O}_Q) = \mathbb{C}$, we obtain an element $u \in I^*$ such that $t' := t - D \circ \alpha^t(u)$ vanishes on X_{red}. Since, on the other hand, we proved that $(t')_0$ is either empty, a point or a line, it turns out that $X_{red} = (t')_0$ is a line.

Finally, let us suppose that a nonzero section $t \in H^0(K(1))$ has nonempty zero locus and let us consider a point $p \in (t)_0$. As in the previous paragraph, there is an element $u \in I^*$ and a line L containing p such that $t' := t - D \circ \alpha^t(u)$ vanishes exactly on L. But now $D \circ \alpha^t(u)(p) = 0$ and therefore $u = 0$. Hence the zero locus of $t = t'$ is a line. □

Definition 3.9 A k-instanton bundle E on Q is called a 't Hooft bundle if $E(1)$ has a non-zero section.

Remark 3.10 Notice that this definition of 't Hooft bundle is coherent with the definition of 't Hooft bundle on $\mathbb{P}^3$ and it is slightly more general than the one given by Faenzi (2014; Definition 2).

Example 3.11 Let $L = l_{[p_1]} \cup \cdots \cup l_{[p_k]}$ be the union of $k \geq 2$ disjoint lines in Q. The rank two vector bundle E_L on Q given by the Hartshorne-Serre construction fits into an exact sequence:

$$0 \to \mathcal{O}_Q(-1) \to E_L \to I_{L,Q} \to 0$$

where $I_{L,Q}$ is the ideal sheaf of L in Q. By construction, E_L is a 't Hooft bundle with $c_2(E_L) = k$. Moreover, a general 't Hooft bundle is of this form.

In the next result, we give an explicit description of the monad associated to a general 't Hooft bundle.

Proposition 3.12 *Given k-disjoint lines* $l_{[p_1]}, \ldots, l_{[p_k]} \subset Q$ *and non-zero integers* $\alpha_i \in \mathbb{C}, 1 \leq i \leq k-1$ *we define the* $(k-1) \times (k)$*-matrix of linear forms on U:*

$$A = \begin{pmatrix} B^*(p_1) & 0 & \dots & 0 & \alpha_1 B^*(p_k) \\ 0 & \ddots & & 0 & \vdots \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & & & B^*(p_{k-1}) & \alpha_{k-1} B^*(p_k) \end{pmatrix}. \tag{3.5}$$

Then, A defines a monad

$$I^* \otimes \mathcal{O}_Q(-1) \xrightarrow{\alpha^t} W^* \otimes S \xrightarrow{\alpha} I \otimes \mathcal{O}_Q,$$

whose cohomology bundle E is a 't Hooft bundle. Moreover, there exists a section of $E(1)$ vanishing on the set of k-disjoint lines $l_{[p_1]}, \ldots, l_{[p_k]} \subset Q$. Conversely, any general 't Hooft bundle can be described by a monad

$$I^* \otimes \mathcal{O}_Q(-1) \xrightarrow{\alpha^t} W^* \otimes S \xrightarrow{\alpha} I \otimes \mathcal{O}_Q,$$

with α given by a matrix as in (3.5).

Proof Since $A \cdot A^t = 0$, the matrix A defines a morphism

$$\alpha : W^* \otimes S \to I \otimes \mathcal{O}_Q$$

such that $\alpha \circ \alpha^t = 0$. Let us see that α^t is a monomorphism of vector bundles, that is, for any point $x \in Q$,

$$\alpha^t(x) : I^* \to S(1)_x \otimes W^*$$

is injective. To this end, for any point $x \in Q$, we consider the factorization

$$\begin{array}{ll} I^* & \xrightarrow{A^t} U^* \otimes W^* \\ {\scriptstyle \alpha^t(x)} \searrow & \downarrow_{\pi_x \otimes id} \\ & S(1)_x \otimes W^* \end{array}$$

where π_x is part of the short exact sequence

$$0 \to S_x \xrightarrow{\beta_x} U^* \otimes \mathcal{O}_{Q,x} \xrightarrow{\pi_x} S(1)_x \to 0 \tag{3.6}$$

obtained by restricting to $x \in Q$ the universal exact sequence

$$0 \to S \xrightarrow{\beta} U^* \otimes \mathcal{O}_Q \xrightarrow{\pi} S(1) \to 0.$$

Assume that there exists $w \in I^*$ such that $\alpha^t(x)(w) = 0$. Then,

$$(\pi_x \otimes id)(A^t(w)) = 0$$

which implies that $A^t(w) \in \ker(\pi_x \otimes id)$. Since (3.6) is an exact sequence this means that any component of $A^t(w)$ is in $\mathrm{im}(\beta_x)$. Identifying U with U^*, we observe that β_x is the natural injection of the point $x = \langle u, v\rangle \in Q$ into the corresponding subspace $\langle u, v\rangle \subset U$. If we denote by $w = (w_1, \ldots, w_{k-1})$,

$$A^t(w) = \begin{pmatrix} w_1 B^*(p_1) \\ w_2 B^*(p_2) \\ \vdots \\ w_{k-1} B^*(p_{k-1}) \\ (\sum_{i=1}^{k-1} w_i \alpha_i) B^*(p_k) \end{pmatrix}.$$

Therefore, the fact that any component of $A^t(w)$ is in $im(\beta_x)$ is equivalent to

$$\begin{array}{ll} w_i p_i \in \langle u, v\rangle & 1 \le i \le k-1 \\ (\sum_{i=1}^{k-1} w_i \alpha_i) p_k \in \langle u, v\rangle. & \end{array} \tag{3.7}$$

Finally, since $x = \langle u, v\rangle \in Q, B(u, v) = 0$ and the condition (3.7) implies that if there exists an s such that $w_s \neq 0$, then $B(p_s, p_k) = 0$ which contradicts the fact that $l_{[p_1]}, \ldots, l_{[p_k]} \subset Q$ are k disjoint lines. Therefore, α^t is a monomorphism of vector bundles.

So, A defines a monad

$$I^* \otimes \mathcal{O}_Q(-1) \xrightarrow{\alpha^t} W^* \otimes S \xrightarrow{\alpha} I \otimes \mathcal{O}_Q$$

whose cohomology bundle E is a k-instanton bundle (Lemma 3.2). Denote by $K = \ker(\alpha)$ and consider the two short exact sequences

$$\begin{gathered} 0 \to K \to W^* \otimes S \xrightarrow{\alpha} I \otimes \mathcal{O}_Q \to 0, \\ 0 \to I^* \otimes \mathcal{O}_Q(-1) \to K \to E \to 0. \end{gathered}$$

Twisting by $\mathcal{O}_Q(1)$ the last exact sequence and taking cohomology we deduce

$$\mathrm{h}^0 E(1) = \mathrm{h}^0 K(1) - (k-1).$$

On the other hand, $w_i := (0, \ldots, 0, \overbrace{B^*(p_i)}^{i^{th}}, 0, \ldots, 0) \in (U^*)^k$, for $1 \le i \le k$, define k independent sections of $K(1)$. Hence, $E(1)$ has at least a non-zero section s and thus E is a 't Hooft bundle. Moreover, by Proposition 3.8, $E(1)$ has a section vanishing on the union of the k-disjoint lines $l_{[p_1]}, \ldots, l_{[p_k]} \subset Q$.

Let us prove the converse. Let E be a general 't Hooft bundle. By Proposition 3.8, $E(1)$ has a section vanishing on the union of k-disjoint lines $l_{[p_1]}, \ldots, l_{[p_k]} \subset Q$. So, by the Hartshorne-Serre construction it fits into a non-trivial extension:

$$e : 0 \to \mathcal{O}_Q(-1) \to E \to I_{L,Q} \to 0 \tag{11}$$

being L the union of the k-disjoint lines $l_{[p_1]}, \ldots, l_{[p_k]} \subset Q$. Since E is a rank two vector bundle, the extension e,

$$e = (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_k) \in \mathbb{P}(\mathrm{Ext}^1(I_{L,Q}, \mathcal{O}_Q(-1))) \cong \mathbb{P}^{k-1}$$

satisfies $\tilde{\alpha}_i \neq 0$ for $1 \leq i \leq k$. Denote by $\alpha_i = \frac{\tilde{\alpha}_i}{\tilde{\alpha}_k}$, for $1 \leq i \leq k-1$, and define

$$A = \begin{pmatrix} B^*(p_1) & 0 & \ldots & 0 & \alpha_1 B^*(p_k) \\ 0 & \ddots & & 0 & \vdots \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & & & B^*(p_{k-1}) & \alpha_{k-1} B^*(p_k) \end{pmatrix}.$$

By Hartshorne-Serre correspondence, E is the cohomology of the monad defined by A. □

Remark 3.13 From the fact that any general 't Hooft bundle is given as the cohomology of a monad as stated in Proposition 3.12 and from Faenzi (2014; Theorem C), we deduce that for $k \geq 3$ the 't Hooft bundles define a smooth subvariety of dimension $4k-1$ inside an irreducible component of the moduli space of k-instanton bundles of dimension $6k-6$.

From Serre correspondence and the above Remark 3.13 we deduce

Corollary 3.14 *Let E be a 't Hooft bundle on Q. Then*

$$h^0(E(1)) \leq 2.$$

For $k=1$, E is the Spinor bundle and for $k=2$, any 't Hooft bundle satisfies $h^0(E(1)) = 2$. For $k \geq 3$, generically $h^0(E(1)) = 1$ and $h^0(E(1)) = 2$ if the lines lie in a 2-dimensional quadric.

It follows from the dimension counting in Remark 3.13 that for $k \geq 3$, the closure of the family of 't Hooft bundles inside the moduli space of k-instanton bundles defines a proper closed subset. In the next Example we will explicitly construct a family of k-instanton bundles which are not 't Hooft bundles.

Example 3.15 Let $k \geq 3$ be an integer. First of all let us see that there is an elliptic curve $Y \subset Q$ of degree $k+4$ and $H^0(I_{Y,Q}(2)) = 0$. By Ballico et al. (2014; Lemma 6.2), there is a smooth elliptic curve $C \subset Q$ of degree 7 and $H^0(I_{C,Q}(2)) = 0$. Let

$H \subset \mathbb{P}^4$ be a general hyperplane and denote by $Q_2 = Q \cap H$ the smooth quadric surface obtained by cutting Q with H. Let $p_1, \ldots, p_7$ be the intersection points of C with H and let l be a line in Q_2 passing through one of this points, say p_1, and avoiding the rest of the points $p_2, \ldots, p_7$.

For $k = 4$, take $Y = C \cup l$. For $k \geq 5$, let $l_1, \ldots, l_{k-4}$ be lines lying on Q_2 belonging to the same ruling such that for any j, $1 \leq j \leq k-4$, $l_j \cap l$ is a point and l_j does not pass through p_i, $1 \leq i \leq 7$, and take

$$Y = C \cup l \cup l_1 \cup \cdots \cup l_{k-4}.$$

Notice that since Y is constructed by adding to C lines meeting transversally to C in a point, we have

$$p_a(Y) = p_a(C) = 1,$$

that is, Y is an elliptic curve. Moreover, since $H^0(I_{C,Q}(2)) = 0$ we also have $H^0(I_{Y,Q}(2)) = 0$. Since

$$\mathrm{Ext}^1(I_{Y,Q}(3), \mathcal{O}_Q) \cong H^0(\omega_Y) \cong H^0(\mathcal{O}_Y) \cong \mathbb{C},$$

using Serre's construction, we get that a non-zero global section $s \in H^0(\omega_Y)$ produces a rank 2 vector bundle E on Q given by the extension

$$0 \to \mathcal{O}_Q(-2) \to E \to I_{Y,Q}(1) \to 0.$$

Notice that $c_1(E) = 1 - 2 = -1$ and $c_2(E) = deg(C) - 4 = k$. Since $H^0(I_{Y,Q}(2)) = 0$, it follows from the above exact sequence that $H^0(E(1)) = 0$ (which in particular implies that E is stable) and that $H^1(E(-1)) = H^1(I_{Y,Q}) = 0$. Therefore E is a k-instanton bundle on Q which is not a 't Hooft bundle.

Proposition 3.16 *Fix k an integer and let $i \in \mathbb{Z}, 0 \leq i \leq k-1$. Then, there exists a k-instanton bundle E on Q with a jumping line $l_{[q]} \subset Q$ of order i.*

Proof Let $l_{[q]} \subset Q$ be a line associated to $[q] \in \mathbb{P}(U)$. Notice that

$$\{v \in U | B(u,q) = 0\}$$

defines a linear hyperplane on $\mathbb{P}(\bigwedge^2 U)$. Hence, we can choose points $[p_1], \ldots, [p_k] \in \mathbb{P}(U)$ such that

$$\begin{aligned} &B(p_r, p_s) \neq 0, \quad 1 \leq r, s, \leq k \\ &B(p_r, q) = 0, \quad 1 \leq r \leq i+1 \\ &B(p_r, q) \neq 0, \quad i+2 \leq r \leq k. \end{aligned} \tag{3.9}$$

Let $l_{[p_1]}, \ldots, l_{[p_k]} \subset Q$ be k disjoint lines on Q defined by $[p_1], \ldots, [p_k] \in \mathbb{P}(U)$ and let E be the cohomogy bundle of the monad

$$I^* \otimes \mathcal{O}_Q(-1) \stackrel{D \circ \alpha^t}{\longrightarrow} W^* \otimes S \stackrel{\alpha}{\longrightarrow} I \otimes \mathcal{O}_Q,$$

with

$$\alpha = \begin{pmatrix} B^*(p_1) & 0 & \dots & 0 & B^*(p_k) \\ 0 & \ddots & & 0 & \vdots \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & & & B^*(p_{k-1}) & B^*(p_k) \end{pmatrix}.$$

By Proposition 3.12, E is a k-instanton bundle and it follows from (3.9) that $rk(\alpha(q)) = k - (i+1)$ which by Lemma 3.6 means that $l_{[q]}$ is an i-jumping line of E. □

We will end this section describing the variety of jumping lines of a 't Hooft bundle. To this end, given any k-instanton bundle E, we will denote by $J(E)$ its variety of jumping lines.

Proposition 3.17 *Let E be a general 't Hooft instanton bundle. Then, $J(E)$ is a curve of degree $\binom{k}{2}$ and its singular locus is a set of $\binom{k}{3}$ points.*

Proof Let E be a general 't Hooft instanton bundle. According to Proposition 3.12, E is given by a monad of the following type

$$I^* \otimes \mathcal{O}_Q(-1) \stackrel{\alpha^t}{\longrightarrow} W^* \otimes S \stackrel{\alpha}{\longrightarrow} I \otimes \mathcal{O}_Q,$$

being $l_{[p_1]}, \dots, l_{[p_k]} \subset Q$ k-lines on Q and α is associated to the matrix

$$A = \begin{pmatrix} B^*(p_1) & 0 & \dots & 0 & \alpha_1 B^*(p_k) \\ 0 & \ddots & & 0 & \vdots \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & & & B^*(p_{k-1}) & \alpha_{k-1} B^*(p_k) \end{pmatrix}. \tag{3.10}$$

On the other hand, by Lemma 2.1, $l_{[q]}$ is a jumping line of order i if and only if $rk(\alpha(q)) = k - (i+1)$. Hence, $l_{[q]}$ is a jumping line of E if and only if there exist p_r and p_s such that

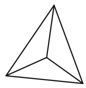
$$B(q, p_r) = 0,$$
$$B(q, p_s) = 0.$$

Thus, $J(E)$ is given by

$$J(E) = \bigcup_{1 \leq r,s \leq k} V(B^*(p_r), B^*(p_s))$$

and therefore it is a curve of degree $\binom{k}{2}$ and its singular locus is a set of $\binom{k}{3}$ points. □

Remark 3.18 The following picture illustrates the curve of jumping lines of a general 't Hooft bundle for $k = 4$:



4 On Instanton Bundles on Higher Dimensional Quadrics. Open Questions

Mimicking the construction of rank 2 instanton bundles on Q_3 and taking into account that they always can be realized as cohomology bundles of a monad as shown in Lemma 3.2, we are led to introduce the following definition of instanton bundle on any odd dimensional quadric Q_{2n+1}:

Definition 4.1 A k-instanton bundle on Q_{2n+1} is a rank $2n$ vector bundle E given as the cohomology bundle of a monad of the type

$$0 \to I \otimes \mathcal{O}_{Q_{2n+1}}(-1) \to W \otimes S \to I^* \otimes \mathcal{O}_{Q_{2n+1}} \to 0$$

where S is the Spinor bundle on Q_{2n+1}, I and W are vector spaces of dimension $k2^{n-1} - n$ and k respectively.

Remark 4.2 Notice that in analogy to Q_3, any k-instanton bundle E on Q_{2n+1} has

$$c_1(E) = -n$$

where we use the identification $Pic(Q_{2n+1}) \cong \mathbb{Z}$. Moreover, since $S^* \cong S(1)$, it is self-dual up to twist:

$$E \cong E^*(-1).$$

Once the definition of k-instanton is settled, the first question that arises is the following:

Question 4.3 Do k-instanton bundles on Q_{2n+1} exist? If so, which are their main features?

We know that they exist on Q_3. Let us now analyze more carefully their existence on Q_{2n+1} for $n \geq 2$. It is well known that $S(1)$ on Q_{2n+1} is generated by its global sections and that $h^0(S(1)) = 2^{n+1}$. Thus, we can consider a vector bundle $G_{k,n}$ given by the exact sequence

$$0 \to \mathcal{O}_{Q_{2n+1}}^{k2^{n-1}-n} \to (S^*)^k \to G_{k,n} \to 0.$$

Conjecture 4.4 *For all $n \geq 1$ and $k \geq 1$, it holds that:*

(a) $G^*_{k,n}(1)$ *is globally generated,*
(b) $c_{2n+1}(G^*_{k,n}(1)) = 0$.

Let us see that if Conjecture 4.4 holds, then we can prove the existence of k-instanton bundles on Q_{2n+1}. To this end, we start recalling the following result (see for instance Tango 1976; Lemma 4).

Lemma 4.5 *Let F be a rank r vector bundle on a smooth projective variety X. Assume that F is generated by its global sections and $c_s(F) = 0$ for some positive integer $s \leq r$. Then F has a trivial vector bundle of rank $r - s + 1$ as a subbundle.*

Since rk $(G_{k,n}) = k2^{n-1} + n$, if Conjecture 4.4 holds, by Lemma 4.5, $G^*_{k,n}(1)$ has a trivial vector subbundle of rank $k2^{n-1} - n$. Thus, there exists a rank $2n$ vector bundle E sitting in the following exact sequence

$$0 \to \mathcal{O}_{Q_{2n+1}}^{k2^{n-1}-n} \to G^*_{k,n}(1) \to E(1) \to 0.$$

In other words, E appears in the following display

$$\begin{array}{ccccccc} & & 0 & & & & \\ & & \downarrow & & & & \\ & & \mathcal{O}_{Q_{2n+1}}(-1)^{k2^{n-1}-n} & & & & \\ & & \downarrow & & & & \\ 0 \to & & G^*_{k,n} & \to S^k \to & \mathcal{O}_{Q_{2n+1}}^{k2^{n-1}-n} & \to 0 \\ & & \downarrow & & & & \\ & & E & & & & \\ & & \downarrow & & & & \\ & & 0 & & & & \end{array}$$

which implies that E is the cohomology of a monad of the type

$$0 \to \mathcal{O}_{Q_{2n+1}}(-1)^{k2^{n-1}-n} \to S^k \to \mathcal{O}_{Q_{2n+1}}^{k2^{n-1}-n} \to 0,$$

that is, E is a k-instanton bundle on Q_{2n+1}. To summarize, we have:

Proposition 4.6 *If Conjecture 4.4 holds, then there exist k-instanton bundles on* Q_{2n+1}.

Example 4.7 Let us construct 2-instantons on Q_5. By Ottaviani (1990) there exists a rank 3 vector bundle F on Q_5 with $c_1(F) = c_2(F) = c_3(F) = 2$ (by abuse of notation in this case we identify the Chern classes with integers), given by the exact sequence

$$0 \to \mathcal{O}_{Q_5} \to S^* \to F \to 0$$

and such that $F^*(1)$ is generated by global sections. On the other hand, S^* on Q_5 is also generated by global sections so we have a rank 6 vector bundle G on Q_5 given by

$$0 \to \mathcal{O}_{Q_5}^2 \xrightarrow{\phi} (S^*)^2 \to G \to 0.$$

Since $(F^*(1))^2$ is globally generated, by a general $\phi, G^*(1)$ is also globally generated. Moreover, since $c_5((F^*(1))^2) = 0$, we have $c_5(G^*(1)) = 0$ and by Lemma 4.5 we obtain a rank 4 vector bundle $E(1)$ given by the exact sequence

$$0 \to \mathcal{O}_{Q_5}^2 \to (G^*(1)) \to E(1) \to 0.$$

It is easy to see that E is a 2-instanton on Q_5.

We have been able to prove the Conjecture 4.4 (b) for values of $n \le 9$ and $k \le 3$ (moreover its veracity for higher concrete values of k and n can be checked using a computer system), but the proof for arbitrary n and k still remains as an open problem. Let us give an example which supports our conjecture:

Example 4.8 Let us see that for $n = 4$ and $k = 1, c_9(G_1^*(1)) = 0$. First of all recall that

$$H^*(\mathbb{Z}, Q_9) \cong \mathbb{Z}e_1 + \mathbb{Z}e_2 + \cdots + \mathbb{Z}e_9$$

with the intersection product given by $e_1^r = e_r$ if $r \le 4$ and $e_1^r = 2e_r$ if $r \le 5$. From the short exact sequence

$$0 \to \mathcal{O}_{Q_9}^4 \to (S^*) \to G_1 \to 0$$

we have $c_i(G_1^*) = c_i(S)$ for $0 \le i \le 9$ and therefore

$$c_9(G_1^*(1)) = \sum_{i=0}^{9} \binom{12-i}{9-i} d_i e_i (e_1)^{9-i}$$

where we denote by $d_i e_i$ the i-th Chern class of the Spinor bundle S on Q_9. From the fact that $S(1) \cong S^*$ together with the short exact sequence

$$0 \to S \to \mathcal{O}_{Q_9}^{2^5} \to S(1) \to 0,$$

we get the following relations between the classes $d_i e_i$:

$$(\sum_{i=0}^{9} d_i e_i t^i)(\sum_{i=0}^{9} (-1)^i d_i e_i t^i) = 1$$

and for $1 \le s \le 4$,

$$-d_{2s+1} e_{2s+1} = c_{2s+1}(S(1)) = \sum_{i=0}^{2s+1} \binom{16-i}{2s+1-i} d_i e_i (e_1)^{2s+1-i}.$$

Using these relations we obtain:

$$\begin{aligned}
d_1 e_1 &= -8 e_1 \\
d_2 e_2 &= \frac{1}{2} d_1^2 e_1^2 = 32 e_2 \\
d_3 e_3 &= -\frac{1}{2} \sum_{i=0}^{2} \binom{16-i}{3-i} d_i e_i (e_1)^{3-i} = -84 e_3 \\
d_4 e_4 &= \frac{1}{2} (2 d_1 d_3 e_1 e_3 - d_2^2 e_2^2) = 160 e_4 \\
d_5 e_5 &= -\frac{1}{2} \sum_{i=0}^{4} \binom{16-i}{5-i} d_i e_i (e_1)^{5-i} = -464 e_5 \\
d_6 e_6 &= \frac{1}{2} (2 d_1 d_5 e_1 e_5 - 2 d_2 d_4 e_2 e_4 + d_3^2 e_3^2) = 528 e_6 \\
d_7 e_7 &= -\frac{1}{2} \sum_{i=0}^{6} \binom{16-i}{7-i} d_i e_i (e_1)^{7-i} = -484 e_7 \\
d_8 e_8 &= \frac{1}{2} (2 d_1 d_7 e_1 e_7 - 2 d_2 d_6 e_2 e_6 + 2 d_3 d_5 e_3 e_5 - d_4^2 e_4^2) = 352 e_8 \\
d_9 e_9 &= -\frac{1}{2} \sum_{i=0}^{8} \binom{16-i}{9-i} d_i e_i (e_1)^{9-i} = -176 e_9.
\end{aligned}$$

Putting altogether we get

$$
\begin{aligned}
c_9(G_1^*(1)) &= \sum_{i=0}^{9} \binom{12-i}{9-i} d_i e_i (e_1)^{9-i} \\
&= [\binom{12}{9}2 - 8\binom{11}{8}2 + 32\binom{10}{7}2 - 84\binom{9}{6}2 + 160\binom{8}{5}2 \\
&\quad - 464\binom{7}{4} + 528\binom{6}{3} - 484\binom{5}{2} + 352\binom{4}{1} - 176\binom{3}{0}]e_9 = 0.
\end{aligned}
$$

We would like to end the paper with some questions related with k-instanton bundles on Q_{2n+1}:

- Do k-instanton bundles on Q_{2n+1} exist?
- Which is the cohomological characterization of k-instanton bundles on Q_{2n+1}?
- Are they simple, semistable or even more stable?
- What can be say about the irreducible component of the Maruyama moduli scheme containing k-instanton bundles?
- What can be said about its variety of jumping lines?
- What can be said about their restriction to maximal linear subspaces of Q_{2n+1}?

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References

Ballico, E., Huh, S. and Malaspina, F., *Globally generated vector bundles of rank 2 on a smooth quadric 3-fold*, Journal of Pure and Applied Algebra **218** (2014), 197–207.

Böhmer, W. and Trautmann, G., *Special instanton bundles and poncelet curves*. Singularities, representation of algebras, and vector bundles (Lambrecht, 1985), 325–336, Lecture Notes in Math. **1273**, Springer, Berlin, 1987.

Coanda, I. and Faenzi, D., *A refined stable restriction theorem for vector bundles on quadric threefolds*. Ann. Mat. Pura Appl. **193** (2014), 859–887.

Ein, L. and Sols, I., *Stable vector bundles on quadric hypersurfaces*, Nagoya Math J. **96** (1984), 11–22.

Faenzi, D., *Even and odd instanton bundles on Fano threefolds of Picard number 1*. Manuscripta Math. **155** (2014), 199–239.

Kuznetsov, A., *Instanton bundles on Fano threefolds*. Central European Journal of Mathematics **10** (2012), 1198–1231.

Loria, G. and Segre, C., *Sur les différentes espèces de complexes du 2éme degré de droites, qui coupent harmoniquement deux surfaces du 2éme ordre*, Mathematische Annalen **XXIII** (1884), 213–234.

Ottaviani, O., *Some extensions of Horrocks criterion to vector bundles on Grassmannians and quadrics*, Ann. Mat. Pura Appl. **155** (1989), 317–341.

Ottaviani, G., *On Cayley bundles on the five dimensional quadric*, Bull U.M.I. (1990), 87–100.

Segre, C., *Note sur les complexes quadratiques dont la surface singulire est une surface du 2me dgr double*, Mathematische Annalen **XXIII** (1884), 235–243.

Tango, H., *An example of indecomposable vector bundle of rank* $n-1$ *on* $\mathbb{P}^n$, J. Math. Kyoto Univ. **16** (1976), 137–141.

Corrado Segre and Nodal Cubic Threefolds

Igor Dolgachev

Abstract
We discuss the work of Corrado Segre on nodal cubic hypersurfaces with emphasis on the cases of 6-nodal and 10-nodal cubics. In particular we discuss the Fano surface of lines and conic bundle structures on such threefolds. We review some of the modern research in algebraic geometry related to Segre's work.

1 Introduction

The following is a detailed exposition of my talk devoted to two memoirs of Corrado Segre on irreducible cubic hypersurfaces in $\mathbb{P}^4$ with d ordinary double points (nodes) (Segre 1886, 1887). We will show in this article how Segre's work is related to the current research in algebraic geometry.

Let X be an irreducible cubic hypersurface in $\mathbb{P}^4$ with a node q. Choose a hyperplane H that intersects X transversally along a nonsingular cubic surface F and consider the projective coordinates in $\mathbb{P}^4$ such that $H = V(t_0)$ and q is equal to the point $[1, 0, 0, 0, 0]$. Then the equation of X can be written in the form

$$X : t_0 a_2(t_1, t_2, t_3, t_4) + a_3(t_1, t_2, t_3, t_4) = 0, \tag{1.1}$$

where a_2 and a_3 are homogeneous forms of degrees 2 and 3, respectively, such that $Q = V(a_2)$ is a nonsingular quadric surface in H and $F = V(a_3)$. The equations $a_2 = a_3 = 0$ define a curve of degree 6 in the hyperplane H. Following (Finkelnberg 1987) we call it the *associated curve* of X with respect to q, it will be denoted

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by $C(X,q)$. The curve $C(X,q)$ is a curve of bidegree (3, 3) lying on Q. We will assume that it is reduced. Let X' be the proper transform of X under the blow-up of $\mathbb{P}^4$ at the point q. Then the projection map $\mathrm{pr}_q : X' \to \mathbb{P}^3$ defines an isomorphism between X' and the blow-up of $\mathbb{P}^3$ with center at $C(X,q)$. The inverse rational map

$$\alpha : \mathbb{P}^3 \mathbb{P}^4$$

is given by the linear system of cubic containing $C(X,q)$. The latter is spanned by the cubic $F = V(a_3)$ and any cubic of the form $V(a_2 l)$, where l is a linear form in $t_1,\ldots,t_4$. It follows that the rational map α is given by the formula $(t_1,\ldots,t_4) \mapsto (a_3, t_1a_2, t_2a_2, t_3a_2, t_4a_2)$, and hence Q is contracted to the point $q = [1,0,0,0,0]$. Also, it is clear that any singular point $q' \neq q$ of X is projected to a singular point of $C(X,q)$. Since we assume that all singular points are ordinary double points, their images are ordinary double points of $C(X,q)$.

The arithmetical genus of a curve of bidegree (3, 3) on Q is equal to 4. If the curve has more than four double points, it must be reducible. A simple analysis shows that the largest possible number k of double points of $C(X,q)$ is equal to 9. Moreover, $k = 9$ happens if and only if $C(X,q)$ is the union of 6 lines on Q. This gives the following.

Proposition 1.1 *The number* $d = k + 1$ *of ordinary double points of an irreducible cubic hypersurface in* $\mathbb{P}^4$ *is less than or equal to 10.*

It follows from the proof of the previous proposition that the curve $C(X,q)$ is reducible if $d > 5$. The number of its irreducible components is equal to the fourth Betti number $b_4(X)$.[1] The number $b_4(X) - 1$ is called the *defect* def(X) of X. The maximal number of linearly independent homology classes of exceptional curves in any small resolution of X is equal to $d - \mathrm{def}(X)$ (see Finkelnberg and Werner 1989). Note that a small resolution of X may not be a projective variety. The number of projective small resolutions of a nodal cubic threefold X can be found in Finkelnberg and Werner (1989).

In this article we will restrict ourselves with two most interesting, in my view, cases when $d = 10$ or $d = 6$. We will start with the case $d = 10$.

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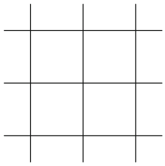
2 Segre 10-Nodal Cubic Primer

By a projective transformation, we can fix the equation $a_2 = 0$ of the quadric Q given in (1.1). The curve $C(X,q)$ has 9 singular points, this forces it to be the union of 9 lines on Q as in the following picture:

[1]This fact was essentially known to Fano (1904a).

$C(X, q)$:



Also, by an automorphism of the quadric, we can fix the curve $C(X, q)$. Two cubic forms defining $F = 0$ and $F' = 0$ that cut out $C(X, q)$ in Q, differ by $a_2 l$, where l is a linear form in $t_1, \ldots, t_4$. Applying the transformation $t_0 \mapsto t_0 + l$, we fix the equation $a_3 = 0$ of the cubic. This shows that

Proposition 2.1 *Two 10-nodal cubic hypersurfaces in* $\mathbb{P}^4$ *are projectively isomorphic.*

We choose one representative of the isomorphism class and denote it by S_3. Any cubic threefold isomorphic to S_3 is called a *Segre cubic primal*. We refer to Dolgachev (2012), 9.4.4 for many beautiful classical facts about such threefolds. Here we add some more.

The polar quadric $V\left(\frac{\partial}{\partial t_0}\right)$ of X at a node q cuts out in X a surface of degree 6 with a point of multiplicity 4 at q which is projected isomorphically outside q onto the quadric $V(a_2)$ and passes through the nodes. This shows that the pre-image of each line component of $C(X, q)$ is a plane in S_3 containing 4 nodes. Also it shows that no three nodes are collinear. Note that a cubic threefold containing a plane can be written in appropriate projective coordinates by equation $xQ_1 + yQ_2 = 0$, where Q_1 and Q_2 are quadratic forms and the plane is given by $x = y = 0$. It follows immediately that the points given by $x = y = Q_1 = Q_2 = 0$ are singular points of the threefold. Their number is less than or equal to 4. Thus we obtain that each node is contained in 6 planes and each plane contains 4 nodes. This gives the following.

Proposition 2.2 *The Segre cubic primal* S_3 *contains 10 nodes and 15 planes that form an abstract configuration* $(15_4, 10_6)$.

This synthetic argument belongs to Segre. Castelnuovo (1891) and later, but independently, Richmond (1902), were able to find the following $\mathfrak{S}_6$-symmetric equations of the Segre cubic (see Dolgachev 2012, 9.4.4).

$$\sum_{i=0}^{5} t_i^3 = \sum_{i=0}^{5} t_i = 0. \tag{2.1}$$

It is checked that the singular points form the $\mathfrak{S}_6$-orbit of the point $[1, 1, 1, -1, -1, -1]$ and the planes form the $\mathfrak{S}_6$-orbit of the plane $t_0 + t_1 = 0, t_2 + t_3 = 0, t_4 + t_5 = 0$.

In his paper Castelnuovo studies linear systems of complexes of lines in $\mathbb{P}^4$, i.e. linear systems of hyperplane sections of the Grassmannian variety $G_1(\mathbb{P}^4)$ in its

Plücker embedding in $\mathbb{P}^9$ (see Dolgachev 2012, 10.2). A 3-dimensional linear system of such complexes defines a rational map

$$f : \mathbb{P}^3\mathbb{P}^4$$

which is given by pfaffians of principal 4×4-matrices of a skew-symmetric matrix of size 5×5 whose entries are linear forms in 4 variables. There are five points of indeterminacy $q_1, \ldots, q_5$ of this map corresponding to skew-symmetric matrices of rank 2. The linear system defining the map f is equal to the linear system $|2h - q_1 - \cdots - q_5|$ of quadrics passing through the points $q_1, \ldots, q_5$, where h is the class of a plane in $\mathbb{P}^3$. If one chooses projective coordinates such that the points q_i become $[1,0,0,0], \ldots, [0,0,0,1], [1,1,1,1]$, then the quadrics from the linear system acquire equations of the following type

$$\sum_{0 \le i < j \le 3} a_{ij} u_i u_j = 0,$$

where $\sum a_{ij} = 0$. The 5-dimensional vector space of such quadrics form the fundamental irreducible representation of the group $\mathfrak{S}_6$. This shows that the image of map f admits a $\mathfrak{S}_6$-symmetry. Castelnuovo and Richmond find a special basis in linear system of quadrics to show that the image of the map f can be given by Eq. (2.1). The images of the 10 lines $\overline{q_i, q_j}$ are the ten nodes of the cubic. The images $\Pi_{i,j,k}$ of the ten planes $\overline{q_i, q_j, q_k}$ and the images Π_i of the five exceptional divisors E_i blown-up from the points q_i are the 15 planes of the cubic.

In an earlier work of Joubert (1867), the Eq. (2.1) appear as the relations between certain six polynomials in roots of a general equation of degree 6. It shows that S_3 should be considered as the set of ordered sets of 6 points in $\mathbb{P}^1$ modulo projective equivalence. Later Coble (1915) made it more precise by proving that the geometric invariant quotient of $(\mathbb{P}^1)^6$ by the group $\mathrm{SL}(2)$ is isomorphic to the Segre cubic

$$\mathrm{S}_3 \cong \mathrm{P}_1^6 := (\mathbb{P}^1)^6 // \mathrm{SL}_2.$$

The rational map $f : \mathbb{P}^3 \mathrm{S}_3$ can be extended to a regular map $\tilde{f} : X \to \mathrm{S}_3$, where $X \to \mathbb{P}^3$ is the composition of the blow-up of the five points $q_1, \ldots, q_5$ and the blow-up of the proper transforms of the lines $\overline{q_i, q_j}$.

$$\begin{array}{ccc} \overline{\mathcal{M}}_{0,6} & \longrightarrow & \mathbb{P}^3. \\ & \searrow \quad \swarrow_{f} & \\ & \mathrm{S}_3 & \end{array} \tag{2.2}$$

In modern times, the variety X appears as the special case of Kapranov's realization of the Knudsen-Mumford moduli space $\overline{\mathcal{M}}_{0,6}$ of stable rational curves with 6 marked ordered points (Kapranov 1993). For any $n \ge 4$, one chooses $n-1$ points in general linear position in $\mathbb{P}^{n-3}$, then start blowing up the points, then the proper transforms of lines joining two points, then proper transforms of planes joining 3 points, and so on. The result is isomorphic to the Knudsen-Mumford moduli space $\overline{\mathcal{M}}_{0,n}$ of stable rational curves with n marked ordered points. If one chooses a general point q in $\mathbb{P}^n$ and passes through it and the points $q_1, \ldots, q_{n+2}$ the unique normal rational curve $C(q)$ of degree n, then one finds n points on $C(q) \cong \mathbb{P}^1$, namely, the points $q_1, \ldots, q_n$ and the point q.

It follows from Kapranov's realization of the variety $\overline{M}_{0,n}$ as a Chow quotient that we have the following commutative triangle of regular maps

$$\begin{array}{ccc} \bar{M}_{0,n} & \xrightarrow{\ \pi\ } & \mathbb{P}^{n-3}, \\ & \searrow \quad \swarrow_{f_n} & \\ & \mathsf{P}_1^n := (\mathbb{P}^1)^n//\mathrm{SL}(2) & \end{array}$$

If $n = 6$, the image of f_n is the Segre cubic S_3.

In the case when $n = 2g + 2$ is even, the geometric invariant theory quotient $\mathrm{P}_1^{2g+2} := (\mathbb{P}^1)^{2g+2}//\mathrm{SL}(2)$ is isomorphic to a compactification of the moduli space of hyperelliptic curves of genus g together with a full 2-level structure, i.e. a choice of a standard symplectic basis in the group of 2-torsion divisor classes. In Coble (1930) shows that the map $f_{2g+2} : \mathbb{P}^{2g-1}\mathrm{P}_1^{2g+2}$ is given by the linear system

$$|gh - (g-1)(q_1 + \cdots + q_{2g+1})|. \tag{2.3}$$

It maps $\mathbb{P}^{2g-1}$ to the projective space $\mathbb{P}^{N-1}$, where $N = \binom{2g}{g} - \binom{2g}{2g-2}$. The image of this map is isomorphic to P_1^{2g+2}.

Also, Coble shows that the projection of P_1^{2g+2} from a general point p defines a degree 2 map onto the Kummer variety $\mathrm{Kum}(\mathrm{Jac}(C_p))$ associated with the Jacobian variety of the hyperelliptic curve C_p corresponding to the point p. It lies in $\mathbb{P}^{2^g-1}$ and embeds there by the map $\mathrm{Jac}(C_p) \to \mathrm{Kum}(\mathrm{Jac}(C_p)) \to \mathbb{P}^{2^g-1}$ given by the linear system $|2\Theta|$, where Θ is the theta divisor of the Jacobian. The locus of singular points of the hypersurfaces from the linear system $|gh - (g-1)(q_1 + \cdots + q_{2g+1}) - q|$, where $f_{2g+2}(q) = p$, is the *Weddle variety* W_g. It maps birationally onto the Kummer variety of C_p. We refer for a modern exposition of Coble's results to Dolgachev (2004) and C. Kumar's paper (2000). Kumar calls the variety P_1^{2g+2} the *generalized Segre variety*. One finds in Kumar's paper a nice relationship between the generalized Segre variety and the theory of vector bundles on hyperelliptic curves.

Note that Segre himself was aware of the relationship between the cubic S_3 and the Kummer quartic surface associated to curves of genus 2. In fact, he shows that the projection of S_3 to $\mathbb{P}^3$ from its nonsingular point is the quartic Kummer surface $\mathrm{Kum}(\mathrm{Jac}(C_p))$. He also shows that the set of nodes of quadrics in $\mathbb{P}^3$ passing through the points $q_1, \ldots, q_5$ and the additional point q is the Weddle quartic surface.

A nice relationship between the generalized Segre variety P_1^{2g+2} and the theory of stable rank 2 vector bundles on not-necessary hyperelliptic curves was recently studied by Alzati and Bolognesi (2016).

Let C_g be a non-hyperelliptic smooth projective curve of genus $g \geq 2$ and let $\mathrm{SU}_{C_g}(2)$ be the moduli space of semi-stable rank 2 bundles on C_g with trivial determinant. Alzati and Bolognesi prove that there exists a rational map

$$\mathrm{SU}_{C_g}(2)\mathbb{P}^g$$

whose fibers are birationally isomorphic to P_1^{2g+2}. If $g = 3$, they are isomorphic to S_3.

Finally note that, according to Finkelnberg (1987), the Segre cubic primal has 1024 small resolutions with 13 isomorphism classes, among them are 332 projective varieties which are divided into 6 isomorphism classes. Since $C(\mathrm{S}_3, q)$ has 6 irreducible components, the rank of the Picard group of a projective small resolution is equal to 5.

3 Cubic Threefolds with 6 Nodes

Let X be a cubic hypersurface in $\mathbb{P}^4$ with six nodes. Choosing one of the nodes of X, we can get the equation of X as in (1.1). Let $Q = V(a_2)$ and $F = V(a_3)$ be the quadric and a cubic surface in the hyperplane $t_0 = 0$ defined by the coefficients a_2 and a_3 of the equation.

Proposition 3.1 *Assume that any five of the nodes of X span $\mathbb{P}^4$. Then $C(X,q) = Q \cap F$ is the union of two rational curves of degree 3 intersecting transversally at 5 points.*

Proof We know that remaining 5 nodes of X are projected to the double points $p_1, \ldots, p_5$ of the curve $C(X,q)$. By assumption, any four of them span $\mathbb{P}^3$. It implies that no four of the points p_i lie on a conic or on a line contained in Q, and hence the curve $C(X,q)$ has no irreducible components of degree ≤ 2. Then an easy computation with the formula for the arithmetic genus of C gives that C consists of two components γ and γ' of degree 3. Since the quadric Q is nonsingular (otherwise q is not an ordinary double point), the curves are curves of types (2, 1) and (1, 2) intersecting at the points $p_1, \ldots, p_5$. These points are the projections of the five nodes of X and if two of them coincide, together with q, they would lie on a line. This contradicts our assumption. □

The following example shows that the condition on the six points is necessary for $C(X,q)$ to be the union of two curves of degree 3.

Example 3.2 Let $Q = V(a_2)$ be a nonsingular quadric and C be a curve on Q equal to the union of a nonsingular conic C_1 and an irreducible curve C_2 of type (2,2) with a node. Let $H = V(l)$ be a plane intersecting Q along C_1 and let $Q' = V(q)$ be a quadric intersecting Q along C_2. Pick up a general plane $H' = V(l')$ such that $F = V(a_2 l' + ql)$ is a nonsingular cubic. Then the cubic threefold given by Eq. (1.1) with a_2, a_3 as above, has 6 nodes with 5 of them spanning $\mathbb{P}^3$ (four of them are projected to the intersection points $C_1 \cap C_2$).

Let X be any 6-nodal cubic such that five of its nodes $q_1, \ldots, q_5$ span a hyperplane H. The intersection $H \cap X$ is a cubic surface with 5 nodes, hence it must be reducible. It is easy to see that this implies that four of the nodes are coplanar. In particular, any five nodes of X span a hyperplane.

We say that a 6-nodal cubic threefold is *nondegenerate* if any subset of five nodes span $\mathbb{P}^4$. Thus Proposition 3.1 applies, and we obtain that the projection from any node gives an Eq. (1.1), where $V(a_2) \cap V(a_3)$ is the union of two rational cubics intersecting at 5 points.

Note that any 5-nodal complete intersection of a quadric and a cubic surfaces in $\mathbb{P}^3$ defines a 6-nodal cubic threefold, and the case from Proposition 3.1 is the only case which gives a nondegenerate 6-nodal cubic threefold.

We shall use the following well-known fact about cubic surfaces several times, so better let us record it.

Lemma 3.3 *Let F be a smooth cubic surface and γ_1, γ_2 be two rational smooth curves of degree 3 on F such that $\gamma_1 + \gamma_2 \in |-2K_F|$. Then the set of 27 lines on F is divided into three disjoint sets:*

- *6 skew lines that do not intersect γ_1 but intersect γ_2 with multiplicity 2,*
- *6 skew lines that do not intersect γ_2 but intersect γ_1 with multiplicity 2,*
- *15 lines that intersect both γ_1 and γ_2 at one point.*

Proof We have $\gamma_i \cdot K_F = -3$, hence $\gamma_i^2 = 1$ and the linear system $|\gamma_i|$ defines a birational morphism $\pi_i : F \to \mathbb{P}^2$. Each of these morphisms blows down six lines that together form a double-six. In the standard (geometric) basis $(e_0, e_1, \ldots, e_6)$ in $\mathrm{Pic}(F)$ defined by π_1, we have $\gamma_1 \sim e_0$ and $\gamma_2 \in |5e_0 - 2(e_1 + \cdots + e_6)|$. The double-six is represented by the classes e_i and $2e_0 - (e_1 + \cdots + e_6) + e_i, i = 1, \ldots, 6$. Each line in the class e_i is blown down by π_1 and intersects γ_2 with multiplicity 2. The remaining lines are represented by the classes $e_0 - e_i - e_j$ which intersect γ_1 and γ_2 with multiplicity 1. □

Let $\mathrm{Bis}(C_i)$ be the surface of bisecant lines of C_i. It is naturally isomorphic to the symmetric product of C_i, and hence to $\mathbb{P}^2$. In the Plücker embedding of the Grassmann variety $G_1(\mathbb{P}^3) \subset \mathbb{P}^5$, it is isomorphic to a Veronese surface. Let $\ell_1, \ldots, \ell_6$ (resp. $\ell'_1, \ldots, \ell_6$) be the set of lines in $\mathrm{Bis}(C_1)$ (resp. $\mathrm{Bis}(C_2)$) corresponding to a double-six of lines on F via the previous lemma. For any $x \in F$, there exists a unique bisecant line of C_1 (resp. C_2) passing through x. This defines two maps $\pi_1 : F \to \mathrm{Bis}(C_1), \pi_2 : F \to \mathrm{Bis}(C_2)$ that blows down the lines $\ell_1, \ldots, \ell_6$ (resp. $\ell'_1, \ldots, \ell'_6$).

Proposition 3.4 *Let X be a nondegenerate 6-nodal cubic hypersurface and let $C(X,q) = Q \cap F$ be the fundamental curve associated to q. Let S be the blow-up of Q at the five nodes $q_1, \ldots, q_5$ of $C(X,q)$. Then S is isomorphic to a nonsingular cubic surface.*

Proof The linear system of curves $|D| = |\mathcal{O}_Q(2) - p_1 - \cdots - p_5|$ of bidegree (2,2) containing the five nodes $p_1, \ldots, p_5$ of $C(X,q)$ is 3-dimensional and defines a regular map $f : S \to S'$ onto a cubic surface $S' \subset |D|^* \cong \mathbb{P}^3$. Since X is nondegenerate, the images of the 5 exceptional curves $E(q_i)$, the 10 lines each passing through one of the points p_i (obviously, no two of them lie on one line), 10 conics passing through three of the points q_i, and the curves C_1, C_2 are the 27 lines on S'. This implies that $S' \cong S$ is a nonsingular cubic surface. □

Let is add some remarks to the previous construction. The lines $\ell_1 = f(C_1)$ and $\ell_2 = f(C_2)$ are skew lines on S', and the images of the exceptional curves $E(q_i)$ is the set of 5 skew lines that intersect both ℓ_1 and ℓ_2. The composition $f' = \sigma \circ f^{-1} : S' \to Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ is given by the two pencils of conics cut out by the pencils of planes $\mathcal{P}_1, \mathcal{P}_2$ in $\mathbb{P}^3$ containing ℓ_1, ℓ_2, respectively. This map factors through an isomorphism $f^{-1} : S' \to S$ which is inverse of the morphism $f : S \to S'$.

For any $x \in S'$ there exists a unique line in $\mathbb{P}^3$ intersecting ℓ_1 and ℓ_2. It is obvious, if $x \notin \ell_1 \cup \ell_2$. If $x \in \ell_1$, we choose the line contained in the plane spanned by ℓ_2 and x and in the plane containing ℓ_1 and tangent to S' at x. Assigning to x the intersection points of this line with ℓ_1 and ℓ_2, we obtain that the surface S', and hence S, is isomorphic to the irreducible surface S'' in $G_1(\mathbb{P}^3)$ of lines in $\mathbb{P}^3$ that intersects both components of $C(X,q)$. The lines passing through the singular points of $C(X,q)$ must be tangent to the quadric at these points. Such lines are 5 lines on S'' that correspond to the skew lines intersecting ℓ_1 and ℓ_2.

Segre proves that any nondegenerate 6-nodal cubic hypersurface can be projectively generated. This means that one can find three projectively equivalent nets of hyperplanes.

$$H(\lambda)_j := \sum_{i=0}^{4} a_i^{(j)}(\lambda) t_i = 0, \quad j = 0, 1, 2,$$

such that

$$X = \{x \in \mathbb{P}^4 : x \in H_1(\lambda) \cap H_2(\lambda) \cap H_3(\lambda) \text{ for some } \lambda\}.$$

Let us rewrite these equations in the following form

$$\lambda_0 l_{0j}(t) + \lambda_1 l_{1j}(t) + \lambda_2 l_{2j}(x) = 0, \; j = 0, 1, 2,$$

where $l_{ij}(t)$ are linear forms in variables t_0, t_1, t_2, t_3, t_4. Then

$$X = \{x \in \mathbb{P}^4 : \det(l_{ij}(x)) = 0\}. \tag{3.1}$$

The six nodes of X are the points x such that $\mathrm{rank}(l_{ij}(x)) = 1$.

We see from formula (3.1) that a projective generation gives a determinantal representation of X. Conversely, the determinantal representation defines a projective generation.

Segre's proof is rather cumbersome and I had a difficulty to follow it. A modern proof was given by Hassett and Tschinkel (2010). They deduce a determinantal representation of X from a determinantal representation of a certain cubic surface associated to X. Let us reproduce a modified version of their proof that, in my opinion, is more straightforward and constructive.

Theorem 3.5 *Let X be a nondegenerate 6-nodal cubic hypersurface in $\mathbb{P}^4$. Then X is isomorphic to the hypersurface $V(\det(A))$, where A is a 3×3-matrix with linear form in coordinates on $\mathbb{P}^4$.*

Proof A normal cubic surface has only double rational points as its singularities. Each cubic surface without singular point of type E_6 is determinantal. This means that there is an embedding of $\mathbb{P}^3$ in the projective space $\mathbb{P}^8$ of 3×3 matrices such that the pre-image of the determinantal cubic hypersurface D_3 in $\mathbb{P}^8$ is equal to X. This was proved first by L. Cremona in 1868 (with a gap related to the assumption on the singularities). C. Segre had filled the gap in 1906. We refer to Dolgachev (2012), 9.3 for the details.

So, our cubic surface $F = V(a_3)$, being nonsingular by our choice of projective coordinates, admits a determinantal representation. Let us recall its construction. We assume that F is a smooth cubic surface in the projective space $|W|$ of lines in a linear vector space W of dimension 4. A determinantal equation of F is defined by a choice of a linear system $|\gamma_1|$ of curves of degree 3 with $\gamma_1^2 = 1$. Let $|\gamma_2| = |-2K_F - \gamma_1|$ be represented by a smooth rational curve γ_2 with $\gamma_2^2 = 1$. The pair of smooth curves (γ_1, γ_2) is a pair from Lemma 3.3. Let $\pi_i : F \to |\gamma_i|^*$ be the corresponding birational morphisms. The birational map $\pi_2 \circ \pi_1^{-1} : |\gamma_1|^* |\gamma_2|^*$ is defined by the linear system of curves of degree 5 with double points at the points $p_i = \pi_1(\ell_i)$, where $(\ell_1, \ldots, \ell_6)$ is the sixer of lines blown down by π_1. Consider the natural map defined by adding the divisors

$$|\gamma_1| \times |\gamma_2| \to |6e_0 - 2e_1 - \cdots - 2e_6| = |-2K_F| \cong |\mathcal{O}_{|W|}(2)|. \tag{3.2}$$

The image of this map is a hyperplane in $|\mathcal{O}_{|W|}(2)|$ orthogonal to a quadric $\mathcal{Q}^*$ in the dual space $|W^\vee|$. It is the dual quadric of the Schur quadric $\mathcal{Q}$ associated to the double-six of lines blown-down by π_1 and π_2 (see Dolgachev 2012, 9.1.3). Composing (3.2) with a linear function defined by $\mathcal{Q}^*$, we can identify the plane $|\gamma_2|$ with the plane $|\gamma_1|^*$, the dual plane of $|\gamma_1|$. Let $|\gamma_1| = |U|$ for some 3-dimensional linear space U. Then the map $\pi_1 \times \pi_2 : F \to |\gamma_1|^* \times |\gamma_2|^*$ can be identified with the linear map

$$j : F \to |U^*| \times |U| \hookrightarrow |U^* \otimes U| = |\mathrm{End}(U)| \cong \mathbb{P}^8.$$

Let D_3 be the determinantal hypersurface in $|\mathrm{End}(U)|$. It is a cubic hypersurface with singular locus equal to $|U^*| \times |U|$ (it parameterizes endomorphisms of rank 1). The determinantal representation of F is defined by the embedding $|W| \hookrightarrow |\mathrm{End}(U^*)| \cong |\mathrm{End}(U)|^*$ such that the pre-image of the determinantal hypersurface is equal to S and the map π_1 (resp. π_2) is defined by taking the kernel of the corresponding endomorphism (resp. its transpose) of U^*. All of this is well-known and can be found in Dolgachev (2012).

Now let X be a nondegenerate 6-nodal cubic hypersurface given by Eq. (1.1) and $C_1 + C_2 = C(X, q)$ be the associated curve with respect to a node q. It follows from the above discussion that the linear systems $|C_1|$ and $|C_2|$ define two determinantal representation of F, each is obtained from another by taking the transpose of the matrix.

Choose a basis in the linear space $U = H^0(F, \mathcal{O}_F(\gamma_1))$ and the dual basis in U^*, then we can identify $\mathrm{End}(U)$ with the space of 3×3-matrices and the determinantal representation of F gives a matrix $B = (b_{ij})$ whose entries are linear forms in $t_1, \ldots, t_4$ such that

$$F = V(\det B).$$

We are looking for a matrix $\tilde{B} = (\tilde{b}_{ij})$ whose entries are linear forms in $t_0, \ldots, t_4$ such that

$$X = V(\det \tilde{B}).$$

If we plug in $t_0 = 0$ in the entries of $\tilde{B}$, we should obtain a matrix equal to B, up to a scalar multiple. This shows that $\tilde{B} = t_0 A + B$, where A is a constant matrix. Write $A = [A_1 A_2 A_3]$ and $B = [B_1 B_2 B_3]$ as the collection of its columns. The usual formula for the determinant of the sum of the matrices shows that

$$\det \tilde{B} = t_0^3 \det A + t_0^2(\det[B_1 A_2 A_3] + \ldots) + t_0(\det[B_1 B_2 A_3] + \ldots) + \det B. \quad (3.3)$$

To make this expression equal to $t_0 a_2 + a_3$ from (1.1), we have to take A with rank equal to 1. So we may assume that the columns of A are equal to some nonzero vector $\mathbf{v} = (\alpha_0, \alpha_1, \alpha_2)$.

Let $\pi_1 : F \to \mathbb{P}^2 = |U|$ and $\pi_2 : F \to |U^*|$ be the two maps defined by the right and the left kernels of the matrix B. Let $\ell = \pi(\gamma)$ and $\ell' = \pi'(\gamma')$. The curve ℓ is a line in $\mathbb{P}^2$, the curve ℓ' is a line in the dual plane. Observe that, for any $y = [t_1, \ldots, t_4] \in F$, the adjugate matrix adj$B$ of B is of rank 1. Thus, for any $x \in F$, the equations $\det[B_1(x)B_2(x)\mathbf{v}] = 0, \det[B_1(x)\mathbf{v}B_3(x)] = 0, \det[\mathbf{v}B_2(x)B_3(x)] = 0$ with unknown vector $\mathbf{v}$, are the equations of the same line $\ell(x)$ in $\mathbb{P}^2$ which we consider as a point in the dual plane. When x runs C_1, the image $\pi_1(C_1)$ is a line in the plane, hence the set of lines $\ell(x), x \in C_1$, is a line in the dual plane. If we take it to be equal to the line equal to $\pi_2(C_2)$, we obtain that the coefficient at t_0 in (3.3) is equal to zero for any $x \in C_1 \cup C_2$. Thus the quadric Q' defined by this coefficient coincides with the quadric Q, and we are done. □

Remark 3.6 Suppose Eq. (1.1) of a nodal cubic threefold can be brought to the form $\det A(t) = 0$. Then, plugging in $t_0 = 0$, we obtain a determinantal representation of the cubic surface $F = V(a_3)$. This shows that F has at most rational double points of type different from E_6. Also, since the discriminant variety D_3 has the double locus of degree 6, we obtain that the singular locus of X is either of dimension ≥ 1, or consists of isolated singular points whose Milnor numbers add up to 6.

Let X be a nondegenerate 6-nodal cubic threefold with nodes $q_1, \ldots, q_6$. The linear system of cubics $|\mathcal{O}_{\mathbb{P}^4}(3) - 2q_1 - \cdots - 2q_6|$ with double points at $q_1, \ldots, q_6$ defines a rational map $f : \mathbb{P}^4\mathrm{S}_3$ to the Segre cubic primal S_3 in $\mathbb{P}^4$. Its fibers are quartic rational normal curves passing through the nodes. In Kapranov's realization of $\mathcal{M}_{0,7}$ this corresponds to the composition of the projection $\mathcal{M}_{0,7} \to \mathcal{M}_{0,6}$ and the map $\mathcal{M}_{0,6} \to \mathrm{S}_3$ from (2.2). This shows that X is birationally isomorphic to the pre-image of a hyperplane section of S_3. Let X' be the blow-up of X at the nodes, followed by the blow-up of the proper transforms of lines joining two nodes. Then f extends to a regular map $X' \to S$, where S is a hyperplane section of S_3. If we use Eq. (2.1) of S_3, then the additional equation $\sum_{i=0} a_i t_i = 0$ defines a cubic surface S given by Cremona's hexahedral equations (see Dolgachev 2012, 9.4.3). By Theorem 9.4.8 from loc.cit., if S is nonsingular, these equations determine uniquely an ordered double-six of lines on S. Conversely, a choice of an ordered double-six of lines defines Cremona's hexahedral equations.

It is an obvious guess that the cubic surface S is isomorphic to the cubic surface from Proposition 3.1. To see this, we consider the blow-up X' of X at any of its singular point q. The exceptional divisor is identified with the quadric Q_i containing $C(X, q_i)$. The pre-image of the linear system $|\mathcal{O}_{\mathbb{P}^4}(3) - 2(q_1 + \cdots + q_6)|$ to X' restricted to the exceptional divisor $E(q)$ consists of quadrics through the 5 points on Q_i corresponding to the lines joining q_i with other nodes of X. It maps $E(q_i)$ to the hyperplane section of S_3 corresponding to X.

We can easily see the double-six of lines on S identified with the blow-up of Q_i at the singular points of $C(X, q_i)$. It consists of the images of the ten lines on Q passing through the singular points and the curves C_1 and C_2. The linear systems $|\gamma_1|$ and $|\gamma_2|$ defining a determinantal representation of S are the images of the curves of bi-degree (3, 1) and (1, 3) passing through the singular points of $C(X, q)$. The order on the 6 nodes of X defines an order on the set of singular points of $C(X, q)$, hence an order of the lines in a sixer, hence a basis of the Picard group of the cubic surface S.

Finally, let us look at the moduli space of nondegenerate 6-nodal cubic hypersurfaces. By a projective transformation we can fix the nodes, to assume that their coordinates form the reference points in $\mathbb{P}^4$. Then the space of cubics with singular points at the reference points consists of cubics with equations

$$\sum_{0 \le i < j < k \le 4} a_{ijk} t_i t_j t_k = 0,$$

where the coefficients a_{ijk} satisfy

$$\sum a_{0jk} = 0, \sum a_{i1k} = 0, \ldots, \sum a_{ij4} = 0.$$

The dimension of the projective space $|V|$ of such cubics is equal to 4. The permutation group $\mathfrak{S}_6$ acts linearly on V via its natural 5-dimensional irreducible permutation. By above, the linear system of such cubics map $\mathbb{P}^4$ onto the Segre cubic primal S_3. Thus its dual space is identified with the space of hyperplane sections of S_3. The action of $\mathfrak{S}_6$ agrees with the natural action of S_3 on S_3. We know that the orbit of a hyperplane section of S_3 corresponds to the moduli space of cubic surfaces together with an unordered double-six. It is the cover of degree 6 of the moduli space of cubic surface. It is known that such variety is rational (see Bauer and Verra 2010). This gives us the following result.

Theorem 3.7 *The moduli space of nondegenerate 6-nodal cubic threefolds is naturally birationally isomorphic to the moduli space of cubic surfaces together with an unordered double-six of lines. It is a rational variety of dimension 4.*

Remark 3.8 Let $X \subset |W| \cong \mathbb{P}^4$ and $W \to \mathrm{End}(U)$ be a linear map defined by a determinantal representation of X. Consider the nondegenerate bilinear form on $\mathrm{End}(U)$ defined by the trace. It allows one to identify $\mathrm{End}(U)$ with its dual space $\mathrm{End}(U)^\vee$. The orthogonal space $W^\perp$ is of dimension 4, and the linear embedding $W^\perp \hookrightarrow \mathrm{End}(U)$ defines a determinantal representation of the cubic surface $S' = |W^\perp| \cap \mathrm{D}_3$. It is proven in Hassett and Tschinkel (2010) that this cubic surface is isomorphic to the cubic surface S.

Conversely, starting with a determinantal representation of a nonsingular cubic $S \subset |V|$ defined by a linear map $V \to \mathrm{End}(U)$, we obtain a determinantal 6-nodal cubic X by taking the intersection of $|V^\perp|$ with the determinantal hypersurface D_3. This is the approach to determinantal representations of 6-nodal cubic threefolds taken in Hassett and Tschinkel (2010). It is analogous to the earlier construction of Beauville and Donagi (1985) that uses pfaffian hypersurface to pair K3 surfaces of genus 8 and 4-dimensional pfaffian cubic hypersurfaces.

The number of small resolutions of a nondegenerate 6-nodal cubic threefold is equal to 64. Among them there are two projective resolutions (Finkelnberg and Werner 1989). Note that a degenerate irreducible 6-nodal cubic does not admit a small projective resolution.

4 The Fano Surface of Lines

Corrado Segre also studied the surface $F(\mathsf{S}_3)$ of lines in S_3. Later on, Gino Fano wrote two papers which establish the basic facts about the surface of lines in any nodal cubic threefold (Fano 1904a, b). For a modern exposition of some of Fano's result see Altman and Kleiman (1977). Here we remind some known facts about the Fano surface $F(X)$ of lines in a nodal cubic threefold, nowadays called the *Fano surface* of X. In the Plücker embedding, the surface $F(X) \subset G_1(\mathbb{P}^4)$ is a locally complete intersection projectively normal surface of degree 45 canonically embedded in the Plücker space $\mathbb{P}^9$. Its class in $H^4(G_1(\mathbb{P}^4), \mathbb{Z}) \cong \mathbb{Z}^2$ with respect to the basis given by the Schubert cycles σ_1 of lines intersecting a general line and σ_2 of lines in a general hyperplane is equal to (18,27).

The Fano surface $F(X)$ is smooth at any point representing a line that does not pass through a singular point of X. The union of lines passing through a node $q \in X$ is projected from q to the curve $C(X, q)$. In particular, $F(X)$ is a non-normal surface. If the number d of nodes is less than or equal to 5, it is an irreducible surface birationally isomorphic to the surface of bisecant lines of the associated curve $C(X, q)$. Starting from $d = 6$, the curve C become reducible, and $F(X)$ becomes reducible too.

Let us start with the Segre cubic primal $F(\mathsf{S}_3)$. Under the map $f : \mathbb{P}^3 \mathrm{S}_3$ given by the linear system of quadrics through 5 points $p_1, \ldots, p_5$, the image of a line ℓ containing the point p_i is a line in $F(\mathsf{S}_3)$. The set D_i of such lines is isomorphic to a del Pezzo surface of degree 5, the blow-up of 4 points in the plane (the points are of course the lines joining p_i with $p_j, j \neq i$). This gives us five isomorphic irreducible components of $F(\mathsf{S}_3)$. The image of a twisted cubic passing through the points $p_1, \ldots, p_5$ is also a line in S_3. The set of such lines is also isomorphic to a del Pezzo surface of degree 5. To see this, we use the Kapranov's realization of the projection map $\overline{\mathcal{M}}_{0,6} \to \overline{\mathcal{M}}_{0,5}$. So we have 15 components isomorphic to $\mathbb{P}^2$, they are lines containing in a plane in $F(\mathsf{S}_3)$. Each of these planes are the images of a conic in the plane $\langle p_i, p_j, p_k \rangle$ through the points p_i, p_j, p_k or the lines in one of the exceptional divisor E_i over the point q_i. Each line intersects five planes. For example, a line from D_1 intersects the plane corresponding to the exceptional divisor E_i and four planes $\langle p_a, p_b, p_c \rangle, 1 \notin \{a, b, c\}$. So, we see that $F(\mathrm{S}_3)$ is highly reducible, it consists of 6 components isomorphic to a del Pezzo surface of degree 5, and 15 components isomorphic to the projective planes. In the Plücker embedding of $G_1(\mathbb{P}^4)$ they are anti-canonically embedded surfaces of degree 5, and the planes.

