Chapter 16
On the Real Forms of the Exceptional Lie Algebra $\mathfrak{e}_6$ and Their Satake Diagrams

Cristina Draper Fontanals and Valerio Guido

Abstract Satake diagrams of the real forms $\mathfrak{e}_{6,-26}$, $\mathfrak{e}_{6,-14}$ and $\mathfrak{e}_{6,2}$ are carefully developed. The first real form is constructed with an Albert algebra and the other ones by using the two paraoctonion algebras and certain symmetric construction of the Freudenthal magic square.

Keywords Lie algebra • Real forms • Satake diagrams

16.1 Introduction

The real simple Lie algebras were classified by Cartan in 1914 in [4]. This paper required a great amount of computations, and the classification was realized without the Cartan involution. Cartan came back to this classification in [6], where he identified the maximal compactly imbedded subalgebra in each case. The numbering appeared in that work is the used here in Table 16.1. Soon he completed the classification by relating Lie algebras and geometry in [5]. This is the paper containing more information about the exceptional Lie algebras. Several authors along the twentieth century provided different classifications trying to simplify Cartan's arguments. Most of them used a maximally compact Cartan subalgebra, but not all, Araki's approach [1] was based on choosing a maximally noncompact Cartan subalgebra. This method for classifying was a considerable improvement in clarity and simplicity. The classification was stated in terms of certain diagrams, called *Satake diagrams*, described by Helgason in [12, p. 534] based on the facts developed by Satake in [17]. Although Satake diagrams were contained in [1] and

C. Draper Fontanals (✉)
Departamento de Matemática Aplicada, Escuela de las Ingenierías, University of Málaga, 29071 Málaga, Spain
e-mail: cdf@uma.es

V. Guido
Dpto. di Matematica E. De Giorgi, Università del Salento, Via Provinciale Lecce-Arnesano, 73100 Lecce, Italy
e-mail: valerio.guido@unisalento.it

the restricted root systems and their multiplicities were given by [5], according to Helgason's words [12, p. 534] Cartan stated the results for exceptional algebras without proof. Since then, many textbooks contain Satake diagrams (more historical notes can be found in [14] and some examples in [3, 16]), but, as far as we know, it is very difficult to find details of how these diagrams were obtained in the nonclassical cases. On the other hand, note that Satake diagrams are called Tits *indices* by some mathematical communities. Tits' famous classification paper [18] uses precisely this terminology, thus containing the indices of algebraic groups of type E_6 [18, pp. 58–59] not only over the field of real numbers but over some special fields, although "a detailed justification of the classification tables would require much space" so that it is omitted. Then Garibaldi and Petersson show explicitly in [10] how Tits indices are connected with an Albert algebra. Their approach and interests do not coincide with ours. Our purpose here is to construct the Satake diagrams starting with concrete models of the real forms, which besides allows to obtain a lot of valuable information.

In our work in progress about gradings on the five real forms of the exceptional Lie algebra of type $\mathfrak{e}_6$, which tries to dive in the structure of such interesting simple real Lie algebras, Satake diagrams have been very useful because they codify some aspects of the structure of the corresponding real semisimple Lie algebras, namely, some questions related to gradings are encoded in the Satake diagram. A result in this line is the following [7, Theorem 3]: A simple Lie algebra $\mathfrak{g}$ admits a $\mathbb{Z}$-grading of the second kind (that is, $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ with dim $\mathfrak{g}_2 = 1$) if and only if there is a long root α of Δ such that the multiplicity $m_{\bar{\alpha}} = 1$. But, when it was applied in [11] to obtain a fine grading by the group $\mathbb{Z}^2 \times \mathbb{Z}_2^3$ on $\mathfrak{e}_{6,-14}$ (as well as on $\mathfrak{e}_{6,2}$ and $\mathfrak{e}_{6,6}$), it was necessary a more precise knowledge of the restricted roots than that one summarized in Table 16.1.

The structure of this paper is as follows. After recalling some basic facts about real forms, we explain how the Satake diagrams are constructed starting with a Cartan decomposition and a Cartan subalgebra adapted in some sense to this decomposition. Before proceeding to compute the Satake diagrams of $\mathfrak{e}_{6,-26}$, $\mathfrak{e}_{6,-14}$ and $\mathfrak{e}_{6,2}$ in Sects. 16.5–16.7 respectively, we have enclosed a section about composition and symmetric composition algebras, Jordan algebras and constructions of exceptional Lie algebras based on these related structures, just because the constructions we have taken of our three real forms make use of these nonassociative algebras.

16.2 Preliminaries About Real Forms

Given a real vector space V, we call the *complexification* of V, and we denote it by $V^{\mathbb{C}}$, to the complex vector space $V \otimes_{\mathbb{R}} \mathbb{C} = V \oplus \mathbf{i}V$ ($\mathbf{i} \in \mathbb{C}$ the imaginary unit). If $\mathfrak{g}$ is a real Lie algebra, then the complexification $\mathfrak{g}^{\mathbb{C}}$ is a complex Lie algebra with the usual extension of the bracket, that is,

$$[x_1 + \mathbf{i}y_1, x_2 + \mathbf{i}y_2] = [x_1, x_2] - [y_1, y_2] + \mathbf{i}([x_1, y_2] + [y_1, x_2])$$

for any $x_1, x_2, y_1, y_2 \in \mathfrak{g}$. In this case, $\sigma \colon \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ given by $\sigma(x + \mathbf{i}y) = x - \mathbf{i}y$ is an order two conjugate-linear antiautomorphism of $\mathfrak{g}^{\mathbb{C}}$, called the *conjugation* related to $\mathfrak{g}$. Note that $\mathfrak{g}$ coincides with the set of elements of $\mathfrak{g}^{\mathbb{C}}$ fixed by this conjugation.

If L is a complex Lie algebra and $\mathfrak{g} \subset L$ is a real subalgebra, it is said that $\mathfrak{g}$ is a *real form* of L when $\mathfrak{g}^{\mathbb{C}} = L$. Two real forms $\mathfrak{g}_1$ and $\mathfrak{g}_2$ of the same complex Lie algebra L (with related conjugations σ_1 and σ_2 respectively) are isomorphic if and only if there is $f \in \mathrm{Aut}(\mathfrak{g})$ such that $\sigma_2 = f\sigma_1 f^{-1}$.

Given $\mathfrak{g}$ a real semisimple Lie algebra, $\mathfrak{g}$ is said to be *split* if it contains a Cartan subalgebra $\mathfrak{h}$ such that $\mathrm{ad}\, h$ is diagonalizable over $\mathbb{R}$ for any $h \in \mathfrak{h}$; and $\mathfrak{g}$ is said to be *compact* if its Killing form is definite (necessarily negative). A well-known result states that any complex semisimple Lie algebra contains both a split and a compact real form.

The importance of the real forms is due to the following. If $\mathfrak{g}$ is a simple real Lie algebra, either $\mathfrak{g}$ is just a complex simple Lie algebra, but considered as a real Lie algebra, or $\mathfrak{g}^{\mathbb{C}}$ is simple, so that $\mathfrak{g}$ is a real form of $\mathfrak{g}^{\mathbb{C}}$.

The real forms of a complex simple Lie algebra L are characterized by the signature of their Killing form. (By abuse of notation, sometimes we speak about the signature of L to refer the signature of the Killing form of L.) For instance, the signature of the split real form coincides with the rank of L and the signature of the compact real form is equal to $-\dim L$. In the case of the complex simple Lie algebra of type E_6, denoted throughout this work by $\mathfrak{e}_6$, besides the compact and the split real forms, there are three more real forms, with signatures -26, -14 and 2.

