

# Chapter 6

## On Commuting $U$ -Operators in Jordan Algebras

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*Dedicated to Professor Amin Kaidi on the occasion of his 65-th anniversary*

**Abstract** Recently Anquela et al. (Trans. AMS 366(11):5877–5902, 2014) proved that for elements  $x, y$  in a non-degenerate Jordan algebra  $J$ , the relation  $x \circ y = 0$  implies that the  $U$ -operators of  $x$  and  $y$  commute:  $U_x U_y = U_y U_x$ . We show that the result may be not true without the assumption on non-degeneracy of  $J$ . We give also a more simple proof of the mentioned result in the case of linear Jordan algebras, that is, when  $\text{char } F \neq 2$ .

**Keywords** Jordan algebras •  $U$ -operators

### 6.1 An Introduction

In a recent paper [1] Anquela et al. have studied the following question for Jordan algebras:

(1) does the relation  $x \circ y = 0$  imply that the quadratic operators  $U_x$  and  $U_y$  commute?

They proved that the answer is positive for non-degenerate Jordan algebras, and left open the question in the general case, not assuming nondegeneracy.

We show that the answer to question (1) is negative in general case. We give also a more simple proof of the result for linear non-degenerate Jordan algebras, that is, over a field  $F$  of characteristic  $\neq 2$ .

Unless otherwise stated, we will deal with associative and Jordan algebras over a field of arbitrary characteristic.

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## 6.2 A Counter-Example

Let us recall some facts on Jordan algebras. We use as general references the books [2, 4, 8], and the paper [3].

Consider the free special Jordan algebra  $SJ[x, y, z]$  and the free associative algebra  $F\langle x, y, z \rangle$  over a field  $F$ . Let  $*$  be the involution of  $F\langle x, y, z \rangle$  identical on the set  $\{x, y, z\}$ . Denote  $\{u\} = u + u^*$  for  $u \in F\langle x, y, z \rangle$ , then  $\{u\} \in SJ[x, y, z]$  [2, 8] (see also [3] for the case of characteristic 2). Below  $ab$  will denote the associative product in  $F\langle x, y, z \rangle$ , so that  $a \circ b = ab + ba$  and  $aU_b = bab$  are the corresponding linear and quadratic operations in  $SJ[x, y, z]$ .

For an ideal  $I$  of  $SJ[x, y, z]$ , let  $\hat{I}$  denote the ideal of  $F\langle x, y, z \rangle$  generated by  $I$ . By Cohn's Lemma [2, lemma 1.1] (see also [3, Corollary to Cohn's Criterion]), the quotient algebra  $J = SJ[x, y, z]/I$  is special if and only if  $I = \hat{I} \cap SJ[x, y, z]$ .

**Lemma 6.2.1.** *The following equality holds in  $SJ[x, y, z] \subseteq F\langle x, y, z \rangle$ :*

$$z[U_x, U_y] = \{(x \circ y)zxy\} - zU_{x \circ y}.$$

*Proof.* We have in  $F\langle x, y, z \rangle$

$$\begin{aligned} z[U_x, U_y] &= yxzxy - xyzyx = (y \circ x)zxy - xyzxy - xyzyx = \\ &= (y \circ x)zxy - xyz(x \circ y) = \{(x \circ y)zxy\} - (x \circ y)z(x \circ y). \end{aligned}$$

□

**Theorem 6.2.1.** *Let  $I$  denote the ideal of  $SJ[x, y, z]$  generated by  $x \circ y = xy + yx$  and  $J = SJ[x, y, z]/I$ . Then for the images  $\bar{x}, \bar{y}$  of the elements  $x, y$  in  $J$  we have  $\bar{x} \circ \bar{y} = 0$  but  $[U_{\bar{x}}, U_{\bar{y}}] \neq 0$ .*

*Proof.* It suffices to show that  $k = z[U_x, U_y] \notin I$ . By lemma 6.2.1,  $k = \{(x \circ y)zxy\} \pmod{I}$ . Now, the arguments from the proof of [2, theorem 1.2], show that  $k \notin I$  when  $F$  is a field of characteristic not 2 (see also [2, exercise 1, p. 12]).

The result is also true in characteristic 2 for quadratic Jordan algebras. In this case, one needs certain modifications concerning the generation of ideals in quadratic case. The author is grateful to T. Cortés and J.A. Anquela who corrected the first "naive" author's proof and suggested the proper modifications which we give below.

