Simple Differential Field Extensions and Effective Bounds

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Abstract. We establish several variations on Kolchin's differential primitive element theorem, and conjecture a generalization of Pogudin's primitive element theorem. These results are then applied to improve the bounds for the effective Differential Lüroth theorem.

Keywords: Differential chow forms · Primitive element theorem · Model theory · Differential Lüroth theorem

1 Notation

Throughout this paper, $\mathcal{U}$ will be a fixed sufficiently large saturated differentially closed field of characteristic zero with a derivation operator δ. An element $c \in \mathcal{U}$ such that $\delta(c) = 0$ is called a constant. In this paper, all the differential fields under discussion are subfields of $\mathcal{U}$ and subscripts denote differentiation.

Let F be a differential subfield of $\mathcal{U}$ and $S \subset \mathcal{U}$. We denote respectively by $F[S]$, $F(S)$, $F\{S\}$, and $F\langle S\rangle$ the smallest subring, the smallest subfield, the smallest differential subring, and the smallest differential subfield of $\mathcal{U}$ containing F and S.

The set S is said to be *differentially dependent* over F if the set $(\delta^k a)_{a\in S, k\geq 0}$ is algebraically dependent over F, and otherwise, S is said to be *differentially independent* over F. In the case $S = \{a\}$, we also say that a is *differentially algebraic* or *differentially transcendental* over F respectively. A maximal subset Ω of S which is differentially independent over F is said to be a *differential transcendence basis* of $F\langle S\rangle$ over F. We use $\text{d.tr.deg}\, F\langle S\rangle/F$ to denote the *differential transcendence degree* of $F\langle S\rangle$ over F, which is the cardinality of Ω.

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Considering F and $F\langle S\rangle$ as algebraic fields, we denote the algebraic transcendence degree of $F\langle S\rangle$ over F by $\text{tr.deg}\, F\langle S\rangle/F$.

We use $F\{y_1,\dots,y_n\}$ to denote the differential polynomial ring over F. Given a differential polynomial $f \in F\{y_1,\dots,y_n\}$, the order of f w.r.t. y_i is the greatest number k such that $y_i^{(k)}$ appears effectively in f, which is denoted by $\text{ord}(f, y_i)$. And if y_i does not appear in f, then we set $\text{ord}(f, y_i) = -\infty$. The *order* of f is defined to be $\max_i \text{ord}(f, y_i)$. A (resp. radical, prime) differential ideal is a (resp. radical, prime) algebraic ideal $\mathcal{I}$ of $F\{y_1,\dots,y_n\}$ satisfying $\delta(\mathcal{I}) \subset \mathcal{I}$.

By affine space, we mean $\mathbb{A}^n = \mathcal{U}^n$. An element $\eta = (\eta_1,\dots,\eta_n) \in \mathbb{A}^n$ is called a differential zero of $f \in F\{y_1,\dots,y_n\}$ if $f(\eta) = 0$. The set of all differential zeros of $\Sigma \subset F\{y_1,\dots,y_n\}$, is called a *differential variety* defined over F, denoted by $\mathbb{V}(\Sigma)$. All the differential varieties in this paper are assumed to be subsets of $\mathbb{A}^n$. For a differential variety V which is defined over F, we denote $\mathbb{I}(V)$ to be the set of all differential polynomials in $F\{y_1,\dots,y_n\}$ that vanish at every point of V.

A differential ideal $\mathcal{I} \subset F\{y_1,\dots,y_n\}$ is prime if and only if it has a generic point, that is, a point $\eta \in \mathbb{V}(\mathcal{I})$ such that for any $f \in F\{y_1,\dots,y_n\}$, $f(\eta) = 0 \Leftrightarrow f \in \mathcal{I}$. Let $\mathcal{I}$ be a prime differential ideal with a generic point $(\eta_1,\dots,\eta_n)$. Then there exist d and h such that for sufficiently large t,

$$\text{tr.deg}\, F(\eta_i^{(k)} : 1 \le i \le n; k \le t)/F = d(t+1) + h.$$

The polynomial $\omega_{\mathcal{I}}(t) = d(t+1)+h$ is called the *Kolchin polynomial* of $\mathcal{I}$ and the corresponding d, h are called the *differential dimension* and *order* of $\mathcal{I}$. When $\bar{a}$ is a tuple in a differential field extension of a differential field K, then we write $\omega_{\bar{a}/K}(t)$ for $\omega_{I(\bar{a}/K)}(t)$ where $I(\bar{a}/K)$ is the differential ideal of all differential polynomials over K which vanish at $\bar{a}$.

2 Introduction

In this note, we discuss various primitive element theorems for ordinary differential field extensions. The oldest such result we consider goes back to Kolchin [7, p. 728], where in fact he proved the primitive element theorem in the more general partial differential settings. Here, we restrict to consider the ordinary differential field extensions, so for convenience, we state Kolchin's primitive element theorem in the ordinary differential case as follows:

Theorem 2.1. *Let F be a differential field containing at least one nonconstant. Let $E = F\langle a_1,\dots,a_n\rangle$ and suppose that* $\text{d.tr.deg}\, E/F = 0$. *Then there is some $b \in E$ such that $E = F\langle b\rangle$.*

Pogudin [12] generalized Kolchin's theorem to the case that F is a constant field, under the assumption that E contains a nonconstant. In this note, we give a mild generalization of Kolchin's theorem, and conjecture a generalization of Pogudin's theorem. We also illustrate how these generalizations are useful for improving the bounds on a problem of effective differential algebra.

Let $F\langle u\rangle$ denote the fraction field of the differential polynomial ring $F\{u\}$ in one variable. Ritt [14] proved the analog of Lüroth's theorem:

Theorem 2.2. *Let K be a differential field such that $F \subset K \subset F\langle u\rangle$. Then there is some element $g \in K$ such that $K = F\langle g\rangle$.*

Ritt's original formulation is for fields of meromorphic functions, but the general theorem follows from this case via Seidenberg's embedding theorem [18] (Kolchin first proved the general theorem in [8,9]). More recent work has focused on computational aspects of the Lüroth's theorem in the case that K is finitely generated over F. To be more precise, suppose $K = F\langle P_1(u)/Q_1(u), \dots, P_n(u)/Q_n(u)\rangle$, then computing a Lüroth generator of K/F and giving order and degree bounds for a Lüroth generator are problems of effective differential algebra. Following Kolchin's idea, if $A(y) \in K\{y\}$ is the minimal differential polynomial of x over K w.r.t. the canonical ranking, then for any pair $(a, b) \in K^2$ of coefficients of the polynomial A satisfying that $a/b \notin F$, this a/b can serve as a Lüroth generator [9]. Thus, using the language of modern differential characteristic sets, a Lüroth generator can be computed in the following way: Given a characteristic set $Q_1(u)y_1 - P_1(u), \dots, Q_n(u)y_n - P_n(u)$ of a prime differential ideal $\mathcal{I} \subset F\{u, y_1, \dots, y_n\}$ w.r.t. the elimination ranking $u < y_1 < \cdots < y_n$, compute a characteristic set $B_1(y_1, y_2), \dots, B_{n-1}(y_1, \dots, y_n), B_0(y_1, \dots, y_n, u)$ of $\mathcal{I}$ w.r.t. the elimination ranking $y_1 < \cdots < y_n < u$. Rewrite $B_0(y_1, \dots, y_n, u) = \sum_i f_i(y_1, \dots, y_n)\theta_i(u)$, if $\zeta = \frac{f_{i_1}(P_1(u)/Q_1(u), \dots, P_n(u)/Q_n(u))}{f_{i_2}(P_1(u)/Q_1(u), \dots, P_n(u)/Q_n(u))} \notin F$, then $K = F\langle\zeta\rangle$. Based on this idea, Gao and Xu [6] gave an algorithmic proof of the differential Lüroth theorem, but did not consider bounds for the degrees or order of the generator. D'Alfonso et al. [3] proved the following effective version of the theorem:

