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Differential and Difference Equations with Applications

ICDDEA, Amadora, Portugal, May 2015, Selected Contributions

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Preface

For 5 days, May 18–22, 2015, more than 170 mathematicians from 50 countries attended the International Conference on Differential and Difference Equations and Applications, held at the Military Academy, Amadora, Portugal.

The scientific aim of this conference was to bring together mathematicians working in various disciplines of differential and difference equations and their applications. There were 6 plenary lectures, 22 main lectures, and 131 communications about the current research in this field. This volume contains 41 selected original papers which are connected to research lectures given at the conference. Each paper has been carefully reviewed.

We take this opportunity to thank all the participants of the conference and the contributors to these proceedings. Our special thanks belong to the Military Academy for the sincere hospitality. We are also grateful to the Scientific and Organizing Committees for all the effort in the preparation of the conference.

These proceedings are dedicated in memory of Professor George Sell (1937–2015). Professor George Sell had been invited to the ICDDEA 2015 as a plenary speaker, but was unable to come and died shortly afterwards.

We hope that this volume will serve researchers in all fields of differential and difference equations.

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Algebraic Properties of the Semi-direct Product of Kac–Moody and Virasoro Lie Algebras and Associated Bi-Hamiltonian Systems

Alexander Zuevsky

Abstract We discuss the semi-direct product of Virasoro and affine Kac–Moody Lie algebras and associated Verma modules, coadjoint orbits, Casimir functions, and bi-Hamiltonian systems.

Mathematics Subject Classification (2000): 53C15, 53C57, 58F05, 58F07

1 The Semi-direct Product of Virasoro Algebra with the Kac–Moody Algebra

This paper is a continuation of the paper [8] where we studied bi-Hamiltonian systems associated to the three-cocycle extension of the algebra of diffeomorphisms on a circle. In this note, we review results showing that certain natural problems (classification of Verma modules, classification of coadjoint orbits, determination of Casimir functions) [3, 5, 7] for the central extensions of the Lie algebra $\mathrm{Vect}(S^1) \ltimes \mathscr{L}\mathscr{G}$ reduce to the equivalent problems for Virasoro and affine Kac–Moody algebras (which are central extensions of $\mathrm{Vect}(S^1)$ and $\mathscr{L}\mathscr{G}$, respectively). Such properties are not true in general for any semi-direct product of Lie algebras. This occurs in this very particular case because the Lie algebras of Virasoro and affine Kac–Moody are related by what is called the Sugawara construction. Let G be a Lie group and $\mathscr{G}$ its Lie algebra. The group $\mathrm{Diff}(S^1)$ of diffeomorphisms of the circle is included in the group of automorphisms of the loop group LG of smooth maps from S^1 to G. The semi-direct product $\mathrm{Diff}(S^1) \ltimes \mathrm{LG}$ of these two groups can thus be constructed. For any pairs $(\phi, \psi) \in \mathrm{Diff}(S^1)^2$ and $(g, h) \in \mathrm{LG}^2$, the composition law of the group $\mathrm{Diff}(S^1) \ltimes \mathscr{L}\mathscr{G}$ is $(\phi, a)\cdot(\psi, b) = (\phi \circ \psi, a.b \circ \phi^{-1})$. The Lie algebra of $\mathrm{Diff}(S^1) \ltimes \mathrm{LG}$ is the semi-direct product $\mathrm{Vect}(S^1) \ltimes \mathscr{L}\mathscr{G}$ of the Lie algebras $\mathrm{Vect}(S^1)$ and $\mathscr{L}\mathscr{G}$.

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Let us recall the Lie algebras that were involved in this paper. Let $\mathcal{G}$ be a Lie algebra and $\langle .,. \rangle$ a nondegenerated invariant bilinear form. $\mathrm{Vect}(S^1)$ is the Lie algebra of vector fields on the circle and $\mathcal{L}\mathcal{G}$ the loop algebra (i.e., the Lie algebra of smooth maps from S^1 to $\mathcal{G}$); $\mathrm{Vect}(S^1)_{\mathbb{C}}$ is the Lie algebra over $\mathbb{C}$ generated by the elements $L_n, n \in \mathbb{Z}$ with the relations $[L_m, L_n] = (n-m)L_{n+m}$. We denote by $\mathcal{L}\mathcal{G}_{\mathbb{C}}$ the Lie algebra over $\mathbb{C}$ generated by the elements $g_n, n \in \mathbb{Z}, g \in \mathcal{G}$ where $(\lambda g + \mu h)_n$ is identified with $\lambda g_n + \mu h_n$ with the relations $[g_n, h_m] = [g, h]_{n+m}$. The semi-direct product of $\mathrm{Vect}(S^1)$ with $\mathcal{L}\mathcal{G}$ is as a vector space isomorphic to $C^\infty(S^1, \mathbb{R}) \oplus C^\infty(S^1, \mathcal{G})$. The Lie bracket of $\mathcal{S}\mathcal{U}(\mathcal{G})$ has the form

$$[(u, a), (v, b)] = \left(uv' - u'v, va' - ub' + [a, b]\right),$$

for any $(u, v) \in C^\infty(S^1, \mathbb{R})^2$ and any $(a, b) \in C^\infty(S^1, \mathcal{G})^2$, where prime denote derivative with respect to a coordinate on S^1. The Lie algebra $\mathrm{Vect}(S^1) \ltimes \mathcal{L}\mathcal{G}$ can be extended with a universal central extension $\mathcal{S}\mathcal{U}(\mathcal{G})$ by a two-dimensional vector space. Two independent cocycles are given by

$$\omega_{\mathrm{Vir}}\big((u, a), (v, b)\big) = \int_{S^1} u'''v,$$
$$\omega_{K-M}\big((u, a), (v, b)\big) = \int_{S^1} \langle a', b \rangle.$$

We denote by (u, a, χ, α) the elements of $\mathcal{S}\mathcal{U}(\mathcal{G})$ with $u \in C^\infty(S^1, \mathbb{R}), a \in C^\infty(S^1, \mathcal{G})$, and $(\chi, \alpha) \in \mathbb{R}^2$. The Lie bracket of $\mathcal{S}\mathcal{U}(\mathcal{G})$ reads (see [3])

$$[(u, a, \phi, \alpha), (v, b, \xi, \beta)] = \left(uv' - u'v, [a, b] - ub' + va', \int_{S^1} u'''v, \int_{S^1} \langle a', b \rangle \right).$$

The algebra $\mathcal{S}\mathcal{U}(\mathcal{G})$ can be also represented as the semi-direct product of Virasoro algebra on the affine Kac–Moody algebra. We denote by c_{Vir} and c_{K-M} the elements $(0, 0, 1, 0)$ and $(0, 0, 0, 1)$, respectively. If $\mathcal{G} = \mathbb{R}$, then the Lie algebra $\mathrm{Vect}(S^1) \ltimes \mathcal{L}\mathbb{R}$ has a universal central extension $\widetilde{\mathcal{S}\mathcal{U}}(\mathbb{R})$ by a three-dimensional vector space. The third independent cocycle is given by

$$\omega_{\mathrm{sp}}\left((u, a), (v, b)\right) = \int_{S^1} \left(ub'' - va''\right).$$

We denote by $(u, a, \chi, \alpha, \gamma, \delta)$ elements of $\widetilde{\mathcal{S}\mathcal{U}}(\mathbb{R})$ with $u \in C^\infty(S^1, \mathbb{R})$, $a \in C^\infty(S^1, \mathcal{G})$, and $(\chi, \alpha, \gamma) \in \mathbb{R}^3$. The Lie bracket of $\widetilde{\mathcal{S}\mathcal{U}}(\mathbb{R})$ is given by

$$\begin{aligned}&[(u, a, \phi, \alpha, \gamma), (v, b, \xi, \beta, \delta)] \\ &\quad = \left(uv' - u'v, [a, b] - ub' + va', \int_{S^1} u'''v, \int_{S^1} \langle a', b \rangle, \int_{S^1} (ub'' - va'')\right).\end{aligned}$$

An algebraic equivalent $\mathcal{SU}_{\mathbb{C}}(\mathcal{G})$ of $\mathcal{SU}(\mathcal{G})$ is defined to be the Lie algebra over $\mathbb{C}$ with generators $L_n, n \in \mathbb{Z}$ and $g_n, n \in \mathbb{Z}, g \in \mathcal{G}$ and two central elements c_{Vir} and c_{K-M} satisfying the relations $[L_m, L_n] = (n-m)L_{n+m} + (n-m)^3\delta_{n,-m}c_{\text{Vir}}$, $[L_m, g_n] = ng_{n+m}$, $[g_m, h_n] = [g, h]_{n+m} + (n-m)\langle g, h\rangle\delta_{n,-m}c_{K-M}$. This is a graded Lie algebra with weights for L_n and g_n equal n and zero weights of c_{Vir} and c_{K-M}. Similarly, an equivalent $\widetilde{\mathcal{SU}}_{\mathbb{C}}(\mathbb{C})$ of $\mathcal{SU}(\mathbb{R})$ is the Lie algebra over $\mathbb{C}$ with generators $L_n, n \in \mathbb{Z}$ and $a_n, n \in \mathbb{Z}$ and three central elements c_{Vir}, c_{sp}, and c_{K-M} satisfying the relations $[L_m, L_n] = (n-m)L_{n+m} + (n-m)^3\delta_{n,-m}c_{\text{Vir}}$, $[L_m, a_n] = na_{n+m} + n^2\delta_{n,-m}c_{\text{sp}}$, $[a_m, a_n] = (n-m)\delta_{n,-m}c_{K-M}$. This Lie algebra is endowed with the grading weights n for L_n and a_n and zero for c_{Vir}, c_{sp}, and c_{K-M}.

2 The Universal Enveloping Algebra of $\mathcal{SU}(\mathcal{G})$

In some very particular cases, the modified generalized enveloping algebra of a semi-direct product $\mathcal{K} \ltimes \mathcal{H}$ of two Lie algebras is isomorphic to the tensor product of some modified generalized enveloping algebras of $\mathcal{K}$ and of $\mathcal{H}$. Let $\widetilde{\mathcal{H}}$ be the central extension of $\mathcal{H}$ with the two-cocycle $\omega_{\mathcal{H}}$. Denote by $\cdot$ the action of the Lie algebra $\mathcal{K}$ on the Lie algebra $\widetilde{\mathcal{H}}$. Let us introduce the semi-direct product $\mathcal{K} \ltimes \widetilde{\mathcal{H}}$ which is a central extension of $\mathcal{K} \ltimes \mathcal{H}$ by a two-cocycle $\omega'_{\mathcal{H}}$ with $\omega'_{\mathcal{H}}\big((0, h_1), (0, h_2)\big) = \omega_{\mathcal{H}}(h_1, h_2)$. A two-cocycle $\omega_{\mathcal{K}}$ on $\mathcal{K}$ defines also a two-cocycle $\omega'_{\mathcal{K}}$ by $\omega'_{\mathcal{K}}\big((g_1, h_1), (g_2, h_2)\big) = \omega_{\mathcal{K}}(g_1, g_2)$ of $\mathcal{K} \ltimes \mathcal{H}$. Let I be the natural inclusion of $\widetilde{\mathcal{H}}$ into $\mathcal{U}^{\mathcal{H}}_{\omega_{\mathcal{H}}}$ and J be the natural inclusion of $\widetilde{\mathcal{H}}$ into $\mathcal{U}^{\mathcal{K}\ltimes\mathcal{H}}_{\omega'_{\mathcal{K}},\omega'_{\mathcal{H}}}$. We call the action of $\mathcal{K}$ on $\mathcal{H}$ realizable in $\mathcal{U}^{\mathcal{H}}_{\omega_{\mathcal{H}}}$ when (1) there exists a map $F : \mathcal{K} \to \mathcal{U}^{\mathcal{H}}_{\omega_{\mathcal{H}}}$ and a two-cocycle α on $\mathcal{K}$ such that for any pair (g_1, g_2) in $\mathcal{K}^2$ $F([g_1, g_2]) = [F(g_1), F(g_2)] + \alpha(g_1, g_2)\mathbf{1}$, (2) the map F satisfies the compatibility condition, for any $g \in \mathcal{K}$ and $h \in \widetilde{\mathcal{H}}$ with the anti-commutator $[F(g), I(h)] = I(g \cdot h)$, of the algebra $\mathcal{U}^{\mathcal{H}}_{\omega_{\mathcal{H}}}$.

Theorem 1. *If the action of $\mathcal{K}$ is realizable in $\mathcal{U}^{\mathcal{H}}_{\omega_{\mathcal{H}}}$, then we have the isomorphism $\mathcal{U}^{\mathcal{K}\ltimes\mathcal{H}}_{\omega'_{\mathcal{K}},\omega'_{\mathcal{H}}} \simeq \mathcal{U}^{\mathcal{K}}_{\omega_{\mathcal{K}}-\alpha} \otimes \mathcal{U}^{\mathcal{H}}_{\omega_{\mathcal{H}}}$.*

2.1 *The Case of $\mathcal{SU}_{\mathbb{C}}(\mathcal{G})$*

Let $\mathcal{G}$ be a simple complex Lie algebra and C_φ its dual Coxeter number. Introduce the $\{K_1, \ldots, K_n\}$ as a basis of $\mathcal{G}$ and the dual basis $\{K_1^*, \ldots, K_n^*\}$ with respect to the Killing form $\langle ., . \rangle$. We apply Theorem 1 for $\mathcal{K} = \text{Vect}(S^1)$, $\mathcal{H} = \mathcal{LG}$, $\omega_{\mathcal{K}} = \xi\omega_{\text{Vir}}$, and $\omega_{\mathcal{H}} = \beta\omega_{K-M}$. In this case, $\omega'_{\mathcal{H}} = \beta\omega_{K-M}$. For $\eta = \beta + C_\varphi \neq 0$, the Sugawara construction [1] delivers a map $F : \text{Vect}(S^1)_{\mathbb{C}} \to \mathcal{U}^{\mathcal{LG}_{\mathbb{C}}}_{\omega_{\mathcal{G}}}$ defined by $F(L_n) = (\beta + \eta)^{-1} \sum_{i\in\mathbb{Z}} \sum_{j=1,\ldots,n} : (K_j)_i(K_j^*)_{n-i} :$, i.e., the action of $\text{Vect}(S^1)$ is realizable in $\mathscr{U}^{\mathscr{L}\mathscr{G}}_{\beta\omega_{K-M}}$, with $\alpha = \beta\omega_{\mathrm{Vir}}/12\eta$ (here, dots denote the normal ordering). Thus, we obtain

Proposition 1. *If $\eta \neq 0$, then $\mathscr{U}^{\mathscr{S}\mathscr{U}_{\mathbb{C}}(\mathscr{G})}_{\xi\omega_{\mathrm{Vir}},\beta\omega_{K-M}} \simeq \mathscr{U}^{\mathrm{Vect}(S^1)_{\mathbb{C}}}_{(\xi-\alpha)} \otimes \mathscr{U}^{\mathscr{L}\mathscr{G}}_{\beta\omega_{K-M}}$.*

The Lie algebra $\mathrm{Vect}_{\mathbb{C}}(S^1)$ acts on the Heisenberg algebra by $L_n.a_m = ma_{n+m} + \delta_{n,-m}m^2c_{K-M}$. In this case, one has $\omega'_{\mathscr{H}} = \beta\omega_{\mathscr{H}} + \gamma\omega_{\mathrm{sp}}$. The map $F : \mathrm{Vect}(S^1)_{\mathbb{C}} \to \widetilde{\mathscr{S}\mathscr{U}_{\mathbb{C}}(\mathbb{C})}$ is defined by $F(L_n) = (2\beta)^{-1}\sum\limits_{i\in\mathbb{Z}} : a_ia_{n-i} : +\gamma\beta^{-1}a_n$, for a cocycle $\alpha' = (\alpha + \gamma^2\beta^{-1})\omega_{\mathrm{Vir}}$. For $\widetilde{\mathscr{S}\mathscr{U}_{\mathbb{C}}(\mathscr{C})}$, we obtain

Proposition 2. *For $\beta \neq 0$, we have $\mathscr{U}^{\widetilde{\mathscr{S}\mathscr{U}_{\mathbb{C}}(\mathbb{C})}}_{\xi\omega_{\mathrm{Vir}},\beta\omega_{K-M},\gamma\omega_{\mathrm{sp}}} \simeq \mathscr{U}^{\mathrm{Vect}(S^1)_{\mathbb{C}}}_{\theta\omega_{\mathrm{Vir}}} \otimes \mathscr{U}^{\mathscr{L}\mathscr{G}}_{\omega_{K-M}}$, with $\theta = \xi - \frac{1}{12} - \frac{\gamma^2}{\beta}$.*

2.2 Representations of $\mathscr{S}\mathscr{U}(\mathscr{G})$

First, we have

Proposition 3. *A positive energy representation V of $\mathscr{S}\mathscr{U}_{\mathbb{C}}(\mathscr{G})$ with nonvanishing βId-action of c_{K-M} results in a pair of commuting representations of Virasoro and affine Kac–Moody Lie algebras.*

Thus, we see that positive energy representations of $\mathscr{S}\mathscr{U}_{\mathbb{C}}(\mathscr{G})$ are representations of Virasoro and affine Kac–Moody Lie algebra with commuting actions. This proposition determines whether a $\mathscr{S}\mathscr{U}_{\mathbb{C}}(\mathscr{G})$ Verma module is a sub-module of another Verma module of $\mathscr{S}\mathscr{U}_{\mathbb{C}}(\mathscr{G})$.

Let $\mathfrak{h}$ be a Cartan algebra of $\mathscr{G}$ with a basis $\{h_1, \ldots, h_k\}$. The Lie subalgebra $\mathfrak{k}$ of $\mathscr{S}\mathscr{U}_{\mathbb{C}}(\mathscr{G})$ is generated by the elements $\{c_{\mathrm{Vir}}, c_{K-M}, u_0, (h_1)_0, \ldots, (h_k)_0\}$. A Verma module $V^{\mathscr{S}\mathscr{U}_{\mathbb{C}}(\mathscr{G})}_{\lambda}$ of $\mathscr{S}\mathscr{U}_{\mathbb{C}}(\mathscr{G})$ is associated to any linear form $\lambda \in \mathfrak{h}^*$. Verma modules V^{Vir}_{ν}, V^{K-M}_{μ} are associated to linear forms ν, μ over the spaces generated by c_{Vir} and u_0, c_{K-M} and $\{(h_1)_0, \ldots, (h_k)_0\}$, correspondingly. For any $\lambda \in \mathfrak{k}^*$, the Verma module $V^{\mathscr{S}\mathscr{U}_{\mathbb{C}}(\mathscr{G})}_{\lambda}$ is a positive energy representation. Thus, $V^{\mathscr{S}\mathscr{U}_{\mathbb{C}}(\mathscr{G})}_{\lambda}$ is Virasoro and affine Kac–Moody algebra module. The generator e of $V^{\mathscr{S}\mathscr{U}_{\mathbb{C}}(\mathscr{G})}_{\lambda}$ brings about a Verma module V^{Vir}_{ν} for Virasoro algebra. It generates also a Verma module V^{Vir}_{ν} for the affine Kac–Moody algebra. The linear form ν satisfies $\nu(u_0)e = \lambda(u_0 - F(u_0))e$, i.e., $(u_0 - (\beta + \eta)^{-1}\sum\limits_{i\in\mathbb{Z}}\sum_{j=1\ldots n} : (K_j)_i(K^*_j)_{-i} :)e = \nu(u_0)e$. Suppose the action of a Casimir element of $\mathscr{G}$ is given by acts by $D(\lambda)Id$ for $D(\lambda) \in \mathbb{C}$. We then have $(u_0 - (\beta + \eta)^{-1}\sum_{i\in\mathbb{Z}}\sum_{j=1\ldots n} : (K_j)_k(K^*_j)_{-i} :).e = (u_0 - (\beta + \eta)^{-1}\sum_{j=1\ldots n} : (K_j)_0(K^*_j)_0 :).e$, $(\lambda(u_0) - \frac{D(\lambda)}{2\eta})e$. This implies $\nu(u_0) = \lambda(u_0) - \frac{D(\lambda)}{2\eta}$. The other values of μ and ν can be computed by the same method.

Proposition 4. *Let* λ *be a linear form over* $\mathfrak{h}$ *with nonvanishing* $\lambda(c_{K-M})$*. Then*

$$V_\lambda^{\mathscr{S}\mathscr{U}_{\mathbb{C}}(\mathscr{G})} \simeq V_\nu^{\mathrm{Vir}} \otimes V_{\mathbb{C}}^{K-M}\mu,$$

where $\mu(e_i) = \lambda(e_i)$*,* $i = 1, \ldots, n$*, defines* μ*,* $\mu(c_{K-M}) = \lambda(c_{K-M})$*, and* $\nu(c_{\mathrm{Vir}}) = \lambda(c_{\mathrm{Vir}}) - \frac{\beta}{12\eta}$ *defines* ν*,* $\nu(u_0) = \lambda(u_0) - \frac{D(\lambda)}{2\eta}$*.*

3 The Kirillov–Kostant Structure of $\mathscr{S}\mathscr{U}(\mathscr{G})$

Now we consider Kirillov–Kostant Poisson brackets of the regular dual of the semi-direct product of Virasoro Lie algebra with the affine Kac–Moody Lie algebra. Let $\mathscr{K}$ be a Lie algebra with a nondegenerated bilinear form $\langle ., . \rangle$. A function $f : \mathscr{K} \to \mathbb{R}$ is called regular at $x \in \mathscr{K}$ if there exists an element $\nabla f(x)$ such that $f(x + \epsilon a) = f(x) + \epsilon \langle \nabla f(x), a \rangle + o(\epsilon)$, for any $a \in \mathscr{K}$. For two regular functions $f,g : \mathscr{K} \longrightarrow \mathbb{R}$, we define the Kirillov–Kostant structure as a Poisson structure on $\mathscr{K}$ with $\{f, g\}(x) = \langle x, [\nabla f(x), \nabla g(x)] \rangle$. Then for any $e \in \mathscr{G}$, the second Poisson structure $\{f, g\}_e(x)$ compatible with the Kirillov–Kostant Poisson structure is defined by $\{f, g\}_e(x) = \langle e, [\nabla f(x), \nabla g(x)] \rangle$. A nondegenerated bilinear form on $\mathscr{S}\mathscr{U}(\mathscr{G})$ and $\widetilde{\mathrm{Vect}(S^1)} \oplus \widetilde{\mathscr{L}\mathscr{G}}$ is defined by

$$\langle (u_1, a_1, \beta_1, \xi_1), (u_2, a_2, \beta_2, \xi_2) \rangle = \int_{S^1} u_1 u_2 + \int_{S^1} \langle a_1, a_2 \rangle + \xi_1 \xi_2 + \beta_1 \beta_2.$$

We denote by $\mathscr{S}\mathscr{U}(\mathscr{G})'$ the subset of $\mathscr{S}\mathscr{U}(\mathscr{G})$ of elements (u, a, ξ, β) with nonvanishing β. Let $u' = u - \frac{\|a\|^2}{2\beta}$. We denote by $(\widetilde{\mathrm{Vect}(S^1)} \oplus \widetilde{\mathscr{L}\mathscr{G}})'$ the subset of $\widetilde{\mathrm{Vect}(S^1)} \oplus \widetilde{\mathscr{L}\mathscr{G}}$ composed of elements (u, a, ξ, β) with $\beta \neq 0$. Let $\mathscr{I}(u, a, \xi, \beta) = (u', a, \xi, \beta)$ be a map from $\mathscr{S}\mathscr{U}(\mathscr{G})'$ to $(\widetilde{\mathrm{Vect}(S^1)} \oplus \widetilde{\mathscr{L}\mathscr{G}})'$. For nonvanishing β, let $\widetilde{\mathscr{I}}(u, a, \xi, \beta, \gamma) = \left(u' - \frac{\gamma}{\beta} a', a, \xi - \frac{\gamma^2}{\beta}, \beta\right)$ be a map from $\mathscr{S}\mathscr{U}(\mathscr{G})$ to $\widetilde{\mathrm{Vect}(S^1)} \oplus \widetilde{\mathbb{L}\mathscr{R}}$.

Theorem 2. $\mathscr{I}$ *and* $\widetilde{\mathscr{I}}$ *are Poisson maps.*

4 Casimir Functions and Coadjoint Orbits for $\mathscr{S}\mathscr{U}(\mathscr{G})$

Now we determine Casimir functions on $\widetilde{\mathscr{S}\mathscr{U}(\mathscr{G})}'$ and $\widetilde{\mathscr{S}\mathscr{U}(\mathbb{R})}$.

Proposition 5. *Let* $\mathscr{C}_{\mathrm{Vir}}$*,* $\mathscr{C}_{K-M}$ $\mathscr{C}_{\mathscr{A}}$ *be Casimir functions for Virasoro, affine Kac–Moody, and the Heisenberg Lie algebras* $\mathscr{A}$ *correspondingly. Let* $\mathscr{S}_P\mathscr{U}(\mathscr{G})$*,* $\widetilde{\mathscr{S}_P\mathscr{U}(\mathbb{R})}$ *be Poisson submanifolds of* $\mathscr{S}\mathscr{U}(\mathscr{G})$ *and* $\widetilde{\mathscr{S}\mathscr{U}(\mathbb{R})}$ *defined by* $\xi = 0$*.*

Then the functions $\mathscr{C}_{\mathrm{Vir}}(u', \xi)$, $c(u, a, \beta, \xi) = \mathscr{C}_{K-M}(a, \beta)$, *and* $\int_{S^1} |u'|^{1/2}$ *are Casimir functions on* $\widetilde{\mathscr{S}\mathscr{U}(\mathscr{G})}'$. *In particular, the functions* $c_{\mathscr{A}}(u, a, \beta, \xi) = \mathbf{C}_{\mathscr{A}}(a, \beta)$, $\mathscr{C}_{\mathrm{Vir}}(u' - \frac{\gamma^2}{\beta} a', \xi)$, *and* $\int_{S^1} |u' - \frac{\gamma^2}{\beta} a'|^{1/2}$ *are Casimir functions on* $\widetilde{\mathscr{S}\mathscr{U}(\mathbb{R})}'$.

Let $\widetilde{\mathscr{H}}$ be a central extension of a Lie algebra $\mathscr{H}$ and H be a Lie group with Lie algebra $\mathscr{H}$. Then H acts on $\widetilde{\mathscr{H}}^\star$ by the coadjoint action along coadjoint orbits.

Proposition 6. *The coadjoint actions of the groups* $\mathrm{Diff}(S^1) \ltimes \mathrm{LG}$ *and* $\mathrm{Diff}(S^1) \ltimes L\mathbb{R}_+^*$ *are given by*

$$Ad^*(\phi, g)^{-1}(u, a, \xi, \beta) = \Big((u \circ \phi)\phi'^2 + \xi S(\phi) + \langle g^{-1}g', a\rangle \phi'^2 + \frac{1}{2}\beta \parallel g^{-1}g' \parallel^2, \phi' Ad(g^{-1}) a \circ \phi + \beta g^{-1} g', \xi, \beta\Big),$$

$$\big((u \circ \phi)\phi'^2 + \xi S(\phi) + \langle g' g^{-1}, a\rangle \phi'^2 + \frac{1}{2}\beta (g'g^{-1})^2 + \gamma g'' g^{-1}, \phi' Ad(g^{-1}) a \circ \phi + \beta g^{-1} g' - \gamma g'' g^{-1}, \xi, \beta, \gamma\big).$$

5 Dispersive Water Wave System and Other Particular Cases

It has been showed in [8] that the dispersive water wave system equation [2, 4, 6] is a bi-Hamiltonian system related to the semi-direct product of Kac–Moody and Virasoro Lie algebras and the hierarchy for this system was found. In this section, some results of [8] are obtained from another point of view. We have

Proposition 7. *The functions* $\left\{\phi_1\left(A(u + B\frac{da}{dx} + C)\right) | \lambda \in \mathbb{R}\right\}$ *commute pairwise for the Sugawara* $\{.,.\}'_{Sug}$ *and e-braket* $\{.,.\}_e$ *with* $e = (1, 0, 0, 2, 0)$, *and* $A = \left(\xi - \frac{\gamma}{\beta - 2\lambda}\right)^{-2}$, $B = -\frac{\gamma}{\beta - 2\lambda}$, $C = -\frac{\|a\|^2}{2\beta - 4\lambda} - \lambda$.

The function $\lambda \mapsto \phi_1\left(A(u + B\frac{da}{dx} + C)\right)$ has an asymptotic development. The coefficients of this development form a hierarchy. The first term of this development is $\int_{S^1} u$, and the second one is $\int_{S^1}(u^2 + \gamma u + \parallel a \parallel^2)$. A linear combination of these two terms gives the Hamiltonian of equations $H(u, a) = \int_{S^1}(u^2 + \parallel a \parallel^2)$.

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Analytical-Numerical Solutions for First-Order Periodic Boundary Value Problems Using the Reproducing Kernel Algorithm

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Abstract This paper proposes an efficient numerical algorithm to obtain an approximate solution of first-order periodic boundary value problems. This new algorithm is based on a reproducing kernel Hilbert space method. Its exact solution is calculated in the form of series in reproducing kernel space with easily computable components. In addition, convergence analysis for this method is discussed. In this sense, some numerical examples are given to show the effectiveness and performance of the proposed method. The results reveal that the method is quite accurate, simple, straightforward, and convenient to handle a various range of differential equations.

Keywords Analytical-numerical solutions • Reproducing kernel algorithm • Periodic boundary condition • Boundary value problems

AMS subject classifications: 34K28, 34K13, 34B15, 47B32.

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1 Introduction

Boundary value problems (BVPs) with periodic boundary conditions have become a focus of research in many fields of physics, engineering, and mathematics, including molecular dynamics, mechanical systems, computer simulations, and composite materials with a periodic microstructure and so on. When such problems are solved numerically, the periodicity condition is often imposed strongly; in other words, the values on periodic edges are required to match exactly. For typical examples, see [18, 19].

The purpose of this paper is to extend the application of the reproducing kernel Hilbert space method (RKHM) to provide approximate solution of a class of first-order periodic BVPs of the following form:

$$u'(x) + g(x)u(x) = f(x, u(x))\,;\quad 0 \le x \le 1, \tag{1}$$

subject to the periodic boundary condition

$$u(0) - u(1) = 0, \tag{2}$$

where $g(x)$ is continuous function, $f(x,u) \in W_2^1[0,1]$, $u = u(x) \in W_2^2[0,1]$ is an unknown function to be determined, $\|f(x,u(x)) - f(x,\overline{u}(x))\|_{W_2^1} \le M\|u(x) - \overline{u}(x)\|_{W_2^1}$ for $x \in [0,1]$, $M \in \mathbb{R}$, $f(x,u)$ is linear or nonlinear function of u depending on the problem discussed, and $W_2^2[0,1]$ and $W_2^1[0,1]$ are reproducing kernel spaces defined in the next section. Throughout this paper, we assume that the BVP models (1) and (2) have a unique smooth solution on the given interval $[0,1]$.

The numerical solvability of BVPs with periodic boundary conditions of different orders has been pursued in literature. To mention a few, Peng [22] has discussed the existence and multiplicity of the positive solutions for first-order periodic BVPs. Al-Smadi et al. [4] have developed an iterative method for systems of first-order periodic BVPs based on the RKHM. Lia [20] has presented the existence of positive solution for fourth-order periodic BVPs. On the other hand, this method has been implemented in several operator, differential, integral, and integrodifferential equations side by side with their theories for instance, singular BVPs [12], singularly perturbed multipantograph delay equations (Geng and Qian, 2014), partial differential equations [17], Fredholm-Volterra integrodifferential equation [2, 5, 6], Fredholm integrodifferential equation ([1, 3, 14]), Volterra integrodifferential equation [7, 8], Fredholm-Volterra integral equation [11], operator equations [21], Fuzzy differential equations [9], and others [10, 15, 16]. The basic motivation of this paper is to apply the RKHM to develop an approach for obtaining the representation of exact and approximate solutions for a class of periodic BVPs (1) and (2), whereas the condition for determining solutions can be imposed in reproducing kernel space. However, this approach is simple, needs less effort to achieve the results, and is effective.

The paper is organized as follows. In Sect. 2, reproducing kernel spaces are presented in order to construct their reproducing kernel functions. In Sect. 3, representations of exact solution for BVPs (1) and (2) together with some essential results are introduced. Meanwhile, an iterative method for solving first-order periodic BVPs is described based on these reproducing kernel spaces. Subsequently, the analysis of the method is discussed in Sect. 4. In Sect. 5, numerical examples are simulated to show the reasonableness of our theory and to demonstrate the high performance of the proposed method. Finally, some conclusions are summarized in the last section.

2 Preliminaries and Materials

In this section, we utilize the reproducing kernel concept to construct the space $W_2^2[0,1]$ in which every function satisfies the periodic boundary condition (2) and formulate its reproducing kernel function. Besides, we present some basic results and remarks in the reproducing kernel theory and its applications.

Definition 1. Let E be a nonempty abstract set. A function $K : E \times E \to \mathbb{R}$ is a reproducing kernel of the Hilbert space $\mathscr{H}$ if:

1. For each $x \in E, K(\cdot, x) \in \mathscr{H}$.
2. For each $x \in E$ and $\varphi \in \mathscr{H}$, $\langle \varphi, K(\cdot, x) \rangle = \varphi(x)$.

The last condition is called *the reproducing property*: the value of the function φ at the point x is reproducing by the inner product of φ with $K(\cdot, x)$.

Remark 1. A Hilbert space $\mathscr{H}$ of functions on a set E is called a reproducing kernel Hilbert space (RKHS) if there exists a reproducing kernel K of $\mathscr{H}$. That is, a Hilbert space which possesses a reproducing kernel is called the RKHS.

Definition 2. The Hilbert space $W_2^m[0,1]$, $m \in \mathbb{N}$, is called a reproducing kernel if for each fixed x in $[0,1]$, there exist $K(x,y) \in W_2^m[0,1]$ such that $\langle u(y), K(x,y) \rangle_{W_2^m} = u(x)$ for any $u(y) \in W_2^m[0,1]$ and $y \in [0,1]$.

Definition 3. The reproducing kernel space $W_2^2[0,1]$ defined as $W_2^2[0,1] = \Big\{ u(x) :$ $u'(x)$ is absolutely continuous real-valued function, $u''(x) \in L^2[0,1]$, and $u(0) = u(1) \Big\}$. The inner product and norm in $W_2^2[0,1]$ are given, respectively, by

$$\langle u(x), v(x) \rangle_{W_2^2} = u(0)v(0) + u'(0)v'(0) + \int_0^1 u''(t)v''(t)dt, \tag{3}$$

and $\|u\| = \langle u, u \rangle^{\frac{1}{2}}$, where $u,\ v \in W_2^2[0,1]$.

Remark 2. The space $W_2^2[0,1]$ is a complete reproducing kernel space, and its reproducing kernel function $K(x,y)$ can be written as

$$k(x,y) = \begin{cases} \sum_{i=1}^{4} c_i(x) y^{i-1}, \ y \leq x, \\ \sum_{i=1}^{4} d_i(x) y^{i-1}, \ y > x, \end{cases} \tag{4}$$

where $c_i(x)$ and $d_i(x), \ i = 1,2,3,4$ will be given by the following assumptions:

Let's assume that $K(x,y) \in W_2^2[0,1]$ satisfies the generalized differential equations

$$\begin{cases} \frac{\partial^4 k(x,y)}{\partial y^4} = \delta(y-x), \frac{\partial^2 k(x,1)}{\partial y^2} = 0, k(x,0) + \frac{\partial^3 k(x,0)}{\partial y^3} + c_1 = 0, \\ \frac{\partial k(x,0)}{\partial y} - \frac{\partial^2 k(x,0)}{\partial y^2} = 0, \frac{\partial^3 k(x,1)}{\partial y^3} + c_1 = 0. \end{cases} \tag{5}$$

where δ is the Dirac delta function.

On the other hand, for $x \neq y$, $K(x,y)$ is the solution of the constant differential equation $\frac{\partial^4 k(x,y)}{\partial y^4} = 0$, subject to the boundary conditions (5). That is, the characteristic equation is given by $\lambda^4 = 0$ and the eigenvalues are $\lambda = 0$ with multiplicity 4. Hence, the general solution can be written as in Eq. (4).

In addition, assume that $K(x,y)$ satisfies the equations $\frac{\partial^m k(x,x+0)}{\partial y^m} = \frac{\partial^m k(x,x-0)}{\partial y^m}$ for $m = 0,1,2$, and $\frac{\partial^3 k(x,x+0)}{\partial y^3} - \frac{\partial^m k(x,x-0)}{\partial y^m} = -1$. Through the last descriptions together with the boundary conditions (5), the unknown coefficients $c_i(x)$ and $d_i(x), \ i = 1,2,3,4$ are uniquely obtained.

However, the representation of the reproducing kernel function $K(x,y)$ in $W_2^2[0,1]$, using Mathematica software package, is provided by

$$K(x,y) = \begin{cases} \frac{1}{48}\left[x^3 y\left(6+3y-y^2\right)+3x^2 y\left(-6-3y+y^2\right)+6xy\left(2+y+y^2\right)-8\left(-6+y^3\right)\right], y \leq x, \\ \frac{1}{48}\left[48+6xy\left(2-3y+y^2\right)+3x^2 y\left(2-3y+y^2\right)-x^3\left(8-6y-3y^2+y^3\right)\right], y > x. \end{cases} \tag{6}$$

Here, it should be noted that the kernel function $K(x,y)$ is unique, symmetric, and nonnegative for any fixed $x \in [0,1]$. For detailed method for obtaining the reproducing kernel function, we refer to [12].

Theorem 1. An arbitrary bounded set of $W_2^2[0,1]$ is a compact set of $C[0,1]$.

Proof Let $\{u_n(x)\}_{n=1}^{\infty}$ be a bounded set of $W_2^2[0,1]$ such that $\|u_n(x)\| < M$, where M is positive constant. From representation of $K(x,y)$, we have $\left|u^{(i)}(x)\right| = \left|\left\langle u(x), \partial_x^i K(x,y)\right\rangle_{W_2^2}\right| \leq \partial_x^i \|K(x,y)\|_{W_2^2} \|u(x)\|_{W_2^2}$. Since $\partial_x^i K(x,y), \ i = 1,2,\ldots$ is uniformly bounded about x and y, we have $\left|u^{(i)}(x)\right| \leq M_i \|u(x)\|_{W_2^2}$. Accordingly, $\|u(x)\|_c \leq M$.

Now, we need to prove that $\{u_n(x)\}_{n=1}^{\infty}$ is a compact set of $C[0,1]$, that is, $\{u_n(x)\}_{n=1}^{\infty}$ are equicontinuous functions. From the property of $K(x,y)$, we have

$$|u_n(x_1) - u_n(x_2)| = \left|\langle u(y), K(x_1,y) - K(x_2,y)\rangle_{W_2^2}\right|$$
$$\leq \|u(x)\|_{W_2^2}\|K(x_1,y) - K(x_2,y)\|_{W_2^2} \leq M\left\|K(x_1,y) - K(x_2,y)_{W_2^2}\right\|.$$

By "mean-value theorem of differentials" and the symmetry of $K(x,y)$, it follows that

$$|K(x_2,y) - K(x_1,y)| = |K(y,x_2) - K(y,x_1)| = \left|\frac{d}{dx}K(y,x)\right|_{x=\eta}|x_2 - x_1| \leq N|x_2 - x_1|.$$

Thus, if $\gamma \leq |x_2 - x_1| \leq \frac{\epsilon}{NM}$, then one can get $|u_n(x_1) - u_n(x_2)| < \epsilon$.

Definition 4. The reproducing kernel space $W_2^1[0,1]$ defined as $W_2^1[0,1] = \Big\{u(x) :$ $u'(x)$ is absolutely continuous real-valued function, $u'(x) \in L^2[0,1]\Big\}$. The inner product and norm in $W_2^1[0,1]$ are given, respectively, by

$$\langle u(x), v(x)\rangle_{W_2^1} = u(0)v(0) + \int_0^1 u'(t)v'(t)dt, \tag{7}$$

and $\|u\| = \langle u,u\rangle^{\frac{1}{2}}$, where $u,\ v \in W_2^1[0,1]$.

