Chapter 10
Some Properties of Mono-correct and Epi-correct Modules

Anta Niane Gueye

Abstract Let $\mathfrak{C}$ be a category. An objet A in $\mathfrak{C}$ is said to be mono-correct (respectively epi-correct) if for any B in $\mathfrak{C}$, and $f : A \longrightarrow B$, $g : B \longrightarrow A$ two monomorphisms (respectively epimorphisms) then $A \simeq B$. In category *Set*, this property is known as the Cantor Bernstein theorem and it's dual. In category of abelian groups, we show that the Cantor Bernstein theorem is not verified. In R-mod, we study some relations between mono-correctness of modules and some algebraic operations as for submodules, direct sum of modules and factor modules.

Keywords Mono-equivalent • Epi-equivalent • Mono-correct • Epi-correct • Hopfian • Co-hopfian

10.1 Introduction

Two objects A, B in a category $\mathfrak{C}$ are called mono-equivalent (respectively epi-equivalent) if there are monomorphisms (respectively epimorphisms) $f : A \longrightarrow B$ and $g : B \longrightarrow A$. The first case is denoted $A \overset{m}{\simeq} B$ and the second $A \overset{e}{\simeq} B$. A and B are called equivalent if there exists an isomorphism $f : A \longrightarrow B$. We denote it by $A \simeq B$.

An object A in a category $\mathfrak{C}$ is said to be mono-correct (respectively epi-correct) if for every B in $\mathfrak{C}$, $A \overset{m}{\simeq} B$ (respectively $A \overset{e}{\simeq} B$) implies $A \simeq B$. A class $\mathcal{C}$ of objects in a category $\mathfrak{C}$ is said to be mono-correct (respectively epi-correct) if for any objects $A,\ B$ in $\mathcal{C}$ $A \overset{m}{\simeq} B$ (respectively $A \overset{e}{\simeq} B$) implies $A \simeq B$. It is well known by the Cantor-Bernstein theorem and it's dual that the category *Set* is mono-correct and epi-correct but when the objects are provided with some algebraic structures the property of being mono-correct or epi-correct is not always conserved by the

A.N. Gueye (✉)
Département de Mathèmatiques et Informatique, Laboratoire d'Algèbre, de Cryptologie, de Géométrie Algébrique et Applications (LACGAA), Dakar, Senegal
e-mail: gantaniane@yahoo.fr

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category. Several papers studying analogues of Cantor-Bernstein theorem in many algebraic structures have been investigated (see [1–4, 6, 7]).

In this paper, we study some properties of mono-correctness and epi-correctness in modules category. We give examples of $\mathbb{Z}$-modules that are mono-equivalent but are not mono-correct, and this show that the Cantor-Bernstein theorem is not verified in the category of abelian groups. Studying mono-correctness, epi-correctness and the algebraic operations on modules, we remark that if the ring R is semisimple or the R-module M is semi simple then all submodules and factor modules are mono-correct. Furthermore we establish that any direct summand of an mono-correct (respectively epi-correct) module is mono-correct (respectively epi-correct) and the direct sum of two modules is mono-correct if and only if each of them is mono-correct. We prove also that if an R-module M contains an injective submodule N such that M/N mono-correct, then M is mono-correct.

10.2 Preliminaries

Let R be an associative ring with identity and R-Mod be the category of left R-modules.

Let M be an R-module. We recall the following definitions and facts:

Definition 10.1. Two R-modules M and N are called mono-equivalent (respectively epi-equivalent) if there are monomorphisms (respectively epimorphisms) $f : M \longrightarrow N$ and $g : N \longrightarrow M$.

We denote M and N mono-equivalent by $M \overset{m}{\simeq} N$, M and N epi-equivalent by $M \overset{e}{\simeq} N$.

