Existence of Mild Solutions for Impulsive Fractional Functional Differential Equations of Order $\alpha \in (1, 2)$

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Abstract This paper investigates the existence result for fractional order functional differential equations subject to non-instantaneous impulsive condition by applying the classical fixed point technique. At last, an example involving partial derivatives is presented to verify the uniqueness result.

Keywords Fractional order differential equation • Functional differential equations • Impulsive conditions • Fixed point theorem

Mathematics Subject Classification (2000): 26A33, 34K05, 34A12, 26A33

1 Introduction

In this paper, we investigate the existence and uniqueness result of mild solutions for the following non-instantaneous impulsive fractional functional differential equation of the form

$$^{C}D_t^{\alpha} y(t) = Ay(t) + f(t, y_{\rho(t,y_t)}),\ t \in (s_i, t_{i+1}] \subset J,\ i = 0, 1, \ldots, N, \tag{1}$$

$$y(t) = g_i(t, y(t)),\ y'(t) = q_i(t, y(t)),\ t \in (t_i, s_i],\ i = 1, 2, \ldots, N, \tag{2}$$

$$y(t) = \phi(t),\ y'(t) = \varphi(t),\ t \in [-d, 0], \tag{3}$$

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where ${}^{C}D_t^{\alpha}$ denotes the Caputo's fractional derivative of order $\alpha \in (1,2)$ and $A : D(A) \subset X \to X$ is the sectorial operator defined on a complex Banach space X. Functions $f : J \times PC_0 \to X$; $\rho : J \times PC_0 \to [-d, T]$ are continuous and satisfy some assumptions, where PC_0 is an abstract space defined in the next section. The map y_t is the element of PC_0 and defined as $y_t(\theta) = y(t+\theta)$, $\theta \in [-d, 0]$. $J = [0, T]$ is operational interval such that $0 = t_0 = s_0 < t_1 \leq s_1 \leq t_2 < \cdots < t_N \leq s_N \leq t_{N+1} = T$ are prefixed numbers. Here y' denotes the derivative of y with respect to t and $g_i, q_i \in C((t_i, s_i] \times X; X)$ for all $i = 1, 2, \ldots, N$. The functions ϕ, φ belong to PC_0 respectively.

The impulsive differential equations have been appeared as in natural description evolution processes. The impulsive effects may be instantaneous or non-instantaneous which is shown in many disciplines. Instantaneous impulse is characterized by abrupt changes of the state at certain moments, but in case of non-instantaneous impulse, it starts abruptly at the fixed moments as the points t_i, and their action continues on the finite interval $[t_i, s_i]$. For the future development and recent update of theory for fractional functional differential equations, we refer the papers [1, 2, 4–6, 9, 10] for state-dependent delay, and for non-instantaneous impulse, one can see the papers [7, 8, 11, 12] and the references therein.

On the available of literature, we found that Hernandez et al. [11] used the first time non-instantaneous impulsive condition for abstract differential equations for order one and established the existence results. Kumar et al. [12] have studied the fractional order problem with non-instantaneous impulse, and by using the Banach fixed point theorem with condensing map, they established the existence and uniqueness results. Motivated by the work [11, 12], we have studied the problem considered in [8] for the order $\alpha \in (0, 1)$ and established the existence results of mild solution of problem. Shu et al. [14] gave the definition of mild solution for fractional differential equations of order $\alpha \in (1, 2)$ and then established the existence results of mild solutions using the Krasnoselskii's fixed point theorem and analytic operator theory.

Inspired by the work [11, 12, 14] and by the survey, we found that there is no literature on fractional functional differential equation with state-dependent delay subject to non-instantaneous impulsive condition of order $(1, 2)$. This is the reason to investigate the problems (1)–(3) and establish the existence of uniqueness result. For further information, we have divided our work in four sections.

2 Preliminary

In this section, we have introduced some notations, basic definitions, and preliminary result, which were required to establish our main results. Let $(X, \|\cdot\|_X)$ be a complex Banach space of functions with the sup-norm $\|u\|_X = \sup_{t\in J}\{|u(t)| : u \in X\}$, and let $L(X)$ denote the space of bounded linear operators from X into X endowed with the natural norm of operators denoted by $\|\cdot\|_{L(X)}$.

