Chapter 11
Strongly Split Poisson Algebras

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Abstract Split Poisson algebras are one of the most known examples of graded Poisson algebras. Since an important category in the class of graded algebras is the one of strongly graded algebras, we introduce in a natural way the category of strongly split Poisson algebras and show that if $(\mathfrak{P}, \{\cdot, \cdot\})$ is a centerless strongly split Poisson algebra, then $\mathfrak{P}$ is the direct sum of split-ideals, each one being a split-simple strongly split Poisson algebra. In case of being $\mathfrak{P}$ infinite dimensional and locally finite, we also show that if $(\mathfrak{P}, \{\cdot, \cdot\})$ is furthermore simple then it is the direct limit of finite dimensional simple (strongly) split Poisson algebras.

Keywords Split Poisson algebra • Strongly graded algebra • Root • Root space • Locally finite algebra • Direct limit • Structure theory

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11.1 Introduction and Previous Definitions

We begin by noting that, unless otherwise stated, all of the Poisson algebras are considered of arbitrary dimension and over an

Definition 11.1. A *Poisson algebra* $\mathfrak{P}$ is a Lie algebra $(\mathfrak{P}, \{\cdot, \cdot\})$ over an arbitrary base field $\mathbb{K}$, endowed with an associative product, denoted by juxtaposition, in such a way that the following *Leibniz identity*

$$\{xy, z\} = \{x, z\}y + x\{y, z\}$$

holds for any $x, y, z \in \mathfrak{P}$.

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Poisson algebras has attracted the interest of many authors in the last years, as consequence in part for their applications in physics and geometry (see for instance [1, 2, 8, 10–12]). A *subalgebra* of $\mathfrak{P}$ is a linear subspace closed by both the Lie and the associative products. An *ideal* I of $\mathfrak{P}$ is a subalgebra satisfying $\{I, \mathfrak{P}\} + I\mathfrak{P} + \mathfrak{P}I \subset I$.

In order to study the structure of arbitrary Poisson algebras, the first author introduced in [6] the concept of split Poisson algebra as a Poisson algebra in which the underlying Lie algebra structure is split. So, let us recall the concept of a split Lie algebra, (see for instance [5] or [16]). A splitting Cartan subalgebra H of a Lie algebra L is defined as a maximal abelian subalgebra, (MASA), of L satisfying that the adjoint mappings ad(h) for $h \in H$ are simultaneously diagonalizable. If L contains a splitting Cartan subalgebra H, then L is called a split Lie algebra. From here:

Definition 11.2. A *split Poisson algebra* is a Poisson algebra $\mathfrak{P}$ in which the Lie algebra $(\mathfrak{P}, \{\cdot,\cdot\})$ is split respect to a MASA H of $(\mathfrak{P}, \{\cdot,\cdot\})$.

This means that we can decompose $\mathfrak{P}$ as the direct sum

$$\mathfrak{P} = H \oplus (\bigoplus_{\alpha \in \Lambda} \mathfrak{P}_\alpha)$$

where $\mathfrak{P}_\alpha = \{v_\alpha \in \mathfrak{P} : \{h, v_\alpha\} = \alpha(h)v_\alpha \text{ for any } h \in H\}$, for a linear functional $\alpha \in H^*$ and $\Lambda := \{\alpha \in H^* \backslash \{0\} : \mathfrak{P}_\alpha \neq 0\}$ is the corresponding *root system*. The subspaces $\mathfrak{P}_\alpha$ for $\alpha \in H^*$ are called *root spaces* of $\mathfrak{P}$ (respect to H) and the elements $\alpha \in \Lambda \cup \{0\}$ are called *roots* of $\mathfrak{P}$ respect to H.

By the other hand, we also recall that a *graded algebra*

$$A = \bigoplus_{g \in G} A_g,$$

that is, A is the direct sum of linear subspaces indexed by the elements in an abelian group $(G, +)$ in such a way that $A_g A_h \subset A_{g+h}$, is called a *strongly graded algebra* if the condition $A_g A_h = A_{g+h}$ holds for any $g, h \in G$, see [9, 13].