16.3   Preliminaries About Satake Diagrams

Let $\mathfrak{g}$ be an arbitrary semisimple Lie algebra over $\mathbb{R}$ and $k \colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ its Killing form, which is nondegenerate. Recall [12, Chap. III, §7] that a decomposition $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$, for a subalgebra $\mathfrak{t}$ and a vector space $\mathfrak{p}$, is called a *Cartan decomposition* if there is a compact real form $\mathfrak{u}$ of $\mathfrak{g}^{\mathbb{C}}$ such that $\mathfrak{t} = \mathfrak{g} \cap \mathfrak{u}$ and $\mathfrak{p} = \mathfrak{g} \cap \mathbf{i}\mathfrak{u}$. There always exists such a decomposition and any two Cartan decompositions are conjugate under an inner automorphism. The automorphism $\theta \colon \mathfrak{g} \to \mathfrak{g}$ which sends $t + p$ to $t - p$ for any $t \in \mathfrak{t}$ and $p \in \mathfrak{p}$, is called a *Cartan involution*. In this case $k|_{\mathfrak{p} \times \mathfrak{p}}$ is positive definite and $k|_{\mathfrak{t} \times \mathfrak{t}}$ is negative definite. Thus, it is said that $\mathfrak{t}$ is a *maximal compactly imbedded subalgebra* of $\mathfrak{g}$. Observe that the signature of the Killing form coincides with $\dim \mathfrak{g} - 2 \dim \mathfrak{t}$.

Take any maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$. Its dimension is called the *real rank* of $\mathfrak{g}$ (which is independent of the choice of $\mathfrak{a}$). For each λ in the dual space $\mathfrak{a}^*$ of $\mathfrak{a}$, let $\mathfrak{g}_\lambda = \{x \in \mathfrak{g} \mid [h, x] = \lambda(h)x \quad \forall h \in \mathfrak{a}\}$. Then λ is called a *restricted root* if $\lambda \neq 0$ and $\mathfrak{g}_\lambda \neq 0$. Denote by Σ the set of restricted roots, which is an abstract root system (not necessarily reduced) according to [3, Proposition 2.3.6], and by $m_\lambda = \dim \mathfrak{g}_\lambda$ the *multiplicity* of the restricted root. Note that the simultaneous diagonalization of $\mathrm{ad}_{\mathfrak{g}}\, \mathfrak{a}$ gives the decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\lambda \in \Sigma} \mathfrak{g}_\lambda$, for $\mathfrak{g}_0 = \mathfrak{a} \oplus \mathrm{Cent}_{\mathfrak{t}}(\mathfrak{a})$.

Now we combine the Cartan decomposition of a semisimple Lie algebra and the root space decomposition of its complexification. Let $\mathfrak{h}$ be any maximal abelian subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}$. Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ (that is, $\mathfrak{h}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$), and, if Δ denotes the root system of $\mathfrak{g}^{\mathbb{C}}$ relative to $\mathfrak{h}^{\mathbb{C}}$ and $\mathfrak{h}_{\mathbb{R}} := \sum_{\alpha \in \Delta} \mathbb{R} h_\alpha$ (where the element $h_\alpha \in [\mathfrak{g}_\alpha^{\mathbb{C}}, \mathfrak{g}_{-\alpha}^{\mathbb{C}}]$ is characterized by $\alpha(h_\alpha) = 2$), we get $\mathfrak{h}_{\mathbb{R}} = \mathfrak{a} \oplus \mathbf{i}(\mathfrak{h} \cap \mathfrak{t})$. The restricted roots are exactly the nonzero restrictions of roots to $\mathfrak{a} \subset \mathfrak{h}^{\mathbb{C}}$, that is: if $\alpha \in \Delta$, denote by $\bar{\alpha} = \alpha|_{\mathfrak{a}} : \mathfrak{a} \to \mathbb{R}$. The roots in $\Delta_0 = \{\alpha \in \Delta \mid \bar{\alpha} = 0\}$ are called the *compact* roots and those in $\Delta \setminus \Delta_0$ the *noncompact* roots. Then $\Sigma = \{\bar{\alpha} \mid \alpha \in \Delta \setminus \Delta_0\}$. Note that $\alpha \in \Delta_0$ if and only if $\alpha(\mathfrak{h}) \subset \mathbf{i}\mathbb{R}$. Again Δ_0 is an abstract root system in the Euclidean space of $\mathfrak{h}_{\mathbb{R}}$ spanned by its elements [3, Proposition 2.3.8].

Take a basis B of the root system Δ adapted to our situation, that is, if we denote by $B_0 = B \cap \Delta_0 = \{\alpha \in B \mid \bar{\alpha} = 0\}$, then the integral linear combinations of elements in B_0 with all the coefficients having the same sign coincide with the elements in Δ_0 (B_0 is a basis of the root system Δ_0). This is equivalent to choose an ordering Δ^+ such that for any $\alpha \in \Delta^+ \setminus \Delta_0$ then $\sigma^* \alpha \in \Delta^+$, where $\sigma^* \alpha(h) = \overline{\alpha(\sigma(h))}$ for σ the conjugation related to $\mathfrak{g}$.

The Satake diagram of the real algebra $\mathfrak{g}$ is defined as follows. In the Dynkin diagram associated to the basis B, the roots in B_0 are denoted by a black circle $\bullet$ and the roots in $B \setminus B_0$ are denoted by a white circle $\circ$. If $\alpha, \beta \in B \setminus B_0$ are such that $\bar{\alpha} = \bar{\beta}$, then α and β are joined by a curved arrow. Observe that the rank of $\mathfrak{g}^{\mathbb{C}}$ coincides with the real rank of $\mathfrak{g}$ plus the number of arrows on the Satake diagram plus the number of black nodes. We enclose a list for each simple Lie algebra $\mathfrak{g}$ which is a real form of $\mathfrak{e}_6$. The table contains the Dynkin diagram of $\bar{B} = \{\bar{\alpha} \mid \alpha \in B \setminus B_0\}$, which is a basis of the root system Σ, joint with the multiplicities m_λ and $m_{2\lambda}$ for $\lambda \in \bar{B}$. This table is extracted from [12, Table VI], that obtained it from [1].

A consistent extension of this notation is to define the Satake diagram of a compact semisimple Lie algebra to be the Dynkin diagram of the complexification $\mathfrak{g}^{\mathbb{C}}$ with all the nodes black.

16.4 Preliminaries About Related Algebras

16.4.1 Composition Algebras

A real *composition algebra* (C, n) is an $\mathbb{R}$-algebra C endowed with a nondegenerate quadratic form (the *norm*) $n : C \to \mathbb{R}$ which is multiplicative, that is, $n(xy) = n(x)n(y)$ for all $x, y \in C$. Denote also by n to the polar form $n(x, y) := n(x + y) - n(x) - n(y)$. A composition algebra is called *split* if its norm is isotropic.

Table 16.1 Satake diagrams, restricted root systems and multiplicities

$\mathfrak{g}$	Satake diagram of (B,θ)	Dynkin diagram of $\bar{B}$	m_λ	$m_{2\lambda}$
EI	(diagram)	(diagram)	1	0
EII	(diagram) $\alpha_1\ \alpha_3\ \alpha_4\ \alpha_5\ \alpha_6$, α_2	(diagram) $\alpha_1\ \alpha_3\ \alpha_2\ \alpha_4$	$\begin{cases} 1\ (i=2,4) \\ 2\ (i=1,3) \end{cases}$	$\begin{matrix}0\\0\end{matrix}$
EIII	(diagram) $\alpha_1\ \alpha_3\ \alpha_4\ \alpha_5\ \alpha_6$, α_2	(diagram) $\alpha_1\ \alpha_2$	$\begin{cases} 8\ (i=1) \\ 6\ (i=2) \end{cases}$	$\begin{matrix}1\\0\end{matrix}$
EIV	(diagram) $\alpha_1\ \alpha_6$	(diagram) $\alpha_1\ \alpha_6$	8	0

The unital composition algebras are called *Hurwitz algebras*. Each Hurwitz algebra satisfies a quadratic equation

$$x^2 - t(x)x + n(x)1 = 0,$$

where the linear map $t(x) := n(x,1)$ is called the *trace*. Besides it has a *standard involution* defined by $\bar{x} := t(x)1 - x$, so that $n(x) = x\bar{x}$.