We have to prove that  $\{(x \circ y)zxy\} \notin I$ . By Nam and McCrimmon [6, (1.9)], the ideal  $I$  is the outer hull of  $F(x \circ y) + U_{x \circ y} \widehat{SJ[x, y, z]}$ , where  $\widehat{J}$  denotes the unital hull of  $J$ . Assume that there exists a Jordan polynomial  $f(x, y, z, t) \in SJ[x, y, z, t]$  with all of its Jordan monomials containing the variable  $t$ , such that  $\{(x \circ y)zxy\} = f(x, y, z, x \circ y)$ . By degree considerations,  $f = g + h$ , where  $g, h \in SJ[x, y, z, t]$ ,  $g$  is multilinear, and  $h(x, y, z, t)$  is a linear combination of  $U_t z$  and  $z \circ t^2$ . On the other hand, arguing as in [2, Theorem 1.2],  $g \in SJ[x, y, z, t] \subseteq H(F\langle x, y, z, t \rangle, *)$ , and

because of degree considerations and the fact that  $z$  occupies inside position in the associative monomials of  $\{(x \circ y)zxy\}$ ,  $g$  is a linear combination of

$$\{xzyt\}, \{xzt y\}, \{tzxy\}, \{tzyx\}, \{yzt x\}, \{yzxt\},$$

and  $h$  is a scalar multiple of  $U_t z$ . Hence  $f$  has the form

$$\begin{aligned} f(x, y, z, t) &= \alpha_1 \{xzyt\} + \alpha_2 \{xzt y\} + \alpha_3 \{tzxy\} \\ &\quad + \alpha_4 \{tzyx\} + \alpha_5 \{yzt x\} + \alpha_6 \{yzxt\} \\ &\quad + \alpha_7 tzt, \end{aligned}$$

and therefore

$$\begin{aligned} \{(x \circ y)zxy\} &= \alpha_1 \{xzy(x \circ y)\} + \alpha_2 \{xz(x \circ y)y\} + \alpha_3 \{(x \circ y)zxy\} \\ &\quad + \alpha_4 \{(x \circ y)zyx\} + \alpha_5 \{yz(x \circ y)x\} + \alpha_6 \{yzx(x \circ y)\} \\ &\quad + \alpha_7 (x \circ y)z(x \circ y), \end{aligned}$$

Comparing coefficients as in [2, Theorem 1.2], we get

$$\begin{aligned} \alpha_1 &= \alpha_2 = \alpha_5 = \alpha_6 = 0, \\ \alpha_3 &= \lambda + 1, \alpha_4 = \lambda, \alpha_7 = -2\lambda, \end{aligned}$$

for some  $\lambda \in F$ . Going back to  $f$ , we get

$$f = (\lambda + 1)\{tzxy\} + \lambda\{tzyx\} - 2\lambda tzt = \{tzxy\} + \lambda\{tz(x \circ y)\} - 2\lambda U_t z,$$

so that  $\{tzxy\} \in SJ[x, y, z, t]$ , which is a contradiction.

In fact, the standard arguments with the Grassmann algebra do not work in characteristic 2, to prove that  $\{tzxy\} \notin SJ[x, y, z, t]$ , but one can check directly (or with aid of computer) that the space of symmetric multilinear elements in  $F\langle x, y, z, t \rangle$  has dimension 12 while the similar space of Jordan elements has dimension 11.

### 6.3 The Non-degenerate Case

Here we will give another proof of the main result from [1] that the answer to question (1) is positive for nondegenerate algebras, in the case of linear Jordan algebras (over a field  $F$  of characteristic  $\neq 2$ ).

Let  $J$  be a linear Jordan algebra,  $a \in J$ ,  $R_a : x \mapsto xa$  be the operator of right multiplication on  $a$ , and  $U_a = 2R_a^2 - R_{a^2}$ .

As in [1], due to the McCrimmon-Zelmanov theorem [5], it suffices to consider Albert algebras. We will need only the fact that an Albert algebra  $A$  is *cubic*, that is, for every  $a \in A$ , holds the identity

$$a^3 = t(a)a^2 - s(a)a + n(a),$$

where  $t(a), s(a), n(a)$  are linear, quadratic, and cubic forms on  $A$ , correspondingly [2]. Linearizing the above identity on  $a$ , we get the identity

$$\begin{aligned} 2((ab)c + (ac)b + (bc)a) &= 2(t(a)bc + t(b)ac + t(c)ab) \\ &\quad -s(a, b)c - s(a, c)b - s(b, c)a + n(a, b, c), \end{aligned}$$

where  $s(a, b) = s(a + b) - s(a) - s(b)$  and  $n(a, b, c) = n(a + b + c) - n(a + b) - n(a + c) - n(b + c) + n(a) + n(b) + n(c)$  are bilinear and trilinear forms. In particular, we have

$$a^2b + 2(ab)a = t(b)a^2 + 2t(a)ab - s(a, b)a - s(a)b + \frac{1}{2}n(a, a, b). \quad (6.1)$$