Theorem 2.3. *Let F be an ordinary differential field of characteristic 0, u differentially transcendental over F and $K = F\langle P_1(u)/Q_1(u), \dots, P_n(u)/Q_n(u)\rangle$, where $P_j, Q_j \in F\{u\}$ are relatively prime differential polynomials of order at most $e \geq 1$ (i.e. at least one derivative of u occurs in P_j or Q_j for some j) and degree bounded by d such that each $P_j/Q_j \notin F$. Then, any Lüroth generator v of K/F can be written as the quotient of two relatively prime differential polynomials $P(u), Q(u) \in F\{u\}$ with order bounded by $\min\{\mathrm{ord}(P_j/Q_j) : 1 \leq j \leq n\}$ and total degree bounded by $\min\{(d+1)^{(e+1)n}, (nd(e+1)+1)^{2e+1}\}$.*

The connection between the differential Lüroth theorem and the primitive element theorem is related to improving the degree bounds. We should note that our manipulations are not designed to attack the problem of bounding the order, but note that as Kolchin proved in [8], any two Lüroth generators ω_1 and ω_2 are related by the formula $\omega_2 = (a\omega_1 + b)/(c\omega_1 + d)$ for some $a, b, c, d \in F$, so any two Lüroth generators should have the same order [3, see the remarks at the end of Subsect. 3.1]. Our technique is most easily employed in the case that the field F posseses a nonconstant element. In this case, the ideas of our techniques essentially derive from a mild generalization of Theorem 2.1. In the case that F is a constant differential field, our ongoing work is related to an attempt to generalize Pogudin's recent primitive element theorem [12].

In the case that F contains a nonconstant element, we improve the degree bounds as follows:

Theorem 2.4. *Let F be an ordinary differential field of characteristic 0 containing a nonconstant element. Let u be differentially transcendental over F and $K = F\langle P_1(u)/Q_1(u), \dots, P_n(u)/Q_n(u)\rangle$, where $P_j, Q_j \in F\{u\}$ are relatively prime differential polynomials of order at most $e \geq 1$ (i.e. at least one derivative of u occurs in P_j or Q_j for some j) and degree bounded by d such that each $P_j/Q_j \notin F$ In Theorem 2.3. Then, any Lüroth generator v of K over F can be written as the quotient of two coprime differential polynomials $P(u), Q(u) \in F\{u\}$ with degree bounded by*

$$\min\{(\lceil n/2 \rceil \cdot d + 1)^{2(e+1)}, (d+1)^{n(e+1)}, (nd(e+1)+1)^{2e+1}\}.$$

In the case that the base field consists of constants, we improve the bound as follows:

Theorem 2.5. *Suppose F is a field of constants, u be differentially transcendental over F and $K = F\langle P_1(u)/Q_1(u), \dots, P_n(u)/Q_n(u)\rangle$ with P_i, Q_i satisfying the same conditions as in Theorem 2.4. Then the total degree of a Lüroth generator of K over F is bounded by*

$$\min\{(d(n+e-1)+1)^{2(e+1)}, (nd(e+1)+1)^{2e+1}\}.$$

The paper is organized as follows. In Sect. 3, we will prove various primitive element theorems for differential fields. In Sect. 4, we will utilize our embedding results to establish the improved bounds for the differential Lüroth's Theorem.

3 Variations of the Primitive Element Theorem for Differential Field Extensions

Throughout this section, all differential fields which appear will be assumed to be subfields of $\mathcal{U}$, the fixed sufficiently large saturated differentially closed field of characteristic zero given in Sect. 1.

Lemma 3.1 *[15, p. 35]. Suppose F contains at least one nonconstant element. If $f \in F\{u\}$ is a nonzero differential polynomial with order r, then for any nonconstant $\eta \in F$, there exists an element $c_0 + c_1\eta + c_2\eta^2 + \cdots + c_r\eta^r$ which does not annul f, where $c_0, \dots, c_r$ are constants in F.*

Remark 3.2. Note that in Lemma 3.1, we can always select the c_i from the rational number field $\mathbb{Q}$. Indeed, let $x_0, \dots, x_r$ be arbitrary constants, i.e., the x_i are algebraically independent over F and $x_i' = 0$. Since $f(\sum_{i=0}^r c_i\eta^r) \neq 0$, $g(x_0, \dots, x_n) = f(x_0 + x_1\eta + x_2\eta^2 + \cdots + x_r\eta^r)$ is a nonzero polynomial in $F\langle\eta\rangle[x_0, \dots, x_r]$. Since $\mathbb{Q}$ is an infinite field, by induction on r, it is easy to show that there exists $(d_0, \dots, d_r) \in \mathbb{Q}^{r+1}$ such that $f(d_0+d_1\eta+d_2\eta^2+\cdots+d_r\eta^r) \neq 0$.

Also, if $f \in F\{u_1, \ldots, u_n\}$ is a nonzero differential polynomial with order bounded by r, then for any nonconstant $\eta \in F$, there exist $c_{ij} \in \mathbb{Q}\,(1 \le i \le n; 0 \le j \le r)$ such that $f(\sum_{i=0}^{r} c_{0i}\eta^i, \ldots, \sum_{i=0}^{r} c_{ni}\eta^i) \neq 0$. We justify this by induction on n. The above paragraph shows that it is valid for $n = 1$. Suppose it holds for $n-1$. Regard $f(u_1, \ldots, u_n)$ as a polynomial in $u_1, \ldots, u_{n-1}$ with coefficients in $F\{u_n\}$, then by the induction hypothesis, there exist $c_{ij} \in \mathbb{Q}\,(1 \le i \le n-1)$ such that $g(u_n) = f(\sum_{j=0}^{r} c_{1j}\eta^j, \ldots, \sum_{j=0}^{r} c_{n-1,j}\eta^j, u_n) \neq 0$. Thus, from the case $n = 1$, there exist $c_{nj} \in \mathbb{Q}$ such that $g(\sum_{j=0}^{r} c_{nj}\eta^j) = f(\sum_{j=0}^{r} c_{1j}\eta^j, \ldots, \sum_{j=0}^{r} c_{nj}\eta^j) \neq 0$.