The group $\mathfrak{S}_6$ of automorphisms of $F(\mathsf{S}_3)$ acts transitively on the set of 6 components D_i of the Fano surface. The stabilizer subgroup is isomorphic to the group of automorphisms of D_i. It also acts transitively on the 15 plane components of the Fano surface. The stabilizer group is isomorphic to $2^3 \rtimes \mathfrak{S}_3$ of order 48.

Remark 4.1 The description of conic bundle structures on the Segre cubic primal can be also found in Gwena (2005).

Next we consider the case when X is a nondegenerate 6-nodal cubic threefold.

Since X has 6 nodes, the Fano surface is singular along the curve parameterizing lines passing through a node. Fix a node q of X and let $C(X,q) = C_1 + C_2 = Q \cap F$ be the associated curve of degree 6. Each line passing through q is projected to a point on $C(X,q)$. Since $C(X,q)$ consists of two curves of degree 3, we see that the lines through q are parameterized by $C(X,q)$ and their union consists of two cubic cones intersecting along 5 common lines.

Let $\mathrm{Bis}(C)$ be the set of lines in $\mathbb{P}^3$ that intersect $C(X,q)$ at ≥ 2 points counting with multiplicity. For any two points $x,y \in C(X,q)$, let $\ell = \overline{x,y}$ be the line spanned by x,y or tangent to $Q = V(a_2)$ if $x = y$. Suppose ℓ intersects C at two points x,y. The linear system of cubics through $C(X,q)$ maps $\mathbb{P}^3$ to X and the image of ℓ is a line $\ell_{x,y}$ on X. If ℓ_x, ℓ_y are the lines through q which are projected to x,y, then $\ell_{x,y}$ is the residual line of the intersection of the plane spanned by ℓ_x, ℓ_y with X (if $x = y$ we take the tangent plane of X along the line ℓ_x). If ℓ intersects $C(X,q)$ at three point, then it must belong to one of the two rulings of Q. Let C_i be the component of $C(X,q)$ such that this ruling intersects C_i at one point z. Then we assign to it the line ℓ_z passing through q. For any line ℓ in X, let $\Pi(\ell)$ be the unique plane containing q and cutting X in 3 lines. If $q \notin \ell$, then $\Pi(\ell)$ is spanned by ℓ and q. Otherwise, $\Pi(\ell)$ is the unique plane cutting X along three lines passing through q. The projection of $\Pi(\ell)$ to $\mathbb{P}^3$ from the point q is a line from $\mathrm{Bis}(C(X,q))$. It is contained in Q if $\Pi(\ell) \cap X$ consists of three lines passing through q. This defines an isomorphism

$$\mathrm{Bis}(C(X,q)) \cong F(X).$$

Obviously, $\mathrm{Bis}(C(X,q))$ consists of three irreducible components, two of which are isomorphic to $P_i = \mathrm{Bis}(C_i)$. The third component P_3 consists of lines intersecting both C_1 and C_2.

For any general point in $\mathbb{P}^3$ there exists a unique bisecant line of a normal rational cubic. Since a general line intersects X at three points, we see that the Schubert cycle σ_1 intersects the components P_1 and P_2 of $F(X)$ at three points. A general hyperplane H in $\mathbb{P}^4$ intersects X along a nonsingular cubic surface. It also intersects each of the cubic cones of lines passing through q along a cubic rational curve γ_i. The curve γ_i can be identified (under the projection map) with a component C_i of $C(X,q)$. Using Lemma 3.3, we see that this defines a set of 6 skew lines, each intersecting one of the curves γ_i with multiplicity 2. These lines correspond to 6 bisecants of C_i. This shows that the bidegree of the surface $\mathrm{Bis}(C_i)$ in

$G_1(\mathbb{P}^4)$ is equal to (3, 6), it is isomorphic to $\mathbb{P}^2$ embedded by the third Veronese map.

The third component P_3 of $F(X)$ must be of bidegree (12,15). As above, 15 corresponds to the 15 lines in a general hyperplane that intersect γ_1 and γ_2, and 12 corresponds to the fact that through each point x on the intersection of a general line with X passes 4 lines whose projection to $\mathbb{P}^3$ intersects C_1 and C_2 at one point.

It follows from Proposition 3.4 that P_3 is isomorphic to the cubic surface S isomorphic to the blow-up of Q at the singular points of $C(X, q)$. Thus we have a map

$$\phi : P_1 \sqcup P_2 \sqcup P_3 \cong \mathbb{P}^2 \sqcup \mathbb{P}^2 \sqcup S \to F(X)$$

that coincides with the normalization map.

Note that $P_1 \cap P_2$ consists of 10 points corresponding to lines $\overline{q_i, q_j}$. The intersection $P_i \cap P_3$ consists of 5 lines corresponding to the lines through q_i intersecting C_i at some other point.

Remark 4.2 It follows from the above that the isomorphism class of the cubic surface isomorphic to the blow-up of Q at singular points of $C(X, q)$ is independent of a choice of a node.

Remark 4.3 The following nice observation is due to Verra. Take any line ℓ on X and consider its image $f(\ell)$ under the map $f : \mathbb{P}^4 \to \mathbb{P}^4$ given by the linear system of cubics through the nodes of X. As we saw in above, the image of X is isomorphic to the cubic surface S. The three components of $F(X)$ are distinguished by the following different kinds of the curves $f(\ell)$. The image of a line from the components P_1 or P_2 is a twisted cubics γ_i such that $|\gamma_i|$ gives one of the two blowing down structures of S defined by a double-six of lines. The image of a line from the component P_3 is cut out by a tangent plane section of the cubic surface.

One can easily describe the Fano surface $F(X)$ of lines on X using a determinant representation $X = V(\det A)$ of X. Let $q_1, \ldots, q_6$ be the nodes of X and $X^{\text{ns}} = X \setminus \{q_1, \ldots, q_6\}$. The nodes correspond to the points at which the matrix X is of rank 1. The right and the left kernel of A define two regular map $\mathfrak{r} : X^{\text{ns}} \to \mathbb{P}^2$ and $\mathfrak{l} : X^{\text{ns}} \to \mathbb{P}^2$. The fiber $\mathfrak{r}^{-1}(p)$ (resp. $\mathfrak{l}^{-1}(p)$) consists of the points $x \in X'$ such that the right kernel (resp. the left kernel) of the matrix $A(x)$ is equal to p. The restriction of these maps to the hyperplane $t_0 = 0$ in Eq. (1.1) define the two blowing-down structures on the cubic surface $F = V(a_3)$ defined by the linear systems $|C_1|, |C_2|$. These are two irreducible components of $F(X)$ isomorphic to $\mathbb{P}^2$. At each node X contains two scrolls $\mathcal{S}_1, \mathcal{S}_2$ of lines passing through q_i. We may assume that $\mathfrak{r}(\mathcal{S}_1)$ and $\mathfrak{l}(\mathcal{S}_2)$ are lines. For any pair of lines $\ell_i \subset \mathcal{S}_i$, the plane spanned by ℓ_1 and ℓ_2 intersects X at the union of ℓ_1, ℓ_2 and a third line ℓ. It belongs to the third irreducible component of $F(X)$. The pair of points $(y_1, y_2) = (\ell \cap \ell_1, \ell \cap \ell)$ corresponds to a point on the cubic surface S associated to X via Proposition 3.1.

Since the components P_1, P_2 of $F(X)$ do not depend on the determinant representation of X, we obtain the following.

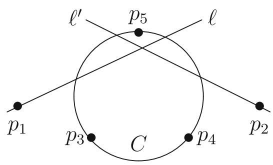
Corollary 4.4 *A nondegenerate 6-nodal cubic threefold admits two non-equivalent determinant representations, one is obtained from another by taking the transpose of the matrix.*

5 Conic Bundles

Let X be a d-nodal cubic hypersurface in $\mathbb{P}^4$ and ℓ be a line on X. We assume that ℓ is not contained in any plane on X. The projection from ℓ defines a conic bundle structure on the blow-up X' of X along ℓ. The discriminant curve Δ_ℓ of this bundle is of degree 5 and has $\geq d$ ordinary nodes or cusps. To see this, one takes a general hyperplane section F of X that contains ℓ. It is know that a line in a nonsingular cubic surface intersects ten more lines forming 5 pairs of lines that are coplanar with ℓ. The restriction of the projection to F defines a conic bundle on F with 5 singular fibers. The projection from ℓ defines a conic bundle on F equal to the pre-image of the conic bundle on X' over a line. This shows that the degree of the discriminant curve is equal to 5.

Let Γ_ℓ be the curve of lines in X intersecting ℓ. For any line $\ell' \in \Gamma$, we have the line ℓ'' such that ℓ, ℓ', ℓ'' are coplanar. This defines an involution $\iota : \Gamma_\ell \to \Gamma_\ell$, and the quotient by this involution is isomorphic to Δ_ℓ. If X is a nonsingular, Fano showed in (1904b) that Γ is a nonsingular curve of genus 11 and degree 15 in $G_1(\mathbb{P}^4)$ (see a modern proof in Clemens and Griffiths 1972).

We start with the case $d = 10$. A line on X which is not contained in a plane in X belongs to one of the six del Pezzo components D_i. We assume also that ℓ is a general line from D_i. Since $\mathfrak{S}_6$ permutes these components, we may assume that $\ell \in D_1$. Consider the projection map $\mathrm{Bl}_{\mathrm{S}_3}(\ell) \to \mathbb{P}^2$, where $\mathrm{Bl}_{\mathrm{S}_3}(\ell)$ is the blow-up of the line ℓ. We consider S_3 as the image of the rational map $\mathbb{P}^3 \mathsf{S}_3$ given by quadrics through 5 points $p_1, \ldots, p_5$. The curve Γ_ℓ consists of 10 components. Four of the components are conics K_i of lines in $D_i, i \neq 1, 6$, represented by lines in $\mathbb{P}^3$ intersecting ℓ. Another four components are lines L_{ijk} represented by conics in the planes $\Pi_{ijk}, 1 \notin \{i, j, k\}$. Also we have the conic K_5 of lines in D_6 represented by rational normal cubics through $p_1, \ldots, p_5$ intersecting ℓ and the pencil L_1 of lines in the exceptional divisor $E(p_1)$ passing through the point corresponding to the line ℓ. The involution $\iota : \Gamma_\ell \to \Gamma_\ell$ pairs the conic K_i with the line $L_{abc}, i \notin \{a, b, c\}$ and the conic K_5 with the line L_1. The following picture shows a component of Δ_ℓ represented by the pair K_2 and L_{345}.



Next we consider the case of a nondegenerate 6-nodal cubic threefold.

Let F be a nonsingular cubic surface obtained by blowing up 6 points $x_1, \ldots, x_6$ in the plane. Let $(e_0, e_1, \ldots, e_6)$ be the corresponding geometric basis of $\mathrm{Pic}(F)$. Let $|\gamma_1| = |e_0|$ and $|\gamma_2| = |5e_0 - 2(e_1 + \ldots + e_6)$ as in Lemma 3.3. We divided all 27 lines on F in three disjoint subsets. The first set of 6 lines intersecting γ_1 will be called lines of type (0,2), they are represented by the exceptional curves $E(x_i)$. The second set of 6 lines will be called lines of type (2,0), they are represented by the conics through 5 points. The remaining 15 lines will be called lines of type (1, 1), they are represented by lines joining a pair of the six points.

Let H be a hyperplane in $\mathbb{P}^4$ that cuts out X along a nonsingular surface F. We fix a node q to identify the linear system of hyperplanes with the linear system of cubics in H containing the fundamental curve $C(X, q) = C_1 + C_2$. Thus, we can divide all lines in F in three sets as above.

Take a general line ℓ in the component P_1 of $F(X)$. Let Γ_ℓ be the curve of lines in X intersecting ℓ. Since no curve from P_1 different from ℓ intersects ℓ, we see that Γ_1 consists of two components $\Gamma_\ell(2)$ and $\Gamma_\ell(3)$ of curves from P_2 or P_3 intersecting ℓ. Take another general curve γ' from P_1 and let F be a cubic surface in X that contains ℓ and ℓ'. We see that ℓ, ℓ' are lines on F of types (2,0). We may represent then by two exceptional curves $E(q_i), E(q_j)$ of the blow-up of 6 points. The number of lines on F of type (0,2) intersecting ℓ and ℓ' is equal to 4, they are represented by conics passing through 5 points including p_i and p_j. This implies that $\Gamma_\ell(2)$ has self-intersection in $P_1 \cong \mathbb{P}^2$ equal to 4. Similarly, we compute the self-intersection of $\Gamma_\ell(3)$. It is equal to 1. We also obtain that $\Gamma_\ell(2) \cdot \Gamma_\ell(3) = 2$. Thus $\Gamma_\ell(2)$ is a conic on $\mathbb{P}^2$ and $\Gamma_\ell(3)$ is a line. In the Plücker embedding of $G_1(\mathbb{P}^4)$, they are curves of degrees 6 and 3.[2] The involution ι_Γ switches the two components. The discriminant curve Δ_ℓ is an irreducible rational curve of degree 5 with 6 nodes.

Next, we take ℓ to be a general line in the component P_3. The curve Γ_ℓ consists of three components $\Gamma_\ell(1), \Gamma_\ell(2), \Gamma_\ell(3)$ of lines from P_1, P_2, P_3, respectively, intersecting ℓ. By similar argument as above, we find that two disjoint lines of type (1, 1) on F are intersected by one line of type (0,2), one line of type (2,0) and three lines of type (1, 1). This intersection matrix of these curves is equal to

$$\begin{pmatrix} 1 & 5 & 3 \\ 5 & 1 & 3 \\ 3 & 3 & 3 \end{pmatrix}.$$

The curves $\Gamma_\ell(1)$ and $\Gamma_\ell(2)$ are rational curves of degree 3 on the cubic surface P_3, and the curve $\Gamma_\ell(3)$ is an elliptic curve of degree 3 on P_3. In the Plücker embedding of $G_1(\mathbb{P}^4)$ the curves are of degrees 3, 3 and 6, respectively, and their sum is a hyperplane section equal to $-3K_{P_3}$. The involution ι switches the first two

[2]To get expected degree 15 one has to add six lines here that come from a choice of one ruling in each exceptional divisor of the resolution $X' \to X$.

components and defines a fixed-point free involution on the third component. The discriminant curve Δ_ℓ consists of the union of a conic and a cubic intersecting at 6 points.

Remark 5.1 It follows that, for any general line from the component P_3, there exists a quadric singular along ℓ that contains the six nodes of X. Note that, counting parameters, we see that it is not true if we replace ℓ with a general line in $\mathbb{P}^4$.

Also, X contains an elliptic curve parameterizing the singular points of the conics corresponding to the degree 3 component of the discriminant curve Δ_ℓ, where $\ell \in P_3$. This elliptic curve contains the six nodes of X. When ℓ varies in P_3, we get a family of elliptic curves (or their degenerations) parameterized by the cubic surface P_3. Is it an elliptic fibration on the blow-up of X at 6 nodes?

Remark 5.2 It is known that the intermediate Jacobian of a smooth cubic threefold is a principally polarized abelian variety of dimension 5. It is also known that it is isomorphic to the Prym variety associated to the discriminant curve of degree 5 of the conic bundle on X defined by the projection from a line on X (see, for example, Beauville 1977). In the case when X is a nondegenerate 6-nodal cubic threefold, the intermediate Jacobian $J(X)$ degenerates to a 5-dimensional algebraic torus which is isomorphic to the generalized Prym variety $\mathrm{Prym}(\tilde{\Delta}/\Delta)$, where Δ is the discriminant curve of the conic bundle on X defined by the projection from a line on X (as was recently shown in a joint work of S. Casalaina-Martin, S. Grushevsky, K. Hulek and R. Laza, this isomorphism does not depend on a choice of a line).

The projection from a line ℓ on X corresponding to the bisecant of $C(X,q)$ joining two points x, y on different component of $C(X,q)$ can be viewed as the conic bundle structure of the blow-up $\tilde{\mathbb{P}}^3$ of $\mathbb{P}^3$ at the reducible curve $C(X,q) \cup \ell$ of degree 7 and arithmetic genus 5 which is given by the linear system of cubics through the curve. In the case of a connected smooth curve of degree 7 of genus 5 in $\mathbb{P}^3$ this conic bundle was considered by V. Iskovskikh (it is a Fano variety with the Picard number equal to 2, No 9 in the list of such varieties that can be found in Iskovskikh and Prokhorov (1999).

6 6-Nodal Cubic Threefolds and Nonsingular Cubic Fourfolds

Let Y be a nonsingular cubic hypersurface in $\mathbb{P}^5$. Suppose Y contains a normal cubic scroll T spanning a hyperplane H in $\mathbb{P}^5$. It is isomorphic to the blow-up of $\mathbb{P}^2$ at one point p and it is embedded in $\mathbb{P}^4$ by the linear system of conics through the point p (see Dolgachev 2012, 8.1.1). Choosing the point p to be $[1,0,0]$ and the basis of the linear system in the form $(x_2^2, x_2x_3, x_1x_2, x_1x_3, x_1^2)$, we can express the equations of the scroll as

$$\mathrm{rank}\begin{pmatrix} t_0 & t_1 & t_2 \\ t_2 & t_3 & t_4 \end{pmatrix} \leq 1.$$

A cubic hypersurface in $\mathbb{P}^4$ containing T can be given by the equation

$$l_1(t_1t_4 - t_2t_3) + l_2(t_2^2 - t_0t_4) + l_3(t_0t_3 - t_1t_2) = 0,$$

For some linear forms l_i in variables $t_0, \ldots, t_4$. For appropriate l_1, l_2, l_3, we find the intersection $X = H \cap X$. Clearly the previous equation gives a determinantal representation of X. As was explained in Remark 3.6, we expect that X is a nondegenerate 6-nodal cubic threefold.

Assume now that Y is a nonsingular cubic fourfold that contains a cubic scroll T whose linear span intersects Y along a nondegenerate 6-nodal cubic threefold X. We know that the degree of the curve in $G_1(\mathbb{P}^4)$ parameterizing the ruling of T is equal to 3 (Dolgachev 2012, 10.4.1). This implies that T is the pre-image of a line under the two rulings of X parameterized by the components P_1, P_2 of $F(X)$. In the determinant representation of X, T is the pre-image of a line under the left or the right kernel maps. Assume that T is the pre-image of a line in the component P_1. We know that $\dim H^4(X, \mathbb{Q}) = 2$. Let $\mathcal{Q}$ be a quadric in $\mathbb{P}^4$ that contains T (it could be taken as one of the minors defining T), it intersects X along the union of T and another cubic scroll T' corresponding to a line in the component P_2.

Let

$$Y \xleftarrow{\ \sigma\ } Z \xrightarrow{\ \tau\ } F(Y)$$

Be the incidence correspondence of lines and points in Y, where $F(Y)$ is the Fano fourfold of lines in Y. It is known to be an irreducible holomorphic symplectic fourfold. The Abel-Jacobi map of integral Hodge structures

$$\Phi : \tau_*\sigma^* : H^4(Y, \mathbb{Z})[2] \to H^2(F(Y))$$

defines an isomorphism of free abelian groups of rank 21:

$$H^{2,2}(Y) \cap H^4(Y, \mathbb{Z}) \to H^{11}(F(Y), \mathbb{Z}) \cap H^2(F(Y), \mathbb{Z})).$$

The group $H^2(F(Y), \mathbb{Z})$ is equipped with the Beauville-Bogomolov quadratic form q_{BB} of signature (1, 20). The isomorphism Φ is a compatible (with the change of the sign) with the cup-product on primitive cohomology of $H^4(Y, \mathbb{Z})$ and the Beauville-Bogomolov quadratic form restricted to the primitive cohomology of $H^2(F(Y), \mathbb{Z})$. Let σ be the class of a hyperplane section of $F(Y)$ in its Plücker embedding in $G_1[\mathbb{P}^5)$. It is known that the degree of $F(Y)$ is equal to 36, so that $\sigma^4 = 36$. However, $q_{BB}(h) = 6$.

For a nonsingular cubic fourfold Y, we have

$$\mathrm{Pic}(F(Y)) = H^2(F(Y),\mathbb{Z}) \cap H^{11}(F(Y),\mathbb{Z}) \cong \mathbb{Z}^r,$$

If Y is general in the sense of moduli (the number of them is equal to 20), then $r = 1$. Let h be the class of a hyperplane section of Y. Then $[X] = h$ and $2h^2 = [T] + [T']$. This shows that $\Phi([T]), \Phi[T']$ belong to $\mathrm{Pic}(F(Y))$ and give $r \geq 2$. We assume that $r = 2$ and the classes $\tau = \Phi([T])$ and σ freely generate $\mathrm{Pic}(F(Y))$. The quadratic lattice $(H^4(Y,\mathbb{Z}), \cup)$ has a basis $(h^2, [T])$ and is defined in this basis by the matrix $\begin{pmatrix} 3 & 6 \\ 3 & 7 \end{pmatrix}$. The quadratic lattice $(\mathrm{Pic}(F(Y)), q_{BB})$ has a basis (σ, τ) and is defined in this basis by the matrix $\begin{pmatrix} 6 & 6 \\ 6 & 2 \end{pmatrix}$.

In their paper Hassett and Tschinkel (2010). B. Hassett and Yu. Tschinkel compute the nef cone of $F(Y)$ in $\mathrm{Pic}(F(Y))_{\mathbb{R}}$ and find that it equals the dual cone of the cone spanned by the classes $\alpha_1 = 7\sigma - 3\tau$ and $\sigma + 3\tau$.

Recall that $F(X)$ contains two planes P_1, P_2. Since, obviously, $F(X) \subset F(Y)$, we see that $F(Y)$ contains two planes (embedded in the Plücker space of $G_1(\mathbb{P}^5)$ as Veronese surfaces spanning $\mathbb{P}^5$).

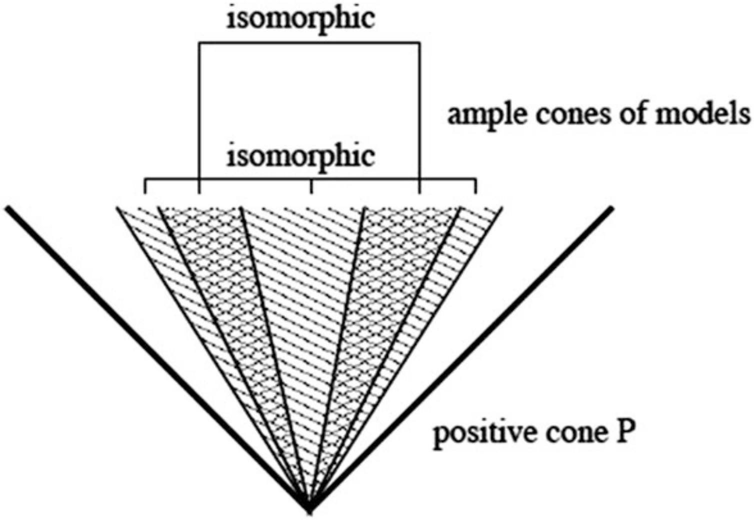
We may perform Mukai flop P_1 (resp. P_2) to obtain new holomorphic symplectic manifolds V_1, V_2 (Mukai 1984). The birational maps between $F(Y)$ and V_1 (resp. V_2) identify their Picard groups but induce an orthogonal transformation R_1 (resp R_2) of their quadratic forms.

The following beautiful result is proven in Hassett and Tschinkel (2010).

Theorem 6.1 *There is an infinite sequence of flops*

$$\cdots F_{-2} \to F_{-1}^{\vee} \to F(Y) \to F_1 \to F_2 \cdots$$

with isomorphisms $F_i \cong F_{i+2}$. The cone of effective divisors on $F(Y)$ can be expressed as the non-overlapping union of nef cones of the F_i's. The birational pseudo-automorphism of $F(Y)$ defined by $F(Y)F_1F_2 \cong F(Y)$ acts on $F(Y)$ by the matrix $\begin{pmatrix} -1 & -2 \\ 6 & 11 \end{pmatrix}$ in the basis σ, τ. The following picture copied from Hassett and Tschinkel (2010):



References

A. Altman, S. Kleiman, *Foundations of the theory of Fano schemes*. Compositio Math. **34** (1977), 3–47.

A. Alzati, M. Bolognesi, *A structure theorem for* $SU_{\mathbb{C}}(2)$ *and the moduli of pointed rational curves*, Journal Alg. Geometry, to appear.

I. Bauer, A. Verra, *The rationality of the moduli space of genus-4 curves endowed with an order-3 subgroup of their Jacobian*. Michigan Math. J. **59** (2010), 483–504.

A. Beauville, *Variétés de Prym et jacobiennes intermédiaires*, Ann. Sci. École Norm. Sup. (4) **10** (1977), 309–391.

A. Beauville, R. Donagi, *La variété des droites d'une hypersurface cubique de dimension 4*. C. R. Acad. Sci. Paris Sér. I Math. **301** (1985), no. 14, 703–706.

G. Castelnuovo, *Ricerche di geometria della rette nello spazio a quattro dimensioni*, Atti. Ist. Veneto, **7** (1891), 855–901.

H. Clemens, Ph. Griffiths, *The intermediate Jacobian of the cubic threefold*. Ann. of Math. (2) **95** (1972), 281–356.

A. Coble, *Point sets and allied Cremona groups*. I, Trans. Amer. Math. Soc. Part I **16** (1915), 155–198.

A. Coble, *A Generalization of the Weddle Surface, of Its Cremona Group, and of Its Parametric Expression in Terms of Hyperelliptic Theta Functions*. Amer. J. Math. **52** (1930), 439–500.

I. Dolgachev, *On certain families of elliptic curves in projective space*. Ann. Mat. Pura Appl. (4) **183** (2004), 317–331.

I. Dolgachev, *Classical algebraic geometry:a modern view*, Cambridge Univ. Press, 2012.

G. Fano, *Sulle superficie algebriche contenute in una variet cubica dello spazio a quattro dimensioni*. Atti R. Acc. Sci. Torino **39** (1904a), 597–613.

G. Fano, *Sul sistema* ∞^3 *di rette contenuto in una quadrica dello spazio a quattro dimensioni*. Atti R. Acc. Sci. Torino **39** (1904b), 778–792.

H. Finkelnberg, *Small resolutions of the Segre cubic*. Nederl. Akad. Wetensch. Indag. Math. **49** (1987), 261–277.

H. Finkelnberg, J. Werner, *Small resolutions of nodal cubic threefolds*. Nederl. Akad. Wetensch. Indag. Math. **51** (1989), 185–198.

T. Gwena, *Degenerations of cubic threefolds and matroids*. Proc. Amer. Math. Soc. **133** (2005), no. 5, 1317–1323 (electronic).

B. Hassett, Yu. Tschinkel, *Flops on holomorphic symplectic fourfolds and determinantal cubic hypersurfaces*. J. Inst. Math. Jussieu **9** (2010), 125–153.

V. Iskovskikh, Yu. Prokhorov, *Fano varieties*. Algebraic geometry, V, 1–247, Encyclopaedia Math. Sci., 47, Springer, Berlin, 1999.

P. Joubert, *Sur l'equation du sixieme degré*, Comptes Rendus hepdomadaires des séances de l'Académie des sciences, **64** (1867), 1025–1029, 1081–1085.

M. Kapranov, *Chow quotients of Grassmannians. I*. I. M. Gelfand Seminar, 29–110, Adv. Soviet Math., 16, Part 2, Amer. Math. Soc., Providence, RI, 1993.

C. Kumar, *Invariant vector bundles of rank 2 on hyperelliptic curves*. Michigan Math. J. **47** (2000), 575–584.

S. Mukai, *Symplectic structure of the moduli space of sheaves on an abelian or K3 surface*. Invent. Math. **77** (1984), 101–116.

H.W. Richmond, *Concerning the locus* $\Sigma(t_r^3) = 0$; $\Sigma(t_r) = 0$ ($r = 1,2,3,4,5,6$), Quart. J. Math. **34** (1902), 117–154.

C. Segre, *Sulle varietà cubica con dieci punti doppi dello spazio a quattro dimensioni*, Atti Accad. Sci. Torino, **22** (1886/87) 791–801 [Opere: v. 4, n. 63].

C. Segre, *Sulle varietá cubiche dello spazio a quattro dimensioni e su certi sistemi di rette e certe superficie dello spazio ordinario*. Mem. Acad. Sci. Torino (2) **39** (1887), 2–48.

Godeaux Surfaces with an Enriques Involution and Some Stable Degenerations

Margarida Mendes Lopes and Rita Pardini

Abstract

We give an explicit description of the Godeaux surfaces S (minimal surfaces of general type with $K_S^2 = \chi(\mathcal{O}_S) = 1$) that admit an involution σ such that S/σ is birational to an Enriques surface; these surfaces give a 6-dimensional unirational irreducible subset of the moduli space of surfaces of general type. In addition, we describe the Enriques surfaces that are birational to the quotient of a Godeaux surface by an involution and we show that they give a 5-dimensional unirational irreducible subset of the moduli space of Enriques surfaces. Finally, by degenerating our description we obtain some examples of non-normal stable Godeaux surfaces; in particular we show that one of the families of stable Gorenstein Godeaux surfaces classified in Franciosi et al. (in preparation) consists of smoothable surfaces.

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1 Introduction

A Godeaux surface is (the canonical model of) a minimal surface of general type with $K_S^2 = \chi(\mathcal{O}_S) = 1$. These surfaces have been intensely studied since the 1970s, but a complete classification is still lacking. A very synthetic summary of the state of the art is as follows:

- the algebraic fundamental group π_1^{alg} of a Godeaux surface is cyclic of order ≤ 5 (Miyaoka 1975); in particular if S is a Godeaux surface then π_1^{alg} is abelian and thus it coincides with the torsion subgroup $\mathrm{Tors}(S)$ of $\mathrm{Pic}(S)$;
- the Godeaux surfaces with π_1^{alg} of order 3, 4, 5 are explicitly described; to each of these possibilities for π_1^{alg} there corresponds an irreducible unirational 8-dimensional connected component of the moduli space (Reid 1978);
- Godeaux surfaces with $\pi_1^{\mathrm{alg}} = 0$ or $\mathbb{Z}_2$ do exist, but little is known about the geometry of the moduli space (Barlow 1984, 1985; Lee and Park 2007; Park et al. 2013).

One of the strategies to overcome the difficulties of the classification is to restrict one's attention to a subclass of Godeaux surfaces with an extra structure, for instance those admitting an involution. This has been done by Keum and Lee (2000) and by Calabri et al. (2007), who described the possibilities for the quotient surface and the fixed locus of the involution.

Here we study in detail the case when the quotient surface is birational to an Enriques surface (in this case, we call σ an "Enriques involution"). Since in this case $\mathrm{Tors}(S) \cong \mathbb{Z}_4$ (Calabri et al. 2007), the universal cover $\tilde{S}$ of the Godeaux surface is a complete intersection in a weighted projective space (Reid 1978). The involution σ lifts to an involution $\tilde{\sigma}$ of $\tilde{S}$ and the action of $\tilde{\sigma}$ on the canonical ring of $\tilde{S}$ can be determined by means of a careful study of linear systems on the quotient Enriques surface, yielding the classification (Theorem 3.2). As a consequence, the locus of Godeaux surfaces with an Enriques involution is irreducible of dimension 6 (Corollary 3.3) and the locus of Enriques surfaces that are birational to the quotient of a Godeaux surface by an involution (Enriques surfaces "of Godeaux-quotient type") is irreducible of dimension 5 (Corollary 4.2). In Sect. 4 we specialize a classical construction of the special Enriques surfaces (Horikawa 1978a, b) to obtain Enriques surfaces of Godeaux-quotient type: since our construction depends on 5 parameters, by the irreducibility of the locus of Enriques surfaces of Godeaux-quotient type it gives the general Enriques surface of Godeaux-quotient type.

The moduli space of (canonical models) of surfaces of general type can be compactified by considering a larger class of surfaces, the so-called stable surfaces (cf. Sect. 6 for the definition). The stable Gorenstein surfaces with $K^2 = 1$ (thus including the stable Gorenstein Godeaux surfaces) are investigated in the series of recent papers (Franciosi et al. 2014a, b, in preparation). In Sect. 6 we give an explicit construction of the general Godeaux surface with an Enriques involution and use it to produce stable Godeaux surfaces. In this way we produce a normal Gorenstein degeneration with an elliptic singularity of degree 4, whose existence was predicted in Franciosi et al. (2014b), and we show the smooth ability of one of the families of non-normal Godeaux surfaces with normalization isomorphic to $\mathbb{P}^2$ (Franciosi et al. 2014b, in preparation). In addition we give examples of stable non-normal Godeaux surfaces with Cartier index equal to 2 whose normalization is not ruled, thus showing that the main result of Franciosi et al. (2014b) does not hold without the Gorenstein assumption.

Finally, a remark on the methods: the constructions of the general Enriques surface of Godeaux-quotient type (Sect. 4) and of the general Godeaux surface with and Enriques involution (Sect. 6) are based:

(a) on the fact that, for a certain involution τ of Y and for a certain double/bidouble cover $p : X \to Y$, τ can be lifted to an involution of X;
(b) on the fact that the 2-divisibility of p^*D for a certain divisor D on Y implies that D is also 2-divisible.

The conditions under which (a) and (b) above hold for a general bidouble cover are investigated in Sect. 3: we believe that this section is of independent interest.

Notation and Conventions: We work over the complex numbers. Following the terminology of Kollár (2013), a variety is called *demi-normal* if it satisfies condition S_2 of Serre and in codimension 1 it is either smooth or double crossings. If X is a demi-normal projective variety, then the dualizing sheaf ω_X is divisorial; we denote by K_X a canonical divisor, that is, a Weil divisor such that $\mathcal{O}_X(K_X) \cong \omega_X$. For a projective variety X we denote by $\mathrm{Tors}(X)$ the torsion subgroup of $\mathrm{Pic}(X)$ and by $\mathrm{Pic}(X)[d]$ the subgroup consisting of the d-torsion elements. We use $\equiv$ to denote linear equivalence of divisors and $\sim$ to denote numerical equivalence of $\mathbb{Q}$-divisors.

Thoughout all the paper G is used to denote the Galois group of a finite cover.

2 Galois Covers and Divisibility

In this section we first summarize the theory of Pardini (1991) and Alexeev and Pardini (2012) for covers with Galois group $\mathbb{Z}_2$ and $\mathbb{Z}_2^2$; the need to cover also the case of non-normal covers arises because in Sect. 6 we consider stable Godeaux surfaces.

Then we present some general results on lift ability of automorphisms to double and bidouble covers that are needed in the rest of the paper. Although these results are probably known to experts, to our knowledge they have not been written down elsewhere and we believe that they are of independent interest.

2.1 Double and Bidouble Covers

Let G be a finite group. A G-cover is a finite map of algebraic varieties $f: X \to Y$ that is the quotient map for a generically faithful G-action, namely such that for every component Y_i of Y the G-action on the restricted cover $X \times_Y Y_i \to Y_i$ is faithful. The cover is *abelian* if G is an abelian group: for the general theory of abelian covers we refer the reader to Pardini (1991) for the case X normal and Y smooth and to Alexeev and Pardini (2012) for a more general treatment.

Here we are mainly interested in the case $G \cong \mathbb{Z}_2$ ("double covers") and $G \cong \mathbb{Z}_2^2$ ("bidouble covers"); for simplicity, we assume throughout that $H^0(Y, \mathcal{O}_Y) = \mathbb{C}$.

Assume first that $f: X \to Y$ is an abelian cover with group G such that X is normal and Y is smooth. Then f is flat and the branch locus is a divisor; we denote by B the branch divisor with reduced structure. For $G = \mathbb{Z}_2$, we have $f_*\mathcal{O}_X = \mathcal{O}_Y \oplus L^{-1}$, where L is a line bundle, G acts on L^{-1} as multiplication by -1 and the multiplication map $L^{-1} \otimes L^{-1} \to \mathcal{O}_Y$ induces an isomorphism $L^{\otimes 2} \cong \mathcal{O}_Y(B)$. The pair (L, B) is called the *building data* of the double cover and it determines $f: X \to Y$ uniquely up to isomorphism of covers, since we assume $H^0(\mathcal{O}_Y) = \mathbb{C}$. We say for short that $f: X \to Y$ is the double cover given by the equivalence relation $2L \equiv B$.

One can reverse this construction: given building data (L, B), i.e. given an effective divisor B and a line bundle L satisfying the relation $2L \equiv B$, one can choose an isomorphism $\phi: L^{\otimes 2} \to \mathcal{O}_Y(B)$, use it to define an associative multiplication on $\mathcal{O}_Y \oplus L^{-1}$, set $X := \mathrm{Spec}(\mathcal{O}_Y \oplus L^{-1})$ and take f to be the natural map $X \to Y$. This construction makes sense more generally for any effective Cartier divisor B (not necessarily reduced) and line bundle L such that $2L \equiv B$ on an arbitrary variety Y. The flat double cover $f: X \to Y$ is called the *standard cover* associated with (L, B); it is not hard to show that every flat double cover is obtained this way, i.e., it is standard.

The situation is similar for bidouble covers. We start again by considering the case X normal and Y smooth. We write χ_1, χ_2, χ_3 for the three non-trivial characters of $G \cong \mathbb{Z}_2^2$ and denote by $g_i \in G$ the generator of $\ker \chi_i$. The branch divisor B decomposes as $B = B_1 + B_2 + B_3$, where B_i is the image of the divisorial part of the fixed locus of g_i and we have a splitting $f_*\mathcal{O}_X = \mathcal{O}_Y \oplus L_1^{-1} \oplus L_2^{-1} \oplus L_3^{-1}$, where G acts on L_i^{-1} as multiplication by the character χ_i. As in the case of double covers, the multiplication in $f_*\mathcal{O}_X$ induces isomorphisms, and therefore equivalence relations:

$$2L_i \equiv B_j + B_k, \quad L_i + L_j \equiv L_k + B_k, \tag{2.1}$$

where (i, j, k) is a permutation of (1, 2, 3). Again, (L_i, B_i), $i = 1, 2, 3$, are called the building data of the bidouble cover and determine $f: X \to Y$ up to isomorphism of $\mathbb{Z}_2^2$-covers. It is easy to see that (2.1) is equivalent to the smaller set of equations:

$$2L_1 \equiv B_2 + B_3, \quad 2L_2 \equiv B_1 + B_3, \quad L_3 \equiv L_1 + L_2 - B_3, \tag{2.2}$$

and in particular L_3 can be recovered from the remaining data. We call $(L_1, L_2, B_1, B_2, B_3)$ the *reduced building data* and we say for short that the cover is given by the relations $2L_1 \equiv B_2 + B_3$, $2L_2 \equiv B_1 + B_3$.

As in the case of double covers, we can perform the reverse construction in greater generality, starting with line bundles L_1, L_2 and effective Cartier divisors satisfying (2.2), and obtain a *standard bidouble cover* of an arbitrary variety Y. Again, the building data determine the standard cover uniquely up to isomorphism of bidouble covers, since we assume $H^0(\mathcal{O}_Y) = \mathbb{C}$. We set $B = B_1 + B_2 + B_3$; observe that B may be non-reduced. We recall the following:

Proposition 2.1 (Alexeev and Pardini 2012, Corollary 1.10) *Let $f : X \to Y$ be a double or bidouble cover with Y smooth* and *X demi-normal. Then f is a standard cover and every component of B has multiplicity at most* 2.

2.2 Lifting Automorphisms to Double and Bidouble Covers

We discuss in detail the case of bidouble covers; the case of double covers can be treated by similar, but simpler, arguments.

Let Y be a variety with $H^0(Y, \mathcal{O}_Y) = \mathbb{C}$, let $f : X \to Y$ be a standard bidouble cover given by relations $2L_1 \equiv B_2 + B_3$ and $2L_2 \equiv B_1 + B_3$ and denote by $G \cong \mathbb{Z}_2^2$ the Galois group of f. Let $\rho \in \mathrm{Aut}(Y)$ be an automorphism such that one of the following holds:

(a) $\rho^* B_i = B_i$, $i = 1, 2, 3$, and $\rho^* L_j \equiv L_j$, $j = 1, 2$
(b) $\rho^* B_1 = B_2$, $\rho^* B_2 = B_1$, $\rho^* B_3 = B_3$, $\rho^* L_1 \equiv L_2$, $\rho^* L_2 \equiv L_1$.

In either case, the automorphism ρ lifts to an automorphism $\tilde{\rho}$ of X. Indeed, consider the following cartesian diagram:

$$\begin{array}{ccc} X' & \xrightarrow{\rho'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{\rho} & Y. \end{array} \tag{2.3}$$

In case (a), f' is a standard bidouble cover given by the same building data as f, hence it is isomorphic to f via an isomorphism compatible with the action of $G \cong \mathbb{Z}_2^2$ and $\tilde{\rho}$ is obtained by composing such an isomorphism with ρ'; in case (b) we modify the G-action on X' by composing with the automorphism of G that switches g_1 and g_2 and argue as in case (a).

Let $\tilde{G}$ be the subgroup of $Aut(X)$ generated by G and by $\tilde{\rho}$. Then there is a short exact sequence of groups:

$$1 \to G \to \tilde{G} \to <\rho> \ \to 1.$$

The group $\tilde{G}$ is abelian in case (a), since $\tilde{\rho}$ preserves the decomposition of $f_*\mathcal{O}_X$ into G-eigensheaves, and it is non abelian in case (b); in particular, if $\rho^2 = 1$ then $\tilde{\rho}^4 = 1$ and, by the classification of groups of order 8, $\tilde{G}$ is isomorphic either to $\mathbb{Z}_2^3$ or $\mathbb{Z}_2 \times \mathbb{Z}_4$ in case (a) and to the dihedral group D_4 in case (b).

In the case of double covers one assumes that $\rho^*B = B$ and $\rho^*L \equiv L$: in this case $\tilde{\rho}$ commutes with the action of $G \cong \mathbb{Z}_2$ and the group $\tilde{G}$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_d$ or to $\mathbb{Z}_{2d}$.

2.3 Divisibility

Recall that a Cartier divisor or line bundle on a projective variety is said to be *even* if its class is divisible by 2 in $\mathrm{Pic}(X)$.

Lemma 2.2 *Let $f: X \to Y$ be a cyclic étale cover of projective varieties and let K be the kernel of $f^*: \mathrm{Pic}(Y) \to \mathrm{Pic}(X)$. Let D be a Cartier divisor on Y such that f^*D is even.*

If $\mathrm{Pic}(X)[2] = 0$, then the class of D is divisible by 2 in $\mathrm{Pic}(Y)/K$.

Proof Let $\tilde{M} \in \mathrm{Pic}(X)$ be a line bundle such that $2\tilde{M} \equiv f^*D$. Denote by g a generator of the Galois group G of f; since D is g-invariant, we have $2g^*\tilde{M} \equiv f^*D \equiv 2\tilde{M}$. Since $\mathrm{Pic}(X)[2] = 0$ it follows that the lines bundles $\tilde{M}$ and $g^*\tilde{M}$ are isomorphic and therefore $\tilde{M}$ admits a G-linearization (G is cyclic). Since f is étale, $\tilde{M}$ descends to a line bundle M on Y. One has $f^*(2M - D) \equiv 0$, hence $D = 2M$ in $\mathrm{Pic}(Y)/K$. □

Lemma 2.3 *Let $f: X \to Y$ be a cyclic étale cover of degree d of projective varieties and let D be an effective Cartier divisor on Y such that f^*D is even. Assume that $\mathrm{Pic}(X)[2] = 0$ and denote by $h: Z \to X$ the flat double cover branched on f^*D. Then*:

(i) *the composite map $f \circ h: Z \to Y$ is a Galois cover with Galois group $\tilde{G}$ isomorphic to $\mathbb{Z}_{2d}$ or to $\mathbb{Z}_2 \times \mathbb{Z}_d$*;
(ii) *D is even iff $\tilde{G}$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_d$.*

Notice that Lemma 2.3 is interesting only if d is even. Indeed, if d is odd then $\tilde{G} \cong \mathbb{Z}_{2d} \cong \mathbb{Z}_2 \times \mathbb{Z}_d$ is cyclic and statement (ii) just says that D is even, as we already know by Lemma 2.2.

Proof

(i) Let $\tilde{L} \in \mathrm{Pic}(X)$ be the only element such that $2\tilde{L} \equiv f^*D$. Let g be a generator of the Galois group G of f; by construction f^*D is G-invariant, hence arguing as in the proof of Lemma 2.3 one sees that $\tilde{L}$ is also G-invariant. Therefore by the discussion of Sect. 2.2 it is possible to lift g to an automorphism $\tilde{g}$ of Z and the subgroup $\tilde{G}$ of $\mathrm{Aut}(Z)$ generated by $\tilde{g}$ and by the involution $\imath$ associated with h is isomorphic to $\mathbb{Z}_{2d}$ or $\mathbb{Z}_2 \times \mathbb{Z}_d$. The former case occurs iff $\tilde{G}$ is generated by $\tilde{g}$ or by $\tilde{g}\imath$. Clearly, $\tilde{G}$ is the Galois group of $f \circ h$.

(ii) Assume that $\tilde{G} \cong \mathbb{Z}_2 \times \mathbb{Z}_d$ and let $\tilde{g}$ be an element of order d that lifts g: then $\tilde{g}$ acts freely on Z by construction and $Z/\tilde{g} \to Y$ is a flat double cover. Since $Z \to Z/\tilde{g}$ is étale, it is easy to see that $Z/\tilde{g} \to Y$ is standard with building data (L, D), for some $L \in \mathrm{Pic}(Y)$, hence D is even. Conversely, assume that D is even and let $L \in \mathrm{Pic}(Y)$ be such that $2L \equiv D$. We have $f^*L = \tilde{L}$ since $\mathrm{Pic}(X)[2] = 0$ and therefore $Z \to Y$ is the fiber product of $f : X \to Y$ and of the double cover given by the relation $2L \equiv D$ and has Galois group isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_d$. □

Let X be a surface and let $p_1, \ldots, p_k \in X$ be A_1 singularities ("nodes"). We say that $p_1, \ldots, p_k$ is an *even set of nodes* of X if there exists a double cover of X branched precisely on $p_1, \ldots, p_k$. Denote by $X' \to X$ the minimal resolution of the singularities $p_1, \ldots, p_k$ and by C_i the exceptional curve over p_i; C_i is a *nodal* curve, i.e., it is smooth rational and $C_i^2 = -2$. The set $\{p_1, \ldots, p_k\}$ is even if and only if $C_1 + \cdots + C_k$ is an even divisor of X'. By using the adjunction formula on X' it is easy to check that an even set of nodes has cardinality divisible by 4.

Lemma 2.4 *Let Y be a smooth projective surface, let B_1, B_2 be even curves of Y meeting transversely at smooth points $q_1, \ldots, q_k$ of Y.*

If $f : X \to Y$ is a flat double cover branched on $B := B_1 + B_2$, then the points $p_1, \ldots, p_k$ lying above $q_1, \ldots, q_k$ are an even set of nodes of X.

Proof The fact that $p_1, \ldots, p_k$ are nodes of X can be checked easily by a local computation. Let $L_3 \in \mathrm{Pic}(X)$ be such that $f_*\mathcal{O}_X = \mathcal{O}_Y \oplus L_3^{-1}$, so that f is given by the relation $2L_3 \equiv B$. Choose $L_1 \in \mathrm{Pic}(X)$ with $2L_1 \equiv B_2$ and set $L_2 := L_3 - L_1$. As explained in Sect. 2.1, the relations $2L_1 \equiv B_2$ and $2L_2 \equiv B_1$ determine a standard bidouble cover $h : Z \to Y$ (we take $B_3 = 0$). For $i = 1, 2$ denote by $g_i \in G \cong \mathbb{Z}_2^2$ the element that fixes $h^{-1}B_i$ pointwise and set $g_3 = g_1 + g_2$. Then Z/g_3 is isomorphic to X and the quotient map $Z \to Z/g_3$ is a double cover branched precisely on $p_1, \ldots, p_k$. □

3 Godeaux Surfaces with an Enriques Involution

In this section we study the following situation:

- S is a numerical Godeaux surface, i.e., a smooth minimal surface of general type with $K_S^2 = 1$ and $p_g(S) = q(S) = 0$
- $\sigma \in \mathrm{Aut}(S)$ is an involution such that $\Sigma := S/\sigma$ is birational to an Enriques surface.

We call the involution σ an *Enriques involution*. Godeaux surfaces with an involution have been studied in Keum and Lee (2000) and in Calabri et al. (2007); in particular, in Calabri et al. (2007) it is proven that a Godeaux surface S with an Enriques involution has $\mathrm{Tors}(S) \cong \mathbb{Z}_4$. In addition, the possible automorphism groups of numerical Godeaux surfaces with torsion of order ≥ 3 have been listed in Maggiolo (2010), but without analyzing the quotient surfaces.

We recall the following example Keum and Lee (2000, Example 4.3):

Example 3.1 Let S be a Godeaux surface with $\mathrm{Tors}(S) \cong \mathbb{Z}_4$ and let $\tilde{S} \to S$ be the universal cover, i.e. the degree 4 cyclic cover given by $\mathrm{Tors}(S)$. By Reid (1978), the minimal model $\tilde{S}_{\mathrm{can}}$ of $\tilde{S}$ is canonically embedded in $\mathbb{P}(1,1,1,2,2)$, with coordinates x_1, x_2, x_3, y_1, y_3, as the zero locus of two homogeneous equations q_0 and q_2 of degree 4.

The equation q_0 involves the monomials:

$$x_1^4, x_2^4, x_3^4, x_1^2x_3^2, x_1x_3x_2^2, x_1x_2y_1, x_2x_3y_3, y_1y_3,$$

and q_2 involves the monomials:

$$x_1^2x_2^2, x_2^2x_3^2, x_1^3x_3, x_1x_3^3, x_1x_2y_3, x_2x_3y_1, y_1^2, y_3^2.$$

We denote by $G \cong \mathbb{Z}_4$ the Galois group of $\tilde{S} \to S$: the group G acts freely also on $\tilde{S}_{\mathrm{can}}$ and the quotient surface is the canonical model S_{can} of S. The action of G extends to the ambient $\mathbb{P}(1,1,1,2,2)$ and there is a generator $g \in G$ that acts by $(x_1, x_2, x_3, y_1, y_3) \mapsto (ix_1, -x_2, -ix_3, iy_1, -iy_3)$.

Now we define an involution $\tilde{\sigma}$ of $\mathbb{P}(1,1,1,2,2)$ by $(x_1, x_2, x_3, y_1, y_3) \mapsto (-x_1, x_2, -x_3, y_1, y_3)$; the involution $\tilde{\sigma}$ commutes with g. We assume from now on that the polynomial q_0 does not involve $x_1x_2y_1$, $x_2x_3y_3$ and the polynomial q_2 does not involve $x_2x_3y_1$, $x_1x_2y_3$, so that q_0 and q_2 are invariant under $\tilde{\sigma}$. Hence $\tilde{\sigma}$ acts on $\tilde{S}_{\mathrm{can}}$ and descends to an involution σ of S_{can} and of its minimal resolution S.

The divisorial part R of the fixed locus σ on S_{can} is the paracanonical curve defined by $x_2 = 0$, hence it is a connected curve of genus 2; if S_{can} is smooth then R is also smooth, and by Corollary 4.8 and Proposition 7.10 of Calabri et al. (2007) it follows that σ is an Enriques involution. Since the quotient of a smooth surface by an involution has canonical singularities, it follows that for every smooth S_{can} as above the involution σ of S is an Enriques involution. Using Bertini's theorem, it is not difficult to see that if q_0 anq q_2 are general the surface $S_{\mathrm{can}} = S$ is smooth.

In this section we characterize the quotient surface S/σ and, exploiting this characterization, we prove the following classification results:

Theorem 3.2 *Let S be a Godeaux surface and let $\sigma \in \mathrm{Aut}(S)$ be an Enriques involution.*

Then S is as in Example 3.1.

The surfaces in Example 3.1 correspond to case R_1 of Table 2 of Maggiolo (2010), hence they form an irreducible unirational subset of dimension 6 of the moduli space of Godeaux surfaces with torsion of order 4. Hence Theorem 3.2 yields immediately:

Corollary 3.3 *The Godeaux surfaces with an Enriques involution give an irreducible unirational subset $\mathcal{GE}$ of dimension 6 of the moduli space of Godeaux surfaces with torsion of order 4.*

A possible strategy for proving Theorem 3.2 would be to use the description given in Maggiolo (2010) of the Godeaux surfaces with torsion of order 4 that admit an involution and decide which involutions are Enriques by looking at the fixed locus, as we have done in Example 3.1. However we prefer to use a more conceptual approach, based on a detailed study of linear systems on the quotient Enriques surfaces, that gives also a description of the family of such Enriques surfaces (cf. Sect. 4).

The rest of the section is devoted to proving Theorem 3.2; we start by fixing some notation.

We denote by $\pi : S \to \Sigma$ the quotient map; by Calabri et al. (2007, Proposition 4.5), the bicanonical map of S is composed with σ and $\mathrm{Fix}(\sigma)$ consists of a smooth curve R and of 5 isolated fixed points $p_1, \ldots, p_5$. We set $q_i = \pi(p_i)$, $i = 1, \ldots, 5$ and $B := \pi(R)$. There is a commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{\epsilon} & S \\ {\scriptstyle\tilde{\pi}}\downarrow & & \downarrow{\scriptstyle\pi} \\ W & \xrightarrow{\eta} & \Sigma \end{array} \tag{3.1}$$

where ϵ is the blow up of S at $p_1, \ldots, p_5$, η is the minimal resolution of Σ and $\tilde{\pi}$ is a flat double cover. For $i = 1, \ldots, 5$ we denote by C_i the exceptional curve over q_i; the C_i are nodal curves, that is, they are smooth rational and $C_i^2 = -2$. By Calabri et al. (2007, Proposition 3.9) and Lemma 4.11, ibidem, there exists a birational morphism $f : W \to Y$ such that:

- Y is a smooth Enriques surface
- the exceptional locus of f is disjoint from the C_i
- there is a flat double cover $p : X \to Y$ fitting in the commutative diagram:

$$\begin{array}{ccccc} X & \xleftarrow{g} & V & \xrightarrow{\epsilon} & S \\ {\scriptstyle p}\downarrow & & {\scriptstyle \tilde{\pi}}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y & \xleftarrow{f} & W & \xrightarrow{\eta} & \Sigma \end{array} \tag{3.2}$$

where X has canonical singularities and g is the minimal resolution.

Also, we abuse notation and we denote by the same letter a curve in V, resp. W, and its image in X, resp. Y. This should not be confusing for the reader, since we will mostly work with the cover $p: X \to Y$ and forget about $\tilde{\pi}: V \to W$. The branch curve $B \subset Y$ has at most negligible singularities and it is disjoint from $C_1, \ldots, C_5$; the flat cover p is given by the linear equivalence $2L \equiv B + C_1 + \cdots + C_5$. For $i = 1, \ldots, 5$, the surface X is smooth above the curve C_i and $p^*C_i = 2\Gamma_i$, with Γ_i a -1-curve. By contracting $\Gamma_1, \ldots, \Gamma_5 \subset X$, one obtains an intermediate object between the minimal surface S and its canonical model S_{can}; in particular p^*B is the pull back of $2K_{S_{\text{can}}}$, hence B is nef and $B^2 = 2$. Since $h^i(B) = h^i(K_Y + (K_Y + B))$, by Kawamata-Viehweg vanishing we have $h^i(B) = 0$ for $i > 0$, so $h^0(B) = 2$. We have $L^2 = -2$, hence $\chi(L) = 0$. Since $h^i(L) = 0$ for $i > 0$ by Kawamata-Viehweg vanishing, we have $h^0(L) = 0$ as well.

Recall cf. Cossec and Dolgachev (1989) that an *elliptic half-pencil* of an Enriques surface Y is an effective divisor E such that $|2E|$ is a free pencil of elliptic curves of Y. One has:

Proposition 3.4 *In the above setting, up to reordering $C_1, \ldots, C_5$, we have*:

(i) *there exists an elliptic half-pencil E of Y such that $B \in |2E + C_5 + K_Y|$*;
(ii) *the divisor $K_Y + C_1 + \cdots + C_4$ is divisible by 2 in* $\operatorname{Pic}(Y)$.

Proof

(i) Let $D \in |B|$ be general. By Calabri et al. (2007, Proposition 5.1), D is irreducible; since $D^2 = 2$, by Bertini's theorem it follows that D is smooth. Consider the system $|M| = |2B|$: the (set-theoretic) base locus of $|M|$ is contained in the (set-theoretic) base locus of $|B|$, which consists of 1 or 2 points. The restriction sequence $0 \to H^0(B) \to H^0(M) \to H^0(2K_D) \to 0$ is exact, since $H^1(B) = 0$; it follows that $|M|$ is free and, in the terminology of Cossec and Dolgachev (1989), it is a *superelliptic* system. By Cossec and Dolgachev (1989, Theorem 4.7.1), $M = 2B'$, where there are two possibilities for B':

(a) there exists elliptic half-pencils E_1, E_2 such that $E_1E_2 = 1$ and $B' = E_1 + E_2$
(b) there exists an elliptic half-pencil E and a nodal curve Z such that $EZ = 1$ and $B' = 2E + Z$.

Since $2B = 2B' = M$, we either have $B = B'$ or $B = B' + K_Y$, and in either case B and B' are numerically equivalent. If case (a) occurs, then $(E_1 + E_2)B = 2$ and $(E_1 + E_2)C_i = 0$ for $i = 1, \ldots, 5$. Since $|2E_i|$ is a free pencil for $i = 1, 2$ and $B^2 > 0$, it follows that $E_iB = 1$ and $E_iC_j = 0$ for $j = 1, \ldots, 5$. So we have $E_i(2L) = E_i(B + C_1 + \cdots + C_5) = 1$, a contradiction. So case (b) occurs. We claim that Z is one of the C_i. Assume by contradiction that this is not the case: then $(2E + Z)C_i = 0$ implies that Z is disjoint from the C_i. The divisor $C_1 + \cdots + C_5 + Z \sim 2L - 2E$ has self-intersection -12, hence $(L - E)^2 = -3$, contradicting the fact that the intersection form on $\mathrm{NS}(Y)$ is even.

So Z is equal to, say, C_5, and we have $B = 2E + C_5 + K_Y$, since $|2E + C_5|$ has C_5 as a fixed component while $|B|$ is an irreducible system.

(ii) follows immediately by (i). □

Lemma 3.5 *Let S be a Godeaux surface with an involution σ of Enriques type. Then* $\mathrm{Tors}(S)$ *is cyclic of order 4 and σ acts as the identity on* $\mathrm{Tors}(S)$.

Proof That $\mathrm{Tors}(S)$ is cyclic of order 4 is proven in Calabri et al. (2007, Proposition 5.3). Here we describe explicitly $\mathrm{Tors}(S)$. Since smooth blow ups do not change the torsion, we may replace S by X. Of course the element of order 2 is p^*K_Y. By Proposition 3.4 there is $N \in \mathrm{Pic}(Y)$ such that $2N \equiv C_1 + \cdots + C_4 + K_Y$; pulling back to X we obtain $2p^*N \equiv 2(\Gamma_1 + \cdots + \Gamma_4) + p^*K_Y$, hence $p^*N - (\Gamma_1 + \cdots + \Gamma_4)$ is a torsion element of order 4 and it is clearly σ-invariant. □

Lemma 3.6 *Let S be a Godeaux surface with an involution σ of Enriques type, let $c : \tilde{S} \to S$ be the canonical cover and let $G =$* $\mathrm{Hom}(\mathrm{Tors}(S), \mathbb{C}^*)$ *be the Galois group of c. Then there is an involution $\tilde{\sigma}$ of $\tilde{S}$ that lifts σ and commutes with G.*

Proof Since the canonical cover is intrinsically associated with S, σ can be lifted to an automorphism h of $\tilde{S}$, so the point is to show that h can be taken to be an involution that commutes with G. We have

$$\tilde{S} = \mathrm{Spec}(\oplus_{\eta \in \mathrm{Tors}(S)} \eta),$$

and by Lemma 3.5 the action of G on $\oplus_{\eta \in \mathrm{Tors}(S)} \eta$ preserves the summands. Thus h commutes with G. Denote by $\tilde{G}$ the subgroup of $\mathrm{Aut}(\tilde{S})$ generated by h and G: it is an abelian group of order 8 with a cyclic subgroup of order 4, hence it is either isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_2$ or to $\mathbb{Z}_8$. To prove the lemma we have to exclude the latter possibility. Assume for contradiction that $\tilde{G}$ is cyclic of order 8: then h generates $\tilde{G}$. So in particular h acts freely on $\tilde{S}$, because G does so. It follows that the group $\tilde{G}$ acts freely on $\tilde{S}$, which is impossible, for instance because $K^2_{\tilde{S}} = 4$ is not divisible by 8. □

By definition, the canonical ring $R(\tilde{S})$ coincides with the paracanonical ring of S:

$$\oplus_{m\in\mathbb{N},\ \eta\in \mathrm{Tors}(S)} H^0(mK_S+\eta).$$

There are two possible choices of $\tilde{\sigma}$ as in Lemma 3.6; each of these choices induces a σ-linearization of the pluricanonical bundles $mK_S+\eta$ compatible with the multiplicative structure of $R(\tilde{S})$ and a $\mathbb{Z}_2$-action on $H^0(mK_S+\eta)$ that lifts σ. So each vector space $H^0(mK_S+\eta)$ splits as a sum of two eigenspaces (corresponding to ± 1), whose dimensions we call the *σ-type* of $mK_S+\eta$.

We determine the σ-type in some cases:

Lemma 3.7 *Let S be a Godeaux surface and $\sigma \in \mathrm{Aut}(S)$ an Enriques involution. Denoting by $1 \in \mathrm{Tors}(S)$ a generator, the σ-type of mK_S+i, for $m = 1,2,4$ and $i \in \mathrm{Tors}(S)$ is shown in row m, column i of Table 1.*

Proof We may replace S by X, since this does not affect the σ-type. We recall the Hurwitz formula $K_X = p^*(K_Y+L)$, where as usual L is the line bundle such that $p_*\mathcal{O}_X = \mathcal{O}_Y \oplus L^{-1}$; in addition, by Lemma 3.5 and its proof, there is a line bundle $N \in \mathrm{Pic}(Y)$ such that $2N \equiv K_Y + C_1 + \cdots + C_4$ and our chosen generator $1 \in \mathrm{Tors}(X) \cong \mathrm{Tors}(S)$ is equal to $p^*N - (\Gamma_1 + \cdots + \Gamma_4)$. So we have:

$$\begin{aligned} &mK_X \equiv p^*(mK_Y+mL)\\ &mK_X+1 \equiv p^*(mK_Y+mL+N)-(\Gamma_1+\cdots+\Gamma_4)\\ &mK_X+2 \equiv p^*((m+1)K_Y+mL)\\ &mK_X+3 \equiv p^*((m+1)K_Y+mL+N)-(\Gamma_1+\cdots+\Gamma_4). \end{aligned} \tag{3.3}$$

Recall also that $h^0(K_X+i)=1$ for $i\neq 0$ and $h^0(mK_X+i) = 1+\frac{m(m-1)}{2}$ for $m\geq 2$ and for every $i\in \mathrm{Tors}(X)$. Using these remarks, the projection formulae for double covers and Kawamata-Viehweg vanishing, it is not hard to obtain Table 1.

As an example, consider $2K_X+1$: using (3.3) and the relation $2L\equiv B+C_1+\cdots+C_5$, gives $2K_X+1 = p^*(B+N)+\Gamma_1+\cdots+\Gamma_4+2\Gamma_5$. Since for $m>0$ and for every i the fixed part of $|mK_X+i|$ contains $m(\Gamma_1+\cdots+\Gamma_5)$, we have $2 = h^0(2K_X+1) = h^0(p^*(B+N))$. The projection formula for double covers gives the following decomposition in $\mathbb{Z}_2$-eigenspaces:

$$H^0(p^*(B+N)) = H^0(B+N)\oplus H^0(B+N-L).$$

Table 1 σ-types of mK_S+i

m\i	0	1	2	3
1	{0, 0}	{1, 0}	{1, 0}	{1, 0}
2	{2, 0}	{1, 1}	{2, 0}	{1, 1}
4	{5, 2}	{4, 3}	{5, 2}	{4, 3}

We have $B+N \sim B+\frac{1}{2}(C_1+\cdots+C_4)$: since B is nef and big, we may apply Kawamata-Viehweg vanishing and we obtain $h^0(B+N)=\chi(B+N)=1$, and thus $2K_X+1$ has σ-type $\{1,1\}$. □

Conclusion of the proof of Theorem 3.2. We follow the steps of Reid's description of the paracanonical ring $R(S)$ taking into account also the action of the cyclic group G (of order 4). So in degree 1 we have generators $x_i \in H^0(K_S+i)$, $i=1,2,3$ and in degree 2 we have two more generators $y_j \in H^0(2K_S+j)$, for $j=1,3$ and the element $g \in G$ acts on these generators as in Example 3.1. In addition, we may assume that all these generators are eigenvectors of $\tilde{\sigma}$, since $\tilde{\sigma}$ and g commute. Finally, up to replacing $\tilde{\sigma}$ by $\tilde{\sigma}g^2$, we may assume that y_1 is $\tilde{\sigma}$ invariant. The space $H^0(2K_S)$ is generated by x_2^2 and x_1x_3: since by Lemma 3.7 the σ-type of $2K_S$ is $\{2,0\}$, it follows that x_1 and x_3 are eigenvectors of $\tilde{\sigma}$ for the same eigenvalue. The space $H^0(2K_S+1)$ is generated by x_2x_3 and y_1 and has type $\{1,1\}$. It follows that x_2 and x_3 have opposite eigenvalues. Similarly, looking at $H^0(2K_S+3)$ we conclude that y_3 is also $\tilde{\sigma}$-invariant. So $\tilde{\sigma}$ has the form $(x_1,x_2,x_3,y_1,y_3)\mapsto(\pm x_1,\mp x_2,\pm x_3,y_1,y_3)$.

