Chapter 13
Localization in a Duo-Ring and Polynomials Algebra

Daouda Faye, Mohamed Ben Fraj Ben Maaouia, and Mamadou Sanghare

Abstract Let A be a duo-ring and S a non-empty subset of A formed regular device items, $\overline{S}$ the saturated multiplicative subset satisfying the left conditions of Ore generated by S, $A\left[(X_s)_{s\in S}\right]$ the polynomials algebra of variables in S, M and M ' two left A- modules, we show that the ring of fractions $S^{-1}A$ exists and is isomorphic to the ring $A\left[(X_s)_{s\in S}\right]$ quotiented by the ideal $\langle 1 - sX_s\rangle_{s\in S}$ and also $S^{-1}A$ is isomorphic to $(\overline{S})^{-1}A$.

We have also shown that the module of fractions $S^{-1}M$ exists and $S^{-1}M$ is isomorphic to $\overline{(S)}^{-1} A \otimes_A M$, $S^{-1}Tor_n^A(M, M')$ is isomorphic to $Tor_n^{S^{-1}A}(S^{-1}M, S^{-1}M')$ and $S^{-1}Ext_A^n\,(M, M')$ is isomorphic to $Ext^n_{S^{-1}A}\left(S^{-1}M, S^{-1}M'\right)$ where n is integer and M is a left A-module of finite type.

Keywords Duo-ring • Regular element • Functor • Ring fraction • Fraction module • Localization

13.1 Introduction

In this paper, A denotes a duo-ring and M a left A -module. Called ring of left fractions of A in respect of a non-empty subset S of A (or localization of A in S), every pair (B, i) where B is a ring and $i : A \to B$ a morphism of rings such as $i(s)$

D. Faye (✉)
Faculty of Sciences and Technologies University, Cheikh Anta Diop de Dakar – UCAD, Laboratory of Algebra, Cryptography, Algebraic Geometry and Application – LACGAA, Dakar, Senegal
e-mail: fayeda2@yahoo.fr

M.B.F. Ben Maaouia
Laboratory Algebra, Codes and Cryptography Applications (ACCA), UFR SAT, University Gaston Berger (UGB), St. Louis, Senegal
e-mail: maaouiaalg@hotmail.com

M. Sanghare
Doctoral School of Mathematics-Computer – UCAD, Cheikh Anta Diop University of Dakar, Dakar, Senegal
e-mail: mamadou.sanghare@ucad.edu.sn

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is invertible in B for every $s \in S$ and satisfied the following universal properties: for any ring B' and for any *morphismf* $: A \to B'$ such that $f(s)$ is invertible in B' for every $s \in S$, then there exists a unique morphism $\overline{f} : B \to B'$ such as $\overline{f}oi = f$.

In their works, many authors have built fractions rings in the commutative case from S a multiplicative subset of a commutative ring A (see [7, 12, 13]). In [13], M. Rotman built, in the commutative case the ring of fractions $S^{-1}A$ and fractions module S^{-1}, where M S is not any empty part A using the saturated part multiplicative $\overline{S}$ generated by S.

In his works, Mr. Ben Maaouia built, where A is not necessarily commutative, the fractions ring $S^{-1}A$ and fractions module $S^{-1}M$ relatively to a multiplicative subset saturated S which satisfies the left Ore conditions (see [1, 2, 4, 5]).

In this paper we build, where A is not necessarily commutative, the fractions ring $S^{-1}A$ and fractions module S^{-1} relatively to a non-empty subset S formed of regular elements of A.

Thus we have shown the following results, if S is a non-empty part of a duo ring A formed of regular elements, then:

1. $\overline{S}$, saturated multiplicative subset generated by S satisfies the left Ore conditions.
2. the ideal $\langle 1 - sX_s \rangle_{s \in S}$ of the polynomials algebra in S variables $A\left[(X_s)_{s \in S}\right]$ generated by the set of polynomials $\{1 - sX_s, s \in S\}$ is two-sided.
3. $S^{-1}A$ exists and $S^{-1}A = {A\left[(X_s)_{s \in S}\right]}/{\langle 1 - sX_s \rangle_{s \in S}}$.
4. $S^{-1}A \cong \overline{(S)}^{-1} A$
5. By asking $S^{-1}M = (\overline{S})^{-1}M$, we have $S^{-1}M \cong \overline{(S)}^{-1} A \otimes_A M$.
   Therefore $S^{-1}M \cong {A\left[(X_s)_{s \in S}\right]}/{\langle 1 - sX_s \rangle_{s \in S}} \otimes_A M$.
6. $S^{-1}Tor_n^A(M, M') \tilde{=} Tor_n^{S^{-1}A}(S^{-1}M, S^{-1}M')$ where n is a integer.
7. $S^{-1}Ext_A^n(M, M') \cong Ext_{S^{-1}A}^n\left(S^{-1}M, S^{-1}M'\right)$ where n is a integer and M is an A -module of finite type.