Lemma 3.3. *Let $F_1 \subset F$ be differential fields. Suppose that F_1 is not a field of constants. Then the Kolchin closure of F_1^n over F is $\mathbb{A}^n$.*

Proof. It suffices to show that $\mathbb{I}(F_1^n)$, the set of all differential polynomials over F which vanish at F_1^n, is the zero differential ideal. Since F contains at least a nonconstant, say η, for any nonzero $f \in F\{y_1, \ldots, y_n\}$, by Remark 3.2, there exist $c_{ij} \in \mathbb{Q}\,(0 \le j \le \mathrm{ord}(f))$ such that $f(\sum_j c_{1j}\eta^j, \ldots, \sum_j c_{nj}\eta^j) \neq 0$. Note that for each i, $\sum_j c_{ij}\eta^j \in F_1$. Thus, $\mathbb{I}(F_1^n) = [0]$. Hence, the Kolchin closure of F_1^n is $\mathbb{V}(0) = \mathbb{A}^n$. $\square$

Note that Kolchin's proof [7, p. 728] for Theorem 2.1 as well as Seidenberg's proof for [17, Theorem 1] implies the following result:

Proposition 3.4. *Let $L = F\langle\alpha_1, \ldots, \alpha_n\rangle$ and* $\mathrm{d.tr.deg}\, L/F = 0$*. Let F be a subfield of F such that F contains a nonconstant. Then, there exist $c_1, \ldots, c_n \in F$ such that $L = F\langle c_1\alpha_1 + \cdots + c_n\alpha_n\rangle$.*

The following result is a straightforward implication of Proposition 3.4 and here we will give a new proof from the geometric point of view.

Proposition 3.5. *Let $K = F\langle\alpha_1, \ldots, \alpha_n\rangle$ with F containing at least one nonconstant and* $\mathrm{d.tr.deg}\, K/F = d > 0$*. Assume without loss of generality that $\alpha_1, \ldots, \alpha_d$ is a differential transcendence basis of K over F. Then for any nonconstant subfield F_1 of F, there exist $c_{d+1}, \ldots, c_n \in F_1$ such that $K = F\langle\alpha_1, \ldots, \alpha_d, \sum_{i=d+1}^{n} c_i\alpha_i\rangle$.*

Proof. Consider the affine differential variety $V \subset \mathbb{A}^{n-d}$ given by the locus of $\alpha_{d+1}, \ldots, \alpha_n$ over $F\langle\alpha_1, \ldots, \alpha_d\rangle$. The variety V is $F\langle\alpha_1, \ldots, \alpha_d\rangle$-definable each of whose points are differentially algebraic over $F\langle\alpha_1, \ldots, \alpha_d\rangle$. Let $\bar{u} = (u_{d+1}, \ldots, u_n)$ be a tuple which are differentially independent over $F\langle\alpha_1, \ldots, \alpha_d\rangle$. Then we claim that the map $\phi_{\bar{u}} : V \to \mathbb{A}^1$ given by $\bar{x} = (x_{d+1}, \ldots, x_n) \to \sum_{i=d+1}^{n} u_i x_i$ is injective. For if not, then there are two points $\bar{a} = (a_{d+1}, \ldots, a_n)$ and $\bar{b} = (b_{d+1}, \ldots, b_n)$ such that $\sum_{i=d+1}^{n} u_i a_i = \sum_{i=d+1}^{n} u_i b_i$. But $\bar{a}$ and $\bar{b}$ are differentially algebraic over $F\langle\alpha_1, \ldots, \alpha_d\rangle$, and so $\bar{u}$ is δ-transcendental over $F\langle\alpha_1, \ldots, \alpha_d, \bar{a}, \bar{b}\rangle$. But now we have a contradiction, because $\sum_{i=d+1}^{n} u_i(a_i - b_i) = 0$, and not all of the $a_i - b_i$ can be zero, since $\bar{a} \neq \bar{b}$.

The injectivity of the map $\phi_{\bar{u}}$ on the $F\langle\alpha_1, \ldots, \alpha_d\rangle$-definable set V is a first order property (over $F\langle\alpha_1, \ldots, \alpha_d\rangle$) of the tuple $\bar{u}$. Since $\bar{u}$ is generic over $F\langle\alpha_1,\ldots,\alpha_d\rangle$, it follows by quantifier elimination that for all $\bar{v}$ in an $F\langle\alpha_1,\ldots,\alpha_d\rangle$-open subset $U \subset \mathbb{A}^{n-d}$, the map $\phi_{\bar{v}}$ is injective. There is a point $\bar{\gamma} = (\gamma_{d+1},\ldots,\gamma_n)$ of F^{n-d} in U by Lemma 3.3 applied to F relative to $F\langle\alpha_1,\ldots,\alpha_d\rangle$.

Now let $W \subset \mathbb{A}^n$ be the locus of $\alpha_1,\ldots,\alpha_n$ over F. Then the map $\pi_\gamma : \mathbb{A}^n \to \mathbb{A}^{d+1}$ given by $(x_1,\ldots,x_n) \mapsto (x_1,\ldots,x_d,\sum_{i=d+1}^n \gamma_i x_i)$ is injective on the fiber above $\alpha_1,\ldots,\alpha_d$ in W (the proper subvariety of W with $x_1=\alpha_1,\ldots,x_d=\alpha_d$). By the genericity of $\bar{\alpha} \in W$ over F, it follows that π_γ is injective on a Kolchin open subset of $\mathbb{A}^n$. So $F\langle\bar{\alpha}\rangle \cong F\langle W\rangle \cong F\langle\pi_\gamma(W)\rangle \cong F\langle\alpha_1,\ldots,\alpha_d,\sum_{i=d+1}^n \gamma_i\alpha_i\rangle$ completing the proof. □

Next, we will establish Proposition 3.5 through the use of differential Chow forms, which enables us to compute β_i effectively. Without assuming a transcendence basis beforehand, we restate the proposition as follows:

Proposition 3.6. *Assume F is a differential field with at least one nonconstant. Suppose that $K = F\langle\alpha_1,\ldots,\alpha_n\rangle$ is a finitely generated differential field extension of F such that the differential transcendence degree of K over F is d. Then for any subfield $F \subset F$ containing a nonconstant, there are $\beta_0,\ldots,\beta_d \in K$ which are F-linear combinations of $\alpha_1,\ldots,\alpha_n$ such that*

$$K = F\langle\beta_0,\ldots,\beta_d\rangle.$$

Proof. Let $\mathcal{I} = \mathbb{I}((\alpha_1,\ldots,\alpha_n)) \subset F\{y_1,\ldots,y_n\}$. Then $\mathcal{I}$ is of differential dimension d. Let

$$L_i = u_{i0} + u_{i1}y_1 + \cdots + u_{in}y_n \,(i=0,1,\ldots,d)$$

be a system of $d+1$ generic differential hyperplanes where all the u_{ij} are differentially independent over F. Denote

$$u_i = (u_{i0},u_{i1},\ldots,u_{in})\,(i=0,\ldots,d) \text{ and } u = \{u_{ij} : i=0,\ldots,d; j\neq 0\}.$$