Now look at $H^0(4K_S)$: the two eigenspaces are spanned by

$$x_1^4, x_2^4, x_3^4, x_1^2x_3^2, x_1x_3x_2^2, y_1y_3 \tag{3.4}$$

and by

$$x_1x_2y_1, x_2x_3y_3.$$

Since by Lemma 3.7 the σ-type of $4K_S$ is $\{5,2\}$, there is a linear relation q_0 involving the monomials (3.4). The same argument shows the existence of a relation q_2 between the monomials:

$$x_1^2x_2^2, x_2^2x_3^2, x_1^3x_3, x_1x_3^3, y_1^2, y_3^2.$$

Finally, we observe that the map $(x_1,x_2,x_3,y_1,y_3)\mapsto(-x_1,-x_2,-x_3,y_1,y_3)$ induces the identity on $\mathbb{P}(1,1,1,2,2)$, so $\tilde{\sigma}$ acts on $\mathbb{P}(1,1,1,2,2)$ as in Example 3.1. □

4 Enriques Surfaces of Godeaux-Quotient Type

Here we apply the results of the previous section to describe the Enriques surfaces that are (birational) quotients of a Godeaux surface by an involution.

We consider Enriques surfaces Y such that Y contains an elliptic half-pencil E and nodal curves $C_1,\ldots,C_5$ such that:

- $EC_5 = 1, EC_1 = \cdots = EC_4 = 0$
- $C_1 + \cdots + C_4 + K_Y$ is divisible by 2 in $\mathrm{Pic}(Y)$.

We call a surface Y as above an Enriques surface of *Godeaux-quotient type*. Proposition 3.4 has a converse:

Proposition 4.1 *In the above setting:*

(i) *the system* $|2E + C_5 + K_Y|$ *is an irreducible pencil*;
(ii) *let* $B \in |2E + C_5 + K_Y|$ *be a curve disjoint from* $C_1, \ldots, C_5$; *then there exists a double cover* $X \to Y$ *branched on* $B + C_1 + \cdots + C_5$ *and the minimal model of* X *is a Godeaux surface with an involution of Enriques type.*

Proof The first assertion follows from the Riemann-Roch theorem and Cossec and Dolgachev (1989 Proposition 3.1.5). For (ii) notice that $B + C_1 + \cdots + C_5$ is even. Let $X \to Y$ be a double cover branched on $B + C_1 + \cdots + C_5$. Standard double cover calculations (cf. for example Mendes Lopes and Pardini 2004, Proposition 2.2) yield the result. □

As a direct consequence of Propositions 4.1, 3.4 and Corollary 3.3, we have:

Corollary 4.2 *The Enriques surfaces of Godeaux-quotient type are an irreducible unirational subset of dimension 5 of the moduli space of Enriques surfaces.*

We now give an explicit construction of Enriques surfaces of Godeaux quotient type.

Example 4.3 Consider the quadric cone $\mathcal{Q} \subset \mathbb{P}^3$ defined by $y_0^2 - y_1 y_2 = 0$ and the involution τ of $\mathcal{Q}$ defined by $[y_0, y_1, y_2, y_3] \mapsto [y_0, -y_1, -y_2, y_3]$. The linear system $|M|$ spanned by the invariant quadrics $y_0^2, y_1^2, y_2^2, y_3^2, y_0 y_3$ embeds the quotient surface $\mathcal{Q}/\tau$ in $\mathbb{P}^4$ as a quartic surface $\mathcal{D}$ defined by $x_0^2 - x_1 x_2 = x_0 x_3 - x_4^2 = 0$. The surface $\mathcal{D}$ ($\mathcal{D}_1'$ in the notation of Cossec and Dolgachev 1989, Chap. 0, Sect. 4) has two singular points of type A_1 at the points $P_1 = [0,1,0,0,0]$ and $P_2 = [0,0,1,0,0]$ (the "simple vertices") and a singularity of type A_3 at the point $P_0 = [0,0,0,1,0]$ (the "A_3-vertex").

An Enriques surfaces is called *special* if it contains a nodal curve C and an elliptic half-pencil E with $EC = 1$. All the special Enriques surfaces can be constructed as follows cf. Horikawa (1978a, b).

Take an element B_0 in the linear system of τ-invariant quartic sections of $\mathcal{Q}$ such that B_0 does not contain the fixed points of τ and has at most negligible singularities. The double cover $\tilde{Y} \to \mathcal{Q}$ is a $K3$ surface with canonical singularities. In particular it has two A_1 points over the vertex $[0,0,0,1] \in \mathcal{Q}$. The involution τ can be lifted to a free involution $\tilde{\tau}$ of $\tilde{Y}$. The quotient surface $\tilde{Y}/\tilde{\tau}$ is an Enriques surface with canonical singularities, and by construction it is a double cover of $\mathcal{D}$ branched over the singular points P_0, P_1, P_2 and on the image B of B_0. The preimage of P_0 is an A_1 singular point, which gives a nodal curve C on the minimal resolution Y of $\tilde{Y}/\tilde{\tau}$; the preimage of the line joining P_0 and P_1 gives an elliptic half-pencil E of Y such that $EC = 1$.

We now specialize this construction in order to get an Enriques surface of Godeaux quotient type. We take $B_0 = D + \tau^* D$, where D is a general quadric section of $\mathcal{Q}$. The curve B_0 has 8 nodes at the intersection points of D and $\tau^* D$, so in this case $\tilde{Y}$ has 10 A_1 points, two occurring over the vertex of $\mathcal{Q}$ and eight occurring over the nodes of B. These last eight points are an even set by Lemma 2.4. As in the proof of Lemma 2.4 consider the bidouble cover $h : Z \to \mathcal{Q}$ given by the relations $2L_1 \equiv D$, $2L_2 \equiv \tau^* D$, where $L_1 = L_2 = \mathcal{O}_{\mathcal{Q}}(1)$. As in Sect. 2.1 we denote by $G = \{1, g_1, g_2, g_3\}$ the Galois group of the bidouble cover and we assume that g_1, respectively g_2, fixes the preimage of D, respectively $\tau^* D$, pointwise, so that $Z/g_3 = \tilde{Y}$. As explained in Sect. 2.2, it is possible to lift τ to an automorphism ρ of Z and the group $\tilde{G} < \mathrm{Aut}(Z)$ generated by the Galois group $G \cong \mathbb{Z}_2^2$ and by ρ is isomorphic to the dihedral group D_4. The subgroup $G < \tilde{G}$ contains two reflections conjugate to one another and the square of a rotation, so we may choose the lift ρ of τ to be a rotation. Since τ switches D and $\tau^* D$, the action of ρ on G by conjugation switches g_1 and g_2 and fixes g_3. It follows that g_1 and g_2 are reflections and $g_3 = \rho^2$. Now let $\tilde{\tau}$ be the automorphism of $\tilde{Y} = Z/\rho^2$ induced by ρ. The fixed locus of ρ^2 on Z is the set of 8 points lying over the nodes of $D + \tau^* D$. Since ρ acts freely on these points, it follows that ρ acts freely on Z and $\tilde{\tau}$ acts freely on $\tilde{Y}$ (the fixed points of $\tilde{\tau}$ correspond to solutions $z \in Z$ of $\rho z = z$ or $\rho z = \rho^2 z$). Let Y be the minimal resolution of the surface $\tilde{Y}/\tilde{\tau} = Z/\rho$. The surface Y is a special Enriques surface that contains, besides $C_5 := C$ as in the general case, four additional disjoint nodal curves $C_1, \ldots, C_4$ arising from the 4 nodes of $\tilde{Y}/\tilde{\tau}$ that are the images of the 8 nodes of $\tilde{Y}$. Since the nodes of $\tilde{Y}$ are an even set, by Lemma 2.2 either $C_1 + \cdots + C_4$ or $C_1 + \cdots + C_4 + K_Y$ is even. Lemma 2.3 tells us that the latter case occurs, and therefore Y is an Enriques surface of Godeaux-quotient type.

Theorem 4.4 *The general Enriques surface of Godeaux-quotient type can be constructed as in Example 4.3.*

Proof Since $\mathrm{Aut}(\mathcal{Q})$ has dimension 3, the construction gives a 5-dimensional family of Enriques surfaces of Godeaux-quotient type and the statement follows by Corollary 4.2. □

5 A Construction of the General Godeaux Surface with an Enriques Involution

We give an alternative description of the general Godeaux surface with an involution of Enriques type, that will be used in Sect. 6 to compute some stable degenerations.

We keep the notation of the previous section (especially of Example 4.3. We take B_1 a general quadratic section of $\mathcal{Q}$, $B_2 = \tau^* B_1$ and B_3 a general hyperplane section containing the two smooth fixed points Q_1 and Q_2 of τ (notice that B_3 is τ-invariant). Consider the minimal resolution $\mathbb{F}_2 \to \mathcal{Q}$, denote by Γ the exceptional curve and use the same letter to denote curves on $\mathcal{Q}$ and their pull-backs to $\mathbb{F}_2$. By Sect. 2.1 there exists a bidouble cover $T_0 \to \mathbb{F}_2$ with branch divisors $B_1, B_2, B_3 + \Gamma$ and by Sect. 2.2 the involution of $\mathbb{F}_2$ induced by τ can be lifted to an automorphism of T_0. The preimage of Γ is the disjoint union of two irreducible -1-curves. Contracting these two curves, one obtains a bidouble cover $q : T \to \mathcal{Q}$, with T smooth, with branch divisors B_1, B_2 and B_3, which is branched also on the vertex $Q_0 = [0, 0, 0, 1]$ of $\mathcal{Q}$. By the Hurwitz formula, one has $K_T \sim \frac{1}{2} B_3$, hence T is smooth minimal of general type with $K_T^2 = 2$. The group $\tilde{G} < \mathrm{Aut}(T)$ generated by the Galois group $G = \{1, g_1, g_2, g_3\} \cong \mathbb{Z}_2^2$ of q and by a lift of τ is isomorphic to D_4 (cf. Sect. 2.2). Denote by $\rho \in D_4$ an element of order 4: then ρ is a lift of τ, ρ^2 is an element of G and commutes with ρ. Since τ exchanges B_1 and B_2, we have $g_3 = \rho^2$ and g_1 and $g_2 = g_1 \rho^2$ are reflections. As in Example 4.3, the surface $\tilde{Y} := T/\rho^2$ is a $K3$ surface with 10 nodes.

Lemma 5.1 *In the above setting:*

(i) *$g_1 \rho$ and $g_1 \rho^3$ induce a fixed point free involution of $\tilde{Y}$;*
(ii) *the surfaces $T/g_1 \rho$ and $T/g_1 \rho^3$ are Godeaux surfaces with an Enriques involution.*

Proof

(i) There are two liftings of τ to $\tilde{Y}$, one induced by ρ and the other one induced by $g_1 \rho$. We know (cf. Example 4.3) that one of these acts freely, while the other one fixes 8 points. Assume for contradiction that ρ induces a fixed point free involution $\tilde{\tau}$ and denote by Y the minimal desingularization of $\tilde{Y}/\tilde{\tau}$. By Example 4.3, Y is an Enriques surface of Godeaux quotient type; in particular $B + C_1 + \cdots + C_5$ is divisible by 2 in $\mathrm{Pic}(Y)$, where we denote by B the strict transform of the image of B_3 and by $C_1, \ldots, C_5$ the nodal curves that arise from the resolution of the images of the 10 nodes of $\tilde{Y}$. On the other hand, arguing as we did at the end of Example 4.3 we see that $B + C_1 + \cdots + C_5 + K_Y$ is divisible by 2 in $\mathrm{Pic}(Y)$. It follows that K_Y is divisible by 2 in $\mathrm{Pic}(Y)$, a contradiction. So the fixed point free involution $\tilde{\tau}$ of $\tilde{Y}$ that lifts τ is induced by $g_1 \rho$. Clearly, also $g_1 \rho^3$ induces the same involution.

(ii) By (i) $g_1\rho$ is a fixed point free involution of $\tilde{Y}$ and the same is true of the conjugate involution $g_1\rho^3$. The surfaces $S_1 := T/g_1\rho$ and $S_2 := T/g_1\rho^3$ are isomorphic; they are smooth minimal of general type with $K^2_{S_i} = 1$ for $i = 1, 2$, hence they are Godeaux surfaces. The involution ρ^2 induces on S_1 and S_2 an Enriques involution with quotient $\tilde{Y}$. □

Proposition 5.2 *The family of surfaces constructed as in Lemma 5.1, (ii) contains a dense open subset of the family of Godeaux surfaces with an Enriques involution.*

Proof By Corollary 3.3, it suffices to count dimensions. □

6 Stable Degenerations of Godeaux Surfaces with an Enriques Involution

At the beginning of this section we recall some facts on stable Godeaux surfaces. Then we describe some examples, obtained by letting the branch divisors in the construction given in Sect. 5 of the general Godeaux surfaces with an Enriques involution acquire singularities or multiple components.

6.1 Non-normal Gorenstein Stable Godeaux Surfaces

The notion of stable surface generalizes that of (canonical model of) minimal surface of general type in the same way as the notion of stable curve generalizes that of smooth curve of genus > 1: there exists a projective coarse moduli space $\overline{\mathcal{M}_{a,b}}$ parametrizing stable surfaces with fixed numerical invariants $K^2 = a$ and $\chi = b$ and the moduli space of surfaces of general type with the same invariants is an open subset $\mathcal{M}_{a,b} \subset \overline{\mathcal{M}_{a,b}}$ (cf. Alexeev 2006 for an exposition of the theory of stable varieties and, more generally, of stable pairs).

We recall the definition: a *stable surface* is a projective surface S such that:

- in the terminology of Kollár (2013) the surface S is *demi-normal*. This means that S satisfies condition S_2 of Serre and there exists an open subset $S_0 \subset S$ such that $S \setminus S_0$ is a finite set and for every $x \in S_0$ the point x is either smooth or double crossings (i.e., S is locally isomorphic to $xy = 0$ in the analytic or étale topology).
- let $\bar{S} \to S$ be the normalization map and let $\bar{D} \subset \bar{S}$ be the *double locus*, that is, $\bar{D}$ the effective divisor defined by the conductor ideal sheaf; then $(\bar{S}, \bar{D})$ is a log-canonical pair.
- there exists an integer m such that $\mathcal{O}_S(mK_S)$ is an ample line bundle.

If S is a stable surface, we denote by $\nu(S)$ the *Cartier index* of S, namely the smallest $m > 0$ such that mK_S is Cartier.

We call a stable surface with $K_S^2 = \chi(S) = 1$ a *stable Godeaux surface*; we say that S is *classical* if it has at most rational double points, i.e., if it is the canonical model of a minimal smooth surface of general type Y with $K_Y^2 = \chi(Y) = 1$. We are mainly interested in the case in which S is Gorenstein. Under this assumption, one has $h^1(\mathcal{O}_S) = h^2(\mathcal{O}_S) = 0$ (Franciosi et al. 2014a, Proposition 4.2) and the possibilities for the pair $(\bar{S}, \bar{D})$ associated to a non-classical Godeaux surface S are quite restricted:

Theorem 6.1 Franciosi et al. (2014b, Theorems 3.7 and 4.1) *Let S be a non-classical stable Godeaux surface and let $(\bar{S}, \bar{D})$ be the corresponding log-canonical pair. If S is Gorenstein, then one of the following cases occurs*:

(N) $S = \bar{S}$, *namely S is normal. Denote by $\epsilon : \tilde{S} \to S$ the minimal desingularization; in this case $\chi(\tilde{S}) = 0$ and the only non canonical singularity of S is an elliptic singularity;*

(P) $\bar{S} = \mathbb{P}^2$, *$\bar{D}$ a quartic;*

(dP) *$\bar{S}$ is a del Pezzo surface of degree 1, with at most canonical singularities, and $D \in |-2K_{\bar{S}}|$;*

(E_+) *$\bar{S}$ is the symmetric product of a curve E of genus 1 and $\bar{D}$ is a stable curve of genus 2 which is a trisection of the Albanese map $\bar{S} \to E$.*

Remark 1 More precisely, in Franciosi et al. (in preparation) it is shown that in case (N) the surface $\tilde{S}$ is either the blow up of a bielliptic surface at a point or a surface ruled over an elliptic curve and the bielliptic case is completely classified. An example with $\tilde{S}$ ruled appears in Lee (2000, Example 2.14); in Sect. 6.2 we give a new one.

The non-normal stable Gorenstein Godeaux surfaces of type (dP) are described in Rollenske (2014), where it is shown that they form an irreducible component of the moduli space, hence in particular they are not smoothable.

The non-normal stable Gorenstein Godeaux surfaces of type (P) and (E_+) are classified in Franciosi et al. (in preparation).

Here we recall the description of one family of surfaces of type (P) such that the general surface in the family has an involution. These surfaces are obtained in Sect. 6.2 as specializations of the Godeaux surfaces with an Enriques involution, and therefore they are smoothable (cf. Proposition 6.4).

Example 6.2 Let $P_1, \ldots, P_4 \in \mathbb{P}^2$ be independent points and let $\phi : \mathbb{P}^2 \to \mathbb{P}^2$ be the projective automorphism such that $\phi(P_i) = P_{i+1}$ for $1 \leq i \leq 4$ (indices are taken modulo 4). The automorphism ϕ induces on the pencil $\mathcal{F}$ of conics through $P_1, \ldots, P_4$ an involution that fixes the reducible conic $L(P_1, P_3) + L(P_2, P_4)$ and a smooth conic $C_0 \in \mathcal{F}$. We take $\bar{S} = \mathbb{P}^2$ and $\bar{D} = C + \phi_* C$, where $C \in \mathcal{F} \setminus \{C_0\}$ is a smooth conic. By Kollár (2013, Theorem 5.13) (cf. also Franciosi et al. 2014b, Theorem 3.2) for the Gorenstein condition) in order to construct a Gorenstein stable surface with $K^2 = 1$ with normalization $(\bar{S}, \bar{D})$, one has to give an involution $\imath$ of the normalization $C \sqcup \phi_* C$ of $\bar{D}$ with the property that $\imath$ acts freely on the eight preimages of $P_1, \ldots, P_4$. We take $\imath$ to be the involution that exchanges C and $\phi_* C$ and identifies C with $\phi_* C$ via ϕ. One has $\chi(S) = 1$ by Franciosi et al. (2014b, Proposition 3.4). The involution ϕ^2 of $\mathbb{P}^2$ commutes with $\imath$ and therefore it induces an involution of S (cf. Franciosi et al. 2014a, Sect. 3.B).

6.2 Degenerating Godeaux Surfaces with an Enriques Involution

A way of obtaining stable degenerations of a Godeaux surface with an Enriques involution is to apply the construction described in Sect. 5 relaxing the assumption that the branch divisors be general. Keeping the notation of Sect. 5, we take B_1 a divisor in $|\mathcal{O}_{\mathcal{Q}}(2)|$, $B_2 = \tau^* B_1$, B_3 a hyperplane section through Q_1 and Q_2 such that the pair $(\mathcal{Q}, \frac{1}{2}(B_1 + B_2 + B_3))$ is log-canonical and we construct the bidouble cover $T \to \mathcal{Q}$ with branch data B_1, B_2, B_3. Observe that ρ induces an isomorphism between the quotient surfaces $T/g_1\rho$ and $T/g_1\rho^3$; we abuse notation and refer to either of these surfaces as to S. By Proposition 5.2 the surface S is a degeneration of the general Godeaux surfaces with an Enriques involution. The next result shows that it is indeed a stable degeneration:

Lemma 6.3 *Consider the setup and notation of Sect. 5 and assume that the pair* $\left(\mathcal{Q}, \frac{1}{2}(B_1 + B_2 + B_3)\right)$ *is log-canonical.*

Then:

(i) *T is a stable Godeaux surface with $\nu(T) = 1$ or 2. If $Q_0 \notin B_1 + B_2 + B_3$ and $B_1 \cap B_2 \cap B_3 = \emptyset$, then T is Gorenstein.*
(ii) *S is a stable Godeaux surface such that $\nu(S)$ divides $2\nu(T)$;*
(iii) *if $B_1 + B_2$ does not contain any of the fixed points Q_0, Q_1, Q_2 of τ on $\mathcal{Q}$, then $T \to S$ is an étale morphism, and in particular $\nu(S) = \nu(T)$.*

Proof

(i) The cover $T \to \mathcal{Q}$ is demi-normal by Alexeev and Pardini (2012, Theorem 1.9). By Proposition 2.5, ibidem, the surface T is slc and $2K_T$ is the pull back of $2K_{\mathcal{Q}} + (B_1 + B_2 + B_3) = H$, where H is the hyperplane section of $\mathcal{Q}$. Hence K_T is ample and 2-Cartier.

If $Q_0 \notin B_1 + B_2 + B_3$, then T is smooth (hence Gorenstein) over Q_0; if $B_1 \cap B_2 \cap B_3 = \emptyset$ then locally over every smooth point of $\mathcal{Q}$, $T \to \mathcal{Q}$ is the composition of two flat double covers and therefore it is Gorenstein.

(ii) Since $g_1\rho$ lifts τ, that has only isolated fixed points, the quotient map $T \to S$ is unramified in codimension 1, hence again by Alexeev and Pardini (2012, Proposition 2.5) we have that S is an slc surface and K_S is ample, since it pulls back to K_T. In addition, the argument in the proof of Alexeev and Pardini (2012, Lemma 2.3) shows that $\nu(S)$ divides $2\nu(T)$. The fact that $K_S^2 = \chi(\mathcal{O}_S) = 1$ follows from the fact that S can be obtained as a flat limit of smooth Godeaux surfaces and so S is a stable Godeaux surface.

(iii) It is enough to show that the involution of $Y := S/\rho^2$ induced by $g_1\rho$ is base point free. If B_1 and B_2 are general, Y is a nodal $K3$ surface and the involution induced by ρ fixes all the preimages of Q_0, Q_1 and Q_2 (cf. proof of Lemma 5.1). By continuity, the involution induced by ρ fixes the preimages of the fixed points of τ for every choice of B_1 and B_2. Since g_1 induces the covering involution of $Y \to \mathcal{Q}$, if $Y \to \mathcal{Q}$ is unramified over Q_0, Q_1, Q_2, then the involution of Y induced by $g_1\rho$ acts freely on the preimages of Q_0, Q_1, Q_2, hence it acts freely on Y. □

6.3 Examples of Degenerations

We examine now some instances of the situation of Sect. 6.2. Recall that T (and S) is normal iff $B_1 + B_2 + B_3$ is a reduced divisor; in general, the normalization $\bar{T}$ of T is a bidouble cover of $\mathcal{Q}$ whose construction is described in Pardini (1991, Sect. 3). For the description of the possible singularities of T we refer the reader to Pardini (1991, Sect. 3).

(1) *B_1 and B_2 intersect at two points R_1, R_2 that are double points of both.*

An example of this type can be constructed as follows. Choose $R_1 \in \mathcal{Q}$ general and set $R_2 = \tau(R_1)$. If $H_1, \ldots, H_4$ are general hyperplane sections containing R_1 and R_2, then $H_1 + H_2$ and $H_3 + H_4$ span a pencil of quadric sections. We take B_1 a general element of this pencil, so that B_1 has ordinary double points at R_1 and R_2 and is smooth elsewhere; as usual, we set $B_2 = \tau^* B_1$. Since $B_1 B_2 = 8$, the divisor $B_1 + B_2$ has ordinary quadruple points at R_1 and R_2. We assume that B_3 is general; by Lemma 6.3, T and S are both Gorenstein. The surface T has two elliptic singularities U_1 and U_2 of degree 4 over R_1 and R_2 (cf. Table 1 of Alexeev and Pardini 2012, Sect. 3). These singularities map in S to one elliptic singularity of the same type, hence $1 = \chi(S) = \chi(\tilde{S}) + 1$, where $\tilde{Y}$ is the minimal desingularization of S. The minimal desingularization $\tilde{T} \to T$ is obtained by blowing up $\mathcal{Q}$ at R_1 and R_2 and taking base change and normalization; the exceptional curves of the blow-up $\hat{\mathcal{Q}} \to \mathcal{Q}$ are not contained in the branch locus of $\tilde{T} \to \hat{\mathcal{Q}}$. Therefore the strict transforms on $\hat{\mathcal{Q}}$ of the plane sections of $\mathcal{Q}$ through R_1 and R_2 meet the branch locus of $\tilde{T} \to \hat{\mathcal{Q}}$ only at two points, and so their preimages in T are pairs of rational curves. So T is ruled and therefore S and $\tilde{S}$ are ruled, too. Since $\chi(\tilde{S}) = 0$, the surface $\tilde{S}$ is ruled over an elliptic curve.

This is a new example of case (N) of Theorem 6.1 with S ruled; the other known example (cf. Lee 2000, Example 2.14) has an elliptic singularity of degree 3.

(2) $B_1 = 2H$, *with* H *a general hyperplane section.*

We have $B_2 = 2\tau^*H$ and we take B_3 general; by Lemma 6.3, T and S are both Gorenstein. In this case, the surface $\tilde{Y} = T/\rho^2$ is the union of two copies of $\mathcal{Q}$ glued along the curve $H + \tau^*H$. The surface T is non-normal and has two irreducible components, both isomorphic to the double cover of $\mathcal{Q}$ branched on the plane section B_3 and on the vertex Q_0 of $\mathcal{Q}$, and therefore both isomorphic to $\mathbb{P}^2$. By Lemma 6.3, the surface S is Gorenstein and therefore irreducible, since $K_S^2 = 1$. So $g_1\rho$ permutes the two components of T and the normalization $\bar{S}$ of S is isomorphic to $\mathbb{P}^2$, namely S is as in case (P) of Theorem 6.1. The surface $\bar{S} = \mathbb{P}^2$ can be naturally identified with one of the irreducible component of T; we denote by $\pi : \bar{S} \to \mathcal{Q}$ the degree 2 map induced by this identification. The double locus $\bar{D} \subset \bar{S}$ is the union of two conics, $C_1 := \pi^*H$ and $C_2 := \pi^*(\tau^*H)$, that are identified with one another by the involution ι of $C_1 \sqcup C_2$ induced by the map $\bar{S} \to S$.

We claim that the surface S belongs to the family constructed in Example 6.2. Let R_1, R_2 be the intersection points of H and τ^*H in $\mathcal{Q}$ and write $\pi^{-1}(R_1) = \{P_1, P_3\}$ and $\pi^{-1}(R_2) = \{P_2, P_4\}$. The points $P_1, \ldots, P_4$ are the base points of the pencil of conics spanned by C_1 and C_2. By construction, the involution ι of $C_1 \sqcup C_2$ lifts the involution of $H + \tau^*H$ given by τ. The involution τ lifts to an automorphism of $\bar{S}$ that exchanges the sets $\pi^{-1}(R_1)$ and $\pi^{-1}(R_2)$ and exchanges the conics C_1 and C_2. Elementary arguments on pencils of plane conics show that such a map is either the that automorphism ϕ that induces a cyclic permutation of $P_1, \ldots, P_4$ or its inverse ϕ^3. So, possibly up to relabelling the P_i, the involution ι of $C_1 \sqcup C_2$ induced by the normalization map $\bar{S} \to S$ switches C_1 and C_2 and identifies C_1 with C_2 via ϕ. Since letting H vary in the pencil of plane sections through R_1 and R_2 we can obtain any conic in the pencil spanned by C_1 and C_2, we have proven the following:

Proposition 6.4 *The surfaces in the family of Example 6.2 are smoothable.*

(3) B_1 *and* B_2 *have a common component which is a hyperplane section.*

Take $B_1 = H_0 + H_1$, where H_0 is a τ-invariant hyperplane section and H_1 is a general one, so that $B_2 = H_0 + \tau^*H_1$, and take B_3 general. Assume that H_0 does not contain the vertex Q_0 of $\mathcal{Q}$, hence H_0 contains the two smooth fixed points Q_1 and Q_2 of τ. By Table 2 of Alexeev and Pardini (2012, Sect. 3) the singularities of T over Q_1 and Q_2 are not Gorenstein, so $\nu(T) = 2$ by Lemma 6.3, and it follows that S is not Gorenstein either.

By Pardini (1991, Sect. 3), the normalization $\bar{T}$ of T is the bidouble cover of $\mathcal{Q}$ branched on H_1, τ^*H_1, B_3+H_0 and the vertex Q_0 of $\mathcal{Q}$. The surface $\bar{T}$ has a pair of singular points of type A_1 over Q_1 and over Q_2 and is smooth elsewhere. By the Hurwitz formula the canonical class $K_{\bar{T}}$ is numerically equivalent to 0. Taking base change of $\bar{T} \to \mathcal{Q}$ with the minimal resolution $\mathbb{F}_2 \to \mathcal{Q}$ one obtains a flat bidouble cover $T_0 \to \mathbb{F}_2$. The standard formulae for double covers give $p_g(T_0) = q(T_0) = 0$, hence $\bar{T}$ is an Enriques surface with four nodes. The involution $g_1\rho$ of T induces an involution of the minimal desingularization $\tilde{T}$ of T, whose fixed locus is contained in the preimages of the points $Q_0, Q_1, Q_2 \in \mathcal{Q}$. The preimage of Q_0 consists of two smooth points, while the preimage of $\{Q_1, Q_2\}$ is the disjoint union of four nodal curves. Assume that one of these nodal curves is preserved by $g_1\rho$; then a local computation shows that this curve is not fixed pointwise by $g_1\rho$. Summing up, the fixed locus of $g_1\rho$ on $\tilde{T}$ is finite. It follows that the quotient surface $\tilde{T}/g_1\rho$ is again an Enriques surface, and so is $\bar{S}$, since it is birational to $\tilde{T}/g_1\rho$.

This example shows that if we remove the assumption that S is Gorenstein, then Theorem 6.1 does not hold any more.

(4) $B_1 = H_1 + 2F_1$, *where* H_1 *is a general hyperplane section and* F_1 *is a general ruling of* $\mathcal{Q}$.

Set $H_2 = \tau^*H_1$, $F_2 = \tau^*F_1$, so that $B_2 = H_2 + 2F_2$. The surface T is singular above F_1 and F_2. The normalization $\bar{T}$ of T is a bidouble cover of $\mathcal{Q}$ branched on the three hyperplane sections H_1, H_2 and B_3, so $K_{\bar{T}}$ is numerically equivalent to the the pull-back of $-\frac{1}{2}H_1$, and $\bar{T}$ is a del Pezzo surface of degree 2. The map $\bar{T} \to \mathcal{Q}$ is unramified over the vertex Q_0, hence the singularities of $\bar{T}$ are four points U_1, U_2, U_3, U_4 of type A_1 occurring above Q_0. The elements $g_1, g_2 = g_1\rho^2, \rho^2$ of D_4 act on U_1, U_2, U_3, U_4 switching them in pairs, so ρ acts as a cyclic permutation of order 4 and $g_1\rho$ switches, say, U_1 and U_3 and fixes U_2 and U_4. Looking at the minimal resolution $\tilde{T}$ of $\bar{T}$, one sees that $g_1\rho$ has two isolated fixed points on each of the nodal curves corresponding to U_2 and U_4, hence the fixed locus of $g_1\rho$ on $\tilde{T}$ is a finite set and the quotient surface $\bar{S} = \bar{T}/g_1\rho$ has canonical singularities (the images of U_2 and U_4 are points of type A_3). Hence $\bar{S}$ is a del Pezzo surface of degree 1.

The double locus $D_T \subset \bar{T}$ is the preimage of $F_1 + F_2$: it consists of two smooth rational curves Γ_1 and Γ_2 meeting transversely at $U_1, \dots, U_4$ and it is an antibicanonical curve. The double locus $\bar{D} \subset \bar{S}$ is the image of D_T: it is an irreducible curve with $p_a = 1$, since it is smooth at the images of U_2 and U_4 and it has a node at the image point of U_1 and U_3. The curve $\bar{D}$ is numerically equivalent to an antibicanonical curve, since it pulls back to D_T, but it is not Cartier since it is smooth at the A_3 points of $\bar{S}$ (notice also the failure of the usual adjunction formula), hence it is not in $|-2K_{\bar{S}}|$. So this case is different from case (dP) of Theorem 6.1. In fact, the surface S is not Gorenstein, since $K_{\bar{S}} + \bar{D}$ is not Cartier.

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References

Valery Alexeev, Higher-dimensional analogues of stable curves. *Proceedings of Madrid ICM2006*, 352(2):515–536, 2006.

Valery Alexeev and Rita Pardini, Non-normal abelian covers. *Compos. Math.*, 148(4):1051–1084, 2012.

Rebecca Barlow, Some new surfaces with $p_g = 0$. *Duke Math. J.*, 51(4):889–904, 1984.

Rebecca Barlow, A simply connected surface of general type with $p_g = 0$. *Invent. Math.*, 79 (2):293–301, 1985.

Alberto Calabri, Ciro Ciliberto, and Margarida Mendes Lopes, Numerical Godeaux surfaces with an involution. *Trans. Amer. Math. Soc.*, 359(4):1605–1632 (electronic), 2007.

François R. Cossec and Igor V. Dolgachev, *Enriques surfaces. I*, volume 76 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1989.

Marco Franciosi, Rita Pardini, and Sönke Rollenske, Stable Gorenstein Godeaux surfaces. in preparation.

Marco Franciosi, Rita Pardini, and Sönke Rollenske, Computing invariants of semi-log-canonical surfaces. preprint http://arxiv.org/abs/1404.3548arXiv:1404.3548, 2014a.

Marco Franciosi, Rita Pardini, and Sönke Rollenske, Log-canonical pairs and Gorenstein stable surfaces with $K_X^2 = 1$. preprint http://arxiv.org/abs/1403.2159arXiv:1403.2159, to appear in Compositio Mathematica, 2014b.

Eiji Horikawa, On the periods of Enriques surfaces. I. *Math. Ann.*, 234(1):73–88, 1978a.

Eiji Horikawa, On the periods of Enriques surfaces. II. *Math. Ann.*, 235(3):217–246, 1978b.

Jonghae Keum and Yongnam Lee, Fixed locus of an involution acting on a Godeaux surface. *Math. Proc. Cambridge Philos. Soc.*, 129(2):205–216, 2000.

János Kollár, *Singularities of the minimal model program*, volume 200 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2013. With a collaboration of Sándor Kovács.

Yongnam Lee, Semistable degeneration of Godeaux surfaces with relatively nef canonical bundle. *Math. Nachr.*, 219:135–146, 2000.

Yongnam Lee and Jongil Park, A simply connected surface of general type with $p_g = 0$ and $K^2 = 2$. *Invent. Math.*, 170(3):483–505, 2007.

S. Maggiolo, Automorphisms of numerical Godeaux surfaces with torsion of order 3, 4, or 5. http://arxiv.org/abs/1002.3494arXiv:1002.3494. 2010.

Yoichi Miyaoka, Tricanonical map of a certain class of surfaces. *Sûrikaisekikenkyûsho Kókyûroku*, (240):31–47, 1975. A Symposium on Complex Manifolds (Kyoto, 1974).

Margarida Mendes Lopes and Rita Pardini, A new family of surfaces with $p_g = 0$ and $K^2 = 3$. *Ann. Sci. École Norm. Sup. (4)*, 37(4):507–531, 2004.

Rita Pardini, Abelian covers of algebraic varieties. *J. Reine Angew. Math.*, 417:191–213, 1991.

Heesang Park, Jongil Park, and Dongsoo Shin, Surfaces of general type with $p_g = 1$ and $q = 0$. *J. Korean Math. Soc.*, 50(3):493–507, 2013.

Miles Reid, Surfaces with $p_g = 0$, $K^2 = 1$. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 25(1):75–92, 1978.

Sönke Rollenske, A new irreducible component of the moduli space of stable Godeaux surfaces. preprint http://arxiv.org/abs/1404.7027arXiv:1404.7027, 2014.

Geometry of Lines and Degeneracy Loci of Morphisms of Vector Bundles

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Abstract

Corrado Segre played a leading role in the foundation of line geometry. We survey some recent results on degeneracy loci of morphisms of vector bundles where he still is of profound inspiration.

Keywords

Grassmannian · Congruence · Focal locus · Skew-symmetric matrices · Constant rank · Instanton bundles

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1 Grassmannians of Lines: Linear Sections and Focal Properties

1.1 Classical Point of View: Work of Corrado Segre on Grassmannians

The geometry of families of lines in the projective space, classically called "line geometry", has been one of the first objects of investigation of Corrado Segre, and a *fil rouge* of his research throughout his career. His graduation thesis was published

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in 1883, in two articles, in “Memorie dell’Accademia delle Scienze di Torino”. In the first article (Segre 1883a), corresponding to the first two chapters of the thesis, Segre systematically studies the (hyper)quadrics, in the second one (Segre 1883b) he develops the geometry of the Klein quadric in $\mathbb{P}^5$, i.e. the Grassmannian $\mathbb{G}(1,3)$ of lines in $\mathbb{P}^3$. He considers linear and quadratic sections of the Grassmannian, called linear and quadratic complexes when of dimension three, and linear and quadratic congruences when of dimension two, and moreover ruled surfaces. In particular he studies the notion of focal locus of a congruence, roughly speaking the set of points where two “infinitely near” lines of the family meet.

We want to mention then a short article published by Segre (1888), where he considers the line geometry in a projective space of any dimension n. Here Segre states a few basic facts on the focal points of a congruence of lines in $\mathbb{P}^n$, meaning now a family of dimension $n-1$.

Many years later, in 1910, Corrado Segre comes back to this circle of ideas in the foundational article (Segre 1910), entirely devoted to the projective-differential geometry of families of linear spaces, and in particular he develops the theory of their focal loci.

Moving in a different direction, in 1917 Segre publishes the article (Segre 1917), where there is a systematic study of the Grassmannian $\mathbb{G}(2,5)$ of 2-planes in $\mathbb{P}^5$, for instance the classification of the orbits of the natural action of the projective group $PGL(6)$ on $\mathbb{P}(\Lambda^3\mathbb{C}^6)$.

An interesting article on the line geometry, possibly inspired by Corrado Segre, was published in 1891 by Castelnuovo (1891).

Here Castelnuovo studies the linear sections of all dimensions, both general and special, of the Grassmannian of lines in $\mathbb{P}^4$, $\mathbb{G}(1,4)$, giving a geometric description of their “singular” (i.e. focal) varieties in $\mathbb{P}^4$. In the first cases they are a conic, a projected Veronese surface (in the case of a congruence of lines), and a Segre cubic hypersurface.

This last hypersurface had been considered by Segre in the article (Segre 1886–87), which contains a description of its beautiful and rich geometry. The point of view developed by Castelnuovo allows to recover the geometrical properties of this threefold with a completely different but natural method. As Castelnuovo says, his article gives a prototype of the geometry of lines in spaces of even dimension.

Let us look more closely at the particular case of a general linear congruence of lines in $\mathbb{P}^4$, let us call it Γ. In other words Γ is a 3-dimensional general linear section of $\mathbb{G}(1,4)$. Castelnuovo proves that the elements of Γ are the trisecant lines of its focal variety, which results to be a projected Veronese surface S in $\mathbb{P}^4$. This implies that Γ can be reconstructed from S. On the other hand, Castelnuovo proves that all projected Veronese surfaces in $\mathbb{P}^4$ arise from such a construction.

To conclude this short historical overview, we want to mention that Palatini (1900) considered similar constructions in $\mathbb{P}^5$. Here a general linear congruence gives raise, as singular variety, to an interesting three-dimensional smooth scroll X, after him named “Palatini scroll”. The interest comes from a “non-normality” property of this threefold, being in some sense the analogous of one of the characteristic properties of the Veronese surface. More precisely, the famous Severi theorem asserts that the projected Veronese surface is the only non degenerate non linearly normal surface of $\mathbb{P}^4$. It easily results from the construction that Palatini scroll X is smooth and not quadratically normal, meaning that the linear series cut on X by the quadric hypersurfaces is not complete. An important conjecture by Christian Peskine states that this is the only smooth and non quadratically normal threefold in $\mathbb{P}^5$ (Schneider 1990).

1.2 Modern Point of View

In this section we will expose how the classical constructions mentioned in Sect. 1.1 can be rephrased in the modern language of the theory of vector bundles. The main references are Bazan and Mezzetti (2001), De Poi (2003), Ottaviani (1992), Tanturri (2016).

Let k be an algebraically closed field, with char $(k) = 0$. We fix integer numbers $2 \leq m \leq n$ and two k-vector spaces U, V, with $\dim U = m$, $\dim V = n+1$.

We consider a general bundle map $\phi : U^* \otimes \mathcal{O}_{\mathbb{P}(V)} \rightarrow \Omega_{\mathbb{P}(V)}(2)$. From the Euler sequence, it follows that $H^0(\Omega_{\mathbb{P}(V)}(2)) \simeq (\Lambda^2 V)^*$. Hence ϕ is defined by m general skew-symmetric matrices $A_1, \ldots, A_m$. Let X_ϕ be the degeneracy locus of ϕ. Then

- ϕ is injective;
- $\dim(X_\phi) = m - 1$;
- $\dim(Sing(X_\phi)) = 2m - n - 4$;
- X_ϕ is set-theoretically the set of points $x \in \mathbb{P}^n$ such that $(y_1 A_1 + \ldots + y_m A_m)x = 0$, for some $[y_1, \ldots, y_m] \in \mathbb{P}(U)$;
- for n even, X_ϕ is a unirational variety parametrized by $\mathbb{P}(U)$;
- for n odd, X_ϕ is a scroll over the hypersurface Z in $\mathbb{P}(U)$ defined by the equation: Pfaffian $(y_1 A_1 + \ldots + y_m A_m) = 0$.

The skew-symmetric matrices $A_1, \ldots, A_m$ correspond bijectively to hyperplanes $H_1, \ldots, H_n$ in the embedding space of the Grassmannian $\mathbb{G}(1, n)$, so they define a linear section $\Gamma := \mathbb{G}(1, n) \cap H_1 \cap \cdots \cap H_n$, which is nothing else than the family of lines of the classical construction, and the degeneracy locus X_ϕ coincides with the classical focal locus.

For $n = 4$, this construction is precisely the one described by Castelnuovo. If $m = 2$ X_ϕ is a plane conic, if $m = 3$ a projected Veronese surface, if $m = 4$ a cubic Segre hypersurface. For $n = 5$, if $m = 3$ we get an elliptic surface scroll in $\mathbb{P}^5$, if $m = 4$ a Palatini scroll.

Having in mind the case of the Veronese surface in $\mathbb{P}^4$, the following natural question arises.

Question 1 *Can* Γ *be recovered from* X_ϕ*?*

The example of an elliptic surface scroll in $\mathbb{P}^5$ shows that the answer is in general negative. Indeed such a surface is obtained as degeneracy locus from four different maps ϕ. This had been observed by Fano (1930) (see Bazan and Mezzetti 2001 for a modern proof), and follows from the structure of the Picard group of elliptic curves.

On the other hand, a Palatini scroll X is obtained from only one map ϕ (Fania and Mezzetti 2002). The geometrical interpretation of the congruence Γ in this case is that the lines of Γ are the 4-secant lines of X not contained in X, so unicity follows.

1.3 General Linear Sections: Hilbert Schemes

In this section we reformulate Question 1 in more precise terms, and we describe the recent progress on it by Faenzi and Fania (2010) and Tanturri (2013, 2015, 2016).

Let us denote by $\mathcal{H}$ the union of the irreducible components of the Hilbert scheme containing the degeneracy loci of general maps $\phi : U^* \otimes \mathcal{O}_{\mathbb{P}(V)} \to \Omega_{\mathbb{P}(V)}(2)$.

We consider the natural rational map $\rho : \mathbb{G}(m, \Lambda^2 V) \to \mathcal{H}$, taking a bundle map to its degeneracy locus. The discussion above motivates the following:

Question 2 *Let* n, m *be integer numbers with* $2 \leq m \leq n + 1$.

(1) *Is* ρ *dominant?*
(2) *Is* ρ *generically injective?*
(3) *If* ρ *is not injective, describe its fibres.*

The results of the above quoted authors give a complete answer to Question 2 for $2 \leq m \leq n$. They are the content of the following theorem.

Theorem 1.1 *In the above notation*,

- *if* $m \geq 4$ *or* $(m, n) = (3, 4)$, *then* ρ is *birational and* $\mathcal{H}$ *is generically smooth;*
- *if* $(m, n) = (3, 5)$, *then* ρ *is dominant and* $4 : 1$;
- *if* $m = 3$ *and* $n \neq 5$, *then* ρ *is generically injective*;
- *if* $m = 3$ *and* n *is even, a general element of the image of* ρ *is a special projection of the Veronese surface* $v_{\frac{n-1}{2}}(\mathbb{P}^2)$. *The centre of projection is the linear span of the partial derivatives of order* $\frac{n-5}{2}$ *of a non-degenerate polynomial of degree* $n - 3$;
- *if* $m = 3$ *and* n is *odd, a general element of the image is a projective bundle* $\mathbb{P}(\mathcal{G})$, *with* $\mathcal{G}$ *a general stable rank two bundle on a general plane curve* C *of degree* $\frac{n}{2}$, *with* $\det(\mathcal{G}) = \mathcal{O}_C(\frac{n-2}{2})$;

- *if* $m = 2$, *then* ρ *is dominant. Its fibres have positive dimension and can be explicitly described.*

The proof of the theorem relies on the method of Kempf–Lascoux–Weyman, in particular on applications of Eagon–Northcott complex, and on the structure theorem of Buchsbaum-Eisenbud and apolarity theory in the case $m = 3$.

2 Linear Systems of Skew-Symmetric Matrices of Constant Rank

Up to now we considered *general* linear sections of Grassmannians of lines, or in other words, general bundle maps $\phi : U^* \otimes \mathcal{O}_{\mathbb{P}(V)} \to \Omega_{\mathbb{P}(V)}(2)$. But, according to a general idea by Ch. Peskine and F. Zak, we expect that degeneracy loci of *special* maps ϕ should produce interesting varieties. In this section we will explore some special cases and related problems.

As we observed in Sect. 1.2, giving the map ϕ is equivalent to giving an m-dimensional linear subspace $\Lambda := \langle A_1, \ldots, A_m \rangle$ in $\Lambda^2 V^*$. The family of lines Γ is the intersection of $\mathbb{G}(1, n)$ with $\mathbb{P}(\Lambda^*)$, the projectivized dual of Λ. Hence, classifying the maps ϕ, for fixed m, is equivalent to classifying the projective subspaces of dimension $m - 1$ in $\mathbb{P}(\Lambda^2 V)^*$ under the action of $PGL(n+1)$, also describing their possible positions with respect to the natural filtration by the rank of skew-symmetric tensors.

Since a complete classification appears to be out of reach, we will focus on a natural special case, which is interesting and has been studied in linear algebra since the classical work of Gantmacher (1959).

By $\sigma_k \mathbb{G}(1, n)$ we denote the kth secant variety, i.e. the Zariski closure of the union of the linear spans of general k-tuples of points in $\mathbb{G}(1, n)$.

Question 3 *Classify the orbits of linear subspaces contained in a quasi-projective variety of the form* $\sigma_k \mathbb{G}(1, n) \setminus \sigma_{k-1} \mathbb{G}(1, n)$ *for some n and k, i.e. the orbits of linear systems of skew-symmetric matrices of constant rank.*

This question is of course a particular case of the more general problem of classifying linear systems of matrices of constant rank of any size without skew-symmetry conditions. An analogous problem refers to symmetric matrices. A different generalization regards linear systems of matrices of bounded rank. For a bibliography on these problems and their applications in algebraic geometry, see Ilic and Landsberg (1999).

From now on, when speaking of dimension of a linear system we will mean its *projective* dimension.

The linear systems of skew-symmetric matrices of constant rank 2 can be interpreted as linear spaces contained in a Grassmannian $\mathbb{G}(1, n)$, a well-known case. Those of maximal dimension correspond either to the lines contained in a

2-plane or to a star of lines; they are a $\mathbb{P}^2$ and a $\mathbb{P}^{n-1}$ respectively. So the first interesting case to analyze is that of linear systems of 6×6 skew-symmetric matrices of constant rank 4. This situation has been studied in Manivel and Mezzetti (2005). The result is the following classification:

- the maximal dimension of a linear system of skew-symmetric 6×6 matrices of constant rank four is two,
- the space of 1-dimensional linear systems of matrices of this type is irreducible of dimension 22, with an open $PGL(6)$-orbit of general lines, and a codimension one orbit of special lines,
- there are four $PGL(6)$-orbits of 2-dimensional linear systems, all of the same dimension 26 and homogeneous under the action of $PGL(6)$.

The above classification of the orbits of 2-planes has found application to the classification of the degenerations of Palatini scroll (De Poi and Mezzetti 2005), such that the cubic surface, which is the base of the scroll, splits as union of a 2-plane and a quadric surface. As a further application, we mention the construction of new examples of non quadratically normal threefolds in $\mathbb{P}^5$ (De Poi and Mezzetti 2008). These threefolds are singular, so the smoothness assumption in the conjecture of Peskine mentioned in Sect. 1.1 results to be necessary.

2.1 Maximal Dimension

For linear systems of $(n+1) \times (n+1)$ skew-symmetric matrices of constant rank r, with $r < n+1$ even, the first step in the study of Question 3 is finding their maximal dimension.

Let us introduce the following notation: $l(r, n+1) := \max\{\dim$ of a linear system of skew-symmetric matrices of rank r and size $n+1\}$.

The following inequality is due to Sylvester (1986) and Westwick (1987): if $r \geq 2$ is an even number, $r < n+1$, then

$$n+1-r \leq l(r, n+1) \leq 2(n+1-r). \tag{2.1}$$

In particular, if $r = n-1$, i.e. if the corank is two, then $2 \leq l(n-1, n+1) \leq 4$.

We mention that the same inequality is valid also for linear systems of symmetric matrices. The main result in Ilic and Landsberg (1999) states that the maximal dimension of a linear system of symmetric matrices of constant rank is $n+1-r$, the lower bound above. We will see that in the skew-symmetric case the situation is different, in particular $l(r, n+1)$ depends on n, and not only on $n-r$, and its precise value is still unknown in general.

To explain why and to underline the deep connections of this problem with algebraic geometry, let us remark that a linear system of dimension $d \leq l(r, n+1)$ of skew-symmetric matrices of constant rank defines an exact sequence of vector bundles:

$$0 \to K \to \mathcal{O}_{\mathbb{P}^d}^{n+1} \to \mathcal{O}_{\mathbb{P}^d}^{n+1}(1) \to N \to 0, \tag{2.2}$$

where:

- the kernel K and the cokernel N are rank r vector bundles,
- $N \simeq K^*(1)$,
- K^* is globally generated and defines an embedding in the appropriate Grassmannian.

In particular, if $r = n - 1$, then K and N are rank 2 bundles and $c_1(K^*) = \frac{r}{2}$. For instance, for $n = 5$ and $r = 4$, the four orbits of 2-planes of 6×6 matrices of constant rank 4 correspond to the four rank 2 bundles on $\mathbb{P}^2$ with $c_1 = 2$ defining an embedding in $\mathbb{G}(1, 5)$, according to their classification given in Sierra and Ugaglia (2006).

This can be restated introducing the definition of *m*-effective bundle. We say that a globally generated rank r bundle on $\mathbb{P}^d$ is *m*-effective if its dual appears as kernel K in an exact sequence of the form (2.2). So all globally generated rank 2 bundles on $\mathbb{P}^2$, with $c_1 = 2$ and defining an embedding in $\mathbb{G}(1, 5)$, are *m*-effective.

A similar result is true for linear systems of 8×8 skew-symmetric matrices of constant rank 6. In Fania and Mezzetti (2011) it is proved that every globally generated rank 2 bundle on $\mathbb{P}^2$, with $c_1 = 3$ and defining an embedding in $\mathbb{G}(1, 7)$, is *m*-effective. This gives a picture of the 2-planes. In the same article it is proved that also for these size and rank there are no linear systems of dimension bigger than 2.

2.2 Spaces of Dimension 1 and 2

Turning to general n, we mention that in Fania and Mezzetti (2011) a complete classification of the orbits of 1-dimensional spaces is given, for every pair (n, r).

As for 2-planes, we focus on the case n odd and $r = n - 1$, corresponding to rank 2 bundles on $\mathbb{P}^2$. The following result has been proved in Boralevi and Mezzetti (2015).

Theorem 2.1 *Let n be an odd integer number.*

- *There exists an m-effective bundle E with Chern classes c_1, c_2 for every pair (c_1, c_2) such that $c_1 = \frac{n-1}{2}$ and c_2 is in the stable range*;
- *there exist unstable non m-effective globally generated rank 2 bundles.*

The proof of the theorem rests on the recent complete description of the possible Chern classes of rank two globally generated vector bundles on the projective plane by Ellia (2013).

The assumption that $r = n - 1$, or that the corank is two, is not so restrictive, because every 2-plane of skew-symmetric matrices of constant rank r can be isomorphically projected to a 2-plane of $(r+2) \times (r+2)$ matrices (for more details see Fania and Mezzetti 2011, Corollary 5.9).

2.3 3-Planes of Skew-Symmetric Matrices of Corank Two

According to Theorem 2.1, there are "many" orbits of 2-planes of skew-symmetric matrices of constant corank two. But the upper bound on $l(n-1, n+1)$ given by inequality (2.1) is 4. The first example of a linear system of dimension bigger than two was exhibited by Westwick (1996). It is a 3-space of 10×10 skew-symmetric matrices of constant rank 8.

Since this example is given without any explanation, it is natural to ask which are the corresponding bundles and to look for other examples.

These problems have been considered in Boralevi et al. (2013). The main result is a method to construct a skew-symmetric matrix A of linear forms on $\mathbb{P}^3$, having constant rank $r = n - 1$, i.e. corank 2, starting from a normalized rank 2 bundle E on $\mathbb{P}^3$ with allowed Chern classes, i.e. $c_1(E) = 0$ and $c_2(E) = \frac{r(r+4)}{48}$. The matrix A will have E as kernel, up to a twist by a line bundle. One needs also a class $\beta \in Ext^2(E(\frac{r}{4} - 1), E(-\frac{r}{4} - 2))$.

Then the main idea is the following: the cone of β, interpreted as a morphism $E(\frac{r}{4} - 1) \to E(-\frac{r}{4} - 2)$ [2] in the derived category $D^b(\mathbb{P}^3)$, is a 2-term complex. Using Beilinson's theorem, necessary and sufficient conditions are found ensuring that this 2-term complex is of the desired form (2.2) and the matrix in the middle is skew-symmetrizable.

A useful remark is that the conditions can be simplified if E has natural cohomology. Since general instantons have natural cohomology, and their minimal graded free resolution is known, as well as that of their cohomology module, it is possible to apply the construction to general instantons. It turns out that the necessary and sufficient conditions are satisfied for low values of $c_2(E)$.

Theorem 2.2

(1) *Any* 2-*instanton on* $\mathbb{P}^3$ *induces a* 3-*dimensional space of* 10×10 *skew-symmetric matrices of constant rank* 8;

(2) *any* 4-*instanton* E *on* $\mathbb{P}^3$ *with natural cohomology and such that* $E(2)$ *is globally generated induces a* 3-*dimensional space of* 14×14 *skew-symmetric matrices of constant rank* 12;

(3) *Westwick's example corresponds to a* 2-*instanton belonging to the most special orbit of the moduli space* $M_{\mathbb{P}^3}(2; 0, 2)$ *under the natural action of* $SL(4)$.

Hence in particular there exists a continuous family of examples of 3-spaces of 10×10 skew-symmetric matrices of rank 8, all non-equivalent to Westwick's one.

Explicit constructions of all these matrices of linear forms can be found in Boralevi et al. (2015).

2.4 Open Problems

To end this survey we list a few of the many open problems related to linear systems of matrices of constant rank.

- Do there exist other examples of 3-spaces of skew-symmetric matrices of constant corank 2, besides those given by 2-instantons and 4-instantons?
- Do there exist linear systems of dimension 4 of skew-symmetric matrices of constant corank 2?
- What kind of variety is the union of the orbits of spaces of matrices of fixed constant rank?

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References

D. Bazan and E. Mezzetti, *On the construction of some Buchsbaum varieties and the Hilbert scheme of elliptic scrolls in* $\mathbb{P}^5$, Geom. Dedicata **86** (2001), no. 1–3, 191–204.

A. Boralevi, D. Faenzi, and E. Mezzetti, *Linear spaces of matrices of constant rank and instanton bundles*, Adv. Math. **248** (2013), 895–920.

A. Boralevi, D. Faenzi, and P. Lella, *Truncated modules and linear presentations of vector bundles*, arXiv:1505.04204, 2015.

A. Boralevi and E. Mezzetti, *Planes of matrices of constant rank and globally generated vector bundles*, Ann. Inst. Fourier **65** (2015), no. 5, 2069–2089.

G. Castelnuovo, *Ricerche di geometria della retta nello spazio a quattro dimensioni*, Atti R. Istituto Veneto (1891), 855–901.

P. De Poi, *Threefolds of* $\mathbb{P}^5$ *with one apparent quadruple point*, Comm. Algebra **31** (2003), no. 4, 1927–1947.

P. De Poi and E. Mezzetti, *Linear congruences and hyperbolic systems of conservation laws*, Projective varieties with unexpected properties, Walter de Gruyter GmbH & Co. KG, Berlin, 2005, pp. 209–230.

P. De Poi and E. Mezzetti, *Congruences of lines in* $\mathbb{P}^5$, *quadratic normality, and completely exceptional Monge–Ampère equations*, Geom. Dedicata **131** (2008), no. 1, 213–230.

Ph. Ellia, *Chern classes of rank two globally generated vector bundles on* $\mathbb{P}^2$, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **24** (2013), no. 2, 147–163.

G. Fano, *Reti di complessi lineari dello spazio* S_5 *aventi una rigata assegnata di rette-centri*, Rend. Reale Accad. Naz. Lincei **11** (1930), 227–232.

D. Faenzi and M.L. Fania, *Skew–symmetric matrices and Palatini scrolls*, Math. Ann. **347** (2010), no. 4, 859–883.

M.L. Fania and E. Mezzetti, *On the Hilbert scheme of Palatini threefolds*, Adv. Geom. **2** (2002), no. 4, 371–389.

M.L. Fania and E. Mezzetti, *Vector spaces of skew-symmetric matrices of constant rank*, Linear Algebra Appl. **434** (2011), 2383–2403.

F.R. Gantmacher, *The Theory of Matrices*, vol. 2, Chelsea Publishing Company, New York, 1959.

B. Ilic and J.M. Landsberg, *On symmetric degeneracy loci, spaces of symmetric matrices of constant rank and dual varieties*, Math. Ann. **314** (1999), no. 1, 159–174.

L. Manivel and E. Mezzetti, *On linear spaces of skew-symmetric matrices of constant rank*, Manuscripta Math. **117** (2005), no. 3, 319–331.

G. Ottaviani, *On 3–folds in* $\mathbb{P}^5$ *which are scrolls*, Ann. Scuola Norm. Sup. Pisa Cl. Sc. (4) **19** (1992), no. 3, 451–471.

F. Palatini, *Sui sistemi lineari dei complessi lineari di rette nello spazio a cinque dimensioni*, Atti del R. Ist. Veneto di Scienze, Lettere e Arti **60** (1900), 371–383.

M. Schneider, *Vector bundles and low codimensional submanifolds of projective space: a problem list*, Topics in algebra, Part 2 (PWN, ed.), vol. 26, Banach Center Publ., 1990, pp. 209–222.

C. Segre, *Studio sulle quadriche in uno spazio lineare ad un numero qualunque di dimensioni*, Mem. R. Acc. Sc. Torino (1883a), 3–86.

C. Segre, *Sulla geometria della retta e delle sue serie quadratiche*, Mem. R. Acc. Sc. Torino (1883b), 87–157.

C. Segre, *Un'osservazione sui sistemi di rette degli spazi superiori*, Rend. Circ. Mat. Palermo (1888), 167–168.

C. Segre, *Preliminari di una teoria delle varietà luoghi di spazi*, Rend. Circ. Mat. Palermo (1910), 87–121.

C. Segre, *Sui complessi lineari di piani nello spazio a cinque dimensioni*, Ann. Mat. Pura Appl. (1917), 75–123.

C. Segre, *Sulla varietà cubica con dieci punti doppi dello spazio a quattro dimensioni*, Atti R. Acc. Scienze Torino **22** (1886–87), 791–801.

J.C. Sierra and L. Ugaglia, *On double Veronese embeddings in the Grassmannian* $\mathbb{P}(1,N)$, Math. Nachr. **279** (2006), no. 7, 798–804.

J. Sylvester, *On the dimension of spaces of linear transformations satisfying rank conditions*, Linear Algebra Appl. **78** (1986), 1–10.

F. Tanturri, *Degeneracy loci of morphisms between vector bundles*, Ph.D. thesis, SISSA, Trieste, 2013.

F. Tanturri, *Degeneracy loci of twisted differential forms and linear line complexes*, Arch. Math. (Basel) **105** (2015), no. 2, 109–118.

F. Tanturri, *On the Hilbert scheme of degeneracy loci of twisted differential forms*, Trans. Amer. Math. Soc. **368** (2016), no. 7, 4561–4583.

R. Westwick, *Spaces of matrices of fixed rank*, Linear and Multilinear Algebra **20** (1987), no. 2, 171–174.

R. Westwick, *Spaces of matrices of fixed rank. II*, Linear Algebra Appl. **235** (1996), 163–169.

Corrado Segre, Guido Castelnuovo and the Riemann-Roch Theorem

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Abstract
We reproduce a self-contained proof of the Riemann-Roch Theorem, originally given by G. Castelnuovo and inspired by ideas of C. Segre.

1 Introduction

Almost all the mathematical work of C. Segre has been collected in four volumes published by Unione Matematica Italiana between the 1950s and the 1960s. The ordering of the material is not chronological, but obeys a criterion of homogeneity decided by the Editors. In particular the first volume collects those papers considered by them as the most characteristic and important. This is confirmed by F. Severi in the Preface to volume I (Severi 1957). This is a short but intense contribution in which he explains carefully the originality of Segre's work. In his energetic and colourful style Severi is generous in giving praises to his master, as well as disapproval to unmentioned addressees. One example will suffice here. On p. VII Severi writes:

> Quando i concetti hanno vissuto quasi un secolo è difficile per chi li possiede d'imaginare lo sforzo che i pionieri dovettero compiere per impadronirsene. Occorre all'uopo sobbarcarsi ad una fatica di ricostruzione critica, alla quale oggi si dà scarsissimo peso, essendo diffuso il vezzo di ridurre quasi a zero la bibliografia e la prospettiva storica, le quali evidentemente costano molto lavoro di consultazione, di comparazione e di riflessione, che pur sarebbe sempre onesto ed utile di compiere!

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It is interesting to compare Severi's view with today's perception of Segre's legacy. It is true that what was called *the hyperspatial method* ("il metodo iperspaziale") in algebraic geometry is so natural and obvious today that it does not even deserve to be called a "method". It is less obvious to recall that Segre pioneered the study of vector bundles on algebraic curves and of rational ruled varieties. It is worth observing how important and fashionable these varieties are today, since they play a central and yet not completely disclosed role in all those investigations dealing with equations and syzygies of projective varieties. Perhaps they represent today the most effective sign of Segre's impact.

It is also interesting to learn from Severi that one of the main goals, if not achievements, of the early geometers, including Brill and Noether, was to geometrize algebra rather than considering geometry as a part of algebra. This clear cut statement is somehow disputable today, when we have such a ramified and complex interaction of algebraic geometry with algebra in a wide sense. Perhaps Segre himself would agree to a compromise, after seeing the astonishing interplay between equations, syzygies and geometry that is being discovered today.

In this note I want to focus on the hyperspatial method and to explain how the theory of algebraic curves was conceived from this point of view. In 1873 the epoch making memoir (Brill 1873) of Brill and Noether had appeared; here they introduced the notion of linear series and used plane curves to give a rigorous foundation to the theory. After the early work of Veronese and Bertini, the possibility of interpreting the theory of linear series on a curve as the geometry of curves in a projective space of arbitrary dimension gradually emerged with C. Segre. One of the outcomes of this hyperspatial point of view was finally a projective proof of the Riemann-Roch theorem, which used as main tools the geometry of ruled varieties and some enumerative formulas, originally due to Chasles and Zeuthen, to replace what in Brill and Noether was represented by the so-called "Fundamentalsatz" $Af + B\varphi$. This project of Segre is explained in detail in Segre (1894) and was brought to a final form by Castelnuovo (1889), where he used a formula of Schubert's as the only enumerative tool. The proof includes the definition of the canonical series, of the genus, and the deduction of the Riemann-Roch formula, including the duality statement in its numerical form, i.e. the identification of the index of speciality as the vector space dimension of the residual series. Even the proof of the enumerative formula was given from scratch. Here I will reproduce the elegant one given by Enriques (1919), but a modern proof can be also found in Arbarello (1984).

For the convenience of the reader I will recount the entire story from the beginning, using the modern language of sheaves and cohomology. Despite the apparently technical language, nothing besides elementary projective geometry will be used.

2 The Canonical Linear Series and the Genus

We will denote by C a projective irreducible and nonsingular curve defined over $\mathbb{C}$. Let $D = \sum n_i p_i \in \mathrm{Div}\,(C)$ be a divisor on C.

A vector subspace $V \subset H^0(C, \mathcal{O}(D))$ of dimension $r+1 \geq 1$ defines a set of effective divisors linearly equivalent to D

$$|V| = \{\mathrm{div}(\sigma) : \sigma \in V,\ \sigma \neq 0\}$$

which is canonically identified with the projective space $\mathbb{P}(V)$; we call $|V|$ the *linear series* defined by V, of degree $n = \deg\,(D)$ and dimension r. If $V = H^0(C, \mathcal{O}(D))$ then $|V|$ will be denoted by $|D|$ and called the *complete linear series* associated to D: it is the set of all effective divisors which are linearly equivalent to D. For an invertible sheaf L we will write $|L|$ to denote the complete linear series $\left|H^0(C, L)\right|$. If Δ is an effective divisor and $V \subset H^0(L)$ we let

$$V(-\Delta) = V \cap H^0(L(-\Delta))$$

Similarly, $V(\Delta) \subset H^0(L(\Delta))$ denotes the image of V under $H^0(L) \to H^0(L(\Delta))$. The symbol g^r_n is a synonymous for "linear series of dimension r and degree n".