Definition 10.2. An R-module M is said to be injective if for any monomorphism of R-modules $f : P \longrightarrow Q$ and any homomorphism $g : P \longrightarrow M$, there exists a homomorphism $h : Q \longrightarrow M$ such that $g = h \circ f$.

Remark 10.2.1. The property of being an injective module is preserved under isomorphism.

Definition 10.3. Let M be an R-module. An element m of M is said to be divisible if, for every non zero divisor r in R, there exists $m' \in M$ such that $m = m'r$. M is said to be a divisible R-module if every element of M is divisible.

Proposition 10.2.2 ([5]). *An abelian group is injective if and only if it is divisible.*

10.3 The Main Results

Proposition 10.3.1. *Let E and F be the $\mathbb{Z}$-modules $E = \bigoplus_{n=1}^{\infty} \mathbb{Q}_n$ and $F = \mathbb{Z} \oplus \left(\bigoplus_{n=1}^{\infty} \mathbb{Q}_n\right)$ where $\mathbb{Q}_n = \mathbb{Q}$ the set of all rational numbers, for each integer $n \geq 1$. Then E and F are mono-equivalent.*

Proof. Consider the natural injection:

$$f : \bigoplus_{n=1}^{\infty} \mathbb{Q}_n \longrightarrow \mathbb{Z} \oplus \left(\bigoplus_{n=1}^{\infty} \mathbb{Q}_n\right)$$

and the morphism

$$g = \bigoplus_{n=0}^{\infty} g_n : \mathbb{Z} \oplus \left(\bigoplus_{n=1}^{\infty} \mathbb{Q}_n\right) \longrightarrow \bigoplus_{n=1}^{\infty} \mathbb{Q}_n$$

defined as follows:
g_0 is the canonical injection

$$g_0 : \mathbb{Z} \longrightarrow \mathbb{Q}_1$$

and for $n \geq 1$

$$g_n : \mathbb{Q}_n \longrightarrow \mathbb{Q}_{n+1}$$

where g_n is identity of $\mathbb{Q}$. It is clear that f and g are monomorphism, hence E and F are mono-equivalent.

Definition 10.4. An R-module M is said to be mono-correct (respectively epi-correct) if for any R-module N, $M \overset{m}{\simeq} N$ (respectively $M \overset{e}{\simeq} N$) implies $M \simeq N$.

Example 10.3.2. $\mathbb{Z}$ is epi-correct and mono-correct as $\mathbb{Z}$-module.

Proof. Let N be a $\mathbb{Z}$-module such that $\mathbb{Z} \overset{e}{\simeq} N$. That is, there are two epimorphisms $f : \mathbb{Z} \longrightarrow N$ and $g : N \longrightarrow \mathbb{Z}$. Then $g \circ f : \mathbb{Z} \longrightarrow \mathbb{Z}$ is a surjective endomorphism of $\mathbb{Z}$. As $\mathbb{Z}$ is noetherian, $g \circ f : \mathbb{Z} \longrightarrow \mathbb{Z}$ is bijective and also f is bijective. Then $\mathbb{Z}$ is epi-correct. For monocorrectness of $\mathbb{Z}$, see [3].

Proposition 10.3.3. *The category of abelian groups is not mono-correct.*

Proof. We have seen that

$$\bigoplus_{n=1}^{\infty}\mathbb{Q}_n \overset{m}{\simeq} \left(\mathbb{Z}\oplus\left(\bigoplus_{n=1}^{\infty}\mathbb{Q}_n\right)\right)$$

but there exist any isomorphism between $\bigoplus_{n=1}^{\infty}\mathbb{Q}_n$ and $\mathbb{Z}\oplus\left(\bigoplus_{n=1}^{\infty}\mathbb{Q}_n\right)$ by Remark 10.2.1 and Proposition 10.2.2. Note that $\bigoplus_{n=1}^{\infty}\mathbb{Q}_n$ is a divisible abelian group and $\mathbb{Z}\oplus\left(\bigoplus_{n=1}^{\infty}\mathbb{Q}_n\right)$ is not a divisible abelian group.