In 2006, Lin and Cui have proved that the space $W_2^1[0,1]$ is a complete reproducing kernel and its reproducing kernel is given by

$$G(x,y) = \begin{cases} (1+y),\, y \leq x, \\ (1+x),\, y > x. \end{cases} \tag{8}$$

3 Adaptation of Reproducing Kernel Algorithm

In this section, the formulation of a linear differential operator and the implementation method are presented in $W_2^2[0,1]$. After a while, the construction of orthogonal function systems is introduced based on the use of the Gram-Schmidt orthogonalization process in order to obtain exact and approximate solutions of periodic BVPs (1) and (2). To do this, we define a differential operator $L : W_2^2[0,1] \to W_2^1[0,1]$ such that $Lu(x) = u'(x) + g(x)u(x)$. Thus, the periodic BVPs (1) and (2) can be converted into the form

$$\begin{cases} Lu(x) = f\,(x, u(x))\,,\ 0 \le x \le 1, \\ u(0) - u(1) = 0, \end{cases} \tag{9}$$

where $u(x) \in W_2^2\,[0,1]$ and $f\,(x,y) \in W_2^1\,[0,1]$ as $y = y(x)\big) \in W_2^2\,[0,1]\,,\ y \in (-\infty,\infty)\,, x \in [0,1]\,.$

Corollary 1 *The operator L:* $W_2^2\,[0,1] \to W_2^1\,[0,1]$ *is a bounded linear operator.*

Proof It is so easy to see that L is a linear operator. Thus, it is enough to show that L is a bounded operator. From Definition 4, we have

$$\|Lu\|_{W_2^1}^2 = \langle Lu, Lu\rangle_{W_2^1} = [(Lu)(0)]^2 + \int_0^1 [(Lu)'(x)]^2 dx.$$

By reproducing property of $K(x,y)$, we have

$$.\begin{cases} u(x) = \langle u(y),\ K\,(x,y)\rangle_{W_2^2}, \\ (Lu)(x) = \Big\langle u,\ LK\,(x,y)\,\Big)\Big\rangle_{W_2^2}, \\ (Lu)'(x) = \Big\langle u,\ LK\,(x,y)\,\Big)'\Big\rangle_{W_2^2}. \end{cases}$$

By Schwarz inequality, we get

$$|(Lu)(x)| = \left|\Big\langle u,\ LK\,(x,y)\,\Big)\Big\rangle_{W_2^2}\right| \le \|LK\,(x,y)\|_{W_2^2}\|u\|_{W_2^2} = M_1\|u\|_{W_2^2},$$

and

$$|(Lu)'(x)| = \left|\langle u, (LK\,(x,y))\,'\rangle_{W_2^2}\right| \le \|(LK\,(x,y))\,'\|_{W_2^2}\|u\|_{W_2^2} = M_2\|u\|_{W_2^2},$$

where $M_1, M_2 > 0$ are positive constants.

Thus $[(Lu)(0)]^2 \le M_1^2\|u\|_{W_2^2}^2$, $[(Lu)'(x)]^2 \le M_2^2\|u\|_{W_2^2}^2$ and $\int_0^1 [(Lu)'(x)]^2 dx \le M_2^2\|u\|_{W_2^2}^2$. That is,

$$\|(Lu)(x)\|_{W_2^1}^2 = [(Lu)(0)]^2 + \int_0^1 [(Lu)'(x)]^2 dx \le \left(M_1^2 + M_2^2\right)\|u\|_{W_2^2}^2 = M\|u\|_{W_2^2}^2,$$

where $M = M_1^2 + M_2^2 > 0$.

Now, we construct an orthogonal system of functions $\{\psi_i(x)\}_{i=1}^{\infty}$ of $W_2^2[0,1]$ by setting $\Phi_i(x) = G\,(x, x_i)$ and $\Psi_i(x) = L^*\Phi_{\rm i}(x)$, where $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$

and L^* is the conjugate operator of L. Consequently, in terms of the properties of $G(x,y)$, one obtains $\langle u(x), \Psi_i(x)\rangle_{W_2^2} = \langle u(x), L^*\Phi_{\mathrm{i}}(x)\rangle_{W_2^2} = \langle Lu(x), \Phi_i(x)\rangle_{W_2^1} = Lu\,(x_i)\,,\ i = 1, 2, \ldots$

Lemma 1 The fact $\Psi_i(x) = \frac{d}{dy}K\,(x,y)\,|_{y=x_i}, i = 1, 2, \ldots$ holds.

Proof From reproducing property of, we can obtain that $\Psi_i(x) = \langle \Psi_i(y), K\,(x,y)\rangle_{W_2^2} = \langle L^*\Phi_{\mathrm{i}}(x), K\,(x,y)\rangle_{W_2^2} = \langle \Phi_{\mathrm{i}}(x), LK\,(x,y)\rangle_{W_2^1} = LK\,(x,x_i) = \frac{d}{dy}K\,(x,y)\,|_{y=x_i}$.

Lemma 2 If $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$; then $\{\Psi_i(x)\}_{i=1}^{\infty}$ is a complete system of $W_2^2[0,1]$.

Proof For each fixed $u(x) \in W_2^2\,[0,1]\,$, let $\langle u(x), \Psi_i(x)\rangle_{W_2^2} = 0$. That is, $\langle u(x), \Psi_i(x)\rangle_{W_2^2} = \langle u(x), L^*\Phi_{\mathrm{i}}(x)\rangle_{W_2^2} = \langle Lu(x), \Phi_i(x)\rangle_{W_2^1} = Lu\,(x_i) = 0,\ i = 1,\ 2, \ldots$. Therefore, $Lu(x) = 0$ from the density of $\{x_i\}_{i=1}^{\infty}$ on $[0,1]$, as well as $u(x) = 0$ from the existence of L^{-1} and the continuity of $u(x)$.

The orthonormal system functions $\left\{\overline{\Psi}_i(x)\,\right\}_{i=1}^{\infty}$ of $W_2^2[0,1]$ can be derived from Gram-Schmidt orthogonalization process of $\{\Psi_i(x)\}_{i=1}^{\infty}$ as follows:

$$\overline{\Psi}_i(x) = \sum_{k=1}^{i}\beta_{ik}\Psi_k(x), \tag{10}$$

where β_{ik} are orthogonalization coefficients $\beta_{ii} > 0,\ i = 1, 2, \ldots, n\Big)$ that are given by

$$\beta_{ij} = \frac{1}{\|\Psi_i\|}, \quad \text{for} \quad i = j = 1,$$

$$\beta_{ij} = \frac{1}{\sqrt{\|\Psi_i\|^2 - \sum_{k=1}^{i-1}\left(\left\langle\Psi_i, \overline{\Psi}_k\right\rangle_{W_2^2}\right)^2}} \quad \text{for} \quad i = j \neq 1, \quad \text{and}$$

$$\beta_{ij} = \frac{-\sum_{j=k}^{i-1}\left\langle\Psi_i, \overline{\Psi}_k\right\rangle_{W_2^2}\beta_{jk}}{\sqrt{\|\Psi_i\|^2 - \sum_{k=1}^{i-1}\left(\left\langle\Psi_i, \overline{\Psi}_k\right\rangle_{W_2^2}\right)^2}} \quad \text{for} \quad i > j.$$

Theorem 2. For each $u(x)$ in $W_2^2[0,1]$, the series $\sum_{i=1}^{\infty}\left\langle u(x), \overline{\Psi}_i(x)\right\rangle\overline{\Psi}_i(x)$ is convergent in the sense of the norm $\|\cdot\|_{W_2^2}$. On the other hand, if $\{x_i\}_{i=1}^{\infty}$ is dense on [0, 1] and $u(x) \in W_2^2\,[0,1]$ is the solution of problem model (9), then $u(x)$ satisfy the following form:

$$u(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}f\left(x_k, u\left(x_k\right)\right)\overline{\Psi}_i(x), \tag{11}$$

and the approximate solution can be obtained by

$$u_n(x) = \sum_{i=1}^{n}\sum_{k=1}^{i}\beta_{ik}f\left(x_k, u\left(x_k\right)\right)\overline{\Psi}_i(x), \tag{12}$$

where $u_0(x) \in W_2^2[0,1]$ (u_0 fixed).

Proof Since $u(x) \in W_2^2[0,1]$, $u(x)$ can be expanded in the form of Fourier series about $\left\{\overline{\Psi}_i(x)\right\}_{i=1}^{\infty}$ as $u(x) = \sum_{i=1}^{\infty}\left\langle u(x), \overline{\Psi}_i(x)\right\rangle\overline{\Psi}_i(x)$, and since the space $W_2^2[0,1]$ is the Hilbert space, then the series $u(x) = \sum_{i=1}^{\infty}\left\langle u(x), \overline{\Psi}_i(x)\right\rangle\overline{\Psi}_i(x)$ is convergent in the norm $\|.\|_{W_2^2}$. From the Fourier series expansion and by Eq. (7), $u(x)$ can be written as

$$\begin{aligned}
u(x) &= \sum_{i=1}^{\infty}\left\langle u(x), \overline{\Psi}_i(x)\right\rangle_{W_2^2}\overline{\Psi}_i(x) = \sum_{i=1}^{\infty}\left\langle u(x), \sum_{k=1}^{i}\beta_{ik}\Psi_k(x)\right\rangle_{W_2^2}\overline{\Psi}_i(x)\\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle u(x), \Psi_k(x)\rangle_{W_2^2}\overline{\Psi}_i(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\left\langle u(x), L^*\Phi_k(x)\right\rangle_{W_2^2}\overline{\Psi}_i(x)\\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\langle\beta_{ik}Lu(x), \Phi_k(x)\rangle_{W_2^1}\overline{\Psi}_i(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}Lu\left(x_k\right)\overline{\Psi}_i(x)\\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}f\left(x_k, u\left(x_k\right)\right)\overline{\Psi}_i(x).
\end{aligned}$$

Therefore, the form in Eq. (11) is the exact solution of Eq. (9). By truncating the series in Eq. (11), we obtain the th-truncated series approximate solution as in Eq. (12). So, the proof of the theorem is complete.

Lemma 3. If $u(x) \in W_2^2[0,1]$, then there exists a positive constant M such that $\left\|u^{(i)}(x)\right\|_c \le M\|u(x)\|_{W_2^2}$, $i = 0, 1$, where $\|u(x)\|_c = \max_{0<x<1}|u(x)|$.

Proof For any $x_1, x_2 \in [0,1]$, we have $u^{(i)}\left(x_1\right) = \left\langle u\left(x_2\right), \partial_{x_1}^i K\left(x_1, x_2\right)\right\rangle_{W_2^2}, i = 0, 1$. By the expression form of $K(x,y)$, it follows that $\left\|\partial_x^i K(x,y)\right\|_{W_2^2} \le M_i, i = 0, 1$. Thus, $\left|u^{(i)}\left(x_1\right)\right| = \left|\left\langle u\left(x_2\right), \partial_{x_1}^i K\left(x_1, x_2\right)\right\rangle_{W_2^2}\right| \le \left\|\partial_{x_1}^i K\left(x_1, x_2\right)\right\|_{W_2^2}\left\|u\left(x_2\right)\right\|_{W_2^2} \le \cdots M_i\|u(x)\|_{W_2^2}, i = 0, 1$. Hence, $\left\|u^{(i)}(x)\right\|_c \le \max_i\left\{M_i\right\}\|u(x)\|_{W_2^2}, i = 0, 1$. The proof is complete.

Corollary 2. *The approximate solution $u_n(x)$ and its derivative $u'_n(x)$ are converging uniformly to the exact solution $u(x)$ and its derivative $u'(x)$ as $n \to \infty$, respectively.*

Proof Form Lemma 3, for any $x \in [0,1]$, it easy to see that $\left|u_n^{(i)}(x) - u^{(i)}(x)\right| = \left|\left\langle u_n(x) - u(x), \partial_x^i K(x,x)\right\rangle_{W_2^2}\right| \le \left\|\partial_x^i K(x,x)\right\|_{W_2^2} \|u_n(x) - u(x)\|_{W_2^2} \le M_i \|u_n(x) - u(x)\|_{W_2^2}$, $i{=}0,1$.

Hence, if $\|u_n(x) - u(x)\|_{W_2^2} \to 0$ as $n \to \infty$, then the approximate solution $u_n(x)$ and its derivative $u'_n(x)$ are converging uniformly to the exact solution $u(x)$ and its derivative $u'(x)$ as $n \to \infty$, respectively. So, the proof of the theorem is complete.

Remark 3. In order to solve Eq. (1) numerically using the RKHS technique, we have the following two cases:

Case 1: If Eq. (1) is linear, then the exact and approximate solutions can be obtained directly from Eqs. (11) and (12), respectively.
Case 2: If Eq. (1) is nonlinear, then in this case the exact and approximate solutions can be obtained by using the following algorithm:

Algorithm 1 According to Eq. (11), the representation of the solution of problem (1) can be denoted by

$$u(x) = \sum_{i=1}^{\infty} B_i \overline{\Psi}_i(x), \tag{13}$$

where $B_i = \sum_{k=1}^{i} \beta_{ik} f(x_k, u_{k-1}(x_k))$. In fact, $B_i, i = 1, 2, \ldots,$ in Eq. (13) are unknown, so we will approximate them using the known A_i as follows: For a numerical computations, let the initial function $u_0(x_1) = 0$, set $u_0(x_1) = u(x_1)$, and define the n-term approximation to $y_s(x)$ by

$$u_n(x) = \sum_{i=1}^{n} A_i \overline{\Psi}_i(x), \tag{14}$$

where the coefficients A_i of $\overline{\Psi}_i(x), i = 1, 2, \ldots, n$, are given by

$$\begin{cases} A_1 = \beta_{11} f(x_1, u_0(x_1)), u_1(x) = A_1 \overline{\Psi}_1(x), \\ A_2 = \sum_{k=1}^{2} \beta_{2k} f(x_1, u_{k-1}(x_k)), u_2(x) = \sum_{i=1}^{2} A_i \overline{\Psi}_i(x), \\ u_{n-1}(x) = \sum_{i=1}^{n-1} A_i \overline{\Psi}_i(x), A_n = \sum_{k=1}^{n} \beta_{nk} f(x_1, u_{k-1}(x_k)). \end{cases} \tag{15}$$

Consequently, the unknown coefficients $B_i, i = 1, 2, \ldots,$ in Eq. (13) will be approximate using the known coefficients $A_i, i = 1, 2, \ldots,$ given in Eq. (14).

However, in the iterative process of the series (14), we can guarantee that the approximation $u_n(x)$ satisfies the periodic boundary condition (2).

4 Convergence Analysis of the Method

In this section, we will prove that the iterative formula (14) is convergent to the exact solution of Eq. (9) in the sense of the norm of $W_2^2[0, 1]$. In fact, this result is fundamental in the RKHS theory and its applications. The remaining lemmas are collected in order to prove the pre-recent theorem.

Lemma 4 If $\|u_n(x) - u(x)\|_{W_2^2} \to 0,\ x_n \to y,\ (n \to \infty)$, and $f(x, z)$ is continuous in $[0, 1]$ with respect to x, z for $x \in [0, 1]\,,\ z \in (-\infty, \infty)$, then the following are held in the sense of the norm of $W_2^2[0, 1]$:

(a) $u_{n-1}(x_n) \to u(y)$ *as* $n \to \infty$.
(b) $f(x_n, u_{n-1}(x_n)) \to f(y, u(y))$, *as* $n \to \infty$.

Proof For part (a), note that

$$\begin{aligned}|u_{n-1}(x_n) - u(y)| &= |u_{n-1}(x_n) - u_{n-1}(y) + u_{n-1}(y) - u(y)| \\ &\leq |u_{n-1}(x_n) - u_{n-1}(y)| + |u_{n-1}(y) - u(y)|\,.\end{aligned}$$

By reproducing property of $K(x, y)$, we have $u_{n-1}(x_n) = \langle u_{n-1}(x), K(x_n, x)\rangle_{W_2^2}$ and $u_{n-1}(y) = \langle u_{n-1}(x), K(y, x)\rangle_{W_2^2}$. Thus,

$$\begin{aligned}|u_{n-1}(x_n) - u_{n-1}(y)| &= \left|\langle u_{n-1}(x), K(x_n, x) - K(y, x)\rangle_{W_2^2}\right| \\ &\leq \|u_{n-1}(x)\|_{W_2^2}\|K(x_n, x) - K(y, x)\|_{W_2^2}.\end{aligned}$$

From the symmetry of $K(x, y)$, it follows that $\left\|K(x_n, x) - K(y, x)_{W_2^2}\right\| \to 0$ as $x_n \to y, n \to \infty$. Hence, $|u_{n-1}(x_n) - u_{n-1}(y)| \to 0$ as soon as $x_n \to y,\ (n \to \infty)$. On the other hand, for any $x \in [0, 1]$, by using Corollary 2, it holds that $|u_{n-1}(y) - u(y)| \to 0$ as $n \to \infty$. Therefore, $u_{n-1}(x_n) \to u(y)$ in the sense of $\|\cdot\|_{W_2^2}$ as $x_n \to y$ and $n \to \infty$. Thus, for part (b), by means of the continuation of $f(\cdot)$, it is obtained that $f(x_n, u_{n-1}(x_n)) \to f(y, u(y))$ as $x_n \to y$ and $n \to \infty$.

Lemma 5 For the approximate solution $u_n(x)$ in iterative formula (14), the following relations hold:

(a) $Lu_n(x_j) = f(x_j, u_{j-1}(x_j))\,, j \leq n$,
(b) $Lu_n(x_j) = Lu(x_j)\,, j \leq n$.

Proof For part (a), the proof will be obtained by mathematical induction. For $j \leq n$, we have

$$Lu_n\left(x_j\right)=\sum_{i=1}^{n}A_iL\overline{\Psi}_i(x)=\sum_{i=1}^{n}A_i\big\langle L\overline{\Psi}_i(x),\Phi_{\mathrm{j}}(x)\big\rangle_{W_2^1}=\sum_{i=1}^{n}A_i\big\langle\overline{\Psi}_i(x),L^*\Phi_{\mathrm{j}}(x)\big\rangle_{W_2^2}$$
$$=\sum_{i=1}^{n}A_i\big\langle\overline{\Psi}_i(x),\Psi_j(x)\big\rangle_{W_2^2}.$$

That is,

$$Lu_n\left(x_j\right)=\sum\nolimits_{i=1}^{n}A_i\big\langle\overline{\Psi}_i(x),\Psi_j(x)\big\rangle_{W_2^2}. \tag{16}$$

Multiplying both sides of Eq. (15) by β_{jl}, summing for l from 1 to j, and using the orthogonality of $\left\{\overline{\Psi}_i(x)\right\}_{i=1}^{\infty}$ yields that

$$\sum_{l=1}^{j}\beta_{jl}Lu_n\left(x_l\right)=\sum_{i=1}^{n}A_i\left\langle\overline{\Psi}_i(x),\sum_{l=1}^{j}\beta_{jl}\Psi_l(x)\right\rangle_{W_2^2}=\sum_{i=1}^{\infty}A_i\big\langle\overline{\Psi}_i(x),\overline{\Psi}_j(x)\big\rangle_{W_2^2}$$
$$=A_j=\sum_{l=1}^{j}\beta_{jl}f\left(x_l,u_{l-1}\left(x_l\right)\right).$$

If $j=1$, then $Lu_n\left(x_1\right)=f\left(x_1,u_0\left(x_1\right)\right)$. Besides, if $j=2$, then $\beta_{21}Lu_n\left(x_1\right)+\beta_{22}Lu_n\left(x_2\right)=\beta_{21}f\left(x_1,u_0\left(x_1\right)\right)+\beta_{22}f\left(x_2,u_1\left(x_2\right)\right)$, that is, $Lu_n\left(x_2\right)=f\left(x_2,u_1\left(x_2\right)\right)$. Thus $Lu_n\left(x_j\right)=f\left(x_j,u_{j-1}\left(x_j\right)\right)$ for $j\le n$.

For part (b), from Corollary 2 as well as by taking limits in Eq. (14), we have $u(x)=\sum\nolimits_{i=1}^{\infty}A_i\overline{\Psi}_i(x)$. Thus, $u_n(x)=P_nu(x)$, where P_n is an orthogonal projector from the space $W_2^2[0,1]$ to Span$\{\Psi_1,\Psi_2,\ldots,\Psi_n\}$. Therefore, $Lu_n\left(x_j\right)=\big\langle Lu_n(x),\Phi_j(x)\big\rangle_{W_2^1}=\big\langle u_n(x),L^*\Phi_j(x)\big\rangle_{W_2^2}=\big\langle P_nu(x),\Psi_j(x)\big\rangle_{W_2^2}=\big\langle u(x),P_n\Psi_j(x)\big\rangle_{W_2^2}=\big\langle u(x),\Psi_j(x)\big\rangle_{W_2^2}=\big\langle u(x),L^*\Phi_j(x)\big\rangle_{W_2^2}=\big\langle Lu(x),\Phi_j(x)\big\rangle_{W_2^1}=Lu\left(x_j\right)$. So, the proof of the lemma is complete.

Lemma 6 The sequence $\{u_n(x)\}_{n=1}^{\infty}$ in the iterative formula (14) is monotone increasing in the sense of $\|\cdot\|_{W_2^2}$.

Theorem 3. Suppose that $\{x_i\}_{i=1}^{\infty}$ is dense on a compact interval $[0,1]$ and $\|u_n(x)\|_{W_2^2}$ is bounded in formula (14), then the n-term approximate solution $u_n(x)$ in the iterative formula (14) is convergent to the exact solution $u(x)$ of Eq. (9) in the space $W_2^2[0,1]$ and $u(x)=\sum\nolimits_{i=1}^{\infty}A_i\overline{\Psi}_i(x)$, where $A_i, i=1,2,\ldots$ are given by Eq. (14).

Proof First of all, we will prove the convergence of $u_n(x)$. From iterative formula (14), we infer that $u_{n+1}(x)=u_n(x)+A_{n+1}\overline{\Psi}_{n+1}(x)$. By the orthogonality of $\left\{\overline{\Psi}(x)\right\}_{i=1}^{\infty}$, it follows that $\|u_{n+1}\|_{W_2^2}^2=\|u_n\|_{W_2^2}^2+\left(A_{n+1}\right)^2=\|u_{n-1}\|_{W_2^2}^2+\left(A_n\right)^2$

$+ (A_{n+1})^2 = \cdots = \|u_0\|^2_{W_2^2} + \sum_{i=1}^{n+1}(A_i)^2$. From Lemma 6, the sequence $\|u_n\|_{W_2^2}$ is monotone increasing, and from the boundedness of $\|u_n\|_{W_2^2}$, we have $\sum_{i=1}^{\infty}(A_i)^2 < \infty$, that is, $\{A_i\}_{i=1}^{\infty} \in l^2\,(i = 1, 2, \ldots)$. Hence, $\|u_n\|_{W_2^2}$ is convergent as $n \to \infty$.

Let $m > n$; for $(u_m - u_{m-1}) \perp (u_{m-1} - u_{m-2}) \perp \cdots \perp (u_{n+1} - u_n)$, it follows that

$$\begin{aligned}\|u_m(x) - u_n(x)\|^2_{W_2^2} &= \|u_m(x) - u_{m-1}(x) + u_{m-1}(x) - \cdots + u_{n+1}(x) - u_n(x)\|^2_{W_2^2} \\ &\le \|u_m(x) - u_{m-1}(x)\|^2_{W_2^2} + \cdots + \|u_{n+1}(x) - u_n(x)\|^2_{W_2^2} \\ &= \sum_{i=n+1}^{m}(A_i)^2 \to 0,\ (n \to \infty)\,.\end{aligned}$$

Considering the completeness of $W_2^2[0,1]$, there exists $u(x) \in W_2^2\,[0,1]$ such that $u_n(x) \to u(x)$ as $n \to \infty$ in sense of $\|.\|_{W_2^2}$.

Secondly, we will prove that $u(x)$ is the solution of Eq. (9). Since $\{x_i\}_{i=1}^{\infty}$ is dense on compact interval [0, 1], thus for any $x \in [0,1]$, there exists subsequence $\{x_{n_j}\}$ such that $x_{n_j} \to x$, as $j \to \infty$. From Lemma 5, $Lu_n\left(x_{n_j}\right) = f\left(x_{n_j}, u_{j-1}\left(x_{n_j}\right)\right)$. Hence, let $j \to \infty$; we have $Lu(x) = f\,(x, u(x))$. That is, $u(x)$ is solution of Eq. (9). The proof is complete.

Theorem 4. Assume that $u_n(x) \in W_2^2\,[0,1]$ is the solution of BVP (9) and $r_n(x) = \|u(x) - u_n(x)\|_{W_2^2}$ is an error function, where $u_n(x)$ is the approximate solution that is given by iterative formula (14). Then the sequence of number $\{r_n\}$ is monotone decreasing in the sense of $\|.\|_{W_2^2}$ and $r_n \to 0$ as $n \to \infty$.

Proof Based on the previous results, it is obvious that

$$\begin{aligned}\|r_n(x)\|^2_{W_2^2} = \|u(x) - u_n(x)\|^2_{W_2^2} &= \left\| \sum_{i=n+1}^{\infty}\sum_{k=1}^{i}\beta_{ik}F\left(x_k, u_{k-1}\left(x_k\right), Tu_{k-1}\left(x_k\right)\right)\overline{\Psi}_i(x)\right\|^2_{W_2^2} \\ &= \left\|\sum_{i=n+1}^{\infty} A_i\overline{\Psi}_i(x)\right\|^2_{W_2^2} = \sum_{i=n+1}^{\infty}(A_i)^2,\end{aligned}$$

and $\|r_{n-1}(x)\|^2_{W_2^2} = \sum_{i=n}^{\infty}(A_i)^2$. Thus, $\|r_n(x)\|_{W_2^2} \le \|r_{n-1}(x)\|_{W_2^2}$. Consequently, the error r_n is monotone decreasing in the sense of $\|.\|_{W_2^2}$. The proof is complete.

5 Applications and Test Problems

In this section, some numerical examples are studied to demonstrate the performance, accuracy, and applicability of the present method for both linear and nonlinear problems. Results obtained are compared with the exact solution of each example and are found to be in good agreement with each other. In the process of computation, all the symbolic and numerical computations performed by using Mathematica software package.

Example 1 Consider the following linear equation

$$u'(x) + u(x) = x^2 + x - 1, 0 \leq x \leq 1, \tag{17}$$

subject to periodic boundary condition

$$u(0) - u(1) = 0 \tag{18}$$

The exact solution is $u(x) = x\,(x-1)$.

Using RKHS method, taking $x_i = \frac{i-1}{n-1}$, $i = 1, 2, \ldots, n$. The numerical results at some selected grid points for $n = 51$ are given in Table 1.

To show the accuracy of the present method for our tested problems, we report two types of error. The first one is the absolute error, $\mathrm{Abs}_n(x)$, and the second one is the relative error, $\mathrm{Rel}_n(x)$, which are defined, respectively, by $\mathrm{Abs}_n(x) = |u(x) - u_n(x)|$, $\mathrm{Rel}_n(x) = \frac{\mathrm{Abs}_n(x)}{|u(x)|}$, where $x \in [0, 1]$, $u_n(x)$ is the n-term approximation of $u(x)$ obtained by the RKHS method, and $u(x) \in W_2^2\,[0, 1]$ is the exact solution.

Example 2 Consider the following nonlinear equation

$$u'(x) + u(x)e^{-u(x)} = \frac{2x - 1 + \ln\left(x^2 - x + 1\right)}{x^2 - x + 1}, 0 \leq x \leq 1, \tag{19}$$

subject to periodic boundary condition

$$u(0) - u(1) = 0 \tag{20}$$

Table 1 Numerical results for Example 1

x_i	$u(x)$	$u_{51}(x)$	$\mathrm{Abs}_{51}(x)$	$\mathrm{Rel}_{51}(x)$
0.16	−0.1344	−0.13440010447668405	1.04477×10^{-7}	7.77356×10^{-7}
0.32	−0.2176	−0.21760010031925470	1.00319×10^{-7}	4.61026×10^{-7}
0.48	−0.2496	−0.24960009873547984	9.87355×10^{-8}	3.95575×10^{-7}
0.64	−0.2304	−0.23040009968473152	9.96847×10^{-8}	3.95575×10^{-7}
0.80	−0.1600	−0.16000010319136138	1.03191×10^{-7}	6.44946×10^{-7}
0.96	−0.0384	−0.03840010934533122	1.09345×10^{-7}	2.84753×10^{-6}

Table 2 Numerical results for Example 2

x_i	$u(x)$	$u_{51}(x)$	$\text{Abs}_{51}(x)$	$\text{Rel}_{51}(x)$
0.16	−0.1443323708899199	−0.1443322995528097	7.13371×10^{-8}	4.94256×10^{-7}
0.32	−0.2453891602615295	−0.2453890219359414	1.38326×10^{-7}	5.63699×10^{-7}
0.48	−0.2871488812901222	−0.2871487102608685	1.71029×10^{-7}	5.95612×10^{-7}
0.64	−0.2618843796306403	−0.2618842287680727	1.50863×10^{-7}	5.76066×10^{-7}
0.80	−0.1743533871447777	−0.1743532975615985	8.95832×10^{-8}	5.13802×10^{-7}
0.96	−0.0391567152011939	−0.0391566982754902	1.69257×10^{-8}	4.32255×10^{-7}

The exact solution is $u(x) = \ln\left(x^2 - x + 1\right)$.

Using RKHS method, taking $x_i = \frac{i-1}{n-1}$, $i = 1, 2, \ldots, n$. The numerical results at some selected grid points for $n = 51$ are given in Table 2.

6 Conclusion

The main concern of this work has been to propose an efficient algorithm for the solutions of first-order periodic BVPs. The goal has been achieved by introducing the RKHS method to solve this class of differential equations. We can conclude that the RKHS method is a powerful and efficient technique in finding approximate solution $u_n(x)$ for linear and nonlinear problems. In the proposed algorithm, the solution $u(x)$ and the approximate solution $u_n(x)$ are represented in the form of series in $W_2^2[0, 1]$. Moreover, the approximate solution and its derivative converge uniformly to the exact solution and its derivative, respectively.

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Existence of the Mild Solutions for Nonlocal Fractional Differential Equations of Sobolev Type with Iterated Deviating Arguments

Alka Chadha and Dwijendra N. Pandey

Abstract This paper investigates a nonlocal differential equation of Sobolev type of fractional order with iterated deviating arguments in Banach space. The sufficient condition for providing the existence of mild solution to the nonlocal Sobolev-type fractional differential equation with iterated deviating arguments is obtained via technique of fixed-point theorems and analytic semigroup method. Finally, an example is given to explain the applicability of the abstract results developed.

Keywords Fractional calculus • Caputo derivative • Fractional differential equation • Nonlocal conditions • Deviated argument

Mathematics Subject Classification (2010): 26A33, 34K37, 34K40, 34K45, 35R11, 45J05, 45K05.

1 Introduction

Recently, the investigation of fractional differential equation has been picking up much attention from researchers. This is due to the fact that fractional differential equations have various applications in engineering and scientific disciplines, for example, fluid dynamics, fractal theory, diffusion in porous media, fractional biological neurons, traffic flow, polymer rheology, neural network modeling, viscoelastic panel in supersonic gas flow, real system characterized by power laws, electrodynamics of complex medium, sandwich system identification, nonlinear oscillation of earthquake, models of population growth, mathematical modeling of the diffusion of discrete particles in a turbulent fluid, nuclear reactors, and theory of population dynamics. Also, the fractional differential equation is an important

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tool to describe the memory and hereditary properties of various materials and phenomena. The details on the theory and its applications may be found in books [11, 14, 16, 18] and references cited therein. In addition, there is considerably interest on the part of mathematics in the examination of differential equation with a deviated argument, both in connection with problems in the hypothesis of control system and because of the intrinsic richness and beauty of such equations. Differential equations with deviated argument have extraordinary applications in the hypothesis of self-oscillating systems, problems connected with combustion in rocket motion, the hypothesis of automatic control, a series of biological problems, and the problem of long-range planning in economics and in numerous other fields of sciences and technology, the quantity of which is consistently extending. For more studies of such types of equations, we refer to monograph [7] and papers [8, 19, 20] and references cited therein.

On the other hand, the abstract evolution equations with nonlocal conditions have been studied by many authors. The existence of a solution for abstract Cauchy differential equation with nonlocal conditions in a Banach space has been considered first by Byszewski [3]. In physical science, the nonlocal condition may be connected with better effect in applications than the classical initial condition since nonlocal conditions are normally more exact for physical estimations than the classical initial condition. For the study of nonlocal evolution equation, we refer to [3–5, 9, 10] and references cited therein.

Our main aim of this paper is to examine the Sobolev-type nonlocal differential equation of fractional order with iterated deviating arguments in Banach space $\mathbb{X}$ illustrated by

$$^{c}D_t^{\beta}[\mathbb{E}\mathbb{B}z(t)] = \mathbb{L}z(t) + \mathbb{H}(t, z(t), z(d_1(t, z(t)))),\ 0 \le t \le T_0, \tag{1}$$

$$z(0) = z_0 + h(z),\ z_0 \in \mathbb{X} \tag{2}$$

where $d_1(\tau, z(\tau)) = b_1(\tau, z(b_2(\tau, \cdots, z(b_m(\tau, z(\tau)))\cdots))),\ m \in \mathbb{N}$, $^{c}D_t^{\beta}$ is the fractional derivative in Caputo derivative of order β, $\beta \in (0, 1)$, and $T_0 \in (0, \infty)$. In (1), we assume that the operators $\mathbb{L} : D(\mathbb{L}) \subset \mathbb{X} \to \mathbb{Z}$, $\mathbb{B} : D(\mathbb{B}) \subset \mathbb{X} \to \mathbb{Y}$, and $\mathbb{E} : D(\mathbb{E}) \subset \mathbb{Y} \to \mathbb{Z}$ are closed operators, where $\mathbb{X}, \mathbb{Y}$, and $\mathbb{Z}$ are the Hilbert spaces such that $\mathbb{Z}$ is continuously and densely embedded in $\mathbb{X}$; the state $y(\cdot)$ takes its values in $\mathbb{X}$. Thus, function $\mathbb{H} : [0, T_0] \times \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ is an appropriate function, and h is a map from some space of functions satisfying some conditions to be stated later. For more studies of Sobolev-type differential equations, we refer to papers [1, 2, 5, 9, 10, 12, 13, 17] and references cited therein.

We divide the article into three parts. Section 2 presents some basic definitions, lemmas, and theorems. Section 3 focuses on existence result of mild solution to consider system by virtue of the theory of semigroup via fixed-point technique. Section 4 considers an application for illustrating the discussed abstract results.

2 Preliminaries

In this section, some essential facts about semigroup theory, fractional calculus, theorems, and lemmas which will be required to obtain our result are stated.

Definition 2.1 ([16]). Let $z \in L^1([0,T_0],\mathbb{R}^+)$. The fractional integral $(\mathbb{J}^{\beta}_{0,t})$ of function z with order β is defined by

$$\mathbb{J}^{\beta}_{0,t}z(t) = \frac{1}{\Gamma(\beta)}\int_0^t (t-\xi)^{\beta-1}z(\xi)d\xi, \tag{3}$$

where Γ denotes the classical gamma functions. We can also write $\mathbb{J}^{\beta}z(t) = (g_\beta * z)(t)$, here

$$g_\beta(t) := \begin{cases} \frac{1}{\Gamma(\beta)}t^{\beta-1}, & t>0, \\ 0, & t \le 0. \end{cases} \tag{4}$$

The notation $*$ stands the convolution of functions and $\lim_{\beta\to 0} g_\beta(t) = \delta(t)$, δ means delta Dirac function.

Definition 2.2 ([16]). The fractional derivative in Riemann-Liouville sense is given by

$${}^{RL}D^{\beta}_{0,t}z(t) = \frac{1}{\Gamma(n-\beta)}\frac{d^n}{dt^n}\int_0^t (t-\xi)^{n-\beta-1}z(\xi)d\xi,\ t>0,\ \beta>0,\ \beta\in(n-1,n),\ n\in\mathbb{N}, \tag{5}$$

and $z \in C^{n-1}([0,T_0],\mathbb{X})$.

Definition 2.3 ([16]). The Caputo fractional derivative is given by

$${}^{c}D^{\beta}_{0,t}z(t) = \frac{1}{\Gamma(n-\beta)}\int_0^t (t-\xi)^{n-\beta-1}z^n(\xi)d\xi,\quad n-1<\alpha<n, \tag{6}$$

where $z \in C^{n-1}((0,T_0);\mathbb{X}) \cap L^1((0,T_0);\mathbb{X})$.

Throughout the paper, we assume that $(\mathbb{X},\|\cdot\|)$, $(\mathbb{Y},\|\cdot\|)$, and $(\mathbb{Z},\|\cdot\|)$ are Banach spaces. The symbol $C([0,T_0],\mathbb{X})$ represents the space of continuous functions $z : [0,T_0] \to \mathbb{X}$ which is a Banach space with the following norm:

$$\|z\|_{[0,T_0]} = \sup\{\|z(t)\| : t \in [0,T_0]\}.$$

The notation $L(\mathbb{X})$ stands for the Banach space of bounded linear operators $f : \mathbb{X} \to \mathbb{X}$ endowed with the norm $\|f\|_{L(\mathbb{X})} = \sup\{\|f(y)\| : \|y\| = 1\}$. Now, we impose the following data on operators $\mathbb{L}$ and $\mathbb{E}$ and $\mathbb{B}$:

(C1) $\mathbb{E} : D(\mathbb{E}) \subset \mathbb{Y} \to \mathbb{Z}$ and $\mathbb{B} : D(\mathbb{B}) \subset \mathbb{X} \to \mathbb{Y}$ are linear operators and $\mathbb{L} : D(\mathbb{L}) \subset \mathbb{X} \to \mathbb{Z}$ is closed.
(C2) $D(\mathbb{B}) \subset D(\mathbb{L})$, $Im(\mathbb{B}) \subset D(\mathbb{E})$, and $\mathbb{E}, \mathbb{B}$ are bijective operators.
(C3) The operators $\mathbb{E}^{-1} : \mathbb{Z} \to D(\mathbb{E}) \subset \mathbb{Y}$ and $\mathbb{B}^{-1} : \mathbb{Y} \to D(\mathbb{B}) \subset \mathbb{X}$ are assumed to be linear, bounded, and compact operators.

By the hypothesis ($C3$), it follows that $\mathbb{B}^{-1}\mathbb{E}^{-1}$ is closed and injective. Thus, its inverse is also closed, i.e., $\mathbb{E}\mathbb{B}$ is closed. By the hypothesis ($C1$)–($C3$) and closed graph theorem, we conclude the boundedness of the linear operator $\mathbb{L}\mathbb{B}^{-1}\mathbb{E}^{-1}$. Therefore, $\mathbb{L}\mathbb{B}^{-1}\mathbb{E}^{-1}$ generates a semigroup $\{\mathcal{S}(t), t \geq 0\}$, $\mathcal{S}(t) := e^{-\mathbb{L}\mathbb{B}^{-1}\mathbb{E}^{-1}}$. Thus, without loss of generality, we may assume that $\mathbb{N}_0 := \sup_{t\geq 0} \|\mathcal{S}(t)\| < \infty$ and $\mathbb{W}_1 = \|\mathbb{E}^{-1}\|, \mathbb{W}_2 = \|\mathbb{B}^{-1}\|$.

According to previous definitions, the system (1)–(2) is equivalent to the following integral equation

$$[\mathbb{E}\mathbb{B}]z(t) = [\mathbb{E}\mathbb{B}]z(0) + \int_0^t \frac{(t-\xi)^{\beta-1}}{\Gamma(\beta)}[\mathbb{L}z(\xi) + \mathbb{H}(\xi, z(\xi), z(h_1(\xi, y(\xi))))]d\xi, \quad (7)$$

provided the integral in (7) exists for a.e. $t \in [0, T_0]$.

In this work, $\mathbb{M} = \mathbb{L}\mathbb{B}^{-1}\mathbb{E}^{-1} : D(\mathbb{M}) \subset \mathbb{Z} \to \mathbb{Z}$ is assumed to be a generator of a compact analytic semigroup $\{\mathcal{S}(t),\ t \geq 0\}$ of uniformly bounded linear operators. Thus, it follows that there exists a positive constant $\mathbb{N}_0 \geq 1$ such that $\|\mathcal{S}(t)\| \leq \mathbb{N}_0$ for each $t \geq 0$. We assume that $0 \in \rho(\mathbb{M})$, $\rho(\mathbb{M})$ means resolvent set of $\mathbb{M}$. Therefore, we may determine the fractional power $\mathbb{M}^\alpha$ for $\alpha \in (0, 1]$ as a closed linear operator with domain $D(\mathbb{M}^\alpha)$ with inverse $\mathbb{M}^{-\alpha}$. Moreover, the subspace $D(\mathbb{M}^\alpha)$ is a dense subset of $\mathbb{X}$ with the norm $\|z\|_\alpha = \|\mathbb{M}^\alpha z\|$ for $z \in D(\mathbb{M}^\alpha)$. Thus, it is not difficult to show that $D(\mathbb{M}^\alpha)$ is a Banach space with supremum norm. Hence, we signify the space $D(\mathbb{M}^\alpha)$ by $\mathbb{X}_\alpha$ endowed with the α-norm ($\|\cdot\|_\alpha$). We also have that $\mathbb{X}_\eta \hookrightarrow \mathbb{X}_\alpha$ for $0 < \alpha < \eta$ which implies the continuity of embedding mapping. Thus, we may define $\mathbb{X}_{-\alpha} = (\mathbb{X}_\alpha)^*$ for each $\alpha > 0$, dual space of $\mathbb{X}_\alpha$, is a Banach space endowed with the norm

$$\|z\|_{-\alpha} = \|\mathbb{M}^{-\alpha}z\|, \quad \text{for} z \in \mathbb{X}_{-\alpha}.$$

For more details on the fractional powers of closed linear operators, we refer to the book by Pazy [15].