Let $|V|$ be a base-point-free (shortly bpf) g^1_n and let J_V be its *jacobian divisor*, namely the ramification divisor of the corresponding morphism $\varphi_V : C \to \mathbb{P}^1$.

Proposition 2.1 *(i) If $|D|$ is a bpf g^r_n, $r \geq 2$, then the jacobian divisors of all the bpf g^1_n's contained in it are linearly equivalent.*

(ii) If $|D|$ is a bpf g^r_n and $|E|$ is a bpf g^s_m then

$$|J_{D+E}| = |J_D + 2E| = |J_E + 2D|$$

where J_D, J_E, J_{D+E} are the jacobian divisors of bpf pencils arbitrarily chosen in $|D|$, $|E|$, $|D+E|$ respectively.

Proof (i) It suffices to prove it for any two bpf pencils contained in a g^2_n. Let $\Gamma \subset \mathbb{P}^2$ be the image of C under the g^2_n. The two pencils are cut on Γ by the lines through two points $P, Q \in \mathbb{P}^2 \backslash \Gamma$. Each of them corresponds to a line in the dual plane $\mathbb{P}^{2\vee}$, that cuts the dual curve $\Gamma^\vee$ in a divisor whose pullback on C is the corresponding jacobian divisor: then they are linearly equivalent.

(ii) By choosing a base-point-free g^1_n in $|D|$ and a base-point-free g^1_m in $|E|$ we can use them to define a birational morphism $C \to \mathbb{P}^1 \times \mathbb{P}^1$. By embedding $\mathbb{P}^1 \times \mathbb{P}^1$ as a quadric $Q \subset \mathbb{P}^3$ and then projecting from a general $O \in Q$ we can map C birationally to a plane curve Γ of degree $n+m$ with two ordinary multiple points A, B, of multiplicity n and m respectively. The pencil of lines through A (respectively B) cuts on Γ the g^1_m (respectively the g^1_n). Consider the dual curve $\Gamma^\vee \subset \mathbb{P}^{2\vee}$. Then the lines of $\mathbb{P}^{2\vee}$ cut on $\Gamma^\vee$ divisors of $|J_{D+E}|$. Consider in particular the line

r corresponding to the pencil through A. It cuts the divisor $J_E + 2D'$, where D' is the divisor corresponding on C to the branches of Γ at A, because r must be tangent to $\Gamma^\vee$ along D'; by construction $D' \in |D|$. Similarly the line s of $\mathbb{P}^{2\vee}$ corresponding to the pencil of lines through B cuts the divisor $J_D + 2E'$, where $E' \in |E|$ is the divisor corresponding on C to the branches of Γ at B. Therefore $J_E + 2D'$, $J_D + 2E' \in |J_{D+E}|$ and this concludes the proof. □

From the proposition it follows that the jacobian divisors of the bpf pencils contained in $|D|$ belong to the same complete linear series $|J_D|$ which is called the *jacobian series* of $|D|$. Moreover part (ii) of the proposition implies that for every bpf linear series $|D|$ the linear equivalence class $|J_D - 2D|$ is independent of $|D|$. It is called *the canonical series* and denoted by $|K|$. One defines the *genus* $g = g(C)$ by the identity

$$\deg(K) = 2g - 2 \tag{1}$$

It is not obvious that g is an integer, i.e. that J_D has even degree: this will be shown in a moment. Of course this definition agrees with the topological one, thanks to Hurwitz formula, but the approach taken here is purely algebraic.

In modern terms, if $\varphi_V : C \to \mathbb{P}^1$ is the morphism defined by $|V|$, the ramification divisor is the degeneracy divisor of the induced homomorphism $T_C \to \varphi_V^* T_{\mathbb{P}^1}$, and therefore it has degree

$$v = \deg(\varphi_V^* T_{\mathbb{P}^1}) - \deg(T_C) = 2n + 2(g-1)$$

and this of course agrees with (1).

The next step is to take as a birational model of C a plane curve Γ of degree n with δ nodes and no other singularities and to compute the genus using the g^1_n defined by the pencil of lines through a general point $O \in \mathbb{P}^2$. Now the degree v of the branch divisor equals the degree of the dual curve, and this is computed by the *first Plucker formula* which gives:

$$v = n(n-1) - 2\delta$$

and this is an even number. Therefore we obtain:

$$g = \binom{n-1}{2} - \delta \tag{2}$$

This is the so-called *Clebsch formula.* The jacobian divisor of the g^1_n defined by the pencil of lines through a general point $O \in \mathbb{P}^2$ is cut on Γ by the polar curve Γ_O of Γ with respect to O. More precisely, Γ_O passes through the nodes with multiplicity one, and cuts elsewhere on Γ the jacobian divisor. Γ_O is a particular *adjoint curve* (of degree $n-1$). Therefore the adjoints of degree $n-3$ cut on Γ canonical divisors. It follows that

$$\dim(|K|+1) \geq \binom{n-1}{2} - \delta = g$$

i.e. $|K|$ is a $g_{2g-2}^{g-1+\alpha}$, with $\alpha \geq 0$. We will prove that $\alpha = 0$ (Corollary 4.3).

If the irreducible plane curve Γ of degree n has ordinary multiple points of multiplicities $m_1, \ldots, m_\delta$ respectively, then an argument similar to the case of nodes involving the polar curves gives the following generalization of the Clebsch formula:

$$g = \binom{n-1}{2} - \sum_{1}^{\delta} \binom{m_i}{2} \tag{3}$$

We can now prove the following:

Theorem 2.2 (Riemann-Roch, weak form) *If $|D|$ is a complete g_n^r then $r \geq n - g$.*

Proof Take a plane model $\Gamma \subset \mathbb{P}^2$ of C having degree m and δ nodes and no other singularities. The adjoint curves of degree $d \geq m$ cut on C a linear series $g_{m_d}^{r_d}$ with $m_d = md - 2\delta$ and

$$r_d \geq \binom{d+2}{2} - \delta - \binom{d-m}{2} - 1 = m_d - g$$

The given g_n^r can be obtained as the series cut on C by the adjoints of a sufficiently high degree d passing through a suitable fixed divisor F of degree k. But then $n = m_d - k$ and $r \geq r_d - k$. Comparing with the previous inequality we get $r \geq n - g$. □

3 An Application of a Formula of Schubert's

Consider the symmetric product C_n, namely the quotient of the cartesian product C^n of C by itself n times modulo the action of the symmetric group σ_n permuting the factors. It is well known and elementary that C_n is a nonsingular n-dimensional projective variety that can be set-theoretically identified with the set of all effective divisors of degree n on C. Let

$$\mathcal{D}_n = \{(D, x) : x \in \mathrm{Supp}(D)\} \subset C_n \times C$$

Then $\mathcal{D}_n$ is a divisor such that

$$\mathcal{D}_{n\,|\,\{D\}\times C} = D$$

for every $D \in C_n$. We will call $\mathcal{D}_n$ the *universal (or tautological) divisor* on $C_n \times C$. Let$C_n \xleftarrow{q_n} C_n \times C \xrightarrow{p} C$

be the projections. Given an invertible sheaf A on C such that $\deg(A) \geq n$, the nth *secant bundle* of A (on C_n) is $E_A = q_{n*}(p^* A_{|\mathcal{D}_n})$; it is a vector bundle of rank n on C_n. Given $U \subset H^0(A)$ of $\dim(U) = \ell + 1 \geq 2$, we have a natural evaluation map:

$$e_{U,n} : U \otimes \mathcal{O}_{C_n} \longrightarrow E_A$$

If $0 \leq s \leq n$ we let $V_n^s(U) \subset C_n$ denote the subscheme defined by the condition $\mathrm{rank}(e_{U,n}) \leq s$. It is supported on the divisors $D \in C_n$ which impose $\leq s$ conditions to $|U|$. By general facts about determinantal varieties we know that every component of $V_n^s(U)$ has codimension $\leq (\ell + 1 - s)(n - s)$. In some cases equality holds.

Proposition 3.1 *(i) Assume that $U \subset H^0(A)$ generates A, i.e. that $|U|$ is a base-point-free $g^{\ell}_{\deg(A)}$. Then $V^{\ell}_{\ell+1}(U) \subset C_{\ell+1}$ is a divisor.*

(ii) Assume that $\dim(U) = 2$ *and U generates A, i.e. that $|U|$ is a base-point-free pencil. Then $V_n^1(U)$ is a curve for every* $1 \leq n \leq \deg(A)$.

Proof Left to the reader. □

The following is the key enumerative result we will use:

Theorem 3.2 (Schubert 1874) *Suppose that $V \subset H^0(L)$ defines a g^r_n, $r \geq 1$, and $W \subset H^0(M)$ defines a g^1_m on C, with $m \geq r + 1$. Let g be the genus of C. Then the number of effective $D \in C_{r+1}$ that are simultaneously contained in a divisor of the g^r_n and in a divisor of the g^1_m is:*

$$Z_{r,n;m} = \binom{m-1}{r}(n-r) - \binom{m-2}{r-1}g \tag{4}$$

Here each D counted by the formula must be taken with an appropriate multiplicity, which is one when D is reduced and contained in unique reduced divisors of $|V|$ and of $|W|$. In modern terms $Z_{r,n;m}$ is the intersection multiplicity of $V^1_{r+1}(W)$ with $V^r_{r+1}(V)$ in C_{r+1}. The proof of this theorem will be given in Sect. 5. The following observation is the key step for the proof of the Riemann-Roch Theorem.

Proposition 3.3 (Castelnuovo 1889) *Suppose that $V \subset H^0(L)$ defines a g^r_n, $r \geq 1$, and $W \subset H^0(M)$ defines a g^1_m on C. If $n - r < g$ and $m \leq r + 1$ then every divisor of $|W|$ is contained in a divisor of $|V|$ (shortly $|W|$ is contained in $|V|$). Moreover the divisors of $|W|$ impose $\leq m - 1$ conditions to $|V|$.*

Proof Assume first that $m = r + 1$. The assumptions imply that $Z_{r,n;r+1} < 0$. This can only happen if $V^1_{r+1}(W) \subset V^r_{r+1}(V)$ because $V^1_{r+1}(W) = |W| \cong \mathbb{P}^1$ is irreducible. Since $V^1_{r+1}(W)$ consists of all divisors in $|W|$, the first part of the proposition is true. Moreover it is clear that all divisors of $|W|$ impose at most $r = m - 1$ conditions to $|V|$.

Assume now that $m < r + 1$ and fix an arbitrary $F \in C_{r+1-m}$. Then, the linear series $F + |W|$ is a g^1_{r+1}, and $F + |W| = V^1_{r+1}(F + |W|) \subset C_{r+1}$. Since we again have $Z_{r,n;r+1} < 0$, we conclude as before that $F + |W| \subset V^r_{r+1}(V)$ because it is isomorphic to $\mathbb{P}^1$, hence it is irreducible. Moreover, since $F + |W|$ imposes at most r consitions to $|V|$ and F is arbitrary, it follows that $|W|$ imposes $\leq m - 1$ conditions to $|V|$. □

Remark 3.4 The proof can be easily modified to show that the same conclusion holds if $W \subset H^0(M)$ is a g^s_m, with $s \geq 2$.

Corollary 3.5 *Suppose $|V|$ is a g^r_n and $|W|$ a g^s_m on C. Assume that $n - r < g$ and $m - s + 1 \leq r + 1$. Then $|W|$ is contained in $|V|$. Moreover $|W|$ imposes at most $m - s$ conditions to $|V|$.*

Proof We may assume that $s \geq 2$. Let H be a general divisor of $|W|$. We have to show that it is contained in a divisor of $|V|$. For every choice of an effective divisor $G < H$ of degree $m - s + 1$, the series $|G|$ is a $g^{1+\epsilon}_{m-s+1}$, $\epsilon \geq 0$, and by Proposition 3.3 and Remark 3.4, it imposes $m - s - \delta$ conditions to $|V|$, for some $\delta \geq 0$. Choose G so that δ is minimum and let $x \in \mathrm{Supp}\,(G)$ be such that $G - x$ imposes the same number $m - s - \delta$ of conditions to $|V|$: such an x exists because G does not impose independent conditions to $|V|$. By the choices made, for every $y \in \mathrm{Supp}\,(H - G)$, the divisor $G - x + y$ imposes at most $m - s - \delta$ conditions to $|V|$. This implies that every $D \in |V|$ containing G also contains y, and therefore there is a $D \in |V|$ containing G and all $y \in \mathrm{Supp}\,(H - G)$. Thus H is contained in D. The last assertion is clear. □

Remark 3.6 This proof is essentially the same given in Segre (1894), n. 71, in Castelnuovo (1889) and in Severi (1926), p. 293–294.

Observe that using the enumerative formula (4) we have deduced an inclusion of linear series, i.e. a functional relation, from a numerical condition. Now we can easily proceed to the proof of the Riemann-Roch Theorem.

4 The Theorem of Riemann-Roch

Definition 4.1 *A divisor D (or a complete linear series $|D|$) is called special If $|D|$ is contained in the canonical series $|K|$.*

We have the following

Corollary 4.2 *Assume* $g \geq 2$. *If* D *is an effective divisor such that* $\dim(|D|) > \deg(D) - g$ *then* $|D|$ *is special and imposes at most* $\deg(D) - \dim(|D|)$ *conditions to* $|K|$.

Proof Suppose that $|D|$ is a g_n^r. We may assume $r \geq 1$ because otherwise $n < g$ and the conclusion is obvious. Applying Corollary 3.5 with the g_n^r and the g_m^s replaced by $|K|$ and $|D|$ respectively the conclusion follows immediately. □

Corollary 4.3 $|K|$ *is a* g_{2g-2}^{g-1} *and it is the only* g_{2g-2}^{g-1} *on* C.

Proof The first assertion is clearly true if $g = 0, 1$. Assume $g \geq 2$ and assume by contradiction that $\dim(|K|) \geq g$. Then the same proof of Corollary 4.2 applies to the linear series $|D|$ of dimension $\deg(D) - g$. This would imply the absurd fact that all linear series are special because, by Theorem 2.2, every D satisfies $\dim(|D|) \geq \deg(D) - g$. The uniqueness follows from Corollary 4.2 applied to any other g_{2g-2}^{g-1}. □

Theorem 4.4 (Riemann-Roch) *If* $|D|$ *is a complete* g_n^r *on* C *then*

$$r = n - g + i$$

where $i = H^0(K - D) = \dim(|K - D|) + 1$.

Proof By Corollary 4.2

$$i - 1 = \dim(|K - D|) \geq g - 1 - (n - r)$$

i.e. $i = g - n + r + \epsilon$, for some $\epsilon \geq 0$. By applying the same argument to $|K - D|$ we obtain:

$$r = \dim(|D|) = g - 1 - (2g - 2 - n - i + 1) + \epsilon'$$

for some $\epsilon' \geq 0$. By adding the two estimates we obtain:

$$i + r = (g - n + r + \epsilon) + (-g + n + i + \epsilon') = r + i + \epsilon + \epsilon'$$

which gives $\epsilon + \epsilon' = 0$. But then $\epsilon = \epsilon' = 0$. □

5 Schubert's Formula

In this section we will prove Theorem 3.2. The formula (4) has been proved again by Castelnuovo (1889), n. 8. A modern version of it can be found in Arbarello (1984), p. 345. The proof given in Arbarello (1984) uses modern intersection theory, including Porteous formula and intersection theory on the symmetric product. Here we will reproduce the elegant simple proof given by Enriques (1919).

If $r = 1$ we want to count the number $Z_{1,n;m}$ of pairs of points that are simultaneously contained in a divisor of a given g^1_n and a given g^1_m. Let us assume first that both pencils are bpf and let us make the simplifying assumption that their common pairs consist of distinct points and have disjoint supports. Then they define a morphism $\varphi : C \to \mathbb{P}^1 \times \mathbb{P}^1$ whose image is a curve of bidegree $(n,\ m)$ having $Z_{1,n;m}$ nodes. Identifying $\mathbb{P}^1 \times \mathbb{P}^1$ with a nonsingular quadric $Q \subset \mathbb{P}^3$ and projecting from a general point of Q the curve C is mapped birationally to a plane irreducible curve of degree $n + m$ having an ordinary m-fold point, an ordinary n-fold point and $Z_{1,\ n;\ m}$ further nodes. By applying (3) we therefore have:

$$g = \binom{n+m}{2} - \binom{n}{2} - \binom{m}{2} - Z_{1,\ n;\ m}$$

which gives

$$Z_{1,\ n;\ m} = (n-1)(m-1) - g \tag{5}$$

and this is precisely (4) for $r = 1$. Now assume that $|V|$ is a g^1_m with a fixed point P. Then it has in common with the g^1_n the $Z_{1,\ n;\ m-1}$ pairs common to the g^1_n and $|V(-P)|$, plus the pairs consisting of P and any point different from P of the divisor of the g^1_n that contains P. Therefore we have again

$$Z_{1,\ n;\ m-1} + n - 1 = Z_{1,\ n;\ m}$$

common pairs. By iterating this argument one takes care of pairs of pencils with any number of base points.

Let us now consider a g^r_n and a g^1_m, both bpf. Let us add a point P to the g^r_n as a fixed point and denote by $g^r_n + P$ the g^r_{n+1} thereby obtained. Let us assume that P has been chosen not in any of the $(r+1)$-tuples common to the g^r_n and the g^1_m. Then the number of $(r+1)$-tuples common to $g^r_n + P$ and to the g^1_m are counted as follows. There are $Z_{r,\ n;\ m}$ of them that are those common to the g^r_n and the g^1_m. Then there are $\binom{m-1}{r}$ that consist of P plus any r points of the unique divisor of the g^1_m containing P (and are therefore contained in a unique divisor of the g^r_n). Hence $g^r_n + P$ and the g^1_m have

$$Z_{r,\ n+1;\ m} = Z_{r,\ n;\ m} + \binom{m-1}{r}$$

$(r+1)$-tuples in common. By iterating we obtain:

$$Z_{r,\,n+h;\,m} = Z_{r,\,n;\,m} + h\binom{m-1}{r} \tag{6}$$

as the number of $(r+1)$-tuples common to $g^r_n + P_1 + \cdots + P_h$ and the g^1_m, where $P_1, \ldots, P_h \in C$ are subject to the same condition as P was.

Now we compute the number of $(r+1)$-tuples that are simultaneously contained in a divisor of the g^r_n and in a divisor of the g^1_m as follows. Denote by $|W|$ the g^r_n and consider a bpf pencil $|V|$ contained in it. Let $r|V|$ be the g^r_{rn} which is the minimal sum of $|V|$ with itself r times; equivalently $r|V|$ is the pullback of the composition of $\varphi_V : C \to \mathbb{P}^1$ with the rth Veronese embedding of $\mathbb{P}^1$ in $\mathbb{P}^r$. Also consider the series $|W| + D_1 + \cdots + D_{r-1}$, where $D_1, \ldots, D_{r-1} \in |W|$ are general divisors. Then $|W| + D_1 + \cdots + D_{r-1}$ is another g^r_{rn}. Since both $r|V|$ and $|W| + D_1 + \cdots + D_{r-1}$ are contained in the same complete series $|(r-1)W|$ we may assume that they have the same number of $(r+1)$-tuples in common with the g^1_m.[1]

The number of $(r+1)$-tuples common to $|W| + D_1 + \cdots + D_{r-1}$ and the g^1_m is given by the formula (6) with $h = (r-1)n$. On the other hand the $(r+1)$-tuples common to $r|V|$ and to the g^1_m are all obtained by choosing a pair $Q_1 + Q_2$ common to $|V|$ and to the g^1_m plus any $r-1$ of the remaining points of the unique divisor of the g^1_m that contains $Q_1 + Q_2$. Therefore they are in number of

$$Z_{1,\,n;\,m} \cdot \binom{m-2}{r-1}$$

After recalling (5) and comparing with (6) with $h = (r-1)n$ we obtain:

$$Z_{r,\,n;\,m} + (r-1)n\binom{m-1}{r} = [(n-1)(m-1) - g]\binom{m-2}{r-1}$$

and this is clearly equivalent to (4). □

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References

Arbarello E., Cornalba M., Griffiths P. and Harris J., *Geometry of Algebraic Curves I*, Springer Grundlehren b. 267 (1984)

Brill A. and Noether M., Ueber die algebraischen Functionen und ihre Anwendung in der Geometrie. *Math. Ann.* 7 (1873), 269–310

[1] This is the key enumerative assumption made by Enriques (1919), which can be fully justified in terms of intersection theory on the symmetric product C_{r+1}.

Castelnuovo G., Ricerche di geometria sulle curve algebriche, *Atti R. Accad. Sci. Torino* 24 (1889), 196–223

Enriques F., Questioni numerative e loro significato nella geometria sopra le curve algebriche. *Rend. Acc. Naz. Lincei* (5) 28 (1919), 370–374

Schubert H., *Kalkul der Abzahlenden Geometrie*, Teubner (Leipzig), 1874

Segre C., Introduzione alla geometria sopra un ente algebrico semplicemente infinito. *Annali di Matematica Pura e Applicata* (2) 23 (1894), 41–142

Severi F., *Trattato di Geometria Algebrica—Volume I Parte I: Geometria delle serie lineari.* Zanichelli (Bologna), 1926

Severi F., Prefazione. In *Corrado Segre: Opere*, vol. I. Cremonese (Roma), 1957

Part III
The Unpublished Notebook of Corrado Segre on Geometry on Simply Infinite Algebraic Entities (1890–1891)

Introduction to the Handwritten Notebook on Geometry on Simply Infinite Algebraic Entities (1890–91)

Alberto Conte

> In the 1890–91 academic year Segre repeated with D'Ovidio in Torino the excellent co-teaching experiment made by Brioschi, Casorati and Cremona in 1869 in Milan. While D'Ovidio gave a course of lessons on *Functions of complex variables and Abelian integrals*, he [Segre] taught *Geometry on a simply infinite algebraic variety* from three points of view, *hyperspatial, algebraic and functional*.[1]

Thus wrote Federico Amodeo recalling Segre's course during the 1890–1891 academic year, a course that was destined to leave a mark on the history of Italian algebraic geometry.

The geometry on a simply infinite algebraic entity, founded by Riemann (*Theorie der Abel'schen Functionen*, 1857) was successively developed along three important lines: the functional, which derives from Riemann; the algebraic-geometrical, the work above all of Brill and Noether (*Über die algebraischen Funktionen und ihre Anwendung in der Geometrie*, 1874); and the algebraic-arithmetical of Kronecker, Dedekind and Weber.

[1]Amodeo (1945, 245): *Nell'anno scolastico 1890–91 Segre ripetette con D'Ovidio a Torino la eccellente prova fatta da Brioschi, Casorati e Cremona nel 1869 a Milano. Mentre D'Ovidio faceva un corso di lezioni sulle* Funzioni di variabile complessa e sugli integrali abeliani, *egli esponeva la* Geometria su di una varietà algebrica semplicemente infinita *sotto il triplice aspetto* iperspaziale, algebrico e funzionale.

The notebook is kept in the *Fondo Segre*, Biblioteca Speciale di Matematica "Giuseppe Peano", Università di Torino, and it can be accessed at http://www.corradosegre.unito.it/Quaderni/Quad3/1_3.php.

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When Segre took his first steps in this research, the Italian School was so strongly dominated by studies in projective line of research that the importance of the problems addressed and partly resolved by Max Noether was hardly understood. Guido Castelnuovo wrote, “It was necessary for us to rediscover those problems ourselves in a form that was better suited to our mentality”.[2]

One line of thinking, which spread in Italy through Klein, led to the extension of projective geometry to hyperspaces; credit goes to Segre for having immediately intuited the possible applications of hyperspace geometry to the theory of algebraic curves. In the introduction to the fundamental memoir of 1894, Segre wrote:

> […] in conducting research – for several years now – on algebraic ruled [surfaces], and in general on varieties composed of ∞^1 spaces, I have needed to make use of the properties of the linear series studied in the Memoir by BRILL-NOETHER, so I became aware how, resorting instead to ruled [surfaces] and to those varieties of spaces, and representing those linear series by hyperspace curves in the sense already mentioned, it was possible to find (at least in part) those properties by means of simple geometric reasonings, avoiding the algebraic calculations or considerations of functions that are necessary to establish the theorem of NOETHER that was fundamental for that Memoir.[3]

It was precisely the geometric method that Segre illustrated in the course of 1890–1891,[4] a method that had no need of “considerations of functions, nor algebraic developments: the only way with which the algebraic nature of the entities appears is with the principle of correspondence for the simply rational forms”.[5]

In his course, Segre’s treatment is preceded by a brief introduction to hyperspaces according to a purely analytical method (Chap. 2), in which, among other things, he underlines the dual order of advantages that are obtained by the use of these. First and foremost, the maximum generality “in as much as it can be said that all the geometric and several analytic (algebraic) entities enter therein. And not only linear systems or varieties, but all types of varieties. Thus the variety of lines in

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[2]Castelnuovo (1929, 192): *Era necessario che quei problemi li ritrovassimo noi stessi sotto una forma più adatta alla nostra mentalità.*

[3]Segre (1894a, *Opere* I, 199): *Ora, nel fare, son già vari anni, delle ricerche sulle rigate algebriche, e in generale sulle varietà composte di spazi, avendo io avuto bisogno di valermi delle proprietà delle serie lineari studiate nella Memoria BRILL-NOETHER, mi accorsi come ricorrendo invece alle rigate ed alle dette varietà di spazi, e rappresentando quelle serie lineari mediante curve iperspaziali nel senso già accennato, si potessero ritrovare (almeno in parte) quelle proprietà mediante semplici ragionamenti geometrici, evitando i calcoli algebrici o le considerazioni funzionali che occorrono per stabilire il teorema del NOETHER fondamentale per quella Memoria*. In citing Segre’s writings (articles and reports), reference is always made to the *Bibliography of the works by Corrado Segre*, at the end of this volume.

[4]Segre (1890–91), in Giacardi (2013).

[5]Segre (1894a, *Opere* I, 200): […] *considerazioni funzionali, né sviluppi algebrici: unico modo con cui compare l'algebricità degli enti è con il principio di corrispondenza per le forme semplici razionali.*

ordinary space is a M_4^2 of S_5".[6] In the second place, the flexibility as a tool for research for the various applications that these might receive in geometry.

In the third chapter, after having defined the point of view that he will adopt to study the curves, that is, that of the properties that are invariant for birational transformations, Segre introduces linear series, emphasising that studying linear series is equivalent to studying the curves that they represent:

> The *neutral* groups of the series correspond to the multiple points and secant spaces to the curve. The groups with points that are variously coincident, to the points and singular hyperplanes of the curve. And also to the varieties of a linear system variously tangent to the curve. As the dimension of the series gives the number of the varieties containing the curve.[7]

The chapter concludes with a brief historical and bibliographical excursus on the geometry on the algebraic entity from Riemann to the most recent works of Castelnuovo. In the next sections of the notebook Segre goes on to develop the geometry of linear series on a curve according to the hyperspace method. Further, alongside the geometric method, Segre also taught the algebraic method of Brill and Noether (p. 144) and the Riemannian functional method (p. 157), in the conviction, which he expressed more than once, that all methods merit study because each allows the problem to be seen from a different point of view.

In the summer of 1890 Bertini had approached the methods of Segre and Castelnuovo during several days of vacation spent in the company of Segre[8] and he was the one who insisted that Segre publish the lithograph of the course of 1890–91. At his friendly insistence, Segre had first thought of using the notes of his lectures taken by his student Gino Fano and had begun to revise them, but finding them 'quite lacking', he later abandoned the idea.[9]

Only in 1894, Segre decided to publish the lengthy and important memoir "Introduzione alla geometria sopra un ente algebrico semplicemente infinito", which, as Francesco Severi wrote, contains 'the roots' of Italian algebraic geometry and in which,

> the synthesis in this field achieved its maximum efficiency. Admirable, for example, are the proofs of the Riemann-Roch theorem, and the Cayley-Brill principle of correspondence.[10]

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[6]Segre (1890–91, 16–17): […] *in quanto si può dire che tutti gli enti geometrici e vari analitici (algebrici) vi rientrano. E non solo i sistemi o varietà lineari, ma ogni specie di varietà. Così la varietà delle rette dello spazio ord.* ^{o}*è una* M_4^2 *di* S_5.

[7]Segre (1890–91, 64): *I gruppi* neutri *della serie corrispondono ai punti multipli e spazi secanti della curva. I gruppi con punti variamente coincidenti ai punti ed iperpiani singolari della curva. Ed anche le varietà di un sist. lineare variamente tang.* ^{i} *alla curva. Come la dimens. della serie dà il numero delle varierà contenenti la curva.*

[8]See C. Segre to G. Castelnuovo, Turin 28 July 1890, ANL-Castelnuovo, in Gario (2010).

[9]See C. Segre to G. Castelnuovo, Turin 8 August 1891, Ibidem.

[10]Severi (1957, X): *la sintesi in questo terreno ha raggiunto la sua efficienza massima. Mirabili ad esempio le dimostrazioni del teorema di Riemann-Roch e del principio di corrispondenza di Cayley-Brill.*

Segre's student Alessandro Terracini, recalling the *Maestro* many years later, wrote:

> His *Introduzione alla geometria sopra un ente algebrico semplicemente infinito* published in 1894 in the *Annali di Matematica* […] was like a kind of Magna Carta that became a point of reference for the geometry on the curve according to Segre's ideas. That *Introduzione* is the fruit of a course taught by Segre here in Turin the 1890–91 academic year, in which – Segre was careful to tell us – he had set forth not only the geometric method, due to him and to Castelnuovo, but also the pre-existing methods: with particular attention to the algebraic methods of Brill and Noether and the transcendent method of Riemann.[11]

In the 1894 memoir Segre limited himself to presenting only the geometric method, but he explained:

> The argument in fact is such that it is not well treated if not developed from several aspects. Thus my decision to expound it from the geometric point of view should not be interpreted in the sense of a preference that in my opinion should be given to this method with respect to the others. All merit being studied; each has its special advantages; per each there are questions in which it goes further, or at least turns out to be more revealing than the others.[12]

While it is true that here Segre only presented the geometric method, nevertheless, by his own express desire, the same volume of the *Annali di matematica pura ed applicata* also contained the work of Eugenio Bertini, entitled "La geometria delle serie lineari secondo il metodo algebrico" (Bertini 1894), in which the author expounded the algebraic method of Brill and Noether, thus offering a different point of view with respect to that presented by his friend. Some years later, in the preface to the textbook *Introduzione alla geometria proiettiva degli iperspazi* (Bertini 1907, V–VI), Bertini would pay homage to Segre, writing that he had made ample use of 'the lengthy handwritten summaries' of his lessons.

In the 1894 memoir and in some cases already in his course, in addition to presenting with admirable clarity a method that was as yet little known, Segre took advantage of the occasion to describe in detail some fundamental concepts, such as that of algebraic variety and of algebraic correspondence between two varieties (p. 48–49). In particular, this last is considered as an algebraic variety contained in the variety of the ordered pairs of elements of the two given varieties (Segre 1894a,

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[11]Terracini (1961, 12): *La sua* Introduzione alla geometria sopra un ente algebrico semplicemente infinito *pubblicata nel 1894 sugli* Annali di Matematica [...] *è stata come la magna charta che ha fatto testo per la geometria sulla curva secondo le idee di Segre. Quell'* Introduzione *è il frutto di un corso tenuto da Segre qua a Torino nell'anno accademico 1890–91, nel quale – Segre ci teneva a dirlo – egli aveva esposto non solo il metodo geometrico, dovuto a lui e a Castelnuovo, ma anche quelli preesistenti: segnatamente il metodo algebrico di Brill e Noether e quello trascendente di Riemann.*

[12]Segre (1894a, *Opere* I, 200): *L'argomento in fatti è tale che non è ben trattato se non si sviluppa sotto più aspetti. Ond'è che l'aver io qui preso ad esporlo dal punto di vista geometrico non va interpretato nel senso di una preferenza che a mio avviso si debba dare a questo metodo rispetto agli altri. Tutti meritano di essere studiati; ognuno ha i suoi pregi speciali; per ciascuno vi sono questioni, in cui esso va più in là, od almeno riesce più luminoso degli altri.*

n. 6). Beginning from this definition, some ten years later his student Severi would construct a synthetic theory of algebraic correspondences between curves and, successively, a general theory of the correspondence between varieties. Segre also prepared, as he himself underlined in the introduction to the 1894 memoir, the bases and the research tools for the creation of the geometry on an algebraic surface, which would be carried forward in his School, with remarkable creative energy, by Castelnuovo and by Federigo Enriques.

Attending the course of 1890–91 were, among others, the brilliant students Gino Fano and Federico Amodeo, who came from Naples expressly to work with Segre, who was already by that time held to be the leader of the School of Italian algebraic geometry. Both undertook the challenge of finding the solution to the problem that Segre proposed during a lesson:

> Define the space S_r not by means of coordinates, but with a series of properties from which the representation with coordinates can be deduced as a consequence'.[13]

In spite of Segre's invitation to work together, each published a separate article. Fano, in his paper of 1892 entitled "Sui postulati fondamentali della geometria proiettiva in uno spazio lineare a un numero qualunque di dimensioni", was concerned, among other things, with proving the independence of the postulates through the search for adequate models and thus arrived to the creation of new (finite) geometries, which a decade or so later would be developed by Oswald Veblen.[14]

The importance of the course of 1890–1891 does not therefore lie only in the scientific relevance of Segre's approach to the study of geometry on a curve, but also in the fact that it was in this moment that his role as the leader of a school was consolidated: Segre suggested readings and topics of research to Castelnuovo, Fano and Amodeo, and illustrated the new methods to Bertini, but above all he was completely aware of the importance of the new line of research and of the existence of a group of researchers who shared that awareness as well as the desire to disseminate their own ideas.

Translated from the Italian by Kim Williams

References

Amodeo, Federico, *Sintesi storico-critica della geometria delle curve algebriche*, Napoli: Conte, 1945.

Avellone, Maurizio, Brigaglia, Aldo and Zappulla, Carmela 2002, The Foundations of Projective Geometry in Italy from De Paolis to Pieri, *Archive for History of Exact Sciences* 56, 5 (2002): 363–425.

Bertini, Eugenio, La geometria delle serie lineari sopra una curva piana secondo il metodo algebrico, *Annali di matematica pura ed applicata*, 2, 22 (1894): 1–40.

[13]Fano (1892, 106): *Definire lo spazio S_r non già mediante coordinate, ma con una serie di proprietà dalle quali la rappresentazione con coordinate si possa dedurre come conseguenza.*

[14]See Avellone et al. (2002, 391).

Bertini, Eugenio, *Introduzione alla geometria proiettiva degli iperspazi*, Pisa: Spoerri, 1907.

Castelnuovo, Guido, La geometria algebrica e la scuola italiana, *Atti del Congresso internazionale dei matematici*, *Bologna 3–10 settembre 1928*, Bologna: Zanichelli, I, 1929, 191–201.

Fano, Gino, Sui postulati fondamentali della geometria proiettiva in uno spazio lineare a un numero qualunque di dimensioni, *Giornale di Matematiche*, 30 (1892): 106–132.

Gario, Paola (ed.), *Lettere e Quaderni dell'Archivio di Guido Castelnuovo*: http://operedigitali.lincei.it/Castelnuovo/Lettere_E_Quaderni/menu.htm (2010).

Giacardi, Livia (ed.), *Corrado Segre e la Scuola Italiana di Geometria Algebrica*: http://www.corradosegre.unito.it/ (2013).

Segre, Corrado, *Introduzione alla geometria sugli enti algebrici semplicemente infiniti*, BMP-Segre, Quaderni. 3 (1890–91).

Segre, Corrado, Introduzione alla geometria sopra un ente algebrico semplicemente infinito, *Annali di Matematica pura ed applicata*, 2, 22 (1894a): 41−142. (*Opere*, 1, 198−304)

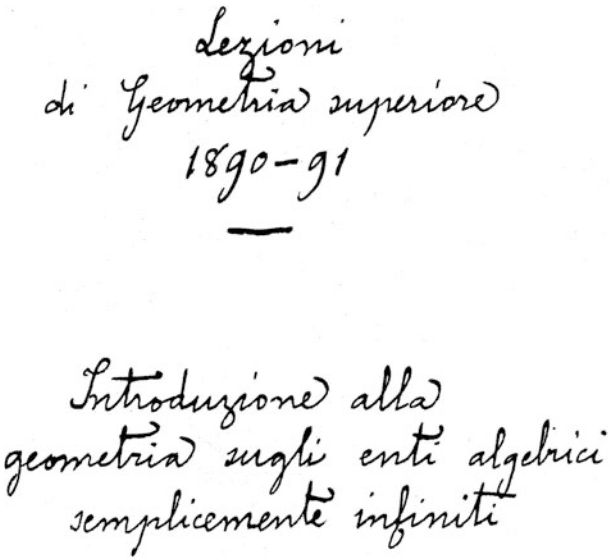
Segre, Corrado, *Opere*, Roma: Ed. Cremonese, 4 vols., 1957−1963, http://www.bdim.eu/item?id=GM_Segre

Severi, Francesco, Prefazione, in Segre, *Opere* 1, pp. V–XII (1957).

Terracini, Alessandro, Parole del Prof. Terracini, in *Atti del Convegno internazionale di geometria algebrica*, *Torino, 24–27.5.1961*, Torino: Rattero, 1961, 9–14.

Lezioni di Geometria Superiore 1890–91 Introduzione alla Geometria sugli Enti Algebrici Semplicemente Infiniti

Corrado Segre


Lezioni
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Indice

Introduzione alla geometria sulle ∞^1 algebriche

Cap. 1°. Preliminari

Considerazioni generali sui legami e le coincidenze fra la Geometria e l'Analisi. Metodo delle coordinate. Due modi di fondare l'edifizio geometrico, o partendo dai postulati relativi allo spazio, a punti, rette, piani, ecc., ovvero introducendo per via puramente analitica questi elementi, come gruppi di numeri (coordinate), ecc. Conseguenze che ne derivano. Elementi complessi: ragione della loro introduzione.

Cenno sulle coordinate projettive omogenee. Altri enti che si posson determinare

2

con coordinate sono quelli dati da un'equazione, cioè le curve piane, superficie, ecc., che ora accenniamo.

Curve piane algebriche. Luoghi ed inviluppi. Ordine e classe. Punti multipli e tangenti multiple. Cenno sulle polari. Definizione di curve <u>aggiunte</u> ad una data. Le prime polari sono aggiunte. Le multiplicità $s_1, s_2 \dots$ dei punti singolari di una curva irriduttibile d'ord. n sono tali che $\frac{1}{2}(n-1)(n-2) - \Sigma \frac{1}{2} s(s-1) \geq 0$. Invero si potrà condurre (almeno) una curva aggiunta d'ordine $n-1$ per $\frac{1}{2}(n-1)(n+2) - \Sigma \frac{1}{2} s(s-1)$ punti semplici della curva [numero che è positivo, perchè uguale ad $n-1$ aumentato della metà di $n(n-1) - \Sigma s(s-1)$, e questa differenza si vede esser positiva considerando (*) le

(*) Cfr. Bertini, Rendiconti Ist. Lomb. 1888. Del resto la considerazione che quel numero è positivo non è necessaria: in caso opposto la disuguaglianza si avrebbe a fortiori.

3

intersezioni della curva con una curva aggiunta d'ordine $n-1$, ad es. la prima polare di un punto qual.] ed il numero complessivo delle intersezioni che così si avranno $\Sigma s(s-1) + \frac{1}{2}(n-1)(n+2) - \Sigma \frac{1}{2} s(s-1)$ dovrà essere $\leq n(n-1)$

Genere di una curva piana dotata di sole singolarità ordinarie $\frac{1}{2}(n-1)(n-2) - \Sigma \frac{1}{2} s(s-1)$. Caso che vi siano solo punti doppi. Cuspidi e flessi. Cenno sulla deduzione delle formole di Plücker dalla teoria delle polari. Dalla 1ª formola si trae $\nu + r - 2n = 2p - 2$, espressione utile del genere. Il genere corrisponde per dualità a se stesso [perchè da $\rho - r = 3(\nu - n)$ si trae $\nu + r - 2n = n + \rho - 2\nu$]

Superficie d'ordine n: numero dei coefficienti o dei punti che le determinano. Classe, rango, punti e linee multiple

Complessi di rette; grado. Numero

4

dei coefficienti; alterabilità dell'equazione; non si può alterare se si tien conto della generazione del complesso, della polarità rispetto a questo (cioè all'equazione) ecc.. Complessi speciali. Complessi lineari: cenno sulle loro proprietà; caso dei complessi lineari speciali.

Connessi in generale di elementi qualunque tolti a forme fondamentali qualunque: esempi particolari. Coefficienti.

Per tutte le specie di enti enumerate abbiamo nei coeff.ⁱ delle loro equazioni delle coordinate nello stesso senso che per gli elementi. La specie più vasta è quella dei connessi: essa abbraccia le precedᵗⁱ, anche i gruppi di n elemⁱ di una forma fond. 1ª sp., coll'equaz. $f(x)=0$. Anche le determinazⁱ degli elemⁱ rientrano in questa, poichè le loro coord. sono i

5

coeffⁱ delle loro equazⁱ.

Diremo che queste varietà (ed altre che si vedranno in seguito) sono <u>lineari</u>, e di <u>dimensione</u> data dal numero delle coord. non omogenee. Entro una varietà lineare stanno altre varietà che possiamo definire*) in due modi diversi. Le coord. $x, y, \ldots$ sian funzⁱ date di $\underline{r}$ parametri indipⁱ $t, u, \ldots$ sicchè $x = f(t, u, \ldots)$, $y = g(t, u, \ldots)$, ..., funzⁱ le quali mutino in generale valore mutando $t, u, \ldots$, e più precisamente assumano dati valori solo per una serie discreta di gruppi di valori di $t, u, \ldots$; la varietà si dice ∞^r se corrisponde ai valori <u>reali</u> di $t, u, \ldots$ e per questi le funzⁱ son definite ed hanno in generale le derivate prime. Ma noi diremo la varietà ∞^r quando corrisponde ai valori <u>complessi</u> di $t, u, \ldots$ pei quali quelle

*) riguardo alla <u>dimensione</u>.

6

funz.ⁱ s'intendono definite e funz.ⁱ nel senso di Riemann (monogene di Cauchy), cioè aventi in generale le prime derivate. — Esempi: curve, superficie, rigate, sistemi e complessi di rette. —

Definiz.ᵉ di funz.ⁱ algebriche e di irriduttibilità: le funz.ⁱ ad un sol valore son razionali. <u>Varietà algebriche</u>: quando le $f, g \ldots$ son funz.ⁱ algebriche +); varietà <u>riduttibili</u>. Da una o più equaz.ⁱ algebriche fra $x, y, \ldots$ è definita una varietà algebrica: ma questa definizione non vale per tutte *). Una varietà algebrica ∞^{k-1} entro la varietà lineare ∞^k è però sempre data da un'equaz. alg. fra le coord. $x, y, \ldots$ (eliminando i param.ⁱ $t, u, \ldots$): così le curve piane, le superf. ecc. <u>Varietà razionali</u>: quando le $f, g, \ldots$ son funz.ⁱ alg.ᵉ ad un sol valore, cioè razionali, ed

+) Legate eventualmente da una o più equaz.ⁱ date.

*) Se con sole $k-2$ equaz.ⁱ si vuol definire una ∞^2. Ma con un numero conveniente di equaz.ⁱ si definisce ogni varietà. Kronecker Festschrift p. 30.

un elemento della varietà corrisponde in generale ad un sol gruppo di valori dei param^i $t, u, \ldots$ Vedremo poi che per le ∞^1 razionali questa 2^a condiz^e non occorre. Curve, superf., ecc. razionali: come il loro studio si riduca per varie questioni a quello delle varietà lineari in cui $t, u, \ldots$ son le coord^e.

Varietà lineari o sistemi lineari di forme si hanno, più in particolare, quando le coord $x, y, \ldots$ son funz^i lineari (fratte col denom. comune) di $t, u, \ldots$, ossia quando le equaz^i degli enti si posson scrivere: $\lambda_0 f_0 + \cdots + \lambda_r f_r = 0$ ove le λ sono i param^i. Se le forme f sono linearm^e indip^i, cioè non legate da alcuna relaz. $a_0 f_0 + \cdots + a_r f_r \equiv 0$ lineare a coeff^i costanti non tutti nulli, allora la varietà è ∞^r poichè l'ente $\Sigma \lambda_i f_i = 0$ coincide coll'ente $\Sigma \mu_i f_i = 0$ solo quando le λ sian pro-

V. anche Wahlen nel Crelle 108 p. 346.

8

porzionali alle μ. (i rapporti delle λ soddisfano alla condiz. posta pei parami $t, u, \ldots$ a pag. 5 nella definizne generale di ∞^r).
L'indipendenza lineare delle f significa non stare in un sistema lineare di dimens. $< r$, cioè che non accade che il sistema determo. da alcune di esse contenga pure le altre (da cui si potrebbe quindi prescindere).
Se $\varphi_i \equiv \Sigma a_{ik} f_k = 0$ sono forme qualunque del sist., il sistema lineare da esse determinato $\Sigma \mu_i \varphi_i = 0$ ossia $\sum_k f_k \sum_i a_{ik} \mu_i = 0$ è tutto contenuto in quello. Se le φ sono $r+1$ cioè $\varphi_0, \ldots \varphi_r$, si può porre $\Sigma \mu_i \varphi_i \equiv \equiv \Sigma \lambda_k f_k$, cioè $\sum_i a_{ik} \mu_i = \lambda_k$ sempre che non sia $|a_{ik}| = 0$ *), il che esigerebbe che esistessero delle μ non tutte nulle per cui $\sum_i a_{ik} \mu_i = 0$, ossia $\Sigma \mu_i \varphi_i \equiv 0$, cioè che le φ fossero legate linearme. Se dunque queste

*) In altri termini si possono ricavare dalle $\varphi_i \equiv \Sigma a_{ik} f_k$ le f come combinazi lineari delle φ.

9

sono indip^i linearm^e il sistema lineare ∞^2 da esse determ^o coincide con quello determ^o dalle f: ciò mostra che le f son forme qualunque linearm^e indip^i del sistema. —
Dal fatto che se un sist^a lin^e contiene alcune forme, contiene pure il sist^a lin^e determ^o da queste si traggono conseguenze importanti. L'intersez. di due sist^i lin^i è pure un sist^a lineare (ove esista). Due sist^i lin^i $\infty^k, \infty^{k'}$ che non abbiano alcuna forma comune stanno in un sist^a lin^e $\infty^{k+k'+1}$ tale che ogni sist^a l^e che contenga quei due deve pur contenere questo (se i due sist^i son determ. dalle forme $f_0, f_1, \dots f_k$; $f'_0, f'_1, \dots f'_{k'}$, queste saran tutte linearm. indip^i, poichè se fosse $a_0 f_0 + \dots + a_k f_k + a'_0 f'_0 + \dots + a'_{k'} f'_{k'} \equiv 0$ ne seguirebbe che i 2 sist^i avrebbero una forma comune; ora tutte quelle $k+k'+2$ forme

10

determinano appunto il sist^a $\infty^{k+k'+1}$).
Due tali sist^i ∞^k, $\infty^{k'}$ si dicono linearm.
indip^i; entro un sist^a lin. $\infty^{k+k'+1}$ si
determinano facilm. due tali sist^i indip^i ecc.
Due sist^i lin^i ∞^k, $\infty^{k'}$, A e A$'$ abbiano
comune uno ∞^i B; entro A$'$ prendiamo
un sist^a l.^{e} $\infty^{k'-i-1}$ C linearm. indip. da B:
esso determinerà con A da cui sarà indip.
un sist^a l.^{e} $\infty^{k+k'-i}$ contenente A e A$'$, e
contenuto in ogni sist^a lin^e passante per
A, A$'$, cioè il minimo sist^a lin^e in cui
quei due stiano insieme. In altri termini
entro un sist^a ∞^r due sist^i ∞^k, $\infty^{k'}$,
che non stiano in un sist^a minore, si tagliano
in un sist^a $\infty^{k+k'-r}$ (in uno
super^e se stanno in un sist^a infer.).
Caso di $k'=r-1$; caso di $k'=r-1$ e
$k=1$: se ne trae la costruzione di un

11

sist^a lin^e ∞^r mediante uno ∞^{r-1} ed una forma esterna (che si congiunge a quello con fasci); esempi.

Considerazioni speciali relative ai sistemi lineari di gruppi di n p^i di una retta (involuzioni), di curve piane e sup. d'ord. n. Se in un tal sistema ∞^r si fissan più punti con date multiplicità (ed anche con qualche tangente) si hanno, ove esistano, le forme di un sist.^a lin^e: poichè quelle condiz^i son lineari nelle λ. Caso che si diano r punti semplici. Caso che il sist^a dato si componga di <u>tutte</u> le curve o sup. d'ord. n. — Nel sist^a lin^e determ^o da più forme ogni p. s-plo per queste sarà s-plo per tutte; e se una retta è tang. ivi a quelle, sarà tg. a tutte*). — Punti base o fondamentali di un sist^a;

*) Se le forme che determinano il sistema hanno una parte comune, questa si stacca da tutto il sist^a, e la parte rimanente descrive un sist^a lineare. Sistemi degeneri.

12

ordinari e straordinari. I fasci di curve o sup. d'ord. n e le reti di sup. hanno sempre punti base: proprietà che vi si collegano.*) — Ogni sist^a lin^e $\lambda_0 f_0 + \dots + \lambda_r f_r = 0$ è segato da una retta o da un piano in un sist^a lin^e $\lambda_0 f_0^{(0)} + \dots + \lambda_r f_r^{(0)} = 0$, il quale però può esser di dimens^e minore, e cioè sarà ∞^{r-h-1} se la retta od il piano fan parte di un sist^a ∞^h contenuto nel dato.

13

Cap. 2°. Degl'iperspazi.

Il nostro non essendo un corso sugl'iperspazi, ma un corso che si varrà di questi sia per raggiunger la massima generalità, sia come strumento di ricerca, ci limiteremo a dire intorno ad essi le cose fondamentali.

Concetto d'iperspazio. — Ricordiamo (pag. 1) la via puram^e analit^a d'introdurre i punti come gruppi di 3 numeri, i piani ecc. come equazioni ecc.: perchè fermarsi a 3 numeri? D'altra parte le varietà lineari considerate nel Cap. preced., i sist^i lin^i in cui le forme eran determ^e dai param^i omog^i $\lambda_0\, \lambda_1 \ldots \lambda_r$ ci conducevano ad enti geometrici determinabili… con gruppi non solo di 3 ma di più coord^e. E si poteva continuare, considerando entro ad un sist. lineare di forme ∞^r la varietà ∞^{r-1} data da un' equ. di

14

grado n fra le $\lambda_0 \ldots \lambda_r$: i coeff. si posson assumere come coord. della varietà ∞^{r-1} entro al sistema di tutte ...; e così via. Si complica l'ente geometrico, ma la rappresentaz. analit^a^ è sempre un gruppo di numeri, variabili ad arbitrio. A questa stessa rappresentaz^e^ anal. si giunge se nello spazio si conservano tutti i postulati, come quello che per 2 punti passa una retta, ecc., ma si toglie quello delle 3 dimensioni: allora i punti della retta (che son determ. da 1 numero) congiunti ad un punto esterno dànno il piano (i cui p^i^ saran determ. da 2 numeri), e congiungendo i p^i^ di questo ad uno esterno si ha uno spazio a 3 dimens^i^, e congiungendo ad 1 p. esterno (supposto esistere) si ha uno spazio a 4 dimens^i^; ecc. ecc. e si giunge allo spazio ad r dimens^i^ definito

15

da r numeri. Valore matematico che avrebbe lo studio dello spazio con tali postulati, quand'anche contraddicessero all'esperienza. — Queste consideraz.i ci suggeriscono un'estensione del significato delle parole <u>spazio</u> e <u>punto</u>. Il <u>punto</u> sarà per noi: 1° un gruppo di valori di r numeri; 2° un elemento di una varietà lineare ∞^r qualunque in cui quei numeri son le coord.e; 3° un punto ordinario nell'ipotesi che lo spazio effettivo sia ad r dimensioni. Il 1° significato è puramente analitico; il 2° ed il 3° son geometrici, ma il 3° più vincolato a postulati (più fisico) che il 2°. Noi non facciamo però distinzione alcuna: il <u>punto</u> è un ente <u>matematico</u>; lo <u>spazio</u> è la varietà di tutti i punti. Ponendo $r = 3$ ed attenendosi al 3° significato si avrà lo spa-

16

zio <u>ordinario</u>. (*)

I vantaggi che si ottengono dall'introduz. degl' iperspazi sono appunto i due di cui si fe' cenno al principio di questo Cap°. Anzitutto la massima generalità, in quanto si può dire che tutti gli enti geometrici e vari analitici (algebrici) vi rientrano. E non solo i sistemi o varietà lineari, ma ogni altra specie di varietà. Così la varietà delle rette

(*) Le definiz^{i} di pag. 5, 6 delle varietà ∞^k, in partic. di quelle algebriche e razionali si conservano per le varietà contenute nello spazio di dim. r, che s'indicherà con S_r. Notazioni per le varietà stesse: le M_{r-1} algebr. son date da equaz^{i}; varietà lineari o spazi entro l'S_r: s'indicano con S_k. L'analogia porta ai nomi di <u>curva</u>, <u>superf</u>., ..., <u>retta</u>, <u>piano</u>, entro l'S_r. Punti reali e complessi.

17

dello spazio ord° è una M_4^2 di S_5 ; i complessi, sistemi, rigate, ne sono le M_3, le superf., le curve, ecc.; il punto di S_5 fuori della M_4^2 non ha un'interpretaz. (solo anal^a) e non occorre. Così si potrebbero studiare le rappresi in iperspazi della varietà delle coppie di punti tolti da due rette o piani, ecc. — L'altra classe di vantaggi degl'iperspazi si ha nelle varie applicaz^i che essi posson ricevere nella geometria; alcune ne vedremo; una è quella del considerare gli enti di uno spazio, ad es. quello ordinario, come projezioni di enti di spazi superiori. Già nella geom. ordinaria si hanno esempi in cui uscendo dalla forma fondam^le di cui si tratta la si studia più facilmente: così le involuz^i di 2° (e 3°) grado di una retta, ricorrendo alla conica (o alla cubica sghemba);

18

il teor. di Desargues sui triangoli omologici; le coniche studiate, a mo' degli antichi, come sezioni del cono circolare; ecc.

Qui però si può obbiettare che se per studiare una figura dello spazio ordinario lo si considera come contenuto in un iperspazio, s'introduce con ciò un nuovo postulato: l'essere lo spazio ordinario contenuto in uno superiore, cioè l'esistenza di punti fuori dello spazio ordinario, ecc.. Ma così non è: poichè basta sostituire al punto dello spazio ord° il gruppo delle coordinate, e poi aumentare questo gruppo con nuovi numeri, e così si avrà l'iperspazio che occorre. — Si può replicare che questo procedimento è puramente analitico. Quantunque noi non diamo una grande importanza all'attenersi piuttosto a concetti geo-

19

metrici che ad analitici, pure possiamo tradurre geometricamente ciò che ora s'è detto, e in due modi diversi. Noi possiam riferire univocamente e linearmente lo spazio ordinario ad un sistema lineare di forme ∞^3, ad es° ad una invol. di grado n e di 3ª specie di una forma fondam° semplice. Questa invol. è contenuta, per n abbastanza grande, in una di quella dimensione che si vuole, e che occorre per i ragionamenti che si voglion fare. Dopo, si ritorna dal sistema lineare ∞^3 di forme allo spazio ordinario, mediante la corrispondenza. Oppure, possiamo direttamente considerare lo spazio ordinario, ai cui punti, come inviluppi di 1ª classe, s'aggiunga una superf. fissa di classe $n-1$, come contenuto in uno spazio lineare superiore

20

i cui punti sarebbero superf. di classe n; le rette ed i piani ordinari sarebbero (sempre aggiungendo a tutti i loro punti la superf. fissa) pure rette S_1 e piani S_2 di quell'iperspazio (v. pag. 11, sist^i lin^i degeneri).

Cenno storico e bibliografico sulla geometria, specialmente projettiva, degl'iperspazi. In quali direzioni convenga coltivarla, e quali siano quelle facili ricerche che posson servire come esercizi, ma non come lavori scientifici.

Spazi contenuti in un iperspazio.

Definizione degli spazi contenuti in S_2 (definiz. tale che se questo è un sistema lineare quelli siano appunto i sist^i lin^i contenuti in esso) mediante le $x_i = \lambda_0 x_i^0 + \lambda_1 x_i' + \cdots + \lambda_k x_i^{(k)}$.

21

Questo spazio, determinato dai p.ᵗⁱ $x^0\, x' \ldots x^{(k)}$, è un S_k, cioè dalle $\sum_\rho \lambda_\rho x_i^{(\rho)} \equiv \sum_\rho \mu_\rho x_i^{(\rho)}$ segue la proporzionalità delle λ alle μ, se non vi son valori (non tutti nulli) delle λ per cui quelle espressioni s'annullino, cioè se $k \leq r$ e i determinanti... delle coord.ᵉ di $x^0\, x' \ldots x^{(k)}$ non son tutti nulli, cioè quei p.ᵗⁱ linearm.ᵉ indipendenti. Se fossero legati linearm.ᵒ, starebbero in uno spazio minore. Lo spazio determ.ᵒ dai punti $\sum \lambda_\rho x^{(\rho)}$, $\sum \mu_\rho x^{(\rho)}$, ... sta tutto su quell'S_k. Sicchè questo contiene infiniti spazi, e se su esso si chiamano coordinate i param.ⁱ λ, si ha che anche su esso gli spazi minori son rappres.ⁱ mediante combinaz.ⁱ lineari delle coord. dei p.ᵗⁱ che li determinano, ossia con forme lineari. – La proposiz. preced. significa che se uno spazio passa per 2 o più punti, passa pure per lo spazio

22

che questi determinano. Ne segue che l'intersez. di due spazi, se esiste, è un punto oppure uno spazio. Tutto ciò è analogo, anzi identico a cose viste sui sist[i] lineari (pag. 7 e segu[i]) Così si ha che due spazi S_k, $S_{k'}$ con un S_i comune stanno in un S_r ove $r \leq k+k'-i$: questo S_r è determ[o] da $i+1$ p[i] dell'S_i ed altri $k-i$, $k'-i$ risp. di S_k, $S_{k'}$.

Dalla rappresentaz. param[a] di un S_{r-1} od iperpiano si trae che esso è pur rappresentabile con un'equaz. $\Sigma \xi_i x_i = 0$; e viceversa; e si trae insieme la condiz. perchè $r+1$ punti stiano in un'iperpiano. Dualità fra punti ed iperpiani. Sistemi lineari (fasci, reti, ecc.) d'iperpiani determinati da h indipendenti (cioè coi determ.[i] delle coord. non tutti nulli) $\Sigma \xi_i x_i = 0$, $\Sigma \eta_i x_i = 0$, ...; da queste equaz[i] si traggono h delle x

come forme lineari delle altre, onde la base di quel sistema lineare ∞^{h-1} ossia l'inters. di h iperpiani indip.^{i} sarà un S_{r-h}. Viceversa un S_k si può considerare come l'inters. di $r-k$ iperpiani indip., ossia la base di un sist. lineare ∞^{r-k-1}. Così gli spazi entro S_r godono di proprietà duali; spazi duali S_k ed S_{r-k-1}. Considerando un S_k e un $S_{k'}$ come inters.^{i} di $r-k$, $r-k'$ iperpiani, si ha che si tagliano in un S_i ove $i \geq k+k'-r$. Se dunque non stanno in uno spazio inferiore ad S_r sicchè (p. 22) $r \leq k+k'-i$, varrà il segno d'uguaglianza. Se $k+k'<r$ non vi è in generale intersez., a meno che i due spazi, i quali stanno sempre in $S_{k+k'+1}$, stiano in uno spazio minore. Se $k+k' \geq r$ essi si tagliano in generale in un $S_{k+k'-r}$; ma si ta-

24

gliano in uno spazio maggiore S_i quando stanno in uno spazio $S_{k+k'-i}$ inferiore ad S_i. Esempi: due spazi duali; uno spazio qualunque ed un iperpiano; ecc.

Rappresentazione degli S_k di S_r con coordinate. Considerandone i pᵗⁱ d'incontro con $k+1$ S_{r-k} si trovano $(k+1)(r-k)$ coordᵉ indipᵗⁱ, cioè che gli S_k di S_r sono $\infty^{(k+1)(r-k)}$. Ma coordᵉ più simmetriche e che non danno eccezioni si hanno nei determinᵗⁱ della matrice delle coord. dei pᵗⁱ o degl'iperpiani da cui l'S_k è determᵒ.: v. Clebsch "Ueber eine Fundamentalaufgabe der Invariantentheorie" (Abhandl. kön. Ges. d. W. zu Göttingen. Bd 17, 1872), e poi D'Ovidio "Le funzioni metriche fondamentali negli spazi ecc." (Memorie Acc. Lincei, ser. 3ᵃ vol. 1°, 1877). Così gli S_k di S_r ap-

25

paiono come formanti una varietà razionale di dimens. $(k+1)(r-k)$ nello spazio di dimens. $\binom{r+1}{k+1}-1$. La condizione d'incidenza di due spazi duali S_k, S_{r-k-1} viene bilineare nelle coord^e^ dei due spazi.

Elementi fondamentali; punti, iperpiani, ecc. aventi tutte le coordinate nulle meno una.

Projezione e sezione entro S_r; forme geometriche fondamentali di questo spazio. Come quelle operazioni servano a riferire fra loro queste forme. Projettività. Definizione diretta della corrispondenza lineare o projettiva fra due S_r mediante le equazioni $x'_i = \sum_k a_{ik} x_k$ ove $|a_{ik}| \neq 0$. Ponendo per le x_k delle forme lineari si vede che gli spazi subordinati dei due S_r si corrispondono pure linearmente. [Collineazione e reciprocità.

26

Punti e iperpiani uniti della collineazione generale entro S_r. Caso particolare dell'omologia e in genere delle collineaz.^i (assiali) $x'_0 = \rho x_0 \dots x'_k = \rho x_k$, $x'_{k+1} = \sigma x_{k+1}, \dots x'_r = \sigma x_r$, che hanno due spazi fondam.^i S_k, S_{r-k-1} di p.^i uniti. La corrispondenza lineare fra due forme lascia inalterati i birapporti delle forme semplici omologhe, (definiti analiticam. $(abcd) = \frac{a-c}{a-d} : \frac{b-c}{b-d}$). Posto $P(x_0 \dots x_r)$, $U(1, \dots 1)$, nel fascio d'iperpiani $\lambda x_0 + \mu x_i = 0$ si trova che il gruppo che projetta $(0\ 1\ UP)$ ha per birapporto $x_0 : x_i$; e così le coord. in S_r si posson definire come birapporti. Le formule $x'_i = \Sigma a_{ik} x_k$ possono anche servire per una trasformazione di coordinate; e si trova come dianzi che le x'_i si possono ancora considerare come birapporti rispetto ad un nuovo sistema di riferimento. Qui conviene osservare che i birapporti furon

27

da noi definiti ammettendo già le coord. Ma nello spazio ordinario essi si posson definire, grazie a Staudt, senza coord. o misure, in modo puramente grafico. Analogamente si potrà fare in S_r quando questo si definisca sinteticamente con alcune proprietà (postulati) relative agli S_k subordinati: con ciò si stabiliranno poi le coord^e in S_r e l'equazione lineare per gl'iperpiani. Trovare quali sono quei postulati che caratterizzano sinteticamente l'S_r (cioè le varietà lineari di enti qualunque)

*) [pag. 25, 26] Cenno sulle omografie involutorie.+) — Reciprocità in S_r $\xi_i' = \sum a_{ik} x_k$,

*) Ritornando alle projettività, se gli elem^i fondam^{li} si corrispondono le equaz^i diventano $x_i' = a_i x_i$, e se inoltre si corrispondono i p^i unit^i $x_i' = x_i$. La proj^à è determ. da $r+2$ coppie p^i omol^i. Proj^à degeneri: si riducono a proj^à tra forme minori. Proj^à fra spazi sovrapposti. Collineazione.

+) V. Bertini, Rendic^i Ist. Lomb^o 1886

28

ossia $\Sigma a_{ik} x'_i x_k = 0$, $\xi_i = \Sigma a_{ki} x'_k$. Punti uniti $\Sigma a_{ik} x_i x_k = 0$. Polarità (elementi involutori $\Sigma a_{ik} x_k = \rho \Sigma a_{ki} x_k$; tutti se questa vale qualunque sia x, ossia $a_{ik} = \rho a_{ki} = \rho^2 a_{ik}$, onde $a_{ik} = a_{ki}$, oppure $a_{ik} = -a_{ki}$): polarità ordinaria e sistema nullo; questo è sempre degenere quando r è pari: definisce un complesso lineare di rette $\Sigma a_{ik} r_{ik} = 0$. Quadriche di S_r: la generaz. come luogo dei p.i uniti di una reciprocità si riduce, se questa è degenere di 1ª, 2ª, ... specie alla generazione con forme inferiori reciproche: i sostegni di queste stanno sulla quadrica; se s'incontrano questa è un cono

29

<u>Varietà di dim. $r-1$ di S_r</u>. Le chiamiamo brevemente <u>varietà</u>. Varietà $f(x)=0$ d'ord. n: numero dei coeff.^{i} $=\binom{n+r}{n}=\binom{n+r}{r}$, onde tante sono le var.^{à} d'ord. n di S_r quante quelle d'ord. r di S_n. Conseguenze dello sviluppo $f(\lambda x+\mu y)=\lambda^n f(x)+\lambda^{n-1}\mu\sum y_i\frac{\partial f}{\partial x_i}+\frac{1}{2}\lambda^{n-2}\mu^2\sum y_i y_k\frac{\partial^2 f}{\partial x_i \partial x_k}+\dots+\mu^n f(y)$. Punto semplice ed iperpiano tang; punto doppio, …, punto s-plo e varietà-cono d'ord. s tang. Se il punto è quello fondamentale 0, l'equ. sarà $a_s x_0^{n-s}+a_{s+1}x_0^{n-s-1}+\dots+a_n=0$. Casi di $s=n-1$ ed $s=n$. — Polarità: definiz. e proprietà delle varietà polari.