As usual, $PC_0 = C([-d,0],X)$ (with $[-d,0] \subset$ R) is the space formed by all the continuous functions defined from $[-d,0]$ into X, endowed with the norm

$$\|u(t)\|_{PC_0} = \sup_{t\in[-d,0]} \{|u(t)|_X\}.$$

In the case of impulsive conditions, we consider

$$PC_T = PC([-d,T];X),\ 0 < T < \infty,$$

which is a Banach space of all such functions $u : [-d,T] \to X$, which are absolutely continuous everywhere except for a finite number of points $t_i \in (0,T),\ i = 1,2,\ldots,N$, at which $u(t_i^+)$ and $u(t_i^-) = u(t_i)$ exists and endowed with the norm

$$\|u\|_{PC_T} = \sup_{t\in[-d,T]} \{\|u(t)\|_X, u \in PC_T\}.$$

For a function $u \in PC_T$ and $i \in \{0,1,\ldots,N\}$, we introduce the function $\bar{u}_i \in A([t_i,t_{i+1}];X)$ given by

$$\bar{u}_i(t) = \begin{cases} u(t), \text{ for } t \in (t_i,t_{i+1}], \\ u(t_i^+), \text{ for } t = t_i. \end{cases}$$

For further analysis, again consider

$$PC_T^1 = PC([-d,T];X),\ 0 < T < \infty,$$

which is a Banach space of all such functions $u : [-d,T] \to X$, which are absolutely continuously differentiable everywhere except for a finite number of points $t_i \in (0,T),\ i = 1,2,\ldots,N$, at which $u'(t_i^+)$ and $u'(t_i^-) = u'(t_i)$ exists and endowed with the norm

$$\|u\|_{PC_T^1} = \sup_{t\in[-d,T]} \left\{ \sum_{j=0}^{1} \|u^j(t)\|_X, u \in PC_T^1 \right\}.$$

For a function $u \in PC_T^1$ and $i \in \{0,1,\ldots,N\}$, we introduce the function $\bar{u}_i \in C^1([t_i,t_{i+1}];X)$ given by

$$\bar{u}_i(t) = \begin{cases} u'(t), \text{ for } t \in (t_i,t_{i+1}], \\ u'(t_i^+), \text{ for } t = t_i. \end{cases}$$

Definition 1. [13] Caputo's derivative of order $\alpha > 0$ with lower limit a, for a function $g : [a, \infty) \to \mathrm{R}$ such that $g \in C^n([a, \infty), X)$ is defined as

$$ {}_{a}^{C}D_t^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} g^{(n)}(s)ds = {}_a J_t^{n-\alpha} g^{(n)}(t), $$

where $n - 1 < \alpha < n,\ a \geq 0,\ n \in N$.

Definition 2. The Riemann–Liouville fractional integral operator of order $\alpha > 0$ with lower limit a, for a function $g \in L^1_{\mathrm{loc}}([a, \infty), X)$ is defined by

$$ {}_aJ_t^0 g(t) = g(t),\ {}_aJ_t^{\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} g(s)ds, \quad \alpha > 0,\ t > 0, $$

where $a \geq 0,\ n \in N$ and $\Gamma(\cdot)$ denotes the Gamma function.

Definition 3. Let $A : D(A) \subset X \to X$ be a closed and linear operator and $\alpha, \beta > 0$. We can say that A is the generator of (α, β) operator function if there exists $\omega \geq 0$ and a strongly continuous function $W_{\alpha,\beta} : \mathrm{R}^+ \to L(X)$ such that $\{\lambda^{\alpha} : \mathrm{Re}\lambda > \omega\} \subset \rho(A)$ and

$$ \lambda^{\alpha-\beta}(\lambda^{\alpha} I - A)^{-1} u = \int_0^{\infty} e^{-\lambda t} W_{\alpha,\beta}(t) u dt,\ \mathrm{Re}\lambda > \omega,\ u \in X. $$

Here $W_{\alpha,\beta}(t)$ is called the operator function generated by A.

Remark 1. The operator function $W_{\alpha,\beta}(t)$ is a general case of α-resolvent family and solution operator. In the case $\beta = 1$, operator function corresponds to solution operator $S_{\alpha}(t)$ by Definition 2.1 in [2], whereas in the case $\beta = \alpha$, operator function corresponds to α-resolvent family defined in [3] in Definition (2.3), and operator function corresponds to $K_{\alpha}(t)$ in [14] in the case $\beta = 2$.

The following result is based on Definition 2.1 in [11].