Now, we present the following lemma follows from the results [6, 21] which will be used to establish the required result.

Lemma 2.1 ([15]). *Let us assume that* $\mathbb{M}$ *generates an analytic semigroup* $\mathcal{S}(t),\ t \geq 0$, *and* $0 \in \rho(\mathbb{M})$. *Then,*

(a) $\mathcal{S}(t) : \mathbb{X} \to D(\mathbb{M}^\alpha)\ \forall\ t > 0, \alpha \geq 0$.
(b) $\mathcal{S}(t)\mathbb{M}^\alpha z = \mathbb{M}^\alpha \mathcal{S}(t)z$ *for each* $z \in D(\mathbb{M}^\alpha)$.
(c) *For each* $t > 0$,

$$\| \frac{d^j}{dt^j}\mathcal{S}(t)\| \leq \mathbb{N}_j, \;\; j = 1, 2, \tag{8}$$

where $\mathbb{N}_j,\ j = 1, 2$ *are some positive constants.*

(*d*) *The operator* $\mathbb{M}^{\alpha}\mathcal{S}(t)$ *is bounded and* $\|\mathbb{M}^{\alpha}\mathcal{S}(t)\| \leq \mathbb{N}_{\alpha}t^{-\alpha}e^{-\delta t}$ *for each* $t > 0$.

(*e*) *For each* $\alpha \in (0, 1]$ *and* $z \in D(\mathbb{M}^{\alpha})$, *then* $\|\mathcal{S}(t)z - z\| \leq C_{\alpha}t^{\alpha}\|\mathbb{M}^{\alpha}z\|$.

Remark 2.1 ([5]). The operator $(\mathbb{M})^{-\alpha}$ is a linear bounded operator in $\mathbb{X}$ such that $D(\mathbb{M}^{\alpha}) = Im(\mathbb{M}^{-\alpha})$.

We denote by $\mathcal{C}^{\alpha}_{T_0} = C([0, T_0], \mathbb{X}_{\alpha})$ Banach space of all continuous function $z : [0, T_0] \to \mathbb{X}_{\alpha}$ endowed with the following norm

$$\|z\|_{\mathcal{C}^{\alpha}_{T_0}} = \sup_{0 \leq t \leq T_0} \|z(t)\|_{\alpha} \;\; \forall \;\; z \in \mathcal{C}^{\alpha}_{T_0}.$$

Now, we consider the space

$$\mathcal{C}^{\alpha-1}_{T_0} = \{z \in \mathcal{C}^{\alpha}_{T_0} : \; \|z(\tau_1) - z(\tau_2)\| \leq \mathcal{L}|\tau_1 - \tau_2|, \;\; \forall \;\; \tau_1, \tau_2 \in [0, T_0]\}, \tag{9}$$

which is Banach space with norm $\| \cdot \|_{\mathcal{C}^{\alpha}_{T_0}}$.

According to Definition 2.4 in [5], we provide the definition of mild solution to system (1)–(2).

Definition 2.4. A function $z \in C([0, T_0], \mathbb{X})$ is called a mild solution of system (1)–(2) if $z(0) = z_0 + h(z)$ and following integral equation

$$\begin{aligned} z(t) &= \mathbb{U}_{\beta}(t)[\mathbb{E}\mathbb{B}][z_0 + h(z)] \\ &\quad + \int_0^t (t-\vartheta)^{\beta-1}\mathbb{V}_{\beta}(t-\vartheta)\mathbb{H}(\vartheta, z(\vartheta), z(d_1(\vartheta, z(\vartheta))))d\vartheta, \;\; t \in [0, T_0], \end{aligned} \tag{10}$$

is satisfied by $z(\cdot)$, where

$$\begin{aligned} \mathbb{U}_{\beta}(t) &= \int_0^{\infty} \mathbb{B}^{-1}\mathbb{E}^{-1}\varphi_{\beta}(\zeta)\mathcal{S}(t^{\beta}\zeta)d\zeta, \\ \mathbb{V}_{\beta}(t) &= \int_0^{\infty} \beta\mathbb{B}^{-1}\mathbb{E}^{-1}\zeta\varphi_{\beta}(\zeta)\mathcal{S}(t^{\beta}\zeta)d\zeta, \\ \varphi_{\beta}(\zeta) &= \frac{1}{\beta}\zeta^{-1-\frac{1}{\beta}}\psi_{\beta}(\zeta^{\frac{-1}{\beta}}) \geq 0, \\ \psi_{\beta}(\zeta) &= \frac{1}{\pi}\sum_{k=1}^{\infty}(-1)^{k-1}\zeta^{-\beta k-1}\frac{\Gamma(k\beta + 1)}{k!}\sin(k\pi\beta), \;\; \zeta \in (0, \infty), \end{aligned}$$

and $\varphi_{\beta}(\zeta)$ denotes probability density function defined on $(0, \infty)$, i.e., $\varphi_{\beta}(\zeta) \geq 0$, $0 < \zeta < \infty$ with $\int_0^{\infty} \varphi_{\beta}(\zeta)d\zeta = 1$.

Remark 2.2 ([21]). For each $\nu \in [0,1]$

$$\int_0^\infty \zeta^\nu \varphi_\beta(\zeta)d\zeta = \int_0^\infty \zeta^{-\beta\nu}\psi_\beta(\zeta)d\zeta = \frac{\Gamma(1+\nu)}{\Gamma(1+\beta\nu)}. \tag{11}$$

Lemma 2.2 ([5]). *Let us assume that $\mathcal{S}(t), t \geq 0$ is a semigroup of uniformly bounded linear operators generated by operator $\mathbb{M}$. Then, the operator $\mathbb{U}_\beta(t)$ and $\mathbb{V}_\beta(t)$, $t \geq 0$, are bounded linear operators such that*

(1) *We have $\|\mathbb{U}_\beta(t)z\| \leq \mathbb{W}_1\mathbb{W}_2\mathbb{N}_0\|z\|$ and $\|\mathbb{V}_\beta(t)z\| \leq \frac{\mathbb{W}_1\mathbb{W}_2\mathbb{N}_0}{\Gamma(\beta)}\|z\|$ for each $z \in \mathbb{X}$.*
(2) *The families $\{\mathbb{U}_\beta(t),\ t \geq 0\}$ and $\{\mathbb{V}_\beta(t),\ t \geq 0\}$ are strongly continuous, i.e., for $0 \leq \tau_1 < \tau_2 \leq T_0$ and $z \in \mathbb{X}$, we have $\|\mathbb{U}_\beta(\tau_2)z - \mathbb{U}_\beta(\tau_1)z\| \to 0$ and $\|\mathbb{V}_\beta(\tau_2)z - \mathbb{V}_\beta(\tau_1)z\| \to 0$ as $\tau_2 \to \tau_1$.*
(3) *The $\mathbb{U}_\beta(t)$ and $\mathbb{V}_\beta(t)$, $t \geq 0$, are both compact operators if $\mathcal{S}(t)$, $t \geq 0$, is compact.*
(4) *For each $z \in \mathbb{X}$, $0 < \eta < 1$, and $0 < \alpha < 1$, we have $\mathbb{M}\mathbb{V}_\beta(t)z = \mathbb{M}^{1-\eta}\mathbb{V}_\beta\mathbb{M}^\eta z$ for $t \in [0, T_0]$. We also have $\|\mathbb{M}^\alpha\mathbb{V}_\beta(t)\| \leq \frac{\beta\mathbb{W}_1\mathbb{W}_2\mathbb{N}_\alpha\Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))}t^{-\alpha\beta}$ for each $t \in (0, T_0]$.*
(5) *For any $z \in \mathbb{X}_\alpha$ and fixed $t \geq 0$, we have $\|\mathbb{U}_\beta(t)z\|_\alpha \leq \mathbb{W}_1\mathbb{W}_2\mathbb{N}_0\|z\|_\alpha$ and $\|\mathbb{V}_\beta(t)z\|_\alpha \leq \frac{\mathbb{W}_1\mathbb{W}_2\mathbb{N}_0}{\Gamma(\beta)}\|z\|_\alpha$.*

Lemma 2.3. *For each $\phi \in L^p([0, T_0], \mathbb{X})$ and $p \in [1, \infty)$,*

$$\lim_{\epsilon \to 0} \int_0^{T_0} \|\phi(\vartheta + \epsilon) - \phi(\vartheta)\|^p d\vartheta = 0, \tag{12}$$

where $\phi(s) = 0$ for $s \notin [0, T_0]$.

3 Main Result

In this segment, the sufficient condition for providing the existence of the α-mild solution for system (1)–(2) is derived. To prove the required result, we have to impose the following assumptions on the data of the system (1)–(2).

$(J1)$ The nonlinear function $\mathbb{H} : [0, T_0] \times \mathbb{X}_\alpha \times \mathbb{X}_{\alpha-1} \to \mathbb{X}$ is a Hölder continuous function, and there exist constants $L_{\mathbb{H}} > 0$ and $\theta_1 \in (0, 1]$ such that

$$\begin{aligned} \| \mathbb{H}(t, z_1, w_1) - \mathbb{H}(s, z_2, w_2)\| \leq L_{\mathbb{H}}(|t-s|^{\theta_1} + \| z_1 - z_2\|_\alpha \\ + \| w_1 - w_2\|_{\alpha-1}), \end{aligned} \tag{13}$$

for each $(t, z_1, w_1), (s, z_2, w_2) \in [0, T_0] \times \mathbb{X}_\alpha \times \mathbb{X}_{\alpha-1}$.

$(J2)$ The functions $b_i : [0, \infty) \times \mathbb{X}_{\alpha-1} \to [0, \infty)$, $(i = 1, \cdots, m)$ are continuous functions, and there are positive constants L_{b_i} and $0 < \theta_2 \leq 1$ such that

$$|b_i(t, z_1) - b_i(s, z_2)| \leq L_{b_i}(|t - s|^{\theta_2} + \| z_1 - z_2\|_{\alpha-1}), \tag{14}$$

for all $(t, z_1),\ (s, z_2) \in [0, T_0] \times \mathbb{X}_{\alpha-1}$.

$(J3)$ $h \in C(\mathbb{X}_\alpha, \mathbb{X}_\alpha)$ is a nonlinear function, and there exists positive constant L_h such that

$$\|h(w_1) - h(w_2)\|_\alpha \leq L_h\|w_1 - w_2\|_\alpha, \tag{15}$$

for each $w_1, w_2 \in \mathbb{X}_\alpha$.

Now, we consider the following space

$$\mathcal{S}^\alpha = \{z \in \mathcal{C}^\alpha_{T_0} \cap \mathcal{C}^{\alpha-1}_{T_0} : \|z\|_\alpha \leq R\}, \tag{16}$$

where $R > 0$ is a constant to be defined later. It is clear that $\mathcal{S}^\alpha$ is a closed and bounded subset of $\mathcal{C}^{\alpha-1}_{T_0}$ which is complete.

Theorem 3.1. *If the assumptions* $(J1)$–$(J3)$ *are fulfilled and* $z_0 \in D(\mathbb{M}^\alpha)$ *with*

$$K^* = \mathbb{W}_1\mathbb{W}_2\mathbb{N}_0\|\mathbb{E}\mathbb{B}\|L_h + \frac{\mathbb{W}_1\mathbb{W}_2\mathbb{N}_\alpha\Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))}L_F(2+\mathcal{L}L_b)\frac{{T_0}^{\beta(1-\alpha)}}{\beta(1-\alpha)} < 1, \tag{17}$$

then the system (1)–(2) admits at least one α-mild solution on $[0, T_0]$.

Proof. Firstly, we consider the map $\Upsilon : \mathcal{S}^\alpha \to \mathcal{S}^\alpha$ defined by

$$\begin{aligned}(\Upsilon z)(t) &= \mathbb{U}_\beta(t)[\mathbb{E}\mathbb{B}](z_0 + h(z)) \\ &\quad + \int_0^t (t - \vartheta)^{\beta-1}\mathbb{V}_\beta(t - \vartheta)\mathbb{H}(\vartheta, z(\vartheta), z(d_1(\vartheta, z(\vartheta))))ds,\ t \in [0, T_0].\end{aligned} \tag{18}$$

Clearly, it is easy to show that $\Upsilon : \mathcal{C}^\alpha_{T_0} \to \mathcal{C}^\alpha_{T_0}$ by using the fact that $\mathbb{H}$ and b_i are continuous functions. Now, it remains to show that $\Upsilon z \in \mathcal{C}^{\alpha-1}_{T_0}$. To this end, let $\tau_1, \tau_2 \in [0, T_0]$ with $\tau_1 < \tau_2$. Then, we get

$$\begin{aligned}\|(\Upsilon z)(\tau_2) - (\Upsilon z)(\tau_1)\|_{\alpha-1} &\leq \|(\mathbb{U}_\beta(\tau_2) - \mathbb{U}_\beta(\tau_1))[\mathbb{E}\mathbb{B}](z_0 + h(z))\|_{\alpha-1} \\ &\quad + \| \int_0^{\tau_2} (\tau_2 - \vartheta)^{\beta-1}\mathbb{V}_\beta(\tau_2 - \vartheta)\mathbb{H}(\vartheta, z(\vartheta), z(d_1(\vartheta, z(\vartheta))))d\vartheta \\ &\quad - \int_0^{\tau_1} (\tau_1 - \vartheta)^{\beta-1}\mathbb{V}_\beta(\tau_1 - \vartheta)\mathbb{H}(\vartheta, z(\vartheta), z(d_1(\vartheta, z(\vartheta))))d\vartheta\,\|_{\alpha-1}, \\ &\leq \|(\mathbb{U}_\beta(\tau_2) - \mathbb{U}_\beta(\tau_1))[\mathbb{E}\mathbb{B}](z_0 + h(z))\|_{\alpha-1} \\ &\quad + \| \int_{\tau_1}^{\tau_2} (\tau_2 - \vartheta)^{\beta-1}\mathbb{V}_\beta(\tau_2 - \vartheta)\mathbb{H}(\vartheta, z(\vartheta), z(d_1(\vartheta, z(\vartheta))))d\vartheta\,\|_{\alpha-1}\end{aligned}$$

$$+ \| \int_0^{\tau_1} (\tau_2 - \vartheta)^{\beta-1} \mathbb{V}_\beta(\tau_2 - \vartheta)\mathbb{H}(\vartheta, z(\vartheta), z(d_1(\vartheta, z(\vartheta))))d\vartheta$$
$$- \int_0^{\tau_1} (\tau_1 - \vartheta)^{\beta-1} \mathbb{V}_\beta(\tau_1 - \vartheta)\mathbb{H}(\vartheta, z(\vartheta), z(d_1(\vartheta, z(\vartheta))))d\vartheta \|_{\alpha-1}. \tag{19}$$

From the first term of the above inequality, we have

$$[\mathbb{U}_\beta(\tau_2) - \mathbb{U}_\beta(\tau_1)]\mathbb{M}^{\alpha-1}[\mathbb{E}\mathbb{B}](z_0 + h(z))$$
$$= \int_0^\infty \mathbb{B}^{-1}\mathbb{E}^{-1}\varphi_\beta(\zeta)[\mathcal{S}(\tau_2^\beta \zeta) - \mathcal{S}(\tau_1^\beta \zeta)]\mathbb{M}^{\alpha-1}[\mathbb{E}\mathbb{B}](z_0 + h(z))d\zeta.$$

Also, we have that for each $z \in \mathbb{X}$

$$[\mathcal{S}(\tau_2^\beta \zeta) - \mathcal{S}(\tau_1^\beta \zeta)]z = \int_{\tau_1}^{\tau_2} \frac{d}{ds}\mathcal{S}(s^\beta \zeta)zds = \int_{\tau_1}^{\tau_2} \beta\zeta s^{\beta-1}\mathbb{M}\mathcal{S}(s^\beta \zeta)zds.$$

Therefore, we estimate the first term as

$$\int_0^\infty \mathbb{B}^{-1}\mathbb{E}^{-1}\varphi_\beta(\zeta)\|\mathcal{S}(\tau_2^\beta \zeta) - \mathcal{S}(\tau_1^\beta \zeta)\|\| \mathbb{M}^{\alpha-1}[\mathbb{E}\mathbb{B}](z_0 + h(z))\|d\zeta$$
$$\leq \int_0^\infty \varphi_\beta(\zeta)\|\mathbb{B}^{-1}\mathbb{E}^{-1}\|[\int_{\tau_1}^{\tau_2} \|\frac{d}{ds}\mathcal{S}(s^\beta \zeta)\|ds]\|[\mathbb{E}\mathbb{B}]\|\| z_0 + h(z)\|_{\alpha-1}d\zeta,$$
$$\leq \int_0^\infty \varphi_\beta(\zeta)\mathbb{W}_1\mathbb{W}_2[\mathbb{N}_1(\tau_2 - \tau_1)]\|[\mathbb{E}\mathbb{B}]\| \times \|z_0 + h(z)\|_{\alpha-1}d\zeta,$$
$$\leq K_1(\tau_2 - \tau_1)\int_0^\infty \varphi_\beta(\zeta)d\zeta,$$
$$= K_1(\tau_2 - \tau_1), \tag{20}$$

where $K_1 = \mathbb{W}_1\mathbb{W}_2\mathbb{N}_1\|[\mathbb{E}\mathbb{B}]\|\|z_0 + h(z)\|_{\alpha-1}$. The second term can be estimated

$$\int_0^{\tau_1} \| (\tau_1 - \vartheta)^{\beta-1}\mathbb{V}_\beta(\tau_1 - \vartheta) - (\tau_2 - \vartheta)^{\beta-1}\mathbb{V}_\beta(\tau_2 - \vartheta)\|_{\alpha-1}\|$$
$$\times \mathbb{H}(\vartheta, z(\vartheta), z(d_1(\vartheta, z(\vartheta))))\|d\vartheta,$$
$$\leq \|\mathbb{B}^{-1}\mathbb{E}^{-1}\| \int_0^{\tau_1} \int_0^\infty \varphi_\beta(\zeta)\| [\frac{d}{d\varsigma}\mathcal{S}((\varsigma - \vartheta)^\beta \zeta)|_{\varsigma=\tau_2}$$
$$-\frac{d}{d\varsigma}\mathcal{S}((\varsigma - \vartheta)^\beta \zeta)|_{\varsigma=\tau_1}]\mathbb{M}^{\alpha-2}\|$$
$$\times \|\mathbb{H}(\vartheta, z(\vartheta), z(d_1(\vartheta, z(\vartheta))))\|d\zeta d\vartheta,$$

$$\leq \mathbb{W}_1\mathbb{W}_2 \int_0^{\tau_1}\int_0^{\infty} \varphi_\beta(\zeta)[\int_{\tau_1}^{\tau_2} \| \mathbb{M}^{\alpha-2}\frac{d^2}{d\varsigma^2}\mathcal{S}((\varsigma-\vartheta)^\beta\zeta)\|d\varsigma]N_{\mathbb{H}}d\zeta d\vartheta,$$

$$\leq \mathbb{W}_1\mathbb{W}_2 \int_0^{\tau_1}\int_0^{\infty} \varphi_\beta(\zeta)\left[\| \mathbb{M}^{\alpha-2}\|\mathbb{N}_2(\tau_2-\tau_1)\right]N_{\mathbb{H}}d\zeta d\vartheta,$$

$$\leq K_2(\tau_2-\tau_1), \tag{21}$$

where $K_2 = \mathbb{W}_1\mathbb{W}_2\| \mathbb{M}^{\alpha-2}\|\mathbb{N}_2 N_{\mathbb{H}}T$. The third integral is estimated as

$$\int_{\tau_1}^{\tau_2} \|(\tau_2-\vartheta)^{\beta-1}\mathbb{V}_\beta(\tau_2-\vartheta)\|_{\alpha-1}\|\mathbb{H}(\vartheta, z(\vartheta), z(d_1(\vartheta, z(\vartheta))))\|d\vartheta$$

$$\leq \int_{\tau_1}^{\tau_2}\int_0^{\infty} \varphi_\beta(\zeta)\|\mathbb{B}^{-1}\mathbb{E}^{-1}\|\| \,[\beta(\tau_2-\vartheta)^{\beta-1}\zeta\mathbb{M}\mathcal{S}((\tau_2-\vartheta)^\beta\zeta)]\mathbb{M}^{\alpha-2}\|$$
$$\times\| \mathbb{H}(\vartheta, z(\vartheta), z(d_1(\vartheta, z(\vartheta))))\|d\zeta d\vartheta,$$

$$\leq \mathbb{W}_1\mathbb{W}_2 \int_{\tau_1}^{\tau_2}\int_0^{\infty} \varphi_\beta(\zeta)\| \frac{d}{d\varsigma}\mathcal{S}((\varsigma-\vartheta)^\beta\zeta)|_{\varsigma=\tau_2}\mathbb{M}^{\alpha-2}\|N_{\mathbb{H}}d\zeta d\vartheta,$$

$$\leq K_3(\tau_2-\tau_1), \tag{22}$$

where $K_3 = \mathbb{N}_1\mathbb{W}_1\mathbb{W}_2\| \mathbb{M}^{\alpha-2}\|N_{\mathbb{H}}$.

Thus, from the inequality (19) to (22), we obtain that

$$\| (\Upsilon y)(\tau_2) - (\Upsilon y)(\tau_1)\|_{\alpha-1} \leq \mathcal{L}(\tau_2-\tau_1), \tag{23}$$

for a positive suitable constant $\mathcal{L} = \sum_{l=1}^{3} K_l$. Therefore, we conclude that $(\Upsilon y) \in \mathcal{C}_{T_0}^{\alpha-1}$. Hence, we deduce that the operator $\Upsilon : \mathcal{C}_{T_0}^{\alpha-1} \to \mathcal{C}_{T_0}^{\alpha-1}$ is a well-defined map.

Next, we prove that $\Upsilon : \mathcal{S}^\alpha \to \mathcal{S}^\alpha$. For $0 \leq t \leq T_0$ and $z \in \mathcal{S}^\alpha$, we get that $\| (\Upsilon z)(t)\|_\alpha$

$$\leq \| \mathbb{U}_\beta(t)[\mathbb{E}\mathbb{B}](z_0 + h(z))\|_\alpha$$
$$+ \int_0^t \| (t-\vartheta)^{\beta-1}\mathbb{V}_\beta(t-\vartheta)\mathbb{H}(\vartheta, z(\vartheta), z(d_1(\vartheta, z(\vartheta))))\|_\alpha d\vartheta,$$

$$\leq \mathbb{W}_1\mathbb{W}_2\mathbb{N}_0\|\mathbb{E}\mathbb{B}\|\cdot\|y_0 + h(y)\|_\alpha + \frac{\mathbb{W}_1\mathbb{W}_2\mathbb{N}_\alpha N_{\mathbb{H}}\Gamma(2-\alpha){T_0}^{\beta(1-\alpha)}}{(1-\alpha)\Gamma(1+\beta(1-\alpha))}. \tag{24}$$

We choose $R = \mathbb{W}_1\mathbb{W}_2\mathbb{N}_0\|\mathbb{E}\mathbb{B}\|\cdot\|z_0 + h(z)\|_\alpha + \frac{\mathbb{W}_1\mathbb{W}_2\mathbb{N}_\alpha N_{\mathbb{H}}\Gamma(2-\alpha){T_0}^{\beta(1-\alpha)}}{(1-\alpha)\Gamma(1+\beta(1-\alpha))}$ such that

$$\| (\Upsilon y)\|_{\mathcal{C}_{T_0}^{\alpha}} \leq R.$$

Therefore, we conclude that $\Upsilon(\mathcal{S}^\alpha) \subset \mathcal{S}^\alpha$. Next, we will show that Υ is a contraction mapping. For $y,\ z \in \mathcal{S}^\alpha$, and $0 \leq t \leq T_0$, we get that $\|(\Upsilon y)(t) - (\Upsilon z)(t)\|_\alpha$

$$\leq \|\mathbb{U}_\beta(t)[\mathbb{E}\mathbb{B}][h(y)-h(z)]\|_\alpha + \|\int_0^t (t-\vartheta)^{\beta-1}\mathbb{V}_\beta(t-\vartheta)[\mathbb{H}(\vartheta, y(\vartheta), y(d_1(\vartheta, y(\vartheta))))$$
$$-\mathbb{H}(\vartheta, z(\vartheta), z(d_1(\vartheta, z(\vartheta))))]d\vartheta\|_\alpha,$$
$$\leq \mathbb{W}_1\mathbb{W}_2\mathbb{N}_0\|\mathbb{E}\mathbb{B}\|L_h\|y-z\|_\alpha + \frac{\beta\mathbb{W}_1\mathbb{W}_2\mathbb{N}_\alpha\Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))}\int_0^t (t-\vartheta)^{\beta(1-\alpha)-1}$$
$$\times\|\mathbb{H}(\vartheta, y(\vartheta), y(d_1(\vartheta, y(\vartheta)))) - \mathbb{H}(\vartheta, z(\vartheta), z(d_1(\vartheta, z(\vartheta))))\|d\vartheta. \tag{25}$$

Now, we estimate

$$\|\ \mathbb{H}(\vartheta, y(\vartheta), y(d_1(\vartheta, y(\vartheta)))) - \mathbb{H}(\vartheta, z(\vartheta), z(d_1(\vartheta, z(\vartheta))))\|$$
$$\leq L_{\mathbb{H}}[\|y(\vartheta)-z(\vartheta)\|_\alpha + \|y(d_1(\vartheta, y(\vartheta))) - z(d_1(\vartheta, z(\vartheta)))\|_{\alpha-1}]$$
$$\leq L_{\mathbb{H}}[\|y(\vartheta)-z(\vartheta)\|_\alpha + \|\mathbb{M}^{-1}\|\cdot\|y(d_1(\vartheta, z(\vartheta))) - z(d_1(\vartheta, z(\vartheta)))\|_\alpha$$
$$+\|y(d_1(\tau, y(\tau))) - y(d_1(\tau, z(\tau)))\|_{\alpha-1}]. \tag{26}$$

Let

$$d_j(\vartheta, z(\vartheta)) = b_j(\vartheta, z(b_{j+1}(\vartheta, \cdots, z(\vartheta, b_m(\vartheta, z(\vartheta)))\cdots))),\ j=1,2,\cdots,m,\ z\in\mathcal{S}^\alpha,$$

with $d_{m+1}(\vartheta, z(\vartheta)) = \vartheta$ [20, p. 2183]. Thus, we obtain

$$\begin{aligned}|d_1(\vartheta, y(\vartheta)) - d_1(\vartheta, z(\vartheta))| &= |b_1(\vartheta, y(d_2(\vartheta, y(\vartheta)))) - b_1(\vartheta, z(d_2(\vartheta, z(\vartheta))))|,\\
&\leq L_{b_1}\|y(d_2(\vartheta, y(\vartheta))) - z(d_2(\vartheta, z(\vartheta)))\|_{\alpha-1},\\
&\leq L_{b_1}[\|y(d_2(\vartheta, y(\vartheta))) - y(d_2(\vartheta, z(\vartheta)))\|_{\alpha-1}\\
&\quad +\|y(d_2(\vartheta, z(\vartheta))) - z(d_2(\vartheta, z(\vartheta)))\|_{\alpha-1}],\\
&\leq L_{b_1}[\mathcal{L}|b_2(\vartheta, y(d_3(\vartheta, y(\vartheta))) - b_2(\vartheta, z(d_3(\vartheta, z(\vartheta)))|\\
&\quad +\|\mathbb{M}\|^{-1}\|y-z\|_{\mathcal{C}^\alpha_{T_0}}],\\
&\quad \cdots\\
&\leq [L^{m-1}L_{b_1}\cdots L_{b_m} + L^{m-2}L_{b_1}\cdots L_{b_{m-1}} + \cdots + \mathcal{L}L_{b_1}L_{b_2}\\
&\quad +L_{b_1}]\|\mathbb{M}\|^{-1}\|y-z\|_{\mathcal{C}^\alpha_{T_0}}.\end{aligned} \tag{27}$$

Therefore, we get

$$\|\ \mathbb{H}(\vartheta, y(\vartheta), y(d_1(\vartheta, y(\vartheta)))) - \mathbb{H}(\vartheta, z(\vartheta), z(d_1(\vartheta, z(\vartheta))))\|$$
$$\leq L_{\mathbb{H}}(2+\mathcal{L}L_b\|\mathbb{M}\|^{-1})\|y-z\|_{\mathcal{C}^\alpha_{T_0}},$$
$$\leq L_{\mathbb{H}}(2+\mathcal{L}L_b)\|y-z\|_{\mathcal{C}^\alpha_{T_0}}, \tag{28}$$

where $L_b = [L^{m-1}L_{b_1}\cdots L_{b_m} + L^{m-2}L_{b_1}\cdots L_{b_{m-1}} + \cdots + \mathcal{L}L_{b_1}L_{b_2} + L_{b_1}] > 0$.

Thus, from the inequalities (25) and (28), we obtain $\|(\Upsilon y)(t) - (\Upsilon z)(t)\|_\alpha$

$$\leq [\mathbb{W}_1\mathbb{W}_2\mathbb{N}_0\|\mathbb{E}\mathbb{B}\|L_h + \frac{\beta\mathbb{W}_1\mathbb{W}_2\mathbb{N}_\alpha\Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))}L_{\mathbb{H}}(2+\mathcal{L}L_b)\frac{T^{\beta(1-\alpha)}}{\beta(1-\alpha)}]\|y-z\|_{\mathcal{C}^\alpha_{T_0}},$$

$$= K^*\|y-z\|_{\mathcal{C}^\alpha_{T_0}}. \tag{29}$$

Taking supremum of t over $[0, T_0]$ and getting

$$\|(\Upsilon y) - (\Upsilon z)\|_{\mathcal{C}^\alpha_{T_0}} \leq K^*\|y-z\|_{\mathcal{C}^\alpha_{T_0}}. \tag{30}$$

Since $K^* = \mathbb{W}_1\mathbb{W}_2\mathbb{N}_0\|\mathbb{E}\mathbb{B}\|L_h + \frac{\mathbb{W}_1\mathbb{W}_2\mathbb{N}_\alpha\Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))}L_{\mathbb{H}}(2+\mathcal{L}L_b)\frac{T^{\beta(1-\alpha)}}{(1-\alpha)} < 1$. It implies that Υ is a contraction mapping on $\mathcal{S}^\alpha$ with constant $K^* < 1$. Therefore, there exists a fixed point of the mapping Υ by Banach fixed-point theorem which is just a mild solution for the problem (1)–(2). □

4 Example

We consider the following nonlocal differential problem

$$^cD_t^\beta[w_{uu}(t,u) - w_{uuuu}(t,u)] + \frac{\partial^2 w(t,u)}{\partial u^2} = \widetilde{H}(u, w(t,u)) + \widetilde{G}(t,u,w(t,u)),$$

$$u \in \mathbb{S},\ t \in [0,T], \tag{31}$$

$$w(0,u) = \sum_{s=1}^{n} C_s w(t_s, u), \quad x \in [0,\pi], \tag{32}$$

$$w(t,0) = w(t,\pi) = 0, \quad 0 < t \leq 1, \tag{33}$$

where $^cD_t^\beta$ denotes the fractional derivative in Caputo sense of order $\beta \in (0,1]$, $C_s > 0$ are constants for $s = 1, \cdots, n$.

Take $\mathbb{X} = \mathbb{Y} = \mathbb{Z} = L^2[0,\pi]$ and $\mathbb{S} = [0,\pi]$. Let us consider the operator $\mathbb{E}, \mathbb{B}, \mathbb{L}$ on domains and ranges which is contained in $L^2[0,\pi]$ defined by

$$\mathbb{E}w = w'', \quad \mathbb{B}w = w - w''\ (\mathbb{E}\mathbb{B}w = w'' - w''''), \quad \mathbb{L}w = -w'', \tag{34}$$

and domains $D(\mathbb{E}),\ D(\mathbb{B}), D(\mathbb{L})$ which are given by

$$\{w \in \mathbb{X} : w, w',\ w'',\ w''''\ \text{are absolutely continuous},\ w'''' \in \mathbb{X},\ w(0) = w(\pi) = 0\}. \tag{35}$$

Thus, the operators have the following expression:

$$\mathbb{E}w = \sum_{m=1}^{\infty} m^2(w, w_m)w_m, \quad \mathbb{B}w = \sum_{m=1}^{\infty}(1+m^2)(w, w_m)w_m, \tag{36}$$

and $\mathbb{L}w = \sum_{m=1}^{\infty}(-m^2)(w, w_m)w_m$ with $w_m(t) = (\sqrt{2/\pi})\sin(mt), \quad m = 1, \cdots$, as the orthogonal set of eigenfunctions of $\mathbb{L}$. Moreover, we have

$$\mathbb{B}^{-1}\mathbb{E}^{-1}w = \sum_{m=1}^{\infty} \frac{1}{m^2(1+m^2)}(w, w_m)w_m, \tag{37}$$

$$\mathbb{L}\mathbb{B}^{-1}\mathbb{E}^{-1}w = \sum_{m=1}^{\infty} \frac{-1}{1+m^2}(w, w_m)w_m, \tag{38}$$

$$\mathcal{S}(t)z = \sum_{m=1}^{\infty} \exp(\frac{-At}{1+m^2})(z, w_m)w_m. \tag{39}$$

Clearly, the operator $\mathbb{B}^{-1}\mathbb{E}^{-1}$ is bounded and compact such that $\|\mathbb{B}^{-1}\mathbb{E}^{-1}\| \le 1$. It is also well known that $\mathbb{M} = \mathbb{L}\mathbb{B}^{-1}\mathbb{E}^{-1}$ generates a strongly continuous semigroup $\mathcal{S}(t)$ on $L^2[0, \pi]$ with $\|\mathcal{S}(t)\| \le e^{-t} \le 1$.

Let $w(t) = w(t, \cdot)$, $h(w) = \sum_{s=1}^{n} C_s w(t_s, u)$. Now, we define the function $\mathbb{H} : \mathbb{R}_+ \times \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ as

$$\mathbb{H}(t, \vartheta, \zeta)(u) = \widetilde{H}(u, \zeta) + \widetilde{G}(t, u, \vartheta), \text{ for } u \in (0, \pi), \tag{40}$$

where $\widetilde{H} : [0, 1] \times \mathbb{X} \to \mathbb{X}$ is defined as

$$\widetilde{H}(u, \zeta) = \int_0^u \mathcal{K}(u, y)\zeta(y)dy, \tag{41}$$

and $\widetilde{G} : \mathbb{R}_+ \times [0, 1] \times \mathbb{X} \to \mathbb{X}$ satisfies following condition

$$\|\widetilde{G}(t, u, \vartheta)\| \le W(u, t)(1 + \|\vartheta\|_{1/2}), \tag{42}$$

where Q is continuous in t and $Q(\cdot, t) \in \mathbb{X}$. Now, from the definition of $\mathbb{H}$ and h, it can be easily shown that $\mathbb{H}$ and h satisfy the assumption $(J1)$–$(J3)$. Applying the result of Theorem 3.1, we can get that the system (31)–(33) admits a unique mild solution on $[0, T]$.

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On Source Identification Problem for Telegraph Differential Equations

Allaberen Ashyralyev and Fatma Çekiç

Abstract In the present paper, the source identification problem for a telegraph equation with unknown parameter p

$$\begin{cases} \frac{d^2u(t)}{dt^2} + \alpha\frac{du(t)}{dt} + Au(t) = p + f(t)\ (0 \le t \le T), \\ u(0) = \varphi, u'(0) = \psi, u(T) = \xi \end{cases} \tag{1}$$

in a Hilbert space H with the self-adjoint positive definite operator A is investigated. Operator approach permitted us to establish stability estimates for the solution of the problem (1). In applications, three source identification problems for telegraph equations are investigated.

Keywords Inverse problem • Telegraph equation • Stability

AMS subject classifications: 35R30, 35L20, 35B35

1 Introduction

The differential equations with parameters play a very important role in many branches of science and engineering. Some examples were given in temperature over-specification by Dehghan [12], robotic chemistry (chromatography) by Kimura and Suzuki [19], and physics (optical tomography) by Gryazin, Klibanov, and Lucas [18].

The source identification problem for partial differential equations has been studied extensively by many researchers (see [1–11, 13–16, 20–22, 24, 26–28] and the references therein). However, such problems were not well investigated in general.

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Our goal in this paper is to investigate telegraph equations with parameter. It is known that various boundary value problems for telegraph equations with parameter can be reduced to source identification problem for the differential equation (1) in a Hilbert space H with self-adjoint positive definite operator A and $A \geq \delta I$. Here $\delta > 0, \alpha > 0$, and

$$\delta > \frac{\alpha^2}{4}. \tag{2}$$

The pair $\{u(t), p\}$ is called a solution of problem (1) if the following conditions are satisfied:

(i) $u(t)$ is twice continuously differentiable function on $[0, T]$. The derivatives at the endpoints of the segment are understood as the appropriate unilateral derivatives.
(ii) The element $u(t)$ belongs to $D(A)$ for all $t \in [0, T]$, and the function $Au(t)$ is continuous on $[0, T]$.
(iii) $u(t)$ satisfies the equation and boundary conditions (1), $p \in H$.

It is clear that for finding a solution $u(t)$ of problem (1), it is useful to apply the substitution

$$u(t) = v(t) + A^{-1}p, \tag{3}$$

where $v(t)$ is the solution of the following nonlocal boundary value problem for the differential equation

$$\begin{cases} \frac{d^2v(t)}{dt^2} + \alpha\frac{dv(t)}{dt} + Av(t) = f(t)\ (0 \leq t \leq T), \\ v(T) = v(0) + \xi - \varphi, v_t(0) = \psi \end{cases} \tag{4}$$

and p is the unknown element defined by formula

$$p = A\left(\xi - v(T)\right). \tag{5}$$

The present paper is organized as follows. Section 1 is introduction. In Sect. 2, the main theorem on stability of problem (1) is established. In applications, theorems on the stability inequalities for the solution of three source identification problems for the telegraph equations are established.

2 The Main Theorem

Let H be a Hilbert space, A be a positive definite self-adjoint operator with $A \geq \delta I$, where $\delta > 0$. Let $\alpha > 0$ and

$$\delta > \frac{\alpha^2}{4}. \tag{6}$$

Throughout this paper, $\{c(t), t \geq 0\}$ is a strongly continuous cosine operator function defined by the formula

$$c\,(t) = \frac{e^{itB^{1/2}} + e^{-itB^{1/2}}}{2}.$$

Then, from the definition of the sine operator function $s\,(t)$

$$s(t)u = \int_0^t c(s)u\,ds,$$

it follows that

$$s\,(t) = B^{-1/2}\frac{e^{itB^{1/2}} - e^{-itB^{1/2}}}{2i}.$$

Here $B = A - \frac{\alpha^2}{4}I$. For the theory of cosine operator function, we refer to [17] and [23].

Now, let us give some lemmas that will be needed below.

Lemma 1. *The estimates hold:*

$$\|c(t)\|_{H\to H} \leq 1,\ \left\|B^{1/2}s(t)\right\|_{H\to H} \leq 1,\ \left\|B^{-1/2}\right\|_{H\to H} \leq \frac{1}{\sqrt{\delta - \frac{\alpha^2}{4}}}. \tag{7}$$

Lemma 2. *Assume that*

$$1 > \left(1 + \frac{\frac{\alpha}{2}}{\sqrt{\delta - \frac{\alpha^2}{4}}}\right) e^{-\frac{\alpha}{2}T}. \tag{8}$$

Then, the operator

$$I - \left(c(T) + \frac{\alpha}{2}s(T)\right) e^{-\frac{\alpha}{2}T}$$

has inverse

$$P = \left\{I - \left(c(T) + \frac{\alpha}{2}s(T)\right) e^{-\frac{\alpha}{2}T}\right\}^{-1},$$

and the following estimate

$$||P||_{H\to H} \leq M \tag{9}$$

holds, where $M = M(\delta, \alpha) > 0$.

Firstly, the solvability of problem (1) in the space $C(H)$ of the continuous H-valued functions $\varphi(t)$ defined on $[0,T]$, equipped with the norm

$$\|u\|_{C(H)} = \max_{0\leq t\leq T} ||\varphi(t)||_H$$

is investigated. We will prove the following main theorem on continuous dependence of the solution on the given data.