Intersez.^{i} di varietà: r varietà d'ord. $n_1, n_2, \dots n_r$ si tagliano in $n_1 n_2 \dots n_r$ punti. Un num. minore <u>i</u> di varietà si tagliano in una M_{r-i} che è incontrata in generale

30

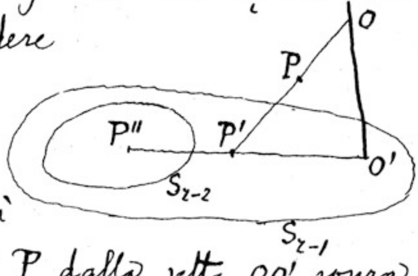
da un S_i in $n_1 n_2 \ldots n_i$ punti.

Sistemi lineari $\lambda_0 f_0 + \ldots + \lambda_k f_k = 0$ di varietà d'ord. n. Si estendono senz'altro le cose viste a pag. 7–11. Danno una nuova specie di varietà lineari o spazi. Così pure si ha che un p. s-plo per le f è tale per tutto il sistema; se le f hanno una parte comune questa si stacca da tutte e rimane un sist. lineare. Le forme del sistema che passan per dati p^i con date multipl^{à} formano, ove esistano, un sist^a lin^e.; ecc. ecc. Quest'ultima proprietà vale anche se i p^i dati sono ∞ e ad esempio si dà una curva, o sup.,… per cui le varietà debban passare con una certa multiplicità: poichè se ne passa più di una, passa tutto il fascio determ. da due, la rete determ. da tre, ecc.… Riguardo

31

poi alla dimens. del sistema che rimane, si noti che un p. s-plo impone $\frac{s(s+1)\dots(s+r-1)}{r!}$ condiz.^{i} lineari, e se si ha un num. finito di passaggi imposti, sommando tutti i num.^{i} di condiz. si avrà un limite super. del numero delle condiz.^{i} complessive.

Le varietà alg. M_k di dim. $k < r-1$, rappres. da equaz.^{i} $F_1(x_1\, t\, u \dots) = 0$, $F_2'(x_2\, t\, u \dots) = 0$, … $F_r(x_r\, t\, u \dots) = 0$, per projez.^{i} in spazi infer.^{i} dànno var.^{a} algebr. Si noti, anche pel seguito, che una proj. su un S_i da un S_{r-l-1} si può scindere in più projezioni centrali da centri $O, O' \dots$ dell'S_{r-i-1} (così



per la projezione di P dalla retta OO' sopra l'S_{r-2}). Ora la proj. centrale della M_k dal p. fondam. 1 si ha ponendo $x_1 = 0$ e conservan-

32

do $x_2 \dots x_r$. A ciò si può obbiettare che il p. fondam. 1 è particolare in quanto che da esso la M_k si projetta più volte, quanto è il grado di F_1' risp. ad x_1; ma si faccia una trasformaz. di coord. e le equaz.[i] (più generali) che si avranno per la M_k proveranno di nuovo l'asserto. — Che la proj. della M_k da 1 p., ... da un S_{r-k-2} sia in generale univoca si vede segando la M_k con un S_{r-k} in un certo num. finito di p.[i] (dalle equaz.[i] lineari che danno l'S_{r-k}) e tirando le rette che congiungono questi: se il p. non sta su queste, ..., se l'S_{r-k-2} non le incontra (ma stanno sull'S_{r-k}), la projez. è univoca. — Due S_{r-k} incontrano in generale la M_k lo stesso num. di volte: se $r = k+1$ è già noto; se $r > k+1$ si diminuisce r projettando la M_k da 1 p. comune ai due S_{r-k}

33

(p. che si può sempre ammettere): ordine di una M_k. Ogni projez. da p^i esterni ha lo stesso ordine ....

Rappresentaz^e analitica delle varietà algebriche più semplice che quella di definiz. Sia una curva e l'S_{r-3} $(34 \ldots r)$ la projetti univocam. in $f(x_1 x_2) = 0$. Su ogni S_{r-2} projettante le $x_1\, x_2$ son fisse e le $x_3 \ldots x_r$ relative a p^i della curva devono essere individuate. E precisamente dalle equaz^i analoghe $g(x_1 x_3) = 0$, $h(x_2 x_3) = 0$ si trae la soluz. comune in x_3 razionalmente in $x_1 x_2$, cioè $x_3 = f_3(x_1 x_2)$, ... $x_r = f_r(x_1 x_2)$. Analogamente in generale per una M_k si ha $f(x_1 \ldots x_{k+1}) = 0$, $x_{k+2} = f_{k+2}(x_1 \ldots x_{k+1})$ .... $x_r = f_r(x_1 \ldots x_{k+1})$. In seguito a trasformaz. di coord., che può esser stata necessaria per render univoca la projez. quando le

34

equaz.ⁱ erano $F_1(x, t\, u \dots) = 0, \dots F_r(x_r t\, u \dots) = 0$ oppure semplicem. per simmetria si può dire che questa M_k si può sempre rappresentare così: $x_1 = f_1(y_1 \cdots y_{k+1}), \dots \; x_r = f_r(y_1 \cdots y_{k+1})$ ove $f(y_1 \cdots y_{k+1}) = 0$, essendo le f_i razionali e tali che assumono ogni gruppo di valori solo per <u>una</u> soluz. della $f(y) = 0$. Così le r irrazionalità o funz.ⁱ algebriche di $t\, u \dots$ che prima s'avevano son ridotte ad una sola, ad es. alla y_{k+1}, come funz. algebr. di $y_1 \dots y_k$. Si può dire che una M_k di S_r quando $r > k+1$ si può sempre riferire razionalmente ad una M_k di S_{k+1} *). Alle formule di trasformaz. si può introducendo l'omogeneità dar la forma

$$x_0 : x_1 : \dots : x_r = \varphi_0(y) : \varphi_1(y) : \dots : \varphi_r(y),$$ con $f(y_0 \dots y_{k+1}) = 0$, ove le φ e f son forme nelle y.

*) Kronecker, Festschrift p. 31.

35

Intersez. di due M_k^n, $M_{k'}^{n'}$: se $k+k' \geq r$ è una $M_{k+k'-r}$ d'ord. prodotto. Basta dimostrare per $k'=r-k$. Se la $M_{k'}$ si spezza in tanti $S_{k'}$. Principio dell'invarianza del numero. Nel caso di una curva e di una $M_{r-1}^{n'}$ l'invariabilità di quel numero seguirà da ciò che alla curva se ne può riferire univocam. un'altra su cui i relativi iperpiani dànno la serie corrisp. a quella data sulla prima dalle $M_{r-1}^{n'}$. Nel caso generale v. dimostraz.i che non ricorrono a quel principio in Halphen, (Bulletin soc. math. de France t. II) ed in Pieri (1888, Giornale di mat. vol. XXVI). — Intersez.i di più varietà.

Varietà luogo di una ∞^k algebrica di S_i: è una M_{k+i} se… Casi particolari. Coni di varie specie e dimensioni

Curve e loro caratteri. Tangenti, piani

36

e spazi osculatori. Classe e ranghi diversi. Cenno sulle formole di Cayley e di Veronese. Punti multipli.

Cenno sulle varietà superiori, superficie, ecc. di S_r rispetto alle tangenti nei punti semplici e multipli (v. Del Pezzo, Sugli spazi tangenti ecc. Rendic' Accad. Napoli 1886).

Quando è che diciamo che una varietà <u>appartiene</u> ad S_r, e quando che è <u>normale</u> per questo spazio.

Una curva d'ordine n non può appartenere ad uno spazio superiore ad S_n, se è irriduttibile.*) E similmente una M_k^n non può appartenere a spazi super' di S_{n+k-1} senza spezzarsi. – Ne segue che le M_k^n di S_{n+k-1} sono normali. – Caso di $n=2$

Per vedere l'esistenza e le prime proprietà di tali varietà osserviamo che segando

*) Clifford, On the Classification of Loci 1878

37

una C^n di S_n cogl' iperpiani per $n-1$ punti fissi risulta la C^n razionale; e due tali fasci d' iperpiani risultano riferiti projettivamente (l'una e l'altra cosa per principii su cui ritorneremo nel Cap. 3°). Ne segue la generaz. con fasci proj^i $\lambda A_1 + \mu B_1 = 0, \ldots, \lambda A_n + \mu B_n = 0$ e se queste equ^i s'imaginano risolte risp. alle coord., queste vengono forme di grado n in λ, μ sicchè la curva è d'ord. n.*) La si può anche generare colle stelle collineari $\rho_1 A_1 + \cdots + \rho_n A_n = 0$, $\rho_1 B_1 + \cdots + \rho_n B_n = 0$ i cui centri son presi ad arbitrio sulla curva e così, segandole con un iperpiano e considerando i p^i uniti della collineazione che si avrà su questo, si scorge di nuovo che la curva è d'ord. n; e similmente si vede che ogni S_{n-2} intersez. di due iperpiani omologhi incon-

*) Una curva raz^{le} così rappresentata appartiene ad S_n se non è nullo il determ. dei coeff^i delle forme.

38

tra C^n in $n-1$ p.^{i} La C^n è individuata da $n+3$ p.^{i}

Similmente $n-1$ fasci proj. d'iperpiani in S_n generano una sup. rigata che è pur generabile mediante due forme aventi per sostegni due gener.^{i} qual., e collineari: ne segue che essa è d'ord. $n-1$ e gli S_{n-2} intersez.^{i} degl'iperpiani omologhi e quindi quelli base di quei fasci segano la rigata secondo delle C^{n-2} *) — E in generale $n-i$ fasci proj. d'iperpiani in S_n generano una M^{n-i}_{i+1} luogo di una ∞^1 razionale di S_i, la quale si può pure generare mediante due forme collineari aventi per sostegni due qualunque di quegli S_i: gli S_{n-2} intersez.^{i} degl'iperpiani omologhi e quindi quelli base dei fasci segano la varietà secondo delle M^{n-i-1}_i luoghi di ∞^1 S_{i-1}. Ogni

*) Ogni rigata d'ord. $n-1$ di S_n si genera in tal modo.

39

M^{n-i}_{i+1} di S_n luogo di ∞^1 S_i si può generare così, poichè un' iperpiano per un S_i dà ancora una M^{n-i-1}_i che dovrà stare in un S_{n-2} (non essendo in generale riduttibile) ed allora il fascio d' iperpiani per questo determina gli ∞^1 S_i, ecc.

Applicazioni delle cose dette in generale sugl' iperspazi ad alcune varietà. 1° Alla geometria della retta, cioè di una M^2_4 di S_5: come su questa si rappresentino i complessi e le congruenze lineari, i fasci di rette e le stelle ed i piani di rette; le rigate. Applicazione del teorema di Clifford alle rigate di 3° e 4° grado. — 2° alla geometria dei sistemi lineari di forme (involuzioni su una forma semplice, sistemi lineari di varietà in un S_r). <u>Indice</u> di una ∞^k

40

di forme: è il numero di quelle che passano per k punti, ed è l'ordine della M_k che rappresenta coi suoi punti quella varietà. Un sistema ∞^k d'indice n sta in un sistema lineare di dimens. $\leq n+k-1$; e se appartiene per $k=1$ ad un sist. lineare ∞^n, esso è razionale, cioè rappresentabile con... Due sist^i ∞^k, $\infty^{k'}$ di forme, d'indici n, n', contenuti in una ∞^r lineare per $k+k' \geq r$ hanno comune una $\infty^{k+k'-r}$ d'indice nn'. E così altre proposizioni, ad esempio quella che una F^3 o appartiene ad S_4 ed è rigata, o sta in S_3 e contiene in generale 27 rette, si traducono pei sist^i lin^i (*).

(*) È appunto per questi che il Cayley introduce gl'iperspazi

41

Un altro modo di rappresentare e studiare i sist^i lin^i di forme mediante gl'iperspazi, il quale coincide poi in sostanza con quello, consiste nel rappresentare le forme del sistema ∞^r $\lambda_0 f_0 + \lambda_1 f_1 + \cdots + \lambda_r f_r = 0$ cogl'iperpiani (di coord. λ) di S_r invece che coi punti. Sia S_k lo spazio di quelle forme (varietà M_{k-1}). Supporremo $r \geq k$. Alle forme passanti per un p. x di S_k corrisponderanno gl'iperpiani passanti pel p. $y_i = f_i(x)$ di S_r. Il p. y può così corrispondere ad un solo p. x od anche a parecchi: quest'ultimo caso quando il passaggio di una forma del sistema per un p. tragga il passaggio per altri $\mu - 1$ p^i. Noi supporremo che μ sia finito, cioè che ad un p. y non corrispondano ∞ p^i x. Allora la varietà descritta dal p. $y_i = f_i(x)$ di S_r sarà una M_k, in corrispondenza $(1, \mu)$ coll'S_k. Così ogni varietà M_k

42

riferibile razionalmente all'S_k ha le sue sezioni cogl'iperpiani rappresentate da un sist. lineare di forme. — Nel caso di $k=1$ se $\mu>1$ i gruppi associati di μ punti formano un'involuzione ∞^1, (pel teor. che vedremo poi, che una ∞^1 di gruppi di p^i tale che ogni p. individui un gruppo è un'involuz.^{e}) e detti φ, ψ due gruppi di questa, le f_0, f_1, … f_r saranno forme di grado $\frac{n}{\mu}$ (n essendo il loro ordine) nelle φ, ψ; viceversa partendo da tali forme si avrà un'involuz. ∞^r in cui ogni gruppo di n p^i si spezza in $\frac{n}{\mu}$ gruppi di un'invol. ∞^1 di grado μ, sicchè il passaggio per un p. trae il passaggio per $\mu-1$. — Analogam. in S_k si hanno delle <u>involuz.^{i}</u> di gruppi di μ punti, e datane una si formano subito i sist.^{i} lineari ∞^r corrisp.^{i}. — Un caso ovvio in cui $\mu>1$ si ha in ge-

43

nerale quando $r = k$: allora si ha la rappresentaz. dei due S_k descritti da x, y l'uno sull'altro mediante le $y_i = f_i(x)$. La corrispondenza sarà univoca in ambi i sensi quando $\mu = 1$, cioè k varietà del sistema lin. abbiano solo 1 inters. variab.: sistema <u>omaloidico</u>; le x sono pure funz.ni raz.li delle y. Senza fermarci su queste corrisp. biraz.li fra due S_k, ci limitiamo a ricordare che per $k = 2$ esse si posson considerare come prodotti di trasformaz.i <u>quadratiche</u> (proposiz. che non si sa estendere a $k > 2$ in modo opportuno), e che in generale si posson costruire i sistemi omaloidici di S_k partendo dalla rappresentazione di una varietà razionale di questo sull' S_{k-1} corrisp., come Cremona mostrò per $k = 3$. —

Ordine della M_k rappresentativa di un

44

sist. lin. ∞^2 di S_k: per $k=1$ si ha una curva il cui ord. è l'ordine dell'invol., astraendo dai p.i fissi che questa può avere; e per $k>1$ è il numero delle inters.i variabi di k forme qual. del sistema. Però nel caso che $\mu>1$ bisogna dividere per μ. Si può togliere questa condiz. contando allora μ volte ogni p. y della M_k, cioè considerando una M_k μ-pla. Questa convenzione rende univoche le corrispondenze multiple; è importante in tutta la matematica ed ha condotto ad es°, come vedremo, alla rappresentaz. di una funz. algebr. di 1 variab. mediante la sup. di Riemann.

Teor.: se un sist lin ∞^r di forme $\lambda_0 f_0 + \dots + \lambda_r f_r = 0$ è contenuto in uno $\infty^{r'}$ $\lambda_0 f_0 + \dots + \lambda_{r'} f_{r'} = 0$ $(r'>r)$, la M_k rappresentaz. di quello, cioè $y_0 = f_0(x) \dots y_r = f_r(x)$, è proj. della M_k rappres. di questo, cioè $y'_0 = f_0(x)$, ... $y'_{r'} = f_{r'}(x)$. — Così potendo sulla retta ogni involuz. di grado $\leq n$, completata con punti fissi, considerarsi come contenuta in quella di tutti i gruppi di n p^i, ogni curva raz^{le} d'ord. $\leq n$ sarà proj. di quella $y_0 = \lambda^n$, $y_1 = \lambda^{n-1}\mu$, ... $y_n = \mu^n$, raz^{le} norm^{le} di S_n. Solo queste dunque son le curve raz^{li} normali; e se ne ha una rappres^e canonica che servirà poi e che si dedurrebbe anche dalla rappres. gener. con forme binarie di grado n risolvendole risp. a $\lambda^n, \lambda^{n-1}\mu, \dots \mu^n$ e trasformando le coord. Segue che due C^n di S_n son projettive. — Analogam. le superf. rappresentate sul piano da sist^i lin^i d'ord. n o minore sono proj. di una F^{n^2} di $S_{\frac{n(n+3)}{2}}$. Caso di $n = 2$ (v. Veronese, La sup. omal. normale del 4° ord. ecc. Mem. Acc. Lincei 1883-84): si ha una F^4

46

con ∞^2 coniche, ecc., che, con le rigate razionali già considerate, dà tutte le F^{n-1} di S_n (v. Del Pezzo, Sulle sup. d'ord. n immerse in S_n Rendic. Acc. Napoli 1885; Sulle projezioni di una sup. e di una varietà di S_n. Rendic. Acc. Napoli 1886; Sulle sup. d'ord. n immerse in S_n, Rendic. Circ. Palermo t. I 1887); essa ha per projez. su S_3 la sup. di Steiner.

47

Cap. 3°.
Oggetto della Geometria su una ∞^1 algebrica. Corrispondenze algebr.e Serie lineari.

Gli enti che studieremo: le ∞^1 algebriche, cioè le curve. Ma il punto di vista, cioè la specie delle proprietà, va ancor definita. Cenno sulle trasformazioni come caratterizzanti gl'indirizzi (Klein, Vergl. Betrachtungen..): geom. elem.e; geom. projettiva. Così la geom. su una M_k è la geom. delle trasformaz.i birazionali di questa. Le trasformaz.i che caratterizzano gl'indirizzi servono a sostituire agli enti dati dei trasformati più semplici. Così alla M_k qual. nella geom. sulla varietà si può sostituire una M_k di S_{k+1}.

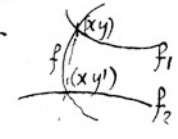
La definiz. data conduce alle corrisp.

48

algebriche: del resto lo studio di una corrisp. su una M_k fa già parte di quest'indirizzo. Se fra due M_k alg. si ha una corrisp. (α, α') sì che le coord. dei p.^{i} dell'una sian date funz.^{i} algebr. di quelle dell'omologo dell'altra (sì che tutti i p.^{i} della prima M_k aventi per coord. quei valori si considerino come omologhi al p. della seconda) la corrisp. si dice <u>algebrica</u>. Corrisp. <u>birazionale</u> (funz.^{i} razionali, indici = 1). Se due varietà sono in corrisp. algebr. o birazionale con una 3^a, anche fra loro. — La definiz. di corrisp. algebr. produce equaz.^{i} algebr. fra le coord. $x\,y\ldots$, $x'y'\ldots$ di p.^{i} omologhi. Dato un numero qual. di tali equaz.^{i}, e quindi un sistema qual. di legami geometrici o costruzioni, traducibili algebricamente, e tali da definire una corrisp.

49

(α, α') fra le due M_k, si possono sempre ricavare $x, y \ldots$ come funz. algebr. ad α valori di $x'y' \ldots$, e viceversa… Dimostriamolo per le curve: possiamo supporle piane $f(xy) = 0$, $f'(x'y') = 0$, e le relaz. $F(xy\,x'y') = 0$, $G(xy\,x'y') = 0, \ldots$ Osserviamo che quando più equaz. in x hanno α soluz. comuni, queste si hanno da un'equ. di grado α razionale nei coeff. di quelle. Se si hanno poi tre o più equaz. in xy con α soluz. com. $f(xy) = 0$, $f_1(xy) = 0$, $f_2(xy) = 0$ … dalle 1ª e 2ª, 1ª e 3ª, … si hanno eliminando y equaz. $\varphi_1(x) = 0$, $\varphi_2(x) = 0$ con le α soluz. comuni e non di più, perchè se vi fosse una x tale che $f(xy) = 0$, $f_1(xy) = 0$ ed in pari tempo $f(xy') = 0$, $f_2(xy') = 0$, ove $y \neq y'$, l'asse delle y sarebbe parallelo alla congiungente di un p. ff_1



50

con un p. ff_2, il che si può evitare. Dunque le x, y son radici di equaz. di grado α razli nei coeffi delle $f(xy)=0$, $f_1(xy)=0$, $f_2(xy)=0$ ... Applicando ciò alle equazi della corrispondenza oltre la $f(xy)=0$ si ha il teorema.

In partice una corrisp. algebr. <u>univoca</u> fra due M_k è birazle e quindi, introducendo l'omogeneità si rappresenta con $y_i = \varphi_i(x)$, e similmente $x_l = \psi_l(y)$. Si noti una differenza fra il caso delle trasformazi univoche di un S_k e quello delle trasformazi univoche di una M_k di uno spazio super.: se le $y_i = \varphi_i(x)$ devon trasformare univocam. tutto lo spazio descritto da x, vedemmo che il sist. lineare di varietà $\sum \lambda_i \varphi_i(x) = 0$ dev'essere omaloidico (pag. 43). Invece se devon trasformare univocam. solo una varietà minore di quello spazio basta che questa non corrisponda a se stessa nell' involuz. (pag. 42) determinata da quel sist. lineare. Se gli spazi di x e y hanno la stessa dim., si vede che la corrisp. univoca fra le due M_k si può considerare contenuta in corrisp. $(1, \mu)$ fra i due spazi: non in corrisp. univoche in generale.

Varietà razionali. Una M_k in corrisp. univoca con una razionale è razionale. Due M_k razionali si posson sempre riferire univocam. La geom. su una M_k razionale è la geom. sull' S_k, cioè la geom. delle trasformaz.[i] biraz.[li] di questo. Caso di due ∞^1 razionali in corrisp. (α, α'): se x e x' sono i param.[i] omologhi, la corrisp. sarà data (p. 49, 50) da un' equ. $f(\overset{\alpha}{x}, \overset{\alpha'}{x'}) = 0$, che è in sostanza il principio di corrisp. di

52

Chasles. Per la corrisp. univoca si ha una relaz. bilineare, e quindi una projettività se si tratta di forme fondam^li (principio di corrisp. univoca di Chasles); e sempre, se diciamo projettive due ∞^1 razionali riferite univocam^e. La geom. su una ∞^1 raz^le si riduce alla geom. projettiva su una forma sempl. fondam^le, ad es. sulla retta.

La corrisp. univoca fra due curve γ, γ' data da $y_i = \varphi_i(x)$ ci conduce alle serie lineari, poichè muta la serie data su γ' dagl'iperpiani $\sum \lambda_i y_i = 0$ nella $\sum \lambda_i \varphi_i(x) = 0$.

In un S_k data una γ e un sist. lineare $\sum \lambda_i \varphi_i(x) = 0$ di varietà, dicesi serie lineare di gruppi di p^i di γ quella segata da quella varietà, togliendo o no quanti si vogliano fra i punti fissi. Serie lineari su una ∞^1 algebrica qualunque.

53

Per trasformaz.ⁱ univoche $x_\rho = f_\rho(y)$ la serie lineare diventa $\Sigma \lambda_i \Phi_i(y) = 0, \ldots,$ rimane una serie lineare. Ordine della serie. Dimensione: la serie composta di un sol gruppo si dice di dimensione 0. In generale poi si osservi che se un gruppo della serie è dato da più di una varietà del sistema, vi sono in questo delle varietà contenenti γ. Se per γ passano ∞^t (ove $t \geq 0$) varietà del sist., le quali formeranno pure un sist. lineare (pag. 30), un'altra varᵃ determinerà con questo il sist. ∞^{t+1} di tutte quelle passanti pel gruppo G dato da quella (poichè ogni varᵃ per G dà con quella un fascio che contiene una varᵃ dell'∞^t). Se dunque il sist. dato è ∞^d si può prendere entro esso un sist ∞^r, ove $r = d - t - 1$, che non incontri l'∞^t: ed esso darà da se tutta

54

la serie, ogni gruppo una volta sola. La serie sarà ∞^r, e se n è il suo ord. s'indicherà con g_n^r. Da r p^i sarà in generale individuato un gruppo: basterà prendere il 1° punto fuori dei p^i base del sist. ∞^r, e così il 2° fuori dei p^i base del sist. ∞^{r-1} ottenuto, ... Sarà $r \leq n$. Se $r = n$, siccome fissati $r-1$ punti si avrebbe un fascio di varietà che staccherebbe da γ i singoli punti, γ sarebbe razionale. Se γ non è razionale, è sempre $r < n$, e quindi da alcune (r) fra le inters. di γ con una varietà di un dato sist. lin. son sempre individuate le rimanenti $(n-r)$.*) Nello studio delle serie lineari una delle principali questioni sarà la determinaz. di n (dei punti mobili fra gli n) e specialmente di r: la relaz. $r = d-t-1$ darà poi t se in una ∞^d che dia quella serie si cercano le varietà

*) In tutto si può intendere che n siano i p^i variab^i.

55

passanti per γ.

Come esempi di serie lineari consideriamo quelle staccate su una γ^m piana da tutte le curve aggiunte d'ord $\nu = m-3+\alpha$. Tutte le serie lineari su γ si posson considerare come date da tali curve (aggiungendo una curva agg^{te} fissa). La dimens. del sistema di tutte le curve agg^{te} d'ord. ν è

$$d \geq \frac{\nu(\nu+3)}{2} - \sum \frac{s(s-1)}{2},$$

ossia, introducendo il genere (pag. 3)

$$p = \frac{m-1 \cdot m-2}{2} - \sum \frac{s(s-1)}{2}$$

è

$$d \geq \frac{\nu(\nu+3)}{2} - \frac{(m-1)(m-2)}{2} + p$$

$$d \geq m\alpha + \frac{\alpha(\alpha-3)}{2} + p - 1$$

Se $\nu < m$ sarà $d = r$ dimensione della serie. Ma se $\nu \geq m$ vi saranno delle curve agg^{te} d'ord. ν contenenti γ, e saranno ∞^t ove

$$t = \frac{\nu-m \cdot \nu-m+3}{2} = \frac{\alpha(\alpha-3)}{2},$$ sicchè (pag. 54

$$r = d - t - 1 \geq m\alpha + p - 2$$

56

Per $\alpha = 1, 2$ questa formola vale ancora. Si ha poi sempre per l'ordine della serie

$$n = m\nu - \Sigma s(s-1) = m\alpha + 2p - 2 .$$

Si ha dunque

$$n - r \leq p \qquad \text{se } \nu > m - 3$$

e per le curve agg^{te} d'ord. $m-3$ si ha $d = r \geq p-1$, $n = 2p-2$, e in generale per $\nu < m-3$ si ha $n - r < p$. Noi vedremo più tardi che per $\nu \geq m-3$ si ha proprio $d =$ al valore trovato (sicchè i passaggi pei p^{i} multipli di γ^m son condiz^{i} indip^{i} per le curve agg^{te} d'ord. $\geq m-3$) e che gli n punti di ogni gruppo son variabili tutti. Osserviamo fin d'ora che per $\nu \geq m-3$ è sempre $d > 0$, tranne per $p = 0$ nel qual caso non esistono curve agg^{te} d'ord. $m-3$ (n è negativo) e per $p = 1$ ne esiste una sola ($n = 0$). Per $p = 0$ abbiamo $n = m\alpha - 2$

57

e $r \geq m\alpha - 2$: varrà dunque (pag. 54) il segno inferiore, tutti gli n p.i saranno mobili e concludiamo che <u>le curve di genere 0 son razionali</u>. Invertiremo più tardi questa proposizione.

Ritornando alle serie lineari si è già notato (pag. 52) come si presentino nello studio della trasformaz. univoca di una curva γ di S_k. Viceversa abbiasi su questa una g_n^r, ove <u>n</u> sono i punti <u>variabili</u> di ogni gruppo (oltre ai quali ve ne posson essere dei fissi), sicchè $r > 0$, e sia data da $\sum_0^r \lambda_i \varphi_i(x) = 0$. Riferendo linearmente le varietà di questo sistema agl'iperpiani (di coord. λ_i) di un S_r, i punti di questo corrisponderanno ai sist.i lin.i ∞^{r-1} contenuti in quello; e così ai p.i x di γ fuori dei p.i base, cioè ai sist.i ∞^{r-1} di varietà per essi corrisponderanno in S_r punti y

58

formanti una curva C sì che $y_i = \varphi_i(x)$. Questa C appartiene ad S_r perchè non v'è nel sistema ∞^r alcuna varietà che contenga γ. — Noi supporremo anzitutto che per la g_n^r il passaggio per un punto non tragga sempre il passaggio per altri, onde $r > 1$. La corrispondenza fra γ e C sarà univoca e la g_n^r sarà segata su C dagl' iperpiani. Inoltre si potranno avere su γ e quindi su C dei punti fissi. Se le φ di un sist. ∞^{r-1} hanno s punti comuni su γ, sicchè il passaggio di un gruppo della g_n^r per uno di quelli trae il passaggio per gli altri, il p. y omologo conterà s volte fra le n inters. di un iperpiano di S_k con C, cioè sarà s-plo. Questo accade in particolare se γ ha un p. s-plo x che non sia p. base per le φ: darà in C un p. y s-plo (almeno). Ma se il p. s-plo x

59

di γ è p. base per le φ, per veder come si muta, si considerino le φ passanti per un p. di γ che s'avvicina indefinitamente ad x e si vede che in generale ai vari rami di γ per x corrispondono altrettanti punti su C, semplici in generale, ma che si vede anche quando saranno multipli. — Esempio: trasformazione di una curva piana in una piana*); il punto doppio, nodo o cuspide, si scioglie mediante una rete di φ passanti per esso, ma il tacnodo no. Per la geom. su una curva siamo condotti a considerare un p. s-plo di una γ come la riunione di s punti distinti, che possono anche essere ∞^{te} vicini se le tgi non son distinte... Questo dà una prima spiegazione della scelta delle curve aggiunte per la geom. sulla curva γ^m. Se su questa si vuol dare una serie lineare di dimens. abbas-

*) Colla rete delle 1e polari si muta la curva nella reciproca.

60

tanza elevata con un sist. di curve che nel punto s-plo x di γ non siano aggiunte, cioè non l'abbiano almeno per $(s-1)$plo, il passaggio per gli s punti che cadono in x importerà solo 1 condiz., o 2, o 3, ... o $(s-1)$ secondo che x non è p. base, od è semplice, doppio .... $(s-2)$-plo. Per le curve con sole singolarità ordinarie le curve agg^{te} danno serie non particolari risp. ai p^i multipli.

Se la g_n^r è tale che il passaggio per 1 punto tragga sempre il passaggio per $\mu-1$, allora C sarà in corrisp. $(1, \mu)$ con γ e sarà solo d'ord. $\frac{n}{\mu}$: così per $r=1$. Poichè C appartiene ad S_r segue $\frac{n}{\mu} \le r$, $\mu \le \frac{n}{r}$. Esempio di una tal serie si ha su una curva che incontri in μ punti variab^i le gener. di un cono: allora con un sistema d'iperpiani o di coni aventi comune il vertice con quello vi si sega una tal serie.

61

Del resto quando su γ si ha la ∞^1 di gruppi di μ punti tale che da 1 p. è individuato il gruppo, ogni serie lineare composta con quella si rappresenterà con una curva C in corrisp. univoca con quella ∞^1 di gruppi e quindi basta prendere una tal curva C e le serie lineari su questa avranno per corrispondenti su γ tutte le serie composte con quella ∞^1 (chè le $y_i = \varphi_i x$ mutano le serie lineari di C in serie lineari di γ; e due C diverse essendo in corrisp. univoca, alle serie lineari dell'una corrispondono le serie lineari dell'altra). — Quando una g_n^r è così composta con una ∞^1 di gruppi di μ punti, sicchè si rappresenta con una C d'ordine $\frac{n}{\mu}$, conviene dire chè ancora C, contando ogni suo p. come μ-plo, è d'ord. n: così per $r=1$ la retta n-pla (cfr. pag. 44). E così, anche quando fra due curve γ, γ'

62

si ha una corrisp. (μ, μ'), si può, contando i p^i di γ μ' volte e quelli di γ' μ volte parlare di corrisp. univoca; la cosa si rende sensibile con la rigata delle congiungenti i p^i omologhi (o con una sua sezione piana).

La definiz. di una g_n^r su γ mediante il sistema ∞^r delle φ mostra che entro quella serie stanno delle serie di minor dimens. e dello stesso ordine; e fissando dei p^i su γ, anche delle serie d'ord. minore. Per $r > 1$ si ha un'immagine conveniente di ciò sulla C^n di S_r.

E non vi son altre serie lineari d'ord. n nella g_n^r; e più precisamente: se una serie di gruppi di n p^i presi tra i gruppi di una g_n^r è tale che per r' punti di γ e quindi di C passi un sol gruppo, allora gl'iperpiani $\infty^{r'}$ che la determinano su C formeranno un inviluppo di 1ª classe, cioè una forma fondam.le,

63

e però quella serie sarà <u>lineare</u> e di quelle già considerate entro la g_n^r (per le forme razionali ne trarremo poi una conseguenza speciale). — Segue che due C^n rappresentanti una stessa g_n^r sono projettive, poichè ad una forma fondam. d'iperpiani di un S_r corrisponde idem…

Segue inoltre che se mutano le φ con cui si determina una g_n^r su γ, non mutano però le serie lineari minori contenute in quella. — Se una $g_n^{r'}$ è contenuta nella g_n^r si può supporre che le due serie sian date risp. da $\sum_0^{r'} \lambda_i \varphi_i = 0$ e $\sum_0^{r} \lambda_i \varphi_i = 0$, e quindi le curve che le rappresentano da $y_i = \varphi_i$ $(i = 0 \dots r')$; $y_i = \varphi_i$ $(i = 0, \dots r)$, sicchè saranno projezioni l'una dell'altra*); viceversa se C' è proj. di C, la serie rappres. da C' è segata su C dalla forma fondam. d'iperpiani projettanti ecc. — Osservando che $r \leq n$

*) Risulta anche dal teor. di pag. 44.45, se ad es. γ si assume piana sicchè le $y_i = \varphi_i(x)$ rappresentano superf. se x si lascia libero nel piano. Così si vengono a considerare per le curve delle superf. rappresentabili che le contengono.

64

dicesi <u>completa</u> (o normale) una g_n^r quando non sta in una d'ugual $\underline{n}$..., <u>parziale</u> in caso contrario: vedremo poi che è unica la serie completa d'ord. $\underline{n}$ in cui sta una data g_n^r. La curva che rappresenta una serie è normale o no secondo che la serie è completa o parziale.

Importanza delle serie lineari di gruppi di punti. La loro determinazione ed il loro studio equivalgono a quelli delle curve che le rappresentano. I gruppi <u>neutri</u> della serie corrispondono ai punti multipli e spazi secanti della curva. I gruppi con punti variamente coincidenti ai punti ed iperpiani singolari della curva. Ed anche alle varietà di un sist lineare variamente tang: alla curva. Come (pag. 54) la dimens. della serie dà il numero delle varietà contenenti la curva.

Cenno storico. La nozione di geometria

sull'ente algebrico è dovuta a Riemann (Theorie der Abel'schen Functionen), che riunisce in una classe (§ 12) tutte le equaz^i algebr. fra due variab^i che si equivalgono birazionalm^e, dimostra l'invariabilità del genere, determina il numero dei moduli, e considera delle serie lineari, specialmente ∞^1, valendosene per ridurre l'equaz. ai minimi gradi, ecc.. Clebsch (Crelle 63, 64), Clebsch e Gordan (Th. d. Ab'sch. F.) adoperano più geometricam. le serie lineari. Brill e Nöther (Math. Ann. VII) fanno una teoria completa senza trascendenti basandosi su un teorema fondam^{le} ($Af + B\varphi$, Restsatz); e specialm^e Nöther vi ritorna, particolarm^e nella Mem^a sulle curve sghembe (Zur Grundlegung der Theorie der alg. Raumcurven, Berl. Abhandl. 1882). Segre da studi sulle rigate condotto a questioni di geom. sulla curva, prima per $p = 1$, poi per

66

p qual. (Math. Ann. XXX, Rendic. Lincei 1887 Rendic. Ist. Lomb. 1888) si vale di curve di ogni spazio, e di rigate... Castelnuovo (Studio dell'involuzione generale sulle curve razionali mediante la loro curva normale dello spazio ad n dimensioni, Ist. Ven° 1886; Geometria sulle curve ellittiche, Atti Torino 1888 vol. 24; e specialmente Ricerche di geometria sulle curve algebriche, 1889 stesso vol.e) prosegue e risolve nuove questioni importanti; così determina i gruppi neutri di una serie o spazi secanti di una curva in due casi molto generali (Una applicazione della geometria enumerativa alle curve algebriche, Rend. Palermo III 1889) ed il numero delle serie lineari su un ente (Numero delle involuzioni razionali giacenti sopra una curva di dato genere, Rend. Lincei 1889). Del resto le curve degl'iperspazi furon

67

usate altre volte da altri, benchè non così metodicamente. Klein (e Fricke) nelle Vorles. ü. ell. Modulfunctionen Bd I rappresenta le serie lineari con tali curve. — È opportuno studiare le serie lineari coi 3 metodi e così noi faremo a suo tempo. Si noti poi che la geom. sulla curva comprende anche altri argomenti: le serie non lineari; le corrispondenze; ecc.; e si posson considerare corrispondenze fra due o più curve, ecc. ecc. Cenno sul programma ulteriore del corso.

68

Cap. 4°
Geometria sugli enti razionali

Gli enti razionali ∞^1; le loro rappresentazioni sulle forme fondam^{li}, p.e. sulla retta le loro rappresentaz^{i} con coord. omogenee o no sugli enti stessi. Come ad essi si estenda tutta la geom. projettiva della retta, il birapporto, ecc. Come si estenda in partic. la definiz. di involuzione $\sum \lambda_i \varphi_i(t) = 0$, ove $t_1\, t_2$ son le 2 coord. omog^{e},...

Una serie lineare g_n^r sull'ente razionale $x_l = f_l(t)$, data da $\sum \lambda_i \varphi_i(x) = 0$ non è che un'involuzione $\sum \lambda_i \varphi_i(ft) = 0$. Viceversa dalla definizione di g_n^r (sulla retta) è evidente che un'involuzione I_n^r è una g_n^r. Se una serie alg. di gruppi di n elem^{i} è tale che da r qualunque è in generale indivi-

69

duato il gruppo, essa è una I_n^r: poichè essa sta nella I_n^n costituita da tutti i gruppi di n elemi della forma e quindi (pag. 62, 63) segue ecc.. — In particolare se le coord. degli elementi di una varietà qual. son funz^i raz^{li} di t e ad ogni elem. corrispondono μ valori di t, questi genereranno una I_μ^1; e poichè questa è raz^{le}, sarà pur tale la varietà (e questa si potrà riferire univocam^e al parametro λ della I_μ^1) *)

<u>Principio di corrispondenza</u> (α, α'). È attribuito a Chasles che primo lo formulò in generale nel 1864; però già nel 1861 Jonquières e Cremona lo applicavano ripetutamente alla determinazione degli ordini di luoghi geometrici. La corrisp. $(\alpha . \alpha')$ su un ente raz^{le} è

*) V. Lüroth, Beweis eines Satzes über rationale Curven (Math. Ann. IX).

70

(pag. 51) data da $\sum_0^{\alpha,\alpha'} a_{ii'} x^i y^{i'} = 0$; e però vi sono $\alpha + \alpha'$ elem.^{i} uniti $\sum a_{ii'} x^{i+i'} = 0$. Soluzioni multiple: se $x = 0$ è unito ($a_{00} = 0$) e due omologhi cadono in esso, e ciò in ambi i modi ($a_{01} = a_{10} = 0$), esso conta due volte; questa condiz.^{e} è anzi necessaria quando la corrisp. è simmetrica od involutoria ($a_{01} = a_{10}$). Come in molti casi si riconosca la multiplicità di una soluz.^{e}. Enunciato generale dovuto a Zeuthen *) : «il numero delle coincidenze di x e y che hanno luogo in un punto o della retta su cui si considera la corrisp.^{a} è uguale alla somma degli ordini dei segmenti infinitesimi xy fra un punto x la cui distanza da o sia infinitesima di 1° ordine ed i corrispondenti punti y (l'ordine di una distanza finita essendo uguale a zero)». Legame

*) Note sur le principe de correspondance (Bulletin des sciences math. V, 1873 p. 186)

71

colla singolarità di una curva piana, imagine della corrispondenza.

Applicazioni del principio di corrispondenza. — In una corrisp. (α, α') vi sono $2\alpha(\alpha'-1)$ elemi di diramazione della 1ª forma (ed altrettanti elem.i doppi corrisp.i nella 2ª). Due corrisp. (α, α'), (β, β') tra due forme hanno $\alpha\beta' + \beta\alpha'$ coppie comuni; se sono involutorie, la metà. Una corrisp. (α, α') su una forma ha $\frac{\alpha.\alpha-1}{2} + \frac{\alpha'.\alpha'-1}{2}$ coppie involutorie.

Curva generata da due fasci di rette di un piano in corrisp. (α, α'); proposiz. inversa. Due γ^n, $\gamma^{n'}$ (di spazi qual.) in corrisp. (α, α') generano una rigata d'ord. $\alpha n' + \alpha' n - x$ (se x sono i p.i uniti): caso di $\alpha = \alpha' = 1$ (a cui ogni altro si può ridurre). Se le curve son sovrapposte $(\alpha + \alpha')n - x$;

72

e se la corrisp. è simmetrica, la metà $\alpha n - \frac{x}{2}$ (onde x pari).

Gli elementi doppi di una I_n^1 sono $2(n-1)$ (Jacobiano). Elementi multipli di un'involuzione qual.: imponendo un punto ν-plo ad un gruppo si danno in generale ν condiz.i o $\nu-1$ secondo che il punto è dato o no. Elementi $(r+1)$pli di una I_n^r: dicendone $[n, r]$ il numero, si ha dal principio di corrisp.: $[n, r] = [n-1, r-1] + (n-r)$, $= [n-2, r-2] + 2(n-r) = \ldots = [n-r+1, 1] + (r-1)(n-r) = (r+1)(n-r)$. In generale formola di Jonquières *): in una I_n^r il numero dei gruppi con elem.i multipli secondo $\nu_1\ \nu_2 \ldots \nu_t$, ove $r = \Sigma\nu_i - t$ +) è uguale a

$$\nu_1 \nu_2 \ldots \nu_t (n-r)(n-r-1) \ldots (n-r-t+1)$$

*) Crelle 66 (1866): Mémoire sur les contacts multiples des courbes de degré r avec une courbe..

+) e $n \geq \Sigma\nu_i$ cioè $n \geq r+t$

La formola di Jonquières si ritrova in un lavoro di Lerch (Sitzb. k. Böhm. Ges. d. W. 1885)

73

diviso per $\alpha!\beta!\dots\delta!$ se fra le ν ve ne sono α uguali, β uguali, … δ uguali. Infatti, supposte le ν in ordine decrescente di grandezza, considerando gli ∞^1 gruppi con elementi multipli secondo $\nu_1 \dots \nu_{t-1}, \nu_t - 1$ e la corrispondenza fra quest'ultimo e gli $n - r - t + 1$ elem' semplici, si ha, indicando con $\{n; r; \nu_1 \dots \nu_t\}$ il numero cercato, la formola ricorrente:

per $\nu_t > 2$

$$\{n; r; \nu_1 \dots \nu_t\} = \frac{1}{\delta}\Big[\{n-1; r-1; \nu_1 \dots \nu_{t-1}, \nu_t - 1\} + (n-r-t+1)\{n-\nu_t+1; r-\nu_t+1; \nu_1 \dots \nu_{t-1}\}\Big]$$

e per $\nu_t = 2$

$$\{n; r; \nu_1 \dots \nu_t\} = \frac{1}{\delta} 2(n-r-t+1)\{n-1; r-1; \nu_1 \dots \nu_{t-1}\}$$

Ora queste formole sono appunto verificate da quell'espressione generale. Ne segue che questa è vera se ha luogo diminuendo successivamente le multiplicità ν di numero e di grandezza. Ma per $t = 1$ si riduce alla preced.: dunque è

74

vera sempre. — Caso particolare:

$$\{n; r; 2, 2, \ldots 2\} = \frac{2^r (n-r) \ldots (n-2r+1)}{r!}$$

— Applicazioni della formola di Jonquières. La $[n, r] = (r+1)(n-r)$ ci dà il num. degl'iperpiani stazionari di una γ^n raz.le di S_r; e in generale che quel rango che è dato dal num. degl' iperpiani a contatto $(\rho+1)$-punto passanti per un $S_{r-\rho-1}$, cioè dall'ordine della $M_{\rho+1}$ luogo degli S_ρ osculatori, è $(\rho+1)(n-\rho)$.*) Altri caratteri della curva dati dagl' iperpiani tangenti multipli.+) Altre applicaz.i alle varietà di un sist. lineare aventi dati contatti con una curva razionale; cerchi osculatori o sfere osculatrici per un punto o a contatto quadri-o

*) Se l' $S_{r-\rho-1}$ incontra γ in un punto, quel numero si riduce a $(\rho+1)(n-\rho-1)$, cioè diminuisce di $\rho+1$; γ è dunque $(\rho+1)$pla per la varietà degli S_ρ osculatori. Ciò vale sempre.

+) I caratteri si modificano per curve particolari.

75

5-punto o tang' altrove; punti sestattici di una curva piana raz^le (si devon togliere i flessi); ecc.

Per lo studio+) dei gruppi ed invol. armoniche od apolari e teor. della polarità convien ricorrere alla C^n raz^le norm. (pag. 45)

$$x_0 = \lambda^n, \ldots x_i = \lambda^{n-i}, \ldots x_n = 1.$$

L'iperpiano osculatore in x sarà $\sum \xi_i y_i = 0$ ove $\sum \xi_i \sigma^{n-i} = 0$ abbia $\sigma = \lambda$ per soluz. n-pla sicchè $\xi_0 = 1, \ldots \xi_i = (-1)^i \binom{n}{i} \lambda^i, \ldots \xi_n = (-1)^n \lambda^n$ ossia confrontando: $\xi_i = (-1)^i \binom{n}{i} x_{n-i}$.

Questa è una reciprocità involutoria in cui il coniugio di due punti od iperpiani è dato da

$$\sum (-1)^i \binom{n}{i} x_i y_{n-i} = 0$$

$$\sum (-1)^i i!(n-i)! \xi_i \eta_{n-i} = 0.$$

Polarità rispetto ad una quadrica, o sistema nullo*), secondo che n è pari o dispari (Clif-

+) V. fra gli altri Castelnuovo, Studio dell'involuz. ecc. e poi Deruyts, Bulletin Acad. Belgique 1887.

*) Gli n p.ti singolari della C^n di ξ_{n-1}, per n dispari.

76

ford). Dicendo coniugate od armoniche due n-ple quando l'una sta nella I_n^{n-1} determinata dagli n elem.^{i} n-pli presi nell'altra, la polarità rispetto a C^n mostra che quella relaz. è reciproca; che se n è impari ogni n-pla è coniugata a se stessa; se n è pari solo se s'annulla un invariante quadratico. Le n-ple coniugate ad una o più date formano un'involuz.; involuz.^{i} I_n^ρ, $I_n^{n-\rho-1}$ coniugate (quando i loro assi $S_{n-\rho-1}$, S_ρ sono polari): gli stessi $(\rho+1)(n-\rho)$ elem.^{i} sono $(\rho+1)$pli per l'una e $(n-\rho)$-pli per l'altra, giacchè ad un S_ρ osculatore a C^n è polare l'$S_{n-\rho-1}$ osculatore nello stesso punto.

Al coniugio od apolarità fra n-ple di elementi si collega la polarità risp. ad una n-pla fissa $a_1 \dots a_n$ (rappres. dall'iperpiano α o

77

dal punto A). Se due elem^i, uno x r-plo, l'altro y $(n-r)$plo fanno una n-pla coniugata alla A, dicesi che y è polo o centro armonico r-esimo, o d'ordine $n-r$ di x; viceversa sarà x… Se i param^i di x e y sono risp. 0 e ∞, l'equaz. di questa n-pla sarà $\lambda^r = 0$ e la condiz. di coniugio colla data $\Sigma \alpha_i \lambda^{n-i} = 0$ diventa $\alpha_r = 0$: il che mostra la coincidenza coll' ordinaria definiz^e dei centri armonici. Una teoria più generale si ha ponendo in luogo di x r p^i distinti $X_1 \ldots X_r$: gli $n-r$ p^i y $(n-r)$-pli in n-ple coniugate alla A e contenenti quegli r X formano il gruppo <u>polare misto</u> di questi r p^i. Si hanno pure prendendo il 1° gruppo polare di X_1 risp. ad A; poi il 1° polare di X_2 risp. a quello, ecc.; caso di $r = n-1$. — Si può domandare un gruppo di r elem^i X <u>apolare</u> ad A, cioè tale che ogni

78

elem. ne sia polo. Dovrà A giacere sull'S_{r-1} congiungente gli X, sicchè l'equaz. del gruppo A sarà $\sum \lambda_i X_i^n = 0$ o $\sum X_i^n = 0$, forma canonica, e viceversa. Allora ogni n-pla contenente tali r elemi sarà coniugata alla A. Perchè ciò accada basta che per l'S_{r-1} passino non solo $n-r$ ma $n-r+1$ iperpiani con contatti $(n-r)$-punti altrove. Assunto un elem. y come $(n-r)$-plo in una n-pla coniugata ad A, i G_r residui fanno una I_r^{r-1}! Si hanno così $n-r+1$ I_r^{r-1} le quali avranno comune un'involuz. che sarà in generale (considerando la C_r^r di S_r) di dimens. $r-(n-r+1) = 2r-n-1$. Dunque se $2r \geq n+1$ i gruppi di r elemi apolari ad un gruppo di n formano in generale una I_r^{2r-n-1}. In partic. se n è impari vi è in generale un determ. gruppo apolare

79

di $\frac{n+1}{2}$ elem.^{i}; e se n è pari ∞^1 di $\frac{n}{2}+1$ elem. La questione in ogni caso coincide con quella degli S_{r-1} r-secanti passanti per A: si potrebbe anche risolvere projettando da A e ricorrendo alle rigate F^{n-2} contenenti la C^n di S_{n-1} ed alle loro direttrici minime; inoltre quella consideraz. degli spazi secanti pone un limite infer. all'ordine dei gruppi apolari tolto quello d'ordine minimo. — Cfr. per la riduzione di una forma binaria a somma di potenze il Cap. sulle forme canoniche nella Algebra del Salmon.

Le involuzioni d'ordine $n+1$ sulla C^n conducono a notevoli proprietà di questa. I gruppi di una tale involuz. dànno i vertici di piramidi iscritte: le facce di queste (iperpiani) formano per la I^r_{n+1} un inviluppo ∞^r di classe $n-r+1$ (perchè r punti

80

della C^n individuano la piramide e quindi $n+1-r$ facce). E poichè la I^r_{n+1} è individuata da $r+1$ gruppi, segue che le facce $(r+1)(n+1)$ di $r+1$ piramidi iscritte nella C^n sono in un inviluppo ∞^r di classe $n-r+1$ (che contiene ∞^r tali piramidi iscritte nella C^n). — Caso di $r=1$: si ha un' altra C^n a cui le ∞^1 piramidi sono circoscritte. Se ad ogni vertice si fa corrispondere la faccia opposta, si ha fra le due C^n una corrisp. univoca: reciprocità; e poichè vi sono piramidi tali che ai vertici corrisp. le facce opposte sarà una polarità risp. ad una quadrica: ecc. ecc. — Caso di $r=n-1$. — Esempi: coniche e cubiche sghembe.

Cenno sulla questione della determinaz. di un' I^r_n mediante gruppi di $i>r$ elem^{i}

81

contenenti in essa: vale a dire determinaz di spazi S_{n-r-1} incidenti a spazi dati S_{i-1}. Così Schubert (Math. Ann. 26 p. 47) dimostra che gli S_{n-2} incidenti a $2n-2$ rette sono $\frac{1}{n-1}\binom{2n-2}{n-2}$, e ne segue che tante sono appunto le I'_n con un dato Jacobiano: proposizione già enunciata dal Meyer (loc. cit. e Math. Ann. 21 p. 132) e trovata pure dallo Stephanos. Altri casi furono determinati da Schubert e Castelnuovo: non si ha però ancora quello generale.*)

*) Per le I'_n, cioè per gli S_{n-2}, v. i Beiträge zur Liniengeom. in n Dimens. (Haml. Mitth. III 1892), ove la questione è risolta completam.^e per le rette.

82

Cap. 5°
Serie lineari ∞^1. Genere degli enti algebrici.

Importanza delle serie lineari ∞^1. Sono caratterizzate dalle due proprietà: 1° di esser razionali, 2° che un elemento individua un gruppo. Invero dalla 1ª segue che i gruppi corrispondono algebr. ad un param. λ; e dalla 2ª che (pag. 49, 50) se x è l'elemento dell'ente la corrisp. fra i gruppi e gli elem.i che li compongono è data da $\lambda = f(x)$ ove f è funz. raz.le, ossia $\lambda = \frac{\varphi(x)}{\psi(x)}$, $\varphi(x) - \lambda\psi(x) = 0$ che prova che la serie è lineare. Si vede inoltre come le <u>funz.i razionali dell'ente algebrico</u> *), cioè delle coord. di elem.i dell'ente, corrispondono alle g^1_n, essendo i gruppi di una tal serie i gruppi di elem.i in cui una

*) Cfr. Weierstrass, Vorl. ü. Abel'sche Functionen p. 22

funz. raz^{le} assume uno stesso valore: il numero $\underline{n}$ degli elem^i variab^i (che non annullano φ e ψ), cioè il numero degli zeri o degl'infiniti della funz., dicesi grado od ordine di questa. Dire che due gruppi di n elem^i stanno in una stessa serie lin. (g_n^r e quindi in una g_n^1) equivale a dire che sono gli zeri e gl'infiniti di una stessa funz. raz^{le}. Ecc. ecc. (V. pag. 94)

Su un ente algebr. γ si considerino due serie $g_n^1\ g_{n'}^1$ e siano S, S': abbiano risp. ν, ν' gruppi od elem^i di diramaz. cioè coincidenze di 2 elem^i di un gruppo. Considerando nella serie S come omologhi due gruppi quando contengono due elem^i x, y di un gruppo di S', si ha:

$$2n(n'-1) = \nu' + 2z\,, \quad (*)$$

ove z è il numero delle coppie comuni ad

(*) Cfr. Riemann pag. 105 lin. 3 da sotto

84

S, S'. Similmente .... e sottraendo

$$\nu - 2n = \nu' - 2n',$$

sicchè sull'ente è costante la differenza fra il numero degli elem.i di diramaz. e il doppio dell'ordine di una ser. lin. ∞^1. Dalla 1ª uguaglianza segue che ν è pari (v. anche pag. 72). L'espressione $\frac{\nu}{2} - n + 1$ dicesi <u>genere</u> dell'ente algebrico: chiamandolo p si ha

$$\frac{\nu}{2} - n + 1 = p$$

$$\nu - 2n = 2p - 2$$

$$\nu = 2(n + p - 1).$$

In questa definiz. di genere, come nel ragionamento preced., si noti che vi posson essere p.i di diramaz. da contar più volte nel numero ν: e precisamente se in un gruppo della g^1_n vi sono i elem.i coinc.i in uno, questo conta $i-1$ volte fra gli elem.i di diramaz., è $(i-1)$-plo come elem.o diram. (ciò si vede approssimativam.^e se si considerano i elem.^i consecutivi, come una retta a contatto i-punto conta come $i-1$ tang.^i da un suo punto: analiticam. si può vedere in modo completo; se l'ente è razion.^le, si vede subito sul Jacobiano)

Gli enti razionali hanno ($\nu = 2(n-1)$) il genere $p=0$.

È evidente che due enti in corrisp. univoca hanno lo stesso genere: donde l'importanza di questo nella geom. delle trasformaz.^i biraz.^ti. Cenno sui <u>moduli</u> ($3p-3$ se $p>1$ ecc.). Il genere e i moduli di una curva, rigata, ∞^1 di S_i non mutano per projez., per sezione, ecc.: così il genere ed i moduli di una ∞^1 di curve o superf. per sezione ecc. ecc.

L'ente algebrico potendosi sempre ri–

86

ferire ad una γ^n piana, vediamo come il genere di questa dipenda dai suoi caratteri. Per g_n^1 si prende quella determinata su γ da un fascio di rette. Se i punti multipli son tutti ordinari, le ν rette di diramaz. sono quante la classe: $\nu = n(n-1) - \Sigma s(s-1)$, e quindi $p = \frac{\nu}{2} - n + 1 = \frac{n-1.n-2}{2} - \Sigma \frac{s.s-1}{2}$ che coincide colla definiz. di pag. 3. Se poi si ha una singolarità superiore, Nöther ha dimostrato (Math. Ann. IX) che essa si può considerare come la riunione di p^i multipli ordinari e di <u>p^i di diramaz</u>. della curva. E precisamente si dice che in un p. P di γ cadono $i-1$ p^i di diramaz., quando, qualunque sia S, la retta SP conta $i-1$ volte fra le rette di diramaz. della g_n^1, cioè i punti di un gruppo di questa cadono in P (sicchè $i \leq$ multiplicità di P). Allora Nö-

87

ther dimostra che la classe è $n(n-1) - \Sigma s(s-1) - x$, ove la somma va estesa a tutte le singolarità ordinarie che equivalgono alle singolarità di γ, e x è il numero complessivo dei p.ti di diramaz. di γ. Ne segue $\nu =$ classe $+ x$, cioè $\nu = n(n-1) - \Sigma s(s-1)$, sicchè rimane valida in ogni caso la formola

$$p = \frac{1}{2}(n-1)(n-2) - \Sigma \frac{1}{2} s(s-1).$$

Caso di soli p.ti doppi: nodi e cuspidi. Il caso di soli d nodi è il più generale per una γ^n piana di gen. p (in questo senso che un p. s-plo impone $\frac{s.s+1}{2} - 2$ condiz.i che sono più che le $\frac{s.s-1}{2}$ condiz. imposte da altrettanti p.ti doppi). Si ha allora

$$d = \frac{1}{2}(n-1)(n-2) - p$$

come numero delle coppie neutre di una g_n^2 sull'ente di genere p. E dalle formole di Plücker pel numero dei flessi e delle

88

g^i doppie $\quad \rho = 3(n+2p-2)$

$$\delta = 2(n+p-2)(n+p-3) - 4p$$

si hanno in una g_n^2 $3(n+2p-2)$ elem^i tripli ecc.

Questo risultato si estende: in una g_n^r su un ente di genere p vi sono

$$(r+1)(n+rp-r)$$

gruppi dotati di elementi $(r+1)$-pli (v. per gli enti razionali pag. 72). Rappresentando la g_n^r con una C^n di gen. p. di S_r si tratta di trovarne gl'iperpiani stazionari od iperosculatori. Essi son dati da una delle formole di Veronese (Math. Ann. XIX, pag. 201, quella che dà $w^{(n-2)}$, ponendovi $R = w_1 = w_1^{(1)} = \ldots = w_1^{(n-3)} = 0$); che però possiamo ricavare direttamente seguendo la via del Castelnuovo (Ricerche, n. 7). Sceghiamo un piano π cogli spazi osculatori alla C^n: se ricordiamo che un S_{r-2} osculatore è intersez. di due iperpiani oscul^i inf. vicini, ossia

89

un iperpiano oscul. $123\ldots r$ congiunge due S_{r-2} oscul^i inf. vicini $123\ldots(r-1)$ e $23\ldots r$; e che un iperpiano iperoscul. $123\ldots r(r+1)$ congiunge tre S_{r-2} consecutivi $12\ldots(r-1)$, $23\ldots r$, $34\ldots(r+1)$; ed uno S_{r-3} oscul. è l'intersez. di tre iperpiani oscul^i consecutivi, vediamo che su π si avrà una curva luogo delle traccie degli S_{r-2} oscul^i e inviluppo degl'iperpiani oscul^i, la quale avrà per tang^i staz^e le traccie degli spazi iperoscul^i e per punti staz^i le traccie degli S_{r-3} oscul^i incidenti a π. Indicando dunque con N_r il numero dei gruppi con elem^i $(r+1)$-pli in una g^r_n sull'ente di gen. p, una nota formola di Plücker darà:

$$N_r - N_{r-3} = 3N_{r-1} - 3N_{r-2}$$

Di qui si trae appunto $N_r = (r+1)(n+rp-r)$ poichè questa formola verifica quella relazione ricorrente e vale per $r=1$ ed $r=2$ (sicchè

90

si può già applicare ad $r = 3$, poichè allora $N_0 = n$ è il numero dei p.i d'incontro di π con C). (*) — Applicaz.i di questa formola a-

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(*) Se la g_n^r è una serie composta mediante i gruppi di μ elem.i di una ∞^1, sicchè $n = \mu m$, la curva C che la rappresenta in S_r sarà d'ord. m (da contare μ volte) e di un certo genere π (il genere di quella ∞^1) ed essendo col dato ente algebr. in corrisp.a $(1, \mu)$, avrà per una formola che vedremo in seguito $y = 2(p-1) - 2\mu(\pi-1)$ elem.i di diramaz.. Gl'iperpiani stazionari a C rappresentano ognuno un gruppo della g_n^r dotato di μ elem.i $(r+1)$-pli e sono $(r+1)(m + r\pi - r)$. Ogni iperpiano poi che in un p. di diramaz. abbia con la C contatto i-punto rappresenta un gruppo della g_n^r dotato di elem.o $(2i)$plo. Ogni p. di diramaz. viene così ad assorbire $\frac{r(r+1)}{2}$ elem.i $(r+1)$pli della g_n^r composta, se vogliamo che anche per naloghe a quelle di pag. 74

Ritornando alla curva piana come imagine dell'ente algebrico, si può sempre con trasf. biraz. del piano ridurla a sole singolarità ordinarie: v. Nöther, Math. Ann. IX e XXIII, e Bertini, Rend. Ist. Lomb. 1888. Anzi si può sempre riferire univocam. a curva con soli p.[i] doppi (v. ad es. Bertini, Rivista di mat. t. I). Noi ci varremo solo della prima riduzione.

Rappresentando così l'ente algebrico segue subito che (pag. 2) $p \geq 0$, e che (pag. 57) se $p = 0$ l'ente è razionale.

Sulla γ^m piana una serie lineare g_n^r si

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questa rimanga valida l'espressione trovata del numero degli elem.[i] $(r+1)$-pli: giacchè questi verranno così ad essere

$$\mu(r+1)(m+r\pi-r)+\frac{r(r+1)}{2}\left[2(p-1)-2\mu(\pi-1)\right]$$
$$=\mu m(r+1)+r(r+1)(p-1)=(r+1)(n+rp-r)$$

92

determina (pag. 55) mediante un sist. di curve agg.^{te} φ^ν. Considerando tutte le φ^ν agg. passanti pegli stessi p.^{i} residui di γ^m si ha una g_n che contiene quella. Ora, posto $\nu = m - 3 + \alpha$, tutte le φ^ν segano una serie d'ord. $m\alpha + 2p - 2$ e dimens. $\geq m\alpha + p - 2$. Fissando quegli $m\alpha + 2p - 2 - n$ p.^{i} residui si ha dunque una g_n di dimens. $\geq n - p$ *) Segue che per una g_n^r completa si ha $r \geq n - p$, ossia $n - r \leq p$. In caso contrario la serie è parziale; una γ^n di uno S_r ove $r < n - p$ non è normale: è proj. di una di S_{n-p}. Una g_n^r per cui $n - r < p$ dicesi __speciale__: così pure una serie contenuta in una dello stesso ordine, la quale sia special. Dunque per le g_n^r complete e le γ_n di S_r normali, quando non son speciali è $n - r = p$; altrimenti $n - r < p$. Se $p = 0$, o $p = 1$, non vi possono essere serie speciali (pag. 54). Se

*) Se $n > p$ si ha sempre per tal modo una g_n^r ove $r \geq n - p$ che contiene un gruppo dato qualunque di n elem.^{i}.

93

$p > 1$ si ottengono serie speciali segandole con curve agg^{te} d'ord. $m-3$ su γ^m (curve agg^i di ord. minore si completano con una curva fissa in tali): poichè quelle dànno una g_{2p-2} di dimens. $\geq p-1$ (pag. 56) e facendole passare per $2p-2-n$ p^i si ha una g_n^r ove $r \geq n-p$. Vedremo poi che viceversa tutte le serie speciali si possono ottenere così, sicchè il loro ordine è sempre $\leq 2p-2$; e che vi è una sola g_{2p-2}^{p-1} (la quale non ha elem^i fissi).*)

Si può sempre rappresentare l'ente di gen. p in ∞ modi (nota a pag. 92) con una curva appartenente ad S_r e d'ordine $n \leq p+r$; o, se appartiene ad S_{p-1}, d'ordine $\leq 2p-2$. In particolare con una curva piana d'ordine $\leq p+2$. Così per $p=1$ si ha la cubica; e per $p=2$ la quartica con 1 punto doppio o la conica doppia (in ambo

*) È opportuno notare che se una serie speciale ha elementi fissi, la serie che rimane astraendo da questi sarà pure speciale.

94

i casi la g_2^1 speciale è evidente). Per $p=3$ la quartica piana generale o la conica doppia.

In queste rappres^i piane dell'ente algebrico si cercava su questo una g^2: la g^2 del minimo ordine dava la curva piana del minimo ord. rappres^e dell'ente. — Si può invece ricorrere a due g^1 cioè a due funz^i raz^{li} dell'ente (v. pag. 82).