Definition 4. A function $y : [-d, T] \to X$ s.t. $y \in PC^1_T$ is called a mild solution of the problems (1)–(3) if $y(0) = \phi(0), y'(0) = \varphi(0),\ \ y(t) = g_j(t, y(t)), y'(t) = q_j(t, y(t))$ for $t \in (t_j, s_j]$ for each $j = 1, 2, \ldots, N$, and satisfying the following integral equation

$$ y(t) = \begin{cases} \phi(0)S_{\alpha}(t) + \varphi(0)K_{\alpha}(t) \\ + \int_0^t T_{\alpha}(t) f(s, y_{\rho(s,y_s)})ds, & t \in [0, t_1], \\ g_i(s_i, y(s_i))S_{\alpha}(t - s_i) + q_i(s_i, y(s_i))K_{\alpha}(t - s_i) \\ + \int_{s_i}^t T_{\alpha}(t - s) f(s, y_{\rho(s,y_s)})ds, & t \in [s_i, t_{i+1}], \end{cases} $$

for $i = 1, 2, \ldots, N$.

3 Main Results

In this section, we have established the existence result of solution for the problems (1)–(3). Let A be a sectorial operator and then strongly continuous functions $\|S_\alpha(t)\| \leq M; \|K_\alpha(t)\| \leq M; \|T_\alpha(t)\| \leq M$. Let us assume the function $\rho : [0,T] \times PC_0 \to [-d,T]$ is continuous. Now, we introduce the following assumption:

(H_1) The function f is continuous and $\exists$ positive constants L_{f1} such that

$$\|f(t,\psi) - f(t,\xi)\|_X \leq L_{f1}\|\psi - \xi\|_{PC_0}, \ \forall \ \psi, \xi \in PC_0.$$

(H_2) The functions g_i, q_i are continuous and $\exists$ positive constants L_{g_i}, L_{q_i} such that

$$\|g_i(t,x) - g_i(t,y)\|_X \leq L_{g_i}\|x-y\|_X; \ \|q_i(t,x) - q_i(t,y)\|_X \leq L_{q_i}\|x-y\|_X$$

for all $x, y \in X, t \in (t_i, s_i]$ and each $i = 1, 2, \ldots, N$.

Theorem 1. *Let the assumptions (H_1) and (H_2) hold and are constant:*

$$\Delta = \max\{MTL_{f_1}, L_{g_i}M + L_{q_i}M + MTL_{f1}\} < 1,$$

for $i = 1, \ldots, N$. Then there exists a unique mild solution $y(t)$ of problems (1)–(3) on J.

Proof. We convert problems (1)–(3) in to the fixed point problem. Consider $\mathscr{B} = \{y : y \in PC_T^1, y(0) = \phi(0), y'(0) = \varphi(0)\}$. Define an operator $\mathscr{P} : \mathscr{B} \to \mathscr{B}$ as

$$\mathscr{P}y(t) = \begin{cases} \phi(0)S_\alpha(t) + \varphi(0)K_\alpha(t) \\ + \int_0^t T_\alpha(t-s)f(s, y_{\rho(s,y_s)})ds, & t \in [0, t_1], \\ g_i(s_i, y(s_i))S_\alpha(t-s_i) + q_i(s_i, y(s_i))K_\alpha(t-s_i) \\ + \int_{s_i}^t T_\alpha(t-s)f(s, y_{\rho(s,y_s)})ds, & t \in [s_i, t_{i+1}]. \end{cases} \tag{4}$$

It is obvious that $\mathscr{P}$ is well defined. Now, we will express that the operator $\mathscr{P}$ has a unique fixed point. So let $y(t), y^*(t) \in \mathscr{B}$ and $t \in [0, t_1]$; we get

$$\begin{aligned} \|\mathscr{P}y - \mathscr{P}y^*\|_X &\leq \int_0^t \|T_\alpha(t-s)\|_{L(X}\|f(s, y_{\rho(s,y_s)}) - f(s, y^*_{\rho(s,y^*_s)})\|_X ds \\ &\leq TML_{f1}\|y - y^*\|_X. \end{aligned}$$

For $t \in [s_i, t_{i+1}]$, we have

$$\begin{aligned} \|\mathscr{P}y - \mathscr{P}y^*\|_X \leq & \|g_i(s_i, y(s_i)) - g_i(s_i, y^*(s_i))\|_X\|S_\alpha(t-s_i)\|_{L(X} \\ & + \|q_i(s_i, y(s_i)) - q_i(s_i, y^*(s_i))\|_X\|K_\alpha(t-s_i)\|_{L(X} \end{aligned}$$

$$+\int_{s_i}^{t} \|T_\alpha(t-s)\|_{L(X}\|f(s, y_{\rho(s,y_s)}) - f(s, y^*_{\rho(s,y^*_s)})\|_X ds$$
$$\leq (L_{g_i}M + L_{q_i}M + TML_{f1})\|y - y^*\|_X.$$