Let $\mathcal{P} = [\mathcal{I},L_0,\ldots,L_d] \subset F\{y_1,\ldots,y_n,u_0,\ldots,u_d\}$. Assume $G(u_0,\ldots,u_d)$ is the differential Chow form of $\mathcal{I}$ and $\mathrm{ord}(G)=h$. Then by the property of the differential Chow form [5, Lemma 4.10],

$$\frac{\partial G}{\partial u_{00}^{(h)}}y_j - \frac{\partial G}{\partial u_{0j}^{(h)}} \in \mathcal{P}\ (j=1,\ldots,n).$$

Since $\xi = (\alpha_1,\ldots,\alpha_n;-\sum_{j=1}^n u_{0j}\alpha_j,u_{01},\ldots,u_{0n};\ldots;-\sum_{j=1}^n u_{dj}\alpha_j,u_{d1},\ldots,u_{dn})$ is a generic point of $\mathcal{P}$ and $\frac{\partial G}{\partial u_{00}^{(h)}} \notin \mathcal{P}$, $\frac{\partial G}{\partial u_{00}^{(h)}}(\xi) \neq 0$. Now regard $\frac{\partial G}{\partial u_{00}^{(h)}}(\xi)$ as a differential polynomial in u with coefficients in K, which is nonzero. By Lemma 3.1, there exists $a_{ij} \in F$ (for any $u_{ij} \in u$) such that

$$\frac{\partial G}{\partial u_{00}^{(h)}}(\alpha_1,\ldots,\alpha_n;-\sum_{j=1}^n a_{0j}\alpha_j,a_{01},\ldots,a_{0n};\ldots;-\sum_{j=1}^n a_{dj}\alpha_j,a_{d1},\ldots,a_{dn}) \neq 0.$$

For each $k = 0, 1, \ldots, d$, let g_k be the differential polynomial in $F\{u_{00}, \ldots, u_{d0}\}$ obtained from $\frac{\partial G}{\partial u_{0k}^{(h)}}$ by replacing $u_{ij} \in u$ by a_{ij}. Then g_0 is a nonzero differential polynomial which satisfies $g_0(-\sum_{j=1}^{n} a_{0j}\alpha_j, \ldots, -\sum_{j=1}^{n} a_{dj}\alpha_j) \neq 0$.

Let $\beta_i = -\sum_{j=1}^{n} a_{ij}\alpha_j$ for $i = 0, \ldots, d$. We claim that $K = F\langle \beta_0, \ldots, \beta_d \rangle$. Let $\bar{L}_i = u_{i0} + a_{i1}y_1 + \cdots + a_{in}y_n$ $(i = 0, \ldots, d)$ and

$$\mathcal{P}_1 = [\mathcal{I}, \bar{L}_0, \ldots, \bar{L}_d] \subset F\{y_1, \ldots, y_n, u_{00}, \ldots, u_{d0}\}.$$

Clearly, $\mathcal{P}_1$ is a prime differential ideal with a generic point $(\alpha_1, \ldots, \alpha_n, \beta_0, \ldots, \beta_d)$. Since $\frac{\partial G}{\partial u_{00}^{(h)}} y_j - \frac{\partial G}{\partial u_{0j}^{(h)}} \in \mathcal{P}$, it is clear that $g_0 y_j - g_j \in \mathcal{P}_1$ for each $j = 1, \ldots, n$. Thus, $g_0 y_j - g_j$ vanishes at $(\alpha_1, \ldots, \alpha_n, \beta_0, \ldots, \beta_d)$, which implies

$$\alpha_j = g_j(\beta_0, \ldots, \beta_d)/g_0(\beta_0, \ldots, \beta_d).$$

Hence, $K = F\langle \beta_0, \ldots, \beta_d \rangle$. □

We use the following two examples to illustrate the method given in the proof of Proposition 3.6 to compute the generators of the required forms. In the both examples, differential Chow forms can be computed using either the characteristic set method as described in [5, Remark 4.4] or the algorithms for computing differential Chow forms given in [10].

Example 3.7. Let $F = \mathbb{Q}(x)$ with derivation $\delta = \frac{d}{dx}$. Let $K = \mathbb{Q}(x)\langle \alpha_1, \alpha_2 \rangle$ where α_1, α_2 are the generic solutions of $y' + 1, z'$ respectively. Let $\mathcal{I} = \mathbb{I}((\alpha_1, \alpha_2)) \subset F\{y, z\}$. Then $\mathcal{I}$ is of differential dimension 0. Take $L_0 = u_0 + u_1 y + u_2 z$. Then the differential Chow form of $\mathcal{I}$ is

$$G = (u_1u_2' - u_2u_1')u_0'' - 2u_1'(u_1u_2' - u_1'u_2) - u_1''(u_0u_2' - u_2u_0') + u_2''(u_0u_1' - u_0'u_1 + u_1^2).$$

So the separant of G is $S_G = u_1u_2' - u_2u_1'$. We can take $u_1 = -1$ and $u_2 = -x$ which does not annul S_G. Hence, $\alpha_1 + x\alpha_2$ is a primitive element of K/F, that is, $K = F\langle \alpha_1 + x\alpha_2 \rangle$.

Example 3.8. Let $F = \mathbb{Q}(x)$ with $\delta = \frac{d}{dx}$. Let $K = \mathbb{Q}(x)\langle u + x, u' + x, u'' \rangle$ where u is differentially transcendental over F. Clearly, the differential transcendence degree of K over F is 1. Let $u_i = (u_{i0}, u_{i1}, u_{i2}, u_{i3})$ $(i = 0, 1)$. Then we can compute the differential Chow form $G(u_0, u_1)$ of $\mathbb{I}((u + x, u' + x, u'')) \subset F\{y_1, y_2, y_3\}$, which is a differential polynomial of order 2 and differential degree 6. The separant of G is

$$\begin{aligned} S_G = u_{13}(&u_{11}u_{12}u_{03}^2 - u_{11}u_{13}u_{02}u_{03} - u_{12}u_{11}u_{03}^2 + u_{12}'u_{13}u_{01}u_{03} - u_{03}'u_{11}u_{13}u_{02} \\ &+ u_{03}'u_{12}u_{13}u_{01} + u_{13}'u_{11}u_{02}u_{03} - u_{13}'u_{12}u_{01}u_{03} + u_{02}'u_{11}u_{13}u_{03} - u_{02}'u_{13}^2u_{01} \\ &- u_{01}'u_{12}u_{13}u_{03} + u_{01}'u_{13}^2u_{02} - u_{11}^2u_{03}^2 + u_{11}u_{12}u_{02}u_{03} + 2u_{11}u_{13}u_{01}u_{03} \\ &- u_{11}u_{13}u_{02}^2 - u_{12}^2u_{01}u_{03} + u_{12}u_{13}u_{01}u_{02} - u_{13}^2u_{01}^2). \end{aligned}$$

We can take $u_{01} = 1, u_{02} = 0, u_{03} = x, u_{11} = x, u_{12} = 1, u_{13} = 1$ which does not annul S_G. Hence, $\beta_1 = (u + x) + tu'', \beta_2 = x(u + x) + (u' + x) + u''$ is a set of generators of K/F, that is, $K = F\langle \beta_1, \beta_2 \rangle$.