Ricordiamo (ivi) che una funz. raz^{le} $z = \frac{\varphi(x)}{\psi(x)}$ dell'ente definisce una g_n^1 ove n è il grado (secondo Weierstrass) della funz.*) Viceversa data una g_n^1 $\varphi(x) - z\psi(x) = 0$ si avranno ∞ funz^i raz^{li} $\frac{a\varphi(x)+b\psi(x)}{c\varphi(x)+d\psi(x)}$, cioè le trasformaz^i lineari $\frac{az+b}{cz+d}$ di una $z = \frac{\varphi(x)}{\psi(x)}$. E non se ne avranno altre, poichè se $\Phi(x) - Z\Psi(x) = 0$ dà la stessa g_n^1, fra le z e Z che corrispondono ad uno stesso elem. x dell'ente vi sarà corrisp. univoca, sicchè

*) Qui e nelle prime pag. segu^i si considerano solo serie non degeneri.

Z sarà funz. lin. fratta di z.

Siano $z = \frac{\varphi(x)}{\psi(x)}$, $s = \frac{\varphi_1(x)}{\psi_1(x)}$ due funz.i raz.i dell'ente, risp. di gradi n, m tali che non ogni gruppo di dato z abbia più elem.i com.i con un gruppo di dato s, cioè che le g_n^1, g_m^1 … non abbiano ∞ coppie comuni. Fra z ed s vi sarà una corrispondenza (m, n), cioè un'equ. $F(\overset{n}{s}, \overset{m}{z}) = 0$, che si avrebbe per eliminaz.e, ed i gruppi (s, z) saranno in corr.isp. univoca coll'ente; le coord. x di questo saranno viceversa funz.i raz.li di s, z: quella equaz. rappresenta l'ente. Così a rappres. un ente alg. si può prendere in generale la equaz. che lega due sue funz.e raz.li qual. Assumendo per una od entrambe le funz.i di grado minimo dell'ente si riduce la F = 0 a gradi minimi m, n. Notiamo ancora che se sull'ente vi è una g_n^1 (n elem.i variab.i),

96

le coord. dell'ente saranno funz^i raz^{li} di un parametro z e della radice s di un'equaz. di grado n a coeff^i raz^{li} in z (z sarebbe la funz. raz^{le} di grado n rappres^e la g_n^1).

Considerando la $F(\overset{n}{s}, \overset{m}{z}) = 0$ come equ. di una curva piana si cade nella seguente rappres^e. Due g_n^1, g_m^1 ... si rappresentino con due fasci di raggi S, S' : questi saranno in corrisp. (m, n) e genereranno in generale una γ^{m+n} su cui il fascio S stacca la g_n^1... Di qui seguirebbe di nuovo il teor. del principio di questo Cap^o (pag. 82), assumendo ad arbitrio una g_m^1 ausiliaria. — Un p. s-plo di γ fuori di S, S' corrisponde ad s elem^i dell'ente, comuni a due gruppi delle g_m^1, g_n^1, e viceversa. Ugnagliando a p il genere di γ si ha la relaz. $p = (m-1)(n-1) - \sum \frac{s.s-1}{2}$ che si trova in sostanza alla fine del n. 7 (p. 107) del-

97

la memoria di Riemann; v. anche Castelnuovo n. 4. — Se due g^1_n hanno un G_n comune, stanno in una g^2_n (coniche pei 3 pi n-pli della γ^{2n}). Se hanno un G_n ed un G_{n-1} comuni, l'ente è razionale. Se hanno due G_n comuni, coincidono. Vedremo poi proposizioni più generali.

Facendo corrisp. la r. SS' a se stessa, γ si riduce all'ordine $m+n-1$; e in generale se le g^1_m, g^1_n hanno un G_t comune, si può ridurre γ all'ord. $m+n-t$ (osservaz. di Bertini): com'è evidente per $t=2$. Basandosi su ciò la rappres. di un ente con una g^1_n mediante una curva piana ed un fascio di raggi S si può semplificare, ottenendo una γ^{p+2} col p. S $(p+2-n)$-plo $(n \leq p+1)$: basta per $n-1$ pi di un G_n della serie ed altri $p+1-(n-1)$ $=p+2-n$ tra cui non vi sia l'$n$esimo condurre una g^1_{p+1} ecc. (Bertini)

Enti dotati di una g^1_2. Valendosi di una g^1_m ove $m \leq p+1$, che contenga un gruppo in cui non vi sia alcuna coppia della g^1_2 (nota a pag. 92)., si ottiene una γ^{m+1} con p. $(m-1)$-plo, ove si può supporre $m+1 \leq p+2$; ma sarà precisamente $=$ poichè il gen. è p (e 0 p.i doppi). Se $p=0$ è evidente che vi sono ∞^2 g^1_2 (nella g^2_2) e che due qualunque hanno una coppia comune. Se $p=1$ ogni coppia determina (nota a p. 92) una g^1_2; si ritrova per imagine la cubica valendosi di due g^1_2 e $p=0$ o $p=1$ (p. doppio o no) secondo che vi è o no una coppia comune: sicchè per $p=1$ ogni coppia <u>individua</u> la g^1_2. Se $p=2$ si ha sempre la quartica con p. doppio. E sempre se $p>1$ non vi può essere che una sola g^1_2.

Gli enti con una g^1_2, valendosi di una funz. raz^{le} di 2° grado z si possono rappresentare (v. il principio di p. 96) così: $x_i = f_i(z, s)$ ove le f_i son raz^{li} ed $s^2 = R(z)$ è un polinomio in z. La g_2^1 ha i gruppi corrisp^i ai vari valori di z, per ognuno dei quali se ne hanno due di s: elem^i di diramaz^e son dunque quelli per cui $R(z) = 0$. Ma sono (p. 84) $2p+2$: dunque questo è il grado di $R(z)$. Elem^i di diramaz. multipli non ve ne possono essere (fine di p. 84); sicchè al più potrà accadere che $R(z)$ si abbassi al grado $2p+1$ per essere $z = \infty$ elem. di diramaz. Cenno sugl'integrali ellittici ed iperellittici. Gli enti $p = 1$ si dicono anche <u>ellittici</u> e quelli… <u>iperellittici</u>. Noi però usiamo questo nome per tutti gli enti con una g_2^1. Per questi enti particolari cominciamo a risolvere certe questioni che poi tratteremo in generale.

100

Moduli degli enti iperellittici $p>0$. Osserviamo anzitutto che $R(z)$ è un polinomio *arbitrario* (senza radici doppie) di grado $2p+2$. Dato un altro ente iperell. di gen. p rappres° da $s'^2=R'(z')$, se mediante una trasformaz. lin. $z'=\frac{az+b}{cz+d}$ si ha $R'(z')=\frac{R(z)}{(cz+d)^{2p+2}}$, aggiungendo $s'=\frac{s}{(cz+d)^{p+1}}$ oppure $s'=-\frac{s}{(cz+d)^{p+1}}$ si muterà in $s^2=R(z)$. Dunque *basta* che i due gruppi di $2p+2$ elem^i diram^i siano projettivi perchè i due enti iperell^i si equivalgano biraz^e. Ma ciò è anche necessario se $p>1$, poichè allora le due uniche g_2^1 devono corrispondersi ... Dunque $2p-1$ *moduli* *) (birapporti)

Rimane il caso $p=1$: allora vi sono $\infty\, g_2^1$ tali che da una coppia, e in partic. da un elem. diram., è individuata la serie. Ma allora se A, A' sono elem. di diram. di due g_2^1, la corrisp

*) Segue che la geom. proj. della *r. doppia* con $2p+2$ p^i diram. che rappresenta l'ente iperell. equivale completamente alla geom. su quest'ente. — Resta anche risolta la questione delle *corrispondenze univoche sull'ente iperellittico*.

univoca determinata dalla g_2^1 che contiene la coppia AA' muterà l'una nell'altra quelle due g_2^1, e però i birapporti delle due quaterne di elem.ⁱ di diramaz. saranno uguali: notevole teorema del birapporto di un ente ellittico. Segue poi che anche per $p=1$ il birapporto di due enti in corrisp. univoca è lo stesso: esso è il modulo. — Le g_2^1 sulla γ^m ellittica si hanno coi fasci di curve agg.^{k} d'ord $m-3+\alpha$ per $m\alpha-2$ p.ⁱ semplici: così sulla cubica piana da fasci d'ord. α per $3\alpha-2$ p.ⁱ; in partic. da fasci di rette coi centri sulla cubica. Segue che tutte le coppie di una g_2^1 ottenuta ad es. da un fascio d'ord. α sono allineate con 1 p. fisso. Che in una corrisp. univoca fra due cubiche alle coppie allineate con 1 p. corrispondono coppie allineate con 1 p.; ecc.

Serie speciali sugli enti iperellittici $p>1$.

102

Lasciamo da parte gli elem. fissi. Abbiasi una g_n^r che non sia composta colla g_2^1: rappres. una γ^n di S_r, semplice o multipla, ma in cui le coppie della g_2^1 sono di elem^i distinti. La rigata (p. 72) delle congiung^i sarà d'ord. $n - \frac{x}{2}$ ossia $n-p-1$, onde (p. 36) $n-p-1 \geq r-1$, $n-p \geq r$. Segue che una g_n^r speciale è necessariamente composta con la g_2^1, e però non è altro che un' invol. nella serie raz^{le} che ha per elem^i le coppie della g_2^1 (cfr. pag. 61); per una serie completa così composta è $n = 2r$: sarà speciale se $n-r<p$ cioè $r \leq p-1$ e quindi $n \leq 2p-2$ (Se poi $r>p$ una serie così composta non può più essere completa poichè $n-r>p$). Si vede ora che (nel caso iperellittico) le curve agg^{ti} d'ord. $m-3$ di una γ^m piana sono precisamente ∞^{p-1} e che la g_{2p-2}^{p-1} che esse staccano

103

non ha p.ⁱ fissi. Segue pure che vi è una sola g_{2p-2}^{p-1} e che per ogni g_{2r}^{r} speciale vi è una $g_{2p-2-2r}^{p-1-r}$ residua tale che ogni gruppo dell'una con ogni gruppo dell'altra dà un gruppo della g_{2p-2}^{p-1}, ecc. — Di passaggio si ha che una γ^n semplice di gen. p del piano se $n-p<2$, $n<p+2$, non può esser iperellitt. (e se $n=p+2$ la curva iperell. ha un p. p-plo), cioè non può esser iperell. una γ^n con d p.ⁱ doppi se $d<\frac{n-2 . n-3}{2}$: esempi $n=4, 5$.

104

Cap. 6°
Formola di Zeuthen.
Varietà ∞^1 di spazi e loro applicazioni.
Le serie speciali

Dati due enti γ_p, $\gamma'_{p'}$ in corrisp. $(1, \mu)$ ad una g^1_n di γ con ν elem.i di diram. corrisponde su γ' una $g^1_{\mu n}$ composta che ha per elem.i diram. i $\mu\nu$ che corrisp. a quei ν, e poi gli y elem.i doppi corrisp. agli y elem.i di diram. di γ nella corrispondenza. Segue (p. 64)

$$\nu = 2n + 2(p-1)$$

$$y + \mu\nu = 2\mu n + 2(p'-1),$$

donde:

$$y = 2(p'-1) - 2\mu(p-1).$$

Importanza di questa formola. Ne segue $\mu(p-1) \leq p'-1$. Quindi se $p = p' > 1$ sarà $\mu = 1$, la corrisp. è necessariam. univoca (per

$p=p'=1$ ciò non varrebbe più). Se $p'=0$ anche $p=0$, donde segue di nuovo che sull'ente raz^{le} una ∞^1 di gruppi di μ elem^i tale che ogni elem. stia in un sol gruppo è raz^{le} (cfr. p. 69).

Prima di passare alle applicazioni alle rigate ecc. osserviamo che quella formola si può generalizzare. Siano $\gamma_p, \gamma'_{p'}$ in corrisp. (x, x') con y, y' elem^i di ram. e consideriamo l'ente Γ_π costituito dalle coppie di elem^i omol^i di γ, γ' (rappres^o ad es. dalla rigata generata da γ, γ' se questi son curve). Sarà γ con Γ in corrisp. $(1, x')$ onde
$$y = 2(\pi-1) - 2x'(p-1).$$
Similmente
$$y' = 2(\pi-1) - 2x(p'-1),$$
e sottraendo
$$y - y' = 2x(p'-1) - 2x'(p-1),$$
<u>formola di Zeuthen</u> (Math. Ann. III). La si può anche avere direttam. senza passare per

106

quel caso partic. assumendo su γ e γ' due g_n^1, $g_{n'}^1$ risp. con ν, ν' elem^i diram: si avranno così su Γ due $g_{nx'}^1$, $g_{n'x}^1$, risp. con $y+\nu x'$ e $y'+\nu' x$ elem^i diram., onde (p. 84)

$$y+\nu x'-2nx' = y'+\nu' x-2n'x$$

che, sostituendo $\nu-2n=2(p-1)$, $\nu'-2n'=2(p'-1)$ dà appunto la formola di Zeuthen. Del resto da quell'uguaglianza, ponendo $x=x'=1$ e quindi $y=y'=0$ si trae la $\nu-2n=\nu'-2n'$: ora essa si può dimostrare direttamente come si dimostrò questo suo caso particolare, cioè cercando il numero z delle coppie della g_n^1 di γ a cui corrispondono su γ' coppie della $g_{n'}^1$ (mediante il princ. di corrispondenza sulla g_n^1) ed uguagliandolo a quello ottenuto analogam colla $g_{n'}^1$.

Data su una curva γ di gen. p una ∞^1 di gen. π di gruppi di m p^i sì che ogni

107

p. stia in un sol gruppo, da pag. 104 abbiamo

(1) $y = 2(p-1) - 2m(\pi - 1)$,

che vale anche se γ è <u>multipla</u> di gen. p… In particolare se si è in S_r e gli spazi S_k cui appartengono i gruppi sono infer. ad S_r …; caso di $k = r-1$; in generale si hanno le curve tracciate sulla rigata di gen. π, sulla ∞^1 di piani, … E anche qui si ha (cfr. p. 105) che se $p=0$ anche $\pi = 0$, sicchè una rigata o varietà … non razionale non può contenere alcuna curva direttrice razionale, e più in generale il genere delle curve dev'esser $p \geq m(\pi - 1)$, ecc.

Il num. y si può esprimere, anzi che coi generi, cogli ordini n di γ e ν della rigata o varietà e qualche altro carattere (ν cessa di aver senso quando $k = r$). Così poi eliminando y si ha una relazione importante fra i caratteri di γ e, se $k \leq r-1$, quelli della M_{k+1} luogo degli ∞^1 S_k.

Se $k=1$ si ha, per la rigata, applicando il princ. di corrisp. ad un fascio d'iperpiani projettanti la corrisp. su γ in cui si considerano come omologhi due p.i A, A' quando sono su una stessa gener.

$$(2) \qquad y = 2(m-1)n - m(m-1)\nu - 2z,$$

ove z è il num. di quei p.i doppi di γ che riuniscono coppie AA' (esempi). Dalle (1) e (2) segue

$$(3) \qquad (m-1)n - p = \frac{m.m-1}{2}\nu - m\pi + m - 1 + z$$

formola da me data nella nota "Intorno alla geometria su una rigata algebrica" (Rendic.i Lincei, 1887, 2° sem.e pag. 3), e poi nelle "Recherches générales sur les courbes et surfaces réglées algébriques, 1.e P.e" (Math.-Ann.-t. 30)

Notiamo che essa vale anche se γ è multi-

109

pla, cioè una curva d'ord. $\frac{n}{\mu}$ da contarsi μ volte, poichè allora vale ancora la (2), senza modificazioni.

Sia k qualunque, $k \leq r$. Introdurremo gli ordini $x_1, x_2, \dots x_i \dots$ delle varietà costituite dalle rette, piani, … S_i … congiungenti 2, 3, … $i+1$, … punti di un gruppo. L'ultimo di questi numeri si ha per $i = k$ se $k \leq r-1$, ed è $x_k = \binom{m}{k+1}\nu$. Si ha anzitutto, similmente alla (2):

$$2(m-1)n = 2x_1 + y ;$$

però se vi son dei p' doppi di γ che contino due volte in gruppi della ∞^1, il loro numero raddoppiato si deve aggiungere al 2° membro. Ora da x_1 si passerebbe analogam. ad x_2 e da questo ad x_3, … considerando similmente su un S_{r-i} la curva luogo delle tracce degli ∞^1 S_i e la rigata luogo di quelle degli ∞^1 S_{i+1}.

110

Ciò equivale ad applicare ancora il princ.° di corrisp., considerando nella nostra figura come omologhi due S_i che congiungano gli stessi $\underline{i}$ p.i P di un gruppo risp. a 2 p.i A, A' di questo; ed in un fascio di S_{r-i-1} di quell'S_{r-i} considerando come omologhi due spazi incidenti a due tali S_i. Si va fino ad $i = k-1$, cioè a segare con un S_{r-k+1}; se $k = r$, il fascio si riduce ad una retta punteggiata nel caso estremo. — La corrisp. nella ∞^1 di S_i è simmetrica d'indice $(i+1)(m-i-1)$; onde nel fascio si avranno $2(i+1)(m-i-1)x_i$ coincidenze. Queste si hanno: $\underline{1}°$ essendo i due S_i distinti ma incidenti all'S_{r-i} nello stesso p., che sarà dunque sull'S_{i-1} dei P: sono così $(m-i)(m-i-1)x_{i-1}$ coincidenze; $\underline{2}°$ essendo i due S_i distinti ed incidenti all'S_{r-i} in due p.i distinti: questi essendo in un S_{r-i-1} del fascio,

111

la loro retta incontra il sostegno S_{r-i-2}, onde l'S_{i+1} di quei due S_i è incidente a questo: sono $(i+1)(i+2)\,x_{i+1}$; e per $i=k-1$ bisogna mettere $x_k = \binom{m}{k+1}\nu$. Però questo caso estremo non ammette questa 2ª specie di coincidenze se $k=r$; dovremo dunque allora mettere $\nu=0$. 3° coincidendo i due S_i pel coincidere di A, A': ciò dà $\binom{m-2}{i}y$ coincidenze. 4° coincidendo i due S_i senza che coincidano A, A', cioè essendovi un S_i con $i+2$ p.ⁱ di un gruppo (ogni tal S_i darebbe $(i+1)(i+2)$ coincidenze): l'esistenza di un gruppo siffatto impone a questo $k-i$ condiz.ⁱ e quindi si può dire che in generale ha luogo solo nel caso estremo $i=k-1$, cioè che vi sono solo dei gruppi di $k+1$ p.ⁱ appartenenti ad S_{k-1}, e siano z. Però nel seguito c'importerà solo di sapere (tranne per $k=r$) che i primi mem_

112

bri delle uguaglianze seguenti sono <u>almeno</u> uguali ai 2^i, sicchè non escluderemo neppure quelle particolarità eccezionali. Abbiamo dunque:

$$2(m-1)n = \qquad 1.2\,x_1 + y$$

$$2.2(m-2)x_1 = (m-1)(m-2)n + 2.3\,x_2 + (m-2)y$$

$$2.3(m-3)x_2 = (m-2)(m-3)x_1 + 3.4\,x_3 + \binom{m-2}{2}y$$

. . . .

$$2.i(m-i)x_{i-1} = (m-i+1)(m-i)x_{i-2} + i(i+1)x_i + \binom{m-2}{i-1}y$$

$$2(i+1)(m-i-1)x_i = (m-i)(m-i-1)x_{i-1} + (i+1)(i+2)x_{i+1} + \binom{m-2}{i}y$$

$$2(i+2)(m-i-2)x_{i+1} = (m-i-1)(m-i-2)x_i + (i+2)(i+3)x_{i+2} + \binom{m-2}{i+1}y$$

. . . .

$$2(k-1)(m-k+1)x_{k-2} = (m-k+2)(m-k+1)x_{k-3} + (k-1)k\,x_{k-1} + \binom{m-2}{k-2}y$$

$$2k(m-k)x_{k-1} = (m-k+1)(m-k)x_{k-2} + k(k+1)\left[\binom{m}{k+1}\nu + z\right] + \binom{m-2}{k-1}y$$

Moltiplicando queste equaz.ⁱ risp. pei fattori: $k.(m-2)\dots(m-k)$, $(k-1)(m-3)\dots(m-k)$, $\dots$, $(k-1)$ e quella generica intermedia per $(k-i)\,i!\,(m-i-2)\dots(m-k)$ e sommando si vede dalle tre di mezzo che scompare x_i, dalle due ultime scompare anche

113

x_{k-1} e, riducendo, rimane

(4) $y = 2\frac{m-1}{k}n - 2\frac{m(m-1)}{k(k+1)}\nu - 2z : \binom{m-2}{k-1}$,

che dà l'espressione cercata del numero y … ;
essa (con quel calcolo) è dovuta a Schubert.
Se $k = r$ bisogna porvi $\nu = 0$. Confrontando
poi con la (1) (pag. 107) si ha:

(5) $\frac{m-1}{k}n - p = \frac{m.m-1}{k.k+1}\nu - m\pi + m - 1 + z : \binom{m-2}{k-1}$,

formola generale della mia Nota « Sulle varietà
algebriche composte di una serie semplicemente
infinita di spazi » (Rendic.ⁱ Lincei 1887, 2° sem.),
in cui si trova anche quel calcolo di Schubert.
Essa vale anche se γ è multipla …

Nel caso estremo $k = r$ si ha dalla (5)

(6) $z = \binom{m-2}{r-1}\left\{\frac{m-1}{r}n - p + m\pi - m + 1\right\}$

come numero dei gruppi di $r+1$ punti comuni ad una g^r_n e ad un' involuz.ᵉ ∞^1 di
ordine m e genere π, sopra un ente di genere
p. Se $\pi = 0$ sicchè l'invol. è una g^1_m si ha

114

$$(7) \qquad z = \binom{m-1}{r}(n-r) - \binom{m-2}{r-1}p\,,$$

che Castelnuovo trova direttam. nel suo n. 8 senza notare che rientra nella (5). Il Castelnuovo ha del resto risolta la questione più generale di trovare il numero dei gruppi di $r+r'$ elem[ent]i comuni ad una g_n^r ed una $g_{n'}^{r'}$ (nel lavoro cit° a pag. 66, Rend. Pal. III).

Si noti che il teor. contenuto nella (6) abbraccia a sua volta la (5): basta applicarlo sostituendo alla g_n^r la g_n^k che su γ è data dagl' iperpiani passanti per un S_{r-k-1}; allora il numero dei gruppi di $k+1$ p[un]ti comuni a quella g_n^k ed alla serie ∞^1... è dato da $\binom{m}{k+1} + z$, ecc..

Il caso particolare della (5) che più importa nel seguito è quello in cui i gruppi di m punti della ∞^1 si compongano in generale di p[un]ti indip[enden]ti, cioè $k = m-1$; diventa:

(8) $\quad n-p = \nu - m\pi + m - 1 + z$,

ossia (8′) $\quad n-p = \nu - (k+1)\pi + k + z$.

Una prima applicazione si fa subito per le rigate e varietà M_{k+1} di ∞^1 S_k di gen. π. Su esse si posson tracciare ∞ curve che incontino ogni S_k in $k+1$ p.i indip.i, sicchè $z=0$; così segando la rigata con quadriche non tang.i e la M_{k+1} con un cono di dim. $r-k$ projettante da un S_{r-k-2} una C^{k+1} raz.le norm.le Se una tal γ è proj. di una γ dello stesso ord. di uno spazio super.e, la M_{k+1} di questo altro spazio sarà con questa nuova curva nella stessa relaz. $z=0$, e non sarà incontrata dallo spazio centrale di projez., perchè altrimenti sulla γ primitiva si avrebbero gruppi di p.i non indip.i; o perchè sempre vale la (8′) con $z=0$. In base dunque alla pag. 92 si ha: una ∞^1 di gen. π e d'ord. ν di S_k ha per spazio

116

normale quello di dim. $\nu-(k+1)\pi+k$, od uno super.; secondo che le γ suddette tracciate su essa son tutte non speciali, o tutte speciali. Nel 2° caso si potrebbe chiamare _speciale_ la varietà, come io ho già fatto per le rigate. In partic. per $\pi=0$ si ha che le varietà ∞^1 razli d'ord. ν di S_k son normali per $S_{\nu+k}$ (cfr. pag. 38): il che si poteva vedere anche altrimenti. — Si noti che tutto ciò vale anche se la varietà è in S_{k+1}. Applicazioni che così si avrebbero del problema delle curve _direttrici_, quando fosse risolto per le varietà normali.

Applicando la (8) ad una g^1_m, $\pi=0$, si ha
$$n-p=\nu+m-1+z$$
nell'ipotesi che i gruppi di m p.i appartengano in generale ad S_{m-1}. Ora se $m-1=r$ il 2° membro è $\geq r$; e se $m-1<r$, siccome

117

il massimo spazio per una M_m d'ord. ν è appunto $S_{\nu+m-1}$ sarà pure quel 2° membro $\geq r$. Dunque sempre $n-p \geq r$. Segue che <u>quando $n-r<p$ i gruppi di m p.i di una g^1_m sulla γ^n di S_r appartengono a spazi di dim. $\leq m-2$</u>.

Abbiasi ora sulla stessa curva e nella stessa ipotesi $n-r<p$ una g^q_Q: dico che i suoi gruppi G_Q stanno in spazi di dim. $Q-q-1$ (Castelnuovo n. 14). Invero se in un G_Q si fissano $q-1$ p.i qual. i rimanenti $Q-q+1$ formano un gruppo di una serie ∞^1 e quindi per la propos. preced. appartengono ad un $S_{Q-q-1-\delta}$. Potrà forse in ogni G_Q il numero δ mutare a seconda della scelta dei $q-1$ p.i: in tal caso si prenda il minimo, e $Q-q$ p.i dell'$S_{Q-q-1-\delta}$ tali che lo determinino; ogni altro p. di G_Q insieme con quei $Q-q$ ne darà $Q-q+1$ che dovranno stare in quello spazio. Dunque tutti

118

i Q punti gli appartengono.

Poichè una γ^n speciale è sempre proj. di una per cui $n - r < p$, segue che su ogni curva speciale i gruppi di una g^q_Q stanno in S_{Q-q-1}. Evidentemente ciò si estende anche al caso che la g^q_Q abbia dei punti fissi.

Applichiamo questo teorema anzitutto alla g^r_n speciale segata sulla γ^n speciale dagl'iperpiani. I suoi gruppi staranno in S_{n-r-1} e però $n - r - 1 \geq r - 1$, ossia

$$n \geq 2r,$$

teorema dovuto a Clifford, per le g^r_n speciali (per Clifford, le γ^n di S_r per cui $n - p < r$). Da esso segue subito che la serie segata su una γ^m piana dalle φ^{m-3} agg^{te} è precisamente ∞^{p-1}, complete, coi $2p-2$ p^i tutti variabili: poichè se, astraendo da x p^i fissi si riducesse ad una g^{p-1+y}_{2p-2-x}, $2p-2-x \geq 2(p-1+y)$, $x=y=0$.

119

La relazione $n \geq 2r$ per le g_n^r speciali varrà anche se vi sono elem.ⁱ fissi, poichè astraendo da questi rimane una serie speciale. Ora se si considera una serie completa speciale, od almeno una per cui $n-r \leq p-1$, sommando questa (tale quale, o raddoppiata) colla $2r \leq n$ si ha

$$r \leq p-1 \,, \quad n \leq 2p-2 \,.$$

Segue che una g_n^r per cui $r > p-1$, o per cui $n > 2p-2$ è certo non speciale; una curva d'ordine $> 2p-2$ non è certo speciale, come pure se appartiene ad uno spazio superiore ad S_{p-1}: e però non potrà appartenere ad uno spazio super. ad S_{n-p} (teor. di Clifford, almeno in parte). Le serie complete d'ordine $n > 2p-2$ sono di dim. $n-p$; le curve normali d'ordine $n > 2p-2$ appartengono ad S_{n-p}. –

120

Segue che una sup. a sezioni spaziali di gen. p, d'ord. $n > 2p-2$ può appartenere al più ad S_{n-p+1}; e similmente una M_k^n ... ad $S_{n-p+k-1}$.

Il numero $n-r$ è quello degli elem.i di un gruppo di una g_n^r che son determ.i da r arbitrari. Se $r > p-1$, od $n > 2p-2$, abbiamo dunque che $n-r \geq p$, ossia almeno p elem.i di un gruppo son conseguenza dei rimanenti. Quindi nel problema di trovare quante fra le varietà di una ∞^d lineare contengono una data γ, se le intersez. sono $n > 2p-2$, basta fissare $n-p+1$ punti di γ e però saranno almeno $\infty^{d-n+p-1}$ quelle varietà (del resto da pag. 53 si ha, essendo $r \leq n-p$, $t \geq d-n+p-1$).

Applicando il lemma di Castelnuovo (pag. 117) ad una g_{2p-2}^{p-1} su una C^{2p-2} di S_{p-1}, si ha che i suoi gruppi dovranno stare in S_{p-2}, e però saranno quelli segati su C dagl'iperpiani. Segue che esiste una sola g_{2p-2}^{p-1}. In altri termini le C^{2p-2} di S_{p-1} rappresentanti uno stesso ente, cioè riferite univocam. fra loro, sono projettive: si posson chiamare curve canoniche di genere p. La geom. sull'ente si riduce alla geom. projettiva della curva canonica che lo rappresenta. Questa curva sarebbe data da $y_0 : y_1 : \dots : y_{p-1} = \varphi_0(x) : \varphi_1(x) : \dots : \varphi_{p-1}(x)$, ove $f(x) = 0$ è una curva piana d'ord. m di cui le φ sono aggte d'ord. $m-3$. La curva canonica si riduce ad una curva razionale normale doppia d'ord. $p-1$ nel caso iperellittico (pag. 102); tolto questo caso, la curva canonica e quindi la g_{2p-2}^{p-1} è semplice, cioè le curve aggte d'ord. $m-3$ …. (*)

Il lemma di Castelnuovo conduce più

(*) Intorno alle curve canoniche v. la mia nota « Sulle curve normali di genere p dei vari spazi » (Rend. Ist. Lomb. 1888), nella quale si trova anche la dimostriz. del teor. di Clifford ecc.

in generale al <u>teorema di Riemann e Roch</u>. Applicandolo in fatti alla curva canonica e ad una g_n^r speciale su questa, si ha che i gruppi di questa stanno in S_{n-r-1}, sicchè se $n-r \leq p-1$ (come accade certo se la serie speciale è completa), per ogni gruppo passano almeno ∞^R S_{p-2}, ove $R = p-n+r-1$. Chiamando <u>speciale</u> un gruppo di n elem.ⁱ quando fa parte di una serie lineare speciale di dim. >0, abbiamo che la posiz. di tali elem.ⁱ è veramente speciale. Mentre n elem.ⁱ qualunque, quando $n>p$, formano sempre un gruppo di una serie di dim >0 (nota a pag. 92), sono sempre gl' infiniti (o gli zeri) di funzioni razionali dell' ente, per $n \leq p$ ciò non accade più in generale, ed n elem.ⁱ qualunque stanno solo nella g_n^0 da essi costituita, non sono più gl' infiniti di alcuna funz.

raz.le di grado n dell'ente. Perchè essi sian tali, stiano in una g_n^r ove $r>0$, occorre che gli n p.i imagini sulla curva canonica C stiano in un S_{n-r-1} (anzi che in un S_{n-1}), stiano in ∞^R gruppi della g_{2p-2}^{p-1}, anzi che in soli ∞^{p-n-1}; presentino cioè <u>al più</u> n − r condizioni, anzi che n, a quei gruppi, cioè sulla curva piana d'ord. m alle agg.te d'ord m − 3. In altri termini degli n p.i di un gruppo speciale, quando esso faccia parte di una g_n^r speciale non data, si posson prendere ad arbitrio <u>al più</u> n − r punti, giacchè l'S_{n-r-1} congiungente di questi dovrà contenere gli altri. – Questo teor. per r = 1 si trova in sostanza nel n. 5 della memoria di Riemann, e per r qual. nella Nota di Roch (Crelle 64, 1864) « Ueber die Anzahl der willkürlichen Constanten in algebraischen Functionen»: lo completeremo in seguito precisandolo meglio sì da ottenere anche il risultato di Roch enunciato in quel titolo (Nel confronto si osservi che Riemann e Roch rappresentando l'ente con $F(\overset{n}{s},\overset{m}{z})=0$ hanno per agg^{te} le $\varphi(\overset{n-2}{s},\overset{m-2}{z})$).

Intanto il teor. ci dà già la ragione del nome di serie <u>speciali</u>. Esso abbraccia tutti i risultati precedenti sulle serie speciali. Così, poichè per un gruppo di una g_n^r speciale si posson prendere al più $n-r$ elem^{i}, mentre quando è data se ne posson prendere r, segue $n-r \geq r$, $n \geq 2r$ (pag. 118). Osserviamo poi che se si prendono appunto $n-r$ p^{i} ad arbitrio su C l'S_{n-r-1} che li congiunge non incontra più in generale in r punti C, se si toglie il caso che esso sia un iperpiano S_{p-2}, cioè $n-r=p-1$; oppure che C sia doppia, cioè l'ente iperellittico.*) Tolto dunque questo

*) Cfr. Noether, Raumcurven teor. III" alla fine di p. 10

caso e supposto $n-r<p-1$, sono meno di $n-r$ gli elem.ⁱ arbitrari di un gruppo di una g_n^r speciale non data, e si ha quindi $n>2r$. Segue che sopra un ente che non sia iperell.° la sola g_n^r speciale con $n=2r$ è la g_{2p-2}^{p-1}.

Terminiamo questo Cap.° con un'applicazione delle proprietà viste delle serie lineari alle rigate speciali (*) Abbiamo visto (pag. 116) che una rigata d'ord. n gen p non speciale ha S_{n-2p+1} per spazio normale; mentre una speciale ha per normale uno spazio superiore. Se $n>2p-2$ si può andare solo fino ad S_{n-p+1} (pag. 120).

Dico anzitutto che la rigata F di gen. $p>0$ e d'ord. $n\geq 2p$ di S_{n-p+1} è un cono.

(*) V. specialmente il mio lavoro sulle curve e rigate algebriche, 2ª P.te (Math. Ann. t. 34).

Poichè un iperpiano per una gener. dovrà contenere altre gener., altrimenti la γ^{n-1} d'inters. residua starebbe in un S_{n-p-1} (anche se si riducesse ad una curva multipla) e gl'iperpiani per questo mostrano che F sarebbe razionale. Projettando F su S_3 comunque ogni piano per una gener. ne conterrebbe altre, cioè le gener. si tagliano a due a due: cono. Ecc. Del resto più tardi ritroveremo questa proposiz. con minori restrizioni. (pag. 133)

Sia ora F una rigata irriduttibile non conica di $S_{n-p-i+1}$, ove $1 \leq i \leq p-1$. Un iperpiano per $i+1$ gener. — che si può sempre tirare se (1) $n \geq p+3i+1$, taglia ancora F in una γ^{n-i-1}, che non può stare in un $S_{n-p-i-1}$, altrimenti gl'iperpiani per questo darebbero su F una g^1_{i+1} di cui farebbe parte il gruppo di $i+1$ gener.

127

che s'era scelto ad arbitrio, il che contraddirebbe al teor. di Riemann. Quindi supposto che essa si spezzi in una γ^m direttrice irriduttibile, semplice o multipla, ed in $n-m-i-1 (\geq 0)$ gener.^{i}, dovrà lo spazio S_h a cui appartiene γ^m essere di dim. $h > m-p$; altrimenti con $n-m-i-1$ gener. qual. darebbe un $S_{n+h-m-i-1}$ e quindi un $S_{n-p-i-1}$ con una γ^{n-i-1}. Segue che nella ipotesi (1) vi è certo su F una <u>curva direttrice speciale</u>.

Si posson assegnare dei limiti superiori ad m ed h considerando la $g_{n-m}^{n-p-i-h}$ che su F è segata dagl' iperpiani per S_h, e supponendo che quella serie <u>non</u> sia speciale. Basta perciò imporre ad es.° la condizione $n-m > 2p-2$, cioè (2) $n \geq 2p+m-1$. Allora sarà $(n-m)-(n-p-i-h) \geq p$, ossia $m \leq h+i$. Questa poi con la $2h \leq m$

128

che segue dalla g^h_m rappres.^{a} da γ^m trae $h \leq i$. Abbiamo dunque nelle ipotesi (1), (2):

$$(3) \quad h \leq i, \quad 2h \leq m \leq h+i.$$

Per $h = i$ si ha $m = 2h$. In ogni caso se $m = 2h$ (v. pag. 124, 125) la γ^m si riduce ad una γ^h razionale normale doppia e la rigata è iperellittica: tolto solo il caso di $h = p-1$ e quindi anche $i = p-1$, cioè rigata di S_{n-2p+2} chè allora la γ^m può anche essere una curva canonica non degenere.

Poichè $m \leq 2p-2$, la condiz. (2) è certo soddisfatta se

$$n \geq 4p-3;$$

ma volendo abbracciare anche la (1), contemplando anche il caso estremo $i = p-1$, si deve porre

$$(4) \quad n \geq 4p-2.$$

Con questa sola ipotesi varranno le (3) ecc. In particolare, rigate di S_{n-p} $(p>1)$, di

129

S_{n-p-1} $(p>2)$, di S_{n-p-2} $(p>3)$. (v. loc. cit°)

Cap. 7°
Serie complete. Serie residue. Curve aggiunte. Applicazioni

Come nel Cap° preced. abbiamo derivato tutto da un teor. fondam.: la formola di Zeuthen, o la mia formola; così nel presente Cap° sarà fondam^{le} il teor. segu.

Se due serie l^i d'ord. n su un ente algebrico hanno un gruppo comune, esse son contenute in una stessa serie l^e d'ord. n.

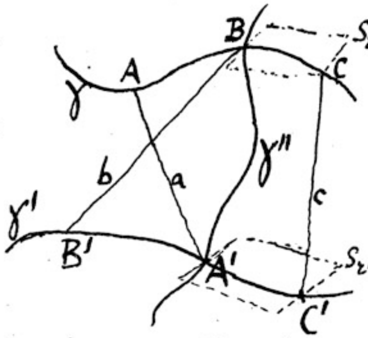
Per dimostrarlo consider. anzitutto il caso di 2 serie non degeneri g_n^r, $g_n^{r'}$. Saran

130

rappres^e da una γ^n di S_r e una γ'^n di $S_{r'}$, e questi 2 spazi li supporremo indipi', cioè congiunti da $S_{r+r'+1}$. Condotto per gli S_{r-1}, $S_{r'-1}$ conten' i due gruppi omologhi di γ, γ', un iperpiano, questo sega la rigata d'ord. $2n$ delle congiungenti i p' omologhi di γ, γ' in n gener. ed in una curva direttr. irriduttib. γ'' d'ord. n. Sulla rigata le due serie g_n^r, $g_n^{r'}$ son date dagl' iperpiani passanti per $S_{r'}$, S_r, e su γ'' da sez' con iperpiani, onde quelle serie saran contenute in quella rappres^a da γ''.

Se le due serie g_n^r, $g_n^{r'}$ hanno degli elem' fissi comuni, basta dimostrare il teor. nelle serie che rimangono togliendo questi elem. Possiamo dunque supporre che i k elem' fissi della g_n^r sian diversi dai k' elem' fissi della $g_n^{r'}$. Queste due serie saran rappres. da curve γ, γ' di S_r, $S_{r'}$ d'ord. $n-k$, $n-k'$ con k

131



p^i fissi A su γ, e k' p^i fissi B' su γ': gli omologhi A' degli A su γ' saran distinti dai B' e così gli omol^i B su γ ai B' saran distinti dagli A; e l'ipotesi che vi sia un gruppo comune alle due serie significa che un S_{r-1} taglia γ oltre che nei k' p^i B in $n-k-k'$ (≥ 0) p^i C, ed un $S_{r'-1}$ taglia γ' oltre che nei k p^i A' in $n-k-k'$ p^i C' sì che i C ed i C' si corrispondono. Sulla rigata d'ord. $2n-k-k'$ generata da γ, γ' le due serie son date: dalle gener. a fisse insieme colle sez. fatte dagl'iperpian. per $S_{r'}$, e dalle b fisse colle sez... Un iperpiano pei due S_{r-1}, $S_{r'-1}$ sega quella rigata nelle $n-k-k'$ gener. c ed in una γ''^{n} irriduttib. E su questa quelle due serie saran segati appunto da quegl'iperpiani, poichè essa contiene i p.ti A' e B.

Il teor. fondam.le si completa subito aggiungendo (pag. 10 e 23) che se una g_n^r ed una $g_n^{r'}$ hanno comune precisamente una g_n^i $(i \geq 0)$, esse stanno in una $g_n^{r+r'-i}$, e non in una serie minore.

Ma per le nostre applicaz.i basta il teor. fondam.le. Esso prova che se due g_n hanno 1 gruppo comune, o l'una sta nell'altra, oppure entrambe stanno in una superiore. Questo 2° caso non si può presentare se l'una almeno delle due g_n è <u>completa</u>: l'altra vi starà.

Dunque: due g_n complete aventi 1 gruppo comune coincidono. Da un gruppo di n elem.i qualunque è individuata la g_n completa che lo contiene (e così una g_n qual. sta in una sola g_n completa): che può ridursi ad una g_n^0 (e vi si ridurrà certo se

133

$n \leq p$ e gli n elem.^{i} si assumono ad arbitrio).
— Quindi se due curve normali d'ord. n sono in corrisp. univoca sì che ad un particolare gruppo di n p.^{i} dell'una, sez. con un iperpiano, corrisponda un analogo gruppo nell'altra, la corrisp.^{a} è una collineazione. —
Segue che una rigata d'ord n e gen. $p > 0$ a sezioni spaziali non speciali (per es. di ordine $n \geq 2p-1$) appartenente ad S_{n-p+1} è un cono (cfr. pag. 125); poichè due sezioni spaziali saranno in collineaz. con n p.^{i} uniti su un S_{n-p-1} ed $n > n-p$, quindi omologia.
Se una g_n^r è completa, è tale anche la g_{n-h}^ρ $(\rho \geq r-h)$ dei gruppi residui di h elem.^{i} fissi risp. ad essa. Poichè se questa g_{n-h}^ρ stesse in una $g_{n-h}^{\rho'}$, questa cogli h elem.^{i} fissi darebbe una $g_n^{\rho'}$ avente comune colla g_n^r completa essa g_n^ρ, e però contenuta nella g_n^r

134

e coincidente quindi colla g^ρ_n. — Quest'osservazione ha per conseguenza che projettando una curva normale da h suoi punti qualunque si ottiene ancora una curva normale.

— <u>Applicazione alle curve agg^{te} di una γ^m piana</u>. Le agg^{te} d'ord. $m-3+\alpha > m-3$ segano una $g_{m\alpha+2p-2}$ di dim. $\geq m\alpha+p-2$ (pag. 56); poichè l'ordine è $> 2p-2$, la serie non è speciale e la differenza fra l'ordine e la dim. sarà $\geq p$. Segue che la dim. è precisamente $m\alpha+p-2$, cioè (loc. cit.) i passaggi pei p^i multipli di γ impongono condiz^i tutte indip^i a quelle agg^{te}; ed inoltre quella serie è completa. È pure completa (pag. 118) quella segata dalle agg^{te} d'ord. $m-3$. E vedremo subito anche quella segata dalle agg^{te} d'ord. minore. — È dunque (pel teor. preced., pag 133, 134) completa la serie segata su γ da tutte le agg^{te} d'ord ν

assoggettate a passare per p^i semplici dati ad arbitrio su γ.

Poichè le agg^{te} d'ord. $m-1$ segano su γ una serie completa, lo stesso sarà per le agg^{te} d'ord $m-1-a$ (se esistono): poichè una curva d'ord. a non passante per p^i multipli di γ le completa in agg^{te} d'ord. $m-1$, sicchè la serie da esse segata è completata dagli am p^i d'inters. di γ con quella curva ausiliaria in serie data da φ^{m-1} agg^{te}; e viceversa ogni φ^{m-1} agg^{ta} passante per quegli am p^i d'incontro conterrà tutta la curva d'ord. a... dunque la serie data dalle φ^{m-1-a}, resto degli am p^i risp. a quella completa data dalle φ^{m-1}, sarà pure completa.

Dal fatto che le agg^{te} di un ord. qual. passanti per dati p^i di γ vi segano, fuori di questi e dei p^i multipli, una serie completa,

136

segue un modo semplice di costruire la serie completa che è individuata su γ da un dato gruppo qual. G_n. Si conduca per questo una curva agg^{te} (il che si può sempre, prendendola d'ord. abbastanza elevato) e sia G_N il resto: le agg^{te} dello stesso ord. passanti per G_N daranno la g_n completa. Mutando quella prima curva agg^{te} muterà G_N (anche N se muta l'ord.); ma si avrà sempre la stessa g_n; si ha così (almeno in parte) il <u>teor. del resto</u> (Brill e Nöther, pag. 273): se due G_n son <u>resti</u> o <u>corresiduali</u> risp. ad un gruppo, son pure tali risp. ad ogni resto dell'uno dei due.

Se il sist. delle agg^{te} d'ord. ν sega su γ^m una serie completa, lo stesso non varrebbe pel sistema di tutte le curve d'ord. ν: si verifica subito che la condiz. più naturale che si presenti per render completa la serie è di as-

137

soggettare quelle curve ad avere in ogni p. s-plo di γ un p. $(s-1)$-plo: donde una nuova ragione dell'uso delle curve aggte. Ma anche se s'impone la stessa multiplicità di γ si ha, fuori dei p.i multipli, una serie completa: poichè per le φ aggte l'avere in un p. s-plo di γ un p. s-plo equivale al passaggio per gli s p.i di γ che cadono in quel p. s-plo, sicchè si può applicare il teor. di pag. 133.

Segue che in un <u>sist. lineare di ∞^k</u> γ^m <u>gen. p</u> <u>determinato dai p.i base</u>, su ogni curva le altre dànno una <u>serie caratteristica</u> g_n^{k-1} <u>completa</u>, onde $n-k+1 \leq p$ $k \geq n-p+1$; ed ha sempre luogo il segno $=$ se la serie non è speciale, ad es.o se $n > 2p-2$, oppure $k > p$; il segno di ineguaglianza se la serie è speciale.

138

In altri termini una sup. razionale d'ord. n a sez. spaziali di gen. p, la quale sia normale, ha per sezioni spaziali delle curve normali, ed appartiene ad S_{n-p+1}, o a spazi superi se quelle sez. sono speciali. Così per $p=0, 1$ si hanno delle F^n di S_{n+1} o di S_n; e poichè queste superf. son note (Del Pezzo), e son note le loro rappresentazioni piane, così se ne trae la riducibilità di tutti i sistemi lineari di genere 0 ed 1, tolti i fasci, a certi tipi.

— Serie residue. Data una g_ν^ρ completa, i resti o residui di un G_n rispetto ad essa formano una $g_{n'}^{r'}$ completa (pag. 133), ove $n+n'=\nu$. E similmente i residui di qualunque gruppo $G'_{n'}$ di questa formano una g_n^r completa: quella determinata da G_n. Segue che ogni gruppo

139

della g_n^r ed ogni gruppo della $g_{n'}^{r'}$, sono residui risp. alla g_ν^ρ: le due serie son residue risp. a questa; per ogni gruppo della g_n^r passano precisamente $\infty^{r'}$ gruppi della g_ν^ρ, ecc. Rappres.° la g_ν^ρ con una C^ν normale di S_ρ, abbiamo che su questa la g_n^r ha i suoi gruppi in spazi $S_{\rho-r'-1}$ e la $g_{n'}^{r'}$ in spazi $S_{\rho-r-1}$.

Applichiamo ciò in partic. alla serie canonica g_{2p-2}^{p-1} ed alle serie speciali residue (rispetto a questa); con che completeremo il teor. Riemann-Roch. Questo diceva che se un gruppo G_n determina una g_n^r completa speciale, sicchè $n-r \leq p-1$, per esso passano $\infty^{r'}$ gruppi della serie canonica sì che $r' \geq p-n+r-1$: vedremo che ha luogo il segno $=$ nell'ipotesi che la g_n^r sia

140

completa. Invero la $g^{r'}_{n'}$ residua, ove $n' = 2p-2-n$, avrà ogni gruppo in ∞^r gruppi della serie canonica, e però sarà similmente $r \geq p-n'+r'-1$; sicchè sommando con la disuguaglianza preced. si trae che deve aver luogo il segno $=$ in entrambe le relaz.ⁱ Così si precisa meglio il Teor. di R. e R.: se un gruppo G_n determina una serie completa di dim. r, e sta in $\infty^{r'}$ gruppi della serie canonica, si ha $r' = p-n+r-1$, ossia $r = n-p+1+r'$; od in altri termini, esso impone precisamente $n-r$ condiz.ⁱ ai gruppi della serie canonica che devon contenerlo. In pari tempo si ha il teorema di reciprocità con cui Brill e Nöther completano il teor. di Riemann e Roch: se due gruppi G_n, $G'_{n'}$ son residui risp. alla serie canonica (sicchè $n+n'=2p-2$)

141

e per G' passano ∞^r gruppi della serie canonica, i quali così dànno come resti di G' la g_n^r completa che contiene G, e similmente per G passano $\infty^{r'}$ gruppi della serie canonica …, si ha $r' = p - n + r - 1$, ecc.; o raddoppiando
$$n' - n = 2(r' - r).$$

Veramente Riemann e Roch non parlano della g_n^r completa determinata da G_n: così considerano il <u>numero delle costanti da cui dipende una funz. raz^le dell'ente</u> che abbia tutti i suoi infiniti (semplici) fra gli elementi di G_n. Se la g_n^r completa è data sull'ente da: $\lambda_0 \varphi_0 + \lambda_1 \varphi_1 + \dots + \lambda_r \varphi_r = 0$, ed il gruppo G_n da $\varphi_0 = 0$, quelle funz^i raz^li dell'ente si avranno dividendo quella forma generale per φ_0, cioè saranno $\lambda_0 + \lambda_1 \frac{\varphi_1}{\varphi_0} + \dots + \lambda_r \frac{\varphi_r}{\varphi_0}$ sicchè contengono <u>linearmente</u> $r+1$ costanti. Il Teor. di R. e R. si enuncia allora così: le

142

funz.i raz.li dell'ente i cui infiniti sono fra gli n elem.i di un dato gruppo G_n pel quale passano $\infty^{r'}$ gruppi della serie canonica, dipendono (linearm.) da $n-p+2+r'$ costanti. – Si noti che il teor. vale anche quando G_n non sta in alcun gruppo della serie canonica, se si mette $r'=-1$, poichè allora è $r=n-p$.

Rappres.o l'ente con la curva canonica o con una γ^m piana si hanno enunciati speciali ...

Applichiamo il teor. Riemann-Roch alla questione: se una γ^n_p piana data sia projez. di una C^n appartenente ad S_r: Se $n \geq p+r$, certamente, poichè la dim. di una ser. g_n completa è $\geq n-p$ e quindi $\geq r$; e la questione consiste appunto nel vedere se la g^2_n segata su γ dalle rette

143

del piano è contenuta in una g^r_n, cioè determina una serie completa di dim. $\geq r$.
Se $n < p + r$, la g^r_n, e quindi la g^2_n, sarà speciale. Per un G_n dato su γ da una retta qual. dovranno passare $\infty^{p-n+r-1}$ curve agg^te d'ord. $n-3$, cioè γ dovrà ammettere altrettante curve agg^te d'ord. $n-4$, e ciò è anche sufficiente. Ora contando le costanti si avrebbe che le agg^te d'ord. $n-4$ sono ∞^{p-n+1} in generale: nel caso attuale dunque essendo $\infty^{p-n+r-1}$ devono $r-2$ delle condiz.^i imposte alle curve agg^te d'ord. $n-4$ dai passaggi pei p.^i multipli di γ esser conseguenza delle rimanenti.

*

144

Cap. 8°
Il metodo algebrico di Brill e Nöther

Teor. algebrico fondamentale di Nöther*) Nel caso che a noi importa dice che perchè una curva piana F si possa rappresentare con $A\varphi + B\psi \equiv F$, ove φ e ψ son curve date tali che in ogni punto P comune, s-plo per φ, s'-plo per ψ cadano solo ss' intersez.i, cioè non vi sian tang.i com.i, basta che F abbia ivi la multiplicità $s+s'-1$; ed allora A e B avranno la multipl.à $s'-1$, $s-1$.

Ciò posto, sulla f_m piana abbiasi una g^r_n data dal sist. lineare $\psi=0$ di curve qualun-

*) Ueber einen Satz aus der Theorie der algebraischen Functionen, Math Ann. VI p. 351; v. anche (per dimostrazioni algebriche) Voss Math Ann. 27, Stickelberger e Nöther Math. Ann XXX.

145

que e pel gruppo G_n dato da $\psi_0 = 0$ si tiri una curva agg^{ta} $\varphi_0 = 0$ d'ord. ν. Si avrà $\varphi_0 \psi \equiv Af + \psi_0 \varphi$, poichè $\varphi_0 \psi$ in un p. comune ad $f = 0$, $\psi_0 = 0$ che sia risp. s-plo, s'-plo, ha multipl^{à} $s + s' - 1$. Inoltre se le tg^{i} in quei p^{i} a f e ψ_0 sono distinte, φ vi avrà multipl^{à} $s-1$ cioè sarà agg^{ta}. Dunque ogni gruppo della g_n^r cioè dato da $f = 0$, $\psi = 0$ fuori di $\psi_0 = 0$ dà $\varphi = 0$: la g_n^r è segata da un sist. lin. di curve agg^{te} d'ord. ν. Un esame speciale completerebbe il caso che la g_n^r abbia p^{i} fissi. — Se il sistema primitivo ψ era già di curve agg^{te}, lo si sarà sostituito con un altro φ (e supponendo che ψ_0 e ψ fossero due curve agg^{te}, anche di ord^{i} diversi, si ottiene il Restsatz sotto la sua forma più generale; v. Brill e Nöther pag. 273). — In conclusione tirata per G la φ_0 agg^{ta} d'ord. ν che dia un

146

resto Γ, la g_n^r sta in quella segata dalle φ^ν agg^{te} per Γ. Si noti che il carattere di curva agg^{te} ci servì nella dimostraz.

Segue che se la g_n^r è completa essa è segata da tutte le φ^ν agg^{te} per Γ, e viceversa. Una serie completa è individuata da un suo gruppo G. Due serie con un gruppo comune stanno nella serie completa determinata da questo, ecc. (*)

Le serie complete essendo così ridotte a curve agg^{te} d'ord. ν... ricordiamo (pag. 55, 56) che si ottiene $n-r \leq p$, e se $\nu = m-3$ una g_{2p-2} di dim. $\geq p-1$, la quale dà delle g_n^r per cui $n-r < p$. Il numero $n-r$ dice, quando è data la g_n^r, quanti sono i p^i determinati dai rimanenti, sicchè se

(*) Per quanto precede v. Nöther, M. A. 23 p. 348. E così per l'ordinamento di quanto segue si tien conto, oltre che della mem. Brill e Nöther, di un passo di una lettera di Nöther a Bertini (26 Marzo 1889)

147

la serie è completa sono al più p. — Vedremo che se $n-r<p$ la serie è sempre segata da φ^{m-3} agg^{te} (supposto, a partir <u>di qui</u>, f <u>irriduttibile</u>)

Premettiamo il <u>teor. di riduzione</u> (Reductionssatz) di Nöther (*): Se un gruppo G_n di una g_n^r completa sta su φ^{m-3} agg^{te} ed il p. P non è su tutte queste φ^{m-3} passanti per G, la g_{n+1} completa determinata da G_n+P ha P per p. <u>fisso</u>, ossia non è altro che la g_{n+1}^r ottenuta aggiungendo ad ogni gruppo della g_n^r il p. P. — Invero per determinare la g_{n+1} completa contenente G_n+P conduciamo per questo una φ^{m-2} agg^{ta}, composta di una φ^{m-3} per G_n e di una retta <u>r</u> per P: il resto Γ non conterrà P, ma gli $m-1$ p^i residui d'inters di f con r,..; le φ^{m-2} agg^{te} per Γ cioè per

(*) Crelle's J. 97 p. 224, M. A. 37 p. 424; e la lettera citata a Bertini.

148

quegli $m-1$ p^i, ... si spezzeranno in r e φ^{m-3} e però conterranno tutte P.

Se $n < p$, per n p^i passano certo delle φ^{m-3}. Per dimostrare che se $n-r < p$, per ogni gruppo di una g_n^r passano delle φ_{m-3} basterà dunque provare, che se ciò è vero, è pur vero pei gruppi di una g_{n+1}^{r+1}. Invero se P è un p. non fisso per questa, i suoi residui formano una g_n^r; per un G_n di questa passano per ipotesi delle φ^{m-3}: e queste conterranno anche P, altrimenti pel teor. di riduzione la serie completa g_{n+1} determ. da $G_n + P$ avrebbe P fisso, mentre la g_{n+1}^{r+1} non ha P fisso. Dunque per 1 gruppo $G_n + P$ della g_{n+1}^{r+1} passano delle φ^{m-3}.

Così il concetto di G_n o di g_n^r dati da φ^{m-3} agg^{te} coincide con quello di gruppi o serie per le cui g_n^r <u>complete</u> è $n-r < p$. Li diciamo

149

gruppi o serie <u>speciali</u>. Sarà dunque $n \leq 2p-2$. Inoltre $r \leq p-1$, cioè le φ^{m-3} non possono essere più di ∞^{p-1}, altrimenti darebbero una g^p_{2p-2} a cui aggiungendo 1 punto si avrebbe una g^p_{2p-1} speciale poichè $n-r<p$, e sarebbe $n>2p-2$, assurdo. Di più nessun punto A può esser fisso per la g^{p-1}_{2p-2} data dalle φ^{m-3}, altrimenti astraendo da esso si avrebbe una g^{p-1}_{2p-3} ed aggiungendo a questa 1 punto P diverso da A una nuova g^{p-1}_{2p-2}, ed i gruppi di questa dovrebbero stare su φ^{m-3} che così avrebbero su f $2p-1$ punti, assurdo.

Le serie complete d'ord. n non speciali, ad es° quelle per cui $n>2p-2$ sono di dim. $n-p$. Questo basterebbe per provare che se due curve di gen' p, p' sono in corrisp. univoca si ha $p=p'$: considerando cioè due g_n complete omologhe con $n>2p-2$, $2p'-2$.

150

Dal teor. di riduzione segue subito il teor. di Riemann e Roch. Supposto il gruppo G_n speciale, cioè sito (precisamente) in $\infty^{r'}$ φ, o determinante una serie completa g_n^r (ove $n-r<p$), si vuole una relazione fra n, r, r'. Presi successivam. i punti $P_1, P_2, \ldots P_{r'+1}$ in guisa da rappresentare altrettante condiz.^{i} distinte per le φ che contengono G_n, il teor. di riduz.^{e} dice che la serie completa determ.^{a} da G_n+P_1, poi quella determ. da $G_n+P_1+P_2$, ... ed infine quella determ. da $G_n+P_1+\ldots+P_{r'+1}$ sono ∞^r (la g_n^r con aggiunti i p.^{i} P). Ma quest'ultima non è speciale, poichè per quel suo gruppo non passa più alcuna φ. Dunque $r = n-p+1+r'$: che è appunto il teor. Riemann-Roch. — Da esso si passa subito al teor. di reciprocità di Brill e Nöther, poichè le φ per G_n daranno la $g_{n'}^{r'}$ residua

151

della g_n^r. — Inoltre poichè a determinare un gruppo nella g_n^r si posson prendere ad arbitrio r punti, e poi altri r' punti arbitrari si posson prendere per una φ contenente quel gruppo, segue $r + r' \leq p-1$, e sommando con la relaz. preced.: $2r \leq n$, cioè il teor. di pag. 118; dal quale s'è pure dedotto che le φ^{m-3} danno una g_{2p-2}^{p-1} senza p.i fissi.

Il teor. di riduzione di Nöther ed il suo inverso, che trarremo subito dal teor. Riemann-Roch, stabiliscono le condizioni perchè una g_{n+1}^r con un punto fisso P sia completa. Osserviamo che la g_n^r residua di P dev'essere completa. È speciale, perchè se non fosse speciale sarebbe $n-r=p$, e quindi $(n+1)-r > p$, sicchè la g_{n+1}^r non sarebbe certo completa. — Tutto si riduce dun

152

questo a vedere quando è che aggiungendo ad una g_n^r speciale completa un punto P si ha una g_{n+1}^r completa. Il teor. di riduz.e dice che basta che P non stia su tutte le φ passanti per un gruppo G_n della g_n^r, cioè non sia un p. fisso della serie residua di questa. Viceversa questa condiz. è necessaria, perchè se P stesse su tutte quelle φ, nella formola $r = n - p + 1 + r'$, dal gruppo G_n al $G_n + P$ non muterebbe r', mentre n aumenterebbe di 1, sicchè anche la dimens.e da r dovrebbe diventare $r+1$ per la g_{n+1} completa determinata da $G_n + P$.

Rappresentando la g_n^r con una C^n di S_r, si ha che la g_{n+1}^r è completa cioè non contenuta in una g_{n+1}^{r+1} se la C^n non è projez. di una C^{n+1} da 1 suo p.o. Ed abbiamo

153

che una C^n normale <u>non</u> speciale è sempre proj. di una C^{n+1}. Ma se la C^n è normale speciale, essa si può ottenere come proj. di una C^{n+1} di S_{r+1} solo quando la residua della g_n^r risp. alla serie canonica ha almeno 1 p° P fisso (e questo p° sarà imag^e del centro di proj. della C^{n+1}).

Il teor. di riduz^e con l' inverso si può enunciare sotto forma analitica così (v. Nöther nel Crelle 97): affinchè le funz^i raz^{li} dell' ente algebr° i cui infiniti (semplici) sono fra gli $n+1$ elem^i $P, P_1, \dots P_n$ siano tutte finite in P è necessario e sufficiente che esista un gruppo della serie canonica (od una φ) che contenga $P_1, \dots, P_n$ ma non P (v. anche Math. Ann. 37 p. 424). Poichè quelle funz^i raz^{li} restan finite in P appunto quando

154

le g'_{n+1} che esse rappresentano e che son determinate dal gruppo $PP_1 \ldots P_n$ hanno P fisso.

Abbiansi n p.i di f formanti un G_n in cui non s'annulli alcuna φ. Siano anzitutto $P_1 \ldots P_\mu P_{\mu+1}$ tali che formino un gruppo $G_{\mu+1}$ di una $g'_{\mu+1}$, mentre μ qual. fra essi non formino ancora un gruppo di una serie infinita. S'indichino con $P_{\mu+2}, \ldots$ i punti di G_n per cui passano tutte le φ che contengono $G_{\mu+1}$. Poi con P_{μ_1+1} un nuovo p.° di G_n e con $P_{\mu_1+2}, \ldots$ i p.i per cui passano le φ che contengono $P_1 \ldots P_{\mu_1+1}$; ecc. Il teor. di riduzione con l'inverso mostrano che delle serie di gruppi di m p.i <u>mobili</u> ottenibili da $G_m = (P_1 \ldots P_m)$ ove $m = 1, 2, \ldots n$ mancheranno quelle corrisp.i ai valori seg.i di m: $1, 2 \ldots \mu, \mu_1+1, \mu_2+1, \ldots$, in tutto pre

155

cisamente p, cioè p saranno le funz.

ciSamente p, cioè p saranno le funz^i raz^{li} dell'ente ~~mancanti~~ fra quelle che hanno per infiniti i gruppi G_m. – Questo teorema è una estensione dovuta al Nöther (Crelle 97) di un teorema che Weierstrass dà nelle sue lezioni sulle funz^i abeliane (Lückensatz). Anzitutto in un cap^o sulle funz^i raz^{li} dell'ente che hanno un solo infinito (multiplo, s'intende) il Weierstrass mostra che se questo p^o non ha una posizione <u>particolare</u>, l'ordine d'infinità, cioè il grado della funz^e, dev'essere $>p$ e per ogni val. $>p$ esistono funz^i siffatte. Considerando il p. m-plo come la riunione di m p^i inf. vicini, ciò si può derivare dal teor. Riemann-Roch (v. la fine della pag. 122); bisogna cioè che il p^o che si vuol sia infinito m-plo per funz. raz^{li} di grado m, ove $m \leq p$, sia tale che le φ che vi hanno

156

contatto $(m-1)$-punto con f vengano in conseguenza ad avere contatto m-punto: il che non accade che per punti particolari (*). Ma considerando poi un p.° qualunque si ha (Lückensatz) che fra i numeri $1, 2, 3, \ldots$ ve ne sono precisamente p che non si possono assumere come gradi di funz.^{i} raz.^{li} aventi tutti gl'infiniti coincidenti in quel p.°. Questa proposiz. ha anche servito (qualche volta) al Weierstrass per definire il rango (genere) dell'ente algebr.°. E serve pure elegantemente per ottenere equaz.^{i} particolari dei vari enti di gen. p (v. le Lez.^{i} di Weierstrass).

(*) Per una dimostraz. dettagliata v. Nöther Crelle 92 p. 301

Cap. 9°
Rappres.ⁱ reali dell'ente algebr.°
Il metodo funzionale di Riemann

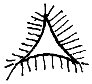
L'ente algebrico come una varietà ∞^2 (v. pag. 5) di elem.ⁱ; ad es.° la varietà delle coppie di numeri complessi (s, z) soddisfacienti alla $F(s, z) = 0$. Questa varietà è continua, chiusa e tale che da ogni elem.° si può andare in ∞ direz.ⁱ. Un'imagine reale si ha in una superficie (reale): si tratta di rappresentarlo effettivam.ᵉ su una superf. in guisa che la corrispondenza sia univoca e continua. Alle superf. si potranno dunque far subire (per ora) deformaz.ⁱ continue qual.; essendo equivalenti le superf. che sono in corrisp. univoca continua. Converrà che le sup. siano chiuse, come l'ente alg.°.

158

Per l'ente razionale si ha subito la rappres^e distendendo il parametro $z = x + iy$ da cui esso dipende sul piano di Argand e Gauss (con 1 sol p^o all'∞), o sulla sfera (di Riemann).