There is a standard process to construct these algebras starting from algebras of lower dimension, the so-called *Cayley-Dickson doubling process*. Let A be a Hurwitz algebra with norm n and let be $0 \neq \alpha \in \mathbb{R}$. Then the product $A \times A$ is endowed with the following product:

$$(a,b)(c,d) = (ac + \alpha \bar{d}b, da + b\bar{c}). \tag{16.1}$$

This new algebra, denoted by $CD(A,\alpha)$, has $(1,0)$ as a unit, it contains a copy of A ($\equiv \{(a,0) \mid a \in A\}$) and the element $u = (0,1)$ satisfies $u^2 = \alpha 1$, so that it can be identified with $A \oplus Au$. In particular $\dim(CD(A,\alpha)) = 2\dim A$. Moreover $CD(A,\alpha)$ is endowed with the quadratic form $n((a,b)) = n(a) - \alpha n(b)$, being a Hurwitz algebra if and only if A is associative.

It can be easily proved that there are seven real Hurwitz algebras up to isomorphism, of dimensions 1, 2, 4 and 8, namely:

- the ground field $\mathbb{R}$ (the involution is the identity);
- the split algebra $\mathbb{R} \oplus \mathbb{R} \cong CD(\mathbb{R},1)$ (with componentwise product and the exchange involution);
- the algebra of complex numbers $\mathbb{C} = \langle 1,i \rangle \cong CD(\mathbb{R},-1)$ (the involution is the conjugation);

- the matrix algebra $\mathrm{Mat}_{2\times 2}(\mathbb{R}) \cong CD(\mathbb{C}, 1)$ (the norm is given by the determinant);
- the quaternion division algebra $\mathbb{H} = \langle 1, i, j, k \rangle \cong CD(\mathbb{C}, -1)$, with $i^2 = j^2 = k^2 = ijk = -1$ (the fundamental formulas discovered by Hamilton in 1843);
- the octonion division algebra $\mathbb{O} = \langle 1, i, j, k, l, il, jl, kl \rangle \cong CD(\mathbb{H}, -1)$, where the multiplication table is obtained from Eq. (16.1) for $CD(\mathbb{H}, -1) = \mathbb{H} \oplus \mathbb{H}l$ with $l^2 = -1$;
- and the split octonion algebra $\mathbb{O}_s \cong CD(\mathbb{H}, 1)$. This algebra has a *standard basis* $\{e_1, e_2, u_1, u_2, u_3, v_1, v_2, v_3\}$ where e_1 and e_2 are orthogonal idempotents ($e_1 + e_2 = 1$),

$$
\begin{aligned}
e_1 u_j = u_j = u_j e_2, \ u_i v_i = -e_1, \ u_i u_{i+1} = v_{i+2} = -u_{i+1} u_i, \\
e_2 v_j = v_j = v_j e_1, \ v_i u_i = -e_2, \ v_i v_{i+1} = u_{i+2} = -v_{i+1} v_i,
\end{aligned}
\tag{16.2}
$$

all the remaining products of basis elements are 0, and the polar form of the norm of two basis elements is zero except for $n(e_1, e_2) = 1 = n(u_i, v_i)$, $i = 1, 2, 3$.

16.4.2 Symmetric Composition Algebras

A real composition algebra (C, n) is called a *symmetric composition algebra* if the (nondegenerate multiplicative) quadratic form satisfies $n(xy, z) = n(x, yz)$ for any $x, y, z \in C$. The multiplication is usually denoted by $*$ instead of by juxtaposition, specially if there is some ambiguity.

Again its dimension is forced to be 1, 2, 4 or 8. The only examples are para-Hurwitz and pseudo-octonion algebras.

- If C is a Hurwitz algebra, the same vector space with new product

$$
x * y = \bar{x}\bar{y}
$$

for any $x, y \in C$ (and the same norm) is a symmetric composition algebra called the *para-Hurwitz algebra* attached to the Hurwitz algebra C. We will denote it by pC.
- Consider the algebra of 3×3 traceless matrices $\mathfrak{sl}(3, \mathbb{C})$ with the involution $x^* = p\bar{x}^t p^{-1}$ given by certain regular matrix p. Note that the (real) subspace of the antihermitian matrices $S = \{x \in \mathfrak{sl}(3, \mathbb{C}) \mid x^* = -x\}$ is closed for the product

$$
x * y = \omega xy - \omega^2 yx - \frac{\omega - \omega^2}{3} \, \mathrm{tr}(xy) I_3,
$$

where $\omega = e^{\frac{2\pi i}{3}} \in \mathbb{C}$ is a primitive cubic root of 1 (so $\bar{\omega} = \omega^2$) and I_3 denotes the identity matrix. The obtained real algebra $(S, *)$, endowed with the norm $n(x) = -\frac{1}{2} \mathrm{tr}(x^2)$, turns out to be a symmetric composition algebra, called *pseudo-octonion* algebra or Okubo algebra. We will use the notation $\mathcal{O}$ in case

$p = I_3$ and $\mathcal{O}_s$ when $p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. Thus $\mathcal{O}$ coincides with $\mathfrak{su}(3)$ as a vector space and $\mathcal{O}_s$ with $\mathfrak{su}(2, 1)$. Observe that the norm $n: \mathcal{O} \to \mathbb{R}$ is definite, while $n: \mathcal{O}_s \to \mathbb{R}$ is isotropic.

These pseudo-octonion algebras are not isomorphic to any para-Hurwitz algebra. They were introduced by Susumu Okubo (see [15, (4.35) and (4.9)], where the definition appears with a minor modification) while he was working in Particle Physics. In particular, there are 4 real symmetric composition algebras of dimension 8: $p\mathbb{O}$ and $p\mathbb{O}_s$ (also called para-octonion algebras), $\mathcal{O}$ and $\mathcal{O}_s$.

16.4.3 Constructions of Exceptional Lie Algebras Based on Symmetric Composition Algebras

Let $(S, *, q)$ be a symmetric composition algebra and let

$$\mathfrak{o}(S, q) = \{d \in \text{End}(S) \mid q(d(x), y) + q(x, d(y)) = 0 \ \forall x, y \in S\}$$

be the corresponding orthogonal Lie algebra. Consider the subalgebra of $\mathfrak{o}(S, q)^3$ defined by

$$\mathfrak{tri}(S, *, q) = \{(d_0, d_1, d_2) \in \mathfrak{o}(S, q)^3 \mid d_0(x * y) = d_1(x) * y + x * d_2(y) \ \forall x, y \in S\},$$

which is called the *triality Lie algebra* in [8]. The order three automorphism ϑ given by

$$\vartheta: \mathfrak{tri}(S, *, q) \longrightarrow \mathfrak{tri}(S, *, q), \quad (d_0, d_1, d_2) \longmapsto (d_2, d_0, d_1),$$

is called the *triality automorphism*. Take the element of $\mathfrak{tri}(S, *, q)$ (denoted by $\mathfrak{tri}(S)$ when there is no ambiguity) given by

$$t_{x,y} := \left(\sigma_{x,y}, \frac{1}{2} q(x, y) id - r_x l_y, \frac{1}{2} q(x, y) id - l_x r_y \right), \tag{16.3}$$

where $\sigma_{x,y}(z) = q(x, z)y - q(y, z)x$, $r_x(z) = z * x$, and $l_x(z) = x * z$ for any $x, y, z \in S$.

Let $(S, *, q)$ and $(S', \star, q')$ be two real symmetric composition algebras. Take nonzero scalars $\varepsilon_0, \varepsilon_1, \varepsilon_2 \in \mathbb{R}$ and consider $\varepsilon = (\varepsilon_0, \varepsilon_1, \varepsilon_2)$ and the following vector space,

$$\mathfrak{g}_\varepsilon(S, S') := \mathfrak{tri}(S, *, q) \oplus \mathfrak{tri}(S', \star, q') \oplus \left(\bigoplus_{i=0}^{2} \iota_i(S \otimes S') \right),$$

where $\iota_i(S \otimes S')$ is just a copy of $S \otimes S'$ ($i = 0, 1, 2$), and the anticommutative product on $\mathfrak{g}_\varepsilon(S, S')$ is determined by the following conditions:

- $\mathfrak{tri}(S, *, q) \oplus \mathfrak{tri}(S', \star, q')$ is a Lie subalgebra of $\mathfrak{g}_\varepsilon(S, S')$,
- $[(d_0, d_1, d_2), \iota_i(x \otimes x')] = \iota_i(d_i(x) \otimes x')$,
- $[(d_0', d_1', d_2'), \iota_i(x \otimes x')] = \iota_i(x \otimes d_i'(x'))$,
- $[\iota_i(x \otimes x'), \iota_{i+1}(y \otimes y')] = \varepsilon_{i+2}\iota_{i+2}((x * y) \otimes (x' \star y'))$,
- $[\iota_i(x \otimes x'), \iota_i(y \otimes y')] = \varepsilon_{i+1}\varepsilon_{i+2}(q'(x', y')\vartheta^i(t_{x,y}) + q(x, y)\vartheta'^i(t'_{x',y'}))$,

for any $(d_0, d_1, d_2) \in \mathfrak{tri}(S)$, $(d_0', d_1', d_2') \in \mathfrak{tri}(S')$, $x, y \in S$, $x', y' \in S'$, $i = 0, 1, 2$ indices taken modulo 3, and where ϑ and ϑ' denote the corresponding triality automorphisms.