Since by [6, Lemma 1] we know that in any split Poisson algebra $\mathfrak{P}$ we have

$$H = \mathfrak{P}_0, \quad \{\mathfrak{P}_\alpha, \mathfrak{P}_\beta\} \subset \mathfrak{P}_{\alpha+\beta}, \quad \mathfrak{P}_\alpha \mathfrak{P}_\beta \subset \mathfrak{P}_{\alpha+\beta} \tag{11.1}$$

for any $\alpha, \beta \in \Lambda \cup \{0\}$, we get that $\mathfrak{P}$ becomes a graded Poisson algebra by means of the abelian free group generated by Λ. Taking into account the above observations we introduce the category of strongly split Poisson algebras as follows.

Definition 11.3. A split Poisson algebra $\mathfrak{P}$ with set of nonzero roots Λ is called a *strongly split Poisson algebra* if $H = \sum\limits_{\alpha \in \Lambda} (\{\mathfrak{P}_\alpha, \mathfrak{P}_{-\alpha}\} + \mathfrak{P}_\alpha \mathfrak{P}_{-\alpha})$ and given $\alpha, \beta \in \Lambda$ such that $\alpha + \beta \in \Lambda$ then we have $\{\mathfrak{P}_\alpha, \mathfrak{P}_\beta\} + \mathfrak{P}_\alpha \mathfrak{P}_\beta = \mathfrak{P}_{\alpha+\beta}$.

As examples of strongly split Poisson algebras we can consider the finite dimensional semisimple Poisson algebras, the Poisson algebras associated to L^*-algebras and to semisimple locally finite split Lie algebras and the split Poisson algebras considered in [5, Sect. 2] among other classes of Poisson algebras (see [14–16]).

Let us focuss for a while on the concept of split-ideal in the framework of split Poisson algebras. Observe that the set of linear mappings $\{\mathrm{ad}(h) : h \in H\}$, where $\mathrm{ad}(h) : \mathfrak{P} \to \mathfrak{P}$ is defined by $\mathrm{ad}(h)(v) = \{h, v\}$, is a commuting set of diagonalizable Lie endomorphisms. Hence, given any ideal I of $\mathfrak{P}$, since I is invariant under this set we get that we can write

$$I = (I \cap H) \oplus (\bigoplus_{\alpha \in \Lambda} (I \cap \mathfrak{P}_\alpha)). \tag{11.2}$$

From here, if $I \cap H \neq 0$, then I adopts a split like expression (respect to $I \cap H$). This motivate us to introduce the concept of split-ideal as follows. An ideal I of a split Poisson algebra $\mathfrak{P}$ is called a *split-ideal* if $I \cap H \neq 0$. A split Poisson algebra $\mathfrak{P}$ will be called *split-simple* if $\{\mathfrak{P}, \mathfrak{P}\}, \mathfrak{P}\mathfrak{P} \neq 0$ and it has no proper split-ideals. Finally, we recall that a root system Λ is called *symmetric* if it satisfies that $\alpha \in \Lambda$ implies $-\alpha \in \Lambda$. Throughout the paper Λ will be always supposed symmetric.

11.2 Main Results

In the following, $\mathfrak{P}$ denotes a strongly split Poisson algebra and

$$\mathfrak{P} = H \oplus (\bigoplus_{\alpha \in \Lambda} \mathfrak{P}_\alpha)$$

the corresponding root spaces decomposition.

Definition 11.4. Let $\alpha \in \Lambda$ and $\beta \in \Lambda$ be two nonzero roots. We say that α is *connected* to β if there exists a family $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Lambda$ satisfying the following conditions:

1. $\alpha_1 = \alpha$.
2. $\{\alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \ldots, \alpha_1 + \cdots + \alpha_{n-1}\} \subset \Lambda$.
3. $\alpha_1 + \alpha_2 + \cdots + \alpha_n = \epsilon\beta$ for some $\epsilon \in \{\pm 1\}$.

We also say that $\{\alpha_1, \ldots, \alpha_n\}$ is a *connection* from α to β.

It is straightforward to verify that the relation connection is an equivalence connection, see [5] or [6]. So we can consider the quotient set

$$\Lambda/\sim = \{[\alpha] : \alpha \in \Lambda\}.$$

Now, for any $[\alpha] \in \Lambda / \sim$ we are going to introduce the linear subspace

$$\mathfrak{P}_{[\alpha]} := H_{[\alpha]} \oplus V_{[\alpha]}$$

where

$$H_{[\alpha]} := \sum_{\beta \in [\alpha]} (\{\mathfrak{P}_\beta, \mathfrak{P}_{-\beta}\} + \mathfrak{P}_\beta \mathfrak{P}_{-\beta}) \subset H \text{ and } V_{[\alpha]} := \bigoplus_{\beta \in [\alpha]} \mathfrak{P}_\beta.$$