In generale per l'ente $F(\overset{n}{s}, \overset{m}{z}) = 0$, fissato un p. z_0 sul piano o sulla sfera, a cui corrispond. i valori $s_0' \dots s_0^{(n)}$ di s, si faccia muovere con continuità ma senza attraversare un taglio che passa per tutti i p^i di diram^e, o i tagli che congiungono 1 p. fisso a questi p^i diram^e; allora congiungendo i valori delle varie radici che si hanno dai due lati di ogni taglio si ottiene la superf. di Riemann n-pla, connessa o no secondo che l'ente è o no irriducibile. Si noti che così si mette in evidenza una g_n^1 qual. ad elem^i tutti mobili (v. pag. 95). Caso dell'ente iperell^o, $n = 2$.

Superf. di Klein: dei p^i reali delle tg^i

imag. della curva piana che rappresenta l'ente (Klein, Ueber eine neue Art der Riemann'schen Flächen, Math. Ann. VII e X): la connessione fra i vari piani sovrapposti si fa nei p.i reali della curva, lungo le tg.i reali isolate o di flesso (e nei p.i reali delle tg.i di flesso imag.e se la curva è imag.a). Esempi: la conica , da cui si ritorna alla sfera (ellissoide); le curve reali di 3a classe   che conducono all'anello.

Si posson avere direttam.e, senza l'astrazione delle sup. multiple, superf. semplici che rappresentino l'ente. Così prendendo per questo una curva imag. che non stia in un piano reale, le rette reali che ne contengono i vari p.i, cioè li congiungono ai coniugati, formano una congruenza reale che rappres.a l'ente (nel caso di una 2a imag.a di 2a sp. si ha

160

la rappres^e di Staudt) e che si può sostituire con la sup. luogo dei p^i medi fra le coppie di p^i coniugati. O si ricorre ad S_4 ed al cono che projetta da 1 r^a imag^a la curva: il luogo dei p^i reali dei vari piani generatori sarà una sup. imagine dell'ente.

Con tagli da una sup. chiusa si può passare ad una con orli o ad un aggregato di superf. Viceversa una tal sup. od un aggregato si considererà come una superf. chiusa, purchè sia fissata una corrispondenza univoca fra i punti degli orli, fra loro accoppiati in un determinato modo. (Esempio: un poligono piano curvilineo, un parallelogrammo che si muta in un anello, ecc.); così che si abbia ancora in abstracto una varietà chiusa. E se la superf. è unilatera o doppia (esempi, di Möbius ecc.), si può pure usare considerandola

161

come una sup. bilatera chiusa, purchè in ogni suo p° se ne considerino <u>due</u>; lo stesso artifizio si può usare per una sup. qualunque aperta, considerando gli orli come linee di passaggio dall'uno all'altro foglio (*)

Lo studio delle proprietà di una sup. che non mutano per trasformaz^i continue costituisce l'Analysis situs o Topologia (Leibniz, Riemann). In essa compajono i tagli <u>trasversi</u> (Querschnitte) e <u>chiusi</u> o rientranti (Rückkehrschnitte). Definiz^e di superf. connessa; semplicem. connessa. In una superf. chiusa (connessa) si posson fare un certo numero N di tagli, chiuso il 1° e trasversi gli altri, che non la spezzano ma la <u>rendono semplicem. connessa</u>, mentre N+1 tagli

(*) V., anche per altre cose del seguito, Klein Ueber Riemann's Theorie der alg. Functionen und ihrer Integrale; e i Neue Beiträge … nei Math. Ann. XXI.

162

la spezzerebbero. Il num. (fondam^le^) N è pari, = 2p ove p è il massimo numero di tagli chiusi che non spezzano la superf. e dicesi genere di questa. La sup. secondo Riemann è N+1 = 2p+1 volte connessa (mentre Klein e Neumann dicono N = 2p la connessione). Si dimostra, con pure consideraz^i^ di posiz^e^ (v. Neumann p. 168-171), che per una sup. sferica di Riemann d'ord. n con ν p^i^ di diram., o ν per somma degli ord^i^ di questi p^i^ si ha per quel num. N = 2p di tagli ν − 2n + 2 : dunque p è il genere dell'ente algebr^o^. Per trasformaz^i^ univoche continue non muta il genere di una sup. : ne segue di nuovo che 2 enti algebr^i^ in corrisp. univoca hanno lo stesso genere. Anzi con analoghe consideraz^i^ di Analysis situs si dimostra che se 2 sup. chiuse di gen. p, p' in corrisp. univoca continua (x, x') hanno y, y' punti uniti.., si ha $y-y'=2x(p'-1)-2x'(p-1)$ (*) ; e se ne trae in particolare la formola di Zeuthen per una corrisp. fra 2 enti algebrici.

L'uguaglianza del genere non è solo necessaria, ma anche sufficiente perchè due sup. chiuse si possan riferire univocam. con continuità (**). Ne segue che a rappresentare un ente alg° di gen. p si posson assumere certe sup. normali di gen. p, ad es° una sfera con p manichi (per $p=1$ l'anello), come fa Klein (op. cit.).

Funzioni complesse del punto sulla sup. Dato un sistema qual. di parami o coord. curvilinee

(*) V. De Paolis, Teoria dei gruppi geometrici e delle corrispondenze ecc. Mem. socª XL, t. VII ser. 3ª 1890

(**) V. ivi, e Möbius, Theorie der elementaren Verwandtschaft (Werke, B. II); Jordan, Sur la déformation des surfaces, J. d. Liouville, 11_2 (1866)

164

p, q sulla sup., non conviene prendere per funz^i compl. del p. ogni funz. di p, q; ma conviene far sì che due qual. funz^i siano funzioni l'una dell'altra nel senso di Riemann (funz^i monogene di Cauchy). Ciò si ottiene se, assunta ad arbitrio una $z = x + iy$ come funz. compl. del punto, si assume poi come tale ogni funz. $W = u + iV$ di z, cioè ogni funz. delle 2 variab^i reali x, y tale che

$$(1) \qquad i\frac{\partial W}{\partial x} = \frac{\partial W}{\partial y} \qquad \text{ossia} \quad \left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2 = 0$$

cioè (2) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. Ora queste condiz^i si posson anche esprimere in altro modo; poichè si sa che esse equivalgono a dire che $\frac{du + idv}{dx + idy}$ non dipende da dx, dy, è una funz. di x, y; e quindi anche $\frac{du - idv}{dx - idy}$; e moltiplicando si ha

3) $du^2 + dv^2 = E\,(dx^2 + dy^2)$ ove E è funz. di x, y [si trova $E = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$], la qual identità equivale alle (2) purchè si aggiunga che $du + idv$

165

deve mutarsi (a meno di un fattore) in $dx + i\,dy$ e non in $dx - i\,dy$. Così allorquando si assume una certa $z = x + iy$ come funz. complessa del punto sulla sup., le altre devono esser quelle $w = u + iv$ tali che la equaz. $dx^2 + dy^2 = 0$ si muti (con la detta restrizione) nella $du^2 + dv^2 = 0$, cioè non cambi.

Sia ora data ad arbitrio nelle coord. curvilinee p, q una forma differenz.le quadratica <u>definita</u>
$$ds^2 = E\,dp^2 + 2F\,dp\,dq + G\,dq^2$$
di discriminante $H^2 = EG - F^2 > 0$. Avremo, spezzandola in due fattori lineari, complessi coniugati:
$$ds^2 \equiv (A\,dp + B\,dq)(\bar{A}\,dp + \bar{B}\,dq)$$
(ove il segno $\equiv$ indica uguaglianza, a meno di un fattore, funz. di p, q); e supposto $\lambda(p, q)$ un fattore integrante di $A\,dp + B\,dq$, sicchè
$$\lambda(A\,dp + B\,dq) = dx + i\,dy$$
e quindi $\bar{\lambda}(\bar{A}\,dp + \bar{B}\,dq) = dx - i\,dy$,

166

sarà $ds^2 \equiv (dx+idy)(dx-idy) \equiv dx^2+dy^2$.
Se dunque si assume $z = x+iy$ come funz. complessa del luogo, sarà la forma $ds^2 = 0$ data, quella che da tutte le funz.i complesse del luogo sarà ridotta alla forma du^2+dv^2.
Fissato il sistema di coord p, q, per ogni forma differ. (o meglio equaz. differ.le) $ds^2 = 0$, si ha un sistema di funz.i complesse del luogo. Si hanno subito le condiz.i caratteristiche di queste mediante E, F, G. Pongasi

$$\Delta_1 u = \frac{1}{H^2}\left\{E\left(\frac{\partial u}{\partial q}\right)^2 - 2F\frac{\partial u}{\partial p}\frac{\partial u}{\partial q} + G\left(\frac{\partial u}{\partial p}\right)^2\right\}$$

$$\Delta_2 u = \frac{1}{H}\left\{\frac{\partial}{\partial p}\frac{1}{H}\left(G\frac{\partial u}{\partial p} - F\frac{\partial u}{\partial q}\right) + \frac{\partial}{\partial q}\frac{1}{H}\left(E\frac{\partial u}{\partial q} - F\frac{\partial u}{\partial p}\right)\right\};$$

quando la forma ds^2 si riduce a dx^2+dy^2 si ha $\Delta_1 u = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$, $\Delta_2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$;
e nel caso che ds indichi l'elem.° lineare essi sono quelli che Lamé chiamò param.i differenz.li di 1° e 2° ord. della funz.° u (v. ad. es.° le Leçons sur les coord. curvil.; veramente Lamé

167

chiama param. differ^le di 1° ord. $\sqrt{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2}$

Beltrami (*) conserva questi nomi alle espress^i che corrispondono all'espressione generale dell'elem^o lineare ds e dimostra che queste $\Delta_1 u$, $\Delta_2 u$ (ed un param. differ^le intermedio, relativo a due funz^i) sono invariab^li per trasformaz^i di coord., cioè che se si passa ad altre coord. p', q', sicchè $ds^2 = E'dp'^2 + \ldots$, le $\Delta_1 u$, $\Delta_2 u$ diventano uguali alle analoghe espressioni fatte con p', q', $E', \ldots$ Ciò vale all'infuori del significato speciale geometrico che egli dà al ds^2. Ne segue dunque che l'equ. caratteristica (1) (pag. 164) di una funz. W del luogo diventa $\Delta_1 W = 0$; e le equaz^i differ^li $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ a cui

(*) Ricerche di Analisi applicata alla Geometria, Giornale di mat t. II (1864) e III (1865); e più specialmente Delle variabili complesse sopra una superficie qualunque, Annali di mat, ser. 2 t. I (1867–8)

168

soddisfano u e v diventano $\Delta_2 u = 0$, $\Delta_2 v = 0$ (onde $\Delta_2 w = 0$). Si trova poi che le (2) (pag. 164) diventano

$$(2') \qquad \frac{\partial v}{\partial p} = \frac{1}{H}\left(F\frac{\partial u}{\partial p} - E\frac{\partial u}{\partial q}\right)$$
$$\frac{\partial v}{\partial q} = \frac{1}{H}\left(G\frac{\partial u}{\partial p} - F\frac{\partial u}{\partial q}\right),$$

e queste caratterizzano la $w = u + iv$ (e sono compatibili se $\Delta_2 u = 0$ ossia $\Delta_2 v = 0$). Due funz^i qual. w soddisfacienti a quelle equ^i son funz^i l'una dell'altra, e viceversa; ecc..

La forma $ds^2 \equiv E dp^2 + 2F dp\, dq + G dq^2$ uguagliata a 0, è rappresentata geometricam. da coppie di direz^i imag. uscenti da ogni p^o della sup., e quindi da un sist. ∞ di 2^o grado (o da due sist^i di 1^o) di linee imag^e, integrali dell'equ. $ds^2 = 0$. Se $u + iv$ è funz. complessa risp. a quella ds^2, poichè sarà $ds^2 \equiv du^2 + dv^2$, le direz^i $du = 0$, $dv = 0$, ossia $u = \text{cost}$, $v = \text{cost}$. saranno coniugate armoniche risp. a $du \pm i dv = 0$, cioè a quelle due.

169

E variando la funz. $u+iv$ che si sceglie, le direz.i delle $u=\text{cost}$, $v=\text{cost}$. passanti per i dati p.° rimarranno sempre coniug. arm. risp. a quelle formeranno un' invol. I sistemi $u=\text{cost}$, $v=\text{cost}$ saranno ortogonali se ds è l'elem. lineare; saranno coniugati se $ds^2=0$ è l'equaz. delle asintotiche. Quando ds è l'elem. lineare, il sist.a $u=\text{cost}$. per cui $\Delta_2 u=0$ dicesi di curve isoterme. Esso col suo ortogonale rende $ds^2=\lambda(Udu^2+Vdv^2)$, ove U e V sono risp. funz.i di u e v. Ma sostituendo ad u e v delle loro funz.i, il che non muta i due sistemi ortog.li di curve isoterme si rende subito $ds^2=\lambda(du^2+dv^2)$; ed allora i param.i u, v si dicono isometrici (isometrico dicesi il sistema di curve $u=\text{cost}$, ed il suo ortog.le $v=\text{cost}$, perchè insieme con questo divide la sup. in

170

quadrati infinitesimi). Le equaz. (2') caratterizzano i parami isometrici ortogonali u, v. E l'artifizio usato a pag. 165 è quello noto per dedurre da una data espressione dell'elemento lineare di una sup. un sistema isotermo. — (Si passi a pag. 172)

[Queste cose si collegano strettamente colla fisica matematica. Abbiasi un fluido che scorre sulla superficie con un potenziale di velocità (Geschwindigkeitspotential secondo Helmholtz) u funz. del punto, ma non del tempo t, cioè un fluido tale che le componenti della sua veloc.^{a} in un punto qual. $\frac{dx}{dt}, \frac{dy}{dt}$, sian date da $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$. In un dato intervallo di tempo dt l'incremento di fluido nell'area $dxdy$ sarà dato da $\Delta_2 u \cdot dxdy \cdot dt$, sicchè affinchè esso sia sempre nullo occorre

171

e basta che $\Delta_2 u = 0$. Quest'equaz. caratterizza dunque il fluido <u>incompressibile</u>, e coll'indipendenza di u da t il flusso <u>staziona-rio</u>. Così nel caso della propagazione del calore u è la temperatura, e si ha l'equilibrio calorifico; mentre nel caso del fluido elettrico u è il potenziale elettrostatico, e si ha un flusso elettrico stazionario. In ogni caso le curve per cui $u = \text{cost}$. sono <u>curve di livello</u>, o curve isoterme, ecc.; mentre quelle $v = \text{cost}$. (ad esse ortog^{li}) sono le <u>linee di flusso</u>. E passando su una di queste, cioè sulla normale n, da una curva di livello u alla $u + du$, siccome per significato geometrico del param° differ^{le} di 1° ord. si ha $\Delta_1 u = \left(\frac{du}{dn}\right)^2$, si conchiude che $\frac{du}{dn} = \sqrt{\Delta_1 u}$ è la <u>velocità</u> del fluido in quel punto.

172

Si hanno così dei mezzi fisici, ad es° l'elettricità, per generare delle funzi complesse su una superf. Si costruisce infatti il potenziale u, ed allora dalle equ.i (2') (pag. 168) è determinata v, e quindi $u+iv$, a meno di una cost.e additiva complessa.]

Se su 2 superf. si considerano come omologhi due p.i $(p.q)$, $(p'.q')$ quando due funz.i complesse risp. delle 2 sup. vi hanno lo stesso valore $u+iv$, funz.i relative risp. alle forme definite $ds^2 \equiv E dp^2 + 2F dp\, dq + G dq^2$, $ds'^2 \equiv E' dp'^2 + 2F' dp' dq' + G' dq'^2$, allora nei p.i omologhi sarà $ds^2 \equiv ds'^2$ $(\equiv du^2 + dv^2)$, le equaz.i differ.li $ds^2 = 0$, $ds'^2 = 0$ si muteranno l'una nell'altra, cioè si corrisponderanno i loro sist.i di curve integrali. E viceversa se si ha fra le 2 sup. una corrispondenza per cui $ds^2 \equiv ds'^2$, le funz.i complesse sull'una

173

risp. a ds^2 si muteranno in funz.ⁱ complesse dell'altra risp. a ds'^2. In partic.ᵉ su una stessa sup. si hanno ∞ corrisp.ᵉ risp. ad una data forma $ds^2=0$ mediante le ∞ funz.ⁱ complesse relative a queste. — Nella rappres.ᵉ di Gauss di $x+iy$ sul piano, la relativa forma differ.ˡᵉ quadratica $ds^2=dx^2+dy^2$ dà l'elemento lineare. In generale quando per ds, ds' si prendono gli elem.ⁱ lin.ⁱ delle superf., le rappresentazioni in cui essi si corrispondono (a meno di un fattore) diventano le rappres.ⁱ <u>conformi</u> (se invece $ds^2=0$ è l'equaz. delle asintotiche si hanno rappres.ⁱ pure notevoli.–) Abbiamo così il modo di riferire 2 superf. qual. in corrispondenza conforme od isogonale (*), in

(*) Questo modo coincide con quello dato da <u>Gauss</u> «Soluzione generale del problema: rappresentare le parti di una sup. data su un'altra sup. data in guisa che la rappresentaz. riesca nelle sue parti infinitesime simile alla figura rappresentata» (trad. da Beltrami, Annali di Mat.ᵗ t. IV, 1861)

174

partic. una sup. qual. ad un piano. Però in ciò bisogna limitarsi a parti delle superf. in cui quelle funz.ⁱ complesse del luogo non abbiano infiniti, ed inoltre, se si vuol che la corrisp.ᵃ sia univoca, siano monovalenti ecc. Nel seguito ci atteniamo a quel ds^2 per definire le funz.ⁱ complesse: quindi due sup. chiuse saranno atte a rappres. le stesse funz.ⁱ complesse quando, e solo quando siano in corrisp.ᵃ univoca conforme. Così se su una sup. si ha una funz. complessa $x + iy$ che prenda ogni valore in n p.ⁱ, essa si rappresenta conform.ᵉ su un piano n-plo di Riemann. (Segue il tratto fra [ ] a pag. 170-2)

Premesse queste generalità intorno alle funzioni complesse su di una superficie, veniamo a quelle funzioni che possono ser-

175

vire per lo studio dell'ente algebrico. Le funz^i razionali dell'ente, che ci han dato analiticam. le serie lineari su queste, son funz^i univoche degli elem^i, legate algebricam^e fra loro, continue, all'infuori di un certo numero di poli (infiniti algebrici), numero che è il <u>grado</u> della funzione (<u>Werthigkeit</u> secondo Klein)(valenza). Orbene abbiasi viceversa su una sup. chiusa qual. una funz. complessa sì fatta, cioè univoca e continua all'infuori di n poli, e che prenda ogni val. in n p^i. Distendendola sul piano complesso z, si ha (pag. 174) un piano n-plo di Riemann in corrispondenza univoca conforme con la superf. Un'altra funz. complessa s univoca e continua all'infuori di poli, sulla sup. si rappresenterà sul piano in una simile funz. di z e però per un noto teor. (v. ad

176
es° Neumann pag. 119) sarà una funz. algebrica di z, cioè $F(s, z) = 0$. Se s prende in gener. valori diversi dove z ne prende uno stesso, la sup. rappresenterà univocam. l'ente algebr° $F(s, z) = 0$. Ogni altra funz. s' univoca sulla sup. e con soli poli sarà similmente algebrica in z; ma siccome la corrispondenza è tale che ad ogni coppia (s, z) corrisponde una sola coppia (s' z), così s' sarà funz. raz^{le} di s, z (v. pag. 50).
Cioè le funz^{i} univoche sulla sup. con soli poli sono precisamente le funz^{i} raz^{li} di s, z, cioè le funz^{i} raz^{li} dell'ente (*)

(*) Questo teor. è di Riemann che al n. 8 vi giunge mostrando algebricamente che le funz^{i} raz^{li} di s, z con m' poli dipendono linearmente da $m' - p + 1$ cost^{i} (con un calcolo che coincide con quello con cui, mediante le curve agg^{te} di $F(s, z) = 0$

177

È appunto quel carattere che nella teoria funzionale di Riemann si sostituisce all'espressione razionale; come in generale (seguendo le orme di Dirichlet) nella definiz. delle funzioni Riemann ricorre a caratteri... anzi che ad espressi analitiche.

Ma per costruire e studiare quelle funzi algebriche univoche della sup. (invece che univoche Riemann dice diramate come la superficie) occorrono nella teor. di Riemann anche i loro integrali, cioè quelle funzi W la cui derivata $\frac{dW}{dz}$ è funz. univoca con soli infi algebrici (cioè senza singolarità essen-

si prova che un $G_{m'}$ sta (almeno) in una $g_{m'}^{m'-p}$);

mentre al n. 5 aveva stabilito che appunto da tante cost^i dipendono le funz^i univoche con dati m' poli.

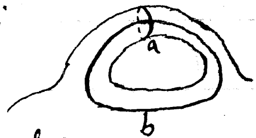
In tutto però vi è la lacuna relativa alle serie special.

Per un'altra dimostraz. v. Klein-Fricke pag. 499

178

ziali), sicchè $W=\int_{s_0 z_0}^{sz} R(s,z)\,dz$ hanno per sole discontinuità sulla sup. dei poli e degli inf.i logaritmici: poichè ~~∞~~ in prossimità di un p. qual. z' la funz. R si può rappres. con $\frac{a_\nu}{(z-z')^\nu}+\cdots+\frac{a_1}{z-z'}+a_0$ ed integrando verrà un termine $a_1\log(z-z')$, [a_1 dicesi <u>residuo</u> (logaritmico) della funz. R nel punto z': Cauchy] e gli altri algebrici. (Se $z'=\infty$ in luogo di $z-z'$ si scriva $\frac{1}{z}$, e se z' è p.° di diramaz.e ν-plo in luogo di $z-z'$ si scriva $(z-z')^{\frac{1}{\nu}}$).

Integrali Abeliani. Distinzione in 3 specie. Loro moduli di periodicità. Sistema normale di tagli (Riemann n. 19, Neumann p. 182, Klein-Fricke p. 495):



sulla sup. normale son rappresent.i dai meridiani a dei manichi e dai paralleli b. Gli integr.i Abel.i son caratterizzati dall'avere solo discontinuità al-

179

gebr. o logaritmiche e dall' essere determ. a meno di periodi, oppure dall'esser univoci sulla sup. tagliata e con differenze costanti lungo i tagli: poichè la derivata di una funz. siffatta sarà algebrica. Assumeremo questo carattere come definiz. degl'integr.i (cfr. p. 177).

Ciò posto, data ad arbitrio una sup. chiusa, ad es. una sup. di Riemann, esistono su essa delle funz.i complesse che presentino i detti caratteri, cioè funz.i algebr.e ed integrali? È un risultato capitale della teoria di Riemann, che la distingue da tutte le altre, quello dell'esistenza di tali funz.i, cioè (Riemann. n. 3): su qual. superf. (di Riemann) di gen. p è determinata a meno di una costante complessa additiva una funz. che 1° nei $2p$ tagli a e b abbia moduli di periodic.à costanti le cui parti reali siano

180

date ad arbitrio, 2° in dati punti abbia date discontinuità algebrico-logaritmiche, 3° dal lato positivo di linee, che (entro la sup. tagliata) congiungano un p. fisso O ai p.ti per cui il residuo logaritmico a_k (pag. 178) non è nullo, superi di $-2\pi i a_k$ il valore che ha dall'altro lato; e del resto sia univoca e continua. — La risoluz. di questo problema si può ridurre alla costruz. della parte reale u della funz. complessa $u+iv$, cioè di un <u>potenziale</u> con date discontinuità e dati valori nei contorni. E perciò Riemann ricorre ad un principio che Dirichlet (completando un analogo ragionam.° della mem. di Gauss "Allgemeine Lehrsätze...") dava nelle sue Lezioni sui potenziali (continui in ogni punto)*), e che

*) V. Vorlesungen di Dirichlet, 2e Auflage, §32 p. 127

181

egli estende al caso di discontinuità. Lacune nelle dimostraz.i del principio di Dirichlet (rilevate da Weierstrass, Kronecker ecc.). Come furono colmate da Schwarz e Neumann (Cap.i 16–18 delle Vorlesungen; v. anche in Klein-Fricke pag. 508 e segu.i un abbozzo del metodo di combinazione di Neumann). Metodo fisico per costruire i potenziali e quindi (p. 172) le funzioni cercate su qualunque superficie chiusa: il Klein (op.° citato) ricorre ad una batteria elettrica: i due poli messi a contatto colla superf. danno due inf.i logaritmici nel potenziale, sorgenti di elettr.à di copia $2a\pi$ (se a è il residuo di u nell'un punto), e facendoli coincidere darebbero un inf.° algebrico; un tratto di curva, od una curva chiusa che non spezzi la sup. e sia sede di una forza elettromotrice costante

182

Le funz^i di cui il teor. d'esistenza di Riemann (pag. 179–80) stabilisce l'esistenza si diranno integrali (Abeliani) di 1^a, 2^a, 3^a specie a seconda dei casi. Se $w_1, w_2, \ldots w_p$ sono integr^i di 1^a sp., le $2p$ costanti reali disponibili nel teor. d'esistenza per tali integr^i mostrano, che quelli si possono assumere linearmente indip^i, e che allora ogni altro è dato da $\alpha_1 w_1 + \ldots + \alpha_p w_p + \text{cost}$. Si vede pure che si possono scegliere in modo che w_k abbia moduli di periodicità nulli in tutte le a tranne a_k ove il mod^o sia πi (così Riemann al n. 20, per introdurre le w come argomenti nelle Θ; e Neumann. — Clebsch e Gordan, e quindi Lindemann, prendono $2\pi i$. — Klein-Fricke ed altri prendono 1): integrali normali. Fra gl'int. 2^a sp. si considerano quelli elementari t_c con un sol polo c : son determ^i

183

a meno di un fattore e di un int. 1ª sp. additivo. Il fattore si determina fissando che il coeffic. di $\frac{1}{z-z_0}$ nello sviluppo in prossimità di $c(z_0 \zeta_0)$ sia 1; l'int. di 1ª sp. può servire ad annullare i mod^i di per^à lungo tutte le a, e si ha così l'int. normale di 2ª sp., determinato a meno di 1 cost. add.. Indicando con t integr^i normali, avremo che ogni integr. di 2ª sp. coi poli $c_1 \dots c_n$ sarà rappres. da

$$\beta_1 t_{c_1} + \dots + \beta_n t_{c_n} + \alpha_1 w_1 + \dots + \alpha_p w_p + \text{cost.}$$

In partic. ogni funz. univoca s che non abbia altre discontinuità che quei poli, e quindi non abbia differenze nelle a sarà

$$s = \beta_1 t_{c_1} + \dots + \beta_n t_{c_n} + \text{cost}.$$

ove le β dovranno ancora esser tali da annullare i periodi lungo le b_k cioè $\tau_c^{(k)}$

$$\beta_1 \tau_{c_1}^{(1)} + \dots + \beta_n \tau_{c_n}^{(1)} = 0$$
$$\dots\dots\dots\dots\dots$$
$$\beta_1 \tau_{c_1}^{(p)} + \dots + \beta_n \tau_{c_n}^{(p)} = 0$$

184

Sono p equaz.ⁱ lineari omog. fra $\beta_1 \dots \beta_n$, onde se $n > p$ è possib. in ∞ modi costruire funz.ⁱ siffatte. Due qual. son legate da un'equ. alg. $F(s, z) = 0$ (pag. 176) che ci riconduce al punto di vista delle pag. 174 e segu.ⁱ. Così tutte saran funz.ⁱ raz.ˡⁱ di due s, z. In partic. le derivate dei nostri integr.ⁱ risp. ad una di esse z saran funz.ⁱ siffatte, sicchè quegl' integr.ⁱ sono effettivam. del tipo $\int R(s, z)\, dz$. Per gl' integr.ⁱ 1ª sp. si dimostra (v. per es. Riemann n. 9) che si può assumere $w = \int \frac{\varphi(s, z)}{\frac{\partial F}{\partial s}}$, ove φ è una funz. di gradi $n-2$, $m-2$ agg.ᵗᵃ ad $F(s, z)$, sicchè le $\varphi = 0$ segano su $F = 0$ la g_{2p-2}^{p-1}.*)

Le equaz. lineari fra le β si posson scrivere altrimenti tenendo conto che (v. Klein – Fricke pag. 532, Lindemann pag. 805, Clebsch e Gordan p. 121) il periodo $\tau_c^{(k)}$ dell'int. normale di 2ª sp. t_c lungo b_k è $\tau_c^{(k)} = -2\left(\frac{dw_k}{dz}\right)_c$

*) Ai p integr.ⁱ $w_1 \dots w_p$ corrisp.º le p φ linearm. indip.ⁱ

185

$= -2W_k'(c)$ (v. anche Riemann pag. 131 ultime linee)

$$\beta_1 W_1'(c_1) + \cdots + \beta_n W_1'(c_n) = 0$$
$$\cdots\cdots\cdots$$
$$\beta_1 W_p'(c_1) + \cdots + \beta_n W_p'(c_n) = 0$$

Poniamo per maggior generalità che queste p equaz.i si riducano a sole $p-\tau$, ove $\tau \geq 0$ siano le identità lineari indip.i che le legano, cioè i sistemi di λ linearm.e indip.i tali che

$$\lambda_1 W_1'(c_1) + \cdots + \lambda_p W_p'(c_1) = 0$$
$$\cdots\cdots\cdots$$
$$\lambda_1 W_1'(c_n) + \cdots + \lambda_p W_p'(c_n) = 0 .$$

Sostituendo $W_k'(c) = \dfrac{\varphi_k(c)}{\left(\frac{\partial F}{\partial s}\right)_c}$ diventano

$$\lambda_1 \varphi_1(c_1) + \cdots + \lambda_p \varphi_p(c_1) = 0$$
$$\cdots\cdots\cdots$$
$$\lambda_1 \varphi_1(c_n) + \cdots + \lambda_p \varphi_p(c_n) = 0 ,$$

cioè τ sono le φ linearm. indip.i che contengono tutto il gruppo $c_1 \ldots c_n$. In tale ipotesi abbiamo che le funz.i raz.li dell'ente i cui poli sono fra questi punti dipendono da $n+1-(p-\tau)$ costanti, e ciò è il teor. di Riemann-Roch (v. pag. 141, ove $\tau' = \tau - 1$), dimostrato appunto

186

col metodo avviato da Riemann (n. 5) e completato da Roch (Crelle 64).

Si può giungere allo stesso risultato col teorema d'Abel. Abbiasi una funz. raz^le qualunque z dell'ente algeb° e siano $x_1 \dots x_n$ i p^i in cui essa prende un dato valore qual. z. Sarà $W(x_1) + \dots + W(x_n)$ una funz. di z univoca e finita per tutti i valori di z (all'infuori di differenze costanti lungo linee corrisp^i alle a e b): e però una costante*) (Questo è in sostanza il ragionam° di Riemann n. 14; che si estende subito, com'egli osserva al teor. relativo agl'integr^i di 2^a e 3^a sp.; e che d'altra parte si può estendere alle corrispondenze algebriche fra due enti: Hurwitz, Math. Ann. 28). Quindi se $c_1 \dots c_n$ è un altro gruppo, sarà:

*) a meno di periodi di W.

187

$$W(x_1)+\dots+W(x_n)\equiv W(c_1)+\dots+W(c_n)$$

ossia $\int_{c_1}^{x_1}+\int_{c_2}^{x_2}+\dots+\int_{c_n}^{x_n}$ presi convenientem. dànn 0. Viceversa se queste p somme d'$\int$ son nulle $c_1\dots c_n$ ed $x_1\dots x_n$ son due gruppi di una stessa g_n^1. Supponendo il gruppo $x_1\dots x_n$ inf vicino a $c_1\dots c_n$ ed indicando con z una variab. indip. (funz. raz^{le} dell'ente) diventa

$$W_1'(c_1)\,dz_1+\dots+W_1'(c_n)\,dz_n=0$$
$$\dots\dots\dots$$
$$W_p'(c_1)\,dz_1+\dots+W_p'(c_n)\,dz_n=0$$

e ad ogni soluz. $dz_1:\dots:dz_n$ di questo sist^{a} corrisponderà una g_n^1 contenente il gruppo $c_1\dots c_n$ e viceversa; sicchè se fra quelle p equaz^{i} sono $p-\tau$ le indip^{i}, e quindi $n-p+\tau-1$ le dz arbitrarie, la g_n completa determinata da quel gruppo conterrà $\infty^{n-p+\tau-1}$ g_n^1, cioè sarà di dim. $r=n-p+\tau$. D'altronde quelle equaz^{i} sono appunto le p del principio della

188

pag. 185; solo che in luogo delle β vi son le dz. Dunque τ ha il significato che ivi si trova, cioè il num. delle φ indip. per $c_1 \dots c_n$. Si ritrova così il teorema R.-R.

Questo teorema (ed in particolare i gruppi speciali) si collega pure al problema d'inversione di Jacobi (v. Neumann, p. 350 e 382): date le p equaz.ⁱ $w_k(x_1)+\dots+w_k(x_p)\equiv V_k$ determinare i pⁱ x mediante le V. Il teor. d'Abel prova che se vi sono due soluz.ⁱ, il problema è indeterm., cioè tutti i gruppi della g_p^1 che congiunge quei due gruppi soddisfano quelle equ.: e si ha (indicando con $d_1 \dots d_{p-2}$ un gruppo residuo di quei gruppi speciali) $-V_k \equiv w_k(d_1)+\dots+w_k(d_{p-2})$. Il problema d'inversione fu risolto da Riemann mediante le funz.ⁱ ϑ. Si esprimono le funz.ⁱ

189

raz.^{li} simmetriche di $z_1 \dots z_p$ mediante prodotti di rapporti del tipo $\frac{\vartheta(w_k\alpha - V_k)}{\vartheta(w_k\beta - V_k)}$ ove α e β son costanti date. Ora si dimostra che $\vartheta(w_1 z - V_1, w_2 z - V_2 \dots, w_p z - V_p)$ è una funz. di z che s'annulla solo per p punti $\eta_1 \dots \eta_p$ tali che $w_k(\eta_1) + \dots + w_k(\eta_p) \equiv V_k$, e però nel caso detto è identicam. nulla, qual. sia z. E così l'essere $c_1 \dots c_n$ un gruppo di una g_n^1 quando $n < p$ si può esprimere dicendo che $\vartheta(w_k z - w_k(c_1) - \dots - w_k(c_n) - w_k(\eta_{n+1}) - \dots - w_k(\eta_p))$ è identicam. nulla qual. siano z ed $\eta_{n+1} \dots \eta_p$. Dalla consideraz. dell'annullarsi identico delle ϑ si può ritrovare il teor. R.-R (v. ad es. Clebsch-Lind. p. 857-62)

Il teor. Riemann-Roch vale anche per $p=0$ e $p=1$: allora è sempre $\tau=0$. La dimostraz. cogl' integr.^{i} di 2^a sp. vale ancora; av-

19°

vertendo che per $p=0$ essi si riducono alle funz.i raz.li, $t_c = \frac{1}{z-c}$, non vi son più periodi, ecc.

Dai teor.i d'esistenza di Riemann traghiamo che assunti ad arbitrio su un piano **z** (o sfera) $\nu = 2(n+p-1)$ punti e costruito un piano n-plo con quei p.i di diram. e quindi di gen. p esistono su esso delle funz.i uniroche con soli poli e quindi esistono enti algebr.i in corrisp. univoca con quella sup. e su cui quelle funz. son raz.li: ai vari valori di **z** corrispondono su un tal ente i gruppi di una g^1_n. Dunque: esistono sempre degli enti di gen. p contenenti una g^1_n per cui i $2(n+p-1)$ gruppi di diram. corrispondano a dati valori della funz. raz.le **z** che rappresenta quella g^1_n, ossia i birap-

191

posti indip.ⁱ di quei $2(n+p-1)$ gruppi singolari abbiano dati valori. Tali enti si dividono in un numero finito di sistemi sì che due enti di uno stesso sistema sono in corrisp. univoca (corrispondendosi quelle due g'_n). Quegli enti si posson tutti rappres.ᵉ con curve piane su cui quella g'_n è data da un dato fascio di rette: allora il teor. si riferisce all'esistenza, e alla distinz. in un num. finito di classi, delle curve di gen. p incontranti quelle rette in n p.ⁱ variab. e con $\nu = 2(n+p-1)$ rette di diramaz. date ad arbitrio (tang.ⁱ e rette che vanno alle cuspidi).

Infine rileviamo ancora un importante risultato che recentemente s'è dedotto dalla teoria di Riemann. S'è dimostrato che qualunque ente algebr.ᵒ cioè la sup. che lo rappresenta si può riferire univocam. nel senso

192

solito (conformemente) ad un poligono curvilineo piano (o sferico), i cui lati sono fra loro accoppiati (v. pag. 160), si da dare una sup. idealmente chiusa di gen. p. Introducendo la variab. complessa **z** al modo solito nel piano, abbiamo che le funz^i raz^{li} dell'ente son funz^i anal^e uniformi di **z**, definite però solo nell'interno di quel poligono. Ma si può fare che i p^i omologhi dei lati corrispond^i si corrispondano per una trasformaz. lineare di z : allora un principio anal^o, quello del proseguimento delle funz^i analitiche, fa sì che per estendere quella funz. al di là di un lato si deve ammettere che si trasformi in se stessa per quella sostituz. lin.. Così continuando, perchè quella funz. sia univoca e definita in tutto il piano si vede che deve ammettere un gruppo infinito (discontinuo) di sostituz^i lineari.

193

Casi di $p = 0$ (funzi razli), $p = 1$ (funzi ellittiche biperiodiche). In generale si ha un sistema di funzi del Poincaré, Fuchsiane o Kleiniane: esse dànno in funzi anal. uniformi di un param. tutte le funzi razli dell'ente; ciò che con le funzi Abeliane per $p > 1$ non si poteva fare.

194

Cap. 10°
I moduli. Le serie lineari sugli enti generali.

La questione della possibilità di corrisp. univoche fra due enti algebrici. – Fra parentesi si osservi che una corrisp. univoca <u>analitica</u> fra due enti (∞^1) algebri[ci] è certo <u>algebrica</u>, poichè produce una corrisp. univoca conforme fra le due superf. imagini e mediante questa le funzi[oni] raz[ionali] dell'un ente son funzi[oni] raz[ionali] dell'altro (funzi[oni] univoche e continue sulla sup. a meno di poli). — Se vi sono più corrisp. univoche fra i due enti, vuol dire che ve ne sono altrettante sull'uno; e viceversa.

Caso del genere 0, non vi son moduli, e ∞^3 corrisp[on]e univoche. Caso del genere 1: un modulo (pag. 100), e ∞^1 corrisp. univoche,

195

di cui due restano individuate quando, dati 2 p.i omol.i, nelle g_2^1 che li hanno per elem.i doppi si dia la corrisp.a proj.a che fa corrispondere anche gli altri elem.i diram.i; sicchè due dati elem.i sono omologhi in 2, 4, 6 ecc..

Se $p>1$ vi sono sw1ente di gen. p degli elem.i particolari, a cui si può ricorrere per la questione: ad es.o elem.i p-pli per g_p^1; cioè sulla curva canonica punti singolari, d' iperosculazione.*) Per un tal p.o sia m il minimo numero tale che esista una g_m^1 di cui esso sia m-plo: allora vi sarà una sola g_m^1 siffatta, altrimenti se vi fosse una g_m^2, il resto di quel p. in questa sarebbe una g_{m-1}^1, di cui quel p. sarebbe $(m-1)$-plo. Insomma vi sia un S_{m-2} iperosculatore (cioè con contatto m-punto) e non uno spazio infer.e; cioè m sia il 1.o numero non mancante nel

*) Dalla formula pag. 88 per $r=p-1$, $n=2p-2$ si ha che tali elem.i sono $(p-1)p(p+1)$.

196

Lückensatz di Weierstrass (pag. 156). La g_m^1 determinata da quell'elem. m-plo avrà in tutto $2(m+p-1)$ elem^i diram^e: al più in uno ne possono coincidere $m-1$, come appunto in quello; ma $m \leq p$ e quindi $p > m-1$, sicchè $2(m+p-1) > 4(m-1)$, e vi sono più di 4 elem^i di diramaz. distinti. Ciò posto se fra due enti, distinti o no, γ_p, γ'_p vi è una corrisp^a univoca alla g_m^1 di γ deve corrisp. un'analoga g_m^1 di γ' ed agli elem^i di diram. gli elem^i di diram., sicchè essendo quelli distinti più di due, le projettività possibili fra le due g_m^1 sono in numero finito. Quando poi è fissata la proj^{à} fra le due g_m^1 le corrisp^e univoche possibili sono pure in num. finito: ciò risulta ad es^o da ciò che il num. delle coppie di elem^i omol^i comuni a due corrispondenze è necessariam. finito (e

197

$\leq 2p+2$). Dunque fra due enti di gen $p>1$ o sopra un ente $p>1$ non vi può essere che un numero finito di corrisp. univoche. È di Schwarz (Crelle 87, 1875) il Teor. che un ente di gen. $p>1$ non può ammettere un'infinità continua (analitica) di corrisp.^{e} univoche. Che non possa neppure ammetterne un'infinità discreta fu poi provato dal Klein (in una lettera dell'82 al Poincaré: v. la dimostraz. di quest'ultimo nella Nota «Sur un théorème de M. Fuchs» Acta math. VII, 1884, p. 1-32; v. pag. 16); ma la 2^a dimostraz. che Nöther (Math. Ann. 21 p. 138 1882) dava del teor. di Schwarz vale pure se la serie infinita supposta è discontinua: su essa è ricalcata quella ora esposta. — Segue in partic. che ogni corrispondenza univoca sopra un ente $p>1$ è periodica. V. anche la Nota di Hurwitz sulle corrisp.^{e} univoche Math. Ann. 32 e quella dei Math. Ann. 41 p. 403 (1892)

198

Segue anche subito il num. dei moduli: Dei $2(m+p-1)$ elem^i di ram. della g^1_m abbiamo $m-1$ coinc^i; dunque in tutto $2p+m$; e però $2p+m-3$ birapporti. Dati questi ad arb^o è determ. in un num. finito di modi la classe di enti (pel teor. pag. 190-1): sicchè quelli sono i moduli. Nel caso generale $m=p$, e però $3p-3$ moduli. Nel caso partic^e $m=2$ l'ente è iperell^o e si ritrova il num. $2p-1$ di moduli.

Il num. $3p-3$ di moduli ($p>1$) è dovuto a Riemann, che lo ottiene come segue (n. 12). Sopra un dato ente il num. delle costanti da cui dipende una funz. raz^{le} di grado $n>2p-2$ è $2n-p+1$, mentre sono $2(n+p-1)$ i valori di z di diram^e per quella funz. <u>Ammesso</u> che questi val^i di diram. siano funz^i indip^i di quelle $2n-p+1$ cost^i,

199

si potrà disporre di queste in modo che altrettanti di quelli assumano valori fissati ad arbitrio, ed allora saran determ.i i rimanenti $3p-3$. Viceversa se oltre a quei val.i fissi si prendono questi $3p-3$ ad arbitrio esiste una classe di enti con funz.i raz.li aventi quei p.ti diram. Onde $3p-3$ moduli. — Ma volendo assicurarsi di quanto sopra si è ammesso si hanno difficoltà, e per toglierle Riemann ricorre poi, invece che alle funz.i raz.li dell'ente, agl'integrali di 1ª specie. — Se $p=0,1$ alle ∞^ρ trasformaz. univoche ($\rho=3,1$) corrisponde il fatto che sono ∞^ρ le funz.i raz.li cogli stessi val.i z di diram.; quelle con val.i distinti rimangono $\infty^{2n-p+1-\rho}$, e fissati $2n-p+1-\rho$ val.i diram.e di z ne rimangono $3p-3+\rho$. Quindi per $p=0,1$ si hanno risp. $0,1$ mod.i (v. anche l'opuscolo di Klein p. 65). — Natural-

200

mente in tutto il ragionam° di Riemann in luogo della funz. raz^le si può parlare di g_n^1 e in luogo di val^i, di birapporti.

I risultati preced^i dànno (op^e di Klein) notevoli proposiz^i sulle rappres^i conformi delle superf. chiuse. Due sup. $p=0$ si posson sempre riferire univoc. conform^e sì che a 3 dati p^i corrisp.° 3 dati p^i; due sup. $p=1$ solo se ...; e per $p>1$ si hanno $6p-6$ cost^i reali (i $3p-3$ moduli compl^i) da cui dipende la classe di sup. Sulle sup. $p=0,1$ vi sono infinite rappres^i conformi di 1^a specie (cioè che non mutano il segno agli angoli); sulle sup. $p>1$ non ve ne può essere che un num. finito.

Infine rivolgiamoci alla questione delle serie lineari g_n^r esistenti sopra un ente di gen. p

201

Basta determinare le serie complete. Quelle non speciali ($n > p$, $r = n - p$) son determ. da un gruppo qual. di n elem^i, di cui $r = n - p$ fissi e i riman^i p variab^i ad arbitrio; onde sono ∞^p per ogni n. Quelle speciali si compongono di gruppi di n p^i $x' \ldots x^n$ sulla curva piana $f_m(x) = 0$ che impongono solo $n - r$ condiz^i alle φ_{m-3} agg^{te}, cioè $\lambda_1 \varphi_1 + \ldots + \lambda_p \varphi_p = 0$; sicchè per un tal gruppo, oltre alle $f(x') = 0, \ldots f(x^n) = 0$, devono le equ^i

$$\lambda_1 \varphi_1(x') + \cdots + \lambda_p \varphi_p(x') = 0$$
$$\cdots\cdots$$
$$\lambda_1 \varphi_1(x^n) + \cdots + \lambda_p \varphi_p(x^n) = 0$$

ridursi a sole $n - r$, cioè nella matrice $\begin{vmatrix} \varphi_1(x') \ldots \varphi_p(x') \\ \cdots \\ \varphi_1(x^n) \ldots \varphi_p(x^n) \end{vmatrix}$ devono annullarsi tutti i determ^i d'ord. $n - r + 1$, ed in partic^e

$$\begin{vmatrix} \varphi_1(x') & \cdot & \varphi_{n-r}(x') & \varphi_i(x') \\ \cdot & \cdot & \cdot & \cdot \\ \varphi_1(x^{n-r}) & \cdot & \varphi_{n-r}(x^{n-r}) & \varphi_i(x^{n-r}) \\ \varphi_1(x^k) & \cdot & \varphi_{n-r}(x^k) & \varphi_i(x^k) \end{vmatrix}$$

$i = n-r+1, \ldots, p$; $k = n-r+1, \ldots, n$.

202

In tutto sono $r(p-n+r)$ condiz.^{i} indip.^{i} (dal complesso delle loro soluz.^{i} si devono però toglie-re delle soluz.^{i} estranee) per gli n p.^{i}; sicchè tali gruppi sono in gener. $\infty^{n-r(p-n+r)}$ [Lo stesso si vede sulla curva canonica ove si tratta di determinare $n-r$ p.^{i} tali che l'S_{n-r-1} che li congiunge la incontri in altri r p.^{i}, il che importa $r(p-n+r-1)$ condiz.^{i}, onde il num. $\infty^{n-r-r(p-n+r-1)}$]. Ciò posto le g_n^r saranno $\infty^{n-r-r(p-n+r)} = \infty^{(r+1)(n-r)-rp}$, ove

$$(1) \qquad (r+1)(n-r)-rp \geq 0$$
$$n \geq r + \frac{rp}{r+1} \quad ; \quad p \leq \frac{(r+1)(n-r)}{r}.$$

Il num. $(r+1)(n-r)-rp$ dà <u>la dimens. del sistema di tutte le g_n^r se i moduli son generali</u>. Se son speciali, il caso più generale è che siano complete (*). Se non son speciali le

(*) Se la g_n^r sta in una $g_n^{r'}$, la dimens.^{e} di tutte le g_n^r siffatte è $(r'+1)(n-r')-r'p+(r'-r)(r+1)$, che è $<(r+1)(n-r)-rp$ poichè $n-r'<p$.

g_n^r stanno in g_n^{n-p} e per ognuna di queste sono $\infty^{(r+1)(n-p-r)}$, sicchè in tutto $\infty^{p+(r+1)(n-p-r)} = \infty^{(r+1)(n-r)-rp}$. Allora la condiz. (1) è soddisfatta sempre, poichè $n-r \geq p$.

Se ne trae il num. delle γ_p^n di S_r; perchè abbian moduli generali deve aver luogo la (1); ed allora (v. pag. 63) al num. dei moduli e poi delle g_n^r vi sarà solo da aggiungere il num. delle collineaz. di S_r, sicchè in tutto si avrà per dimens. di tutte le γ_p^n di un dato S_r

$$3p-3+(r+1)(n-r)-rp+(r+1)^2-1$$
$$(2) \qquad = (r+1)n-(r-3)(p-1).$$

Per $p=0,1$ in luogo di $3p-3$ moduli si deve scrivere $3p-3+\rho$ (v. pag. 199), ma allora le ∞^ρ trasformaz. univoche dell'ente diminuiscono di ρ il num. d'inf. delle serie non equivalenti, che dànno origine a curve distinte; sicchè la formola (2) vale sempre. Per modu-

204

li partic.i può accadere che senza la (1) esistano delle g_n^r, γ_p^n di S_r: allora si trova che la dimens. di queste curve è $\geq$ all'espress. (2). Per $r=3$ si hanno ∞^{4n} γ_p^n di S_3, ecc.

La (1) dà sull'ente generale le serie di minimo ord. n per una data dimens. r. Così per $r=1$ abbiamo, se p è pari un num. finito di $g^1_{\frac{p}{2}+1}$, se p è impari ∞^1 $g^1_{\frac{p+3}{2}}$. Ciò fu già notato da Riemann, che ne profitta per ridurre l'equaz. $F(s,z)=0$ ai minimi gradi risp. ad s, z (n. 13); v. pag. 96. Escluso $p=2$ per cui si ha una sola g_2^1, si può ricorrere a due g_n^1 minime. Anzi, siccome il num. delle coppie comuni a queste è (pag. 96) $(n-1)^2-p$, ossia, a seconda che p è pari od impari, $\frac{p \cdot p-4}{4}$ o $\left(\frac{p-1}{2}\right)^2$, così esclusi i casi $p=1, 2, 4$ si può profittare della coppia comune che sempre esiste in ogni

205

altro caso per ridurre la $F(\overset{n}{s}, \overset{n}{z}) = 0$ all'ordine $(2n-2$, cioè$)$ p con due punti $\left(\frac{p}{2}-1\right)^{pli}$, oppure $p+1$ con due punti $\left(\frac{p-1}{2}\right)^{pli}$. La g_n^1 minima coi suoi $2(n+p-1)$ ossia $3p$ se p è pari, $3p+1$ se p è impari, gruppi diram^e serve a ritrovare i $3p-3$ moduli come birapp^i di quei gruppi (ove nel 2° caso fra le ∞^1 g_n^1 si scelga ad es. una con un elem^o diram. doppio). (*)

Per $r=2$ la (1) dà $p \leq \frac{3(n-2)}{2}$, e quindi dicendo d il num. dei p^i doppi della γ_p^n piana $d = \frac{n-1 \cdot n-2}{2} - p \geq \frac{n-2 \cdot n-4}{2}$ cioè: perchè una curva piana d'ord. n rappresenti un ente di moduli generali, deve avere almeno $\frac{n-2 \cdot n-4}{2}$ punti doppi. Si ha poi per g_n^2 minime $n = \frac{2p+6}{3}, \frac{2p+7}{3}, \frac{2p+8}{3}$, in num. finito, o ∞^1, o ∞^2. Questo sarà l'ordine minimo delle γ^n piane.

(*) Se vi è una g_n^1, ove $n < \frac{p}{2}+1$, si avranno similmente $2n+2p-5$ moduli, cioè $p-2n+2$ di meno che in generale. Tante son dunque le condizioni per l'esistenza di g_n^1.

206

E poichè il num. degl' invarianti assoluti di una γ_p^n piana è $\frac{n.n+3}{2} - \frac{n-1.n-2}{2} + p - 8 = 3n + p - 9$, così si trae di nuovo il numero $3p-3$ di moduli.

Analogam., per ogni valor di r si ha dalla (1) qual è il minimo ord. n di una g_n^r, cioè di una γ_p^n, per moduli generali. E la (1) mostra che queste serie minime o curve di minimo ord. son sempre speciali, tolti al più alcuni valori di p.

Va notato però che l'esistenza di g_n^r con la condizione (1) fu provata coll'enumeraz. delle costanti. Per assicurarsi della validità di questo ragionam° ed anzi trovare il numero delle g_n^r minime nel caso estremo che sia finito, si può procedere come segue. Si consideri una C^{n+p} normale di gen. p di S_n. Un gruppo di una g_n^r dà un iperpiano S_{n-1}

207

con un resto di p punti, e se quella g_n^r è completa, per questi p punti passeranno precisamente ∞^r iperpiani (pag. 138), cioè un S_{n-r-1}. Trovare le serie speciali d'ord. n significa dunque trovare gli spazi p-secanti di quella C^{n+p}, e tante sono quelle serie in generale quanti sono questi spazi. Il Castelnuovo (v. pag. 66) seguì questo concetto, e come C^{n+p} di gen. p notò che si può prendere una C^n raz.le normale con p corde, e che allora gli S_{n-r-1} sono quelli secanti queste p rette. Quando $(r+1)(n-r) = rp$ il numero di tali spazi è in generale finito e, ponendo $r' = p-n+r-1$, uguale a $\dfrac{2!\,3!\ldots r!\;2!\,3!\ldots r'!\;p!}{2!\,3!\ldots(r+r'+1)!}$. Questo è dunque il numero delle g_n^r nel detto caso (Castelnuovo). — In casi particolari si può trovare altrimenti le serie lineari. Così per $p=6$ supposto che veramente esista una

208

g_4^1, la residua sarà una g_6^2 e però l'ente si rappresenterà con una sestica piana con 4 punti doppi. Un gruppo qual. di una g_6^2 su questa curva imporrà solo 4 condiz.i alle cubiche pei 4 p.i doppi; onde tali cubiche saranno ∞^1 (e segheranno la g_4^1 residua): ma avendo 10 p.i comuni si spezzeranno in una conica ed una retta. Se è la retta che sta fissa, la g_6^2 è quella data dalle rette del piano; se no si hanno altre 4 g_6^2 date dalle coniche per 3 p.i doppi: e così 5 g_4^1.

209

Indice

Appendix 1
Corrado Segre's Archives at the University of Turin

Livia Giacardi, Erika Luciano, Chiara Pizzarelli, Clara Silvia Roero

After Corrado Segre's death in May 1924, his wife Olga Michelli donated the husband's offprints and portraits,[39] and in March 1926 his forty Notebooks and others documents to the Special Mathematics Library of the University of Turin, at the Faculty of Sciences.[40]

On the occasion of the celebrations for the 150th anniversary of the birth of Segre,[41] in 2013 Paola Gario donated to the same Mathematics Library thirty-four plates of Descriptive and Projective Geometry, which Segre drew in 1878–79, when he attended the 'G. Sommeiller' Technical Institute in Turin. She had received these plates from Segre's heirs in 1989.

In January 2014 and in October 2015 the University of Turin, through L. Giacardi and C. S. Roero, received the donation of a vast collection of correspondence, plates, obituaries and documents of various types, previously kept in Ancona, from Segre's grand-nephews Silvano and Daniele Fuà.[42]

[39]Cf. Giacardi, Livia and Roero, Clara Silvia, Biblioteca Speciale di Matematica "Giuseppe Peano", in Roero, Clara Silvia (ed.), *La Facoltà di Scienze Matematiche Fisiche e Naturali di Torino 1848–1998*, I, *Ricerca, Insegnamento, Collezioni scientifiche*, Torino: Deputazione Subalpina di Storia Patria 1999: 446–447.

[40]Cf. Conte, Alberto and Giacardi, Livia, and Novaria, Paola, *Corrado Segre (1863–1924). A 150 anni dalla nascita Catalogo delle Mostre documentarie-Novembre 2013*, Torino: KWB, 2013: 46. At present, these documents form the *Fondo Segre*, which was catalogued in 1996: see Giacardi, Livia and Varetto, Tiziana, Il Fondo Corrado Segre della Biblioteca "G. Peano" di Torino, *Quaderni di Storia dell'Università di Torino*, I (1996): 207–246; Giacardi, Livia, The Corrado Segre Archive, *Historia Mathematica*, 28 (2001): 296–301. The *Fondo Segre* can be accessed at the website edited by L. Giacardi: http://www.corradosegre.unito.it/.

[41]The conference *Homage to Corrado Segre (1863–1924)* took place in Turin from 28th to 30th November 2013: see the website http://ricerca.mat.uniroma3.it/GVA/Segre150/segre150.html. Cf. also Luciano, Erika, Celebrazioni di Corrado Segre (1863–1924) a 150 anni dalla nascita, *Rivista di Storia dell'Università di Turino*, II (2013): 133–135.

[42]Part of these documents have been viewed by us (before the donation) at the website http://users.mat.unimi.it/users/gario, Section "Corrado Segre Archivio di Ancona". Cf. Gario, Paola, Su alcune carte di Corrado Segre recentemente rinvenute, *Atti dell'Accademia delle Scienze di Torino*, 123 (1989): 187–198. All Segre's archives will be soon available at the website http://www.corradosegre.unito.it/

G. Casnati et al. (eds.), *From Classical to Modern Algebraic Geometry*, Trends in the History of Science, DOI 10.1007/978-3-319-32994-9

We are presenting here only a preliminary description of the entire archives. A detailed analysis of the documents will be required for the final cataloguing. Segre's Archives can be divided into the following ten Series.

UTo-ACS Università di Torino, Archivi di Corrado Segre (University of Turin, Corrado Segre's Archives)

I. **Documenti di Carriera (Career Documents)**

- Corrispondenza istituzionale (Institutional Correspondence)
- Attestati e lettere di Accademie e Società scientifiche (Academies and Scientific Societies' certificates and letters)
- Premi di studio, diplomi e onorificenze (Certificates of awards, diplomas and honours)

II. **Documenti di Famiglia (Family Documents)**

- Indirizzario di Segre (Segre's Address-book)
- Lettere di Segre a O. Michelli (Segre's letters to O. Michelli)
- Lettere di Mario Segre a O. Michelli (Mario Segre's letters to O. Michelli)
- Lettere di condoglianze alla famiglia (Condolence letters to the family)[43]
- Miscellanea di documenti di famiglia (Miscellany of family documents)
- Ritratti (Portraits)

III. **Tavole (Plates)**

- Tavole di Geometria Descrittiva, Istituto tecnico 1878–79 (Descriptive Geometry Plates, Technical Institute, 1878–79)
- Tavole di Geometria Proiettiva, Istituto tecnico 1878–79 (Projective Geometry Plates, Technical Institute, 1878–79)
- Tavole di Geometria Descrittiva, Università di Torino, 1880 (Descriptive Geometry Plates, University of Turin, 1880)
- Tavole di Geometria Proiettiva, Università di Torino, 1881 (Projective Geometry Plates, University of Turin, 1881)

IV. **Appunti e Resoconti (Notes and Reports)**

- Miscellanea di appunti (Miscellany of notes)
- Resoconti di scritti letti (Reports of readings)

V. **Manoscritti di pubblicazioni e appunti di ricerca (Autographs and research notes)**

- Tesi di laurea di Segre (Segre's Degree Dissertation)[44]

[43]Some of these are edited in Annexes of the essay by 68–72 of Luciano and Roero in this volume.

[44]Segre's dissertation is kept in BMP, Fondo Segre, *Scritti* 1; see Giacardi and Varetto 1996: 362.

- Autografi di Segre (Segre's Autographs)[45]
- Recensioni di Segre per il *Jahrbuch über die Fortschritte der Mathematik* (Segre's reviews for *Jahrbuch über die Fortschritte der Mathematik*)
- Appunti su C.G.C. von Staudt (Notes on C.G.C. von Staudt)
- Appunti ed estratti per l'edizione delle *Opere* di L. Cremona (Notes and offprints related to L. Cremona's *Works*)
- Appunti per il discorso inaugurale "Le Previsioni" (Notes for the inaugural address "Le Previsioni")

VI. **Quaderni e documenti relativi all'attività didattica (Notebooks and documents related to teaching)**

- Lista di costruzioni per i corsi di Geometria proiettiva e descrittiva (List of constructions for Projective and Descriptive Geometry courses)
- Registri delle lezioni (Class Registers)
- Quaderni delle lezioni 1–40 (Notebooks of courses 1–40)[46]
- Appunti (Notes)

VII. **Carteggi (Correspondence)**

- Lettere di Segre (Letters by Segre)
- Lettere a Segre (Letters to Segre)
- Corrispondenza relativa a *Annali di Matematica pura ed applicata* (Correspondence related to *Annali di Matematica pura ed applicata*)

VIII. **Ritratti (Portraits)**

IX. **Elenchi di opere, articoli e Schede bibliografiche (Lists of works, articles and Card Index)**

- Elenco delle pubblicazioni di Segre (List of Segre's publications)
- Elenco dei corsi di Segre (List of Segre's lecture courses)
- Elenco di articoli su periodici nazionali e internazionali (Articles published in Italian and foreign journals)
- Elenco di riviste e libri in biblioteche di Torino (List of journals and books in Turin libraries)
- Cataloghi di libri, estratti e riviste della Biblioteca personale di Segre (Catalogues of books, offprints and journals in Segre's library)[47]
- Schede bibliografiche autografe (Segre's Card Index)[48]

[45]See BMP, Fondo Segre, *Scritti* 2–7; see also Giacardi and Varetto 1996: 362–363.

[46]The Notebooks have been catalogued (Giacardi and Varetto 1996), they are kept in BMP, Fondo Segre, *Quaderni* and can be accessed at: http://www.corradosegre.unito.it/quaderni.php.

[47]See Annex 75 of the essay by E. Luciano and C. S. Roero in this volume.

[48]The Card Index is kept in BMP, Fondo Segre, *Scritti*.17. See the essay by A. Conte and L. Giacardi, Appendix 4, in this volume and http://www.corradosegre.unito.it/doc/schedario.pdf.

X. **Necrologi e Commemorazioni (Obituaries and Eulogies)**

- Necrologi di Segre (Segre's Obituaries)
- Commemorazioni di Segre (Segre's Eulogies)

In order to provide an insight into the contents of Segre's Archives, we give a preliminary description of the documents included in the ten series.

I. **Career Documents**

The Series is formed by three folders, which contain official correspondence and documents, in chronological order:

- *Institutional Correspondence*: collection of 48 letters, 46 of which were sent to Segre, 33 from the Ministry of Public Education and 13 from the dean of the University of Turin. The two remaining letters are send to the dean, with Segre in copy, one from the Minister of Public Education, the other from Enrico D'Ovidio. The folder contains a draft of a letter from Segre to the Ministry of Public Education. The collection refers to Segre's institutional career from 1883 to 1913 at the University of Turin. In particular, the letters concern the issuance of the Degree in Mathematics and of the certificate of attendance at lectures at the Teacher Training School (1883); the three-monthly scholarship from the Ministry of Public Education in the academic year 1884–85; the appointments as assistant to the Algebra and Analytic Geometry chair (1883) and to that of Projective and Descriptive Geometry (1885), as professor of Higher Geometry (1888, 1892), as adjunct professor of Mathematical Physics (1895–96), as lecturer at the Mathematics course at the Teacher Training School (1909); and salary increases from 1886 to 1912. There is also a letter from the dean to Segre, with the request of writing Giuseppe Bruno's biography (1893), and letters notifying Segre's designation as Knight (1892) and as *Commendatore* (1917) of the *Ordine della Corona d'Italia*, and as Knight of the *Ordine dei Santi Maurizio e Lazzaro* (1914).
- *Academies and Scientific Societies' certificates and letters*: unit of twenty letters and documents sent to Segre from Italian and foreign Academies, Scientific Societies and Institutes. One letter is related to the awarding of the gold medal of the *Società italiana delle Scienze* for Segre's mathematical works (1884), the remaining nineteen letters concern honorary designations by eleven scientific institutes. These appointments included fellowships of the *R. Accademia delle Scienze, Lettere ed Arti di Modena*, the *Circolo Matematico di Palermo* (as fellow from 1887, as member of the executive board from 1888), the *R. Accademia delle Scienze di Torino* (as fellow from 1889, as Secretary from 1910 to 1914, as director of the Class of Physical Mathematical and Natural Sciences from 1920), the *R. Accademia dei Lincei* (1891), the *R. Istituto Lombardo di*

Scienze e Lettere (1893), the *R. Istituto Veneto di Scienze, Lettere ed Arti* (1906), the *R. Accademia delle Scienze dell'Istituto di Bologna* (1908), and as honorary member of the *Calcutta Mathematical Society* (1915), of the *Renares Mathematical Society* (1920) and of the *Czechoslovakian Mathematicians and Physicists Society of Prague (Jednota Ceskoslovenskych Matematiku a Fysiku*, 1923). There is also a brass plaque relating to Segre's nomination as national member of the *R. Accademia dei Lincei*. Incipit: "Regia Lynceorum Academia, An. a Societate Instituta CCXVIII Conradum Segre inter sodales …".

- *Certificates of awards, diplomas and honors*: collection of six official documents related to the scholarly and institutional career of Segre from 1873 to 1924. It contains the certificate of merit for the first student classified in the first class of the technical School (1873), the scholarship from the *Camera di Commercio ed Arti* for the top student classified in the Industrial and Professional Technical Institute of Turin (1880), the Degree (1883), the Certificate of merit from the *Società di Mutuo Soccorso fra gl'Impiegati Secondari della R. Università di Torino ed altri Istituti Governativi d'Istruzione* (1892), the designation as Knight of the *Ordine della Corona d'Italia* (1892), the Diploma as honorific fellow in Segre's remembrance by the *Ente Ricreatorio e Gruppo Sportivo Cesare Battisti per i figli dei caduti in guerra e dei combattenti* (1924).