Theorem 1. *Suppose that $\varphi, \xi \in D(A)$ and $\psi \in D(A^{\frac{1}{2}})$. Let conditions (6) and (8) be satisfied and $f(t)$ be continuously differentiable function on $[0,T]$. Then, for the solution $(u(t), p)$ of problem (1) in $C(H)\times H$ the following stability inequalities*

$$c\ \|u\|_{C(H)} + \|\ A^{-1}p\ \|_H \leq M(\delta,\alpha)\left[\|\varphi\|_H + \|\xi\|_H + \left\|A^{-\frac{1}{2}}\psi\right\|_H + \|f\|_{C(H)}\right],$$

$$\left\|\tfrac{d^2u}{dt^2}\right\|_{C(H)} + \|Au\|_{C(H)} + \|p\|_H \leq M(\delta,\alpha)\left[\left\|A\varphi\right\|_H\right.$$

$$\left. + \|\varphi\|_H + \left\|A^{\frac{1}{2}}\psi\right\|_H + \|A\xi\|_H + \|\xi\|_H + \max_{0\leq t\leq T}\|f'(t)\|_H + \|f(0)\|_H\right]$$

hold, where $M(\delta,\alpha)$ is independent of $f(t), t\in[0,T]$ and φ,ψ,ξ.

Proof of Theorem 1 is based on formulas (3) and (5) and the following theorem on well posedness of nonlocal boundary value problem (4).

Theorem 2. *Suppose that the assumptions of Theorem 1 hold. Then, for the solution $v(t)$ of problem (4) in $C(H)$ the stability estimates*

$$\|v\|_{C(H)} \leq M(\delta,\alpha)\left[\|\varphi\|_H + \|\psi\|_H + \|\xi\|_H + \|f\|_{C(H)}\right], \tag{10}$$

$$\left\|\frac{d^2v}{dt^2}\right\|_{C(H)} + \|Av\|_{C(H)} \tag{11}$$

$$\leq M(\delta,\alpha)\Big[\ \|A\varphi\|_H + \|\varphi\|_H + \left\|A^{\frac{1}{2}}\psi\right\|_H + \|A\xi\|_H + \|\xi\|_H$$

$$+ \max_{0\leq t\leq T}\left\|f'(t)\right\|_H + \|f(0)\|_H\ \Big]$$

hold, where $M(\delta,\alpha)$ does not depend on $f(t), t\in[0,T]$ and φ,ψ,ξ.

Proof. First, we obtain the formula for solution of problem (4) under the assumption (6). We have the following formula

$$v(t) = e^{-\frac{\alpha}{2}t}c\,(t)\,v(0) + \frac{\alpha}{2}e^{-\frac{\alpha}{2}t}s(t)v(0) + e^{-\frac{\alpha}{2}t}s(t)\psi$$

$$+ \int_0^t e^{-\frac{\alpha}{2}(t-z)}s(t-z)f(z)dz \tag{12}$$

for the mild solution of initial value problem

$$\begin{cases} \frac{d^2v(t)}{dt^2} + \alpha\frac{dv(t)}{dt} + Av(t) = f(t) \ (0 \le t \le T), \\ v(0) \text{isgiven}, v'(0) = \psi. \end{cases}$$

Applying condition $v(T) = v(0) + \xi - \varphi$, and formula (12), we get

$$v(0) = \left(c(T) + \frac{\alpha}{2}s(T)\right)e^{-\frac{\alpha}{2}T}v(0) + e^{-\frac{\alpha}{2}T}s(T)\psi \\ + \int_0^T e^{-\frac{\alpha}{2}(T-z)}s(T-z)f(z)dz + \varphi - \xi. \tag{13}$$

By Lemma 2, under the assumption (8), there exists of inverse

$$P = \left\{I - \left(c(T) + \frac{\alpha}{2}s(T)\right)e^{-\frac{\alpha}{2}T}\right\}^{-1}.$$

Therefore, using (13), we obtain

$$v(0) = P\left\{e^{-\frac{\alpha}{2}T}s(T)\psi + \int_0^T e^{-\frac{\alpha}{2}(T-z)}s(T-z)f(z)dz + \varphi - \xi\right\}. \tag{14}$$

Consequently, the solutions of problem (4) satisfy formulas (12) and (14).

Second, we obtain estimate (10). Using formulas (14) and (12) and estimate (7), we obtain

$$\|v(0)\|_H \le M_1(\delta,\alpha)\left[\|\varphi\|_H + \left\|A^{-\frac{1}{2}}\psi\right\|_H + \|\xi\|_H + \max_{0\le t\le T}\|f(t)\|_H\right],$$

$$\max_{0\le t\le T}\|v(t)\|_H \le M_2(\delta,\alpha)\left[\|v(0)\|_H + \left\|A^{-\frac{1}{2}}\psi\right\|_H + \max_{0\le t\le T}\|f(t)\|_H\right].$$

Estimate (10) follows from these estimates.

Third, we obtain estimate (11). Applying A to formula (14) and estimates (7), we get

$$\|Av(0)\|_H \le \|P\|_{H\to H}\left\{\left\|A^{\frac{1}{2}}B^{-\frac{1}{2}}\right\|_{H\to H}e^{-\frac{\alpha}{2}T}\left\|B^{\frac{1}{2}}s(T)\right\|_{H\to H}\left\|A^{\frac{1}{2}}\psi\right\|_H + \|A\varphi\|_H \right. \\ \left. + \|A\xi\|_H + \left\|AB^{-1}\right\|_{H\to H}\left[\|f(T)\|_H + e^{-\frac{\alpha}{2}T}\|c(T)\|_{H\to H}\|f(0)\|_H\right]\right.$$

$$\times \left\|AB^{-1}\right\|_{H\to H}\int\limits_0^T e^{-\frac{\alpha}{2}(T-z)}\parallel c(T-z)\parallel_{H\to H}\left[\frac{\alpha}{2}\parallel f(z)\parallel_H+\parallel f'(z)\parallel_H\right]dz\right\}$$

$$\leq M_1(\delta,\alpha)\left\{\|A\varphi\|_H+\|A\xi\|_H+\left\|A^{1/2}\psi\right\|_H+\|f(0)\|_H+\int\limits_0^T\left\|f'(t)\right\|_H dt\right\}. \quad (15)$$

Applying A to formula (12) and using an integration by parts, we can write the formula

$$Av(t)=e^{-\frac{\alpha}{2}t}c(t)Av(0)+\frac{\alpha}{2}e^{-\frac{\alpha}{2}t}A^{\frac{1}{2}}s(t)A^{\frac{1}{2}}v(0)+A^{\frac{1}{2}}e^{-\frac{\alpha}{2}t}s(t)A^{\frac{1}{2}}\psi$$
$$+e^{-\frac{\alpha}{2}t}AB^{-1}\left[e^{\frac{\alpha}{2}t}f(t)-c(t)f(0)-\int\limits_0^t e^{\frac{\alpha}{2}z}c(t-z)\left[\frac{\alpha}{2}f(z)+f'(z)\right]dz\right].$$

Using the last formula and estimates (7), we obtain

$$\|Au(t)\|_H\leq\|c(t)\|_{H\to H}e^{-\frac{\alpha}{2}t}\|Av(0)\|_H$$
$$+\parallel B^{\frac{1}{2}}s(t)\parallel_{H\to H}\left\|A^{1/2}B^{-\frac{1}{2}}\right\|_{H\to H}\left|\frac{\alpha}{2e^{\frac{\alpha}{2}t}}\right|\parallel A^{\frac{1}{2}}v(0)\parallel_H$$
$$+\parallel B^{\frac{1}{2}}s(t)\parallel_{H\to H}\left\|A^{1/2}B^{-\frac{1}{2}}\right\|_{H\to H}|e^{-\frac{\alpha}{2}t}|\parallel A^{\frac{1}{2}}\psi\parallel_H$$
$$+\left\|AB^{-1}\right\|_{H\to H}\left[\|f(t)\|_H+e^{-\frac{\alpha}{2}t}\|c(t)\|_{H\to H}\|f(0)\|_H\right]$$
$$+\left\|AB^{-1}\right\|_{H\to H}\int\limits_0^t e^{-\frac{\alpha}{2}(t-z)}\parallel c(t-z)\parallel_{H\to H}\left[\frac{\alpha}{2}\parallel f(z)\parallel_H+\parallel f'(z)\parallel_H\right]dz$$
$$\leq M_3(\delta,\alpha)\left[\|Av(0)\|_H+\parallel A^{\frac{1}{2}}\psi\parallel_H+\|f(0)\|_H+\max_{0\leq t\leq T}\parallel f'(t)\parallel_H\right]$$

for any $t\in[0,T]$. Then, we get

$$\max_{0\leq t\leq T}\parallel Av(t)\parallel_H$$
$$\leq M_3(\delta,\alpha)\left[\parallel Av(0)\parallel_H+\parallel A^{\frac{1}{2}}\psi\parallel_H+\|f(0)\|_H+\max_{0\leq t\leq T}\parallel f'(t)\parallel_H\right]. \quad (16)$$

Estimate

$$\max_{0\leq t\leq T}\|Au(t)\|_H$$
$$\leq M_4(\delta,\alpha)\left\{\|A\varphi\|_H+\|A\xi\|_H+\left\|A^{1/2}\psi\right\|_H+\|f(0)\|_H+\int\limits_0^T\left\|f'(t)\right\|_H dt\right\}$$

follows from estimates (11), (15) and (16). Finally, estimate for $\max_{0\le t\le T}\left\|\frac{d^2u}{dt^2}\right\|_H$ follows from the last estimate and the triangle inequality. Theorem 2 is proved. □

Now, we will consider three applications of Theorem 1.

First, we consider the nonlocal boundary value problem for telegraph equation

$$\begin{cases} u_{tt}(t,x)+\alpha u_t(t,x)-(a(x)u_x)_x+\delta u(t,x)=p(x)+f(t,x), \\ 0<t<T, 0<x<l, \\ u(0,x)=\varphi(x), u_t(0,x)=\psi(x), u(T,x)=\xi(x), 0\le x\le l, \\ u(t,0)=u(t,l), u_x(0,x)=u_x(t,l), 0\le t\le T. \end{cases} \tag{17}$$

Problem (17) has a unique smooth solution $(u(t,x),p(x))$ for the smooth $a(x)\ge a>0, x\in(0,l), \delta>0, a(l)=a(0),\ \varphi(x),\psi(x),\xi(x),(x\in[0,l]$ and $f(t,x)(t\in(0,T), x\in(0,l))$ functions. This allows us to reduce boundary value problem (17) to abstract boundary value problem (1) in a Hilbert space $H=L_2[0,1]$ with a self-adjoint positive definite operator A^x defined by formula

$$A^x u(x)=-(a(x)u_x)_x+\delta u \tag{18}$$

with domain

$$D(A^x)=\{u(x): u(x), u_x(x), (a(x)u_x)_x\in L_2[0,1], u(1)=u(0), u_x(1)=u_x(0)\}.$$

Theorem 3. *Let conditions (6) and (8) be satisfied. Then, for the solution* $\{u(t,x),p(x))\}$ *of problem (17), we have the following stability inequalities*

$$\| u \|_{C(L_2[0,1])}+\| (A^x)^{-1} p \|_{L_2[0,1]} \tag{19}$$

$$\le M(\delta,\alpha)\left[\| \varphi \|_{L_2[0,1]}+\| \psi \|_{L_2[0,1]}+\| \xi \|_{L_2[0,1]}+\max_{0\le t\le T}\|f(t)\|_{L_2[0,1]}\right],$$

$$\max_{0\le t\le T}\left\|u''\right\|_{L_2[0,1]}+\|u\|_{C(W_2^2[0,1])}+\|p\|_{L_2[0,1]} \tag{20}$$

$$\le M(\delta,\alpha)\left[\| \varphi \|_{W_2^2[0,1]}+\| \psi \|_{W_2^1[0,1]}+\max_{0\le t\le T}\left\|f'(t)\right\|_{L_2[0,1]}\right.$$

$$\left.+\|\xi\|_{W_2^2[0,1]}+\|f(0)\|_{L_2[0,1]}\right],$$

where $M(\delta,\lambda)$ *is independent of* $\varphi(x),\psi(x),\xi(x)$, *and* $f(t,x)$. *Here, the Sobolev space* $W_2^2[0,1]$ *is defined as the set of all functions* f *defined on* $[0,1]$ *such that* f *and second order derivative function* f'' *are both locally integrable in* $L_2[0,1]$, *equipped with the norm*

$$\| f \|_{W_2^2[0,1]} = \left(\int_0^1 |f(x)|^2 \, dx \right)^{\frac{1}{2}} + \left(\int_0^1 |f_{xx}(x)|^2 \, dx \right)^{\frac{1}{2}},$$

and the Sobolev space $W_2^1[0,1]$ *is defined as the set of all functions* f *defined on* $[0,1]$ *such that* f *and first order derivative function* f' *are both locally integrable in* $L_2[0,1]$*, equipped with the norm*

$$\| f \|_{W_2^1[0,1]} = \left(\int_0^1 |f(x)|^2 \, dx \right)^{\frac{1}{2}} + \left(\int_0^1 |f_x(x)|^2 \, dx \right)^{\frac{1}{2}}.$$

Proof. Problem (17) can be written in abstract form

$$\begin{cases} \frac{d^2u(t)}{dt^2} + \alpha \frac{du(t)}{dt} + Au(t) = f(t) \ (0 \le t \le T), \\ u(0) = \varphi, u'(0) = \psi, u(T) = \xi \end{cases} \tag{21}$$

in a Hilbert space $L_2[0,l]$ of all square integrable functions defined on $[0,l]$ with self-adjoint positive definite operator $A = A^x$ defined by formula (18). Here, $f(t) = f(t,x)$ and $u(t) = u(t,x)$ are known and unknown abstract functions defined on $[0,l]$ with the values in $H = L_2[0,l]$. Therefore, estimates (19) and (20) follow from estimates of Theorem 1. □

Second, let $\Omega \subset R^n$ be a bounded open domain with smooth boundary $S, \overline{\Omega} = \Omega \cup S$. In $[0,T] \times \Omega$, we consider the nonlocal boundary value problem for the telegraph equation

$$\begin{cases} u_{tt}(t,x) + \alpha u_t(t,x) - \sum_{r=1}^{n} (a_r(x) u_{x_r}) x_r = p(x) + f(t,x), \\ x = (x_1, \dots, x_n) \in \Omega, 0 < t < T, \\ u(0,x) = \varphi(x), \frac{\partial u(0,x)}{\partial t} = \psi(x), u(T,x) = \xi(x), x \in \overline{\Omega}, \\ u(t,x) = 0, x \in S, 0 \le t \le T, \end{cases} \tag{22}$$

where $\alpha_r(x), (x \in \Omega), \varphi(x), \psi(x), \xi(x), \left(x \in \overline{\Omega}\right)$ and $f(t,x), (t \in (0,T)), x \in \Omega$ are given smooth functions and $\alpha_r(x) > 0, \delta \ge 0$. We introduce the Hilbert spaces $L_2(\overline{\Omega})$ of the all square integrable functions defined on $\overline{\Omega}$, equipped with the norm

$$\|f\|_{L_2(\overline{\Omega})} = \left\{ \int \cdots \int_{x \in \overline{\Omega}} |f(x)|^2 dx_1 \cdots dx_n \right\}^{1/2}.$$

Problem (22) has a unique smooth solution $(u(t,x),p(x))$ for the smooth functions $\varphi(x)$, $\psi(x)$, $a_r(x)$ and $f(t,x)$. This allows us to reduce the problem (22) to the abstract boundary value problem (1) in the Hilbert space $H = L_2(\overline{\Omega})$ with a self-adjoint positive definite operator A^x defined by formula

$$A^x u(x) = -\sum_{r=1}^{n} (a_r(x) u_{x_r})_{x_r} \tag{23}$$

with domain

$$D(A^x) = \left\{u(x) : u(x), u_{x_r}(x), (a_r(x)u_{x_r})_{x_r} \in L_2(\overline{\Omega}), 1 \le r \le n, u(x) = 0, x \in S\right\}.$$

Theorem 4. *Let conditions (6) and (8) be satisfied. Then, for the solution* $\{u(t,x),p(x))\}$ *of problem (22) the stability inequalities*

$$\| u \|_{C(L_2(\overline{\Omega}))} + \| (A^x)^{-1} p \|_{L_2(\overline{\Omega})}$$
$$\le M(\delta,\alpha)\left[\| \xi \|_{L_2(\overline{\Omega})} + \| \varphi \|_{L_2(\overline{\Omega})} + \| \psi \|_{L_2(\overline{\Omega})} + \max_{0\le t\le T} \|f(t)\|_{L_2(\overline{\Omega})}\right],$$
$$\max_{0\le t\le T} \left\|u''\right\|_{L_2(\overline{\Omega})} + \|u\|_{C(W_2^2(\overline{\Omega}))} + \|p\|_{L_2(\overline{\Omega})}$$
$$\le M(\delta,\alpha)\left[\| \varphi \|_{W_2^2(\overline{\Omega})} + \| \psi \|_{W_2^1(\overline{\Omega})} + \max_{0\le t\le T} \left\|f'(t)\right\|_{L_2(\overline{\Omega})}\right.$$
$$\left. + \|\xi\|_{W_2^2(\overline{\Omega})} + \|f(0)\|_{L_2(\overline{\Omega})}\right]$$

hold, where $M(\delta,\alpha)$ *does not depend on* $\varphi(x)$, $\psi(x)$, $\xi(x)$ *and* $f(t,x)$. *Here and in future, the Sobolev space* $W_2^2(\overline{\Omega})$ *is defined as the set of all functions* f *defined on* $\bar{\Omega}$ *such that* f *and all second order partial derivative functions* f_{x_r,x_r}, $r = 1,\ldots n$ *are both locally integrable in* $L_2(\overline{\Omega})$*,equipped with the norm*

$$\| f \|_{W_2^2(\overline{\Omega})} = \| f \|_{L_2(\overline{\Omega})} + \left(\int \cdots \int_{x\in\overline{\Omega}} \sum_{r=1}^{n} |f_{x_r,x_r}|^2 \, dx_1 \cdots dx_n\right)^{1/2},$$

and the Sobolev space $W_2^1(\overline{\Omega})$ *is defined as the set of all functions* f *defined on* $\bar{\Omega}$ *such that* f *and all first order partial derivative functions* f_{x_r}, $r = 1,\ldots n$ *are both locally integrable in* $L_2(\overline{\Omega})$*,equipped with the norm*

$$\| f \|_{W_2^1(\overline{\Omega})} = \| f \|_{L_2(\overline{\Omega})} + \left(\int \cdots \int_{x\in\overline{\Omega}} \sum_{r=1}^{n} |f_{x_r}|^2 \, dx_1 \cdots dx_n\right)^{1/2}.$$

The proof of Theorem 4 is based on Theorem 1 and the symmetry properties of the operator A^x defined by formula (23) and the following theorem on the coercivity inequality for the solution of the elliptic differential problem in $L_2(\overline{\Omega})$.

Theorem 5. *For the solutions of the elliptic differential problem [25]*

$$\begin{cases} A^x u(x) = \omega(x), x \in \Omega, \\ u(x) = 0, x \in S, \end{cases}$$

the following coercivity inequality holds

$$\sum_{r=1}^{n} \|u_{x_r x_r}\|_{L_2(\overline{\Omega})} \le M_1 ||\omega||_{L_2(\overline{\Omega})}.$$

Here M_1 does not depend on $\omega(x)$.

Third, in $[0,T] \times \Omega$, the boundary value problem for the multidimensional telegraph equation

$$\begin{cases} u_{tt}(t,x) + \alpha u_t(t,x) - \sum\limits_{r=1}^{n} (a_r(x) u_{x_r}) x_r + \delta u = p(x) + f(t,x), \\ \\ x = (x_1, \ldots, x_n) \in \Omega, 0 < t < T, \\ \\ u(0,x) = \varphi(x), \frac{\partial u(0,x)}{\partial t} = \psi(x), u(T,x) = \xi(x), x \in \overline{\Omega}, \\ \\ \frac{\partial u(t,x)}{\partial \mathbf{n}} = 0, x \in S, 0 \le t \le T \end{cases} \tag{24}$$

with the Neumann condition is considered. Here, $\mathbf{n}$ is the normal vector to S, $a_r(x) \ge a > 0, (x \in \Omega), \varphi(x), \psi(x), \xi(x)$ $(x \in \overline{\Omega})$, and $f(t,x)$ $(t \in (0,T),\ x \in \Omega)$ are given smooth functions and $\delta > 0$. Problem (24) has a unique smooth solution $(u(t,x), p(x))$ for the smooth functions $\varphi(x), \psi(x), \xi(x), a_r(x)$ and $f(t,x)$. This allows us to reduce the problem (24) to the abstract boundary value problem (1) in the Hilbert space $H = L_2(\overline{\Omega})$ with a self-adjoint positive definite operator A^x defined by formula

$$A^x u(x) = -\sum_{r=1}^{n} (a_r(x) u_{x_r})_{x_r} + \delta u \tag{25}$$

with domain

$$D(A^x) = \left\{ u(x) : u(x), u_{x_r}(x), (a_r(x) u_{x_r})_{x_r} \in L_2(\overline{\Omega}), 1 \le r \le n, \frac{\partial u\,(x)}{\partial \mathbf{n}} = 0, x \in S \right\}.$$

Theorem 6. *Let conditions (6) and (8) be satisfied. Then, for the solution $\{u(t,x),p(x))\}$ of problem (24), the following stability inequalities*

$$\| u \|_{C(L_2(\overline{\Omega}))} + \| (A^x)^{-1} p \|_{L_2(\overline{\Omega})}$$
$$\leq M(\delta,\alpha)\left[\| \xi \|_{L_2(\overline{\Omega})} + \| \varphi \|_{L_2(\overline{\Omega})} + \| \psi \|_{L_2(\overline{\Omega})} + \max_{0\leq t\leq T} \|f(t)\|_{L_2(\overline{\Omega})}\right],$$
$$\max_{0\leq t\leq T} \left\|u''\right\|_{L_2(\overline{\Omega})} + \|u\|_{C(W_2^2(\overline{\Omega}))} + \|p\|_{L_2(\overline{\Omega})}$$
$$\leq M(\delta,\alpha)\left[\| \varphi \|_{W_2^2(\overline{\Omega})} + \| \psi \|_{W_2^1(\overline{\Omega})} + \max_{0\leq t\leq T} \left\|f'(t)\right\|_{L_2(\overline{\Omega})}\right.$$
$$\left. + \| \xi \|_{W_2^2(\overline{\Omega})} + \|f(0)\|_{L_2(\overline{\Omega})}\right],$$

hold, where $M(\delta,\alpha)$ does not depend on $\varphi(x), \psi(x), \xi(x)$ and $f(t,x)$.

The proof of Theorem 6 is based on Theorem 1 and the symmetry properties of the operator A^x defined by formula (24) and the following theorem on the coercivity inequality for the solution of the elliptic differential problem in $L_2(\overline{\Omega})$.

Theorem 7. *For the solutions of the elliptic differential problem*

$$\begin{cases} A^x u(x) = \omega(x), x \in \Omega, \\ \frac{\partial u(x)}{\partial \mathbf{n}} = 0, x \in S, \end{cases}$$

the following coercivity inequality holds [25]

$$\sum_{r=1}^{n} \|u_{x_r x_r}\|_{L_2(\overline{\Omega})} \leq M_1(\delta)||\omega||_{L_2(\overline{\Omega})}.$$

Here $M_1(\delta)$ is independent of $\omega(x)$.

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On a One-Equation Turbulent Model with Feedbacks

H.B. de Oliveira and A. Paiva

Abstract A one-equation turbulent model is derived in this work on the basis of the approach used for the k-epsilon model. The novelty of the model consists in the consideration of a general feedback forces field in the momentum equation and a rather general turbulent dissipation function in the equation for the turbulent kinetic energy. For the steady-state associated boundary value problem, we prove the uniqueness of weak solutions under monotonous conditions on the feedbacks and smallness conditions on the solutions to the problem. We also discuss the existence of weak solutions and issues related with the higher integrability of the solutions gradients.

Keywords Turbulence • k-epsilon model • Feedback forces • Uniqueness

Mathematics Subject Classification (2010): 76F60, 93A30, 35Q35, 76D03

1 Introduction

The Navier–Stokes equations were proposed by Navier in 1822, and later on, in 1845, due to the clarifying work made by Stokes, these equations found a phenomenological justification on the basis of the principles of fluid mechanics. Since then, these equations are used to describe Newtonian fluid flows, which, in the case of incompressible and homogeneous fluids, can be written as

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$$\operatorname{div}\mathbf{u} = 0, \tag{1}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{div}(\mathbf{u}\otimes\mathbf{u}) = \mathbf{f} - \frac{1}{\rho}\nabla p + \nu\,\mathbf{D}(\mathbf{u}), \quad \mathbf{D}(\mathbf{u}) = \frac{1}{2}\left(\boldsymbol{\nabla}\,\mathbf{u} + \boldsymbol{\nabla}\,\mathbf{u}^T\right), \tag{2}$$

where $\mathbf{u} = (u_1, u_2, u_3)$ is the velocity field, p is the pressure, ρ is the constant (positive) density, and $\mathbf{f}$ denotes the external forces field. The tensor $\mathbf{D}(\mathbf{u})$ is the symmetric part of $\boldsymbol{\nabla}\,\mathbf{u}$ and accounts for the different strains in the fluid. The positive factor ν expresses the ratio of the internal forces in the fluid, called dynamic viscosity, to the mass density ρ, and is usually called kinematic viscosity. In 1883, Reynolds has succeeded to prove the importance of a threshold value separating the laminar flow regime from the turbulent one within a similar fluid. Nowadays, this value is known as the Reynolds number, and it is usually defined as the ratio of inertial forces to viscous forces $Re = \frac{u(l)\,l}{\nu} \equiv \frac{(\mathbf{u}\cdot\nabla)\mathbf{u}}{\nu\Delta\mathbf{u}}$, where l and $u(l)$ are characteristic length and velocity scales. It was Stokes, even before Reynolds, who observed the inadequacy of (1) and (2) to model certain flow regimes that could probably result from eddies which rendered the motion more chaotic. However, it seems to have been Reynolds the first to study the mechanical significance of the existence of such eddies. The approach made by Reynolds was to assume that the flow has two different scales, leading to the supposition that it is possible to decompose the quantities in the Navier–Stokes equations in average and fluctuating, or aleatory, values (Reynolds hypothesis). The idea associated to this decomposition was to filter the Navier–Stokes equations in time intervals large enough, in comparison to the temporal scale of the flow, but small enough in comparison with the time scale of the average of the flow. Therefore, the velocity of a molecule was decomposed into two components:

$$\mathbf{u} = \overline{\mathbf{u}} + \mathbf{u}', \tag{3}$$

where $\mathbf{u}'$ represents the fluctuating, or relative velocity, and $\overline{\mathbf{u}}$ represents an average velocity. Underlying this decomposition is a filter, or an average, concept that can be mathematically defined as a Reynolds operator [15], i.e., an operator $\mathscr{R} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by $\mathscr{R}(\mathbf{u}) = \overline{\mathbf{u}}$ and satisfying to the following properties:

$$\mathscr{R}(\mathbf{u} + \lambda\mathbf{v}) = \mathscr{R}(\mathbf{u}) + \lambda\mathscr{R}(\mathbf{v}) \quad \forall\,\mathbf{u},\ \mathbf{v} \in \mathbb{R}^3 \quad \forall\,\lambda \in \mathbb{R}; \tag{4}$$

$$\mathscr{R}(\mathscr{R}(\mathbf{u})) = \mathscr{R}(\mathbf{u}) \quad \forall\,\mathbf{u} \in \mathbb{R}^3; \tag{5}$$

$$\mathscr{R}(\partial\mathbf{u}) = \partial\left(\mathscr{R}(\mathbf{u})\right) \quad \forall\,\mathbf{u} \in \mathbb{R}^3; \tag{6}$$

$$\mathscr{R}(\mathbf{u}\otimes\mathbf{v}) = \mathscr{R}(\mathbf{u})\otimes\mathscr{R}(\mathbf{v}) + \mathscr{R}((\mathbf{u} - \mathscr{R}(\mathbf{u}))\otimes(\mathbf{v} - \mathscr{R}(\mathbf{v}))) \quad \forall\,\mathbf{u},\ \mathbf{v} \in \mathbb{R}^3. \tag{7}$$

Observe that (5) implies that $\mathscr{R}(\mathbf{u}') = \mathbf{0}$ for any $\mathbf{u} \in \mathbb{R}^3$, and from (7) we have $\mathscr{R}(\mathbf{u}\otimes\mathscr{R}(\mathbf{v})) = \mathscr{R}(\mathbf{u})\otimes\mathscr{R}(\mathbf{v})$ and $\mathscr{R}(\mathscr{R}(\mathbf{u})\otimes\mathbf{v}) = \mathscr{R}(\mathbf{u})\otimes\mathscr{R}(\mathbf{v})$ which makes $\mathscr{R}(\mathbf{u}\otimes\mathscr{R}(\mathbf{v})) = \mathscr{R}(\mathscr{R}(\mathbf{u})\otimes\mathbf{v})$. Since the tensorial product is not commutative, we have, in general, $\mathscr{R}(\mathbf{u}\otimes\mathscr{R}(\mathbf{v})) \neq \mathscr{R}(\mathbf{v}\otimes\mathscr{R}(\mathbf{u}))$. However, this inequality is not observed when the Reynolds averaged Navier–Stokes equations (RANS) are derived in the scalar form. The definition of Reynolds operator described above is used to filter the Navier–Stokes equations (1) and (2) in a domain $\Omega \subset \mathrm{R}^3$, which represents the volume occupied by the fluid on the time $t \in [0, T]$. After some algebraic manipulations, we obtain the so-called RANS equations

$$\operatorname{div} \overline{\mathbf{u}} = 0, \tag{8}$$

$$\frac{\partial \overline{\mathbf{u}}}{\partial t} + \mathbf{div}(\overline{\mathbf{u}} \otimes \overline{\mathbf{u}}) = \overline{\mathbf{f}} - \frac{1}{\rho}\nabla \overline{p} + \nu \,\mathbf{div}\,\mathbf{D}(\overline{\mathbf{u}}) - \mathbf{div}(\overline{\mathbf{u}' \otimes \mathbf{u}'}). \tag{9}$$

Equation (9) looks the same as the momentum equation (2), with the addition of a term involving the average of a product of fluctuating parts of the velocity. The additional term

$$\mathbf{R} := -\overline{\mathbf{u}' \otimes \mathbf{u}'}\,, \tag{10}$$

often called the Reynolds stress tensor, can be seen as the average of changes in $\mathbf{u}'$ due to the particle transport with the fluid movement. Therefore, the tensor (10) acts like an effective stress and cannot be determined from the classical principles. As we do not have any way to know directly its magnitude, the modeling of its effect is usually done in terms of known quantities or quantities that we can determine. This is known in the literature as the closing problem of turbulence, and, as a result, many schemes have been developed to approximate the Reynolds stress tensor.

2 The $k - \epsilon$ Turbulent Model

Reynolds has made experiments suggesting that the tensor (10) was somehow related with $\nabla\overline{\mathbf{u}}$, which, by reasons of symmetry, can now be considered in the form $\mathbf{R} = \mathbf{F}(\mathbf{D}(\overline{\mathbf{u}}))$. However, the application $\mathbf{F}$ cannot be arbitrarily chosen, because the model should give the same results regardless of the considered referential. In analogy with the Stokes law for laminar flows, Boussinesq has proposed that $\mathbf{R} = \nu_T\,\mathbf{D}(\overline{\mathbf{u}})$ (Boussinesq turbulence hypothesis), where ν_T was denoted by eddy or turbulent viscosity. By a simple comparison of the traces of the last expression with (10), it can be readily seen that the Boussinesq hypothesis must be rewritten in the form

$$\mathbf{R} = -\frac{2}{3}k\mathbf{I} + \nu_T\,\mathbf{D}(\overline{\mathbf{u}})\,, \quad k := \frac{1}{2}\overline{|\mathbf{u}'|^2}\,, \tag{11}$$

where k is called turbulent kinetic energy, a new unknown in the problem that needs also to be modeled. In 1942, Kolmogorov [10] proposed a model in which the turbulence was described by $\nu_T = \rho\frac{k}{f}$ and $l = k^{\frac{1}{2}}/f$, where l is a length scale, suggesting that k and f, the characteristic frequency of the energy-containing movements, should be determined by transport equations. Inspired by the previous model of his own, Prandtl [14] proposed, in 1945, that $\nu_T = \rho k^{\frac{1}{2}} l$, suggesting also that the turbulent kinetic energy was determined from a transport equation, but the length scale l should be algebraically prescribed. Later on, during the 1970 decade, Launder and Spalding [11] observed the importance of the turbulent dissipation $\epsilon := \nu\overline{|\nabla \mathbf{u}'|^2}$, a new quantity, in determining the rate of dissipation of the turbulent kinetic energy in the turbulent flow process, which, again by means of symmetry, can be written as $\epsilon := \nu\overline{|\mathbf{D}(\mathbf{u}')|^2}$. The turbulent dissipation ϵ is determined by the first process in the energy cascade, which consists in the transfer of energy from the largest eddies to the smaller ones. Assuming these large eddies are characterized by length scale l_0, velocity scale u_0, and time scale $t_0 = l_0/u_0$ and have energy of $1/2\rho u_0^2$, then the rate of transfer of energy can be supposed to scale as $u_0^2/t_0 = u_0^3/l_0$. Consequently, ϵ scales as u_0^3/l_0 and independently of ν. Therefore, it is reasonable to model ϵ and consequently the turbulent viscosity (in view of Prandtl's hypothesis), as

$$\epsilon = C_D \frac{k^{\frac{3}{2}}}{l}, \quad \nu_T = \rho k^{\frac{1}{2}} l \Rightarrow \nu_T = C_\mu \frac{k^2}{\epsilon}, \quad \text{where } \epsilon := \nu\overline{|\mathbf{D}(\mathbf{u}')|^2}\,, \tag{12}$$

C_D is a closure constant, and C_μ is a constant related with the kinematic viscosity and determined by experimental measures of k and ϵ. To derive an equation for the transport of the turbulent kinetic energy, we start by considering (1) the velocity field decomposed in the form (3). Then, subtracting (9) to this equation, we obtain

$$\operatorname{div} \mathbf{u}' = 0. \tag{13}$$

Likewise, we subtract the RANS equations (9) to the momentum equation (2), where all the quantities are decomposed as in (3). Then, we multiply the resulting equation by $\mathbf{u}'$ and we apply the filter produced by the Reynolds operator. Using the properties set forth at (4)–(7) and some vectorial calculus together with (10), $(11)_2$, and (13), we obtain

$$\frac{\partial k}{\partial t} + \overline{(\overline{\mathbf{u}} + \mathbf{u}') \cdot \nabla \frac{|\mathbf{u}'|^2}{2}} = \mathbf{R} : \mathbf{D}(\overline{\mathbf{u}}) - \frac{1}{\rho}\operatorname{div}(\overline{p'\mathbf{u}'}) + \nu\, \overline{\mathbf{u}' \cdot \mathbf{div}\, \mathbf{D}(\mathbf{u}')}. \tag{14}$$

By using the hypothesis that convection by random fields produces diffusion for the mean [12], the second term of the left-hand side of (14) can be approximated by $\overline{\mathbf{u}} \cdot \nabla k - \operatorname{div}(\nu_D \nabla k)$, where $\nu_D := \nu_D(k, \epsilon)$ is the turbulent diffusivity. For the last two terms, it is used an ergodicity hypothesis [4] asserting that, over a long period of time, the remaining time on a given region in space is proportional to the region volume. This allows us to use the approximations $\overline{\mathbf{u}' \cdot \mathbf{div}\, \mathbf{D}(\mathbf{u}')} \simeq \overline{|\mathbf{D}(\mathbf{u}')|^2} \simeq \epsilon$ and $\operatorname{div}(\overline{p'\mathbf{u}'}) \simeq 0$. Using this information, we obtain the following transport equation for k

$$\frac{\partial k}{\partial t} + \overline{\mathbf{u}} \cdot \nabla k = \mathrm{div}\,(\nu_D \nabla k) + \nu_T |\mathbf{D}(\overline{\mathbf{u}})|^2 - \epsilon\,, \quad \nu_D := \nu_D(k, \epsilon). \tag{15}$$

The usual process to derive an equation for the evolution of the turbulent dissipation ϵ starts by applying the rotational to the RANS equation (9), and then working with calculus tools, the following is obtained:

$$\frac{\partial \varepsilon}{\partial t} - 2\nu \left(\overline{\mathbf{rot}\,\mathbf{u}' \cdot \mathbf{rot}(\mathbf{u}' \times \mathbf{rot}\,\overline{\mathbf{u}})}\right) - 2\nu \left(\overline{\mathbf{rot}\,\mathbf{u}' \otimes \mathbf{rot}\,\mathbf{u}'} : \nabla \overline{\mathbf{u}}\right)$$
$$- 2\nu \left(\overline{(\mathbf{rot}\,\mathbf{u}' \otimes \mathbf{rot}\,\mathbf{u}')' : \nabla \mathbf{u}'}\right) + \left(\overline{(\overline{\mathbf{u}} + \mathbf{u}') \cdot \nu \nabla |\mathbf{rot}\,\mathbf{u}'|^2}\right) = -2\nu^2 \overline{|\nabla \mathbf{rot}\,\mathbf{u}'|^2}. \tag{16}$$

The second term of the left-hand side can be neglected because the terms involved approximately cancel one each other. Using an arguing similar to the Boussinesq hypothesis, the second term is approximated by $2\nu C k |\mathbf{D}(\overline{\mathbf{u}})|^2$. The last term is approximated by $\overline{\mathbf{u}} \cdot \nabla \varepsilon - \mathrm{div}\,(\nu_D \nabla \varepsilon)$ by the application of the convection-diffusion hypothesis [12] similarly as it was done for the k-equation. Finally, the fourth term on the left-hand side and the term on the right-hand side are usually approximated by $C\frac{\varepsilon^2}{k}$ to avoid the need of another equation in order to close the problem [4, 12]. After all, we arrive in the following evolution equation:

$$\frac{\partial \varepsilon}{\partial t} + \overline{\mathbf{u}} \cdot \nabla \varepsilon = \mathrm{div}\,(\nu_D(k, \varepsilon) \nabla \varepsilon) + C_1 k |\mathbf{D}(\overline{\mathbf{u}})|^2 + C_2 \frac{\varepsilon^2}{k}\,, \tag{17}$$

where C_1 and C_2 are positive constants that can be determined from the experiments.

3 Feedback Forces Fields

In this section we consider, for simplicity, 1-equation models comprised by Eqs. (8), (9), (11), (12), and (15), and we assume the turbulent dissipation ϵ depends only on k. Observe that the consideration of 1-equation models is acceptable in the sense that the equation for ϵ may be discarded by prescribing an appropriate length scale. This assumption has also implications on the turbulent viscosity and on the turbulent diffusivity, defined at (12) and (15), in the sense that now they only depend on k. We consider here the case when the external forces field depends on the own velocity, i.e., we assume that the vector field $\mathbf{f}$, in the momentum equation (2), is replaced by

$$\mathbf{g} - \mathbf{f}(\mathbf{u}). \tag{18}$$

Now, in (18), $\mathbf{g}$ is an external forcing term that depends only on the space and time variables, and $\mathbf{f}(\mathbf{u})$ is the feedback forces field that may have different signs, according to each application where it is considered. Probably the best-known situation happens for fluid flows in a rotating frame, where the Coriolis force

$\mathbf{f}(\mathbf{u}) = 2\Omega \times \mathbf{u}$ must be considered, being Ω, here, the angular velocity vector. Another example is the Lorentz force $\mathbf{f}(\mathbf{u}, \mathbf{B}) = -\mathbf{J} \times \mathbf{B}$ considered to model turbulent flows controlled by a magnetic field $\mathbf{B}$, where $\mathbf{J}$ is the total electric current intensity, given by Ohm's law $\mathbf{J} = \sigma(-\nabla\Phi + \mathbf{u} \times \mathbf{B})$. Here, σ is the conductivity, a material-dependent parameter, and Φ is the electric potential, which in turn satisfies to the Poisson equation $\triangle\Phi = \mathrm{div}(\mathbf{u} \times \mathbf{B})$ (see, e.g., [9]). However, our main motivation comes from the study of flows through porous media. In this field of the applications, it is important to consider the Darcy and Forchheimer terms to model the drag due to the flow through the porous medium.