Proposition 11.1. *Any $\mathfrak{P}_{[\alpha]}$ is a split-ideal of $\mathfrak{P}$. If furthermore $(\mathfrak{P}, \{\cdot, \cdot\})$ is centerless then $\mathfrak{P}_{[\alpha]}$ is split-simple.*

Proof. First, let us show that

$$\{\mathfrak{P}_{[\alpha]}, \mathfrak{P}\} = \{H_{[\alpha]} \oplus V_{[\alpha]}, H \oplus (\bigoplus_{\beta \in [\alpha]} \mathfrak{P}_\beta) \oplus (\bigoplus_{\gamma \notin [\alpha]} \mathfrak{P}_\gamma)\} \subset \mathfrak{P}_{[\alpha]}. \tag{11.1}$$

Clearly, see Eq. (11.1), we have $\{H_{[\alpha]}, H \oplus (\bigoplus_{\beta \in [\alpha]} \mathfrak{P}_\beta)\} + \{V_{[\alpha]}, H\} \subset V_{[\alpha]}$.

Since in case $\{\mathfrak{P}_\delta, \mathfrak{P}_\tau\} \neq 0$ for some $\delta, \tau \in \Lambda$ with $\delta + \tau \neq 0$, the connections $\{\delta, \tau\}$ and $\{\delta, \tau, -\delta\}$ imply $[\delta] = [\delta + \tau] = [\tau]$, we get $\{V_{[\alpha]}, \bigoplus_{\beta \in [\alpha]} \mathfrak{P}_\beta\} \subset \mathfrak{P}_{[\alpha]}$ and

$$\{V_{[\alpha]}, \bigoplus_{\gamma \notin [\alpha]} \mathfrak{P}_\gamma\} = 0. \tag{11.2}$$

Taking now into account the fact $H_{[\alpha]} := \sum_{\beta \in [\alpha]} (\{\mathfrak{P}_\beta, \mathfrak{P}_{-\beta}\} + \mathfrak{P}_\beta \mathfrak{P}_{-\beta})$, Jacobi identity and Leibniz identity together with Eq. (11.2) finally give us

$$\{H_{[\alpha]}, \bigoplus_{\gamma \notin [\alpha]} \mathfrak{P}_\gamma\} = 0 \tag{11.3}$$

and so Eq. (11.1) holds.

Second, let us verify that

$$\mathfrak{P}_{[\alpha]} \mathfrak{P} = (H_{[\alpha]} \oplus V_{[\alpha]})(H \oplus (\bigoplus_{\beta \in [\alpha]} \mathfrak{P}_\beta) \oplus (\bigoplus_{\gamma \notin [\alpha]} \mathfrak{P}_\gamma)) \subset \mathfrak{P}_{[\alpha]}. \tag{11.4}$$

Since $H_{[\alpha]} \subset H = \mathfrak{P}_0$ we have, (take into account Eq. (11.1)), that

$$H_{[\alpha]}(\bigoplus_{\beta \in [\alpha]} \mathfrak{P}_\beta) + V_{[\alpha]} H \subset V_{[\alpha]}.$$

We also have by arguing with the associative product as we did above with the Lie product that

$$V_{[\alpha]}(\bigoplus_{\beta\in[\alpha]}\mathfrak{P}_\beta)\subset\mathfrak{P}_{[\alpha]}$$

and

$$V_{[\alpha]}(\bigoplus_{\gamma\notin[\alpha]}\mathfrak{P}_\gamma)=0. \tag{11.5}$$

Now, by taking also into account the expression of $H_{[\alpha]}$, Eq. (11.5), Leibniz identity and associativity we get

$$H_{[\alpha]}(\bigoplus_{\gamma\notin[\alpha]}\mathfrak{P}_\gamma)=0;$$

while by taking into account the fact $H=\mathfrak{P}_0$, Leibniz identity and associativity we have

$$H_{[\alpha]}H\subset H_{[\alpha]}.$$

We have shown that Eq. (11.4) holds. In a similar way we can prove

$$\mathfrak{P}\mathfrak{P}_{[\alpha]}\subset\mathfrak{P}_{[\alpha]}$$

and so we have showed $\mathfrak{P}_{[\alpha]}$ is an ideal of $\mathfrak{P}$. Since Eq. (11.3) implies $\{H_{[\gamma]},V_{[\alpha]}\}=0$ for any $[\gamma]\neq[\alpha]$, the facts $H=\sum_{[\beta]\in\Lambda/\sim}H_{[\beta]}$ and $\alpha\neq 0$ allow us to get $H_{[\alpha]}\neq 0$. From here, we can also assert that $\mathfrak{P}_{[\alpha]}$ is a strongly split Poisson ideal admitting the split decomposition

$$\mathfrak{P}_{[\alpha]}=H_{[\alpha]}\oplus(\bigoplus_{\beta\in[\alpha]}\mathfrak{P}_\beta).$$