II. **Family Documents**

This Series collects six folders of documents and correspondence of various types:

- *Segre's Address-book*: a small notebook of 34 sheets, 25 of which are numbered and 2 unnumbered, three loose sheets of different sizes and one folded (1r–2v). It may be dated to between 1890 and 1924, and it contains 489 addresses, not alphabetically sorted. On the twenty-seven initial sheets there are the addresses of mathematicians, teachers, students and Institutions of the following nationalities: 128 Italians (four women), the Mathematics Seminar of the Faculty of Sciences of the University of Rome, the Special Library of the Mathematics Faculty in Bologna, the *Scuola Normale* in Pisa and the *Scuola Normale* in Pavia; 216 foreigners, of whom eight English, four Swedish, three Finnish, five Danish, eight Dutch, twenty-four French, six Belgian, four Swiss, seventy-five German, nine Austrian, six Czech, nine Polish, one Portuguese, two Spanish, three Greek, two Hungarian, one Rumanian, one Russian, three Indian, four Japanese, one Canadian, thirty-seven American; the University College of

Wales, the Ceckoslovenskych Matematika Society in Fysiku, the *Rumanian Seminarul Matematic Universitate* in Jassy.

In the loose sheets Segre annotated 138 addresses of Italians, among them numerous women. These were mostly students at the University of Turin between 1910 and 1924, who then became teachers. There is also a list of addresses of students and assistants in the decade 1905–1915.

- *Segre's letters to O. Michelli*: 206 letters, postcards and a telegram, sent by Segre to his wife in the period from 23 December 1892 to 28 April 1916. The places are predominantly linked to Segre's scientific committment: from Turin during the period of his engagement (1892–1893); from Rome where he stayed for a competition for the chair of Descriptive and Projective Geometry (1893) and for that of Algebra and Analytic Geometry (1899), for sittings of the *Accademia dei Lincei* (1895, 1899, 1902–1908, 1912), for the assignment of the Royal Prize for mathematics (1898) and for the awarding of the Guccia Medal to Francesco Severi (1908); from Zurich (1897), Heidelberg (1904) and Rome (1908) for the International Congresses of Mathematicians; and from Parma for the Congress of the Italian Society for the Advancement of the Sciences (1907).
- *Mario Segre's letters to O. Michelli*: five letters sent by Segre's brother Mario to Olga Michelli from 2 August 1903 to 3 August 1905.
- *Condolence letters to the family*: twenty-eight letters sent to Segre's relatives between 19 May 1924 and 2 January 1925: twenty-five to Segre's wife, one to Segre's brother Arturo, one to Segre's daughters Adriana and Elena. The senders are listed in alphabetical order. Among them there are eight foreign mathematicians: Paul Aricò, Julian Coolidge, Jacques Hadamard, Arthur Schönflies, Charles Herschel Sisam, Virgil Snyder, Eduard Study and Grace Chisholm Young;[49] and sixteen Italian mathematicians, intellectuals, colleagues and students: Eugenio Bertini, Vittorio Brondi, Teresita Elter Castelli, Guido Castelnuovo, Giovanni Chevalley, Laura Fuà, Guido Ghersina, Giovanni Gorrini, Giacinto Guareschi, Laura Hidalgo, Gino Loria, Oreste Mattirolo, Salvatore Pincherle, Carlo Somigliana, Alessandro Terracini, Guido Toja and Giulio Vivanti.
- *Miscellany of family documents*: one letter related to the release of Segre's passport (1880); six letters from the Army regarding Segre's military service (1885, 1895, 1899); two documents and a clipping from *L'Ordine. Corriere delle Marche* referring to his wedding to Olga Michelli, whose civil ceremony took place in Ancona on 25 March 1893, followed by a religious wedding on 26 March 1893; four sheets of the manuscript *Riccardo mio!*; three clippings, the first one from *Gazzetta di Saluzzo e del circondario*, entitled *Fellow citizens that honour themselves* (*Concittadini che si fanno onore*), with a tribute to Segre's career (10 December 1893); the second from *Cronaca di Ancona* with Segre's

[49]See Annexes 68–72 of the essay by Erika Luciano and Clara S. Roero in this volume.

death announcement by the family (20 May 1924); and the third from *Il Subalpino*, entitled *A noble scientist who has passed away* (*Una nobile figura di scienziato scomparso*), by Emilio Bissoni (28 May 1924).

- *Portraits*: nine portraits: seven with the Segre's profile, in two of which Segre is young, and four of which are marked '*Florence 1899*'; one with Segre and his wife; one with Segre and his two children; one of Pio Foà; and one of Guido Michelli dated 4 October 1924.

III. **Plates**

The Series consists of three folders containing drawings with Segre's autograph signature, with Roman numbering, sorted numerically:

- *Projective and Descriptive Geometry Plates*, Technical Institute, year IV, 1878–79: unit of thirty-four plates, drawn in black and red ink, collected in a binder with autograph title.[50] They are divided into Projective Geometry Plates (nos. I–XIX) and Descriptive Geometry Plates (nos. I–XV). The two sets are separated by a sheet with the autograph writing 'Descriptive Geometry'. They contain drawings relating to theorems, problems, exercises and constructions, sometimes grouped up to four per plate, accompanied by a title or statement.
- *Projective Geometry Plates*, University of Turin, year II, 1880: unit of seventy-five plates, in black and red ink, dated and authenticated by Donato Levi,[51] who was assistant professor of Projective and Descriptive Geometry. They are grouped by date: nos. I–XIX of 13 January 1880, nos. XX–XXVI of 14 February 1880, nos. XXVII–XXXIX of 11 March 1880, nos. XL–LII of 17 April 1880, nos. LIII–LXX of 25 May 1880, nos. LXXI–LXXV of 11 June 1880.
- *Descriptive Geometry Plates*, University of Turin, year II, 1881: unit of 130 plates, in black and red ink, dated. The first 127 are authenticated by Giuseppe Bruno,[52] professor of Descriptive and Projective Geometry, the remaining three by Bruno's assistant, Giuseppe Savoja. They are grouped by date: nos. I–XXXIV of 11 January 1881, nos. XXXV–XLVI of 1 February 1881, nos. XLVII–LXVIII of 22 February 1881, nos. LXIX–XCII of 22 March 1881, nos. XCIII–CIV of 21 April 1881, nos. CV–CXI of 12 May 1881, nos. CXII–CXXVII of 4 June 1881, nos. CXXVIII of 12 June 1881, nos. CXXIX–CXXX of 16 June 1881.

[50]These are the plates donated by Paola Gario to the Library of the 'G. Peano' Department of Mathematics in November 2013.

[51]Levi (1834–1885). Cf. Navale, Maria Teresa, *Donato Levi*, in Roero 1999, vol. 2: 50.

[52]Bruno (1828–1893). Cf. Roero, Clara Silvia, *Giuseppe Bruno*, in Roero 1999, vol. 2: 484–486.

IV. Notes and Reports

- *Miscellany of notes*[53]: almost 500 papers of various types, some sheets are collected in folders, and generally undated. In some of them Segre dealt systematically with a specific topic or he presented a bibliographic study of it, in others there are reports of works he had read, in some cases with comments, critics and quotes. The five groups of notes, which date back to the period of Segre's training, are entitled "Discussioni di problemi" (Summer 1879, 2 loose sheets), "Il negativo, l'infinito e l'immaginario", "Sui concetti nuovi della Geometria moderna" (Summer 1880, a notebook of 22 pages), "Sulle omografie d'ordine superiore" (July 1881, 29 sheets in a folder), "Sulle stelle di raggi e piani" (November 1881, 8 loose sheets), "Ricerche sui sistemi projettivi di complessi di enti geometrici e sul numero di soluzioni di un sistema di equazioni a più variabili od a più sistemi di variabili" (January and February 1882, 11 loose sheets). The topics of other sheets are, for example, the Geometry of straight line, the Involutions, the Kummer surfaces and the History of Mathematics.
- *Reports of readings*: a folder of 116 loose sheets, with autograph title, which contains notes and bibliographic references about 170 works that Segre read, presumably before 1889. Among the foreign authors there are Alexander Brill, Arthur Cayley, Michel Chasles, William Kingdon Clifford, Ferdinand Georg Frobenius, David Hilbert, Marie Georges Humbert, Ernst Eduard Kummer, Ferdinand von Lindemann, Paul Painlevé, Émil Picard, Theodor Reye, and among the Italians Ferdinando Aschieri, Giuseppe Battaglini, Eugenio Beltrami, Enrico D'Ovidio, Ernesto Padova and Nicola Salvatore Dino. There are also reports on different articles published in the *Bulletin de la Société mathématique de France* from 1872–73 to 1875–76 and various notes are dedicated to History of Mathematics.

V. Manuscripts of Publications and Research notes

- *Segre's Autographs*: three manuscripts of Segre's publications: fifty-seven numbered sheets with autograph title "C.G.C. von Staudt e i suoi lavori", which represented Segre's preface to the Mario Pieri's Italian translation of the *Geometrie der Lage* of von Staudt, published in 1889 in Turin by Bocca; fourteen numbered sheets entitled "Le linee principali di una superficie di S_5 e una proprietà caratteristica della superficie di Veronese, nota I e II", which appeared in 1921 in the *Rendiconti della R. Accademia dei Lincei*; twelve numbered sheets entitled "Le superficie degli iperspazi con una doppia infinità

[53] See http://users.mat.unimi.it/users/gario/Elenco-Segre.html.

di curve piane e spaziali, nota II", marked as 'urgent' and published in 1924 in the *Atti della R. Accademia delle Scienze di Torino*.

- *Segre's reviews for Jahrbuch über die Fortschritte der Mathematik*: collection of twenty-six loose sheets, which contains notes about forty-four reviews by Segre, which were published in the volumes sixteen (1884) and seventeen (1885) of the *Jahrbuch über die Fortschritte der Mathematik*.[54] They are collected in two binders: the first one, entitled "Recensioni 1884", consists of nineteen reviews of Italian mathematicians' works, such as those of Ferdinando Aschieri, Eugenio Bertini, Alberto Brambilla, Ernesto Cavalli, Ernesto Cesàro, Enrico D'Ovidio, Riccardo De Paolis, Alfonso Del Re, Gino Loria, Ulderigo Masoni, Vittorio Martinetti, Nicola Salvatore Dino, Virginio Retali, Theodor Reye and Giuseppe Veronese. The second binder, "Comptes Rendus 1885", consists of twenty-five reviews, including those of Raffaele Badia, Pietro Cassani, Pasquale Del Pezzo, Vincenzo Finamore, Carmelo Intrigila, Giuseppe Jung, Giulio Lazzeri, Vincenzo Mollame, Carlo Maria Piuma, Augusto Porchiesi, Virginio Retali, Gaetano Sforza, J.S. and M.N. Vaněcěk. There is also a list of works, reviewed by other mathematicians in volume seventeen of the *Jahrbuch*.
- *Notes on C.G.C. von Staudt*: collection of twenty-four sheets of different dimensions, regarding biographical information and the scientific works of Carl Georg Christian von Staud.
- *Notes and offprints related to L. Cremona's Works*: collection of twenty-six loose sheets of notes about some of the works of Luigi Cremona, which Segre edited for the publication of the *Opere Matematiche di Luigi Cremona* (1914–1917). In particular, there are notes about the articles "Sulla teoria delle coniche" (1863), "Sopra alcune questioni nella teoria delle curve piane" (1864), and about Curtze's translation of Cremona's work *Grundzüge einer allgemeinen Theorie der Oberflächen in synthetischer Behandlung* (1870).
- *Notes for the inaugural address 'Le Previsioni'*: collection of sixty loose sheets with notes, remarks and quotes from works, which Segre read to prepare the inaugural address of the academic year 1918–19 at the University of Turin, which dealt with theoretical and experimental forecasts. In his notes he quoted more than fifty authors from very different fields of study: for example Luigi Luciani; (Physiology) Pierre Simon Laplace and Jacob Bernoulli (Probability); Gregor Mendel, William Ramsay and Frederick Soddy (Chemistry); Bernard Brunhes, Robert Mayer and Hermann von Helmholtz (Thermodynamics); Henri Poincaré, Victor Puiseux, Svante August Arrhenius, Gustave-Adolphe Hirn and Alphonse Berget (Cosmogony and Astronomy); Tycho Brahe and Isaac Newton (Astrology); and Jean De Bloch (Political History).

[54]See Appendix 4.3, *Bibliography of the Works by Corrado Segre*, in this present volume.

VI. **Notebooks and documents related to teaching**

- *List of construction for Projective and Descriptive Geometry*: notebook undated, with 130 statements for drawing plates of Geometry. It is divided in five groups, entitled "Metodo della projezione centrale" (nos. 1–33), "Metodo dei piani quotati" (nos. 34–46), "Metodo delle projezioni ortogonali" (nos. 47–79), "Rappresentazione dei poliedri" (nos. 80–92), "Curve e superficie" (nos. 93–130).
- *Notes*: collection of loose sheets, ten regarding the course of Higher Geometry in the academic year 1886–87, entitled "Teoria geometrica delle curve piane algebriche"; the index's draft "Complessi Geom. proj. e descr. 1917–18", probably for the course of the academic year 1918–19; undated sheets regarding lectures at the Teacher Training School, which include a plan for organizing a lesson and three lists of topics, with bibliographical references to be consulted and a short list of students. There are also loose sheets about students' dissertations, for example those of the year 1902 defended by Ernesto Laura[55] on the conics normal to given straight lines and planes (Coniche normali a rette e piani dati). The students' names and the academic years are not always registered. Segre's notes generally deal with the rational of the topic, his judgment about the dissertation's originality, his advice on critical points and the literature.
- *Class Registers: collection of fifteen registers of the following lecture courses*: two of Projective Geometry (1885–86, 1887–88), twelve of Higher Geometry (from 1890–91 to 1901–02) and one of Mathematical Physics (1896–97).

VII. **Correspondence**

This Series collects three folders of letters and drafts, sorted in chronological order:

- *Letters by Segre*: 29 drafts of Segre's letters, collected in a binder with Segre autograph title *Lettere a scienziati* (Letters to Scientists), from 1882 to 1884.[56] Specifically the collection contains two letters' drafts to Arthur Cayley, one to Luigi Cremona, one to Jean Gaston Darboux, one to Alfredo De Paolis, one to Carl Friedrich Geiser, one to Thomas Archer Hirst, eleven to Felix Klein (one of which also from Gino Loria, edited in Luciano and Roero, 2012), four to Leopold Kronecker, one to Oscar Xavier Schlömilch, one to Giuseppe

[55] Laura (1879–1949) took his degree in Mathematics at the University of Turin in 1902, he was assistant to Giuseppe Peano to the Infinitesimal Calculus course from 1902 to 1903 and to Giacinto Morera to the Rational Mechanics course from 1903 to 1908. He gave a lecture course of Rational Mechanics from 1908 to 1915. Cf. *Annuario della R. Università degli Studi di Torino*, *ad annum*; ASUT, Facoltà di Scienze MFN, *Registri di carriera scolastica*, 1898–99, n. 62; Barberis, Bruno, *Ernesto Laura*, in Roero 1999, vol. 2: p. 567.

[56] See the Annexes 1–16 of E. Luciano and C.S. Roero in this volume.

Veronese, two to Aurel Voss and one to Karl Weierstrass. There are also two drafts written by Segre, the first one to the dean of the University of Turin on behalf of the Mathematics second year's students (1880–81), the second one to Enrico D'Ovidio signed by Guido Castelnuovo, Francesco Gerbaldi, Gino Loria, Fortunato Maglioli, Giacinto Morera, Enrico Novarese, Giuseppe Peano, Segre and Guido Valle (1889).

- *Letters to Segre*: forty letters and postcards to Segre and one loose sheet. Specifically the collection contains seven letters from Ettore Caporali (from 1884 to 1885), nineteen from Riccardo De Paolis (from 1885 to 1902), five from Alfredo De Paolis (1892) and one from Francesco Vercelli (1918). A group of eight letters and postcards, from 1887 to 1888, concerns Segre's research on Carl Georg Christian von Staudt: two of these are from K. Rudel and six from August Papellier. There is also a loose sheet with notes on two letters from Theodor Reye, with autograph title "Dimostrazione sintetica di un teorema del Reye sulle curve asintotiche della superficie di Kummer".
- *Correspondence related to* Annali di Matematica pura ed applicata[57]: twenty-nine letters sent to Segre, from January 1921 to November 1923, three enclosures and five loose sheets. Specifically the collection contains: one letter from Luigi Bianchi, one from Emilio De Benedetti, two from Guido Fubini, seven from Giuseppe Jung, one from Gino Olivetti, twelve from Salvatore Pincherle and five from Virgil Snyder. Among the enclosures and the loose sheets there are the lists of the subscribers in 1922 and in 1923, the promotion of the new series (the fourth) of the Italian journal in the *Bulletin of the American Mathematical Society* (1923), an announcement by the Editorial Board to the members and one by the Zanichelli publishing house to the editors of the journal, and a letter from Pincherle to Jung (1923).

VIII. **Portraits**

Fifty-five portraits of the following mathematicians and physicists: Eugenio Beltrami, Eugenio Bertini, Luigi Berzolari, Enrico Betti, Luigi Bianchi, Moritz Cantor, Felice Casorati, Guido Castelnuovo, Jean Gaston Darboux, Pasquale Del Pezzo, Enrico D'Ovidio, Wilhelm Fiedler, Lazarus Immanuel Fuchs, Hermann von Helmholtz, Charles Hermite, David Hilbert, Adolf Hurwitz, Klein Felix, Leo Koenigsberger, Sonja Kowalewski, Leopold Kronecker (two portraits), Johann Heinrich Lambert, Sophus Lie, Rudolf Lipschitz, Gino Loria, Gian Antonio Maggi, Vittorio Martinetti, Franz Meyer, Gösta Mittag-Leffler, Onorato Nicoletti, Max Nöther, Ernesto Pascal, Émile Picard, Mauro Picone, Salvatore Pincherle, Henri Poincaré, Luigi Puccianti, Theodor Reye, Carlo Rosati, George Salmon, Achille Sannia, Heinrich Schröter, Hermann Schubert, Friedrich Schur, Francesco

[57]Some of these letters are edited in Annexes 61–67 of the essay by E. Luciano and C. S. Roero in this volume.

Severi, Charles Hershel Sisam, Virgil Snyder, Carlo Somigliana, Stephanus Cyparissos, Eduard C.H. Study, Rudolf Sturm, Giuseppe Veronese, Karl Weierstrass and Hieronymus Georg Zeuthen.
The portraits are labelled and arranged into four pictures frames,[58] which used to be hung on the walls of Segre's office at home. They were probably sent to Segre by Italian and foreign correspondents.

IX. **Lists of works, articles and Card Index**

- *List of Segre's publications*: four lists of Segre's articles. Specifically the folder contains: an handwritten list of 123 Segre's articles from 1883 to 1922; a typewritten copy of 125 Segre's articles from 1883 to 1924, whose last two titles are handwritten; a list of sixty-four Segre's memoirs published from 1886–87 to 1900–01, and another one marked 'missing' ('mancano') with some of Segre's publications (1883–1922).
- *List of Segre's lecture courses*: five copies of the typewritten list of thirty-seven courses, which Segre held at the University of Turin from 1888–89 to 1922–23, with the lithography of notes for his course, entitled "Lectures on Projective Geometry (1885–86)".
- *Articles published in Italian and foreign journals*: collection of nine loose sheets with handwritten lists of articles, published in the following journals: *Journal für die reine und angewandte Mathematik* (1844–1881), *Zeitschrift für Mathematik und Physik* (1856–1864), *Annali di Matematica* (1859, 1860, 1868–1869), *Mathematische Annalen* (1869–1882), *Bulletin de la Société mathématique de France* (1872–73 to 1875–76) and *Giornale di Matematiche ad uso degli studenti delle Università italiane* (1873–1882).
- *List of journals and books in Turin libraries*: collection of five loose sheets with handwritten lists of volumes, journals and memoirs by Italian and foreign authors, which were held in the libraries of the Academy of Sciences, the Special Mathematics Library of the University and the Military Library in Turin.
- *Catalogues of books, offprints and journals in Segre's library*: collection of four groups of loose sheets with the lists of works, treatises, journals and articles, held in Segre's personal library, part of which were sold in December 1924 to the National Insurance Institute.[59] Two of these lists are sorted alphabetically, the other two topographically, with the shelf numbers of the library.

[58]At present, they are hanging on the walls in the office of the head of the Mathematics Department in Turin.

[59]See Annex 75 of the essay by E. Luciano and C. S. Roero in this volume.

X. **Obituaries and Eulogies**

The Series consists of two folders of edited obituaries published after Segre's death and of printed commemorations, which were written for special occasions, such as the centenary year of the birth. They are listed alphabetically.

- *Segre's Obituaries*:

[Announcement by E. Bertini, G. Castelnuovo, E. D'Ovidio, G. Fano, E. Pascal, C. Somigliana], 30 May 1924, *Rendiconti della Reale Accademia Nazionale dei Lincei*, (5) XXXIII, 1 (1924): 459–461.
Announcement, *Rendiconti del Circolo Matematico di Palermo*, XLVIII, 1, (1924): n.n.
Baker, Henry. F., Corrado Segre, October 1926, *Journal of the London Mathematical Society*, I (1926): 263–271.
Castelnuovo, Guido, Commemorazione del socio nazionale Corrado Segre, 2 November 1924, *R. Accademia Nazionale dei Lincei*, (5) XXXIII, 2 (1924): 353–359.
Coolidge, Julian L., Corrado Segre, May–June 1927, *Bulletin of the American Mathematical Society*, 33 (1927): 352–357.
Fano, Gino, Corrado Segre, *Annuario della R. Università di Torino 1924–1925*, (1925): 1–12.
Loria, Gino, L'opera geometrica di Corrado Segre, *Annali di Matematica pura ed applicata*, (4) II (1924): 1–21.
Segre (Fuà), Elena, Un grande geometra ebreo: Corrado Segre, *La Rassegna mensile di Israel*, (3) XVIII, 3 (1952): 125–127.
Terracini, Alessandro, Corrado Segre (1863–1924), *Jahresbericht der deutschen Mathematiker-Vereinigung*, XXXV (1926): 209–250.
Viglezio, Elisa, In Memoria di Corrado Segre, *Rassegna di Matematica e Fisica*, V, 1–2 (1924): 1–2.
La Redazione, Corrado Segre (20 agosto 1863–18 maggio 1924), *Supplemento ai Rendiconti del Circolo Matematico di Palermo*, XV (1926–1928): 40–73.

- *Segre's Eulogies*:

[Unsigned], Illustri matematici dell'Italia moderna. Corrado Segre (1863–1924), November–December 1959, *Archimede. Rivista per gli insegnanti e i cultori di Matematiche pure e applicate*, XI (1959): 302–308.
Boggio, Tommaso, Nel 4° anniversario della morte di Corrado Segre, *Atti della R. Accademia delle Scienze di Torino*, 63 (1927–28): 303–320.
Segre, Beniamino, Nel primo centenario della nascita di Corrado Segre, 20 December 1963, *Rendiconti del Seminario Matematico dell'Università e del Politecnico di Torino*, XXIII (1964): 7–21.

Terracini, Alessandro, I quaderni di Corrado Segre, *Atti del IV Congresso dell'Unione Matematica Italiana*, Taormina 25–31 October 1951, Roma: Cremonese, (1953): 252–262.

Terracini, Alessandro, Parole del prof. Alessandro Terracini, *Atti del Convegno Internazionale di Geometria Algebrica*, Torino 24–27 May 1961, Torino: Rattero (1961): 9–14.

Appendix 2
Bibliography of the Works by Corrado Segre[60]

1. Scientific Papers

1883a	Sur les différentes espèces de complexes du 2e degré des droites qui coupent harmoniquement deux surfaces du second ordre (with G. Loria), *Mathematische Annalen*, 23, pp. 213–234. (*Opere*, 3, 1–24)
1883b	Studio sulle quadriche in uno spazio lineare ad un numero qualunque di dimensioni, *Memorie della R. Accademia delle Scienze di Torino*, 2, 36, pp. 3–86. (*Opere*, 3, 25–126)
1883c	Sulla geometria della retta e delle sue serie quadratiche, *Memorie della R. Accademia delle Scienze di Torino*, 2, 36, pp. 87–157. (*Opere*, 3, 127–217)
1883d	Su una trasformazione irrazionale dello spazio e sua applicazione allo studio del complesso quadratico di Battaglini e di un complesso lineare di coniche iscritte in un tetraedro, *Giornale di Matematiche*, 21, pp. 355–378. (*Opere*, 3, 234–261)
1883–84a	Sulle geometrie metriche dei complessi lineari e delle sfere e sulle loro mutue analogie, *Atti della R. Accademia delle Scienze di Torino*, 19, pp. 159–186. (*Opere*, 3, 262–287)
1883–84b	Sulle rigate razionali in uno spazio lineare qualunque, *Atti della R. Accademia delle Scienze di Torino*, 19, pp. 355–372. (*Opere*, 1, 1–16)
1883–84c	Sulla teoria e sulla classificazione delle omografie in uno spazio lineare ad un numero qualunque di dimensioni, *Atti della R. Accademia Nazionale dei Lincei. Memorie della Cl. Sci. Fis. Mat. Nat.*, 3, 19, pp. 127–148. (*Opere*, 3, 304–333)
1883–84d	Ricerche sui fasci di coni quadrici in uno spazio lineare qualunque, *Atti della R. Accademia delle Scienze di Torino*, 19, pp. 878–896. (*Opere*, 3, 485–501)
1884a	Teorema sulle relazioni tra una coppia di forme bilineari e la coppia delle loro forme reciproche, *Giornale di Matematiche*, 22, pp. 29–32. (*Opere*, 3, 229–233)

[60]This list of Segre's publications is from the website Giacardi (2013). Segre's works are collected in *Corrado Segre, Opere,* Roma: Ed. Cremonese, 4 vols., 1957–1963, and can be accessed at http://www.bdim.eu/item?id=GM_Segre.

G. Casnati et al. (eds.), *From Classical to Modern Algebraic Geometry*,
Trends in the History of Science, DOI 10.1007/978-3-319-32994-9

1884b Note sur les complexes quadratiques dont la surface singulière est une surface du 2° dégré double, *Mathematische Annalen*, 23, pp. 235–243. (*Opere*, 3, 218–228)

1884c Sur les droites qui ont des moments donnés par rapport à des droites fixes, *Journal für die reine und angewandte Mathematik*, 97, pp. 95–110. (*Opere*, 3, 288–303)

1884d Sur les invariants simultanés de deux formes quadratiques, *Mathematische Annalen*, 24, pp. 152–156. (*Opere*, 3, 334–338)

1884e Étude des différentes surfaces du 4.e ordre à conique double ou cuspidale (générale ou décomposée) considérées comme des projections de l'intersection de deux variétés quadratiques de l'espace à quatre dimensions, *Mathematische Annalen*, 24, pp. 313–444. (*Opere*, 3, 339–484)

1884f Sur un cas particulier de la surface de Kummer. Lettre à M.K. Rohn (Vorgelegt von Prof. Klein), *Berichte über die Verhandlungen der Königlich Sächsischen Gesellschaft der Wissenschaften zu Leipzig. Mathematisch-Physische Classe,* 36, pp. 132–135. (*Opere*, 3, 502–505)

1884g Sur les courbes de tangentes principales des surfaces de Kummer (Extrait d'une lettre adressée a M. Th. Reye), *Journal für die reine und angewandte Mathematik*, 98, pp. 301–303. (*Opere*, 3, 545–547)

1884–85 Considerazioni intorno alla geometria delle coniche di un piano e alla sua rappresentazione sulla geometria dei complessi lineari di rette, *Atti della R. Accademia delle Scienze di Torino*, 20, pp. 487–504. (*Opere*, 4, 1–17)

1885a Ricerche sulle omografie e sulle correlazioni in generale e particolarmente su quelle dello spazio ordinario considerate nella geometria della retta, *Memorie della R. Accademia delle Scienze di Torino*, 2, 37, pp. 395–425. (*Opere*, 4, 18–57)

1885b Sur une expression nouvelle du moment mutuel de deux complexes linéaires, *Journal für die reine und angewandte Mathematik*, 99, pp. 169–172. (*Opere*, 4, 58–62)

1885–86a Sulle varietà normali a tre dimensioni composte di serie semplici razionali di piani, *Atti della R. Accademia delle Scienze di Torino*, 21, pp. 95–115. (*Opere*, 1, 17–35)

1885–86b Sugli spazi fondamentali di un'omografia, *Atti della R. Accademia Nazionale dei Lincei. Rendiconti*, 4, 2, sem.1, pp. 325–327. (*Opere*, 4, 78–80)

1885–86c Ricerche sulle rigate ellittiche di qualunque ordine, *Atti della R. Accademia delle Scienze di Torino*, 21, pp. 868–891. (*Opere*, 1, 56–77)

1886a Remarques sur les transformations uniformes des courbes elliptiques en elles-mêmes, *Mathematische Annalen*, 28, pp. 296–314. (*Opere*, 1, 36–55)

1886b Le coppie di elementi imaginari nella geometria proiettiva sintetica, *Memorie della R. Accademia delle Scienze di Torino*, 2, 38, pp. 3–24. (*Opere*, 2, 208–236)

1886c Note sur les homographies binaires et leurs faisceaux, *Journal für die reine und angewandte Mathematik*, 100, pp. 317–330. (*Opere*, 4, 63–77)

1886d Su alcune proprietà metriche delle correlazioni, *Giornale di Matematiche*, 25, pp. 20–24. (*Opere*, 4, 81–86)

1886–87a Nuovi risultati sulle rigate algebriche di genere qualunque, *Atti della R. Accademia delle Scienze di Torino*, 22, pp. 362–363. (*Opere*, 1, 78–79)

1886–87b Sulla varietà cubica con dieci punti doppi dello spazio a quattro dimensioni, *Atti della R. Accademia delle Scienze di Torino*, 22, pp. 791–801. (*Opere*, 4, 88–98)

1887a Sull'equilibrio di un corpo rigido soggetto a forze costanti in direzione ed intensità e su alcune questioni geometriche affini, *Memorie di Matematica e di Fisica della Società Italiana delle Scienze (detta dei XL)*, 3, 6, pp. 1–35. (*Opere*, 3, 506–544)

1887b Recherches générales sur les courbes et les surfaces réglées algébriques (I partie, Courbes algébriques), *Mathematische Annalen*, 30, pp. 203–226. (*Opere*, 1, 80–104)

1887c Sur un théorème de la géométrie à *n* dimensions (Extrait d'une lettre adressée à Mr. F. Klein), *Mathematische Annalen,* 30, p. 308. (*Opere*, 4, 87)

1887d Sui sistemi lineari di curve piane algebriche di genere *p* (Estratto di lettera al dott. G.B. Guccia), *Rendiconti del Circolo Matematico di Palermo*, 1, pp. 217–221. (*Opere*, 1, 105–109)

1887e Intorno alla geometria su una rigata algebrica, *Atti della R. Accademia Nazionale dei Lincei. Rendiconti*, 4, 3, sem. 2 pp. 3–6. (*Opere*, 1, 110–113)

1887f Sulle varietà algebriche composte di una serie semplicemente infinita di spazi, *Atti della R. Accademia Nazionale dei Lincei. Rendiconti*, 4, 3, sem. 2, pp. 149–153. (*Opere*, 1, 114–118)

1888a Alcune considerazioni elementari sull'incidenza di rette e piani nello spazio a quattro dimensioni, *Rendiconti del Circolo Matematico di Palermo*, 2, pp. 45–52. (*Opere*, 4, 160–166)

1888b Sulle curve normali di genere *p* dei vari spazi (Estratto di lettera al Prof. E. Bertini), *Rendiconti. Reale Istituto lombardo di scienze e lettere,* 2, 21, pp. 523–528. (*Opere*, 1, 119–124)

1888c Un'osservazione sui sistemi di rette degli spazi superiori, *Rendiconti del Circolo Matematico di Palermo*, 2, pp. 148–149. (*Opere*, 4, 167–168)

1888–89 Le corrispondenze univoche sulle curve ellittiche, *Atti della R. Accademia delle Scienze di Torino*, 24, pp. 734–756. (*Opere*, 1, 152–172)

1889a Recherches générales sur les courbes et les surfaces réglées algébriques (II partie, Surfaces réglées algébriques), *Mathematische Annalen,* 34, pp. 1–25. (*Opere*, 1, 125–151)

1889b C. G. C. von Staudt e i suoi lavori, (Preface to the Italian translation by M. Pieri of "*Geometrie der Lage*" of C.G. C. Staudt), in *Geometria di posizione*, Torino: Bocca, pp. V–XXI. (*Opere*, 4, 372–386)

1889c Sulle varietà cubiche dello spazio a quattro dimensioni e su certi sistemi di rette e certe superficie dello spazio ordinario, *Memorie della R. Accademia delle Scienze di Torino*, 2, 39, pp. 3–48. (*Opere*, 4, 99–159)

1889–90 Un nuovo campo di ricerche geometriche, Note I–III, *Atti della R. Accademia delle Scienze di Torino*, 25, pp. 180–205; 290–317; 376–396. (*Opere*, 2, 237–303)

1890 [Nota alla Memoria di Guido Castelnuovo: Sulle superficie algebriche le cui sezioni piane sono curve iperellittiche], *Rendiconti del Circolo Matematico di Palermo*, 4, pp. 86–88.

1890–91 Un nuovo campo di ricerche geometriche, Nota IV, *Atti della R. Accademia delle Scienze di Torino*, 26, pp. 35–71. (*Opere*, 2, 303–337)

1891a Su alcuni indirizzi nelle investigazioni geometriche. Osservazioni dirette ai miei studenti, *Rivista di Matematica*, 1, pp. 42–66. (*Opere*, 4, 387–412).

1891b Una dichiarazione [Risposta a G. Peano], *Rivista di Matematica*, 1, pp. 154–156.

1891c Sulle varietà che rappresentano le coppie di punti di due piani o spazi, *Rendiconti del Circolo Matematico di Palermo*, 5, pp. 192–204. (*Opere*, 1, 173–184)

1891d Le rappresentazioni reali delle forme complesse e gli enti iperalgebrici, *Mathematische Annalen*, 40, pp. 413–467. (*Opere*, 2, 338–395)

1892a Intorno alla storia del principio di corrispondenza e dei sistemi di curve, *Bibliotheca Mathematica*, 2, 6, pp. 33–47. (*Opere*, 1, 185–197)

1892b Riccardo De Paolis. [Cenni biografici], *Rendiconti del Circolo Matematico di Palermo*, 6, pp. 208–224. (*Opere*, 4, 413–427)

1892–93 Alcune idee di Ettore Caporali intorno alle quartiche piane, *Annali di Matematica pura ed applicata*, 2, 20, pp. 237–242. (*Opere*, 4, 169–174)

1893–94 Giuseppe Bruno. [Cenni biografici], *R. Università di Torino. Annuario accademico 1893–94*, Torino: Stamperia Reale, pp. 155–166. (*Opere*, 4, 428–437)

1894a Introduzione alla geometria sopra un ente algebrico semplicemente infinito, *Annali di Matematica pura ed applicata*, 2, 22, pp. 41–142. (*Opere*, 1, 198–304)

1894b Nota sul lavoro di R. De Paolis: Teoria generale delle corrispondenze proiettive e degli aggruppamenti proiettivi sulle forme fondamentali a due dimensioni, *Atti della R. Accademia Nazionale dei Lincei. Rendiconti*, 5, 3, pp. 227–229. (*Opere*, 4, 175–178)

1894–95 Cenni del Socio Corrado Segre su "Die Entwickelung der Theorie der algebraischen Funktionen in älterer und neuerer Zeit", von A. Brill und M. Noether, *Atti della R. Accademia delle Scienze di Torino*, 30, pp. 109–111. (*Opere*, 4, 438–440)

1895 Sulla forma Hessiana, *Atti della R. Accademia Nazionale dei Lincei. Rendiconti*, 5, 4, pp. 143–148. (*Opere*, 1, 305–311)

1895–96 Intorno ad un carattere delle superficie e delle varietà superiori algebriche, *Atti della R. Accademia delle Scienze di Torino*, 31, pp. 485–501. (*Opere*, 1, 312–326)

1896–97 Intorno ad una mia Memoria: Sulla scomposizione dei punti singolari delle superficie algebriche, *Atti della R. Accademia delle Scienze di Torino*, 32, pp. 781–789.

1897a Sulla scomposizione dei punti singolari delle superficie algebriche, *Annali di Matematica pura ed applicata*, 2, 25, pp. 2–54. (*Opere*, 1, 327–379)

1897b Su alcuni punti singolari delle curve algebriche, e sulla linea parabolica di una superficie, *Atti della R. Accademia Nazionale dei Lincei. Rendiconti*, 5, 6, pp. 168–175. (*Opere*, 2, 1–8)

1897–98 Su un problema relativo alle intersezioni di curve e superficie, *Atti della R. Accademia delle Scienze di Torino*, 33, pp. 19–23.

1898 Le molteplicità nelle intersezioni delle curve piane algebriche con alcune applicazioni ai principii della teoria di tali curve, *Giornale di Matematiche*, 36, pp. 1–50. (*Opere*, 1, 380–429)

1898–99 Sophus Lie, *Atti della R. Accademia delle Scienze di Torino*, 34, pp. 363–366; also in *Bollettino di bibliografia e storia delle scienze matematiche,* 2, 1898, 68–75. (*Opere*, 4, 441–443)

1899 Intorno ai punti di Weierstrass di una curva algebrica, *Atti della R. Accademia Nazionale dei Lincei. Rendiconti*, 5, 8, pp. 89–91. (*Opere*, 1, 430–432)

1900 Gli ordini delle varietà che annullano i determinanti dei diversi gradi estratti da una data matrice, *Atti della R. Accademia Nazionale dei Lincei. Rendiconti*, 5, 9, pp. 253–260. (*Opere*, 4, 179–187)

1900–01 Un'osservazione relativa alla riducibilità delle trasformazioni Cremoniane e dei sistemi lineari di curve piane per mezzo di trasformazioni quadratiche, *Atti della R. Accademia delle Scienze di Torino*, 36, pp. 645–651. (*Opere*, 1, 433–439)

1902–03 Congetture intorno all'influenza di Girolamo Saccheri sulla formazione della geometria non-Euclidea, *Atti della R. Accademia delle Scienze di Torino*, 38, pp. 535–547. (*Opere*, 4, 444–455)

1904 On some tendencies in geometric investigations, *Bulletin of the American Mathematical Society*, 2, 10, pp. 442–468. English translation by John Wesley Young of Segre (1891a)

1905a La Geometria d'oggidì e i suoi legami coll'analisi, in *Verhandlungen des dritten internationalen Mathematiker-Kongresses in Heidelberg vom 8 bis 13 August 1904*, pp. 109–120; also in *Rendiconti del Circolo Matematico di Palermo*, 19, 1905, pp. 81–93. (*Opere*, 4, 456–468)

1905b Geometrya dzisiejsza i jej związki z Analizą, *Wiadomości Matematyczne*, Warszawa, 9, pp. 7–21. Polish translation by Samuel Dickstein of Segre (1905a).

1906 Sur la génération projective des surfaces cubiques. (Extrait d'une lettre adressée à M. le Prof. R. Sturm), *Archiv der Mathematik und Physik,* 3, 10, pp. 209–215. (*Opere*, 4, 188–196)

1906–07a Le congruenze rettilinee W aderenti a due superficie rigate, *Atti della R. Accademia delle Scienze di Torino*, 42, pp. 539–550. (*Opere*, 2, 9–19)

1906–07b Su una classe di superficie degl'iperspazi, legate colle equazioni lineari alle derivate parziali di 2° ordine, *Atti della R. Accademia delle Scienze di Torino*, 42, pp. 1047–1079. (*Opere*, 2, 20–49)

1907 Monge e le congruenze generali di rette, *Bibliotheca Mathematica*, 3, 8, pp. 321–324. (*Opere*, 4, 469–473)

1907–08 Sulla generazione delle superficie che ammettono un doppio sistema coniugato di coni circoscritti, *Atti della R. Accademia delle Scienze di Torino*, 43, pp. 985–997. (*Opere*, 2, 50–61)

1908 Complementi alla teoria delle tangenti coniugate di una superficie, *Atti della R. Accademia Nazionale dei Lincei. Rendiconti*, 5, 17, pp. 405–412. (*Opere*, 2, 62–70)

1910a Preliminari di una teoria delle varietà luoghi di spazi, *Rendiconti del Circolo Matematico di Palermo*, 30, pp. 87–121. (*Opere*, 2, 71–114)

1910b Aggiunta alla memoria: Preliminari di una teoria delle varietà luoghi di spazi, *Rendiconti del Circolo Matematico di Palermo*, 30, pp. 346–348. (*Opere*, 2, 115–118)

1911–12 Le geometrie proiettive nei campi di numeri duali, Nota I e II, *Atti della R. Accademia delle Scienze di Torino*, 47, pp. 114–133; 164–185. (*Opere*, 2, 397–431)

1913–14 Sulle congruenze rettilinee W di cui una od ambe le falde focali sono rigate, *Atti della R. Accademia delle Scienze di Torino*, 49, pp. 257–269. (*Opere*, 2, 119–129)

1917a Su una generazione dei complessi quadratici di rette del Battaglini, *Rendiconti del Circolo Matematico di Palermo*, 42, pp. 85–93. (*Opere*, 4, 197–209)

1917b Sui complessi lineari di schiere rigate, o regoli, *Atti della R. Accademia Nazionale dei Lincei. Rendiconti*, 5, 26, pp. 341–344. (*Opere*, 4, 210–214)

1917c Commemorazione del socio nazionale Giuseppe Veronese, *Atti della R. Accademia Nazionale dei Lincei. Rendiconti*, 5, 26.2, pp. 249–258. (*Opere*, 4, 474–486)

1918a Sui complessi lineari di piani nello spazio a cinque dimensioni, *Annali di Matematica pura ed applicata,* 3, 27, pp. 75–123. (*Opere*, 4, 215–263)

1918b Sulla geometria delle schiere rigate o regoli, e in particolare sui complessi lineari di tali enti, *Annali di Matematica pura ed applicata*, 3, 27, pp. 151–181. (*Opere*, 4, 264–294)

1918c Su alcune classi particolari di sistemi continui di quadriche, e sui rispettivi inviluppi, in *Scritti matematici offerti ad Enrico D'Ovidio*, Bocca, Torino, pp. 1–21. (*Opere*, 2, 130–148)

1918–19 Discorso inaugurale. Le previsioni, *R. Università di Torino 1918–1919. Inaugurazione dell'anno accademico. Discorsi e dati statistici*, Torino: Stamperia Reale, pp. 11–25

1919 Un principio di riduzione nello studio delle corrispondenze algebriche, *Atti della R. Accademia Nazionale dei Lincei. Rendiconti*, 5, 28, pp. 308–312. (*Opere*, 4, 329–334)

1919–20 H.G. Zeuthen. Cenno commemorativo, *Atti della R. Accademia delle Scienze di Torino*, 55, pp. 327–328. (*Opere*, 4, 487–488)

1920a I connessi bilineari alternati di coppie di rette, *Rendiconti del Circolo Matematico di Palermo*, 44, pp. 139–166. (*Opere*, 4, 295–328)

1920b Sulle corrispondenze quadrilineari tra forme di 1.a specie e su alcune loro rappresentazioni spaziali, *Annali di Matematica pura ed applicata*, 3, 29, pp. 105–140. (*Opere*, 4, 335–371)

1920–21 Le superficie degli iperspazi con una doppia infinità di curve piane o spaziali, *Atti della R. Accademia delle Scienze di Torino*, 56, pp. 78–89. (*Opere*, 2, 163–175)

1921a Sui fochi di 2° ordine dei sistemi infiniti di piani e sulle curve iperspaziali con una doppia infinità di piani plurisecanti, *Atti della R. Accademia Nazionale dei Lincei. Rendiconti*, 5, 30, pp. 67–71. (*Opere*, 2, 149–153)

1921b Le linee principali di una superficie di S_5 una proprietà caratteristica della superficie di Veronese, Nota I e II, *Atti della R. Accademia Nazionale dei Lincei. Rendiconti*, 5, 30, pp. 200–203; 227–231. (*Opere*, 2, 154–162)

1921c Mehrdimensionale Raume, in *Encyklopädie der mathematischen Wissenschaften*, III.2.2 C7, Hälfte A, pp. 769–972.

1921–22a Le superficie degl'iperspazi con una doppia infinità di curve spaziali, *Atti della R. Accademia delle Scienze di Torino*, 57, pp. 307–317. (*Opere*, 2, 176–185)

1921–22b Max Noether e Hermann Schwarz, *Atti della R. Accademia delle Scienze di Torino*, 57, pp. 161–163. (*Opere*, 4, 489–491)

1922 Commemorazione del socio straniero Carlo Teodoro Reye, *Atti della R. Accademia Nazionale dei Lincei. Rendiconti*, 5, 31, pp. 269–272. (*Opere*, 4, 492–496)

1923–24 Le curve piane d'ordine n circoscritte a un $(n + 1)$-latero completo di tangenti ad una conica e una classe particolare di superficie con doppio sistema coniugato di coni circoscritti, *Atti della R. Accademia delle Scienze di Torino*, 59, pp. 145–162. (*Opere*, 2, 192–207)

1924 Sugli elementi curvilinei che hanno comuni la tangente e il piano osculatore, *Atti della R. Accademia Nazionale dei Lincei. Rendiconti*, 5, 33, pp. 325–329. (*Opere*, 2, 186–191)

1.1 Lithograph

1885–86 *Sunti delle lezioni di geometria proiettiva*, 2 vols. Litografia, Torino.

For the sake of completeness, it should be mentioned that Segre collaborated in editing the mathematics works by Luigi Cremona, *Opere matematiche di Luigi Cremona*, 3 vols.: Hoepli, Milano, 1914–1917. In particular he compiled the following notes placed at the end of the three volumes: vol. I notes [21], [22], [40]–[98]; vol. II notes [13]–[16], [19]–[24], [29]–[55], [63]–[71], [81]–[135]; vol. III notes [1]–[42], [51]–[61], [111].

2. Reports by Segre in *Atti dell'Accademia delle Scienze di Torino*

1889–90 Relazione sul saggio storico di G. Loria, "Il periodo aureo della Geometria greca", 25, pp. 208–209 (with E. D'Ovidio)

1890–91 Relazione intorno alla Memoria di Guido Castelnuovo: "Ricerche generali sopra i sistemi lineari di curve piane", 26, pp. 595–602 (with E. D'Ovidio)

1891–92 Relazione sulla Memoria del Prof. Riccardo De Paolis, intitolata: "Le corrispondenze projettive nelle forme geometriche fondamentali di 1ª specie", 27, pp. 366–375 (with E. D'Ovidio)

1892–93a Relazione intorno alla Memoria intitolata: "Sopra le curve di dato ordine e dei massimi generi in uno spazio qualunque" del dott. Gino Fano, 28, pp. 865–866 (with E. D'Ovidio)

1892–93b Relazione intorno alla Memoria di V. Mollame "Sulle equazioni abeliane reciproche le cui radici si possono rappresentare con $x, \theta x, \theta^2 x, \dots \theta^{x-1} x$", 28, p. 781 (with E. D'Ovidio and G. Peano)

1892–93c Relazione intorno alla Memoria intitolata: “Ricerche di geometria sulle superficie algebriche” del dott. F. Enriques, 28, pp. 867–868 (with E. D'Ovidio)

1894–95 Relazione intorno alla Memoria intitolata, “Sulle equazioni del moto dei corpi elastici” del Dott. Giuseppe Lauricella, 30, pp. 884–885 (V. Volterra with Segre)

1895–96 Relazione sulla Memoria del Dott. Gino Fano, intitolata: “Sulle varietà algebriche con un gruppo continuo non integrabile di trasformazioni proiettive in sé”, 31, pp. 623–624 (with E. D'Ovidio and V. Volterra)

1896–97a Relazione sulla Memoria del Dott. Emilio Almansi intitolata: “Sulla deformazione della sfera elastica”, 32, pp. 329–330 (V. Volterra with Segre)

1896–97b Relazione sulla Memoria del Prof. Orazio Tedone avente per titolo: “Sulle vibrazioni dei corpi solidi omogenei ed isotropi”, 32, pp. 654–655 (V. Volterra with Segre)

1897–98a Relazione sulla Memoria M. Pieri, “I principii della Geometria di Posizione composti in sistema logico-deduttivo”, 33, pp.148–150 (G. Peano with E. D’Ovidio and Segre)

1897–98b Relazione sulla Memoria del dott. Beppo Levi intitolata: “Sulla varietà delle corde di una curva algebrica”, 33, pp. 504–505 (with E. D'Ovidio)

1897–98c Relazione sulla Memoria del dott. Gino Fano intitolata: “I gruppi di Jonquières generalizzati”, 33, pp. 796–797 (with E. D'Ovidio)

1898–99a Relazione sulla Memoria del Prof. Tullio Levi-Civita, “Tipi di potenziali che si possono far dipendere da due sole coordinate”, 34, pp. 345–346 (V. Volterra with Segre)

1898–99b Relazione sulla Memoria del prof. M. Pieri, intitolata: “Della Geometria Elementare come sistema ipotetico-deduttivo”, 34, pp. 760–762 (with E. D'Ovidio)

1899–00 Relazione sulla Memoria del Dott. Ermenegildo Daniele: “Sulle deformazioni infinitesime delle superficie flessibili ed estendibili”, 35, pp. 726–727 (V. Volterra with Segre)

1900–01a Relazione sulla Memoria del prof. Gino Fano, intitolata: “Nuove ricerche sulle congruenze di rette del 3° ordine prive di linea singolare”, 36, pp. 278–279 (with E. D'Ovidio)

1900–01b Relazione sulla Memoria del D^{r}. Francesco Severi “Sopra alcune singolarità delle curve di un iperspazio”, 36, pp. 380–381, (with E. D'Ovidio)

1900–01c Relazione sulla Memoria del prof. Emilio Veneroni, “Sui connessi bilineari fra punti e rette nello spazio ordinario”, 36, pp. 615–616 (with E. D'Ovidio),

1901–02a Relazione sulla Memoria del Prof. Angelo Battelli e Dr. Luigi Magri, intitolata “Sulle scariche oscillatorie”, 37, p. 220 (A. Naccari with Segre)

1901–02b Relazione sulla Memoria del D^{r}. Francesco Severi, intitolata: "Sulle intersezioni delle varietà algebriche, e sopra i loro caratteri e singolarità projettive", 37, pp. 267–269 (with E. D'Ovidio)

1901–02c Relazione intorno alla Memoria di G.Z. Giambelli, intitolata: "Risoluzione del problema degli spazi secanti", 37, p. 733 (with E. D'Ovidio)

1902–03a Relazione sulla Memoria del D^{r}. Guido Fubini: "Sui gruppi di trasformazioni geodetiche", 38, pp. 329–331 (L. Bianchi with Segre)

1902–03b Relazione sulla Memoria del Dott. Francesco Severi, intitolata: "Sulle corrispondenze fra i punti di una curva algebrica, e sopra certe classi di superficie", 38, pp. 764–766 (with G. Morera)

1903–04a Relazione intorno alla Memoria del D^{r}. Umberto Perazzo, intitolata: "Sull'incidenza di rette, piani e spazi ordinari in uno spazio a cinque dimensioni, e su alcune corrispondenze birazionali fra piani e spazi ordinari", 39, p. 355 (with G. Morera)

1903–04b Relazione sulla Memoria del D^{r}. Beppo Levi, intitolata: "Fondamenti della geometria projettiva", 39, pp. 716–717 (with G. Morera)

1904–05a Relazione sulla Memoria del prof. Mario Pieri, "Nuovi principî di geometria projettiva complessa", 40, pp. 378–379 (with G. Peano)

1904–05b Relazione sulla Memoria del prof. Ugo Amaldi, intitolata: "I gruppi conformi reali dello spazio", 40, p. 974 (with G. Morera)

1905–06 Relazione intorno alla Memoria del prof. Ugo Amaldi, "Sui gruppi continui infiniti di trasformazioni di contatto dello spazio", 41, pp. 1114–1115 (with C. Somigliana)

1907–08a Relazione intorno alla Memoria del Dott. Umberto Perazzo, intitolata: "Sopra alcune varietà di rette ed in particolare su vari tipi di complessi cubici", 43, p. 252 (with C. Somigliana)

1907–08b Relazione sulla Memoria di G.Z. Giambelli, intitolata: "Risoluzione del problema generale numerativo per gli spazi plurisecanti di una curva algebrica", 43, p. 1165 (with G. Morera)

1908–09 Relazione intorno alla Memoria del D^{r}. A. Comessatti: "Sulle curve doppie di genere qualunque, e particolarmente sulle curve ellittiche doppie", 44, pp. 509–510 (with E. D'Ovidio)

1910–11 Relazione sulla Memoria dell'Ing.^{e} Carlo Luigi Ricci: "L'ellisse di elasticità trasversale e le sue applicazioni nella Scienza delle Costruzioni", 46, pp. 203–205 (C. Guidi with Segre)

1911–12 Relazione sulla Memoria del sig. M. Stuyvaert: "Un complexe cubique de droites", 46, pp. 219–220 (with E. D'Ovidio)

1914–15a Relazione intorno alla Memoria del prof. Cesare Burali-Forti, "Isometrie vettoriali e Moti geometrici", 50, p. 23, (G. Peano with Segre)

1914–15b Relazione intorno alla Memoria del Dott. Ettore Del Vecchio: "Sulle equazioni $\partial^3 z/\partial x^3 - \partial z/\partial x = \varphi(xy)$, $\partial^3 z/\partial x^3 - \partial^2 z/\partial x^2 = \varphi(xy)$", 50, p. 839 (with E. D'Ovidio)

3. Reports by Segre in *Jahrbuch über die Fortschritte der Mathematik*[61]

Volume 16 (1884):

pp. 246–247 G. Tomaselli, *Esercizi sulle equazioni differenziali con introduzione di Francesco Brioschi,* Milano, Hoepli, 1883

pp. 472–476 R. de Paolis, *Elementi di geometria*, Torino, Loescher

pp. 503–505 A. Del Re, La quadrica dei dodici punti e la quadrica dei dodici piani, Nota I, *Giornale di Matematiche*, v. 22, pp. 221–235

pp. 540–543 V. Martinetti, Sopra alcune trasformazioni involutorie del piano, *Annali di Matematica pura ed applicata*, s. 2, v. 13, pp. 53–80

pp. 543–544 G. Loria, Sulle corrispondenze projettive fra due piani e fra due spazi, *Giornale di Matematiche*, v. 22, pp. 1–16

p. 554 V. Retali, Sopra una proprietà focale della parabola, *Giornale di Matematiche*, v. 22, pp. 217–220

pp. 557–560 A. Del Re, Obblique e circoli osculatori alle coniche in relazione tra loro ed in relazione con altri elementi geometrici di cui sono casi particolari, *Giornale di Matematiche*, v. 22, pp. 75–117

p. 573 N. Salvatore Dino, *Le proprietà fondamentali delle superficie di second'ordine stabilite con i principi della geometria projettiva,* Napoli, Morano

pp. 587–592 E. Bertini, Contribuzione alla teoria delle 27 rette e dei 45 piani tritangenti di una superficie di 3° ordine, *Annali di Matematica pura ed applicata*, v. 12, pp. 301–346

pp. 608–610 E. d'Ovidio, *Teoria analitica delle forme geometriche fondamentali,* Torino, Loescher, 1885; E. d'Ovidio, *Le proprietà fondamentali delle curve di second'ordine studiate sulla equazionegenerale di secondo grado in coordinate cartesiane,* 2ª ed., Torino, Loescher, 1883; E. d'Ovidio, *Le proprietà fondamentali delle superficie di second'ordine studiate sulla equazionegenerale di secondo grado in coordinate cartesiane*, Torino, Loescher, 1883 (Auch in einem Bande, unter dem Titel: E. d'Ovidio, *Geometria analitica*, parte prima)

pp. 711–712 A. Brambilla, Sulla curva gobba del quarto ordine dotata di punto doppio, *Rendiconti del Reale Istituto Lombardo di Scienze e di Lettere*, s. 2, v. 17, pp. 857–866

pp. 717–719 G. Loria, Intorno alla geometria su un complesso tetraedrale, *Atti della R. Accademia delle Scienze di Torino*, v. 19, pp. 849–878

pp. 724–728 E. Caporali, Rapporto, *Rendiconto dell'Accademia delle Scienze fisiche e matematiche, Napoli*, v. 22, pp. 144–145

pp. 733–736 G. Battaglini, L. Cremona, Relazione, *Atti della R. Accademia dei Lincei. Transunti*, s. 3, v. 8, pp. 354–355

[61]These reports can be accessed at http://www.emis.de/MATH/JFM/JFM.html.

pp. 781–782 E. Cavalli, Generalizzazione di un teorema di Pappo e conseguenze che ne derivano, *Atti della R. Accademia delle Scienze di Torino*, v. 20. pp. 33–42

Volume 17 (1885):

p. 53 P. Cassani, *Complementi d'algebra,* Torino, Bocca (only the title appears)

p. 64 V. Mollame, Nuova serie di funzioni sostituibili a quelle di Sturm con vantaggio dei calcoli occorrenti per determinare il numero delle radici reali di un'equazione algebrica, *Atti dell'Accademia Gioenia di scienze naturali in Catania*, s. 3, v. 18, pp. 11–28

p. 65 V. Mollame, Sul sistema di equazioni costituito da una forma quadratica con n variabili uguagliata a zero e da 1 od $n-2$ equazioni lineari ed omogenee fra quelle variabili, *Atti dell'Accademia Gioenia di scienze naturali in Catania*, s. 3, v. 18, pp. 53–59

p. 511 P. Cassani, Geometria pura euclidiana ad n dimensioni, *Giornale di Matematiche*, v. 23, pp.1–19

pp. 589–590 G. Jung, Sopra una classe di configurazioni d'indice 3, *Rendiconti del Reale Istituto Lombardo di Scienze e di Lettere*, s. 2, v. 18, pp. 231–238

p. 591 R. Badia, *Lezioni di geometria complementare*, Città di Castello, S. Lapi

p. 613 F. Aschieri, Sulla trasformazione omografica generale di uno spazio lineare di specie qualunque, *Rendiconti del Reale Istituto Lombardo di Scienze e di Lettere*, s. 2, v. 18, pp. 989–998

pp. 620–621 C.M. Piuma, Intorno ai triangoli iscritti in un'ellisse che hanno il centro di gravità in un punto dato della sua superficie, *Giornale di Matematiche*, v. 23, pp. 20–33

p. 627 P. del Pezzo, Sui sistemi di coniche, *Rendiconto dell'Accademia delle Scienze fisiche e matematiche, Napoli*, v. 23. pp. 61–73

p. 627 V. Retali, Sopra una serie particolare di coniche d'indice due, *Memorie della R. Accademia delle Scienze dell'Istituto di Bologna*, s. 4, v. 5, pp. 373–386

pp. 628–629 C. Intrigìla, Studio geometrico sull'ipocicloide tricuspide, *Giornale di Matematiche*, v. 23, pp. 263–284

pp. 667–668 J.S. et M.N. Vaneček, Sur la génération des surfaces et des courbes gauches par les faisceaux de surfaces, *Atti della R. Accademia dei Lincei. Rendiconti*, Roma, s. 4, v. 1, pp. 130–133

pp. 763–764 G. Loria, Nuovi studi sulla geometria della sfera, *Atti della Reale Accademia delle Scienze di Torino*, v. 20, pp. 505–526

p. 779 R. d'Emilio, Le superficie rigate di una congruenza lineare, *Atti del R. Istituto Veneto di scienze, lettere ed arti*, s. 6, v. 3, pp. 1255–1275

pp. 780–781 A. Porchiesi, Una rappresentazione del complesso lineare sullo spazio ordinario, *Atti della R. Accademia dei Lincei. Memorie*, s. 4, v. 1, pp. 610–621 pp. 781–782 G. Lazzeri, La rappresentazione dello spazio rigato sopra un piano connesso e sua applicazione allo studio dei connessi lineo-lineari, *Atti del R. Istituto Veneto di scienze, lettere ed arti*, s. 6, v. 3, pp. 247–268

p. 797 V. Martinetti, Sopra una classe di trasformazioni involutorie dello spazio, *Rendiconti del Reale Istituto Lombardo di Scienze e di Lettere*, s. 2, v. 18, pp. 132–142

p. 805 G. Sforza, *Il campo ternario completo rappresentato sullo spazio rigato,* Reggio Emilia, Stabil. degli Artigianelli

p. 848 E. Cavalli, Le ovali di Cartesio considerate dal punto di vista cinematico, *Atti della R. Accademia delle Scienze di Turino*, v. 20, pp. 1143–1165

p. 1160 G. Battaglini (relatore), L. Menabrea, E. Betti, U. Dini, Relazione sul concorso al premio Reale per la Matematica nell'anno 1883, *Atti della R. Accademia dei Lincei. Rendiconti*. Roma, s. 4, v. 1, pp. 410–419 (only the title appears)

4. Varia

1898 Review of: F. Enriques, *"Lezioni di geometria proiettiva", Bollettino di bibliografia e storia delle scienze matematiche*, 1, pp. 11–15

1909 Relazione del concorso internazionale per la Medaglia Guccia, (with M. Noether and H. Poincaré), in G. Castelnuovo (ed.) *Atti del IV Congresso Internazionale dei Matematici, in Roma, 6–11, Aprile, 1908,* 3 vols. Roma: Accademia Nazionale dei Lincei, vol. I, pp. 209–216; also in *Rendiconti del Circolo Matematico di Palermo*, 26, 1908, pp. 145–151

1909 Relazione sul concorso al premio Reale per la Matematica, del 1907, *Atti R. Accademia Nazionale dei Lincei. Rendiconti delle Adunanze solenni,* 2, pp. 410–424

1916 Relazione sul concorso al premio Reale per la Matematica del 1913, (with E. Bertini, L. Bianchi, S. Pincherle and V. Volterra), *Atti R. Accademia Nazionale dei Lincei. Rendiconti delle Adunanze solenni,* 3, pp. 23–33; also in *Bollettino di Matematica*, 14, 1915–16, pp. 175–181

5. Handwritten Notebooks (BMP-Segre)[62]

Quaderni. 1 *Teoria generale delle curve e superficie algebriche* (1888–89)
Quaderni. 2 *Introduzione alla teoria delle curve e superficie algebriche* (1889–90)
Quaderni. 3 *Introduzione alla geometria sugli enti algebrici semplicemente infiniti* (1890–91)
Quaderni. 4 *Lezioni di Geometria generale* (1891–92)
Quaderni. 5 *Introduzione alla geometria sugli enti algebr.*[i] *sempl. infiniti* (1892–93), *Introduzione alla geometria delle trasformaz.*[i] *biraz.*[li] *del piano* (1893–94)

[62]The Notebooks can be accessed at the website Giacardi (2013).

Quaderni. 6 *Teoria delle singolarità delle curve e superficie algebriche* (1894–95)
Quaderni. 7 *Fisica matematica* (1895–96)
Quaderni. 8 *Lezioni sulle singolarità delle curve e superficie algebriche* (1896–97)
Quaderni. 9–10 *Fisica matematica* (1896–97)
Quaderni. 11 *Lezioni sui gruppi continui di trasformazioni* (1897–98)
Quaderni. 12 *Lezioni sulle curve algebriche dei vari spazî* (1898–99)
Quaderni. 13 *Lezioni di Geometria numerativa* (1899–900)
Quaderni. 14 *Lezioni sulla teoria delle superficie razionali e dei sistemi lineari di curve piane* (1900–01)
Quaderni. 15 *Introduzione alla geometria sopra una superficie algebrica* (1901–02)
Quaderni. 16 *Lezioni di Geometria non euclidea* (1902–03)
Quaderni. 17 *Applicazioni degli integrali Abeliani alla Geometria* (1903–04)
Quaderni. 18 *Lezioni sulla forma delle curve algebriche* (1904–05)
Quaderni. 19 *Introduzione alla classificazione delle curve algebriche sghembe* (1905–06)
Quaderni. 20 *I gruppi in Geometria* (1906–07)
Quaderni. 21 *Capitoli vari di Geometria della retta* (1907–08)
Quaderni. 22 *Rassegna di concetti e metodi della Geometria moderna* (1908–09)
Quaderni. 23 *Superficie del 3° ordine e curve piane del 4° ordine* (1909–10)
Quaderni. 24 *Le curve e le superficie algebriche, dal punto di vista della Geometria delle trasformazioni birazionali* (1910–11)
Quaderni. 25 *Gruppi continui di trasformazioni* (1911–12)
Quaderni. 26 *Enti geometrici legati ai sistemi lineari di coniche e quadriche* (1912–13)
Quaderni. 27 *Capitoli di Geometria degl'iperspazi* (1913–14)
Quaderni. 28 *Teoria degl'invarianti applicata alla Geometria* (1914–15)
Quaderni. 29 *Capitoli di Geometria differenziale* (1915–16)
Quaderni. 30 *Vedute superiori sulla Geometria elementare* (1916–17)
Quaderni. 31 *Applicazioni degl'integrali Abeliani alle curve algebr.*$^{\text{e}}$ (1917–18)
Quaderni. 32 *Complessi di rette di 1° e 2° grado* (1918–19)
Quaderni. 33 *Lezioni sui gruppi d'ordine finito* (1919–20)
Quaderni. 34 *Geometria delle equazioni differenziali* (1920–21)
Quaderni. 35 *Capitoli di Geom.*$^{\text{a}}$ *algebrica* (1921–22)
Quaderni. 36 *Geometria dei cerchi e delle sfere* (1922–23)
Quaderni. 37 *Geometria differenziale* (1923–24)
Quaderni. 38 [Elenco e valutazione degli studenti dal 1883 al 1892; Appunti di geometria proiettiva]
Quaderni. 39 [Miscellanea di geometria superiore]
Quaderni. 40 [Appunti relativi alle lezioni tenute per la Scuola di Magistero][63]

[63]In the list of his courses compiled by Segre himself, recently found among the documents donated by the Family Fuà, this course is called "Appunti e bibliografia per il Corso di Magistero di Matematica".

Name Index

G. Casnati et al. (eds.), *From Classical to Modern Algebraic Geometry*,
Trends in the History of Science, DOI 10.1007/978-3-319-32994-9

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