The proofs of Lemma 3.3 and Proposition 3.5 can be generalized to the case of a differential field with finitely many commuting derivations $\delta_1, \dots, \delta_m$, under the assumption that F contains m elements $\beta_1, \dots, \beta_n$ whose Jacobian, $\det(\delta_i(\beta_j))$, is nonzero [7], but the proof of Proposition 3.6 is not suited for the partial case.

Proposition 3.5 does not hold in the case that the field F is the constant field:

Example 3.9. Let F be the rational number field with the trivial derivation. Let x, y be two constants in a differential extension field of F, which are algebraically independent over F. Consider $K = F\langle x, y\rangle = F(x, y)$. Then K is of differential transcendence degree 0, but the transcendence degree of K over F is 2. Clearly, there is no $a, b \in F$ such that $K = F(ax + by)$.

When F is a constant differential field, although Proposition 3.5 is not valid, we have the following similar result, which can be regarded as a consequence of Seidenberg's proof [17].

Proposition 3.10. *Assume F is a differential field of constants. Suppose that $K = F\langle \alpha_1, \dots, \alpha_n\rangle$ is a finitely generated differential field extension of F such that the differential transcendence degree of K over F is $d > 0$. Suppose α_1 is differentially transcendental over F. Then there exist $\beta_1, \dots, \beta_d \in K$ such that $K = F\langle \alpha_1, \beta_1, \dots, \beta_d\rangle$ and each β_i is an F-linear combination of $\alpha_2, \dots, \alpha_n$ and powers of α_1 bounded by h where h is the order of the differential Chow form of $\alpha_2, \dots, \alpha_n$ over $F\langle \alpha_1\rangle$. In particular, there exist $c_{ijk} \in \mathbb{Q}$ such that $\beta_i = \sum_{j=2}^{n}(\sum_{k=0}^{h} c_{ijk}\alpha_1^k)\alpha_j, i = 1, \dots, d$ and $K = F\langle \alpha_1, \beta_1, \dots, \beta_d\rangle$.*

Proof. Let $K_1 = F\langle \alpha_1\rangle$. Then the differential transcendence degree of K over K_1 is $d-1$. Consider the differential ideal $\mathcal{I} = \mathbb{I}((\alpha_2, \dots, \alpha_n)) \subset K_1\{y_2, \dots, y_n\}$. Suppose the order of $\mathcal{I}$ is equal to h. Suppose $G(u_0, \dots, u_{d-1})$ is the differential Chow form of $\mathcal{I}$, then $\mathrm{ord}(G) = h$. Applying the similar method as in the proof of Proposition 3.6 to K/K_1, by Remark 3.2, we can find $c_{ijk} \in \mathbb{Q}$ such that for $\beta_i = \sum_{j=2}^{n}(\sum_{k=0}^{h} c_{ijk}\alpha_1^k)\alpha_j$, $K = F\langle \alpha_1, \beta_1, \dots, \beta_d\rangle$. □

Also, in the case that F is a constant differential field, one can establish Proposition 3.5 when making an additional assumption on the elements $\alpha_1, \dots, \alpha_n$. The additional assumption uses terminology from model theory, which we will now introduce. Our conventions are designed to deliver the model theoretic notions in the differential algebraic setting, where some of the notions can be given significantly simpler definitions than in the general setting.

We remind the reader that $\mathcal{U}$ is a universal differential field. Let $X\mathbb{A}^n$ be a constructible set in the Kolchin topology over F; that is, X is a boolean combination of affine differential varieties over F. Then we say X is *orthogonal to the constants* if for any differential field extension K of F, any element c of the constant field $\mathcal{C}$ of $\mathcal{U}$, and any $\bar{a} \in X$, we have the equality of the Kolchin polynomials:

$$\omega_{\bar{a}/K\langle c\rangle}(t) = \omega_{\bar{a}/K}(t).$$

This implies that if c is transcendental over K, then c is transcendental over $K\langle a\rangle$.

The notion defined in the previous paragraph is a special case of the general notion defined in [1, see Ziegler's article, page 40 for additional details].

Proposition 3.11. *Suppose that $K = F\langle\alpha_1, \ldots, \alpha_n\rangle$ is a finitely generated differential field extension of F such that the differential transcendence degree of K over F is d. Assume without loss of generality that $\alpha_1, \ldots, \alpha_d$ are differentially independent over F. Suppose that $loc((\alpha_{d+1}, \ldots, \alpha_n)/F\langle\alpha_1, \ldots, \alpha_d\rangle)$ is orthogonal to the constants. Then there is $\beta_{d+1} \in K$ such that $K = F\langle\alpha_1, \ldots, \alpha_d, \beta_{d+1}\rangle$ and β_{d+1} is an $\mathbb{Q}$-linear combination of $\alpha_{d+1}, \ldots, \alpha_n$.*

Proof. First, we claim that the general result follows from the case in which $n = d+2$. This follows inductively, noting that $\mathbb{Q}$-linear combinations of $\mathbb{Q}$-linear combinations of $\alpha_{d+1}, \ldots, \alpha_n$ are again $\mathbb{Q}$-linear combinations of $\alpha_{d+1}, \ldots, \alpha_n$ and $\mathbb{Q}$-linear combinations preserve orthogonality to the constants. So, without loss of generality, assume that $n = d + 2$.

Let $X = loc_{F\langle\alpha_1,\ldots,\alpha_d\rangle}(a, b)$, the Kolchin closure of (a, b) over the ground field $F\langle\alpha_1, \ldots, \alpha_d\rangle$. Let $c_1, c_2 \in \mathcal{C}$ be independent transcendental constants over $F\langle\alpha_1, \ldots, \alpha_d\rangle$. We claim the map

$$\phi_{\bar{c}} : X \to \mathbb{A}^1$$

given by $(x, y) \mapsto c_1x + c_2y$ is injective on a Kolchin open subset of X. If this is not the case, there are $(x_1, y_1), (x_2, y_2) \in X$ such that (x_i, y_i) is generic on X over $F\langle\alpha_1, \ldots, \alpha_d, c_1, c_2\rangle$ (which implies that c_1, c_2 are independent transcendentals over $F\langle\alpha_1, \ldots, \alpha_d, x_i, y_i\rangle$ for $i = 1, 2$) such that

$$c_1x_1 + c_2y_1 = c_1x_2 + c_2y_2.$$

But now taking $K = F\langle\alpha_1, \ldots, \alpha_d, c_1, x_1, y_1\rangle$, we can see that c_2 is not transcendental over $K\langle x_2, y_2\rangle$ as $c_2 = c_1 \cdot \frac{x_2-x_1}{y_1-y_2}$ is in $K\langle x_2, y_2\rangle$. But this implies:

$$\omega_{(x_2,y_2)/K\langle c_2\rangle}(t) \neq \omega_{(x_2,y_2)/K}(t),$$

and this contradicts the assumption that X is orthogonal to the constants.