Here, we gather this drag in the function $\mathbf{f}(\mathbf{u}) = C_D\mathbf{u} + C_F|\mathbf{u}|\mathbf{u}$, where C_D and C_F are the Darcy and Forchheimer parameters, positive constants that depend on the permeability and porosity of the medium. The mathematical modeling of turbulence in porous media considers the simultaneous application of time and volume-average operators. When this procedure is applied to the continuity and momentum equations, they come as in (8) and (9), with the peculiarity that the Darcy and Forchheimer terms come in the form $\mathbf{f}(\overline{\mathbf{u}}) = C_D\overline{\mathbf{u}} + C_F|\overline{\mathbf{u}}|\overline{\mathbf{u}}$. This procedure is being applied to many situations of turbulent fluids through porous media, as is the case of turbulent combustion in porous media or turbulent impinging jets in porous media (see, e.g., [5]). The underlying idea of considering this double-decomposition concept corresponds, in a certain sense, to consider a feedback forces field

$$\mathbf{f}(\mathbf{u}) = \overline{\mathbf{f}(\mathbf{u})} + \mathbf{f}(\mathbf{u})', \quad \overline{\mathbf{f}(\mathbf{u})} = \mathbf{f}(\overline{\mathbf{u}}), \quad \overline{\mathbf{f}(\mathbf{u})' \cdot \mathbf{u}'} = h(|\overline{\mathbf{u}}|)k. \tag{19}$$

The best example of this situation is a drag's force purely Darcy $\mathbf{f}(\mathbf{u}) = C_D\mathbf{u}$ for which $h(|\overline{\mathbf{u}}|) = 2C_D$. A more complex example of a feedback forces field satisfying to (19) is given by the generalized Forchheimer force

$$\mathbf{f}(\mathbf{u}) = h(|\mathbf{u}|)\mathbf{u}, \quad h(|\mathbf{u}|) = |\overline{\mathbf{u}}|^{2n}, \ \ n \in \mathbb{N}. \tag{20}$$

The Reynolds averaged process for the momentum equation, considered with a feedback forces field satisfying to (18) and (19) and assuming the Reynolds stresses, is given by $\mathbf{R} = \nu_T\,\overline{\mathbf{D}}$, and the turbulent viscosity, defined by (12), depends only on k, leads us to

$$\frac{\partial\overline{\mathbf{u}}}{\partial t} + \mathbf{div}(\overline{\mathbf{u}} \otimes \overline{\mathbf{u}}) = \overline{\mathbf{g}} - \mathbf{f}(\overline{\mathbf{u}}) - \frac{1}{\rho}\nabla\overline{p} + \mathbf{div}\left((\nu + \nu_T(k))\mathbf{D}(\overline{\mathbf{u}})\right). \tag{21}$$

The same procedure used to derive the k-equation, and assuming, in addition, the turbulent diffusivity, given by (15), depends also only on k, allows us to write

$$\frac{\partial k}{\partial t} + \overline{\mathbf{u}} \cdot \nabla k = \mathrm{div}\left(\nu_D(k)\nabla k\right) + \nu_T(k)|\mathbf{D}(\overline{\mathbf{u}})|^2 + h(|\overline{\mathbf{u}}|)k - \epsilon(k). \tag{22}$$

Although the term $h(|\overline{\mathbf{u}}|)k$ does not bring any difficulty to our analysis, we may avoid its presence in Eq. (22) by considering a general feedback forces field satisfying to (19), but with $(19)_3$ replaced by

$$\overline{\mathbf{f}(\mathbf{u})' \cdot \mathbf{u}'} = 0. \tag{23}$$

This situation happens, for instance, when we consider the Coriolis force $\mathbf{f}(\mathbf{u}) = 2\Omega \times \mathbf{u}$. In this case, the same procedure used to derive (22) allows us to write, in view of (23),

$$\frac{\partial k}{\partial t} + \overline{\mathbf{u}} \cdot \nabla k = \operatorname{div}\left(\nu_D(k)\nabla k\right) + \nu_T(k)|\mathbf{D}(\overline{\mathbf{u}})|^2 - \epsilon(k). \tag{24}$$

4 A Stationary Problem

In this section, we consider a stationary version of the problem formulated by the Eqs. (8), (21), and (24),

$$\operatorname{div} \mathbf{u} = 0, \tag{25}$$

$$\mathbf{div}(\mathbf{u} \otimes \mathbf{u}) = \mathbf{g} - \mathbf{f}(\mathbf{u}) - \frac{1}{\rho}\nabla p + \mathbf{div}\left((\nu + \nu_T(k))\mathbf{D}(\mathbf{u})\right), \tag{26}$$

$$\mathbf{u} \cdot \nabla k = \operatorname{div}\left(\nu_D(k,\epsilon)\nabla k\right) + \nu_T|\mathbf{D}(\mathbf{u})|^2 + g - \epsilon(k), \tag{27}$$

where for the sake of simplifying the notation, we have omitted the bars over the filtered quantities. Observe also that in the last but one term of Eq. (27), we are considering a more general situation than in (24). We shall consider the problem posed by the Eqs. (25)–(27) in a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, with a compact boundary denoted by $\partial\Omega$. The problem (25)–(27) is supplemented by the following Dirichlet boundary conditions:

$$\mathbf{u} = \mathbf{0}, \quad k = 0 \quad \text{on } \partial\Omega. \tag{28}$$

For the analysis we make in this work, we assume the turbulent viscosity and the turbulent diffusivity are bounded

$$0 \le \nu_T(k) \le C_T, \quad c_D \le \nu_D(k) \le C_D, \tag{29}$$

where C_T, c_D, and C_D are positive constants. The following weak formulation of the problem gives us the notion of the solutions we are interested in to look for.

Definition 1. Let Ω be a bounded domain of $\mathbb{R}^d$, $d = 2, 3$, and assume that both conditions in (29) are fulfilled. In addition, assume that

$$\mathbf{g} \in \mathbf{L}^2(\Omega) \quad \text{and} \quad g \in L^q(\Omega) \text{ with } \frac{2d}{d+2} \le q < d'. \tag{30}$$

We say the couple $(\mathbf{u}, k)$ is a weak solution to the problem (25)–(28), if $\mathbf{u} \in \mathbf{V}$, $k \in W_0^{1,q}(\Omega)$, with $\frac{2d}{d+2} \le q < d'$, $\mathbf{f}(\mathbf{u}) \cdot \mathbf{v} \in \mathbf{L}^1(\Omega)$ and

$$\int_\Omega ((\mathbf{u}\cdot\nabla)\mathbf{u})\cdot\mathbf{v}\,d\mathbf{x}+\int_\Omega(\nu+\nu_T(k))\mathbf{D}(\mathbf{u}):\nabla\mathbf{v}\,d\mathbf{x}+\int_\Omega\mathbf{f}(\mathbf{u})\cdot\mathbf{v}\,d\mathbf{x}=\int_\Omega\mathbf{g}\cdot\mathbf{v}\,d\mathbf{x} \quad (31)$$

for any $\mathbf{v}\in\mathbf{V}\cap\mathbf{L}^d(\Omega)$, $\varepsilon(k)\varphi\in L^1(\Omega)$ and

$$\begin{aligned}&\int_\Omega(\mathbf{u}\cdot\nabla k)\varphi\,d\mathbf{x}+\int_\Omega\nu_D(k)\nabla k\cdot\nabla\varphi\,d\mathbf{x}+\int_\Omega\varepsilon(k)\varphi\,d\mathbf{x}\\&=\int_\Omega\nu_T(k)|\mathbf{D}(\mathbf{u})|^2\varphi\,d\mathbf{x}+\int_\Omega g\,\varphi\,d\mathbf{x}\end{aligned} \quad (32)$$

for any $\varphi\in W_0^{1,q'}(\Omega)$, with $q'>d$, and $k\geq 0$ and $\varepsilon\geq 0$ a.e. in Ω.

In [6, 7] we prove two distinct existence results for the problem (25)–(28) in the sense of Definition 1. For the first existence result, we assume growth conditions both on the feedback $\mathbf{f}(\mathbf{u})$ and on the function $\varepsilon(k)$ that describe the turbulent dissipation (see [6]). For the second, we consider the case in which these terms are strongly nonlinear, i.e., without assuming any restrictions on its growth (see [7]). We have already established local higher integrability results for the gradients of $\mathbf{u}$ and of k.

Both proofs use an iterative scheme to uncouple the Navier–Stokes equations from the equation for the turbulent kinetic energy. The analysis of the decoupled equations follows the approach of [1, 2], with respect to the truncation of the feedbacks, and the arguing of [3] for the treatment of the L^1 terms.

Theorem 1. *Let Ω be a bounded domain of $\mathbb{R}^d$, $d=2,3$, with a Lipschitz-continuous boundary $\partial\Omega$, and let $(\mathbf{u},k)$ be a weak solution to the problem (25)–(28) in the conditions of Definition 1.*

1. *If $\mathbf{g}\in\mathbf{L}^r(\Omega)$, with $r>2$, and $\mathbf{f}(\mathbf{u})\leq C|\mathbf{u}|^s$ for $0\leq s\leq\frac{d+2}{d-2}$ if $d\neq 2$ or any $s\geq 0$ if $s=2$, then there exists $\sigma>2$ such that $\nabla\mathbf{u}\in\mathbf{L}^\sigma(\Omega)$.*
2. *If $g\in L^r(\Omega)$, with $r>d'$, and $|\varepsilon(k)|\leq C|k|^s$ with $0\leq s\leq 2-\frac{2(d-2)}{d(d-1)}$ if $d\neq 2$ or any $s\geq 0$ if $d=2$, then there exists $\tau>\frac{d}{d-1}$ such that $\nabla k\in L^\tau(\Omega)$ as long as $\nabla\mathbf{u}\in\mathbf{L}^\sigma(\Omega)$ for $\sigma>\frac{2d(d-1)}{(d-1)^2+1}$.*

The following result of global higher integrability that, for the reason of lack of space, cannot be shown here will be proved elsewhere.

The proof of this result adapts the arguments of [16] (see also [8]) for the Navier–Stokes equations together with the reasoning of [13] (see also [3]) for the equation for the turbulent kinetic energy.

5 On the Uniqueness

We will prove the uniqueness by imposing conditions on the monotony of $\mathbf{f}(\mathbf{u})$ and $\varepsilon(k)$, as well as by imposing the Lipschitz continuity of $\nu_T(k)$ e $\nu_D(k)$.

Theorem 2. *Let Ω be a bounded domain of $\mathbb{R}^d$, $d = 2, 3$, with a Lipschitz-continuous boundary $\partial\Omega$, and let $(\mathbf{u}, k)$ be a weak solution to the problem (25)–(28) in the conditions of Theorem 1. If the following conditions are fulfilled for all $\mathbf{u}_1, \mathbf{u}_2 \in \mathbf{V}$ and for all $k_1, k_2 \in W_0^{1,q}(\Omega)$,*

$$(\mathbf{f}(\mathbf{u}_1) - \mathbf{f}(\mathbf{u}_2)) \cdot (\mathbf{u}_1 - \mathbf{u}_2) \geq 0, \qquad (\varepsilon(k_1) - \varepsilon(k_2))\,(k_1 - k_2) \geq 0, \tag{33}$$

$$|\nu_T(k_1) - \nu_T(k_2)| \leq C_{L_1}|k_1 - k_2|, \qquad |\nu_D(k_1) - \nu_D(k_2)| \leq C_{L_2}|k_1 - k_2|, \tag{34}$$

where C_{L_1} and C_{L_2} are positive constants and then the weak solution $(\mathbf{u}, k)$ is unique.

Remark 1. As we shall see in the proof, the above result is obtained under smallness assumptions on $\|\nabla\,\mathbf{u}\|_{\mathbf{L}^\sigma(\Omega)}$ and $\|\nabla k\|_{L^\tau(\Omega)}$ for $\sigma, \tau > 2$ *if* $d = 2$ *or* $\sigma, \tau \geq 2$ if $d \neq 2$, when compared with the kinematic viscosity ν and with the turbulent diffusivity lower bound c_D [see $(29)_2$].

Proof. Being $(\mathbf{u}_1, k_1)$ and $(\mathbf{u}_2, k_2)$ two solutions of the problem, we start by subtracting the corresponding equation (31) of the weak formulation where in both is taken $\mathbf{v} := \mathbf{u}_1 - \mathbf{u}_2$ for the test function.

After some algebraic manipulations and using the assumptions $(29)_2$ and $(33)_1$ together with Korn's inequality, we obtain

$$\begin{aligned}\frac{\nu}{C_K^2}\int_\Omega |\nabla(\mathbf{u}_1 - \mathbf{u}_2)|^2\,d\mathbf{x} \leq &-\int_\Omega (\nu_T(k_1) - \nu_T(k_2))\mathbf{D}(\mathbf{u}_2) : \nabla(\mathbf{u}_1 - \mathbf{u}_2)\,d\mathbf{x}\\ &-\int_\Omega ((\mathbf{u}_1 - \mathbf{u}_2)\cdot\nabla)\,\mathbf{u}_2\cdot(\mathbf{u}_1 - \mathbf{u}_2)\,d\mathbf{x} := I_1 + I_2\,,\end{aligned} \tag{35}$$

where C_K is the Korn's inequality constant. To estimate the term I_1, we use Hölder's and Sobolev's inequalities together with assertion 1 of Theorem 1, which states that $\sigma > 2$,

$$\begin{aligned}I_1 &\leq \|k_1 - k_2\|_{L^{2^*}(\Omega)}\|\nabla\mathbf{u}_2\|_{\mathbf{L}^\sigma(\Omega)}\|\nabla(\mathbf{u}_1 - \mathbf{u}_2)\|_{\mathbf{L}^2(\Omega)}\\ &\leq C_1\|\nabla(k_1 - k_2)\|_{L^2(\Omega)}\|\nabla(\mathbf{u}_1 - \mathbf{u}_2)\|_{\mathbf{L}^2(\Omega)}\,, \quad C_1 = C(d, \Omega, \|\nabla\mathbf{u}_2\|_{\mathbf{L}^\sigma(\Omega)})\,.\end{aligned}$$

For I_2, we use Hölder's and Sobolev's inequalities, this in the case of $d \leq 4$, to obtain

$$I_2 \leq \|\mathbf{u}_1 - \mathbf{u}_2\|^2_{\mathbf{L}^{2^*}(\Omega)}\|\nabla\mathbf{u}_2\|_{\mathbf{L}^\sigma(\Omega)} \leq C_2\|\nabla(\mathbf{u}_1 - \mathbf{u}_2)\|^2_{\mathbf{L}^2(\Omega)}\,,$$

where $C_2{=}C(d,\Omega,\|\nabla\mathbf{u}_2\|_{\mathbf{L}^\sigma(\Omega)})$. Now, gathering the estimates of I_1 and I_2 in (35), we obtain, after the use of Cauchy's inequality with suitable ϵ,

$$C_{\mathbf{u}}\|\nabla(\mathbf{u}_1-\mathbf{u}_2)\|^2_{\mathbf{L}^2(\Omega)} \leq C_I\|\nabla(k_1-k_2)\|^2_{L^2(\Omega)}\,,\ C_{\mathbf{u}}=\left(\frac{\nu}{2C_K^2}-C_2\right),\ C_I=\frac{C_1^2}{2\nu}. \tag{36}$$

Next, we subtract the Eq. (32) corresponding to k_1 and k_2 and taking for test function, in both, $\varphi=k_1-k_2$. After some simplifications and using the assumptions $(29)_1$ and $(33)_2$, we obtain

$$\begin{aligned} c_D\int_\Omega|\nabla(k_1-k_2)|^2\,d\mathbf{x} &\leq -\int_\Omega(\mathbf{u}_1\cdot\nabla k_1-\mathbf{u}_2\cdot\nabla k_2)(k_1-k_2)\,d\mathbf{x}\\ &-\int_\Omega(\nu_D(k_1)-\nu_D(k_2))\nabla k_2\cdot\nabla(k_1-k_2)\,d\mathbf{x}\\ &+\int_\Omega(\nu_T(k_1)|\mathbf{D}(\mathbf{u}_1)|^2-\nu_T(k_2)|\mathbf{D}(\mathbf{u}_2)|^2)(k_1-k_2)\,d\mathbf{x}:=J_1+J_2+J_3. \end{aligned} \tag{37}$$

After a simplification of J_1, we use Hölder's and Sobolev's inequalities, observing that $\tau>\frac{d}{d-1}$, to have

$$\begin{aligned} J_1 &\leq \|\mathbf{u}_1\|_{\mathbf{L}^\sigma(\Omega)}\|\nabla(k_1-k_2)\|_{L^2(\Omega)}\|k_1-k_2\|_{L^{2^*}(\Omega)}\\ &\quad+\|\mathbf{u}_1-\mathbf{u}_2\|_{\mathbf{L}^{2^*}(\Omega)}\|\nabla k_1\|_{L^\tau(\Omega)}\|\nabla(k_1-k_2)\|_{L^2(\Omega)}\\ &\leq C_1\|\nabla(k_1-k_2)\|^2_{L^2(\Omega)}+C_2\|\nabla(\mathbf{u}_1-\mathbf{u}_2)\|_{\mathbf{L}^2(\Omega)}\|\nabla(k_1-k_2)\|_{L^2(\Omega)}\\ &\leq C_{J_1}\|\nabla(k_1-k_2)\|^2_{L^2(\Omega)}\quad\text{by (36)},\quad C_{J_1}=C_1+C_2\sqrt{C_I/C_{\mathbf{u}}}\,, \end{aligned}$$

where $C_1=C(d,\Omega,\|\nabla\mathbf{u}_1\|_{\mathbf{L}^\sigma(\Omega)})$ and $C_2=C(d,\Omega,\|\nabla k_1\|_{L^\tau(\Omega)})$. As for the term J_2, we use assumption $(34)_2$ together with Hölder's and Sobolev's inequalities, the last again in the case of $\tau>\frac{d}{d-1}$, in the following way

$$J_2\leq C_{L_2}\|k_1-k_2\|_{L^{2^*}(\Omega)}|\nabla k_2\|_{\mathrm{L}^\tau(\Omega)}\|\nabla(k_1-k_2)\|_{\mathrm{L}^2(\Omega)}\leq C_{J_2}\|\nabla(k_1-k_2)\|^2_{L^2(\Omega)}\,,$$

where $C_{J_2}=C(C_{L_2},d,\Omega,\|\nabla k_2\|_{\mathrm{L}^\tau(\Omega)})$. The term J_3 is firstly simplified, and then we use the assumptions (29) and $(34)_1$ together with Hölder's and Sobolev's inequalities, and yet observing that $\sigma>2$,

$$\begin{aligned} J_3 &\leq C_{L_1}\|k_1-k_2\|^2_{L^{2^*}(\Omega)}\|\nabla\mathbf{u}_1\|^2_{\mathbf{L}^\sigma(\Omega)}\\ &\quad+C_T\|k_1-k_2\|_{L^{2^*}(\Omega)}\|\nabla(\mathbf{u}_1-\mathbf{u}_2)\|_{\mathbf{L}^2(\Omega)}\|\nabla\mathbf{u}_1+\nabla\mathbf{u}_2\|_{\mathbf{L}^\sigma(\Omega)}\\ &\leq C_1\|\nabla(k_1-k_2)\|^2_{L^2(\Omega)}+C_2\|\nabla(k_1-k_2)\|_{L^2(\Omega)}\|\nabla(\mathbf{u}_1-\mathbf{u}_2)\|_{\mathbf{L}^2(\Omega)}\\ &\leq C_{J_3}\|\nabla(k_1-k_2)\|^2_{L^2(\Omega)}\quad\text{by (36)},\quad C_{J_3}=C_1+C_2\sqrt{C_I/C_{\mathbf{u}}}\,, \end{aligned}$$

where $C_1 = C(C_{L_1}, d, \Omega, \|\nabla \mathbf{u}_1\|_{\mathbf{L}^{\sigma}(\Omega)})$ and $C_2 = C(C_T, \Omega, \|\nabla \mathbf{u}_1\|_{\mathbf{L}^{\sigma}(\Omega)}, \|\nabla \mathbf{u}_2\|_{\mathbf{L}^{\sigma}(\Omega)})$. Now, gathering the estimates of J_1, J_2, and J_3 in (37), we obtain $(c_D - C_J) \int_\Omega |\nabla(k_1 - k_2)|^2 \, d\mathbf{x} \leq 0$, where $C_J = \sum_{i=1}^{3} C_{J_i}$. As a consequence, it follows, by Sobolev's inequality, that $k_1 = k_2$ a.e. in Ω, as long as $c_d > C_J$. Consequently it follows from (36) that also $\mathbf{u}_1 = \mathbf{u}_2$ a.e. in Ω, as long as $\nu > 2C_2C_K^2$, where C_2 is the constant from the estimate of I_2.

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On Stationary Solutions of KdV and mKdV Equations

A.V. Faminskii and A.A. Nikolaev

Abstract Stationary solutions on a bounded interval for an initial-boundary value problem to Korteweg–de Vries and modified Korteweg–de Vries equation (for the last one both in focusing and defocusing cases) are constructed. The method of the study is based on the theory of conservative systems with one degree of freedom. The obtained solutions turn out to be periodic. Exact relations between the length of the interval and coefficients of the equations which are necessary and sufficient for the existence of nontrivial solutions are established.

Keywords KdV equation • Stationary solutions

AMS subject classifications: 35Q53, 34B15

Both Korteweg–de Vries equation (KdV)

$$u_t + au_x + u_{xxx} + uu_x = 0$$

and modified Korteweg–de Vries equation (mKdV)

$$u_t + au_x + u_{xxx} \pm u^2 u_x = 0$$

(the sign "+" stands for the focusing case and the sign "−" for the defocusing one) describe propagation of long nonlinear waves in dispersive media. We assume a to be an arbitrary real constant. If these equations are considered on a bounded interval $(0, L)$, then for well posedness of an initial-boundary value problem besides an initial profile, one must set certain boundary conditions, for example,

$$u\big|_{x=0} = u\big|_{x=L} = u_x\big|_{x=L} = 0$$

(see [5, 7, 8, 11] and others).

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It follows from the results of Faminskii and Larkin [9] that such a problem for KdV equation possesses certain internal dissipation: under some relations between a and L and sufficiently small initial data solution decay at large time. Similar properties hold for mKdV equation. In order to answer the question if the smallness is essential, one has to construct non-decaying solutions. The simplest case of such solutions is stationary solutions: $u = u(x)$. In this situation the considered equations are reduced to the following ordinary differential equations:

$$u''' + au' + uu' = 0, \tag{1}$$

$$u''' + au' + u^2u' = 0, \tag{2}$$

$$u''' + au' - u^2u' = 0, \tag{3}$$

and the boundary conditions—to the following ones:

$$u(0) = u(L) = u'(L) = 0. \tag{4}$$

The goal of the present paper is to investigate the existence of nontrivial solutions to these problems under different relations between a and L. The method of the study is based on the qualitative theory of conservative systems with one degree of freedom (see, for example, [4]).

The first example of such a solution by this method for Eq. (1) was constructed in the case $a = 0$ and $L = 2$ in [10]. In the recent paper [6] and also for Eq. (1), such solutions were constructed for $a = 1$ and $L \in (0, 2\pi)$, and exact formulas via elliptic Jacobi functions were obtained. In the present paper, these special functions are not used.

Lemma 1. *If $u \in C^3[0, L]$ is a solution to any problems (1), (4), or (2), (4), or (3), (4), then it is infinitely smooth and periodic with period L.*

Proof. Integrating each of the Eqs. (1)–(3), we obtain that the function u satisfies an equation

$$u'' + F'(u) = 0, \quad F(0) = 0, \quad F \in C^\infty. \tag{5}$$

The following [4] introduces a "full energy" $E(x) \equiv \frac{1}{2}\big(u'(x)\big)^2 + F\big(u(x)\big)$. Then, (5) yields that $E'(x) \equiv 0$, that is, $E(x) \equiv \text{const}$. By virtue of (4) $E(L) = 0$, therefore, $E(0) = 0$ and so $u'(0) = 0$. The end of the proof is obvious. □

Further let a fundamental period for a nontrivial periodic function denotes a minimal possible positive value of a period.

By a symbol $u_{a,T}$, denote a nontrivial solution to any of considered problems with the fundamental period T.

Theorem 1. *If $aL^2 \neq 4\pi^2$, then there exists a unique solution $u_{a,L}$ to problem (1), (4). If $aL^2 = 4\pi^2$, such a solution does not exist.*

Theorem 2. *If $aL^2 < 4\pi^2$, then there exists a unique up to the sign solution $u_{a,L}$ to problem (2), (4). If $aL^2 \geq 4\pi^2$, such solutions do not exist.*

Theorem 3. *If $aL^2 > 4\pi^2$, then there exists a unique up to the sign solution $u_{a,L}$ to problem (3), (4). If $aL^2 \leq 4\pi^2$, such solutions do not exist.*

Remark 1. If $aL^2 \neq 4\pi n^2$ for certain natural $n \geq 2$, then obviously the function $u(x) \equiv n^2 u_{a/n^2,L}(nx)$ is a solution to problems (1), (4) with the fundamental period $T = L/n$. If $aL^2 < 4\pi n^2$ for certain natural n, then the function $u(x) \equiv n u_{a/n^2,L}(nx)$ is a solution to problems (2), (4) with the fundamental period $T = L/n$. In particular, nontrivial solutions to problems (1), (4) and (2), (4) exist for any a and positive L. If $aL^2 \leq 4\pi^2$, then nontrivial solutions to problems (3), (4) do not exist.

Further for convenience we pass from the segment $[0, L]$ to the segment $[-1, 1]$. For $x \in [-1, 1]$ in the case of Eq. (1), make a substitution $y(x) \equiv \frac{L^2}{4} u\big(\frac{L}{2}(x + 1)\big)$, while in the case of Eqs. (2) and (3), substitution $y(x) \equiv \frac{L}{2} u\big(\frac{L}{2}(x + 1)\big)$. Then, for $b = \frac{L^2}{4} a$, these equations transform, respectively, to the following ones:

$$y''' + by' + yy' = 0, \tag{6}$$

$$y''' + by' + y^2 y' = 0, \tag{7}$$

$$y''' + by' - y^2 y' = 0, \tag{8}$$

and consider periodic solutions to these equations with the fundamental period $T = 2$ such that

$$y(-1) = y'(-1) = 0. \tag{9}$$

We apply the following lemma in the spirit of the qualitative theory of conservative systems with one degree of freedom.

Lemma 2. *Consider an initial value problem*

$$y'' + F'(y) = 0, \qquad y(-1) = y'(-1) = 0, \tag{10}$$

where $F \in C^\infty$, $F(0) = 0$. Then, a nontrivial periodic solution to problem (10) with the fundamental period $T = 2$ exists if and only if $F'(0) \neq 0$, and there exists $y_0 \neq 0$ such that $F(y_0) = 0$, $F'(y_0) \neq 0$, $F(y) < 0$ for $y \in (0, y_0)$ if $y_0 > 0$, $F(y) < 0$ for $y \in (y_0, 0)$ if $y_0 < 0$ and

$$\int_0^{y_0} \frac{dy}{\sqrt{-2F(y)}} = 1 \quad \text{if } y_0 > 0, \qquad \int_{y_0}^0 \frac{dy}{\sqrt{-2F(y)}} = 1 \quad \text{if } y_0 < 0. \tag{11}$$

Proof. First of all note that similarly to (5) $E(x) \equiv \frac{1}{2}\big(y'(x)\big)^2 + F\big(y(x)\big) \equiv 0$ if $y(x)$ is a solution to problem (10). Due to uniqueness of solutions to the initial value problem, the condition $F'(0) \neq 0$ is necessary for the existence of nontrivial solutions.

Consider, for example, the case $F'(0) < 0$. If the function F is negative $\forall y > 0$, then it is easy to see that there is no periodic solution to problem (10). Therefore, the existence of positive y_0 such that $F(y_0) = 0$, $F(y) < 0$ for $y \in (0, y_0)$ is necessary.

Uniqueness of the solution implies that the function $y(x)$ is even (if exists). Then, it is easy to see that it possesses the following properties: $y'(x) > 0$ for $x \in (-1, 0)$, $y'(x) < 0$ for $x \in (0, 1)$, and $y(0) = y_0$, $y'(0) = 0$. Again due to the uniqueness, $F'(y_0) \neq 0$.

Therefore, for $x \in [0, 1]$ the function $y(x)$ satisfies the following conditions:

$$\frac{dy}{dx} = -\sqrt{-2F(y)}, \quad y(0) = y_0, \quad y(1) = 0.$$

Integrating we obtain that $\displaystyle\int_0^{y_0} \frac{dy}{\sqrt{-2F(y)}} = 1$.

It is easy to see that under these assumptions, the desired solution exists. The case $F'(0) > 0$ is considered in a similar way (then $y_0 < 0$). □

Now we can prove our theorems.

Proof (Theorem 1). Equation (6) is equivalent to equation

$$y'' + by + \frac{1}{2}y^2 = c \tag{12}$$

for certain real constant c. Therefore, construction of a solution transforms to search of a constant c such for a function

$$F(y) \equiv \frac{1}{6}y^3 + \frac{b}{2}y^2 - cy = \frac{1}{6}y(y^2 + 3by - 6c) \equiv \frac{1}{6}yF_0(y)$$

the hypothesis of Lemma 2 is satisfied. Note that $F'(y) = \frac{1}{2}y^2 + by - c$. Therefore, the condition $F'(0) \neq 0$ implies that $c \neq 0$.

Real simple nonzero roots of the function F_0 exist if and only if $D = 9b^2 + 24c > 0$, and then these roots are expressed by formulas $y_0 = \frac{1}{2}(-3b + \sqrt{D})$ and $y_1 = -\frac{1}{2}(3b + \sqrt{D})$.

It is easy to see that if $c > 0$, then for any b the root $y_0 > 0$, $F(y) < 0$ for $y \in (0, y_0)$, $F'(y_0) \neq 0$. If $c \in (-3b^2/8, 0)$, then for $b > 0$ the root $y_0 < 0$, $F(y) < 0$ for $y \in (y_0, 0)$, $F'(y_0) \neq 0$.

Therefore, we have to find the constant c for which condition (11) is satisfied. Note that

$$-2F(y) = \frac{1}{3}y(y_0 - y)(y - y_1).$$

After the change of variable $y = y_0 t$, each of the Eq. (11) reduces to an equation

$$I(b, c) \equiv \sqrt{3}\int_0^1 \frac{dt}{\sqrt{t(1-t)(y_0 t - y_1)}} = 1.$$

Since $y_0 t - y_1 = \frac{1}{2}(\sqrt{D} - 3b)t + \frac{1}{2}(\sqrt{D} + 3b)$, it is easy to see that for the fixed b, the function $I(b, c)$ monotonically decreases. Moreover, $\lim\limits_{c \to +\infty} I(b, c) = 0$ and for $b > 0$

$$\lim_{c \to -\frac{3}{8}b^2 + 0} I(b, c) = \sqrt{\frac{2}{b}} \int_0^1 \frac{dt}{\sqrt{t}(1-t)} = +\infty, \quad \lim_{c \to 0} I(b, c) = \frac{1}{\sqrt{b}} \int_0^1 \frac{dt}{\sqrt{t(1-t)}} = \frac{\pi}{\sqrt{b}},$$

for $b = 0$

$$\lim_{c \to 0+0} I(b, c) = \lim_{c \to 0+0} \frac{1}{\sqrt{2c}} \int_0^1 \frac{dt}{\sqrt{t(1-t)(t+1)}} = +\infty,$$

for $b < 0$

$$\lim_{c \to 0+0} I(b, c) = \frac{1}{\sqrt{|b|}} \int_0^1 \frac{dt}{t\sqrt{1-t}} = +\infty.$$

Therefore, the desired value of c exists and is unique if $b \neq \pi^2$, while for $b = \pi^2$ such a value does not exist. ◻

Remark 2. The substitution $u(x) = a_0 + v(x - x_0)$ under the appropriate choice of the parameters a_0 and x_0 transforms any periodic solution of Eq. (1) with the period L to solution of an equation $v''' + (a + a_0)v' + vv' = 0$ satisfying conditions $v(0) = v'(0) = v(L) = v'(L) = 0$. Therefore, any solution of Eq. (1) with the fundamental period L can be expressed in this way by the functions $u_{a+a_0,L}$. Solutions similar to functions $u_{a,L}$ were considered also in [13]. In [2] representation of periodic solutions of Eq. (1) is given via elliptic Jacobi functions. The advantage of our approach is that it can give transparent description of solutions.

Consider, for example, the case $b > 0$. Then, for $b \in (0, \pi^2)$ the constructed solution of problems (6), (9) is an even "hill" of the height $y_0 = \frac{1}{2}(-3b + \sqrt{9b^2 + 24c}) > 0$, while for $b > \pi^2$, an even "hole" of the depth $y_0 < 0$. Note that $I_c(b, c) < 0$, $I_b(b, c) < 0$. Therefore, the equation $I(b, c) = 1$ determines a smooth decreasing function $c(b)$. Since $I(\pi^2, 0) = 1$, we have that $c(\pi^2) = 0$. Return to Eq. (1). Let $a > 0$. If $u_0 = \frac{1}{2}(-3a + \sqrt{9a^2 + 384cL^{-2}})$, where $c = c(L^2a/4)$, then for $L < 2\pi/\sqrt{a}$, the solution $u_{a,L}$ to problems (1), (4) is a "hill" of the height $u_0 > 0$ and for $L > 2\pi/\sqrt{a}$, a "hole" of the depth $u_0 < 0$ (the center in both cases is at the point $L/2$). In addition, $u_0 \to +\infty$ as $L \to 0$, $u_0 \to 0$ as $L \to 2\pi/\sqrt{a}$, $u_0 \to 0$ as $L \to +\infty$.

Proof (Theorem 2). Equation (7) is equivalent to equation

$$y'' + by + \frac{1}{3}y^3 = c \tag{13}$$

for certain real constant c. Let

$$F(y) \equiv \frac{1}{12}y^4 + \frac{b}{2}y^2 - cy = \frac{1}{12}y(y^3 + 6by - 12c) \equiv \frac{1}{12}yF_0(y).$$

Note that the substitution $z(x) \equiv -y(x)$ leads to an equation similar to (13), where c is replaced by $(-c)$. Therefore, further it is sufficient to assume that $c > 0$ (if $c = 0$, then $F'(0) = 0$).

Similarly to the proof of Theorem 1, we need to find the roots of the function F_0. We apply Cardano formulas. Let $D = 8b^3 + 36c^2$,

$$p = \sqrt[3]{6c + \sqrt{D}}, \quad q = \sqrt[3]{6c - \sqrt{D}} \quad \text{if } D \geq 0,$$
$$p = \sqrt[3]{6c + i\sqrt{|D|}} = \sqrt{2|b|}e^{\frac{i}{3}\arccos(3c/\sqrt{2|b|^3})}, \quad q = \overline{p} \quad \text{if } D < 0.$$

The function F_0 has a real root $y_0 = p + q > 0$. Moreover, if $D > 0$ there are two complex conjugate roots with negative real parts and if $D \leq 0$ (it is possible only for $b < 0$), two negative real roots y_1 and y_2 ($y_1 = y_2$ if $D = 0$).

According to Viète formulas, $y_1 + y_2 = -y_0$, $y_1 y_2 = 6b - y_0 y_1 - y_0 y_2 = 6b + y_0^2$ and then

$$-2F(y) = \frac{1}{6}y(y_0 - y)(y^2 + y_0 y + y_0^2 + 6b).$$

After the change of variable $y = y_0 t$, the first equation (11) reduces to an equation

$$I(b,c) \equiv \sqrt{6}\int_0^1 \frac{dt}{\sqrt{t(1-t)(y_0^2 t^2 + y_0^2 t + y_0^2 + 6b)}} = 1.$$

It is easy to see that for the fixed b the function $y_0(c)$ monotonically increases and $y_0(c) \to +\infty$ as $c \to +\infty$ (note that $y_0 = \sqrt{8|b|}\cos\left(\frac{1}{3}\arccos(3c/\sqrt{2|b|^3})\right)$ if $D < 0$). Then, for the fixed b the function $I(b,c)$ monotonically decreases and $\lim\limits_{c \to +\infty} I(b,c) = 0$. Moreover, if $c \to 0+0$, then $y_0(c) \to 0$ for $b \geq 0$ and $y_0(c) \to \sqrt{6|b|}$ for $b < 0$. Therefore,

$$\lim_{c \to 0+0} I(b,c) = \frac{1}{\sqrt{b}}\int_0^1 \frac{dt}{\sqrt{t(1-t)}} = \frac{\pi}{\sqrt{b}} \quad \text{if } b > 0,$$

$$\lim_{c \to 0+0} I(b,c) = \lim_{c \to 0+0} \frac{\sqrt{6}}{\sqrt[3]{12c}}\int_0^1 \frac{dt}{\sqrt{t(1-t^3)}} = +\infty \quad \text{if } b = 0,$$

$$\lim_{c \to 0+0} I(b,c) = \frac{1}{\sqrt{|b|}}\int_0^1 \frac{dt}{t\sqrt{1-t^2}} = +\infty \quad \text{if } b < 0.$$

Hence, the desired positive value of c exists and is unique if $b < \pi^2$, while for $b \geq \pi^2$, such a value does not exist. □

Proof (Theorem 3). Equation (8) is equivalent to equation

$$y'' + by - \frac{1}{3}y^3 = c \tag{14}$$

for certain real constant c. Let

$$F(y) \equiv -\frac{1}{12}y^4 + \frac{b}{2}y^2 - cy = -\frac{1}{12}y(y^3 - 6by + 12c) \equiv -\frac{1}{12}yF_0(y).$$

As in the proof of Theorem 2, consider only the case $c > 0$.

Again apply Cardano formulas. Let $D = -8b^3 + 36c^2$,

$$p = \sqrt[3]{-6c + \sqrt{D}}, \quad q = \sqrt[3]{-6c - \sqrt{D}} \quad \text{if } D \geq 0,$$
$$p = \sqrt[3]{-6c + i\sqrt{|D|}} = \sqrt{2b}e^{\frac{i}{3}\left(\pi + \arccos(3c/\sqrt{2b^3})\right)}, \quad q = \overline{p} \quad \text{if } D < 0.$$

If $D > 0$ then the function F_0 has a real root $y_0 = p + q < 0$ and two complex conjugate roots y_1 and y_2. If $D = 0$ then again the function F_0 has a real root $y_0 = p + q < 0$ and a double real root $y_1 = y_2 > 0$. Both these two cases do not satisfy the hypothesis of Lemma 2 since $F'(0) < 0$.

It remains to consider the case $D < 0$ (it is possible only if $b > 0$), then $c \in (0, \frac{\sqrt{2}}{3}b^{3/2})$. Here the function F_0 has three distinct real roots, where a root $y_0 = p + q = \sqrt{8b}\cos\left(\frac{\pi}{3} + \frac{1}{3}\arccos(3c/\sqrt{2b^3})\right) > 0$, a root $y_1 < 0$, a root $y_2 > y_0$. We have that $y_1 + y_2 = -y_0$, $y_2y_2 = -6b + y_0^2$ and then

$$-2F(y) = \frac{1}{6}y(y_0 - y)(6b - y_0^2 - y_0y - y^2).$$

After the change of variable $y = y_0t$, the first equation (11) reduces to an equation

$$I(b, c) \equiv \sqrt{6}\int_0^1 \frac{dt}{\sqrt{t(1-t)(6b - y_0^2(1 + t + t^2))}} = 1.$$

Similarly to the previous theorem for the fixed b, the function $y_0(c)$ monotonically increases; therefore, unlike the previous theorem, the function $I(b, c)$ also monotonically increases. It is easy to see that

$$\lim_{c \to 0+0} I(b, c) = \frac{1}{\sqrt{b}}\int_0^1 \frac{dt}{\sqrt{t(1-t)}} = \frac{\pi}{\sqrt{b}},$$

$$\lim_{c \to \frac{\sqrt{2}}{3} b^{3/2} - 0} I(b,c) = \sqrt{\frac{3}{b}} \int_0^1 \frac{dt}{(1-t)\sqrt{t(t+2)}} = +\infty.$$

Hence, the desired positive value of c exists and is unique if $b > \pi^2$, while for $b \le \pi^2$, such a value does not exist. □

Remark 3. In [1, 3, 12] periodic solutions of Eqs. (2) and (3) were considered in the case when the constant $c = 0$ in Eqs. (13) and (14). Therefore, the periodic solutions constructed in the present paper do not coincide with solutions from that papers.

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A Filippov-Type Existence Theorem for Some Nonlinear q-Difference Inclusions

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Abstract We study two classes of boundary value problems associated to nonlinear q-difference inclusions and our aim is to show that Filippov's ideas can be suitably adapted in order to obtain the existence of solutions for the problems considered. Note that for a differential inclusion defined by a lipschitzian set-valued map with nonconvex values, Filippov's theorem consists in proving the existence of a solution starting from a given "quasi" solution. Moreover, the result provides an estimate between the "quasi" solution and the solution obtained. In this way we improve some existing results in the literature.

Keywords Difference inclusion • Set-valued map • Existence of solution

AMS subject classifications: 34A60

1 Introduction

In the last years, we may see a strong development of the study of boundary value problems associated to q-difference equations and inclusions as one can see in [1–3, 7, 8] etc. A reason is that in numerical analysis instead of the standard discretization of the ordinary differential equations based on the arithmetic progression, it can be used as the q-discretization related to geometric progression. These alternative methods lead to q-difference equations which at limit $q \to 1$ correspond to the classical differential equations. On the other hand, the q-difference equations are also useful in the theory of quantum groups [6].