When one of the involved symmetric composition algebras has dimension 2 (respectively 1) and the other one has dimension 8, the anticommutative algebra $\mathfrak{g}_\varepsilon(S, S')$ defined in this way turns out to be a real form of $\mathfrak{e}_6$ (respectively of $\mathfrak{f}_4$) according to the following table (note that $\mathbb{R} = p\mathbb{R}$):

$(1, 1, 1)$	$\mathbb{R}$	$p\mathbb{C}$	$p(\mathbb{R} \oplus \mathbb{R})$
$p\mathbb{O}, \mathbb{O}$	$\mathfrak{f}_{4,-52}$	$\mathfrak{e}_{6,-78}$	$\mathfrak{e}_{6,-26}$
$p\mathbb{O}_s, \mathbb{O}_s$	$\mathfrak{f}_{4,4}$	$\mathfrak{e}_{6,2}$	$\mathfrak{e}_{6,6}$

$(1, -1, 1)$	$\mathbb{R}$	$p\mathbb{C}$	$p(\mathbb{R} \oplus \mathbb{R})$
$p\mathbb{O}, \mathbb{O}$	$\mathfrak{f}_{4,-20}$	$\mathfrak{e}_{6,-14}$	$\mathfrak{e}_{6,-26}$
$p\mathbb{O}_s, \mathbb{O}_s$	$\mathfrak{f}_{4,4}$	$\mathfrak{e}_{6,2}$	$\mathfrak{e}_{6,6}$

In other words, all the real forms of $\mathfrak{e}_6$ (and of $\mathfrak{f}_4$) can be obtained with this construction (details in [9]). Furthermore, all the real forms of *any* exceptional simple Lie algebra appear by choosing symmetric composition algebras S and S' of various dimensions.

16.4.4   Jordan Algebras

A *Jordan algebra* is a commutative (nonassociative) algebra satisfying the Jordan identity $(x^2y)x = x^2(yx)$. An important example is $H_3(\mathbb{O}) = \{x = (x_{ij}) \in \mathrm{Mat}_{3 \times 3}(\mathbb{O}) \mid x = \bar{x}^t\}$ with the product $x \cdot y = \frac{1}{2}(xy + yx)$, which is an exceptional Jordan algebra denoted by $\mathbb{A}$. We will make use of the subspace of zero trace elements (for $\mathrm{tr}(x) = \sum_i x_{ii} \in \mathbb{R}$), which will be denoted by $\mathbb{A}_0$.

The real forms of the complexified algebra $\mathbb{A}^{\mathbb{C}}$ are called real *Albert algebras*, and they are quite involved in the real forms of the exceptional Lie algebras. There are three of them up to isomorphism: $\mathbb{A}$, $H_3(\mathbb{O}_s)$, and a third one constructed in [13] by doing slight modifications to the construction of $H_3(\mathbb{O})$ (more precisely, considering $x^* = p\bar{x}^tp^{-1}$ for $p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ instead of being p the identity matrix). We follow the approach in [2] for a homogeneous description convenient for our aims.

Given a real symmetric composition algebra $(S, *, q)$ of dimension 8, and a set of three nonzero scalars $\varepsilon = (\varepsilon_0, \varepsilon_1, \varepsilon_2) \in \{\pm 1\}^3$, we define the commutative algebra

$$\mathbb{A}_{(\varepsilon_0, \varepsilon_1, \varepsilon_2)}(S) := \mathbb{R}^3 \oplus \left(\bigoplus_{i=0}^{2} \iota_i(S) \right),$$

where $\iota_i(S)$ is just a copy of S (i=0,1,2) and the product is given by

- $(\alpha_0, \alpha_1, \alpha_2)(\beta_0, \beta_1, \beta_2) = (\alpha_0\beta_0, \alpha_1\beta_1, \alpha_2\beta_2)$,
- $(\alpha_0, \alpha_1, \alpha_2)\iota_i(x) = \frac{1}{2}(\alpha_{i+1} + \alpha_{i+2})\iota_i(x)$,
- $\iota_i(x)\iota_{i+1}(y) = \varepsilon_{i+2}\iota_{i+2}(x * y)$,
- $\iota_i(x)\iota_i(y) = 2\,\varepsilon_{i+1}\varepsilon_{i+2}q(x, y)(E_{i+1} + E_{i+2})$,

where the indices have been taken modulo 3 and $\{E_0, E_1, E_2\}$ denotes the canonical basis of $\mathbb{R}^3$. This algebra $\mathbb{A}_\varepsilon(S)$ is an Albert algebra, and conversely, the three real Albert algebras appear in this way (for suitable ε and S).

This construction of the Albert algebras is related to the construction of the real forms of $\mathfrak{f}_4$ in the above subsection. Namely, there is a Lie algebra isomorphism between the Lie algebra of derivations $\text{Der}(\mathbb{A}_\varepsilon(S))$ and $\mathfrak{g}_\varepsilon(S, \mathbb{R}) = \text{tri}(S) \oplus (\oplus_{i=0}^2 \iota_i(S))$ (where $S \otimes \mathbb{R}$ has of course been identified with S) given by the map

$$\rho: \mathfrak{g}_\varepsilon(S, \mathbb{R}) \to \text{Der}(\mathbb{A}_\varepsilon(S)), \tag{16.4}$$

where if $(d_0, d_1, d_2) \in \text{tri}(S, *, q)$, then $\rho(d_0, d_1, d_2)$ is the derivation of $\mathbb{A}_\varepsilon(S)$ given by $\rho(d_0, d_1, d_2)(\alpha_0, \alpha_1, \alpha_2) = 0$ and $\rho(d_0, d_1, d_2)(\iota_i(x)) = \iota_i(d_i(x))$; and $\rho(\iota_i(x))$ is the derivation of $\mathbb{A}_\varepsilon(S)$ given by $2[l_{\iota_i(x)}, l_{E_{i+1}}]$ for l the left multiplication operator.

16.5 Satake Diagram and Facts on $\mathfrak{e}_{6,-26}$

Consider $\mathfrak{g} = \text{Der}(\mathbb{A}) \oplus \mathbb{A}_0$, which is endowed with a Lie algebra structure for the Lie bracket $[x, y] := [l_x, l_y] \in \text{Der}(\mathbb{A})$ if $x, y \in \mathbb{A}_0$ (l the left multiplication operator) and the natural action of $\text{Der}(\mathbb{A})$ on $\mathbb{A}_0$. It is well known [13] that $\mathfrak{g}$ is a real form of $\mathfrak{e}_6$ of signature -26, but also will be a consequence of the computation of its Satake diagram.

First, the decomposition $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$, for

$$\mathfrak{t} = \text{Der}(\mathbb{A}),$$
$$\mathfrak{p} = \mathbb{A}_0,$$

is a Cartan decomposition, since $k|_\mathfrak{p}$ is positive definite and $k|_\mathfrak{t}$ is negative definite, being $k: \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ the Killing form. (In particular, the signature of $\mathfrak{g}$ is $-52 + 26 = -26$.) The maximal compactly imbedded subalgebra $\mathfrak{t}$ is of type F_4.