So, there is a Kolchin open subset $U \subset X$ such that the map $\phi_{\bar{c}}|_U$ is an injective map. Injectivity is a definable property of the map $\phi_{\bar{c}}|_U$, and it holds for the generic point in $\mathcal{C}^2$, so for some Zariski open (the Kolchin open subsets of $\mathcal{C}$ are Zariski open) subset $U_1 \subset \mathcal{C}$, for all $\bar{c}' \in U_1$, the map $\phi_{\bar{c}'}|_U$ is injective. By the density of $\mathbb{Q}^2$ in $\mathcal{C}^2$, there are $\bar{q} = q_1, q_2$ for which $\phi_{\bar{q}}|_U$ is injective and thus gives an isomorphism between the differential function field of X and its image, completing the proof. □

Remark 3.12. The assumption that $tp(\alpha_{d+1}, \ldots, \alpha_n/F\langle\alpha_1, \ldots, \alpha_d\rangle)$ is orthogonal to the constants is rather difficult to verify in practice. On the other hand, it is folklore of the model theory of differential fields that *most* differential equation of some order ≥ 1 and degree ≥ 2 should be strongly minimal and trivial (which implies orthogonality to the constants). For specific instances of results of this nature, see [2, 13].

There is a considerable literature devoted to verifying this condition for various specific differential equations [2, 4, 13]; proving that a given strongly minimal differential equation has trivial forking geometry (and is thus orthogonal to the constants) is also the key to proving that differential closure is not minimal [16]. As far as we can tell, only the results of [4] provide examples which are defined over a differential transcendental, which is the only case pertinent to the differential Lüroth theorem. To give the reader an idea of the hypothesis, we will give a specific example, in the case of two variables in order to keep the technicalities minimal.

So, let

$$P_1/Q_1 := S(u'+u) + R(u'+u)\cdot((u'+u)')^2,$$

where

$$R(y) = \frac{y^2 - 1968y + 2\ 654\ 208}{2y^2(y-1728)^2},$$

and

$$S(x) = \left(\frac{x''}{x'}\right)' - \frac{1}{2}\left(\frac{x''}{x'}\right)^2$$

is the Schwarzian derivative. Let

$$P_2/Q_2 = u' + u.$$

Then the type $tp(P_2/Q_2/F\langle P_1/Q_1\rangle)$ is the generic solution to the differential equation

$$S(x) + R(x)\cdot((x)')^2 = P_1/Q_1.$$

By the results of [4], this type is strongly minimal. It follows that the type is nonorthogonal to the constants, since the equivalence relation of nonorthogonality refines transcedence degree on strongly minimal sets (the authors of [4] also prove that this set has trivial forking geometry).

Propositions 3.5 and 3.10 are effective in the sense that the degree of the elements which generate the differential field extension are bounded. We further conjecture the following mild strengthening of Pogudin's primitive element theorem:

Conjecture 3.13. *Assume F is a constant differential field. Suppose that $K = F\langle\alpha_1,\ldots,\alpha_n\rangle$ is a finitely generated differential field extension of F such that the differential transcendence degree of K over F is $d > 0$. Assume without loss of generality that $\alpha_1,\ldots,\alpha_d$ are a differential transcendence basis for K over F. Assume that at least one of $\alpha_{d+1},\ldots,\alpha_n$ is a nonconstant. Then there is a polynomial $P \in \mathbb{Q}[x_{d+1},\ldots,x_n]$ such that*

$$K = F\langle\alpha_1,\ldots,\alpha_d, P(\alpha_{d+1},\ldots,\alpha_n)\rangle.$$

The above conjecture is a direct consequence of the following stronger conjecture.

Conjecture 3.14. *Assume F is a differential field which contains at least one nonconstant. Suppose that $K = F\langle\alpha_1,\ldots,\alpha_n\rangle$ is a finitely generated differential field extension of F such that each α_i is differentially algebraic over F and at* *least one* α_i *is a nonconstant. Then there is a polynomial* $P \in \mathbb{Q}[x_1, \ldots, x_n]$ *such that*

$$K = F\langle P(\alpha_1, \ldots, \alpha_n)\rangle.$$

The above conjecture is not true if all α_i are constants. Similar to Example 3.9, if x, y are constants which are independent algebraic indeterminates, $\mathbb{Q}(t)\langle x, y\rangle \neq \mathbb{Q}(t)\langle P(x, y)\rangle$ for any $P \in \mathbb{Q}[x, y]$.

In the current work in progress we hope to establish the conjecture in an effective form (bounding the degree of P); bounding the degree of P might then be used to improve the bounds for the degree of a Lüroth generator while working over a constant differential field.

4 Improving the Bounds in the Differential Lüroth Theorem

In this section, we explain how the results of the previous section can be applied to improve the degree bound for the differential Lüroth theorem.

4.1 The Nonconstant Case

We will work first in the case where F contains some nonconstant element, proving Theorem 2.4. The analysis is simpler in this case and the primitive element style analysis of the previous section yields improved bounds.

Suppose that

$$K = F\langle P_1(u)/Q_1(u), \ldots, P_n(u)/Q_n(u)\rangle$$

where $P_j, Q_j \in F\{u\}$ are relatively prime differential polynomials with order satisfying $e = \max\{\mathrm{ord}(P_i), \mathrm{ord}(Q_j)\} \geq 1$ and total degree bounded by d such that $P_j/Q_j \notin F$ for every $1 \leq j \leq n$.