In this note we consider the following problems:

$$D_q^3 x(t) \in F(t, x(t)), \quad t \in J, \quad x(0) = 0, \quad D_q x(0) = 0, \quad x(1) = 0, \tag{1}$$

$$D_q^2 x(t) \in F(t, x(t)), \quad t \in J, \quad x(0) = \eta x(1), \quad D_q x(0) = \eta D_q x(1), \tag{2}$$

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where D_q, D_q^2, D_q^3 denotes the first, the second, and the third order q-derivative, respectively, $I = [0,1]$, $J = \{q^n, \quad n \in \mathbf{N}\} \cup \{0,1\}$, $q \in (0,1)$, $\eta \neq 1$ and $F : I \times \mathbf{R} \to \mathscr{P}(\mathbf{R})$ is a set-valued map not necessarily convex valued.

Existence results for problems (1) and (2) were obtained in [1] and in [2], respectively, for convex as well as nonconvex set-valued maps. All the results in [1, 2] are obtained using fixed point techniques.

The aim of this note is to show that Filippov's ideas [5] can be suitably adapted in order to obtain the existence of solutions for problems (1) and (2). Recall that for a differential inclusion defined by a Lipschitzian set-valued map with nonconvex values, Filippov's theorem [5] consists in proving the existence of a solution starting from a given "quasi" solution. Moreover, the result provides an estimate between the "quasi" solution and the solution obtained. In this way we improve some results in [1, 2].

The paper is organized as follows: in Sect. 2 we recall some preliminary results that we need in the sequel, and in Sect. 3, we prove our main results.

2 Preliminaries

Let (X,d) be a metric space. Recall that the Pompeiu–Hausdorff distance of the closed subsets $A, B \subset X$ is defined by

$$d_H(A,B) = \max\{d^*(A,B), d^*(B,A)\},\ d^*(A,B) = \sup\{d(a,B); a \in A\},$$

where $d(x,B) = \inf_{y \in B} d(x,y)$.

We denote by $C(J,\mathbf{R})$ the Banach space of all continuous functions from J to $\mathbf{R}$ with the norm $||x(.)||_C = \sup_{t\in J} |x(t)|$ and $L^1(I,\mathbf{R})$ is the Banach space of integrable functions $u(.) : I \to \mathbf{R}$ endowed with the norm $||u(.)||_1 = \int_0^1 |u(t)|dt$.

We recall next some basic facts from q-calculus [6].

For $q \in (0,1)$ the q-derivative of a real-valued function f is defined by

$$D_q f(t) = \frac{f(t) - f(qt)}{(1-q)t}, \quad D_q f(0) = \lim_{t\to 0} D_q f(t).$$

The higher-order q-derivatives are given by

$$D_q^0 f(t) = f(t), \quad D_q^n f(t) = D_q D_q^{n-1} f(t), \quad n \in \mathbf{N}.$$

For example, $D_q(t^k) = [k]_q t^{k-1}$ where $[k]_q = \frac{q^k-1}{q-1}$, $k \in \mathbf{N}$. In particular, $D_q(t^2) = (1+q)t$.

Note that for f differentiable at t, we have $\lim_{q\to 1-} D_q f(t) = f'(t)$.

For $y \geq 0$ denote $J_y = \{yq^n, \quad n \in \mathbf{N}\} \cup \{0\}$ and define the q-integral of the function $f : J_y \to \mathbf{R}$ by

$$I_q f(y) = \int_0^y f(s) d_q s = \sum_{n=0}^{\infty} y(1-q) q^n f(yq^n),$$

provided that the series converges. If $b_1 = yq^{n_1}$, $b_2 = yq^{n_2}$, $n_1, n_2 \in \mathbf{N}$, one defines

$$\int_{b_1}^{b_2} f(s) d_q s = I_q f(b_2) - I_q f(b_1) = (1-q) \sum_{n=0}^{\infty} q^n [b_2 f(b_2) - b_1 f(b_1)].$$

Similarly, one has

$$I_q^0 f(t) = f(t), \quad I_q^n f(t) = I_q I_q^{n-1} f(t), \quad n \in \mathbf{N}.$$

Note that $D_q I_q f(t) = f(t)$ and if f is continuous at $t = 0$, then $I_q D_q f(t) = f(t) - f(0)$. In particular, it follows that if $D_q I_q f(t) = g(t)$, then $f(t) = I_q g(t) + c$ with $c \in \mathbf{R}$ arbitrary.

The product rule and the integration by parts formula are

$$D_q(gh)(t) = D_q g(t) h(t) + g(qt) D_q h(t),$$

$$\int_0^x f(t) D_q g(t) d_q t = [f(t) g(t)]_0^x - \int_0^x D_q f(t) g(qt) d_q t.$$

At the limit $q \to 1-$, the above statements correspond to their counterparts in standard calculus.

We recall the next two technical results are proven in [1] and in [2], respectively.

Lemma 1. *Let $f : J \to \mathbf{R}$ be continuous. The solution of the problem*

$$D_q^3 x(t) = f(t), \quad x(0) = 0, \quad D_q x(0) = 0, \quad x(1) = 0,$$

is given by $x(t) = \int_0^1 G(t,s,q) f(s) d_q s$, where $G(.,.,.)$ is the Green's function given by

$$G(t,s,q) = \begin{cases} qs(1-t)[q^2 s(1+t) - (1+q)t] & \text{if} \quad 0 \le s < t \le 1, \\ t^2(1-qs)(q^2 s - 1) & \text{if} \quad 0 \le t \le s \le 1. \end{cases}$$

Denote $M := \max_{t,s \in I} |G(t,s,q)|$.

Lemma 2. *Let $f : J \to \mathbf{R}$ be continuous. The solution of the problem*

$$D_q^2 x(t) = f(t), \quad x(0) = \eta x(1), \quad D_q x(0) = \eta D_q x(1),$$

is given by $x(t) = \int_0^1 G_1(t,s,q) f(s) d_q s$, where $G_1(.,.,.)$ is the Green function given by

$$G_1(t,s,q) = \frac{1}{(\eta-1)^2} \begin{cases} \eta(\eta-1)(qs-t)+\eta & \text{if} \quad 0 \le t < s \le 1, \\ (\eta-1)(qs-t)+\eta & \text{if} \quad 0 \le s \le t \le 1. \end{cases}$$

Denote $M_1 := \max_{t,s\in I} |G_1(t,s,q)|$.

Definition 1.

a) A function $x(.) \in C(J,\mathbf{R})$ is a solution of problem (1.1) if there exists a function $f(.) \in L^1(J,\mathbf{R})$ that satisfies $f(t) \in F(t,x(t))$ *a.e.* (J) and $x(t) = \int_0^1 G(t,s,q)f(s)d_qs$, where $G(.,.,.)$ is defined in Lemma 1.
b) A function $x(.) \in C(J,\mathbf{R})$ is a solution of problem (1.2) if there exists a function $f(.) \in L^1(J,\mathbf{R})$ satisfying $f(t) \in F(t,x(t))$ *a.e.* (J) and $x(t) = \int_0^1 G_1(t,s,q)f(s)d_qs$, where $G_1(.,.,.)$ is defined in Lemma 2.

3 The Main Results

First we recall a selection result (e.g., [4]) which is a version of the celebrated Kuratowski and Ryll-Nardzewski selection theorem.

Lemma 3. *Consider X a separable Banach space, B is the closed unit ball in X, $H : J \to \mathcal{P}(X)$ is a set-valued map with nonempty closed values, and $g : J \to X, L : J \to \mathbf{R}_+$ are measurable functions. If*

$$H(t) \cap (g(t) + L(t)B) \neq \emptyset \quad a.e.\ (J),$$

then the set-valued map $t \to H(t) \cap (g(t) + L(t)B)$ has a measurable selection.

In order to prove our results, we need the following hypotheses:

Hypothesis.

i) *$F(.,.) : I \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ has nonempty closed values and $F(.,x)$ is measurable for any $x \in \mathbf{R}$.*
ii) *There exists $L(.) \in L^1(I,(0,\infty))$ such that $F(t,.)$ is $L(t)$-Lipschitz in the sense that*

$$d_H(F(t,x_1),F(t,x_2)) \le L(t)|x_1 - x_2| \quad \forall\ x_1,x_2 \in \mathbf{R}.$$

Denote $L_0 = \int_0^1 L(s)ds$.

Theorem 1. *Assume that the hypothesis is satisfied and $ML_0 < 1$. Let $y(.) \in C(J,\mathbf{R})$ be such that $y(0) = 0$, $D_qy(0) = 0$, $y(1) = 0$ and there exists $p(.) \in L^1(I,\mathbf{R}_+)$ verifying $d(D_q^3y(t),F(t,y(t))) \le p(t)$ a.e. (J).*

Then, there exists $x(.)$, a solution of problem (1) satisfying for all $t \in J$

$$|x(t) - y(t)| \leq \frac{M}{1 - ML_0} \int_0^1 p(t)dt. \tag{3}$$

Proof. The set-valued map $t \to F(t, y(t))$ is measurable with closed values and

$$F(t, y(t)) \cap \{D_q^3 y(t) + p(t)[-1, 1]\} \neq \emptyset \quad a.e.\ (J).$$

It follows from Lemma 3 that there exists a measurable selection $f_1(t) \in F(t, y(t))$ $a.e.$ (J) such that

$$|f_1(t) - D_q^3 y(t)| \leq p(t) \quad a.e.\ (J) \tag{4}$$

Define $x_1(t) = \int_0^1 G(t, s, q) f_1(s) d_q s$ and one has

$$|x_1(t) - y(t)| \leq M \int_0^1 p(t) d_q t \leq M \int_0^1 p(t)dt.$$

We claim that it is enough to construct the sequences $x_n(.) \in C(J, \mathbf{R})$, $f_n(.) \in L^1(J, \mathbf{R})$, $n \geq 1$ with the following properties:

$$x_n(t) = \int_0^1 G(t, s, q) f_n(s) d_q s, \quad t \in J, \tag{5}$$

$$f_n(t) \in F(t, x_{n-1}(t)) \quad a.e.\ (J), \tag{6}$$

$$|f_{n+1}(t) - f_n(t)| \leq L(t)|x_n(t) - x_{n-1}(t)| \tag{7}$$

for almost all $t \in J$.

If this construction is realized, then from (4)–(7) we have, for $t \in J$,

$$|x_{n+1}(t) - x_n(t)| \leq M(ML_0)^n \int_0^1 p(t)dt \quad \forall n \in \mathbf{N}.$$

Indeed, assume that the last inequality is true for $n - 1$ and we prove it for n. One has

$$\begin{aligned}
|x_{n+1}(t) - x_n(t)| &\leq \int_0^1 |G(t, t_1, q)|.|f_{n+1}(t_1) - f_n(t_1)| d_q t_1 \\
&\leq M \int_0^1 L(t_1)|x_n(t_1) - x_{n-1}(t_1)| d_q t_1 \\
&\leq M \int_0^1 L(t_1) d_q t_1 M(ML_0)^n \int_0^1 p(t)dt \\
&\leq M \int_0^1 L(t_1) dt_1 M(ML_0)^n \int_0^1 p(t)dt = M(ML_0)^{n+1} \int_0^1 p(t)dt.
\end{aligned}$$

Therefore, $\{x_n(.)\}$ is a Cauchy sequence in the Banach space $C(J,\mathbf{R})$, hence converging uniformly to some $x(.) \in C(J,\mathbf{R})$. Therefore, by (7), for almost all $t \in J$, the sequence $\{f_n(t)\}$ is Cauchy in $\mathbf{R}$. Let $f(.)$ be the pointwise limit of $f_n(.)$.

Moreover, one has

$$|x_n(t)-y(t)| \le |x_1(t)-y(t)| + \sum_{i=1}^{n-1} |x_{i+1}(t)-x_i(t)|$$

$$\le M\int_0^1 p(t)dt + \sum_{i=1}^{n-1}(M\int_0^1 p(t)dt)(ML_0)^i = \frac{M\int_0^1 p(t)dt}{1-ML_0}. \tag{8}$$

On the other hand, from (4), (7), and (8), we obtain for almost all $t \in J$

$$|f_n(t)-D_q^3 y(t)| \le \sum_{i=1}^{n-1} |f_{i+1}(t)-f_i(t)| + |f_1(t)-D_q^3 y(t)| \le L(t)\frac{M\int_0^1 p(t)dt}{1-ML_0} + p(t).$$

Hence, the sequence $f_n(.)$ is integrably bounded and therefore $f(.) \in L^1(J,\mathbf{R})$.

Using Lebesgue's dominated convergence theorem and taking the limit in (5), (6), we deduce that $x(.)$ is a solution of (1). Finally, passing to the limit in (8), we obtained the desired estimate on $x(.)$.

It remains to construct the sequences $x_n(.), f_n(.)$ with the properties in (5)–(7). The construction will be done by induction.

Since the first step is already realized, assume that for some $N \ge 1$, we already constructed $x_n(.) \in C(J,\mathbf{R})$ and $f_n(.) \in L^1(J,\mathbf{R})$, $n = 1,2,\ldots N$ satisfying (5), (7) for $n = 1,2,\ldots N$ and (6) for $n = 1,2,\ldots N-1$. The set-valued map $t \to F(t,x_N(t))$ is measurable. Moreover, the map $t \to L(t)|x_N(t)-x_{N-1}(t)|$ is measurable. By the Lipschitzianity of $F(t,.)$, we have that for almost all $t \in J$

$$F(t,x_N(t)) \cap \{f_N(t) + L(t)|x_N(t)-x_{N-1}(t)|[-1,1]\} \neq \emptyset.$$

Lemma 3 yields that there exists a measurable selection $f_{N+1}(.)$ of $F(.,x_N(.))$ such that

$$|f_{N+1}(t)-f_N(t)| \le L(t)|x_N(t)-x_{N-1}(t)| \quad a.e.\ (J).$$

We define $x_{N+1}(.)$ as in (5) with $n = N+1$. Thus, $f_{N+1}(.)$ satisfies (6) and (7) and the proof is complete.

The assumption in Theorem 1 is satisfied, in particular, for $y(.) = 0$ if we assume that $d(0,F(t,0)) \le p(t)$ a.e. (J). We obtain the following consequence of Theorem 1.

Corollary 1. *Assume that the hypothesis is satisfied,* $d(0,F(t,0)) \le L(t)$ *a.e.* (I) *and* $ML_0 < 1$. *Then, there exists* $x(.)$ *a solution of problem (1) satisfying for all* $t \in J$

$$|x(t)| \leq \frac{ML_0}{1 - ML_0}. \tag{9}$$

Remark 1. A similar result to the one in Corollary 1 may be found in [1], namely, Theorem 5, but without a priori estimates as in (9).

With the same proof as the proof of Theorem, we obtain a similar result for problem (2).

Theorem 2. *Assume that the hypothesis is satisfied and $M_1 L_0 < 1$. Let $y(.) \in C(J, \mathbf{R})$ be such that $y(0) = \eta x(1)$, $D_q y(0) = \eta D_q y(1)$ and there exists $p(.) \in L^1(I, \mathbf{R}_+)$ verifying $d(D_q^2 y(t), F(t, y(t))) \leq p(t)$ a.e. (J).*

Then, there exists $x(.)$ a solution of problem (2) satisfying for all $t \in J$

$$|x(t) - y(t)| \leq \frac{M_1}{1 - M_1 L_0} \int_0^1 p(t)dt.$$

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Complex-Valued Fractional Derivatives on Time Scales

Benaoumeur Bayour and Delfim F.M. Torres

Abstract We introduce a notion of fractional (noninteger order) derivative on an arbitrary nonempty closed subset of the real numbers (on a time scale). Main properties of the new operator are proved and several illustrative examples given.

Keywords Fractional calculus • Calculus on time scales • Complex-valued operator • Hilger derivative of noninteger order

Mathematics Subject Classification (2010): 26A33; 26E70

1 Introduction

The study of fractional (noninteger) order derivatives on discrete, continuous and, more generally, arbitrary nonempty closed set (i.e., a time scale) is a well-known subject under strong current development. The subject is very rich and several different definitions and approaches are available, either in discrete [1], continuous [11], and time-scale settings [2]. In continuous time, i.e., for the time scale $\mathbb{T} = \mathbb{R}$, several definitions are based on the classical Euler Gamma function Γ. For the time scale $\mathbb{T} = \mathbb{Z}$, the Gamma function is nothing else than the factorial, while for the q-scale, one has the q-Gamma function Γ_q [9]. For the definition of Gamma function on an arbitrary time scale $\mathbb{T}$, see [6]. Similarly to [2, 3], here we introduce a new notion of fractional derivative on an arbitrary time scale $\mathbb{T}$ that does not involve Gamma functions. Our approach is, however, different from the ones available in the literature [2–5]. In particular, while in [2–5] the fractional derivative at a point

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is always a real number, here, in contrast, the fractional derivative at a point is, in general, a complex number. For example, the derivative of order $\alpha \in (0,1]$ of the square function t^2 is always given by $t^\alpha + (\sigma(t))^\alpha$, where $\sigma(t)$ is the forward jump operator of the time scale, which is in general a complex number (e.g., for $\alpha = 1/2$ and $t < 0$) and a generalization of the Hilger derivative $(t^2)^\Delta = t + \sigma(t)$.

The text is organized as follows. In Sect. 2 we recall the notion of Hilger/delta derivative. Our complex-valued fractional derivative on time scales is introduced in Sect. 3, where its main properties are proved and several examples given.

2 Preliminaries

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. For $t \in \mathbb{T}$, we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$, and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$. Then, one defines the graininess function $\mu : \mathbb{T} \to [0, +\infty[$ by $\mu(t) = \sigma(t) - t$.

If $\sigma(t) > t$, then we say that t is right scattered; if $\rho(t) < t$, then t is left scattered. Moreover, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called right dense; if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called left dense. If $\mathbb{T}$ has a left-scattered maximum m, then we define $\mathbb{T}^\kappa = \mathbb{T} \setminus \{m\}$; otherwise $\mathbb{T}^\kappa = \mathbb{T}$. If $f : \mathbb{T} \to \mathbb{R}$, then $f^\sigma : \mathbb{T} \to \mathbb{R}$ is given by $f^\sigma(t) = f(\sigma(t))$ for all $t \in \mathbb{T}$.

Definition 1 (The Hilger Derivative [8]). Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}$. We define $f^\Delta(t)$ to be the number (provided it exists) with the property that given any $\epsilon > 0$ there is a neighborhood U of t (i.e., $U = (t-\delta, t+\delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]| \leq \epsilon |\sigma(t) - s|$$

for all $s \in U$. We call $f^\Delta(t)$ the Hilger (or delta) derivative of f at t.

For more on the calculus on time scales, we refer the reader to the books [7, 8].

3 Complex-Valued Fractional Derivatives on Time Scales

Let $f : \mathbb{T} \to \mathbb{R}$ with $\mathbb{T}$ a given time scale. We introduce here a new definition of fractional (noninteger) delta derivative of order $\alpha \in (0,1]$ at a point $t \in \mathbb{T}^\kappa$.

Definition 2 (The Delta Fractional Derivative of Order α). Assume $f : \mathbb{T} \to \mathbb{R}$ with $\mathbb{T}$ a time scale. Let $t \in \mathbb{T}^\kappa$ and $\alpha \in (0,1]$. We define $f^{\Delta^\alpha}(t)$ to be the number (provided it exists) with the property that given any $\epsilon > 0$ there is a neighborhood U of t (i.e., $U = (t-\delta, t+\delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$\left|[f^\alpha(\sigma(t)) - f^\alpha(s)] - f^{\Delta^\alpha}(t)[\sigma(t)^\alpha - s^\alpha]\right| \leq \epsilon\, |\sigma(t)^\alpha - s^\alpha| \qquad (1)$$

for all $s \in U$. We call $f^{\Delta^\alpha}(t)$ the delta derivative of order α of f at t or the delta fractional (noninteger order) derivative of f at t. Moreover, we say that f is delta differentiable of order α on $\mathbb{T}^\kappa$ provided $f^{\Delta^\alpha}(t)$ exists for all $t \in \mathbb{T}^\kappa$. Function f^{Δ^α} : $\mathbb{T}^\kappa \to \mathbb{C}$ is then called the delta derivative of order α of f on $\mathbb{T}^\kappa$.

Remark 1. In (1) we use f^α to denote the power α of f. It is clear that the new derivative coincides with the standard Hilger derivative in the integer order case $\alpha = 1$. Differently from $\alpha = 1$, in general $f^{\Delta^\alpha}(t)$ is a complex number.

Theorem 1. *Assume $f : \mathbb{T} \to \mathbb{R}$ with $\mathbb{T}$ a time scale. Let $t \in \mathbb{T}^\kappa$ and $\alpha \in \mathbb{R}$. Then, the following proprieties hold:*

1. *If f is continuous at t and t is right scattered, then f is delta differentiable of order α at t with*

$$f^{\Delta^\alpha}(t) = \frac{f^\alpha(\sigma(t)) - f^\alpha(t)}{\sigma^\alpha(t) - t^\alpha}. \tag{2}$$

2. *If t is right dense, then f is delta differentiable of order α at t if and only if the limit*

$$\lim_{s \to t} \frac{f^\alpha(t) - f^\alpha(s)}{t^\alpha - s^\alpha}$$

exists as a finite number. In this case

$$f^{\Delta^\alpha}(t) = \lim_{s \to t} \frac{f^\alpha(t) - f^\alpha(s)}{t^\alpha - s^\alpha}. \tag{3}$$

3. *If f is delta differentiable of order α at t, then*

$$f^\alpha(\sigma(t)) = f^\alpha(t) + (\sigma(t)^\alpha - t^\alpha) f^{\Delta^\alpha}(t).$$

Proof. 1. Assume f is continuous at t and t is right scattered. By continuity,

$$\lim_{s \to t} \frac{f^\alpha(\sigma(t)) - f^\alpha(s)}{\sigma^\alpha(t) - s^\alpha} = \frac{f^\alpha(\sigma(t)) - f^\alpha(t)}{\sigma^\alpha(t) - t^\alpha}.$$

Hence, given $\epsilon > 0$, there is a neighborhood U of t such that

$$\left| \frac{f^\alpha(\sigma(t)) - f^\alpha(s)}{\sigma^\alpha(t) - s^\alpha} - \frac{f^\alpha(\sigma(t)) - f^\alpha(t)}{\sigma^\alpha(t) - t^\alpha} \right| \leq \epsilon$$

for all $s \in U$. It follows that

$$\left| f^\alpha(\sigma(t)) - f^\alpha(s) - \frac{f^\alpha(\sigma(t)) - f^\alpha(t)}{\sigma^\alpha(t) - t^\alpha} [\sigma^\alpha(t) - s^\alpha] \right| \leq \epsilon \, |\sigma^\alpha(t) - s^\alpha|$$

for all $s \in U$. Hence, we get the desired result (2).

2. Assume f is differentiable at t and t is right dense. Let $\epsilon > 0$ be given. Since f is differentiable at t, there is a neighborhood U of t such that

$$\mid [f^{\alpha}(\sigma(t)) - f^{\alpha}(s)] - f^{\Delta^{\alpha}}(t)[\sigma^{\alpha}(t) - s^{\alpha}] \mid \leq \epsilon \mid \sigma^{\alpha}(t) - s^{\alpha} \mid$$

for all $s \in U$. Since $\sigma(t) = t$, we have that

$$\mid [f^{\alpha}(\sigma(t)) - f^{\alpha}(s)] - f^{\Delta^{\alpha}}(t)[t^{\alpha} - s^{\alpha}] \mid \leq \epsilon \mid \sigma^{\alpha}(t) - s^{\alpha} \mid$$

for all $s \in U$. It follows that $\left|\frac{f^{\alpha}(t)-f^{\alpha}(s)}{t^{\alpha}-s^{\alpha}} - f^{\Delta^{\alpha}}(t)\right| \leq \epsilon$ for all $s \in U$, $s \neq t$, and we get the desired equality (3). Assume $\lim_{s\to t} \frac{f^{\alpha}(t)-f^{\alpha}(s)}{t^{\alpha}-s^{\alpha}}$ exists and is equal to X and $\sigma(t) = t$. Let $\epsilon > 0$. Then, there is a neighborhood U of t such that

$$\left|\frac{f^{\alpha}(\sigma(t)) - f^{\alpha}(s)}{t^{\alpha} - s^{\alpha}} - X\right| \leq \epsilon$$

for all $s \in U$. Because $\mid f^{\alpha}(\sigma(t)) - f^{\alpha}(s) - X(t^{\alpha} - s^{\alpha}) \mid \leq \epsilon |t^{\alpha} - s^{\alpha}|$ for all $s \in U$,

$$f^{\Delta^{\alpha}}(t) = X = \lim_{s\to t} \frac{f^{\alpha}(t) - f^{\alpha}(s)}{t^{\alpha} - s^{\alpha}}.$$

3. If $\sigma(t) = t$, then $\sigma^{\alpha}(t) - t^{\alpha} = 0$ and

$$f^{\alpha}(\sigma(t)) = f^{\alpha}(t) = f^{\alpha}(t) + (\sigma^{\alpha}(t) - t^{\alpha})f^{\Delta^{\alpha}}(t).$$

On the other hand, if $\sigma(t) > t$, then by item 1

$$\begin{aligned} f^{\alpha}(\sigma(t)) &= f^{\alpha}(t) + (\sigma^{\alpha}(t) - t^{\alpha}) \frac{f^{\alpha}(\sigma(t)) - f^{\alpha}(t)}{\sigma(t)^{\alpha} - t^{\alpha}} \\ &= f^{\alpha}(t) + (\sigma^{\alpha}(t) - t^{\alpha})f^{\Delta^{\alpha}}(t) \end{aligned}$$

and the proof is complete.

Example 1. If $\mathbb{T} = \mathbb{R}$, then (3) yields that $f : \mathbb{R} \to \mathbb{R}$ is delta differentiable of order α at $t \in \mathbb{R}$ if and only if $f^{\Delta^{\alpha}}(t) = \lim_{s\to t} \frac{f^{\alpha}(t)-f^{\alpha}(s)}{t^{\alpha}-s^{\alpha}}$ exists, i.e., if and only if f is fractional differentiable at t. In this case we get the derivative $f^{(\alpha)}$ of [10].

Example 2. If $\mathbb{T} = \mathbb{Z}$, then item 1 of Theorem 1 yields that $f : \mathbb{Z} \to \mathbb{R}$ is delta-differentiable of order α at $t \in \mathbb{Z}$ with

$$f^{\Delta^{\alpha}}(t) = \frac{f^{\alpha}(\sigma(t)) - f^{\alpha}(t)}{\sigma^{\alpha}(t) - t^{\alpha}} = \frac{f^{\alpha}(t+1) - f^{\alpha}(t)}{(t+1)^{\alpha} - t^{\alpha}}.$$

Example 3. If $f : \mathbb{T} \to \mathbb{R}$ is defined by $f(t) \equiv \lambda \in \mathbb{R}$, then $f^{\Delta^\alpha}(t) \equiv 0$. Indeed, if t is right scattered, then by item 1 of Theorem 1 $f^{\Delta^\alpha}(t) = \frac{f^\alpha(\sigma(t))-f^\alpha(t)}{\sigma^\alpha(t)-t^\alpha} = \frac{\lambda^\alpha-\lambda^\alpha}{\sigma^\alpha(t)-t^\alpha} = 0$; if t is right dense, then by (3) we get $f^{\Delta^\alpha}(t) = \lim_{s\to t} \frac{\lambda^\alpha-\lambda^\alpha}{t^\alpha-s^\alpha} = 0$.

Example 4. If $f : \mathbb{T} \to \mathbb{R}$, $t \mapsto t$, then $f^{\Delta^\alpha} \equiv 1$ because if $\sigma(t) > t$ (i.e., t is right scattered), then $f^{\Delta^\alpha}(t) = \frac{f^\alpha(\sigma(t))-f^\alpha(t)}{\sigma^\alpha(t)-t^\alpha} = \frac{\sigma^\alpha(t)-t^\alpha}{\sigma^\alpha(t)-t^\alpha} = 1$; if $\sigma(t) = t$ (i.e., t is right dense), then $f^{\Delta^\alpha} = \lim_{s\to t} \frac{f^\alpha(t)-f^\alpha(s)}{t^\alpha-s^\alpha} = \frac{t^\alpha-s^\alpha}{t^\alpha-s^\alpha} = 1$.

Example 5. Let $g : \mathbb{T} \to \mathbb{R}$, $t \mapsto \frac{1}{t}$. We have $g^{\Delta^\alpha}(t) = -\frac{1}{(t\sigma(t))^\alpha}$. Indeed, if $\sigma(t) = t$, then $g^{\Delta^\alpha}(t) = -\frac{1}{t^{2\alpha}}$; if $\sigma(t) > t$, then

$$g^{\Delta^\alpha}(t) = \frac{g^\alpha(\sigma(t)) - g^\alpha(t)}{\sigma^\alpha(t) - t^\alpha} = \frac{\left(\frac{1}{\sigma(t)}\right)^\alpha - \left(\frac{1}{t}\right)^\alpha}{\sigma^\alpha(t) - t^\alpha} = \frac{\frac{t^\alpha-\sigma^\alpha(t)}{t^\alpha\sigma^\alpha(t)}}{t^\alpha - \sigma^\alpha(t)} = -\frac{1}{t^\alpha\sigma^\alpha(t)}.$$

Example 6. Let $h : \mathbb{T} \to \mathbb{R}$, $t \mapsto t^2$. We have $h^{\Delta^\alpha}(t) = \sigma^\alpha(t) + t^\alpha$. Indeed, if t is right dense, then $h^{\Delta^\alpha}(t) = \lim_{s\to t} \frac{t^{2\alpha}-s^{2\alpha}}{t^\alpha-s^\alpha} = 2t^\alpha$; if t is right scattered, then

$$h^{\Delta^\alpha}(t) = \frac{h^\alpha(\sigma(t)) - h^\alpha(t)}{\sigma^\alpha(t) - t^\alpha} = \frac{\sigma^{2\alpha}(t) - t^{2\alpha}}{\sigma^\alpha(t) - t^\alpha} = \sigma^\alpha(t) + t^\alpha.$$

Example 7. Consider the time scale $\mathbb{T} = h\mathbb{Z}$, $h > 0$. Let f be the function defined by $f : h\mathbb{Z} \to \mathbb{R}$, $t \mapsto (t-c)^2$, $c \in \mathbb{R}$. The fractional derivative of order α of f at t is

$$\begin{aligned} f^{\Delta^\alpha}(t) &= \frac{f^\alpha(\sigma(t)) - f^\alpha(t)}{\sigma^\alpha(t) - t^\alpha} = \frac{((\sigma(t)-c)^2)^\alpha - ((t-c)^2)^\alpha}{\sigma^\alpha(t) - t^\alpha} \\ &= \frac{(t+h-c)^{2\alpha} - (t-c)^{2\alpha}}{(t+h)^\alpha - t^\alpha}. \end{aligned}$$

Remark 2. Examples 5, 6, and 7 show that in general $f^{\Delta^\alpha}(t)$ is a complex number (for instance, choose $\alpha = \frac{1}{2}$ and $t < 0$).

Theorem 2. *Assume $f, g : \mathbb{T} \to \mathbb{R}$ are continuous and delta differentiable of order α at $t \in \mathbb{T}^\kappa$. Then the following proprieties hold:*

1. *For any constant λ, function $\lambda f : \mathbb{T} \to \mathbb{R}$ is delta differentiable of order α at t with $(\lambda f)^{\Delta^\alpha} = \lambda^\alpha f^{\Delta^\alpha}$.*
2. *The product $fg : \mathbb{T} \to \mathbb{R}$ is delta differentiable of order α at t with*

$$(fg)^{\Delta^\alpha}(t) = f^{\Delta^\alpha}(t)g^\alpha(t) + f^\alpha(\sigma(t))g^{\Delta^\alpha}(t) = f^{\Delta^\alpha}(t)g^\alpha(\sigma(t)) + f^\alpha(t)g^{\Delta^\alpha}(t).$$

3. *If $f(t)f(\sigma(t)) \neq 0$, then $\frac{1}{f}$ is delta differentiable of order α at t with*

$$\left(\frac{1}{f}\right)^{\Delta^\alpha}(t) = \frac{-f^{\Delta^\alpha}(t)}{f^\alpha(\sigma(t))f^\alpha(t)}.$$

4. *If* $g(t)g(\sigma(t)) \neq 0$, *then* $\frac{f}{g}$ *is delta differentiable of order* α *at* t *with*

$$\left(\frac{f}{g}\right)^{\Delta^\alpha}(t) = \frac{f^{\Delta^\alpha}(t)g^\alpha(t) - f^\alpha(t)g^{\Delta^\alpha}(t)}{g^\alpha(\sigma(t))g^\alpha(t)}.$$

Proof. 1. Let $\epsilon \in (0,1)$. Define $\epsilon^* = \frac{\epsilon}{|\lambda|^\alpha} \in (0,1)$. Then there exists a neighborhood U of t such that $|f^\alpha(\sigma(t)) - f^\alpha(s) - f^{\Delta^\alpha}(t)(\sigma^\alpha(t) - s^\alpha)| \leq \epsilon^*|\sigma^\alpha(t) - s^\alpha|$ for all $s \in U$. It follows that

$$\begin{aligned} &|(\lambda f)^\alpha(\sigma(t)) - (\lambda f)^\alpha(s) - \lambda^\alpha f^{\Delta^\alpha}(t)(\sigma^\alpha(t) - s^\alpha)| \\ &\quad = |\lambda|^\alpha \,|\, f^\alpha(\sigma(t)) - f^\alpha(s) - f^{\Delta^\alpha}(t)(\sigma^\alpha(t) - s^\alpha) \,| \\ &\quad \leq \epsilon^*|\lambda|^\alpha|\sigma^\alpha(t) - s^\alpha| \leq \frac{\epsilon}{|\lambda|^\alpha}|\lambda|^\alpha|\sigma^\alpha(t) - s^\alpha| = \epsilon|\sigma^\alpha(t) - s^\alpha| \end{aligned}$$

for all $s \in U$. Thus, $(\lambda f)^{\Delta^\alpha}(t) = \lambda^\alpha f^{\Delta^\alpha}(t)$ holds.

2. Let $\epsilon \in (0,1)$. Define $\epsilon^* = \epsilon[1 + |f^\alpha(t)| + |g^\alpha(\sigma(t))| + |g^{\Delta^\alpha}(\sigma(t))|]^{-1}$. Then $\epsilon^* \in (0,1)$ and there exist neighborhoods U_1, U_2, and U_3 of t such that

$$|f^\alpha(\sigma(t)) - f^\alpha(s) - f^{\Delta^\alpha}(t)(\sigma^\alpha(t) - s^\alpha)| \leq \epsilon^*|\sigma^\alpha(t) - s^\alpha|$$

for all $s \in U_1$, $|g^\alpha(\sigma(t)) - g^\alpha(s) - g^{\Delta^\alpha}(t)(\sigma^\alpha(t) - s^\alpha)| \leq \epsilon^*|\sigma^\alpha(t) - s^\alpha|$ for all $s \in U_2$ and such as f is continuous. Then $|f(t) - f(s)| \leq \epsilon^*$ for all $s \in U_3$. Define $U = U_1 \cap U_2 \cap U_3$ and let $s \in U$. It follows that

$$\begin{aligned} &|(fg)^\alpha(\sigma(t)) - (fg)^\alpha(s) - [g^{\Delta^\alpha}(t)f^\alpha(t) + g^\alpha(\sigma(t))f^{\Delta^\alpha}(t)][\sigma^\alpha(t) - s^\alpha]| \\ &= |[f^\alpha(\sigma(t)) - f^\alpha(s) - f^{\Delta^\alpha}(t)(\sigma^\alpha(t) - s^\alpha)](g^\alpha(\sigma(t))) + g^\alpha(\sigma(t))f^\alpha(s) \\ &\quad + g^\alpha(\sigma(t))f^{\Delta^\alpha}(t)(\sigma^\alpha(t) - s^\alpha) - f^\alpha(s)g^\alpha(s) \\ &\quad - [g^{\Delta^\alpha}(t)f^\alpha(t) + g^\alpha(\sigma(t))f^{\Delta^\alpha}(t)][\sigma^\alpha(t) - s^\alpha]| \\ &= |[f^\alpha(\sigma(t)) - f^\alpha(s) - f^{\Delta^\alpha}(t)(\sigma^\alpha(t) - s^\alpha)](g^\alpha(\sigma(t))) \\ &\quad + [g^\alpha(\sigma(t)) - g^\alpha(s) - g^{\Delta^\alpha}(t)(\sigma^\alpha(t) - s^\alpha)](f^\alpha(t)) \\ &\quad + [g^\alpha(\sigma(t)) - g^\alpha(s) - g^{\Delta^\alpha}(t)(\sigma^\alpha(t) - s^\alpha)](f^\alpha(s) - f^\alpha(t)) + f^\alpha(s)g^\alpha(s) \\ &\quad + g^{\Delta^\alpha}(t)f^\alpha(s)(\sigma^\alpha(t) - s^\alpha) + g^\alpha(\sigma(t))f^{\Delta^\alpha}(t)(\sigma^\alpha(t) - s^\alpha) - g^\alpha(s)f^\alpha(s) \\ &\quad + g^{\Delta^\alpha}(t)f^\alpha(s)(\sigma^\alpha(t) - s^\alpha) + g^\alpha(\sigma(t))f^{\Delta^\alpha}(t)(\sigma^\alpha(t) - s^\alpha) - f^\alpha(s)g^\alpha(s) \\ &\quad - [g^{\Delta^\alpha}(t)f^\alpha(t) + g^\alpha(\sigma(t))f^{\Delta^\alpha}(t)][\sigma^\alpha(t) - s^\alpha]| \end{aligned}$$

$$
\begin{aligned}
&\leq |f^{\alpha}(\sigma(t)) - f^{\alpha}(s) - f^{\Delta^{\alpha}}(t)(\sigma^{\alpha}(t) - s^{\alpha})||(g^{\alpha}(\sigma(t)))| \\
&\quad + |g^{\alpha}(\sigma(t)) - g^{\alpha}(s) - g^{\Delta^{\alpha}}(t)(\sigma^{\alpha}(t) - s^{\alpha})||(f^{\alpha}(t))| \\
&\quad + |g^{\alpha}(\sigma(t)) - g^{\alpha}(s) - g^{\Delta^{\alpha}}(t)(\sigma^{\alpha}(t) - s^{\alpha})||f^{\alpha}(s) - f^{\alpha}(t)| \\
&\quad + |g^{\Delta^{\alpha}}(t)||f^{\alpha}(t) - f^{\alpha}(s)||\sigma^{\alpha}(t) - s^{\alpha}| \\
&= \epsilon^{*}|(g^{\alpha}(\sigma(t)))||\sigma^{\alpha}(t) - s^{\alpha}| \\
&\quad + \epsilon^{*}|(f^{\alpha}(t))||\sigma^{\alpha}(t) - s^{\alpha}| + \epsilon^{*}|\sigma^{\alpha}(t) - s^{\alpha}|\epsilon^{*} + \epsilon^{*}|g^{\Delta^{\alpha}}(t)||\sigma^{\alpha}(t) - s^{\alpha}| \\
&\leq \epsilon^{*}|\sigma^{\alpha}(t) - s^{\alpha}|(\epsilon^{*} + |(f^{\alpha}(t))| + |g^{\Delta^{\alpha}}(t)| + |g^{\Delta^{\alpha}}(t)|) \\
&\leq \epsilon^{*}|\sigma^{\alpha}(t) - s^{\alpha}|(1 + |(f^{\alpha}(t))| + |g^{\Delta^{\alpha}}(t)| + |g^{\Delta^{\alpha}}(t)|) = \epsilon|\sigma^{\alpha}(t) - s^{\alpha}|.
\end{aligned}
$$

Thus, $(fg)^{\Delta^{\alpha}}(t) = f^{\alpha}(t)g^{\Delta^{\alpha}}(t) + f^{\Delta^{\alpha}}(t)g^{\alpha}(\sigma(t))$ holds at t. The other product rule follows from this last equality by interchanging functions f and g.

3. We use the delta derivative of a constant (Example 3). Since $\left(f \cdot \frac{1}{f}\right)^{\Delta^{\alpha}}(t) = 0$, it follows from item 2 that $\left(\frac{1}{f}\right)^{\Delta^{\alpha}}(t)f^{\alpha}(\sigma(t)) + f^{\Delta^{\alpha}}(t)\frac{1}{f^{\alpha}(t)} = 0$. Because we are assuming $f(t)f(\sigma(t)) \neq 0$, one has $\left(\frac{1}{f}\right)^{\Delta^{\alpha}}(t) = \frac{-f^{\Delta^{\alpha}}(t)}{f^{\alpha}(\sigma(t))f^{\alpha}(t)}$.
4. For the quotient formula, we use items 2 and 3 to compute

$$
\begin{aligned}
\left(\frac{f}{g}\right)^{\Delta^{\alpha}}(t) = \left(f \cdot \frac{1}{g}\right)^{\Delta^{\alpha}}(t) &= f^{\alpha}(t)\left(\frac{1}{g}\right)^{\Delta^{\alpha}}(t) + f^{\Delta^{\alpha}}(t)\frac{1}{g^{\alpha}(\sigma(t))} \\
&= -f^{\alpha}(t)\frac{g^{\Delta^{\alpha}}(t)}{g^{\alpha}(\sigma(t))g^{\alpha}(t)} + f^{\Delta^{\alpha}}(t)\frac{1}{g^{\alpha}(\sigma(t))} \\
&= \frac{f^{\Delta^{\alpha}}(t)g^{\alpha}(t) - f^{\alpha}(t)g^{\Delta^{\alpha}}(t)}{g^{\alpha}(\sigma(t))g^{\alpha}(t)}.
\end{aligned}
$$

This concludes the proof.