13.2 Preliminary Definitions and Results

A is a ring and M a left A -module, then the left ring of fractions $S^{-1}A$ and the fractions module $S^{-1}M$ exist if and only if S is a saturated multiplicative part of A that satisfies the conditions of Ore (see [6] Chap. 1). Note that the existence of such a party is not evident in any ring.

In a duo-ring, all the regular elements form a saturated multiplicative party checking Ore conditions (see [2] Chap. 2). Thus if S is a part of a duo-ring A formed from regular elements, then S generates saturated multiplicative subset $\overline{S}$ which checks the Ore conditions. Then we can localize from a party formed of regular elements of a duo-ring.

Moreover, several authors have worked on class are not necessarily commutative duo-rings as Rdao in 1970 (see [9]), Brungs in 1975 (see [2]), Mr. Sangharé in 1989 in his thesis of State (see [2]) and in his article "On S-duo-rings" (see [2]), Ben Maaouia in 2003 in his thesis 3rd cycle (see [6]), in 2011 in his article entitled "Localisation in the duo-ring" (see [2]) and in his thesis of State (see [2] Chap. 2).

Definition 2.1. Let A be a ring, A is said:

1. left duo-ring if every left ideal is two-sided,
2. right duo-ring if every right ideal is two-sided
3. duo-ring if it is left duo-ring and right duo-ring.

Example 13.2.

1) Any commutative ring A is a commutative duo-ring.
2) All non-commutative valuation ring is a non-commutative ring duo (see [8, 10, 12]).
3) (see [3, 14, 15]) Let F be the field of polynomial functions with coefficients in Q, where Q is the field of rational numbers. The elements of F are ordered as follows:
$(q_n t^n + \ldots + q_0)(q_m^{'} t^m + \ldots + q_0^{'}) > 0$ if and only if $q_n q_m^{'} > 0$.
Let F be the field of polynomial functions with coefficients in Q, where Q is the field of rational numbers. The elements of F are ordered as follows:.
We consider $R = Q\{\{G^+\}\}$ the ring of formal series $\sum q_\alpha g_\alpha$ où $q_\alpha \in Q$ and $\{g_\alpha\}$ is a suite of well-ordered G with elements $g_\alpha \geqslant (1, 0)$.
Then, R is a duo-ring.

Definition 2.3. Let A be a ring and S a part of A. It is said that :

1. S is a multiplicative part of A if $1_A \in S$ and S is stable by multiplication; ie for all $x, t \in S, , xt \in S, .$
2. S is a saturated multiplicative part if: for all x, $t \in A,\ xt \in S,$ imply $x \in S,$ and $t \in S, .$

Definition 2.4 (Ore Conditions). Let S saturated multiplicative part of a ring A. It is said that S satisfies the left Ore conditions (right respectively) if:

1. $\forall\, a \in A,\ \forall\ s\ \in\ S$, they exist $t \in S,$ and $b \in A,$ such as $ta = bs$ (respectively $at = sb$). It is said that s is left switchable (right respectively).
2. $\forall\, a \in A,\ si\ s\ \in\ S$ such as $as = 0$ (respectively $sa = 0$), then there exists $t \in S,$ such as $ta = 0$ (respectively $at = 0$). It is said that S is left invertible (right respectively).

Example 2.5.

1. The set of regular elements of a duo-ring A is a saturated multiplicative part of A that satisfies the left and right Ore conditions (see [2]).
2. If s is a regular element of a duo-ring A, the set $S = \left\{s^k, k \in \mathbb{N}\right\}$ is a saturated multiplicative subset of A which satisfies left and right Ore conditions.

Definition 2.6. Let A be a ring and S a saturated multiplicative part of A.

Called saturated multiplicative subset generated by S, the smallest saturated multiplicative part containing S denoted $\overline{S}$ if it exists.