Suppose now $(\mathfrak{P},\{\cdot,\cdot\})$ is centerless and let us show $\mathfrak{P}_{[\alpha]}$ is split-simple. Consider a split-ideal I of $\mathfrak{P}_{[\alpha]}$. By Eq. (11.2) we can write $I=(I\cap H_{[\alpha]})\oplus(\bigoplus_{\beta\in[\alpha]}(I\cap\mathfrak{P}_\beta))$ with $I\cap H_{[\alpha]}\neq 0$. For any $0\neq h\in I\cap H_{[\alpha]}$, the fact $(\mathfrak{P},\{\cdot,\cdot\})$ is centerless gives us that there exists $\beta\in[\alpha]$ such that $\{h,\mathfrak{P}_\beta\}\neq 0$. From here we get $\{I\cap H_{[\alpha]},\mathfrak{P}_\beta\}=\mathfrak{P}_\beta$ and so $0\neq\mathfrak{P}_\beta\subset I$.

Given now any $\delta\in[\alpha]\setminus\{\pm\beta\}$, the fact that β and δ are connected allows us to take a connection $\{\alpha_1,\alpha_2,\ldots,\alpha_n\}$ from β to δ. Since $\alpha_1,\alpha_2,\alpha_1+\alpha_2\in\Lambda$ we have $\{\mathfrak{P}_{\alpha_1},\mathfrak{P}_{\alpha_2}\}+\mathfrak{P}_{\alpha_1}\mathfrak{P}_{\alpha_2}=\mathfrak{P}_{\alpha_1+\alpha_2}\subset I$ as consequence of $\mathfrak{P}_{\alpha_1}=\mathfrak{P}_\beta\subset I$. In a similar way $\{\mathfrak{P}_{\alpha_1+\alpha_2},\mathfrak{P}_{\alpha_3}\}+\mathfrak{P}_{\alpha_1+\alpha_2}\mathfrak{P}_{\alpha_3}=\mathfrak{P}_{\alpha_1+\alpha_2+\alpha_3}\subset I$ and we finally get by following this process that $\mathfrak{P}_{\alpha_1+\alpha_2+\alpha_3+\cdots+\alpha_n}=\mathfrak{P}_{\epsilon\delta}\subset I$ for some $\epsilon\in\pm1$. From here we have $H_{[\alpha]}\subset I$ and as consequence, taking also into account that Equation (11.3) allows us to assert $\{H_{[\alpha]}, \mathfrak{P}_\delta\} = \mathfrak{P}_\delta$ for any $\delta \in [\alpha]$, that $V_{[\alpha]} \subset I$. We have showed $I = \mathfrak{P}_{[\alpha]}$ and so $\mathfrak{P}_{[\alpha]}$ is split-simple.

Theorem 11.1. *Any strongly split Poisson algebra* $\mathfrak{P}$ *such that* $(\mathfrak{P}, \{\cdot,\cdot\})$ *is centerless is the direct sum of split-ideals, each one being a split-simple strongly split Poisson algebra.*

Proof. Since we can write the disjoint union $\Lambda = \bigcup_{[\alpha]\in\Lambda\setminus\sim} [\alpha]$ we have $\mathfrak{P} = \sum_{[\alpha]\in\Lambda\setminus\sim} \mathfrak{P}_{[\alpha]}$. Let us now verify the direct character of the sum: given $x \in \mathfrak{P}_{[\alpha]} \cap \sum_{\substack{[\beta]\in\Lambda/\sim \\ \beta\nsim\alpha}} \mathfrak{P}_{[\beta]}$, since by Eqs. (11.2) and (11.3) we have $\{\mathfrak{P}_{[\alpha]}, \mathfrak{P}_{[\beta]}\} = 0$ for $[\alpha] \neq [\beta]$, we obtain

$$\{x, \mathfrak{P}_{[\alpha]}\} + \left\{ x, \sum_{\substack{[\beta]\in\Lambda/\sim \\ \beta\nsim\alpha}} \mathfrak{P}_{[\beta]} \right\} = 0.$$

From here $\{x, \mathfrak{P}\} = 0$ and so $x = 0$, as desired. Consequently we can write

$$\mathfrak{P} = \bigoplus_{[\alpha]\in\Lambda\setminus\sim} \mathfrak{P}_{[\alpha]}.$$

Finally, Proposition 11.1 completes the proof.