For unifying the notation, note the isomorphism $\mathbb{A}_{(1,1,1)}(p\mathbb{O}) \cong \mathbb{A}$ such that

$$(\alpha_0, \alpha_1, \alpha_2) + \iota_0(x) + \iota_1(y) + \iota_2(z) \mapsto \begin{pmatrix} \alpha_0 & 2z & 2\bar{y} \\ 2\bar{z} & \alpha_1 & 2x \\ 2y & 2\bar{x} & \alpha_2 \end{pmatrix}.$$

So, we will work with $\mathfrak{g} \equiv \rho(\mathfrak{g}_{(1,1,1)}(p\mathbb{O}, \mathbb{R})) \oplus (\mathbb{A}_{(1,1,1)}(p\mathbb{O}))_0$, being ρ the map defined in Eq. (16.4).

Second, we find $\mathfrak{h} = \mathfrak{a} \oplus (\mathfrak{h} \cap \mathfrak{t})$ a Cartan subalgebra of $\mathfrak{g}$ such that $\mathfrak{a}$ is a maximal abelian subalgebra of $\mathfrak{p}$. If we fix the basis of the octonion division algebra

$$\{e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7\} := \{1, i, j, k, l, il, jl, kl\}, \tag{16.5}$$

(the bilinear form $\frac{1}{2}q$ relative to this basis is the identity matrix of size 8), then we can take

$$\mathfrak{a} = \langle \{E_0 - E_1, E_2 - E_0\} \rangle \subset \mathbb{A}_0,$$
$$\mathfrak{h} \cap \mathfrak{t} = \langle \{\rho(t_{e_0,e_1}), \rho(t_{e_2,e_3}), \rho(t_{e_4,e_5}), \rho(t_{e_6,e_7})\} \rangle \subset \rho(\mathfrak{tri}(p\mathbb{O})),$$

with the notation for the elements in the triality Lie algebra considered in Eq. (16.3).

Third, we decompose the complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$ relative to its Cartan algebra $\mathfrak{h}^{\mathbb{C}} = \mathfrak{h} \otimes_{\mathbb{R}} \mathbb{C}$. Take as a basis

$$h_1 = \tfrac{1}{2}\rho(t_{e_0,e_1}), \quad h_3 = \tfrac{1}{2}\rho(t_{e_4,e_5}), \quad h_5 = E_2 - E_0,$$
$$h_2 = \tfrac{1}{2}\rho(t_{e_2,e_3}), \quad h_4 = \tfrac{1}{2}\rho(t_{e_6,e_7}), \quad h_6 = E_0 - E_1,$$

and an arbitrary element $h = \sum_{i=1}^{6} w_i h_i \in \mathfrak{h}^{\mathbb{C}}$. A long but straightforward computation shows that $\mathrm{ad}\, h$ diagonalizes $\mathfrak{g}^{\mathbb{C}}$ with set of eigenvalues Δ union of:

- $\{\mathbf{i}(\pm w_j \pm w_k) \mid j \neq k, 1 \leq j, k \leq 4\}$ in $\rho(\mathfrak{tri}(p\mathbb{O}))$,
- $\{\pm \mathbf{i} w_j \pm \frac{w_5 + w_6}{2} \mid 1 \leq j \leq 4\}$ in $\rho(\iota_0(p\mathbb{O})) \oplus \iota_0(p\mathbb{O})$,
- $\{\frac{1}{2}(\mathbf{i}(\varepsilon_1 w_1 + \varepsilon_2 w_2 + \varepsilon_3 w_3 + \varepsilon_4 w_4) \pm (2w_5 - w_6)) \mid \varepsilon_i = \pm 1, \Pi_{i=1}^{4}\varepsilon_i = 1\}$ in $\rho(\iota_1(p\mathbb{O})) \oplus \iota_1(p\mathbb{O})$,
- $\{\frac{1}{2}(\mathbf{i}(\varepsilon_1 w_1 + \varepsilon_2 w_2 + \varepsilon_3 w_3 + \varepsilon_4 w_4) \pm (w_5 - 2w_6)) \mid \varepsilon_i = \pm 1, \Pi_{i=1}^{4}\varepsilon_i = -1\}$ in $\rho(\iota_2(p\mathbb{O})) \oplus \iota_2(p\mathbb{O})$.

Take $B = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\} \subset \Delta$ for $\alpha_i \colon \mathfrak{h}^{\mathbb{C}} \to \mathbb{C}$ given by

$$\alpha_1(h) = \tfrac{1}{2}(\mathbf{i} w_1 - \mathbf{i} w_2 - \mathbf{i} w_3 - \mathbf{i} w_4 + w_5 - 2w_6),$$
$$\alpha_2(h) = \mathbf{i}(w_1 + w_2),$$
$$\alpha_3(h) = \mathbf{i}(-w_1 + w_2),$$
$$\alpha_4(h) = \mathbf{i}(-w_2 + w_3),$$
$$\alpha_5(h) = \mathbf{i}(-w_3 + w_4),$$
$$\alpha_6(h) = \tfrac{1}{2}(-2\mathbf{i} w_4 + w_5 + w_6).$$

Again it is a long and direct computation to check that B is a basis of the root system Δ, since $\Delta \subset \sum_{i=1}^{6} \mathbb{Z}_{\geq 0}\alpha_i \cup \sum_{i=1}^{6} \mathbb{Z}_{\leq 0}\alpha_i$. The choice of B is well adapted to the

situation, since the set of compact roots in the basis, $B_0 = \{\alpha \in B \mid \alpha(\mathfrak{a}) = 0\} = \{\alpha \in B \mid \alpha(h_5) = \alpha(h_6) = 0\}$, that is,

$$B_0 = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5\},$$

turns out to be a basis of the set of compact roots $\Delta_0 = \{\mathbf{i}(\pm w_j \pm w_k) \mid j \neq k, 1 \leq j, k \leq 4\}$ (which is of type D_4). As $\alpha_1(w_5 h_5 + w_6 h_6) = \frac{1}{2}(w_5 - 2w_6) \neq \frac{1}{2}(w_5 + w_6) = \alpha_6(w_5 h_5 + w_6 h_6)$, then the nodes related to α_1 and α_6 are not joined in the Satake diagram, which is
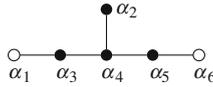
Finally the set of restricted roots is $\Sigma = \pm\{\frac{w_5 + w_6}{2}, \frac{w_5 - 2w_6}{2}, \frac{2w_5 - w_6}{2}\}$, which is a root system of type A_2 (with basis $\bar{B}$), more precisely, $\Sigma = \pm\{\bar{\alpha}_1, \bar{\alpha}_6, \overline{\alpha_1 + \alpha_6}\}$. The multiplicities are

$$m_{\bar{\alpha}_1} = 8, \quad m_{\bar{\alpha}_6} = 8.$$

16.6 Satake Diagram and Facts on $\mathfrak{e}_{6,-14}$

Consider $\mathfrak{g} = \mathfrak{g}_{(1,-1,1)}(p\mathbb{O}, p\mathbb{C})$, which is a real form of $\mathfrak{e}_6$ of signature -14 according to Sect. 16.4.3, but also will be a consequence of the next computations.

Take the decomposition $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$, for

$$\mathfrak{t} = \mathfrak{tri}(p\mathbb{O}) \oplus \mathfrak{tri}(p\mathbb{C}) \oplus \iota_1(p\mathbb{O} \otimes p\mathbb{C}),$$
$$\mathfrak{p} = \iota_0(p\mathbb{O} \otimes p\mathbb{C}) \oplus \iota_2(p\mathbb{O} \otimes p\mathbb{C}),$$

which is a Cartan decomposition since $\mathfrak{t} \oplus i\mathfrak{p} \cong \mathfrak{g}_{(1,1,1)}(p\mathbb{O}, p\mathbb{C}) \cong \mathfrak{e}_{6,-78}$ is compact. Thus $k|_{\mathfrak{p}}$ is positive definite and $k|_{\mathfrak{t}}$ is negative definite, being $k: \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ the Killing form. (In particular, the signature of $\mathfrak{g}$ is $-46 + 32 = -14$.) The maximal compactly imbedded subalgebra $\mathfrak{t}$ is of type $D_5 + Z$.