Proposition 2.7. *Let A be a duo-ring and S a non-empty saturated multiplicative part of A consists of regular elements. Then S generates a saturated multiplicative subset $\overline{S}$ that satisfies the Ore conditions.*

Proof. Let F be the set of all saturated multiplicative parts of A containing S and satisfies the left Ore conditions. F is not empty, indeed, the set of regular elements of A is a saturated multiplicative part of A containing S and verifies the Ore conditions (see [6]). So the smallest saturated multiplicative part of A containing S and vérifies the Ore conditions is called saturated multiplicative subset generated by S, denotes by $\overline{S}$.

13.3 Fractions Rings and Polynomial Algebra

Theorem 3.1. *Let A be a duo-ring, S of a non-empty part of A consists of regular elements, $A\left[(X_s)_{s\in S}\right]$ algebra of polynomials in S variables with coefficients in A.*

Then the application:

$$\varphi : A\left[(X_s)_{s\in S}\right] \longrightarrow \left(\overline{S}\right)^{-1} A$$
$$P \longmapsto \varphi\left(P\right) = P\left((1/s)_{s\in S}\right).$$

is an algebra homomorphism $\ker\varphi = \langle 1 - sX_s\rangle_{s\in S}$ where $\langle 1 - sX_s\rangle_{s\in S}$ is the two-sided ideal of the algebra of polynomials $A\left[(X_s)_{s\in S}\right]$ generated by the set of polynomials $\{1 - sX_s, s \in S\}$.

Moreover, one has the isomorphism $A\left[(X_s)_{s\in S}\right]/\langle 1 - sX_s\rangle_{s\in S} \cong \left(\overline{S}\right)^{-1} A$ où $\overline{S}$ is saturated multiplicative subset generated by the elements $(1 - sX_S)_{s\in S}$.

Proof. Let P and Q be two polynomials; we have :

$$\varphi(P+Q) = (P+Q)((1/s)_{s\in S}) = (P(1/s))_{s\in S\_} + (Q(1/s))_{s\in S} = \varphi(p) + \varphi(Q)$$

$$\varphi(PQ) = (PQ)((1/s)_{s\in S}) = (P(1/s))_{s\in S\_}(Q(1/s))_{s\in S} = \varphi(p)\varphi(Q)$$

$$\varphi(1_{A[(X_s)_{s\in S}]} = \varphi(1_A) = 1/1 = 1_{(\overline{S})^{-1}A}$$

$$\varphi(\lambda P) = (\lambda P)((1/s)_{s\in S}) = \lambda(P(1/s)_{s\in S}) = \lambda\varphi(P)$$

check that $\ker\varphi = \langle 1 - sX_s\rangle_{s\in S}$.

1. if $P = (1 - sX_S)_{s \in S}$ *alors* $P(\frac{1}{s}) = (1 - s.\frac{1}{s})_{s \in S} = 0$, then $\langle 1 - sX_s \rangle_{s \in S} \subseteq \ker \varphi$.
2. if $P \notin \langle 1 - sX_s \rangle$, then $\varphi(P) \neq 0 \Rightarrow P \notin \ker \varphi$.

Therefore $\ker \varphi = \langle 1 - sX_s \rangle_{s \in S}$ and so $\langle 1 - sX_s \rangle_{s \in S}$ is a two-sided ideal.

Hence, according to the universal property of the Rings quotients, we have the isomorphism ${}^{A\,[(X_s)_{s\in S}]}/_{\langle 1 - sX_s \rangle_{s \in S}} \cong \left(\overline{S}\right)^{-1} A$

Theorem 3.2 (Existence of Fractions Ring). *Let A be a duo-ring and S a non-empty part of regular elements of A. Then the ring of left fractions of A on S, $S^{-1}A$ exists.*

Proof. Let $X = \{x_s : s \in S\}$ such that the application:

$$\begin{aligned} \varphi : X &\to S \\ x_s &\mapsto \varphi\left(x_s\right) = s \end{aligned}.$$

is a bijection.

Let $A\,[(X_s)_{s\in S}]$ be the ring of polynomials over A in S variables and $I = \langle 1 - sX_s \rangle_{s \in S}$ the two-sided ideal generated by the set $\{1 - sx_s : s \in S\}$.

Show that $S^{-1}A = {}^{A\,[(X_s)_{s\in S}]}/_{I}$. As I is a two-sided ideal, then $A\,[(X_s)_{s\in S}]\,/I$ is a ring and a same algebra.

Définissons

$$\begin{aligned} i : A &\to A\,[(X_s)_{s\in S}]/_{I} \\ a &\mapsto i\,(a) = a + I \end{aligned}.$$

the canonical map.