Remark 3. The delta derivative of order α of the sum $f + g : \mathbb{T} \to \mathbb{R}$ does not satisfy the usual property, that is, in general $(f+g)^{\Delta^{\alpha}}(t) \neq (f)^{\Delta^{\alpha}}(t) + (g)^{\Delta^{\alpha}}(t)$. For instance, let $\mathbb{T}$ be an arbitrary time scale and f, g be functions defined by $f : \mathbb{T} \to \mathbb{R}$, $t \mapsto t$, and $g : \mathbb{T} \to \mathbb{R}$, $t \mapsto 2t$. One can easily find that $(f + g)^{\Delta^{\alpha}}(t) = \sqrt{3} \neq f^{\Delta^{\alpha}}(t) + g^{\Delta^{\alpha}}(t) = 1 + \sqrt{2}$.

Proposition 1. *Let $\alpha \in \mathbb{R}$ and $m \in \mathbb{N}$, $m > 1$. For g defined by $g(t) = t^m$, we have*

$$
g^{\Delta^{\alpha}}(t) = \sum_{k=0}^{m-1}(t^{\alpha})^{m-k-1}(\sigma^{\alpha})^{k}(t). \tag{4}
$$

Proof. We prove the formula by induction. If $m = 2$, then $g(t) = t^2$ and from Example 6 we know that $g^{\Delta^\alpha}(t) = \sum_{k=0}^{1}(t^\alpha)^{1-k}(\sigma^\alpha)^k(t) = t^\alpha + \sigma^\alpha(t)$. Now assume

$$g^{\Delta^\alpha}(t) = \sum_{k=0}^{m-1}(t^\alpha)^{m-k-1}(\sigma^\alpha)^k(t)$$

holds for $g(t) = t^m$ and let $G(t) = t^{m+1} = t \cdot g(t)$. We use the product rule of Theorem 2 to obtain

$$\begin{aligned} G^{\Delta^\alpha}(t) &= g^\alpha(t) + \sigma^\alpha(t) g^{\Delta^\alpha}(t) = (t^\alpha)^m + \sigma^\alpha(t)\sum_{k=0}^{m-1}(t^\alpha)^{m-k-1}(\sigma^\alpha)^k(t) \\ &= (t^\alpha)^m + \sum_{k=0}^{m-1}(t^\alpha)^{m-k-1}(\sigma^\alpha)^{k+1}(t) = (t^\alpha)^m + \sum_{k=1}^{m-1}(t^\alpha)^{m-k}(\sigma^\alpha)^k(t) \\ &= \sum_{k=0}^{m}(t^\alpha)^{m-k}(\sigma^\alpha)^k(t). \end{aligned}$$

Hence, by mathematical induction, (4) holds.

Example 8. Choose $m = 3$ in Proposition 1. Then $\left(t^3\right)^{\Delta^\alpha} = t^{2\alpha} + (t\sigma(t))^\alpha + \sigma^{2\alpha}(t)$.

The notion of fractional derivative here introduced can be easily extended to any arbitrary real order α.

Definition 3. Let $\alpha > 0$ and $N \in \mathbb{N}_0$ be such that $N < \alpha \leq N + 1$. Then we define $f^{\Delta^\alpha} = \left(f^{\Delta^N}\right)^{\Delta^{\alpha-N}}$, where f^{Δ^N} is the usual Hilger derivative of order N.

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Uniform Stabilization of a Hybrid System of Elasticity: Riesz Basis Approach

M. Driss Aouragh

Abstract A hybrid system, composed of an elastic beam governed by an Euler-Bernoulli beam equation and a linked rigid body governed by an ordinary differential equation, is considered. This paper studies the basis property and the stability of a hybrid system when the usual linear boundary feedback is applied to the end without mass. It is shown that there is a sequence of generalized eigenfunctions of the system, which forms a Riesz basis for the state Hilbert space. As consequence expressions of eigenvalues, the spectrum-determined growth condition and the exponential stability are readily presented. To confirm numerically the asymptotic estimate of eigenvalues, we shall use the spectral method to calculate the eigenvalues.

Keywords Beams • Spectrum • Stabilization of systems by feedback • Riesz basis

AMS subject classifications: 74K10, 47A10, 93O15, 47E05

1 Introduction

Consisting of an elastic beam, linked to a rigid antenna, this dynamical system can be described by the Euler-Bernoulli equation for the vibration of the elastic beam and the Newton-Euler rigid-body equations for all the oscillations of the antenna:

$$\begin{aligned} \partial_{tt}y(x,t) + \partial_{xxxx}y(x,t) &= 0, \quad 0 < x < 1,\ t > 0 \\ M\partial_{tt}y(1,t) - \partial_{xxx}y(1,t) &= 0, \quad t > 0 \\ J\partial_{xtt}y(1,t) + \partial_{xx}y(1,t) &= 0, \quad t > 0 \end{aligned} \tag{1}$$

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where t is the time variable and x the space coordinate along the beam, in its equilibrium position. The function y is the transverse displacement of the beam, M the mass of the antenna, and J the moment of inertia associated with the antenna. For further description concerning the physical structure of the system, we refer to [4]. Our goal is to choose suitable boundary damping at the end $x = 0$ such that the hybrid system can be stabilized uniformly.

We consider the elastic beam with the following boundary feedback [6]:

$$\partial_x y(0,t) = 0, \quad (y + \partial_{xxx} y + a\partial_t y)(0,t) = 0, \quad t > 0, \tag{2}$$

where $a \geq 0$ is a positive constant. Notice that the boundary condition (2) can be realized by means of passive mechanical systems of springer-damper similar to those used in [2].

In the next section, we give a result of a well-posedness of the solution of the system, and the asymptotic expressions of eigenvalues and eigenfunctions are derived. In Sect. 3, we show that there is a sequence of generalized eigenfunctions of system (1)–(2), which forms a Riesz basis for the state Hilbert space and the exponential stability of the system is obtained. Numerical simulation of eigenvalues is presented in Sect. 4.

2 Well-Posedness, Asymptotic Expression of Eigenpairs

The energy space associated to system (1)–(2) is

$$H := \mathscr{V} \times L^2(0,1) \times C^2, \quad \mathscr{V} = \{\phi \in H^2(0,1)/\partial_x\phi(0) = 0\}$$

with the inner product induced norm

$$\|(u, v, \zeta, \delta)\|^2 := \int_0^l [|v|^2 + |\partial_{xx}u|^2]dx + |u(0)|^2 + M^{-1}|\zeta|^2 + J^{-1}|\delta|^2,$$

The system (1)–(2) can be written as

$$\partial_t Y(t) = LY(t), Y(t) = (y(.,t), \partial_t y(.,t), m\partial_t y(1,t), \partial_{xt} y(1,t)), \tag{3}$$

where the associated system operator

$$\begin{aligned} L(\phi, \psi, \zeta, \delta) &= (\psi, -\partial_{xxxx}\phi, \partial_{xxx}\phi(1), -\partial_{xx}\phi(1)), \\ D(L) &= \{(\phi, \psi, \zeta, \delta) \in (\mathscr{H}^4(0,1) \cap \mathscr{V}) \times \mathscr{V} \times C^2/ \\ \phi(0) + \partial_{xxx}\phi(0) &+ a\psi(0) = 0, \zeta = M\psi(1), \delta = J\partial_x\psi(1)\}. \end{aligned} \tag{4}$$

Lemma 1. L^{-1} *exists and is compact on* H. *Hence* $\sigma(L)$, *the spectrum of* L, *consists of isolated eigenvalues only.*

Proof. For any $(u, v, f, g) \in H$, solving

$$L(\phi, \psi, \zeta, \delta) = (\psi, -\partial_{xxxx}\phi, -\partial_{xx}\phi(1), \partial_{xxx}\phi(1)) = (u, v, f, g),$$

produces the unique solution $\psi = u, \quad \zeta = Mu(1), \quad \delta = Ju'(1)$ and

$$\begin{aligned}\phi(x) = \phi(0) + \tfrac{f}{6} - \tfrac{1}{6}\int_0^1 t^3 v(t)dt - \left[\tfrac{f}{6} + \tfrac{1}{2}\int_0^1 t^2 v(t)dt\right]x \\ -\tfrac{g}{2}x^2 + \tfrac{f}{6}(x-1)^3 + \tfrac{1}{6}\int_x^1 (x-1)^3 y(t)dt \qquad (5)\end{aligned}$$

with $\phi(0) = -(f + au(0) + \int_0^1 vdx)$. The compactness follows from the Sobolev embedding theorem [5]. Other conclusions are obvious, and the details are omitted.

Lemma 2. *For any $\lambda = i\tau^2 \in \sigma(L)$, there is a unique eigenfunction (up to a scalar) $(\phi, \lambda\phi, M\lambda\phi(1), J\lambda\partial_x\phi(1))$ where*

$$\begin{aligned}\phi(x) = &-(1 + MJ\tau^4)\cosh\tau x + [2J\tau^3 \sin\tau + (-1 + MJ\tau^4)\cos\tau]\cosh\tau(1-x) \\ &+[2J\tau^3 \sinh\tau - (1 + MJ\tau^4)\cos\tau + (-1 + MJ\tau^4)\cosh\tau]\cos\tau(1-x) \\ &+[(-1 + MJ\tau^4)\sin\tau - 2M\tau\cos\tau]\sinh\tau(1-x) \\ &+[(1 - MJ\tau^4)\sinh\tau - (1 + Mj\tau^4)\sin\tau + 2M\tau\cosh\tau]\sin\tau(1-x) \quad (6)\end{aligned}$$

and the characteristic equation that λ satisfies is

$$\begin{aligned}&(1 + ia\tau^2)[-(1 + MJ\tau^4) + (J\tau^3 - M\tau)\cos\tau\sinh\tau + (-1 + MJ\tau^4)\cos\tau\cosh\tau \\ &-(J\tau^3 + M\tau)\sin\tau\cosh\tau] - \tau^3[2J\tau^3\sin\tau\sinh\tau + (-1 + MJ\tau^4)\cos\tau\sinh\tau \\ &+(-1 + MJ\tau^4)\sin\tau\cosh\tau] = 0, \qquad (7)\end{aligned}$$

Proof. Solving the eigenvalue problem

$$L(\phi, \psi, \zeta, \delta) = (\psi, -\partial_{xxxx}\phi, \partial_{xxx}\phi(1), -\partial_{xx}\phi(1)) = \lambda(\phi, \psi, M\psi(1), J\partial_x\psi(1))$$

one has $\psi = \lambda\phi, \quad \zeta = M\psi(1), \quad \delta = J\psi'(1)$ and

$$\begin{aligned}\partial_{xxxx}\phi + \lambda^2\phi = 0, \\ \partial_x\phi(0) = \phi(0)(1 + \lambda a) + \partial_{xxx}\phi(0) = 0, \\ \partial_{xxx}\phi(1) - M\lambda^2\phi(1) = \partial_{xx}\phi(1) + J\lambda^2\partial_x\phi(1) = 0, \qquad (8)\end{aligned}$$

Let $f(x) = \phi(x-1)$. Then f satisfies

$$\begin{aligned}\partial_{xxxx}f + \lambda^2 f = 0, \\ \partial_x f(1) = f(1)(1 + \lambda a) - \partial_{xxx}f(1) = 0,\end{aligned}$$

$$\partial_{xxx}f(0) + M\lambda^2 f(0) = \partial_{xx}f(0) - J\lambda^2\partial_x f(0) = 0 \tag{9}$$

Let $\lambda = i\tau^2$; it is easily seen that for any $\lambda = i\tau^2$, the general solution of the following equation

$$\partial_{xxxx}f + \lambda^2 f = 0,$$
$$\partial_{xxx}f(0) + M\lambda^2 f(0) = \partial_{xx}f(0) - J\lambda^2\partial_x f(0) = 0,$$

is of the form

$$f(x) = [(d_1-d_2)-MJ\tau^4(d_1+d_2)]\cos h\ \tau x + [(d_1-d_2)+MJ\tau^4(d_1+d_2)]\cos\ \tau x$$
$$+2M\tau[d_1 \sin h\ \tau x + d_2 \sin\ \tau x],$$

where $d_1,\ d_2$ are arbitrary constants. By $\partial_x f(1) = 0$, one has (up to a scalar)

$$d_1 = (1 + MJ\tau^4)\sin h\ \tau + (-1 + MJ\tau^4)\sin\ \tau - 2M\tau\cos\ \tau,$$
$$d_2 = (1 - MJ\tau^4)\sin h\ \tau - (1 + MJ\tau^4)\sin\ \tau + 2M\tau\cos h\ \tau,$$
$$d_1 - d_2 = 2MJ\tau^4 \sin h\ \tau + 2MJ\tau^4 \sin\ \tau - 2M\tau\cos\ \tau - 2M\cos h\ \tau,$$
$$d_1 + d_2 = 2\sin h\ \tau - 2\sin\ \tau - 2M\tau\cos\ \tau + 2M\tau\cos h\ \tau.$$

by $\phi(x) = f(x-1)$ this is (6). In order for f to be a solution of (9), it is necessary and sufficient that $f(1)(1 + \lambda a) - \partial_{xxx}f(1) = 0$, which induces (7), proving the lemma.

Lemma 3. *There is a family of eigenvalues $\{\lambda_n = i\tau_n^2, -i\tau_n^2\}$ of L with the following asymptotic expression*

$$\lambda_n = i\tau_n^2 = -a + i\left(\frac{2}{M} + (m\pi)^2\right) + O(n^{-1}) \tag{10}$$

where $m = n - \frac{1}{4}$, n is a sufficiently large positive integer. A corresponding eigenfunction $\Phi_n = (\phi_n, \lambda_n\phi_n, M\lambda_n\phi_n(1), J\lambda_n\partial_x\phi(1))$, where

$$\phi_n(x) = -(1 + MJ\tau_n^4)\cos h\ \tau_n x$$
$$+[2J\tau_n^3 \sin\ \tau_n + (-1 + MJ\tau_n^4)\cos\ \tau_n]\cos h\ \tau_n(1-x)$$
$$+[2J\tau_n^3 \sin h\ \tau_n - (1+MJ\tau_n^4)\cos\ \tau_n + (-1+MJ\tau_n^4)\cos h\ \tau_n]\cos\ \tau_n(1-x)$$
$$+[(-1 + MJ\tau_n^4)\sin\ \tau_n - 2M\tau_n\cos\ \tau_n]\sin h\ \tau_n(1-x)$$
$$+[(1-MJ\tau_n^4)\sin h\ \tau_n - (1+Mj\tau_n^4)\sin\ \tau_n + 2M\tau_n\cos h\ \tau_n]\sin\ \tau_n(1-x),$$

which is obtained by (6) with $\tau = \tau_n$. The following asymptotic expression holds where $F_n(x) = -\frac{2}{MJ}\tau_n^{-6}e^{-\tau_n}(\phi_n'', \lambda_n\phi_n, M\lambda_n\phi_n(1), J\lambda_n\phi'(1))$,

$$F_n(x)=\begin{bmatrix} e^{-m\pi(1-x)}-e^{-m\pi x}\cos\ m\pi+\cos\ m\pi(1-x)-\sin\ m\pi(1-x) \\ i[e^{-m\pi(1-x)}-e^{-m\pi x}\cos\ m\pi-\cos\ m\pi(1-x)+\sin\ m\pi(1-x)] \\ 0 \\ 0 \end{bmatrix}^{\mathrm{T}} + O\,(n^{-1}) \tag{11}$$

(11) holds uniformly in $x \in [0,1]$. *It is seen that*

$$\lim_{n\to+\infty}\|F_n\|^2_{L^2\times L^2\times C^2}=\lim_{n\to+\infty}\|\frac{2}{MJ}\tau_n^{-6}e^{-\tau_n}\Phi_n\|^2_H=2,$$

Proof. Note that for a large positive integer n, in a uniformly bounded small neighborhood of $m\pi=(n-\frac{1}{4})\pi$

$$|\sin\ \tau|\le C, |\cos\ \tau|\le C, |e^{-\tau}\sinh\tau|\le C, |e^{-\tau}\cosh\tau|\le C,$$

uniformly for all n with some constant C. By multiplying $-e^{-\tau}\tau^{-7}(MJ)^{-1}$ on both sides of (7), we can write (7) in a uniformly bounded small neighborhood of $m\pi=(n-\frac{1}{4})\pi$ for each n to be

$$\sin\ \tau+\cos\ \tau=O(|\tau|^{-1}),\ \text{ or }\ \sin\ \tau+\cos\ \tau=\frac{1}{\tau}\left(ia+\frac{2}{M}\right)\cos\ \tau+O(|\tau|^{-2}) \tag{12}$$

The first equation in (12) can be rewritten as $sin\ 2\tau=-1+O(|\tau|^{-2})$. Applying Rouche's theorem in a small neighborhood of $m\pi=(n-\frac{1}{4})\pi$ where n is a large positive integer, we obtain a solution τ_n which is of the form

$$\tau=\tau_n=m\pi+O(n^{-1}), \tag{13}$$

for sufficiently large n. Substituting (13) into the second equation of (12) yields

$$2O(n^{-1})=\frac{1}{m\pi}\left(ia+\frac{2}{M}\right)+O(n^{-2}),$$

and so $\tau_n=m\pi+\frac{1}{m\pi}\left(\frac{ia}{2}+\frac{1}{M}\right)+O(n^{-2})$.

For the estimation of (11), we treat the first component only because the second component can be treated similarly. Now

$$\begin{aligned}\tau_n^{-2}\phi_n''(x)=&-(1+MJ\tau_n^4)\cosh\tau_n x\\ &+[2J\tau_n^3\sin\ \tau_n+(-1+MJ\tau_n^4)\cos\ \tau_n]\cosh\tau_n(1-x)\\ &-[2J\tau_n^3\sinh\tau_n-(1+MJ\tau_n^4)\cos\ \tau_n\\ &+(-1+MJ\tau_n^4)\cosh\tau_n]\cos\ \tau_n(1-x)\\ &+[(-1+MJ\tau_n^4)\sin\ \tau_n-2M\tau_n\cos\ \tau_n]\sinh\tau_n(1-x)\end{aligned}$$

$$-[(1 - MJ\tau_n^4) \sin h\ \tau_n - (1 + Mj\tau_n^4) \sin\ \tau_n$$
$$+2M\tau_n \cos h\ \tau_n] \sin\ \tau_n(1 - x),$$

Since for any bounded $y > 0$ and $x \in [0, 1]$, it holds uniformly

$$e^{-\tau_n y} = e^{-m\pi y} + O(n^{-1}),\ \sin\ \tau_n x = \sin\ m\pi x + O(n^{-1}),\ \cos\ \tau_n x$$
$$= \cos\ m\pi x + O(n^{-1}),$$

Hence

$$2(MJ)^{-1} e^{-\tau_n} \tau_n^{-6} \phi_n''(x) = -e^{\tau_n(1-x)} + e^{-\tau_n x} \cos\ \tau_n - \cos\ \tau_n(1 - x)$$
$$+ \sin\ \tau_n(1 - x) + O(n^{-1}),$$

Moreover,

$$2(MJ)^{-1}\tau_n^{-6} e^{-\tau_n}\phi_n(1) = O(n^{-1}),\ 2(MJ)^{-1}\tau_n^{-6} e^{-\tau_n}\phi_n'(1) = O(n^{-1}),$$

3 Riesz Basis Property and Exponential Stability

Theorem 1 (Guo [3]). *Let A be a densely defined discrete operator, that is, $(\lambda - A)^{-1}$ is compact for some λ in a Hilbert space H. Let $\{z_n\}_1^{+\infty}$ a Riesz basis for H. If there are an $N \geq 0$ and a sequence of a generalized eigenvectors $\{x_n\}_{N+1}^{+\infty}$ of A such that*

$$\sum_{n=N+1}^{+\infty} \|x_n - z_n\|^2 < +\infty$$

then

- ***(i)*** *There are an $M > N$ and generalized eigenvectors $\{x_{n0}\}_1^M \cup \{x_n\}_{M+1}^{+\infty}$ form a Riesz basis for H.*
- ***(ii)*** *Consequently, let $\{x_{n0}\}_1^M \cup \{x_n\}_{M+1}^{+\infty}$ correspond to eigenvalues $\{\sigma_n\}_1^{+\infty}$ of A. Then $\sigma(A) = \{\sigma_n\}_1^{+\infty}$ where σ_n is counted according to its algebraic multiplicity.*
- ***(iii)*** *If there is an $M_0 > 0$ such that $\sigma_n \neq \sigma_m$ for all $m, n \geq M_0$, then there is an $N_0 > M_0$ such that all $\sigma_n, n > N_0$ are algebraically simple.*

In order to apply theorem 1 to the operator L, we consider the following system:

$$\partial_{tt} y(x,t) + \partial_{xxxx} y(x,t) = 0, \qquad 0 < x < 1,\ t > 0,$$
$$\partial_x y(0,t) = 0, y(0,t) + \partial_{xxx} y(0,t) = 0, \quad t > 0, \tag{14}$$

$$M\partial_{tt}y(1,t) - \partial_{xxx}y(1,t) = J\partial_{xtt}y(1,t) + \partial_{xx}y(1,t) = 0, \qquad t > 0,$$

The system operator L_0 associated with (14) is nothing but the operator L with $a = 0$:

$$L_0(\phi, \psi, \zeta, \delta) = (\psi, -\partial_{xxxx}\phi, \partial_{xxx}\phi(1), -\partial_{xx}\phi(1)),$$

with $D(L_0) = \{(\phi, \psi, \zeta, \delta) \in (H^4(0,1) \cap \mathscr{V}) \times \mathscr{V} \times C^2 / \phi(0) + \partial_{xxx}\phi(0) = 0, \zeta = M\psi(1), \delta = J\partial_x\psi(1)\}$.

Then it is easily checked that L_0 is indeed a discrete skew-adjoint linear operator in H. From Lemma 2, each eigenvalue of L_0 is geometrically simple and hence algebraically simple. Because all eigenvalues of L_0 lie on the imaginary axis and the eigenvalues appear in conjugate pairs, we need to consider only positive solutions of (7) in order to find eigenvalues of L_0. From Lemma 2, we can obtain the unique (up to a scalar) eigenfunction of L_0 associated with $\mu_n = i\omega_n$ to be $\Psi_n = (f_n, \mu_n f_n, M\mu_n f_n(1), J\mu_n f_n'(1))$, where

$$\begin{aligned}
f_n(x) = &-(1 + MJ\omega_n^4)\cos h\ \omega_n x + [2J\omega_n^3 \sin\ \omega_n \\
&+(-1 + MJ\omega_n^4)\cos\ \omega_n]\cos h\ \omega_n(1-x) \\
&+[2J\omega_n^3 \sin h\ \omega_n - (1 + MJ\omega_n^4)\cos\ \omega_n \\
&+(-1 + MJ\omega_n^4)\cos h\ \omega_n]\cos\ \omega_n(1-x) \\
&+[(-1 + MJ\omega_n^4)\sin\ \omega_n - 2M\omega_n \cos\ \omega_n]\sin h\ \omega_n(1-x) \\
&+[(1 - MJ\omega_n^4)\sin h\ \omega_n - (1 + Mj\omega_n^4)\sin\ \omega_n + 2M\omega_n \cos h\ \omega_n]\sin\ \omega_n(1-x),
\end{aligned}$$

and

$$G_n(x) = \begin{bmatrix} e^{-m\pi(1-x)} - e^{-m\pi x}\cos\ m\pi + \cos\ m\pi(1-x) - \sin\ m\pi(1-x) \\ i[e^{-m\pi(1-x)} - e^{-m\pi x}\cos\ m\pi - \cos\ m\pi(1-x) + \sin\ m\pi(1-x)] \\ 0 \\ 0 \end{bmatrix}^{\mathrm{T}} + O(n^{-1}) \tag{15}$$

where $G_n(x) = -\frac{2}{MJ}\omega_n^{-6} e^{-\omega_n}(f_n'', \mu_n f_n, M\mu_n f_n(1), J\mu_n f_n'(1))^{\mathrm{T}}$, Moreover,

$$2(MJ)^{-1}\omega_n^{-6} e^{-\omega_n} f_n(1) = O(n^{-1}), \quad 2(MJ)^{-1}\omega_n^{-6} e^{-\omega_n} f_n'(1) = O(n^{-1}),$$

Since L_0 is a discrete operator, there are only finite number of eigenvalues in any bounded complex region, all with at most other finite number of generalized eigenfunctions (in the sense of w-linearly independent) of L_0 forming a Riesz basis for H. Therefore, we may assume, without loss of generality, that the generalized eigenfunctions of $L_0 = \{-2(MJ)^{-1}\omega_n^{-6} e^{-\omega_n}\Psi_n\} \bigcup \{$ their conjugates $\}$. It follows from (11) and (15) that there exists an N such that

$$\sum_{n=N+1}^{+\infty} \left\|2(MJ)^{-1}\tau_n^{-6}e^{-\tau_n}\Phi_n - 2(MJ)^{-1}\omega_n^{-6}e^{-\omega_n}\Psi_n\right\|_H^2 = \sum_{n=N+1}^{+\infty} O(n^{-2}) < +\infty, \quad (16)$$

The same result is verified for their conjugates. We can now apply Theorem 1 to obtain the main results of the present paper.

Theorem 2. *For any $a \geq 0$*

(i) *There is a sequence of generalized functions properly normalized of L which forms a Riesz basis of the Hilbert space H.*
(ii) *The eigenvalues of L have the asymptotic behavior (10).*
(iii) *All $\lambda \in \sigma(L)$ with sufficiently large modulus are algebraically simple. Therefore, L generates a C_0 semigroup on H. Moreover, for the semigroup e^{Lt} generated by L, the spectrum-determined growth condition holds: $\omega(L) = S(L)$, where*

$$\omega(L) = \lim_{t\to+\infty} t^{-1} \| e^{Lt} \|, \ S(L) = \sup\{\mathrm{Re}\lambda / \ \lambda \in \sigma(L)\}$$

Theorem 3. *Suppose $a > 0$. Then there exists an $\omega > 0$ such that Re $\lambda < -\omega$ for all $\lambda \in \sigma(L)$. Therefore, the C_0 semigroup e^{Lt} generated by L is exponentially stable:*

$$\|e^{Lt}\Phi\| \leq me^{-\omega t}\|\Phi\|^2$$

for any $\Phi \in H$, where $m > 0$ is a constant independent of Φ.

Proof. It suffices to show that Re $\lambda < 0$ for all $\lambda \in \sigma(L)$. We start from the eigenproblem (8). Multiplying $\overline{\phi}$, the conjugate of ϕ, on both sides of the first equation in (8) and integrating from 0 to 1 with respect to x yields

$$\int_0^1 |\phi''(x)|^2 dx + \lambda^2[M|\phi(1)|^2 + J|\phi'(1)|^2 + \int_0^1 |\phi(x)|^2 dx] \quad (17)$$
$$+(1 + a\lambda)|\phi(0)|^2 = 0.$$

Clearly, if λ is a real number, it must have $\lambda < 0$. Notice that $\lambda = 0$ is always not in the spectrum of L. Suppose that $\lambda = \lambda_1 + i\lambda_2(\lambda_2 \neq 0)$. Then comparing the imaginary parts of 7 yields

$$2\lambda_1[\int_0^1 |\phi(x)|^2 dx + M|\phi(1)|^2 + J|\phi'(1)|^2] + a|\phi(0)|^2 = 0. \quad (18)$$

There are two cases. When $\lambda_1 \neq 0$, it is obvious that $\lambda_1 < 0$ as $a > 0$. While as $\lambda_1 = 0$, it must be $\phi(0) = 0$ and so $\phi'''(0) = 0$. In this case, the solution of (8) shall be (we may assume that $\lambda_2 > 0$) $\phi(x) = \cos h \sqrt{\lambda_2}x - \cos \sqrt{\lambda_2}x$. But from

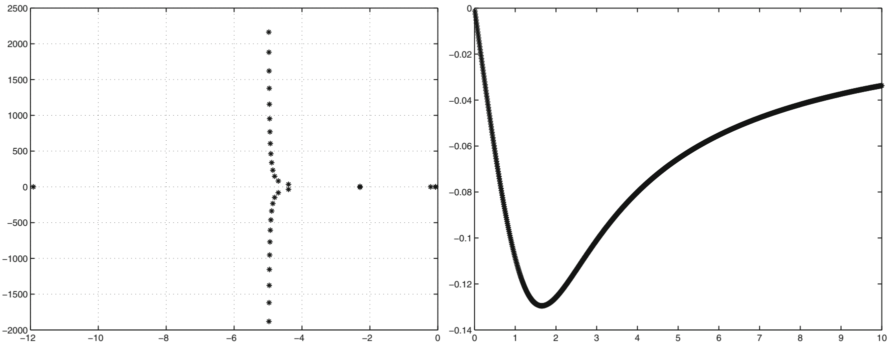

Fig. 1 Distribution of eigenvalues for $a = 5$ (*left*) and functional relation between $S(L)$ and a(*right*)

the boundary condition $\phi'''(1) = -M\lambda_2^2\phi(1)$, we arrive the contradiction that

$$\sin h\ \sqrt{\lambda_2} - \sin\ \sqrt{\lambda_2} = -M\sqrt{\lambda_2}(\cos h\ \sqrt{\lambda_2} - \cos\ \sqrt{\lambda_2}).$$

4 Numerical Simulation of Eigenvalues

In this section, the Legendre polynomial method is used to compute the spectrum of the hybrid system. We refer the procedure to [1] for details. Here we take $N = 100, \quad M = 0.1, \quad J = 0.2$ (Fig. 1).

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Monotone Iterative Technique for Systems of Nonlinear Caputo Fractional Differential Equations

Faten Toumi

Abstract In this work, we deal with the existence of extremal quasisolutions for the following finite system of nonlinear fractional differential equation ${}^{C}D^{q}u(t) + f(t, u(t)) = 0$ in $(0, 1), u(0) - \alpha u'(0) = \lambda, u(1) + \beta u'(1) = \mu$,where $1 < q < 2, \alpha, \beta \in (\mathbb{R}+)^{n}, \lambda, \mu \in \mathbb{R}^{n}$ and $f \in C([0, 1] \times \mathbb{R}^{n}, \mathbb{R}^{n})$ and ${}^{C}D^{q}$ is the Caputo fractional derivative of order q. We shall prove constructive existence results for a class of nonlinear equations by the use of iterative method technique combined with upper and lower quasisolutions. We construct a pair of sequences of coupled lower and upper quasisolutions which converge uniformly to extremal quasisolutions. Then, a uniqueness result is given under additional conditions on the nonlinearity f.

Keywords Nonlinear fractional differential system • Coupled lower and upper soltions • Mixed quasimonotone property • Monotone method

AMS subject classifications: 34A08, 34B34, 34B15, 34B27

1 Introduction

In the present paper, we study the existence of extremal quasisolutions for the following finite system of nonlinear fractional differential equations

$$ {}^{C}D^{q}u(t) + f(t, u(t)) = 0 \text{ in } (0, 1) \tag{1} $$

$$ u(0) - \alpha u'(0) = \lambda, u(1) + \beta u'(1) = \mu, \tag{2} $$

where $1 < q < 2, \alpha, \beta \in (\mathbb{R}+)^{n}, \lambda, \mu \in \mathbb{R}^{n}$ and $f \in \mathscr{C}([0, 1] \times \mathbb{R}^{n}, \mathbb{R}^{n})$ and ${}^{C}D^{q}$ is the Caputo fractional derivative of order q.

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The monotone iterative technique combined with the upper and lower solutions offers an effective method for proving constructive existence for a variety of nonlinear systems of integer order; see [3, 7, 8] and the references therein. Recently, many works investigated nonlinear systems of fractional order by following this approach. More precisely, most of those works treated nonlinear systems of order $0 < q < 1$ (see, for e.g., [5, 9]), while few works studied nonlinear differential equations or systems of order $q \geq 1$ in this setup. Due to comparison results established by Shi and Zhang [10] and Al-refaii [1], some works applied successfully the monotone iterative technique combined with the upper and lower solutions to obtain constructive existence of nonlinear differential equations in the case where the fractional order $q \in (1, 2)$; see [2, 4, 6, 11]. Motivated by the above works, we will introduce a method based on lower and upper quasisolutions to prove existence of minimal and maximal quasisolutions of the problem (1)–(2).

2 Preliminary Results

In this section, we shall first present some definitions and properties related to the Caputo fractional derivatives and the method of lower and upper quasisolutions. We then state a positivity result. Finally, we shall state an existence and uniqueness result for some linear system associated to our main problem.

We recall that for a function $u \in \mathscr{C}^2([0,1], \mathbb{R})$ the Riemann–Liouville fractional integral $I^q u$ and the Caputo fractional derivative ${}^C D^q u$ of order $q \in (1, 2)$ are respectively defined by

$I^q u(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} u(s)\, ds$, and

${}^C D^q u(t) = \frac{1}{\Gamma(2-q)} \int_0^t (t-s)^{1-q} u^{(2)}(s)\, ds = I^{2-q} u^{(2)}(t)$, where $t > 0$ and Γ is the Euler Gamma function.

Lemma 1 ([11]). *Let $p \in \mathscr{C}^2([0,1], \mathbb{R})$ and $M > 0$. If p satisfies the following inequalities,*

$$ {}^C D^q p(t) - Mp(t) \leq 0, t \in (0, 1) \tag{3} $$

$$ p(0) - \alpha p'(0) \geq 0, p(1) + \beta p'(1) \geq 0, \tag{4} $$

where $\alpha, \beta \geq 0$. Then $p(t) \geq 0$ for $t \in [0, 1]$, provided that $\alpha \geq \frac{1}{q-1}$.

Using standard arguments, we state the following:

Lemma 2. *Let $M > 0$ and $h \in \mathscr{C}([0,1], \mathbb{R})$. A function $u \in \mathscr{C}^2([0,1], \mathbb{R})$ is solution of the problem*

$$ {}^C D^q u(t) + h(t) - Mu(t) = 0, t \in (0, 1) \tag{5} $$

$$ u(0) - \alpha u'(0) = \lambda, u(1) + \beta u'(1) = \mu, \tag{6} $$

if, and only if, u is solution of the integro-differential equation

$$u(t) = \lambda + \frac{\alpha(\mu-\lambda)}{\alpha+\beta+1}(1+t) + \int_0^1 G_{\alpha,\beta}(t,s)(h(s)-Mu(s))\,ds \tag{7}$$

where for $t, s \in [0,1]$,

$$G_{\alpha,\beta}(t,s) = \frac{(\alpha+t)(1-s)^{q-2}(1-s+\beta(q-1)) - (\alpha+\beta+1)\left((t-s)^+\right)^{q-1}}{(\alpha+\beta+1)\,\Gamma(q)}. \tag{8}$$

$G_{\alpha,\beta}(t,s)$ *is the Green's function of the boundary value problem* (5)–(6)*. Here, for a real number* r, $r^+ = \max(r,0)$.

Lemma 3. *For all* $t \in [0,1]$, *we have*

$$0 \leq \int_0^1 G_{\alpha,\beta}(t,s)\,ds \leq \frac{\Lambda_{\alpha,\beta}}{\Gamma(q)},$$

where $\Lambda_{\alpha,\beta} = \alpha\left(\frac{1}{q}+\beta(q-1)\right) + \left(\frac{q-1}{q}\right)\frac{\left(\frac{1}{q}+\beta(q-1)\right)^{\frac{q}{q-1}}}{(\alpha+\beta+1)^{\frac{1}{q-1}}}$.

Proof. An elementary calculus yields to this result, so we omit the proof. □

Let r_i and s_i be nonnegative integers for each i, $1 \leq i \leq n$, such that $r_i + s_i = n-1$, so that we can split the vector $u \in \mathbb{R}^n$ into $\left(u_i, [u]_{r_i}, [u]_{s_i}\right)$. Then Eqs. (1)–(2) can be written as

$${}^C D^q u_i(t) + f_i\left(t, u_i(t), [u(t)]_{r_i}, [u(t)]_{s_i}\right) = 0 \text{ in } (0,1) \tag{9}$$

$$u_i(0) - \alpha_i u_i'(0) = \lambda_i, u_i(1) + \beta_i u_i'(1) = \mu_i, \tag{10}$$

for each i, $1 \leq i \leq n$.

Now, recall that for $u, v \in \mathbb{R}^n$, $u \leq v$ implies that $u_i \leq v_i$ for each i, $1 \leq i \leq n$ and define for $u, v \in \mathscr{C}^2([0,1], \mathbb{R}^n)$ the set

$$[u,v] = \left\{h \in \mathscr{C}^2([0,1], \mathbb{R}^n) : u(t) \leq h(t) \leq v(t), t \in [0,1]\right\}.$$

For the sake of simplicity, we set $\varphi_i(u) = u_i(0) - \alpha_i u_i'(0)$ and $\psi_i(u) = u_i(1) + \beta_i u_i'(1)$ for each $i \in \{1, \ldots, n\}$.

Definition 1. A function $f \in \mathscr{C}([0,1] \times \mathbb{R}^n, \mathbb{R}^n)$ is said to possess a mixed quasimonotone property if for each i, $1 \leq i \leq n$, $f_i\left(t, u_i, [u]_{r_i}, [u]_{s_i}\right)$ is monotone nondecreasing in $[u]_{r_i}$ and monotone nonincreasing in $[u]_{s_i}$.

Definition 2. Let $v, w \in \mathscr{C}^2([0,1], \mathbb{R}^n)$, v and w are coupled lower and upper quasisolutions of (9)–(10) if they satisfy

$$^{C}D^{q}v_{i}\left(t\right)+f_{i}\left(t,v_{i}\left(t\right),\left[v\left(t\right)\right]_{r_{i}},\left[w\left(t\right)\right]_{s_{i}}\right)\geq0\text{ in }\left(0,1\right)\tag{11}$$

$$\varphi_{i}\left(v\right)\leq\lambda_{i},\psi_{i}\left(v\right)\leq\mu_{i},\tag{12}$$

and

$$^{C}D^{q}w_{i}\left(t\right)+f_{i}\left(t,w_{i}\left(t\right),\left[w\left(t\right)\right]_{r_{i}},\left[v\left(t\right)\right]_{s_{i}}\right)\leq0\text{ in }\left(0,1\right)\tag{13}$$

$$\varphi_{i}\left(w\right)\geq\lambda_{i},\psi_{i}\left(w\right)\geq\mu_{i},\tag{14}$$

for each $i, 1\leq i\leq n$.

Definition 3. Let $v,w\in\mathscr{C}^{2}\left(\left[0,1\right],\mathbb{R}^{n}\right)$, v and w are coupled quasisolutions of (9)–(10) if they satisfy

$$^{C}D^{q}v_{i}\left(t\right)+f_{i}\left(t,v_{i}\left(t\right),\left[v\left(t\right)\right]_{r_{i}},\left[w\left(t\right)\right]_{s_{i}}\right)=0\text{ in }\left(0,1\right)$$
$$\varphi_{i}\left(v\right)=\lambda_{i},\psi_{i}\left(v\right)=\mu_{i},$$

and

$$^{C}D^{q}w_{i}\left(t\right)+f_{i}\left(t,w_{i}\left(t\right),\left[w\left(t\right)\right]_{r_{i}},\left[v\left(t\right)\right]_{s_{i}}\right)=0\text{ in }\left(0,1\right)$$
$$\varphi_{i}\left(w\right)=\lambda_{i},\psi_{i}\left(w\right)=\mu_{i},$$

for each $i, 1\leq i\leq n$.

In the rest of the paper, we adopt the following assumptions:

$(H1)$ $v^{0},w^{0}\in\mathscr{C}^{2}\left(\left[0,1\right],\mathbb{R}^{n}\right)$ are coupled lower and upper quasisolutions of (9)–(10) such that $v^{0}\leq w^{0}$ on $[0,1]$.

$(H2)$ The function f possesses a mixed quasimonotone property, and there exists $M\in\left(\mathbb{R}_{+}^{*}\right)^{n}$ such that $M_{i}<\frac{\Gamma(q)}{\Lambda_{\alpha_{i},\beta_{i}}}$ and

$$f_{i}\left(t,u_{i},[u]_{r_{i}},[u]_{s_{i}}\right)-f_{i}\left(t,\underline{u}_{i},[u]_{r_{i}},[u]_{s_{i}}\right)\geq-M_{i}\left(u_{i}-\underline{u}_{i}\right)$$

for each $i, 1\leq i\leq n$, whenever $v^{0}\leq\underline{u}\leq u\leq w^{0}$ on $[0,1]$.