When $x \in \mathbb{N}$, let $\lfloor x \rfloor, \lceil x \rceil$ denote the standard floor and ceiling functions, respectively. Let

$$K_1 = F\langle P_1(u)/Q_1(u), \ldots, P_{\lfloor n/2 \rfloor}(u)/Q_{\lfloor n/2 \rfloor}(u)\rangle$$

and consider the differential field extension

$$K = K_1\langle P_{\lfloor n/2 \rfloor+1}(u)/Q_{\lfloor n/2 \rfloor+1}(u), \ldots, P_n(u)/Q_n(u)\rangle.$$

Since each $P_i(u)/Q_i(u) \notin F$ and $\mathrm{d.tr.deg}\, K/F = 1$, $\mathrm{d.tr.deg}\, K/K_1 = 0$. Apply Proposition 3.4 to the extension K over K_1 with K_1 playing the role of F in Proposition 3.4, then we obtain a generator β for K over K_1 which is an F-linear combination of $P_{\lfloor n/2 \rfloor+1}(u)/Q_{\lfloor n/2 \rfloor+1}(u), \ldots, P_n(u)/Q_n(u)$. Note that this β has order at most e and degree at most $\lceil n/2 \rceil \cdot d$. Specifically, the total degree of β is bounded by the sum of the degrees of $P_{\lfloor n/2 \rfloor+1}(u)/Q_{\lfloor n/2 \rfloor+1}(u), \ldots, P_n(u)/Q_n(u)$. Here, It may happen that $K = K_1$

and in this case, the obtained β may be contained in F. If this happens, we reset $\beta = P_n(u)/Q_n(u)$.

Now we have $K = K_1\langle\beta\rangle = F\langle P_1(u)/Q_1(u), \dots, P_{\lfloor n/2\rfloor}(u)/Q_{\lfloor n/2\rfloor}(u), \beta\rangle$. Clearly, d.tr.deg $K/F\langle\beta\rangle = 0$. Applying Proposition 3.4 to the differential field extension

$$K = F\langle P_1(u)/Q_1(u), \dots, P_{\lfloor n/2\rfloor}(u)/Q_{\lfloor n/2\rfloor}(u), \beta\rangle$$

over $F\langle\beta\rangle$, then there is an F-linear combination of

$$P_1(u)/Q_1(u), \dots, P_{\lfloor n/2\rfloor}(u)/Q_{\lfloor n/2\rfloor}(u)$$

which generates K over $F\langle\beta\rangle$. Call this element α and note that α has order at most e and degree at most $\lfloor n/2\rfloor \cdot d$. Specifically, the total degree of α is bounded by the sum of the total degrees of $P_1(u)/Q_1(u), \dots, P_{\lfloor n/2\rfloor}(u)/Q_{\lfloor n/2\rfloor}(u)$.

Now, we have obtained $K = F\langle\alpha, \beta\rangle$ with $\max\{\text{ord}(\alpha), \text{ord}(\beta)\} = e_1 \le e$ and $\deg(\alpha), \deg(\beta) \le \lceil n/2\rceil \cdot d$. Note it may happen that $e_1 = 0$. In the following, we show that applying Theorem 2.3 to the differential field extension $F\langle\alpha, \beta\rangle$ over F, the degree of a Lüroth generator is bounded by $(\lceil n/2\rceil \cdot d + 1)^{(e+1)2}$.

Lemma 4.1. *The degree of a Lüroth generator of $F\langle\alpha, \beta\rangle$ over F is bounded by $(\lceil n/2\rceil \cdot d + 1)^{(e+1)2}$.*

Proof. If $e_1 \ge 1$, applying Theorem 2.3 directly to the differential field extension $F\langle\alpha, \beta\rangle$ over F, the bound can be obtained.

Now suppose $e_1 = 0$ and $\alpha = R_1(u)/S_1(u), \beta = R_2(u)/S_2(u) \in F(u)$. Let $u = z'$, the first derivative of a new element z. Since u is differentially transcendental over F, z is differentially transcendental over F too. Thus, $K = F\langle R_1(u)/S_1(u), R_2(u)/S_2(u)\rangle = F\langle R_1(z')/S_1(z'), R_2(z')/S_2(z')\rangle \subset F\langle z\rangle$. With respect to the new differential indeterminate z, $\max\{\text{ord}(R_i, z), \text{ord}(S_i, z)\} = 1$ which satisfying the conditions in Theorem 2.3. Thus, there exists coprime paris $(P(z), Q(z)) \in F\{z\}^2$ with $\text{ord}(P, z), \text{ord}(Q, z) \le 1$ and $\deg(P), \deg(Q) \le (\lceil n/2\rceil \cdot d + 1)^{(1+1)2} \le (\lceil n/2\rceil \cdot d + 1)^{(e+1)2}$ such that $K = F\langle P(z)/Q(z)\rangle$. Since $K = F\langle R_1(u)/S_1(u), R_2(u)/S_2(u)\rangle$ has a Lüroth generator $T_1(u)/T_2(u)$ and by [8, p. 359], the two Lüroth generators $P(z)/Q(z)$ and $T_1(u)/T_2(u)$ are related by the formula $P(z)/Q(z) = (aT_1(u)/T_2(u) + b)/(cT_1(u)/T_2(u) + d)$ for some $a, b, c, d \in F$, Thus, $P(z), Q(z) \in F[z']$. Replacing z' by u in P and Q, we get a Lüroth generator $P_0(u), Q_0(u)$ satisfying $\text{ord}(P_0, u), \text{ord}(Q_0, u) = 0$ and $\deg(P_0), \deg(Q_0) \le (\lceil n/2\rceil \cdot d + 1)^{(e+1)2}$. □

Combining the degree bound given in Theorem 2.3 with Lemma 4.1, we obtain the degree of a Lüroth generator of $K = F\langle P_1(u)/Q_1(u), \dots, P_n(u)/Q_n(u)\rangle$ over F is bounded by

$$\min\{(\lceil n/2\rceil \cdot d + 1)^{(e+1)2}, (d+1)^{(e+1)n}, (nd(e+1)+1)^{2e+1}\}.$$

This establishes Theorem 2.4.

Remark 4.2. The first quantity of the minimum taken above is $(\lceil n/2 \rceil \cdot d + 1)^{(e+1)2}$. This is almost always smaller than the second quantity $(d+1)^{(e+1)n}$, the only pertinent exceptional case being $n = 3, e = 1$, and $d = 1$. When any of the inputs is larger, $(\lceil n/2 \rceil \cdot d + 1)^{(e+1)2}$ is less than $(d+1)^{(e+1)n}$. It is also true that $(\lceil n/2 \rceil \cdot d + 1)^{(e+1)2}$ is very often smaller than $(nd(e+1)+1)^{2e+1}$, though there are infinitely many exceptional cases (essentially by picking n or d to be sufficiently large compared to e). From a practical standpoint, examples with low order, degree, and number of variables are of particular interest; when $n \leq 10$ and $d \leq 10$, $(\lceil n/2 \rceil \cdot d + 1)^{(e+1)2}$ is the smallest of the above bounds (excluding the exceptional case $n = 3, e = 1$, and $d = 1$).

4.2 The Constant Case

In this subsection, we assume F is a field of constants. Let

$$K = F\langle P_1(u)/Q_1(u), \ldots, P_n(u)/Q_n(u)\rangle$$

where each $P_i(u)/Q_i(u) \notin F$.