We have $\overline{1 - sX_s} = \overline{0} \Rightarrow \overline{sX_s} = \overline{s}.\overline{X_s} = \overline{1}$ likewise $\overline{X_s.s} = \overline{X_s}.\overline{s} = \overline{1}$ because variables commute with the constants (A elements) in $A\,[(X_s)_{s\in S}]$.

So $\overline{s} = i(s)$. And as fractions ring $S^{-1}A$ of A in S exists to isomorphism, one can take $S^{-1}A = {}^{A\,[(X_s)_{s\in S}]}/_{I}$.

Proposition 3.3. *Let S a non-empty rings of a duo-part A consists of regular elements. Let $\overline{S}$ saturated multiplicative subset generated by S. So we have:*

$\left(\overline{S}\right)^{-1} A \cong S^{-1}A$.

Proof. This proposal is the result of **Theorems 3.1 and 3.2**.

Proposition 3.4. *Let A be a duo-ring, M left A -module, S a non-empty subset of A consists of regular elements, s a element of S. While the application:*

$$\begin{aligned} \mu_s : M &\to M \\ m &\mapsto \; sm \end{aligned}.$$

is bijective for every $s \in S$ *if* M *is a* $S^{-1}A$ *-module. Fractions module called left of* M *with respect to* S*, every pair* (P, h_P) *where* P *is a left* $S^{-1}A$ *-module (that is to say,* $\begin{array}{l}\mu_s : P \to P \\ p \mapsto sp\end{array}$ *is bijective for every* $s \in S$ *and* $h_P : M \to P$ *is called canonical morphism, which is the solution of universal problem solution:*

That is to say, if $\varphi : M \to M'$ is a morphism of left A -modules where M is a S^{-1} A-module, then there exists a unique morphism
$\overline{\varphi} : P \to M'$ such as $\overline{\varphi} o h = \varphi$.

Notation. The left fractions module of M on S, if it exists, is denote by $(S^{-1}M, h)$*or* S $^{-1}$ M if there is no confusion.

Remark 3.5. A is a duo-ring and S of a non-empty A consists of regular elements. So if the left fractions module of M on S exists, it is unique up to isomorphism.

Definition 3.6. Let A be a duo-ring, S is a non-empty subset of A consists of regular elements and M a left A -module. Then we define the left module of fractions M on S denote $S^{-1}M$ by $S^{-1}M = (\overline{S})^{-1}M$ where $(\overline{S})^{-1}M$ is the left fractions module of M with respect to the saturated multiplicative part $\overline{S}$ satisfying the left Ore conditions (see [2] and [4]).

Theorem 3.7. *Let* A *a duo-ring,* S *a non-empty part of* A *consists of regular elements and* M *a left* A *-module. So the couple* $((\overline{S})^{-1}A \otimes_A M, h)$ *where* h *is the morphism defined by:*

$$\begin{aligned} h : M &\to (\overline{S})^{-1}A \otimes_A M \\ m &\mapsto 1 \otimes m \end{aligned}$$

is a left fractions module of M *on* S*.*

Proof. Let $\varphi : M \to M'$ be a morphism where M' is a $S^{-1}A$-module

The morphism:

$$\begin{aligned} \overline{(S)}^{-1} A \times M &\to M' \\ (a/\sigma, m) &\mapsto (a/\sigma)\varphi(m) \end{aligned}$$

where $a \in A$ and $\sigma \in \overline{S}$ is A-bilinear. There exists a unique morphism $\overline{\varphi} : (\overline{S})^{-1}A \otimes_A M \to M'$ such as $\overline{\varphi} \circ h = \varphi$. As $h(M)$ generates $(\overline{S})^{-1}A \otimes_A M$, then $\overline{\varphi}$ is the unique morphism making the following diagram commutative.

So according to the universal property of the localization, $(\overline{S})^{-1}A \otimes_A M$ is a module of fractions M with respect to $\overline{S}$.

Corollary 3.8. *Let* A *be a duo-ring,* S *a non-empty part of* A *consists of regular elements,* M *a left* A*- module and* $S^{-1}M$ *a left fractions of module* M *on* S*. Then* $S^{-1}M \cong \overline{(S)}^{-1} A \otimes_A M$ *Consequently* $S^{-1}M \cong {}^{A\,[(X_s)_{s\in S}]}/_{\langle 1 - sX_s\rangle_{s\in S}} \otimes_A M$

Proof. Indeed just take in Proposition 3.7 $(\bar{S})^{-1}A = A[(X_s)_{s\in S}/(1 - sX_s)_{s\in S}])$ applying Propositions 3.1 and 3.2. The proof follows from **Propositions 3.1, 3.2 and 3.7**.