Definition 10.5. An R-module M is said to be hopfian (respectively co-hopfian) if every surjective (respectively injective) endomorphism $f : M \to M$ is an automorphism.

Remark 10.3.4 ([3] and [2]). The following facts are well known:

- For a commutative ring R, any co-hopfian module is mono-correct.
- If R is a strongly Π-regular ring, then any finitely generated module is mono-correct.

Proposition 10.3.5. *Let R be a ring.*

1. *Any hopfian R-module is epi-correct.*
2. *If R is commutative then any finitely generated R-module is epi-correct.*

Proof. Let M be a hopfian module, N an R-module and $f : M \longrightarrow N$, $g : N \longrightarrow M$ two epimorphisms. We have that $g \circ f$ is a surjective endomorphism of M, then f is bijective.

For the second point, remark that if R is a commutative ring, **Vasconcelos** have shown that any finitely generated module is hopfian.

Proposition 10.3.6. *Let M be a mono-correct (respectively epi-correct) module and K be a direct summand of M. Then K is mono-correct (respectively epi-correct).*

Proof. Let $M = K\bigoplus K'$ and let N be a module such that $K \overset{m}{\simeq} N$ (respectively $K \overset{e}{\simeq} N$).

That is, there are two monomorphisms (respectively epimorphisms) $f : K \longrightarrow N$ and $g : N \longrightarrow K$. Then f and g induce monomorphisms (respectively epimorphisms) $f \oplus 1_{K'} : M \longrightarrow N\bigoplus K'$ and $g \oplus 1_{K'} : N\bigoplus K' \longrightarrow M$.

Since M is mono-correct (respectively epi-correct), then $M \simeq N\bigoplus K'$, hence $K \oplus K' \simeq N \oplus K'$, thus $K \simeq N$.

Proposition 10.3.7. *Let M_1, M_2 be R-modules and $M = M_1\bigoplus M_2$. Then M is mono-correct if and only if M_1, M_2 are mono-correct.*

Proof. If M is mono-correct, then by Proposition 10.3.6 M_1, M_2 are mono-correct. Conversely, $M = M_1 \bigoplus M_2$ with M_1, M_2 mono-correct. Let N be an R-module f, g be two monomorphisms as follows: $f : M \longrightarrow N$ and $g : N \longrightarrow M$.
Let $g(N) = N_1 \bigoplus N_2$, hence $N \simeq N_1 \bigoplus N_2$.
Now, consider the following morphisms:

$$f_1 : M_1 \xrightarrow{f|_{M_1}} f(M_1) \xrightarrow{g|_{f(M_1)}} N_1, \; g_1 : N_1 \longrightarrow M_1.$$

and

$$f_2 : M_2 \xrightarrow{f|_{M_2}} f(M_2) \xrightarrow{g|_{f(M_2)}} N_2, \; g_2 : N_2 \longrightarrow M_2.$$

f_1, g_1, f_2, g_2 are monomorphisms and M_1, M_2 mono-correct, thus $M_1 \simeq N_1$, $M_2 \simeq N_2$, hence $M_1 \bigoplus M_2 \simeq N_1 \bigoplus N_2 \simeq N$.

Definition 10.6. Let M be an R-module. An R-module P is said to be generated by M or M-generated if, for every pair of distinct morphisms $f, g : P \longrightarrow Q$, $Q \in R$-Mod, there is a morphism $h : M \longrightarrow P$ and $hf \neq hg$.

Definition 10.7. Let M be an R-module. An R-module N is said to be subgenerated by M if N is isomorphic to a submodule of an M-generated module.

We let $\sigma[M]$ denote the full subcategory of R-Mod whose objects are all R-modules subgenerated by M.