If $tp(P_2(u)/Q_2(u), \ldots, P_n(u)/Q_n(u)/F\langle P_1(u)/Q_1(u)\rangle)$ is orthogonal to the constants, then the analysis from the previous subsection works completely analogously with Proposition 3.11 in place of Proposition 3.5. The criterion also applies with $P_i(u)/Q_i(u)$ exchanging roles with $P_1(u)/Q_1(u)$, for any $i = 2, \ldots, n$. In the following, we do not assume such conditions on the generators P_i/Q_i.

To give the main theorem in this section, we first need several lemmas.

Lemma 4.3 *[10, Theorem 18]. Let $\mathfrak{I}$ be a prime differential ideal of differential dimension d in $F\{y_1, \ldots, y_n\}$, and $\mathcal{A} = \{A_1, \ldots, A_{n-d}\}$ a characteristic set of $\mathfrak{I}$ under an arbitrary ranking. Then $\mathrm{ord}(\mathfrak{I})$ is bounded by the Jacobi number of $\mathcal{A}$. That is,*

$$\mathrm{ord}(\mathfrak{I}) \leq \max_{\sigma} \sum_{i=1}^{n-d} \mathrm{ord}(A_i, y_{\sigma(i)}),$$

where σ runs among all injective maps from $\{1, \ldots, n-d\}$ to $\{1, \ldots, n\}$.

Lemma 4.4 *[5, Theorem 2.11]. Let $\mathfrak{I}$ be a prime differential ideal in $F\{y_1, \ldots, y_n\}$. Then $\mathrm{ord}(\mathfrak{I})$ is the maximum of all the relative orders of $\mathfrak{I}$, that is,*

$$\mathrm{ord}(\mathfrak{I}) = \max_{\mathrm{U}} \mathrm{ord}_{\mathrm{U}}(\mathfrak{I}),$$

where U is any parametric set of $\mathfrak{I}$, that is, U is a maximal subset of variables $\{y_1, \ldots, y_n\}$ such that $\mathfrak{I} \cap F\{U\} = \{0\}$.

With the above preparations, we now prove Theorem 2.5. That is, to show when F is a field of constants, then the total degree of a Lüroth generator is bounded by

$$\min\{(d(n+e-1)+1)^{2(e+1)}, (nd(e+1)(n+e-1)+1)^{2e+1}\}.$$

Proof of Theorem 2.5. Let $\alpha_i = P_i(u)/Q_i(u)\,(i = 1,\ldots,n)$ and $K = F\langle\alpha_1,\ldots,\alpha_n\rangle$. Clearly, the differential transcendence degree of K over F is 1. Also, by the hypothesis, each $\alpha_i \in F\langle u\rangle\backslash F$. Suppose h is the order of the prime differential ideal $\mathbb{I}((\alpha_2,\ldots,\alpha_n))$ over $F\langle\alpha_1\rangle$. By Proposition 3.10, there exist $c_{jk} \in \mathbb{Q}$ such that for $\eta = \sum_{j=2}^{n}(\sum_{k=0}^{h} c_{jk}\alpha_1^k)\alpha_j$, $K = F\langle\alpha_1,\eta\rangle$. The problem is reduced to the case $n = 2$. The order of η is still bounded by e. The degree of η is bounded by $d(n + h - 1)$.

It suffices to give a bound for h. Consider the prime differential ideal

$$\mathcal{I} = \mathbb{I}((u,\alpha_1,\alpha_2,\ldots,\alpha_n)) \subset F\{y_1,\ldots,y_n,z\}.$$

It is easy to show that

$$\mathcal{A} := Q_1(z)y_1 - P_1(z),\ldots,Q_n(z)y_n - P_n(z)$$

is a characteristic set of $\mathcal{I}$ w.r.t. the elimination ranking $z < y_1 < \ldots < y_n$. Since the orders of P_i, Q_i is bounded by e, the order matrix $(s_{ij})_{n\times(n+1)}$ of $\mathcal{A}$ satisfies $s_{ii} = \mathrm{ord}(A_i, y_i) = 0$, $s_{ij} = -\infty\,(i \neq j \leq n)$ and $s_{i,n+1} = \mathrm{ord}(A_i, z) \leq e$. So the Jacobi number of $\mathcal{A}$, $\max_\sigma\{\sum_{i=1}^{n} s_{i\sigma(i)}\}$ for σ running through injective maps from $\{1,\ldots,n\}$ to $\{1,\ldots,n+1\}$, is bounded by e. Thus, by Lemma 4.3, $\mathrm{ord}(\mathcal{I}) \leq e$. Let $\mathcal{I}_1 = \mathbb{I}((\alpha_1,\alpha_2,\ldots,\alpha_n)) = \mathcal{I} \cap F\{y_1,\ldots,y_n\}$. The Kolchin polynomials of $\mathcal{I}$ and $\mathcal{I}_1$ have the following relations: for sufficiently large t,

$$\begin{aligned}
\omega_{\mathcal{I}}(t) &= \mathrm{tr.deg}\, F\big(u^{(k)},\alpha_i^{(k)} : k \leq t, i = 1,\ldots,n\big)/F \\
&= (t+1) + \mathrm{ord}(\mathcal{I}) \\
&= (t+1) + \mathrm{ord}(\mathcal{I}_1) \\
&\quad + \mathrm{tr.deg}\, F\big(u^{(k)},\alpha_i^{(k)} : k \leq t, i = 1,\ldots,n\big)/F\big(\alpha_i^{(k)} : k \leq t, i = 1,\ldots,n\big)
\end{aligned}$$

Thus, $\mathrm{ord}(\mathcal{I}_1) \leq \mathrm{ord}(\mathcal{I}) \leq e$. Note that $h = \mathrm{tr.deg}\, F\langle\alpha_1,\alpha_2,\ldots,\alpha_n\rangle/F\langle\alpha_1\rangle$ is also equal to the relative order of $\mathcal{I}$ w.r.t. the parametric set $\{y_1\}$, by Lemma 4.4, $h \leq \mathrm{ord}(\mathcal{I}) \leq e$.

So the degree of η is bounded by $d(n + e - 1)$. Hence, by Theorem 2.3, the degree of a Lüroth generator is bounded by

$$\min\{(d(n+e-1)+1)^{2(e+1)}, (2d(e+1)(n+e-1)+1)^{2e+1}\}.$$

□

Remark 4.5. In most of the cases, especially when either n or e is large, we have

$$(d(n+e-1)+1)^{2(e+1)} < (2d(e+1)(n+e-1)+1)^{2e+1}$$

and

$$(d(n+e-1)+1)^{2(e+1)} < (nd(e+1)+1)^{2e+1}.$$

Hence, the degree bound given in Theorem 2.5 is smaller than that in Theorem 2.3.

As an experiment to compare the two bounds, we have computed more than 10,000 randomly generated tuples (n, d, e) simulating the pertinent cases of the bounds when each of the variables is less than 30, and our bound gives the better result approximately 94.3 % of the time.

In future work, we hope to prove Conjecture 3.13 in an effective manner and use the result to improve the bounds for the degree in the effective differential Lüroth theorem for the case that the base field is constant.

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