13.4 Properties for Functions $S^{-1}(\ )$, Ext and Tor

In this section, we show that if A is a duo-ring, S a non-empty subset of A consists of regular elements, M a A -module, then the functor $S^{-1}()$ commutes with the functors $Tor(M,)$ and $Ext(M,)$.

Proposition 4.1. *A is a duo-ring, S a non-empty part of A consists of regular elements. Let B and M of A -modules where B is finitely. Then there exists a natural isomorphism $\psi_B : S^{-1}Hom_A(B,M) \to Hom_{S^{-1}A}(S^{-1}B, S^{-1}M))$.*

Proof. Just build the natural isomorphisms $\theta_B : Hom_A(B, S^{-1}M) \to Hom_{S^{-1}B}(S^{-1}B, S^{-1}M)$ et $\varphi_B : S^{-1}Hom_A(B,M) \to Hom_A(B, S^{-1}M)$ and consequently take $\psi_B = \theta_B \circ \varphi_B$.

a) Suppose that $B = A^n$ is an A -module of finite type. Let $\mathrm{b}_1, \mathrm{b}_2, \ldots, (\mathrm{b}_n$ a basis B, then $\frac{b_1}{1}, \frac{b_2}{1}, \ldots, \frac{b_n}{1}$ is a basis of $S^{-1}B = \overline{(S)}^{-1}A \otimes_A A^n$.
   The morphism $\theta_{A^n} : Hom_A(B, S^{-1}M) \to Hom_{S^{-1}A}(S^{-1}B, S^{-1}M)$ defined by $\theta_{A^n}(f) = \tilde{f}$ with $\tilde{f}(\frac{b_i}{\delta}) = \frac{f(b_i)}{\delta}$ is well defined and is an isomorphism.
b) Now let B A $-module$ a finitely generated, then the sequence $A^t \to A^n \to B \to 0$ is exact.

Let us apply this result contravariant functors $Hom_A(-,P)$ et $Hom_{S^{-1}A}(-,P)$ où $P = S^{-1}M$. We obtain the following commutative diagram or lines are exact:

$$\begin{array}{ccccc} 0 \longrightarrow Hom_A(B,P) & \longrightarrow & Hom_A(A^n,P) & \longrightarrow & Hom_A(A^t,P) \\ \downarrow \theta_B & & \downarrow \theta_{A^n} & & \downarrow \theta_{A^t} \\ 0 \longrightarrow Hom_{S^{-1}A}(S^{-1}B,P) & \longrightarrow & Hom_{S^{-1}A}(S^{-1}A^n,P) & \longrightarrow & Hom_{S^{-1}A}(S^{-1}(A)^t,P) \end{array}.$$

Because θ_{A^n} and θ_{A^t} are isomorphisms then according **[11]**, θ_B and there is an isomorphism.

The isomorphism θ_B is defined by:

$$\theta_B(\beta) = \tilde{\beta} \text{ où } \beta \in Hom_A(B,P)$$

and

$$\begin{array}{c} \tilde{\beta} : S^{-1}B \to P = S^{-1}M \\ \frac{b}{\delta} \mapsto \frac{\beta(b)}{\delta}. \end{array}$$

build now φ_B by

$$\varphi_B : S^{-1}Hom_A(B,M) \to Hom_A(B, S^{-1}M)$$

by

$$\varphi_B : \frac{g}{\delta} \mapsto g_\delta.$$

where

$$\begin{aligned} g_\delta &: B \to S^{-1}M \\ b &\mapsto g_\delta(b) = \tfrac{g(b)}{\delta}. \end{aligned}$$

Recall (see [11]) que $S^{-1}Hom_A(B,M) = (\bar{S})^{-1} \otimes_A Hom_A(B,M)$ and φ_B is an isomorphism if B is a free finitely A-module.

En appliquant à la suite exacte $A^t \to A^n \to B \to 0$ les foncteurs contravariants exacts à gauche : $Hom_A(-,M)$ et $Hom_A(-,S^{-1}M)$, on obtient le diagramme commutatif suivant

$$\begin{array}{ccccccc} 0 \longrightarrow & S^{-1}Hom_A(B,M) & \longrightarrow & S^{-1}Hom_A(A^n,M) & \longrightarrow & S^{-1}Hom_A(A^t,M) \\ & \downarrow \varphi_B & & \downarrow \varphi_{A^n} & & \downarrow \varphi_{A^t} \\ 0 \longrightarrow & Hom_A(B,S^{-1}M) & \longrightarrow & Hom_A(A^n,S^{-1}M) & \longrightarrow & Hom_A(A^t,S^{-1}M) \end{array}$$

φ_{A^n} and φ_{A^t} being isomorphisms, according **[11]** φ_B is an isomorphism.