11.3 On Locally Finite Split Poisson Algebras

Throughout this section the base field $\mathbb{K}$ will be algebraically closed and of characteristic 0.

A Lie algebra L is called *locally finite* if every finite subset of L is contained or equivalently generates a finite dimensional subalgebra of L. Since we are working in a split framework, we recall that the class of semisimple locally finite split Lie algebras can be characterized among all split algebras by the property that all its roots are integrable, i.e., corresponding to $sl(2,\mathbb{K})$ subalgebras acting in a locally finite fashion (see [16, III.19]). As a consequence, any nonzero root space L_α of a semisimple locally finite split Lie algebra $(L, [\cdot,\cdot])$ satisfies $dimL_\alpha = dimL_{-\alpha} = 1$ and $\alpha([L_\alpha, L_{-\alpha}]) \neq 0$. By [16, Proposition I.7 (v) and Theorem III.19]) we also know that in any of such an algebras, if $\alpha, \beta, \alpha+\beta \in \Lambda$ then $[L_\alpha, L_\beta] = L_{\alpha+\beta}$ and $H = \sum_{\alpha\in\Lambda} [L_\alpha, L_{-\alpha}]$. Hence the class of semisimple locally finite split Lie algebras is contained in the one of strongly split Lie algebras and so any Poisson structure associate to this family of Lie algebras gives rise to a family of strongly split

Poisson algebras, (in particular the results in our previous sections would apply). We introduce the class of locally finite Poisson algebras in a natural way as follows.

Definition 11.5. A Poisson algebra $\mathfrak{P}$ is called *locally finite* if every finite subset of $\mathfrak{P}$ is contained in a finite dimensional subalgebra of $\mathfrak{P}$.

Our goal in this section is to prove that any infinite dimensional locally finite split Poisson algebra with $(\mathfrak{P}, \{\cdot,\cdot\})$ simple, (by the above comments strongly split), is the direct limit of a family of finite dimensional simple (strongly) split Poisson algebras. Hence, from now on

$$\mathfrak{P} = H \oplus (\bigoplus_{\alpha \in \Lambda} \mathfrak{P}_\alpha)$$

will denote such an algebra.

Let S be a non empty finite subset of Λ, and denote by $\mathfrak{P}_S$ the Poisson subalgebra of $\mathfrak{P}$ generated by the set $\{\mathfrak{P}_\alpha : \alpha \in S \cup -S\}$ which will be called the subalgebra of $\mathfrak{P}$ *associated* to S. Taking into account that the above comments on semisimple locally finite split Lie algebras, Leibniz identity and associativity allow us to write $\mathfrak{P}_S = \widetilde{H_S} + \{\mathfrak{P}_S, \mathfrak{P}_S\}$ where $\widetilde{H_S}$ is the linear spam of the set $\{\mathfrak{P}_\alpha \mathfrak{P}_{-\alpha} : \alpha \in S \cup -S\}$, by arguing as in [4, Proposition 2.7] we can verify $(\mathfrak{P}_S, \{\cdot,\cdot\})$ is a finite dimensional semisimple split subalgebra of the Lie algebra $(\mathfrak{P}, \{\cdot,\cdot\})$. It is well known from the theory of finite dimensional semisimple Lie algebras that $\mathfrak{P}_S$ can be written