For $t \in (t_j, s_j]$, we get

$$\|\mathscr{P}y - \mathscr{P}y^*\|_X \leq L_{g_j}\|y - y^*\|_X,\ j = 1, 2, \ldots, N.$$

Gathering above results, we obtain

$$\begin{aligned}\|\mathscr{P}y - \mathscr{P}y^*\|_X &\leq \max\{MTL_{f_1}, L_{g_i}M + L_{q_i}M + MTL_{f1}\}\|y - y^*\|_X \\ &\leq \Delta\|y - y^*\|_X.\end{aligned}$$

Since $\Delta < 1$, which implies that $\mathscr{P}$ is a contraction map, there exists a unique fixed point which is the mild solution of problems (1)–(3) on J.

4 Example

In this section, we gave an example to illustrate our main result. Consider the following fractional order functional differential equation:

$$\frac{\partial^\alpha u(t,x)}{\partial t^\alpha} = \frac{\partial^2 u(t,x)}{\partial y^2} + \frac{u(t-\sigma(\|u\|), x)}{49}, (t,x) \in \cup_{i=1}^{N}[s_i, t_{i+1}] \times [0,\pi], \quad (5)$$

$$u(t,0) = u(t,\pi) = 0, \quad t \geq 0, \quad (6)$$

$$u(t,x) = \phi(t,x), u'(t,x) = \varphi(t,x), t \in [-d, 0], x \in [0,\pi], \quad (7)$$

$$u(t,x) = G_i(t,y);\ u'(t,x) = H_i(t,y),\ t \in (t_i, s_i]. \quad (8)$$

where $\frac{\partial^\alpha}{\partial t^\alpha}$ denotes the partial Caputo's fractional derivative of order $\alpha \in (1,2), 0 = t_0 = s_0 < t_1 \leq s_1 < \cdots < t_N \leq sN < t_{N+1} = 1$ are prefixed numbers, and $\phi, \varphi \in PC_0$. Let $X = L^2[0,\pi]$ be a Banach space and define the operator $A : D(A) \subset X \to X$ by $Ay = y''$ with the domain $D(A) := \{y \in X : \ y, y'$ to be absolutely continuous, $y'' \in X, y(0) = 0 = y(\pi)\}$. Then

$$Ay = \sum_{n=1}^{\infty} n^2 (y, y_n) y_n,\ y \in D(A),$$

where set $y_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx),\ n \in N$ is the space of eigenvectors of A in which element is orthogonal. It is clear that that the operator A stays the infinitesimal generator of an analytic semigroup operator $(T(t))_{t\geq 0}$ in Banach space X and is defined as

$$T(t)\omega = \sum_{n=1}^{\infty} e^{-n^2 t}(\omega, \omega_n)\omega_n, \text{ for all } \omega \in X, \text{ and every } t > 0.$$

The subordination opinion of solution operator implies that A stays the infinitesimal generator of $K(t), S(t)$. Since $K(t), S(t)$ are strongly continuous operators on interval $[0, \infty)$ by the theorem of uniformly boundedness, there exists a constant $M > 0$ such that $\|S(t)\| \leq M$, $\|K(t)\| \leq M$ for $t \in [0, 1]$. We have for $(t, \phi) \in [0, 1] \times PC_0$.

Setting $u(t)(x) = u(t, x)$, and

$$\rho(t, \phi) = t - \sigma(\|\phi(0)\|), \ (t, \phi) \in J \times PC_0,$$

we have

$$f(t, \phi) = \frac{\phi}{49}; \ g_i(t, y) = G_i(t, y); \ q_i(t, y) = H_i(t, y),$$

then by the above Eqs. (5)–(8) can be composed in the given abstract form as (1)–(3). Furthermore, we can see that for $(t, \phi), (t, \psi) \in J \times PC_0$, we may verify that

$$\|f(t, \phi) - f(t, \psi)\|_{L^2} \leq \left[\int_0^{\pi} \left\{\|\frac{\phi}{49} - \frac{\psi}{49}\|\right\}^2 dy\right]^{1/2} \leq \frac{\sqrt{\pi}}{49}\|\phi - \psi\|.$$

Hence function f satisfies (H_1). Similarly we can show that the functions g_i, q_i satisfy (H_2). All the conditions of Theorem 1 have been satisfied, so we can drive that the system (5)–(8) has a unique mild solution on [0, 1].

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