Lemma 4.2 ([11]). *Let A and B, and two rings* $T : {}_AMod \to {}_BMod$ *an exact and additive functor . So T commutes with the homology functor* H_n *: any complex* (C,d) *of the category* ${}_AComp$ *and any relative integer n, we have :*
$\mathbf{H_n(TC,Td) \cong TH_n(C,d)}$.

Lemma 4.3 ([11]). *A is a ring; for every left A -module B, we have :* $\otimes_A B \cong Tor_0^A(-,B)$ *; that is to say for all A -module right R we have:* $R \otimes_A B \cong Tor_0^A(R,B)$.

Theorem 4.4. *Let A be a duo-ring and S of a non-empty A consists of regular elements. Then for any natural number* $n \geq 0$ *and all A -modules M and* M'*, we have :* $\mathbf{S^{-1}Tor_n^A(M,M') \cong Tor_n^{S^{-1}A}(S^{-1}M,S^{-1}M')}$.

Proof.

a) For $n = 0$ is deduced **Lemmas 4.2 and 4.3** :
$Tor_0^A(M,M') \cong M_k \otimes M'$ et $Tor_0^{S^{-1}A}(S^{-1}M,S^{-1}M') \cong S^{-1}M \otimes S^{-1}M'$, whence $S^{-1}(M \otimes_A M') \cong S^{-1}M \otimes_A S^{-1}M'$.

b) Let now $P_{M'}$ projective resolution of M'.
As the functor S^{-1} keeps productivity (see [2]), then $S^{-1}(\mathrm{P}_{M'})$ is a projective resolution $S^{-1}M'$.

According to the **Theorem 4.1**, proving the existence of isomorphism ψ_B, we deduce the isomorphism complex
$\mathbf{S^{-1}(M \otimes {}_AP_{M'}) \cong S^{-1}M \otimes {}_{S^{-1}A}S^{-1}(P_{M'})}$.

Therefore their homology groups are isomorphic and as the functor $S^{-1}()$ is exact (according to [6]) and by definition the functor *Tor*, we have : $\mathbf{H_n(S^{-1}(M \otimes {}_AP_{M'}) \cong S^{-1}H_n(M \otimes {}_AP_{M'}) \cong S^{-1}Tor_n^A(M,M')}$; as well as $S^{-1}(\mathrm{P}_{M'})$ is a projective resolution $S^{-1}M'$, so
$\mathbf{H_n(S^{-1}M \otimes {}_{S^{-1}A}S^{-1}(P_{M'})) \cong Tor_n^{S^{-1}A}(S^{-1}M,S^{-1}M')}$.

Theorem 4.5. *Let S be a non-empty part formed of regular elements of a Noetherian duo-ring A, and M a A-Module left finitely. So*
$S^{-1}Ext_A^n(M, M') \cong Ext_{S^{-1}A}^n(S^{-1}M, S^{-1}M')$ *for every* $n \geq 0$ *and every A-module left* M'.

Proof. As A is Noetherian and M is finitely generated, according **[11]**, there is a projective resolution P_M of M for which each term is of finite type.

According to the **Theorem 4.1**, there is a natural isomorphism:
$\psi_M : S^{-1}Hom(M, M') \to Hom_{S^{-1}A}(S^{-1}M, S^{-1}M')$
for everything A-module M'.

We deduce the isomorphism complex :
$S^{-1}(Hom_A(P_M, M')) \cong Hom_{S^{-1}A}(S^{-1}(P_M), S^{-1}M')$.

Applying the homology functor H_n we have:
$H_n(S^{-1}(Hom_A(P_M, M'))) \cong S^{-1}H_n(Hom_A(P_M, M')) \cong S^{-1}Ext_A^n(M, M')$ then the functor $S^{-1}()$ is exact therefore we have:
$H_n(Hom_{S^{-1}A}(S^{-1}P_M, S^{-1}M')) = Ext_{S^{-1}A}^n(S^{-1}M, S^{-1}M')$ because $S^{-1}(P_M)$ is a projective resolution.

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