Theorem 10.3.1 ([6]). *For a module M, the following assertions are equivalent:*

1. *The class of all modules in $\sigma[M]$ is mono-correct.*
2. *every module in $\sigma[M]$ is mono-correct.*
3. *M is semisimple.*

Proposition 10.3.8. *Let R be a ring and M an R-module.*

1. *If any proper submodule of M is mono-correct, then M is mono-correct.*
2. *If M is semi simple, then all submodules and all factor modules of M are mono-correct.*
3. *If R is semi simple, then all R-module are mono-correct.*

Proof. Let N be an R-module and $f : M \longrightarrow N$, $g : N \longrightarrow M$ be two monomorphisms. We have $g(N) \simeq N$ and $g(N)$ is a submodule of M. If $g(N)$ is not a proper submodule of M, then $g(N) = M$. If $g(N)$ is a proper submodule of M, then $g(N)$ is mono-correct. Now let us consider:

$$h : M \xrightarrow{f} N \xrightarrow{g} g(N) \text{ and the canonical injection } k : g(N) \longrightarrow M.$$

h, k are monomorphisms and then $M \simeq g(N) \simeq N$

For the second point, remark that all submodules and all factor modules of M belong to $\sigma[M]$, and by Theorem 10.3.1, they are all mono-correct.
For the third point, note that if R is semi simple then $\sigma[R] = R$-Mod

Proposition 10.3.9. *Let M be an R-module and N a submodule of M. If N is an injective R-module and M/N mono-correct then M is mono-correct.*

Proof. Let K be a submodule of M and $f : M \longrightarrow K$ a monomorphism.
We have that $N \simeq f(N)$, $K/f(N) \simeq K/N$ and $K/f(N)$ is a submodule of M/N.
Let us consider the following commutative diagram:

$$\begin{array}{ccccc} N & \xrightarrow{\ i\ } & M & \xrightarrow{\ p\ } & M/N \\ f'\big\downarrow & & f\big\downarrow & & \bar{f}\big\downarrow \\ f(N) & \xrightarrow{\ i'\ } & K & \xrightarrow{\ p'\ } & K/f(N) \end{array}$$

i, i' are canonical injections, p, p' are canonical surjections and f' is the restriction of f on N. Since f is injective, f' is an isomorphism, p is surjective, we have that $\bar{f}$ is injective by the five lemma.

$\bar{f} : M/N \longrightarrow K/f(N)$, $i : K/f(N) \longrightarrow M/N$ are monomorphisms and M/N mono-correct then $M/N \simeq K/f(N)$. As N is an injective R-module, $M = N \bigoplus N'$, and $K = f(N) \bigoplus K'$, hence $N' \simeq K'$, thus $M \simeq K$.

References

1. I.G. Connell, Some rings theoretic Schroder-Bernstein theorems. Am. Math. Soc. **132**, 335–351 (1968). http://dx.doi.org/10.2307/1994844
2. A.N. Guèye, C.T. Guèye, On mono-correct modules. Br. J. Math. Comput. Sci. **3**(4), 598–604 (2013); SCIENCEDOMAIN International
3. A.N. Guèye, C.T. Guèye, M. Sangharè, A new characterization of commutative strongly Π-regular rings. J. Math. Res. **4**(5), 30–33 (2012). Canadian Center of Science and Education, http://dx.doi:10.5539/jmr.V4n5p30
4. S.K. Rososhek, Correctness of modules (russ.). Izvestiya VUZ. Mat. **22**(10), 77–82 (1978); translated in Russian Mathematics, Allerton Press
5. D.W. Sharpe, P. Vamos, *Injective Modules*. Cambridge Tracts in Mathematics and Mathematical Physics, vol. 62 (Cambridge, University Press, 1972)
6. R. Wisbauer, Correct classes of Modules. Algebra Discret. Math. 1–13 (2005)
7. V. Trnkova, V. Koubek, The Cantor-Bernstein Theorem for functors. Commun. Math. Univ. Carol. **14**(2), 197–204 (1973)