$$\mathfrak{P}_S = \bigoplus_{i=1}^{n_S} \mathfrak{P}_{S_i},$$

with any $\mathfrak{P}_{S_i}$ a finite dimensional simple Lie algebra. Let us verify any $\mathfrak{P}_{S_i}$ is actually a (finite dimensional) simple Poisson algebra. By taking into account $\mathfrak{P}_{S_j} = \{\mathfrak{P}_{S_j}, \mathfrak{P}_{S_j}\}$ for any $j \in \{1, \dots, n_S\}$, Leibniz identity gives us $\mathfrak{P}_{S_i}\mathfrak{P}_{S_j} = \mathfrak{P}_{S_i}\{\mathfrak{P}_{S_j}, \mathfrak{P}_{S_j}\} \subset \mathfrak{P}_{S_j}$ whence $i \neq j$. From here, by writing now $\mathfrak{P}_{S_i}\mathfrak{P}_{S_j} = \{\mathfrak{P}_{S_i}, \mathfrak{P}_{S_i}\}\mathfrak{P}_{S_j}$, Leibniz identity allows us to assert $\mathfrak{P}_{S_i}\mathfrak{P}_{S_j} = 0$ if $i \neq j$. This fact implies $\mathfrak{P}_{S_i}\mathfrak{P}_S = \mathfrak{P}_{S_i}(\bigoplus_{i=1}^{n_S} \mathfrak{P}_{S_i}) \subset \mathfrak{P}_{S_i}\mathfrak{P}_{S_i}$. Since Leibniz identity gives us that $\mathfrak{P}_{S_i}\mathfrak{P}_{S_i}$ is a Lie ideal of the semisimple Lie algebra $(\mathfrak{P}_S, \{\cdot,\cdot\})$ then $\mathfrak{P}_{S_i}\mathfrak{P}_{S_i} = \bigoplus_{r=1}^{k} \mathfrak{P}_{S_r}$ with $k \leq n_S$ and for any $r \in \{1, \dots, k\}$ being $\mathfrak{P}_{S_r} = \mathfrak{P}_{S_j}$ for some $j \in \{1, \dots, n_S\}$. If some $\mathfrak{P}_{S_r} \neq \mathfrak{P}_{S_i}$ then, by the one hand Leibniz identity gives $\{\mathfrak{P}_{S_i}\mathfrak{P}_{S_i}, \mathfrak{P}_{S_r}\} = 0$ and by the other hand $\{\mathfrak{P}_{S_i}\mathfrak{P}_{S_i}, \mathfrak{P}_{S_r}\} = \{\bigoplus_{r=1}^{k} \mathfrak{P}_{S_r}, \mathfrak{P}_{S_r}\} = \mathfrak{P}_{S_r} \neq 0$, a contradiction. From here $\mathfrak{P}_{S_i}\mathfrak{P}_{S_i} = \mathfrak{P}_{S_i}$ and so $\mathfrak{P}_{S_i}\mathfrak{P}_S \subset \mathfrak{P}_{S_i}$. In a similar way we get $\mathfrak{P}_S\mathfrak{P}_{S_i} \subset \mathfrak{P}_{S_i}$ and hence $\mathfrak{P}_{S_i}$ is a (simple) Poisson ideal of $\mathfrak{P}_S$.

From here, we can consider the family of finite dimensional simple Poisson subalgebras of $\mathfrak{P}$,

$$\{\mathfrak{P}_{S_i}\}_{S\in\mathcal{F},i\in\{1,\ldots,n_S\}},$$

where $\mathcal{F}$ denotes the family of all non empty finite subsets of Λ. If we now denote by $\{i_{S_i,T_j}\}$ the inclusion mappings, we will show that

$$R := (\{\mathfrak{P}_{S_i}\}_{S\in\mathcal{F},i\in\{1,\ldots,n_S\}}, \{i_{S_i,T_j}\}) \tag{11.1}$$

is a direct system. First, we note that by arguing as in [3, Lemma 1 and Corollary 2] or [4, Sect. 2], we can prove the next lemma:

Lemma 11.1. *Let* $(L, [\cdot,\cdot])$ *be a finite dimensional semisimple split Lie algebra. Then the following assertions hold*

1. *If* α *is a nonzero root, then* L_α *belongs to a simple component* L_j*.*
2. *If* α *and* β *are two connected nonzero roots, then* L_α *and* L_β *belong to the same simple component* L_j*.*