Second, we find $\mathfrak{h} = \mathfrak{a} \oplus (\mathfrak{h} \cap \mathfrak{t})$ a Cartan subalgebra of $\mathfrak{g}$ such that $\mathfrak{a}$ is a maximal abelian subalgebra of $\mathfrak{p}$. With the basis of $p\mathbb{O}$ fixed in Eq. (16.5), in which the two first elements $\{e_0, e_1\}$ can be considered as a basis of the paracomplex algebra $p\mathbb{C}$, we take

$$\mathfrak{a} = \langle\{\iota_0(e_0 \otimes e_1), \iota_0(e_1 \otimes e_0)\}\rangle,$$
$$\mathfrak{h} \cap \mathfrak{t} = \langle\{t_{e_2,e_3}, t_{e_4,e_5}, t_{e_6,e_7}, (0, \sigma'_{e_0,e_1}, -\sigma'_{e_0,e_1})\}\rangle,$$

with the notations $t_{x,y}$ given in Eq. (16.3), and where the primes are again used for the second symmetric composition algebra involved, in this case, $p\mathbb{C}$.

Now we decompose the complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$ relative to its Cartan algebra $\mathfrak{h}^{\mathbb{C}} = \mathfrak{h} \otimes_{\mathbb{R}} \mathbb{C}$. Take as a basis

$$h_1 = \tfrac{1}{2}t_{e_2,e_3}, \; h_3 = \tfrac{1}{2}t_{e_6,e_7}, \qquad\qquad h_5 = \tfrac{1}{2}\iota_0(e_0 \otimes e_1),$$
$$h_2 = \tfrac{1}{2}t_{e_4,e_5}, \; h_4 = \tfrac{1}{4}(0, \sigma'_{e_0,e_1}, -\sigma'_{e_0,e_1}), \; h_6 = \tfrac{1}{2}\iota_0(e_1 \otimes e_0),$$

and an arbitrary element $h = \sum_{i=1}^{6} w_i h_i \in \mathfrak{h}^{\mathbb{C}}$. We get that $\operatorname{ad} h$ acts on

- the eigenvector $t_{e_0,e_2} + \mathbf{i}t_{e_0,e_3} + \iota_0((e_2 + \mathbf{i}e_3) \otimes e_1)$ with eigenvalue $-\mathbf{i}w_1 - w_5$,
- the eigenvector $t_{e_2,e_4} + \mathbf{i}t_{e_3,e_4} + \mathbf{i}t_{e_2,e_5} - t_{e_3,e_5}$ with eigenvalue $-\mathbf{i}w_1 - \mathbf{i}w_2$,
- the eigenvector $\iota_0(e_0 \otimes e_0) - \iota_0(e_1 \otimes e_1) + t_{e_0,e_1} + t'_{e_0,e_1}$ with eigenvalue $w_5 + w_6$,

where that kind of eigenvectors spans $\operatorname{tri}(p\mathbb{O}) \oplus \operatorname{tri}(p\mathbb{C}) \oplus \iota_0(p\mathbb{O} \otimes p\mathbb{C})$, and on

- the eigenvector $\iota_1((e_0 + \mathbf{i}e_1) \otimes (e_0 + \mathbf{i}e_1)) + \iota_2((e_0 - \mathbf{i}e_1) \otimes (\mathbf{i}e_0 + e_1))$ with eigenvalue $\tfrac{1}{2}(\mathbf{i}w_1 + \mathbf{i}w_2 + \mathbf{i}w_3 - \mathbf{i}w_4 - w_5 - w_6)$.

Analogously, it is checked that $\operatorname{ad} h$ diagonalizes $\mathfrak{g}^{\mathbb{C}}$ with eigenvalues

$$\begin{aligned}
\Delta = \; & \{\pm\mathbf{i}w_k \pm w_5, \pm\mathbf{i}w_k \pm w_6 \mid k = 1, 2, 3\} \\
& \cup \{\mathbf{i}(\pm w_1 \pm w_2), \mathbf{i}(\pm w_1 \pm w_3), \mathbf{i}(\pm w_2 \pm w_3)\} \\
& \cup \{\pm w_5 \pm w_6\} \\
& \cup \{\tfrac{1}{2}(\mathbf{i}(\pm w_1 \pm w_2 \pm w_3 \pm w_4) \pm w_5 \pm w_6)\}.
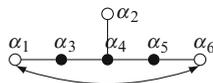\end{aligned}$$

The choice of a basis is usually the most difficult task. In this case, a suitably adapted choice is $B = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\} \subset \Delta$ for $\alpha_i \colon \mathfrak{h}^{\mathbb{C}} \to \mathbb{C}$ given by

$$\begin{aligned}
\alpha_1(h) &= \tfrac{1}{2}(-\mathbf{i}w_1 - \mathbf{i}w_2 - \mathbf{i}w_3 - \mathbf{i}w_4 - w_5 + w_6), \\
\alpha_2(h) &= -\mathbf{i}w_1 + w_5, \\
\alpha_3(h) &= \mathbf{i}(w_2 + w_3), \\
\alpha_4(h) &= \mathbf{i}(w_1 - w_2), \\
\alpha_5(h) &= \mathbf{i}(w_2 - w_3), \\
\alpha_6(h) &= \tfrac{1}{2}(-\mathbf{i}w_1 - \mathbf{i}w_2 + \mathbf{i}w_3 + \mathbf{i}w_4 - w_5 + w_6),
\end{aligned}$$

because it is straightforward to check $\Delta \subset \sum_{i=1}^{6} \mathbb{Z}_{\geq 0}\alpha_i \cup \sum_{i=1}^{6} \mathbb{Z}_{\leq 0}\alpha_i$ (so that B is a basis of the root system Δ) and also $B_0 = \{\alpha \in B \mid \alpha(\mathfrak{a}) = 0\} = \{\alpha \in B \mid \alpha(h_5) = \alpha(h_6) = 0\}$ is

$$B_0 = \{\alpha_3, \alpha_4, \alpha_5\},$$

a basis of $\Delta_0 = \{\mathbf{i}(\pm w_1 \pm w_2), \mathbf{i}(\pm w_1 \pm w_3), \mathbf{i}(\pm w_2 \pm w_3)\}$ (a root system of type A_3). As $\alpha_1(w_5 h_5 + w_6 h_6) = \tfrac{1}{2}(-w_5 + w_6) = \alpha_6(w_5 h_5 + w_6 h_6)$, the nodes related to α_1 and α_6 are joined by an arrow and then the Satake diagram is

In this case the set of restricted roots is $\Sigma = \{\pm w_5, \pm w_6, \pm w_5 \pm w_6, \frac{1}{2}(\pm w_5 \pm w_6)\}$, which coincides with

$$\Sigma = \pm\{\overline{\beta}_1, \overline{\beta}_2, \overline{\beta}_1 + \overline{\beta}_2, 2\overline{\beta}_1 + \overline{\beta}_2, 2\overline{\beta}_1, 2\overline{\beta}_1 + 2\overline{\beta}_2\}$$

for $\beta_1 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$ and $\beta_2 = -\alpha_2$, since $\overline{\beta}_1 = \frac{w_5+w_6}{2}$ and $\overline{\beta}_2 = -w_5$. Thus Σ is a nonreduced root system of type BC_2. The multiplicities of the restricted roots are

$$m_{\frac{w_5+w_6}{2}} = 8,$$
$$m_{w_5+w_6} = 1,$$
$$m_{-w_5} = 6.$$

We can observe that the noncompact (obviously long) roots in the basis (α_1 and α_2) have (restricted) multiplicity different from 1 (so that their root vectors cannot be used to obtain $\mathbb{Z}$-gradings as in [7, Theorem 3], as recalled in Introduction), but the maximal root $\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 = w_5 + w_6$ has (restricted) multiplicity 1 and the $\mathbb{Z}$-grading does appear.