**Lemma 6.3.1.** *Let  $a, b \in J$  with  $ab = 0$ . Then  $[U_a, U_b] = [R_{a^2}, R_{b^2}]$ .*

*Proof.* Linearizing the Jordan identity  $[R_x, R_{x^2}] = 0$ , one obtains

$$[R_{a^2}, R_b] = -2[R_{ab}, R_a] = 0,$$

and similarly  $[R_a, R_{b^2}] = 0$ . Therefore,

$$[U_a, U_b] = [2R_a^2 - R_{a^2}, 2R_b^2 - R_{b^2}] = 4[R_a^2, R_b^2] + [R_{a^2}, R_{b^2}].$$

Furthermore,  $[R_a^2, R_b^2] = [R_a, R_b^2R_a + R_aR_b^2]$ . By the operator Jordan identity [2, (1.O<sub>2</sub>)],

$$R_b^2R_a + R_aR_b^2 = -R_{(ba)b} + 2R_{ab}R_b + R_{b^2}R_a = R_{b^2}R_a,$$

therefore  $[R_a^2, R_b^2] = [R_a, R_{b^2}R_a] = [R_a, R_{b^2}]R_a = 0$ , which proves the lemma.  $\square$

**Theorem 6.3.1.** *Let  $J$  be a cubic Jordan algebra and  $a, b \in J$  with  $ab = 0$ . Then  $[U_a, U_b] = 0$ .*

*Proof.* For any  $c \in J$  we have by Lemma 2 and by the linearization of the Jordan identity  $(x, y, x^2) = 0$

$$c[U_a, U_b] = c[R_{a^2}, R_{b^2}] = (a^2, c, b^2) = -2(a^2b, c, b).$$

By (6.1), we have

$$\begin{aligned} (a^2b, c, b) &= t(b)(a^2, c, b) - s(a)(b, c, b) - s(a, b)(a, c, b) \\ &= -2t(b)(ab, c, a) - s(a, b)(a, c, b) = -s(a, b)(a, c, b). \end{aligned}$$

Substituting  $c = a$ , we get  $(a^2b, a, b) = ((a^2b)a)b = (a^2(ba))b = 0$ , which implies  $0 = s(a, b)(a, a, b) = s(a, b)(a^2b)$ . Therefore,  $s(a, b) = 0$  or  $a^2b = 0$ . In both cases this implies  $c[U_a, U_b] = 0$ .  $\square$

**Corollary 6.3.1.** *In an Albert algebra  $A$ , the equality  $ab = 0$  implies  $[U_a, U_b] = 0$ .*

In connection with the counter-example above, we would like to formulate an open question. Let  $f, g \in SJ[x, y, z]$  such that  $g \in \hat{f}$  but  $g \notin (f)$ , where  $(f)$  and  $\hat{f}$  are the ideals generated by  $f$  in  $SJ[x, y, z]$  and in  $F(x, y, z)$ , respectively. Then the quotient algebra  $SJ[x, y, z]/(f)$  is not special, due to Cohn's Lemma. It follows from the results of [7] that the quotient algebra  $\hat{f}/(f)$  is degenerated. The question we want to ask is the following:

*If  $f = 0$  in a nondegenerate Jordan algebra  $J$ , should also be  $g = 0$ ?*

Of course, there is a problem of writing  $f$  and  $g$  in an arbitrary Jordan algebra, we know only what they are in  $SJ[x, y, z]$ , but in the free Jordan algebra  $J[x, y, z]$  they have many pre-images (up to  $s$ -identities), and one may choose pre-images for which the question has a negative answer. For example, the answer is probably negative for  $f = x \circ y$  and  $g = z[U_x, U_y] + G(x, y, z)$ , where  $G(x, y, z)$  is the Glennie  $s$ -identity [2].

So we modify our question in the following way:

In the situation as above, is it true that there exists  $g' \in J[x, y, z]$  such that  $g - g'$  is an  $s$ -identity and  $f = 0$  implies  $g' = 0$  in non-degenerate Jordan algebras?

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