By returning to Eq. (11.1), we assert that given

$$\mathfrak{P}_{S_i}, \mathfrak{P}_{T_j} \in \{\mathfrak{P}_{S_i}\}_{S\in\mathcal{F},i\in\{1,\ldots,n_S\}},$$

there exists

$$\mathfrak{P}_{Q_{i_0}} \in \{\mathfrak{P}_{S_i}\}_{S\in\mathcal{F},i\in\{1,\ldots,n_S\}}$$

such that $\mathfrak{P}_{S_i}, \mathfrak{P}_{T_j} \subset \mathfrak{P}_{Q_{i_0}}$. Indeed, let us fix $\alpha_0 \in S_i$. Since as consequence of Proposition 11.1 we have that $\mathfrak{P}$ has all of its nonzero roots connected, then we have that for any $\beta \in S_i \cup T_j$ there exists a connection from α_0 to β, which we denote by $C_{\alpha_0,\beta}$. We also have that $Q := \bigcup_{\beta\in S_i\cup T_j} C_{\alpha_0,\beta}$ is a finite set of Λ and therefore we can consider the finite dimensional semisimple subalgebra associated $\mathfrak{P}_Q$. Write $\mathfrak{P}_Q = \bigoplus_{i=1}^{n_Q} \mathfrak{P}_{Q_i}$, each $\mathfrak{P}_{Q_i}$ being a simple Poisson subalgebra of $\mathfrak{P}_Q$. By Lemma 11.1-1, there exists $\mathfrak{P}_{Q_{i_0}}$ such that $\mathfrak{P}_{\alpha_0} \subset \mathfrak{P}_{Q_{i_0}}$. Finally, by Lemma 11.1-2, $\mathfrak{P}_{S_i}, \mathfrak{P}_{T_j} \subset \mathfrak{P}_{Q_{i_0}}$. Therefore, R is a direct system of finite dimensional simple Poisson algebras.

Let us denote by $\varinjlim R = (\mathfrak{P}', \{e_j\}_j)$ the direct limit of this direct system. Since the pair $(\mathfrak{P}, \{i_j\})$, where i_j denotes the inclusion mapping, satisfies the conditions of the direct limit for R, we have that the universal property of the direct limit gives us the existence of a unique monomorphism $\Phi : \mathfrak{P}' \to \mathfrak{P}$ such that $\Phi \circ e_j = i_j$. Since $\mathfrak{P}' = \bigcup_j e_j(\mathfrak{P}_j)$, (see for instance [7]), we have $\Phi(\mathfrak{P}') = \Phi(\bigcup_j e_j(\mathfrak{P}_j)) = \bigcup_j \mathfrak{P}_j$, and therefore Φ is an isomorphism from $\mathfrak{P}'$ onto $\bigcup_j \mathfrak{P}_j$. Let us show that $\mathfrak{P} = \bigcup_j \mathfrak{P}_j$. Indeed, if $x \in \mathfrak{P}$, by Theorem 11.1 we can write $x = \sum_{i=1}^{n} h_{\alpha_i} + \sum_{j=1}^{m} v_{\gamma_j}$ with $\alpha_i, \gamma_j \in \Lambda$, $v_{\gamma_j} \in \mathfrak{P}_{\gamma_j}$ and where any $h_{\alpha_i} \in \{\mathfrak{P}_{\alpha_i}, \mathfrak{P}_{-\alpha_i}\}$. Consider $T = \{\alpha_i : i = 1, \ldots, n\} \cup \{\gamma_j :$

$j = 1, \dots, m\} \subset \Lambda$ and, following the above notation, $T' = \bigcup_{\beta \in T} C_{\delta_0, \beta}$, δ_0 being a fixed element of T. We have T' is a finite set of Λ that gives us the semisimple finite dimensional Poisson algebra associated $\mathfrak{P}_{T'}$. Write $\mathfrak{P}_{T'} = \bigoplus_{i=1}^{r} \mathfrak{P}_{T'_i}$, where $\mathfrak{P}_{T'_i}$, $i = 1, \dots, r$ are simple finite dimensional Poisson algebras. As R is a direct system for the inclusion then there exists a finite dimensional simple Poisson subalgebra $\mathfrak{P}_{P_0}$ such that $\bigcup_{i=1}^{r} \mathfrak{P}_{T'_i} \subseteq \mathfrak{P}_{P_0}$ and therefore $x \in \mathfrak{P}_{P_0}$ as we wished to show. From here, we can state the next result.

Theorem 11.2. *Let $\mathfrak{P}$ be an infinite dimensional locally finite split Poisson algebra with $(\mathfrak{P}, \{\cdot, \cdot\})$ simple. Then there exists a direct system, with the inclusion,*

$$R = (\{\mathfrak{P}_j\}_{j \in J}, \{i_{kj}\}_{j \leq k})$$

of finite dimensional simple Poisson subalgebras of $\mathfrak{P}$ such that

$$\mathfrak{P} = \lim_{\rightarrow} R.$$

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