16.7 Satake Diagram and Facts on $\mathfrak{e}_{6,2}$

Consider $\mathfrak{g} = \mathfrak{g}_{(1,1,1)}(p\mathbb{O}_s, p\mathbb{C})$, which is a real form of $\mathfrak{e}_6$ of signature 2 according to Sect. 16.4.3, although this fact will also be a consequence of the next computations. The description of a Cartan decomposition is more involved than in the previous cases, so we will work with this algebra a little bit before giving one. Recall that $d = \sigma'_{e_0,e_1} \in \mathfrak{o}(p\mathbb{C}, n)$ acts in the paracomplex algebra by sending e_0 to $2e_1$ and e_1 to $-2e_0$, and $\mathfrak{tri}(p\mathbb{C}) = \{(\beta_0 d, \beta_1 d, \beta_2 d) \mid \sum_{i=0}^2 \beta_i = 0\}$ is a two-dimensional abelian algebra. Take

$$h_1 = t_{e_1,e_2},\ h_3 = t_{u_2,v_2},\ h_5 = \tfrac{1}{4}(\sigma'_{e_0,e_1}, \sigma'_{e_0,e_1}, -2\sigma'_{e_0,e_1}),$$
$$h_2 = t_{u_1,v_1},\ h_4 = t_{u_3,v_3},\ h_6 = \tfrac{1}{4}(\sigma'_{e_0,e_1}, -2\sigma'_{e_0,e_1}, \sigma'_{e_0,e_1}),$$

where we are taking the standard basis of $\mathbb{O}_s$ as in Eq. (16.2). It turns out that $\sum_{i=1}^6 \mathbb{C}h_i$ is a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$. More concretely, an arbitrary element $\sum_{i=1}^4 w_i h_i \in \mathfrak{h}_0 = \langle h_1, h_2, h_3, h_4 \rangle$ is ad-diagonalizable with eigenvalues

- 0 in $\mathfrak{tri}(p\mathbb{C}) \oplus \mathfrak{h}_0$,
- $\Phi_l = \{\pm w_j \pm w_k \mid j \neq k, 1 \leq j, k \leq 4\}$ in $\mathfrak{tri}(p\mathbb{O}_s)$ (to be precise, $\Phi_l \cup \{0\}$),
- $\Phi_0 = \{\pm w_j \mid 1 \leq j \leq 4\}$ in $\iota_0(p\mathbb{O}_s \otimes p\mathbb{C})$,
- $\Phi_1 = \{\frac{1}{2}(\varepsilon_1 w_1 + \varepsilon_2 w_2 + \varepsilon_3 w_3 + \varepsilon_4 w_4) \mid \varepsilon_i = \pm 1, \Pi_{i=1}^4 \varepsilon_i = 1\}$ in $\iota_1(p\mathbb{O}_s \otimes p\mathbb{C})$,
- $\Phi_2 = \{\frac{1}{2}(\varepsilon_1 w_1 + \varepsilon_2 w_2 + \varepsilon_3 w_3 + \varepsilon_4 w_4) \mid \varepsilon_i = \pm 1, \Pi_{i=1}^4 \varepsilon_i = -1\}$ in $\iota_2(p\mathbb{O}_s \otimes p\mathbb{C})$;

and $w_5 h_5 + w_6 h_6$ acts with eigenvalues

- 0 in $\mathfrak{tri}(p\mathbb{C}) \oplus \mathfrak{tri}(p\mathbb{O}_s)$,
- $\mp\frac{\mathbf{i}}{2}(w_5 + w_6)$ in $\iota_0(x \otimes (e_0 \pm \mathbf{i}e_1))$ for all $x \in p\mathbb{O}_s$,
- $\pm\mathbf{i}(\frac{1}{2}w_5 - w_6)$ in $\iota_1(x \otimes (e_0 \pm \mathbf{i}e_1)) \in \iota_1(p\mathbb{O}_s \otimes p\mathbb{C}))$,
- $\pm\mathbf{i}(w_5 - \frac{1}{2}w_6)$ in $\iota_2(x \otimes (e_0 \pm \mathbf{i}e_1)) \in \iota_2(p\mathbb{O}_s \otimes p\mathbb{C}))$.

In particular $\mathfrak{g}_0 = \mathfrak{g}_{(1,1,1)}(p\mathbb{O}_s, \mathbb{R}e_0)$, which is a real split subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{f}_{4,4}$, has $\mathfrak{h}_0$ as a Cartan subalgebra with roots $\Phi = \Phi_l \cup \Phi_0 \cup \Phi_1 \cup \Phi_2$ (observe that Φ_l are the long roots of Φ and $\Phi_0 \cup \Phi_1 \cup \Phi_2$ are the short ones, which correspond to $\sum_{i=0}^{2} \iota_i(p\mathbb{O}_s \otimes \mathbb{R}e_0)$).

As $\{\frac{1}{2}(w_1 - w_2 - w_3 - w_4), w_4, w_3 - w_4, w_2 - w_3\}$ is a basis of the root system Φ, then we have an ordering with positive roots $\Phi^+ = \{w_j \pm w_k \mid j < k\} \cup \{w_j \mid 1 \le j \le 4\} \cup \{\frac{1}{2}(w_1 + \varepsilon_2 w_2 + \varepsilon_3 w_3 + \varepsilon_4 w_4) \mid \varepsilon_i = \pm 1\}$. Take, for each $\alpha \in \Phi^+$, elements $e_\alpha \in (\mathfrak{g}_0)_\alpha$ and $f_\alpha \in (\mathfrak{g}_0)_{-\alpha}$ such that $[e_\alpha, f_\alpha] = h_\alpha$ (defined as in Sect. 16.3 by $\alpha(h_\alpha) = 2$). That implies that $k_0(e_\alpha, f_\alpha) > 0$ (and $k_0((\mathfrak{g}_0)_\alpha, (\mathfrak{g}_0)_\beta) = 0$ if $\alpha + \beta \ne 0$) for k_0 the Killing form of $\mathfrak{g}_0$. In particular $\mathbf{i}\mathfrak{h}_0 \oplus \langle\{e_\alpha - f_\alpha \mid \alpha \in \Phi^+\}\rangle \oplus \langle\{\mathbf{i}(e_\alpha + f_\alpha) \mid \alpha \in \Phi^+\}\rangle \cong \mathfrak{f}_{4,-52}$ is a compact real form of $(\mathfrak{g}_0)^{\mathbb{C}}$.

Note that, for each $i = 0, 1, 2$, the map $\Psi_i \colon \mathfrak{g} \to \mathfrak{g}$ given by

- $\Psi_i|_{\mathfrak{tri}(p\mathbb{O}_s) \oplus \mathfrak{tri}(p\mathbb{C})} = \mathrm{id}$,
- $\Psi_i|_{\iota_i(p\mathbb{O}_s \otimes p\mathbb{C})} = -\mathrm{id}$,
- $\Psi_i(\iota_j(x \otimes e_k)) = (-1)^{k+1}\iota_j(x \otimes e_{k+1})$ if $j \ne i$, $x \in p\mathbb{O}_s$ and $k = 0, 1 \pmod 2$,

is an automorphism. In particular $k(\Psi_i(e_\alpha), \Psi_i(f_\alpha)) = k(e_\alpha, f_\alpha)$ (a positive multiple of $k_0(e_\alpha, f_\alpha)$) for each $\alpha \in \Phi^+$. Since $\iota_i(p\mathbb{O}_s \otimes \mathbb{R}e_1) = \Psi_{i+1}(\iota_i(p\mathbb{O}_s \otimes \mathbb{R}e_0))$, then

$$\mathfrak{tri}(p\mathbb{C}) \oplus \mathbf{i}\mathfrak{h}_0 \oplus \langle\{e_\alpha - f_\alpha \mid \alpha \in \Phi^+\}\rangle \oplus \langle\{\mathbf{i}(e_\alpha + f_\alpha) \mid \alpha \in \Phi^+\}\rangle \oplus$$
$$\oplus_{i=0}^{2} \left(\langle\{\Psi_i(e_\alpha - f_\alpha) \mid \alpha \in \Phi^+ \cap \Phi_{i+1}\}\rangle \oplus \langle\{\mathbf{i}\Psi_i(e_\alpha + f_\alpha) \mid \alpha \in \Phi^+ \cap \Phi_{i+1}\}\rangle\right)$$

is a compact real form of $\mathfrak{g}^{\mathbb{C}}$. That means that a Cartan decomposition of $\mathfrak{g}$ is the following:

$$\mathfrak{t} = \mathfrak{tri}(p\mathbb{C}) \oplus \langle\{e_\alpha - f_\alpha \mid \alpha \in \Phi^+\}\rangle \oplus \left(\oplus_{i=0}^{2}\langle\{\Psi_i(e_\alpha - f_\alpha) \mid \alpha \in \Phi^+ \cap \Phi_{i+1}\}\rangle\right),$$
$$\mathfrak{p} = \mathfrak{h}_0 \oplus \langle\{e_\alpha + f_\alpha \mid \alpha \in \Phi^+\}\rangle \oplus \left(\oplus_{i=0}^{2}\langle\{\Psi_i(e_\alpha + f_\alpha) \mid \alpha \in \Phi^+ \cap \Phi_{i+1}\}\rangle\right),$$

and the signature of $\mathfrak{g}$ turns out to be $38 - 36 = 2$. Thus $\mathfrak{h} = \mathfrak{a} \oplus (\mathfrak{h} \cap \mathfrak{t})$ is a suitably adapted Cartan subalgebra for

$$\mathfrak{a} = \mathfrak{h}_0 \subset \mathfrak{p},$$
$$\mathfrak{h} \cap \mathfrak{t} = \mathfrak{tri}(p\mathbb{C}),$$

and we have already done the simultaneous diagonalization of the complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$ relative to $\mathfrak{h}^{\mathbb{C}}$: An arbitrary element $\sum_{i=1}^{6} w_i h_i \in \mathfrak{h}^{\mathbb{C}}$ diagonalizes $\mathfrak{g}^{\mathbb{C}}$ with eigenvalues

$$\Delta = \{\pm w_j \pm w_k \mid j \neq k, 1 \leq j, k \leq 4\} \cup \{\pm w_j \pm \tfrac{1}{2}(w_5 + w_6) \mid 1 \leq j \leq 4\}$$
$$\cup \{\tfrac{1}{2}(\varepsilon_1 w_1 + \varepsilon_2 w_2 + \varepsilon_3 w_3 + \varepsilon_4 w_4) \pm \mathbf{i}(\tfrac{1}{2}w_5 - w_6) \mid \varepsilon_i = \pm 1, \Pi_{i=1}^4 \varepsilon_i = 1\}$$
$$\cup \{\tfrac{1}{2}(\varepsilon_1 w_1 + \varepsilon_2 w_2 + \varepsilon_3 w_3 + \varepsilon_4 w_4) \pm \mathbf{i}(w_5 - \tfrac{1}{2}w_6) \mid \varepsilon_i = \pm 1, \Pi_{i=1}^4 \varepsilon_i = -1\}.$$

Take $B = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\} \subset \Delta$ for $\alpha_i \colon \mathfrak{h}^{\mathbb{C}} \to \mathbb{C}$ given by

$$\alpha_1(h) = \tfrac{1}{2}(w_1 - w_2 - w_3 - w_4 + \mathbf{i}(2w_5 - w_6)),$$
$$\alpha_2(h) = w_2 - w_3,$$
$$\alpha_3(h) = \tfrac{1}{2}(2w_4 - \mathbf{i}w_5 - \mathbf{i}w_6),$$
$$\alpha_4(h) = w_3 - w_4,$$
$$\alpha_5(h) = \tfrac{1}{2}(2w_4 + \mathbf{i}w_5 + \mathbf{i}w_6),$$
$$\alpha_6(h) = \tfrac{1}{2}(w_1 - w_2 - w_3 - w_4 - \mathbf{i}(2w_5 - w_6)).$$

It is straightforward to check that B is a basis of the root system Δ, since $\Delta \subset \sum_{i=1}^6 \mathbb{Z}_{\geq 0}\alpha_i \cup \sum_{i=1}^6 \mathbb{Z}_{\leq 0}\alpha_i$. In this occasion all the roots are noncompact, $\Delta_0 = \{\alpha \in \Delta \mid \alpha(\mathfrak{a}) = 0\} = \{\alpha \in \Delta \mid \alpha(h_i) = 0 \quad \forall i = 1, \ldots, 4\} = \emptyset$, so that $B_0 = \emptyset$ and all the nodes are white. Besides $\overline{\alpha}_1 = \overline{\alpha}_6 \colon \sum_{i=1}^4 w_i h_i \mapsto \tfrac{1}{2}(w_1 - w_2 - w_3 - w_4)$ and $\overline{\alpha}_3 = \overline{\alpha}_5 \colon \sum_{i=1}^4 w_i h_i \mapsto w_4$. Hence, the Satake diagram is:
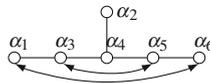
Finally, the set of restricted roots Σ is just Φ, that is, a root system of type F_4 with basis $\{\overline{\alpha}_1, \overline{\alpha}_2, \overline{\alpha}_3, \overline{\alpha}_4\}$. And $\{\alpha \in \Delta \mid \overline{\alpha} = \overline{\alpha}_i\} = \{\alpha_i\}$ if $i = 2, 4$, so that the multiplicities are given by

$$m_{\overline{\alpha}_2} = 1, \; m_{\overline{\alpha}_4} = 1,$$
$$m_{\overline{\alpha}_1} = 2, \; m_{\overline{\alpha}_3} = 2.$$

References

1. S.I. Araki, On root systems and an infinitesimal classification of irreducible symmetric spaces. J. Math. Osaka City Univ. **13**, 1–34 (1962)
2. A.J. Calderón, C. Draper, C. Martín, Gradings on the real forms of $\mathfrak{g}_2$ and $\mathfrak{f}_4$. J. Math. Phys. **51**(5), 053516, 21 pp. (2010)

3. A. Čap, J. Slovák, *Parabolic Geometries I, Background and General Theory*. Mathematical Surveys and Monographs, vol. 154 (American Mathematical Society, Providence, RI, 2009)
4. É. Cartan, Les groupes réels simples finis et continus. Ann. Sci. École Norm. Sup. **31**, 263–355 (1914)
5. É. Cartan, Sur certaines formes riemanniennes remarquables des géométries a groupe fondamental simple. Ann. Sci. École Norm. Sup. **44**, 345–467 (1927)
6. É. Cartan, Sur une classe remarquable d'espaces de Riemann. II. Bull. Soc. Math. Fr. **55**, 114–134 (1927)
7. J.-H. Cheng, Graded Lie algebras of the second kind. Trans. Am. Math. Soc. **302**(2), 467–488 (1987)
8. A. Elduque, The magic square and symmetric compositions. Rev. Mat. Iberoam. **20**, 477–493 (2004)
9. A. Elduque, A new look at Freudenthal's magic square, in *Non-associative Algebra and Its Applications*. Lecture Notes Pure and Applied Mathematics, vol. 246 (Chapman & Hall/CRC, Boca Raton, FL, 2006)
10. S. Garibaldi, H. Petersson, Groups of type E_6 with trivial Tits algebras. Transform. Groups **12**(3), 443–474 (2007)
11. V. Guido, Gradings on $\mathfrak{e}_6$. Ph.D. thesis, Dottorato di Ricerca in Matematica, Università del Salento, 2013–2014
12. S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces.* Corrected reprint of the 1978 original. Graduate Studies in Mathematics, vol. 34 (American Mathematical Society, Providence, RI, 2001), xxvi+641 pp.
13. N. Jacobson, *Exceptional Lie Algebras*. Lecture Notes in Pure and Applied Mathematics (Dekker, New York, 1971)
14. A.W. Knapp, *Lie Groups Beyond an Introduction*, 2nd edn. Progress in Mathematics, vol. 140 (Birkhäuser, Boston, MA, 2002), xviii+812 pp.
15. S. Okubo, *Introduction to Octonion and Other Non-associative Algebras in Physics.* Montroll Memorial Lecture Series in Mathematical Physics, vol. 2 (Cambridge University Press, Cambridge, 1995)
16. A.L. Onishchnik, È.B. Vinberg (eds.) *Lie Groups and Lie Algebras III*. Encyclopedia of Mathematical Sciences, vol. 41 (Springer, Berlin, 1991)
17. I. Satake, On representations and compactifications of symmetric Riemannian spaces. Ann. Math. **71**, 77–110 (1960)
18. J. Tits, Classification of algebraic semisimple groups, in *Algebraic Groups and Discontinuous Subgroups*. Proceedings of Symposium Pure Mathematics, Boulder, CO, 1965 (American Mathematical Society, Providence, RI, 1966), pp. 33–62