To state our main result, we need the following Lemma

Lemma 4. *Assume* $(H1)$ *and* $(H2)$*. Suppose that for each* i, $1\leq i\leq n, \alpha_{i}\left(q-1\right)\geq1$. *Let* η *and* ξ *be two fixed functions belonging to* $\left[v^{0},w^{0}\right]$. *Then, the uncoupled linear fractional differential system*

$$^{C}D^{q}u_{i}\left(t\right)+f_{i}\left(t,\eta_{i}\left(t\right),\left[\eta\left(t\right)\right]_{r_{i}},\left[\xi\left(t\right)\right]_{s_{i}}\right)-M_{i}\left(u_{i}\left(t\right)-\eta_{i}\left(t\right)\right)=0\text{ in }\left(0,1\right)\tag{15}$$

$$\varphi_{i}\left(u\right)=\lambda_{i},\psi_{i}\left(u\right)=\mu_{i},\tag{16}$$

for each $i, 1\leq i\leq n$*, admits a unique positive solution* u *in* $\left[v^{0},w^{0}\right]$.

Proof. First, it is obvious to see from Lemma 2 that $u \in \mathscr{C}^2([0,1],\mathbb{R}^n)$ is a solution of problem (15)–(16) if, and only if, u is solution of the following integro-equation

$$u_i(t) = \lambda_i + \frac{(\mu_i - \lambda_i)}{\alpha_i + \beta_i + 1}(\alpha_i + t) + \int_0^1 G_{\alpha_i,\beta_i}(t,s)(h_i(s) - M_i u_i(s))\,ds, \quad (17)$$

where $h_i(t) = f_i\left(t, \eta_i(t), [\eta(t)]_{r_i}, [\xi(t)]_{s_i}\right) + M_i\eta_i(t)$, on $[0,1]$, for each $1 \le i \le n$.

Let $\mathscr{C}([0,1],\mathbb{R}^n)$ be the Banach space endowed with the norm $\|u\|_\infty = \sup_{t\in[0,1]}\left(\max_{1\le i\le n}(|u_i(t)|)\right)$. Define on $\mathscr{C}([0,1],\mathbb{R}^n)$ the operator T by $Tu = (T_1u_1,\ldots,T_nu_n)$, where

$$T_iu_i(t) = \lambda_i + \frac{(\mu_i - \lambda_i)}{\alpha_i + \beta_i + 1}(\alpha_i + t) + \int_0^1 G_{\alpha_i,\beta_i}(t,s)(h_i(s) - M_i u_i(s))\,ds$$

for each $t \in [0,1]$. Let $u, v \in \mathscr{C}([0,1],\mathbb{R}^n)$; then, by hypothesis on M_i and Lemma 3, we get $\|Tu - Tv\|_\infty < \|u - v\|_\infty$. So by Banach Theorem, the operator T admits a unique fixed point u in $\mathscr{C}([0,1],\mathbb{R}^n)$. Now, let us prove that $u \in \left[v^0, w^0\right]$. By hypothesis $(H1)$, v^0 is a lower quasisolution, so v^0 satisfies (11) and (12) for each $1 \le i \le n$. Set $p = v^0 - u$. Using the fact that u satisfies (15) and (16) and the mixed quasimonotone property of the function f, we obtain

$$\begin{aligned} {}^CD^q p_i(t) &\ge f_i(t,\eta_i,[\eta]_{r_i},[\xi]_{s_i}) - f_i(t,v_i^0,\left[v^0\right]_{r_i},\left[w^0\right]_{s_i}) - M_i(u_i - \eta_i) \\ &\ge -M_i\left(\eta_i - v_i^0\right) - M_i(u_i - \eta_i). \end{aligned}$$

Thus ${}^CD^q p_i(t) - M_ip_i(t) \ge 0$ on $(0,1)$, $\varphi_i(p) \le 0$, $\psi_i(p) \le 0$. Then, by Lemma 1, we deduce that $p_i(t) \le 0$ on $[0,1]$, for each $1 \le i \le n$. That is, $v^0 \le u$. Similarly, we prove that $u \le w^0$, which ends the proof. □

3 Main Results

In this section, we mainly prove the existence of extremal quasisolutions for the nonlinear fractional differential systems (1) and (2) and we state then a uniqueness result.

Theorem 1. *Consider the boundary value problem (9)–(10) with $\alpha(q-1) \ge 1$. Assume that $(H1)$ and $(H2)$ hold. Then there exists $\left(v^k\right)_{k\ge1}, \left(w^k\right)_{k\ge1}$ a pair of monotone sequences of coupled lower and upper quasisolutions of (9)–(10) such that the sequences $\left(v^k\right)_{k\ge1}$ and $\left(w^k\right)_{k\ge1}$ converge monotonically and uniformly to v^* and w^*, respectively with $v^0 \le v^* \le w^* \le w^0$ on $[0,1]$. Moreover, v^* and w^* are minimal and maximal quasisolutions of (9)–(10) in $\left[v^0, w^0\right]$. Further, any solution of (9)–(10) in $\left[v^0, w^0\right]$ satisfies $v^* \le u \le w^*$.*

Proof. Let $\eta, \xi \in \left[v^0, w^0\right]$. Then, by Lemma 4, the uncoupled linear fractional differential system

$$ {}^{C}D^{q}u_i(t) + f_i\left(t, \eta_i(t), [\eta(t)]_{r_i}, [\xi(t)]_{s_i}\right) - M_i\left(u_i(t) - \eta_i(t)\right) = 0 \text{ in } (0,1) \quad (18) $$

$$ \varphi_i(u) = \lambda_i, \psi_i(u) = \mu_i, \quad (19) $$

for each $i, 1 \le i \le n$, admits a unique positive solution $u_{\eta,\xi}$ in $\left[v^0, w^0\right]$. So we can define a map $A : \left[v^0, w^0\right] \times \left[v^0, w^0\right] \to \left[v^0, w^0\right]$ such that $A(\eta, \xi) = u_{\eta,\xi}$. We shall prove that A is mixed monotone operator. Let $\eta^1, \eta^2, \xi \in \left[v^0, w^0\right], \eta^1 \le \eta^2$ be given and suppose that $x^1 = A\left(\eta^1, \xi\right)$ and $x^2 = A\left(\eta^2, \xi\right)$. Then we have ${}^{C}D^{q}\left(x_i^2 - x_i^1\right) - M_i\left(x_i^2 - x_i^1\right) = f_i\left(t, \eta_i^1, \left[\eta^1\right]_{r_i}, [\xi]_{s_i}\right) - f_i\left(t, \eta_i^2, \left[\eta^2\right]_{r_i}, [\xi]_{s_i}\right) - M_i\left(\eta_i^2 - \eta_i^1\right)$. Using the mixed monotone property of f and $(H2)$, we get ${}^{C}D^{q}\left(x_i^2 - x_i^1\right) - M_i\left(x_i^2 - x_i^1\right) \le f_i\left(t, \eta_i^1, \left[\eta^2\right]_{r_i}, [\xi]_{s_i}\right) - f_i\left(t, \eta_i^2, \left[\eta^2\right]_{r_i}, [\xi]_{s_i}\right) - M_i\left(\eta_i^2 - \eta_i^1\right) \le 0$.

On the other hand, we have $\varphi_i\left(x^2 - x^1\right) = \varphi_i\left(x^2\right) - \varphi_i\left(x^1\right) = 0$ and $\psi_i\left(x^2 - x^1\right) = 0$. Thus, by Lemma 1, we conclude that $x_i^2 \ge x_i^1$, for each $i, 1 \le i \le n$. That is, $x^1 \le x^2$, and so $A(\eta, \xi)$ is nondecreasing in the first variable η. At the same manner, we prove that $A(\eta, \xi)$ is nonincreasing in its second variable ξ. Whence, A is mixed monotone operator.

Now define the sequences $\left(v^k\right)_{k\ge1}$ and $\left(w^k\right)_{k\ge1}$ as follows: $v^k = A\left(v^{k-1}, w^{k-1}\right)$ and $w^k = A\left(w^{k-1}, v^{k-1}\right)$ *for* $k \ge 1$. We shall prove that the sequences $\left(v^k\right)_{k\ge1}$ and $\left(w^k\right)_{k\ge1}$ are nondecreasing and nonincreasing on $[0, 1]$, respectively. We will proceed by induction. It is obvious to see that $v^1 = A\left(v^0, w^0\right) \ge v^0$ and $w^1 = A\left(w^0, v^0\right) \le w^0$. Assume that the hypothesis is true up $k \ge 1$, that is, $v^{k-1} \le v^k$ *and* $w^k \le w^{k-1}$. The mixed monotone property of A yields to $v^{k+1} \ge A\left(v^{k-1}, w^k\right) \ge v^k$ and $w^{k+1} \le A\left(w^{k-1}, v^k\right) \le w^k$. Thus the sequences $\left(v^k\right)_{k\ge1}$ and $\left(w^k\right)_{k\ge1}$ are monotone on $[0, 1]$.

Now, we intend to prove that, for each $k \ge 1$, v^k and w^k are coupled of lower and upper quasisolutions of (9)–(10). Let $k \ge 1$. Then $v^k = A\left(v^{k-1}, w^{k-1}\right)$ satisfies

$$ {}^{C}D^{q}v_i^k + f_i\left(t, v_i^{k-1}, \left[v^{k-1}\right]_{r_i}, \left[w^{k-1}\right]_{s_i}\right) - M_i\left(v_i^k - v_i^{k-1}\right) = 0 \text{ in } (0,1) \quad (20) $$

$$ \varphi_i\left(v^k\right) = \lambda_i, \psi_i\left(v^k\right) = \mu_i, \quad (21) $$

for each $i, 1 \le i \le n$.

By adding $f_i\left(t, v_i^k(t), \left[v^k(t)\right]_{r_i}, \left[w^k(t)\right]_{s_i}\right)$ to both sides of (20), we obtain

$$ \begin{aligned} &{}^{C}D^{q}v_i^k + f_i\left(t, v_i^k, \left[v^k\right]_{r_i}, \left[w^k\right]_{s_i}\right) \\ &\quad = f_i\left(t, v_i^k, \left[v^k\right]_{r_i}, \left[w^k\right]_{s_i}\right) - f_i\left(t, v_i^{k-1}, \left[v^{k-1}\right]_{r_i}, \left[w^{k-1}\right]_{s_i}\right) + M_i\left(v_i^k - v_i^{k-1}\right). \end{aligned} $$

Using the fact that $v^{k-1} \leq v^k$, $w^k \leq w^{k-1}$ and the mixed monotone property of the function f, we get ${}^C D^q v_i^k + f_i\left(t, v_i^k, \left[v^k\right]_{r_i}, \left[w^k\right]_{s_i}\right) \geq 0$. Similarly, we show that ${}^C D^q w_i^k + f_i\left(t, w_i^k, \left[w^k\right]_{r_i}, \left[v^k\right]_{s_i}\right) \leq 0$,which together with the fact that $\varphi_i\left(v^k\right) = \varphi_i\left(w^k\right) = \lambda_i$, $\psi_i\left(v^k\right) = \psi_i\left(w^k\right) = \mu_i$ yields to $\left(v^k, w^k\right)$ is a pair of coupled lower and upper quasisolutions of problems (9)–(10). Next, let us prove that $v^k \leq w^k$, for each $k \geq 1$. We use induction argument. For $k = 1$, since $v^0 \leq w^0$ then using the mixed monotone property of A we get $v^1 \leq A\left(w^0, w^0\right) \leq w^1$. Now, suppose that $v^k \leq w^k$, for some $k \geq 1$, then we have $v^{k+1} \leq A\left(w^k, v^k\right) = w^{k+1}$. Whence, $v^k \leq w^k$, for each $k \geq 1$.

Next, since the sequences $\left(v^k\right)_{k\geq 1}$ and $\left(w^k\right)_{k\geq 1}$are uniformly bounded and equicontinuous in $\mathscr{C}\left([0,1], \mathbb{R}^n\right)$, then, by Arzela–Ascoli Theorem, $\left(v^k\right)_{k\geq 1}$ and $\left(w^k\right)_{k\geq 1}$ are relatively compact in $\mathscr{C}\left([0,1], \mathbb{R}^n\right)$. Thus, we deduce the existence of subsequences $\left(v^{k_j}\right)_{j\geq 1}$ and $\left(w^{k_j}\right)_{j\geq 1}$which converge to v^* and w^*, respectively. Using the fact that the sequences $\left(v_i^k\right)_{k\geq 1}$ and $\left(w_i^k\right)_{k\geq 1}$ are monotone, we reach to the convergence of the entire sequences that is $\lim_{k\to+\infty} v^k(t) = v^*(t)$ and $\lim_{k\to+\infty} w^k(t) = w^*(t)$, on $[0,1]$. Using the equicontinuity of the sequences $\left(v^k\right)_{k\geq 1}$ and $\left(w^k\right)_{k\geq 1}$, the pointwise convergence implies the uniform one; then, we have $\left(v^k\right)_{k\geq 1}$ and $\left(w^k\right)_{k\geq 1}$ converge uniformly respectively to v^* and w^* on $[0,1]$.

Next, let us verify that v^* and w^* are quasisolutions of (9)–(10) in $\left[v^0, w^0\right]$. By Lemma 2, we have for each

$$v_i^k(t) = \lambda_i + \frac{(\mu_i - \lambda_i)}{\alpha_i + \beta_i + 1}(\alpha_i + t) + \int_0^1 G_{\alpha_i,\beta_i}(t,s)\left(h_i(s) - M_i v_i^k(s)\right) ds,$$

where $h_i(t) = f_i\left(t, v_i^{k-1}(t), \left[v^{k-1}(t)\right]_{r_i}, \left[w^{k-1}(t)\right]_{s_i}\right) + M_i v_i^{k-1}(t)$, on $[0,1]$.

Using the continuity of the function f and $v^k \in \left[v^0, w^0\right]$, by Lebesgue convergence theorem, we deduce that

$$v_i^*(t) = \lambda_i + \frac{(\mu_i - \lambda_i)}{\alpha_i + \beta_i + 1}(\alpha_i + t) + \int_0^1 G_{\alpha_i,\beta_i}(t,s) f_i\left(s, v_i^*, \left[v^*\right]_{r_i}, \left[w^*\right]_{s_i}\right) ds.$$

Similarly, we obtain

$$w_i^*(t) = \lambda_i + \frac{(\mu_i - \lambda_i)}{\alpha_i + \beta_i + 1}(\alpha_i + t) + \int_0^1 G_{\alpha_i,\beta_i}(t,s) f_i\left(s, w_i^*, \left[w^*\right]_{r_i}, \left[v^*\right]_{s_i}\right) ds.$$

where

$$G_{\alpha_i,\beta_i}(t,s) = \frac{(\alpha_i + t)(1-s)^{q-2}(1-s+\beta_i(q-1)) - (\alpha_i+\beta_i+1)\left((t-s)^+\right)^{q-1}}{(\alpha_i+\beta_i+1)\,\Gamma(q)}.$$

So,

$$v_i^*(t) + I^q f_i\left(., v_i^*, [v^*]_{r_i}, [w^*]_{s_i}\right)(t)$$

$$= \lambda_i + \left(\frac{(\mu_i - \lambda_i)}{\alpha_i + \beta_i + 1} + \int_0^1 \frac{(1-s)^{q-2}(1-s+\beta_i(q-1))}{(\alpha_i+\beta_i+1)\Gamma(q)} f_i\left(s, v_i^*, [v^*]_{r_i}, [w^*]_{s_i}\right) ds\right)(\alpha_i + t).$$

Applying ${}^C D^q$ on both sides of the last equation, it follows that ${}^C D^q v_i^*(t) + f_i\left(t, v_i^*(t), [v^*(t)]_{r_i}, [w^*(t)]_{s_i}\right) = 0$, On the other hand, a simple calculus yields to $v_i^*(0) = \lambda_i + \frac{(\mu_i - \lambda_i)}{\alpha_i + \beta_i + 1}$ and $\left(v_i^*\right)'(0) = \frac{(\mu_i - \lambda_i)}{\alpha_i + \beta_i + 1}$. Thus $\varphi_i(v^*) = \lambda_i$; we show also that $\psi_i(v^*) = \mu_i$. Similarly, we get ${}^C D^q w_i^*(t) + f_i\left(t, w_i^*(t), [w^*(t)]_{r_i}, [v^*(t)]_{s_i}\right) = 0$ and $\varphi_i(w^*) = \lambda_i$, $\psi_i(w^*) = \mu_i$. Whence, v^* and w^* are quasisolutions of (9)–(10) in $\left[v^0, w^0\right]$.

Next, let us show that v^* and w^* are minimal and maximal coupled quasisolutions of (9)–(10) in $\left[v^0, w^0\right]$. Let v and w be coupled quasisolutions of (9)–10 in $\left[v^0, w^0\right]$. We will use induction. We have $v^0 \le v$ and $w \le w^0$. Suppose that for some $k \ge 1$, $v^k \le v$ and $w \le w^k$. For each $i, 1 \le i \le n$, we have ${}^C D^q v_i(t) + f_i\left(t, v_i(t), [v(t)]_{r_i}, [w(t)]_{s_i}\right) = 0, \varphi_i(v) = \lambda_i, \psi_i(v) = \mu_i$. So, using (3) and the mixed monotone property of the function f, it follows ${}^C D^q\left(v_i - v_i^{k+1}\right) - M_i\left(v_i - v_i^{k+1}\right) \le 0$. On the other hand, we have $\varphi_i\left(v - v^{k+1}\right) = \lambda_i, \psi_i\left(v - v^{k+1}\right) = \mu_i$. Thus, by Lemma 1, we deduce that $v^{k+1} \le v$. At the same manner, we show that $w \le w^{k+1}$. Whence, by taking $k \to +\infty$, we reach $v^* \le v$ and $w \le w^*$. Finally, by induction argument, it is easy to show that any solution u of (9)–(10) in $\left[v^0, w^0\right]$ satisfies $v^* \le u \le w^*$. This ends the proof. □

In the following, we state a uniqueness result.

Theorem 2. *Suppose that assumptions* $(H1) - (H2)$ *hold. Then if, for each* $i, 1 \le i \le n$, *there exists* $N_i > 0$ *such that*

$$f_i\left(t, u_i, [u]_{r_i}, [\underline{u}]_{s_i}\right) - f_i\left(t, \underline{u}_i, [\underline{u}]_{r_i}, [u]_{s_i}\right) \le -N_i\left(u_i - \underline{u}_i\right)$$

whenever $v^0 \le \underline{u} \le u \le w^0$ *on* $[0, 1]$, *the problem* (9)–(10) *has unique solution in* $\left[v^0, w^0\right]$.

Proof. By Theorem 1, there exist v^* and w^* minimal and maximal quasisolutions of (9)–(10) in $\left[v^0, w^0\right]$ such that $v^* \le w^*$. By Lemma 1, we prove that $w^* \le v^*$ on $[0, 1]$. Hence $v^* = w^*$ is the unique solution of the problem (9)–(10) in $\left[v^0, w^0\right]$; this ends the proof. □

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Oscillatory Solutions of Boundary Value Problems

Felix Sadyrbaev

Abstract We consider boundary value problems of the form

$$x'' = f(t, x, x'),$$
$$x(a) = A, \quad x(b) = B,$$

assuming that f is continuous together with f_x and $f_{x'}$. We study also equations in a quasi-linear form

$$x'' + p(t)x' + q(t)x = F(t, x, x').$$

Introducing types of solutions of boundary value problems as an oscillatory type of the respective equation of variations, we show that for a solution of definite type, the problem can be reformulated in a quasi-linear form. Resonant problems are considered separately. Any resonant problem that has no solutions of indefinite type is in fact nonresonant. The ways of how to detect solutions of definite types are discussed.

Keywords Gene regulation • attractive sets • dynamical system

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1 Introduction

The classical result in the theory of boundary value problems (BVP) for ordinary differential equations states [1] that the problem

$$x' + A(t)x = f(t,x), \quad x \in R^n, \tag{1}$$

$$B_1 x(a) + B_2 x(b) = 0, \tag{2}$$

where all entries are continuous, is solvable if f is bounded and the homogeneous problem

$$x' + A(t)x = 0, \quad B_1 x(a) + B_2 x(b) = 0 \tag{3}$$

has only the trivial solution $x(t) \equiv 0$.

If homogeneous problem has a nontrivial solution, the situation is different and BVP is then called *resonant.*

These results when interpreted for the second-order problem

$$x'' + p(t)x' + q(t)x = F(t,x,x'), \tag{4}$$

$$x(a) = A,\ x(b) = B, \tag{5}$$

state that problems (4) and (5) are solvable if F is bounded and homogeneous problem

$$x'' + p(t)x' + q(t)x = 0, \quad x(a) = 0,\ x(b) = 0 \tag{6}$$

has only the trivial solution. Otherwise (if homogeneous problem has a nontrivial solution), the problems (4) and (5) are resonant. There is intensive literature on solvability of resonant problems.

In this article, we treat both nonresonant and resonant the second-order BVP through the notion of a *type* of a solution. It is shown that if the second-order problem (resonant or not) has a solution $x(t)$ of certain type (definition will be given soon), then it can be reformulated in the quasi-linear form with the linear part $(l_2x)(t) := x'' + p(t)x' + q(t)$ of the same oscillatory type. This reformulation (reduction) is not possible if the type of a solution $x(t)$ is indefinite (we call this *internal* resonance). We show by constructing examples that there are formally resonant problems that in fact are not resonant: they can be reformulated in a nonresonant form.

2 Types of Linear Parts and Types of Solutions

Consider quasi-linear problem (4), (5). We say that the linear part $(l_2x)(t) = x'' + p(t)x' + q(t)x$ has type i, if a solution $y(t)$ of the Cauchy problem

$$y'' + p(t)y' + q(t)y = 0, \quad y(a) = 0, \ y'(a) = 1 \tag{7}$$

has exactly i zeros in the interval (a, b) and $y(b) \neq 0$.

If $y(b) = 0$, then the linear part $(l_2x)(t)$ is *resonant.*

For instance, the linear part $x'' + k^2x$, where $3\pi < k < 4\pi$, has type 3.

Let us pass to definition of the type of a solution.

Let $\xi(t)$ be a solution of the BVP

$$x'' = f(t, x, x'), \quad x(a) = A, \ x(b) = B. \tag{8}$$

The right-hand side function f may contain a linear part.

Consider the respective equation of variations

$$y'' = f_x(t, \xi(t), \xi'(t))y + f_{x'}(t, \xi(t), \xi'(t))y'. \tag{9}$$

Definition 1. Let $\xi(t)$ be a solution of BVP. We say that the type of $\xi(t)$ is i, if equation of variations (9) with respect to $\xi(t)$ is such that a solution $y(t)$ with the initial conditions

$$y(a) = 0, \quad y'(a) = 1 \tag{10}$$

has exactly i zeros in the interval (a, b) and $y(b) \neq 0$. Denote this:

$$type(\xi) = i.$$

If moreover $y(b) = 0$, denote the intermediate type

$$type(\xi) = (i, i + 1). \tag{11}$$

Remark 1. Therefore, a solution of type $(i, i + 1)$ is a solution $\xi(t)$ such that the respective $y(t)$ has exactly i zeros in (a, b) and $y(b) = 0$.

Remark 2. The study of solutions of BVP in terms of solutions of equations of variations was initiated in early papers [4–6]. The authors have observed the existence of solutions of zero type (in our terminology) in problems satisfying some standard requirements.

Example 1. The trivial solution $\xi(t) \equiv 0$ of the problem $x'' = -x + x^3$, $x(0) = 0$, $x(\pi) = 0$ is of type $(0, 1)$ since a solution $y(t)$ of the Cauchy problem $y'' = -y$ (equation of variations with respect to ξ), $y(0) = 0, \ y'(0) = 1$ has the first zero at $t = \pi$.

Example 2. To illustrate types of solutions, consider the problem

$$x'' = -2x^3, \quad x(0) = 0,\ x(1) = 0. \tag{12}$$

This problem has infinitely many solutions. These solutions can be expressed in terms of the lemniscate sine function $\mathrm{sl}\, t$. This function resembles the usual sine function, and it is periodic with minimal period $4A$, where $A := \int_0^1 \frac{ds}{\sqrt{1-s^4}} \approx 1.311$. The A number is an analogue of $\pi/2$. More about lemniscatic functions and a set of formula similar to usual trigonometric relations can be found in [3].

A solution of the Cauchy problem

$$x'' = -2x^3, \quad x(0) = 0,\ x'(0) = \beta \tag{13}$$

is given by $x(t;\beta) = \sqrt{\beta}\,\mathrm{sl}\,(\sqrt{\beta}t)$. Functions $x(t;\beta)$ satisfy the condition $x(1) = 0$ only for $\beta = \pm(2A)^2, \pm(4A)^2$ and so on. Let us look at first three nontrivial solutions of the problem (13). These solutions are

$$x_1(t) = 2A\,\mathrm{sl}\,(2A\,t), \quad x_2(t) = 4A\,\mathrm{sl}\,(4A\,t),\ x_3(t) = 6A\,\mathrm{sl}\,(6A\,t).$$

In order to detect their types, consider the equations of variations

$$y_1'' = -6x_1^2(t)y_1, \quad y_2'' = -6x_2^2(t)y_2, \quad y_3'' = -6x_3^2(t)y_3$$

along with the initial conditions $y(0) = 0$, $y'(0) = 1$ (Fig. 1).

It follows that $type(x_1) = 1$.

Figure 2 visualizes properties of x_2, x_3 and the respective solutions y_2, y_3 of equations of variations.

The types of x_2 and x_3 are respectively two and three.

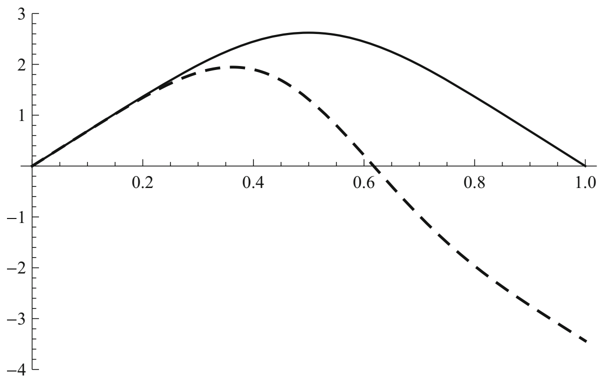

Fig. 1 A solution $x_1(t)$ (*solid line*) and the respective $y_1(t)$(*dashed*)

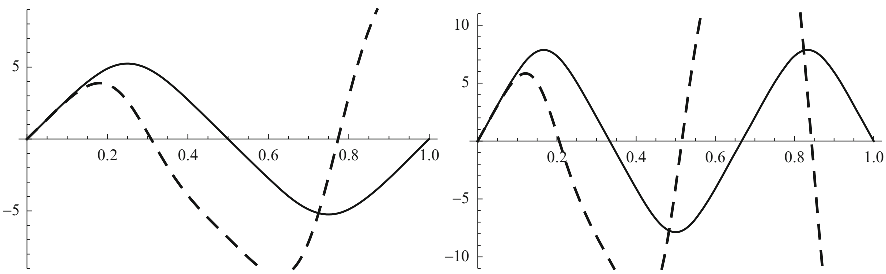

Fig. 2 The solutions $x_2(t)$ and $x_3(t)$ (*solid*) and the respective $y_2(t)$ and $y_3(t)$ (*dashed*)

3 Reduction to Quasi-linear Problem

Return to the problem (7). The right side f may contain linear part. The case of the problem (7) being resonant is not excluded.

Theorem 1. *Suppose the problem (7) has a solution $\xi(t)$ of type i.*

Then the problem can be reformulated in the quasi-linear form (4), (5), where the linear part $x'' + p(t)x' + q(t)x$ in (4) has the type i.

Proof. One has that

$$\begin{aligned} rlx - \xi''(t) &= f(t,x,x') - f(t,\xi(t),\xi'(t)) \\ &= f_x(t,\xi(t),\xi'(t))(x-\xi(t)) + f_{x'}(t,\xi(t),\xi'(t))(x'-\xi'(t)) + \varphi(t,x,x') \end{aligned} \tag{14}$$

or

$$\begin{aligned} lx - f_{x'}(t,\xi(t),\xi'(t))x' - f_x(t,\xi(t),\xi'(t))x = {} & \xi''(t)(t,\xi(t),\xi'(t))\xi'(t) \\ & - f_x(t,\xi(t),\xi'(t))\xi(t) + \varphi(t,x,x') \end{aligned} \tag{15}$$

and finally

$$(l_2x)(t) = h(t) + \tilde{\varphi}(t,x,x'), \tag{16}$$

where $(l_2x)(t) = x'' - f_{x'}(t,\xi(t),\xi'(t))x' - f_x(t,\xi(t),\xi'(t))x$,

$$h(t) = \xi''(t)(t,\xi(t),\xi'(t))\xi'(t) - f_x(t,\xi(t),\xi'(t))\xi(t),$$

$\tilde{\varphi}(t,x,x')$ is a smooth (C^1) bounded function which coincides with $\varphi(t,x,x')$ in some vicinity of $(t,\xi(t),\xi'(t))$, $t \in [a,b]$.

The linear part $(l_2 x)(t)$ has type i since a solution $\xi(t)$ has type i. Function ξ, by construction, is a solution of BVP (15), (5). □

Remark 3. The above reduction is not possible if a solution $\xi(t)$ is of indefinite type $(i, i+1)$.

The converse of Theorem 1 is not true. There exist quasi-linear problems (i.e., problems with equations containing the linear part and bounded nonlinearity) that have linear parts of definite type i and have not a solution of type i.

Indeed, consider the problem

$$x'' = \varphi(x), \quad x(0) = 0, \quad x(\pi) = 0, \tag{17}$$

where

$$\varphi(x) = \begin{cases} 0, & x > 1, \\ -x + x^3, & -1 \leq x \leq 1, \\ 0, & x < -1, \end{cases} \tag{18}$$

Any nontrivial solution $x(t;\gamma)$ of the Cauchy problem $x'' = \varphi(x), x(0) = 0,\ x'(0) = \gamma$ has the first zero $t_1(\gamma)$ (if any) after the point $t = 1$. So the trivial solution $\xi \equiv 0$ is the only solution of the problem (17). The equation of variations for the trivial solution is $y'' = -y$ and the type of ξ is $(0, 1)$.

However, the following is true.

Theorem 2. *Quasi-linear problem (4), (5), where*

1) *F is bounded continuous function with continuous partial derivatives F_x, $F_{x'}$;*
2) *The linear part $x'' + p(t)x' + q(t)x$ is of type i;*
   either has a solution of type i or a solution of type $(i-1, i)$ or a solution of type $(i, i+1)$.

The proof is based on the following lemmas which are stated here for the reader's convenience.

Lemma 1 (Lemma 2.1 in [7]). *A set S of all solutions of BVP (4), (5) is nonempty and compact in $C^1([0, 1])$.*

Lemma 2 (Lemma 2.2 in [7]). *There are elements $x^*(t)$ and $x_*(t)$ in S, which possess the properties ${x^*}'(a) = \max\{x'(a) : x \in S\}$, ${x_*}'(a) = \min\{x'(a) : x \in S\}$.*

Lemma 3 (Lemma 2.3 in [7]). *Suppose that the linear part $x'' + p(t)x' + q(t)x$ in (4) is of type i. Let ξ be any element of S.*

Then for γ large enough the difference $u(t;\gamma) = x(t;\gamma) - \xi(t)$ has exactly i zeros in the interval (a, b) and $u(b;\gamma) \neq 0$.

Here $x(t;\gamma)$ stands for a solution of the Cauchy problem (4) $x(a) = A$, $x'(a) = \gamma$.

Lemma 4 (Lemma 2.4 in [7]). *Let ξ be any element of S and $u(t;\gamma)$ as above.*

Zeros $t_i(\gamma)$ (if any) of the function $u(t;\gamma)$ continuously depend on γ. If $|\gamma| < B < +\infty$, then there exists $\delta(B) > 0$ such that the distance between two consecutive zeros of $u(t;\gamma)$ cannot be less than δ. If for some $\gamma_0 \neq \xi'(a)$ $u(b;\gamma_0) = 0$, then the respective $x(t;\gamma_0)$ solves problem (4), (5).

The proof of Theorem 2 in general repeats that of Theorem 2.1 in [7] (one may consult also the paper [8] for application).

Proof. Consider, for definiteness, a solution $x^*(t)$ of problems (4) and (5), that is, a solution with maximal value of the derivative $x^{*\prime}$ at $t = a$. Suppose this solution is not of type i and it is not of type $(i-1, i)$ and not of type $(i, i+1)$. Consider the respective equation of variations

$$y'' + p(t)y' + q(t)y = F_x(t, x^*(t), x^{*\prime}(t))y + F_{x'}(t, x^*(t), x^{*\prime}(t))y' \tag{19}$$

together with the initial conditions $y(a) = 0, \quad y'(a) = 1$. It follows from our assumptions about $x^*(t)$ and definition of a type of a solution that there are two possibilities for $y(t)$: either (a) $\tau_{i+1} \in (a,b)$ or (b) $\tau_{i-1} \in (a,b)$ and $y(t)$ doe not vanish in $(\tau_{i-1}, b]$.

Consider the first case. Since $y(t)$ is approximation for $u(t,\gamma) = x(t,\gamma) - x^*(t)$, this difference has $(i+1)$-st zero z_{i+1} in the interval (a,b) for $\gamma > x^{*\prime}(a)$ and sufficiently close to $x^{*\prime}(a)$. Let us increase γ. It follows from Lemma 3 that for γ sufficiently large the difference $u(t,\gamma)$ has exactly i zeros in (a,b) and $u(b,\gamma) \neq 0$. Then it follows from Lemma 4 that the zero z_{i+1} left the interval $(a,b]$ following changes in γ and passing through $t = b$ at some $\gamma = \gamma_0 > x^{*\prime}(a)$. The respective $x(t,\gamma_0)$ solves the problem (4), (5) since $\xi(t)$ does.

Therefore, there exists a solution $x(t,\gamma_0)$ of problems (4) and (5) with $x'(a,\gamma_0)$ greater than $x^{*\prime}(a)$. This contradicts the choice of $x^*(t)$ as a solution with maximal value of the derivative at $t = a$.

Other possible cases can be considered similarly. □

4 Resonant Problems

Consider, for simplicity, the problem

$$x'' + k^2 x = f(t,x), \quad x(0) = 0, \quad x(1) = 0. \tag{20}$$

If f is continuous function (together with f_x) and bounded, then the problem above is solvable provided that k is not multiple of π. If $k = i\pi$, where i is an integer, then the homogeneous problem

$$x'' + k^2 x = 0, \quad x(0) = 0, \quad x(1) = 0 \tag{21}$$

has a nontrivial solution, and solvability of the problem (20) is not guaranteed.

The Fredholm alternative gives an answer in case $f = f(t)$. If not, the following approach was proposed. Change the left side in equation so that it is not resonant yet, add the same term to the right, and finally truncate modified right side appropriately obtaining nonresonant quasi-linear problem. This problem has a solution, and if some estimates can be proved for a solution, then this solution may solve also the original problem.

Look how this works. Consider instead of (20) the equivalent problem

$$x'' + k^2 x + \varepsilon^2 x = f(t, x) + \varepsilon^2 x, \quad x(0) = 0, \quad x(1) = 0. \tag{22}$$

Truncate the right side in Eq. (22) so that the truncated right side function $F(t, x)$ coincides with $f(t, x) + \varepsilon^2 x$ for $x \in [-N, N]$ and $t \in [0, 1]$.

The modified problem

$$x'' + k^2 x + \varepsilon^2 x = F(t, x), \quad x(0) = 0, \quad x(1) = 0 \tag{23}$$

has a solution $x(t)$, and the representation

$$x(t) = \int_0^1 G(t, s) F(s, x(s))\, ds$$

is valid, where G is Green's function associated with new (nonresonant) left side. If the key inequality

$$\Gamma \cdot M \le N$$

holds, where Γ and M are respectively bounds (estimate constants) for the Green's function $|G|$ and $|F(t, x)|$, then $|x(t)| \le N$ and $x(t)$ is a solution of (20). Due to Theorem 2, $x(t)$ has definite type induced by the linear part in (23). Therefore, multiple application of this scheme using multiple different linear parts can prove the existence of multiple solutions of the (resonant) problem (20). This scheme was tested on equations of the Emden-Fowler type in [9] (see also [2]).

5 Conclusion

If the second-order BVP is known to have a solution of type i, then the problem can be reduced to a quasi-linear problem "around" a solution, irrespective of either the original problem is resonant or not.

If the second-order formally resonant BVP has not a solution of indefinite type, then either it has not a solution at all or it can be reduced to quasi-linear problem.

It is reasonable to try different quasi-linearizations of a given problem since multiple solutions can be obtained.

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An Issue About the Existence of Solutions for a Linear Non-autonomous MTFDE

M. Filomena Teodoro

Abstract This article is concerned with the existence of solution of a certain non-autonomous linear delayed-advanced differential equation. The main objective is to provide the proof of a theorem introduced in Lima et al. (J. Comput. Appl. Math. 234(9):2732–2744, 2010) about existence of solution of a class of mixed-type functional differential equations (MTFDEs). It is an effort to complete the theoretical basis of some computational methods introduced earlier to solve numerically such equations, which were deduced making use of that theorem.

Keywords Mixed-type functional differential equation • Method of steps • Boundary value problem • Existence of solution

Mathematics Subject Classification (2000): PACS: 02.30 Ks, 02.60 Lj 2000 MSC: 34K06, 34K28, 65Q05, 34K10

1 Introduction

MTFDEs are an important issue under study because they appear in a large number of different areas of knowledge. From theoretical to applied cases, we can find a lot of examples. From economics, the author of [3] models the competitive growth in a life-cycle model; in [2] a short-run dynamics of optimal growth model is analyzed; from biology, numerical schemes approximate a MTFDE from the nervous conduction in a myelinated axon in [1, 16]; some studies in optimal control using MTFDEs can be found in [21]. The authors of [4, 5, 8] approximate traveling wave solutions in discrete media. In [9, 10] some theory about center manifolds for MTFDEs is

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developed. The oscillatory behavior of solutions of linear MTFDEs was recently investigated in [20], where the author has formulated some criteria which guarantee that all the solution of an equation of certain type is oscillatory. In [12], the authors obtain existence and stability results for a class of functional differential equations, where the unknown depends on two variables, being a delay differential equation with respect to one of the variables and a mixed-type equation with respect to the other. Boundary value problems (BVPs) for second-order MTFDEs are analyzed in [19] with the help of a Picard operator technique. Very recently, some results have been obtained in [13], where the Fredholm theory for MTFDEs is extended, developed in [17] to the case of implicitly defined functional differential equations.

This article is concerned with the existence of solution of a linear non-autonomous MTFDE. The main goal of our study is the search for a solution $x(t)$, defined for $t \in [-1, k]$, ($k \in \mathbb{N}$), which verifies this equation almost everywhere on $[0, k-1]$ and assumes known values on $[-1, 0]$ and $(k-1, k]$. In [14], it is provided a discussion of existence and uniqueness theory for the problems under consideration and described the method of steps, which is used in the construction of several numerical algorithms proposed for linear case.

The non-autonomous linear MTFDE under study has the following form:

$$x'(t) = \alpha(t)x(t) + \beta(t)x(t-1) + \gamma(t)x(t+1), \tag{1}$$

where x is the unknown function and α, β, and γ are known functions.

In [11], Eq. (1) was studied for a particular case, presenting existence and uniqueness results. A similar approach has been followed by the authors of [6] where a new approach to the analysis of the Eq. (1) in the autonomous case is proposed, where α, β, and γ are known constants. They considered a boundary value problem (BVP). They looked for a differentiable solution on an interval $[-1, k]$, $k \in \mathbb{N}$, given its values on the intervals $[-1, 0]$ and $(k-1, k]$. Imposing some conditions on boundary conditions which guarantee the existence of solution, they introduced a numerical method which to compute such solution. Based on this work, in [23] new numerical schemes were proposed for the numerical solution of autonomous linear BVP, based on the method of steps, collocation, and least squares. The approach to non-autonomous linear MTFDE (when α, β, and γ are smooth functions of t) was done in [22] and [14], where the solution of such BVPs was computed. In particular, in [14] a discussion about existence and theory of such problems is provided, and a numerical analysis of the introduced numerical algorithms is done. As consequence of such work, in [15], where interesting numerical and analytical results were obtained, the same authors introduced a numerical scheme using the finite element method (FEM). Based on the study of analytical decomposition of solutions of mixed-type equations as sum of “forward” solutions and “backward” solutions, performed in [18], the authors of [7] presented an algorithm using central difference approximation to decompose the solutions of a particular class of MTFDE into growing and decaying components. Knowing that the nonlinear case of such problems is richer in real applications, an important feature is to solve nonlinear problems. Consequently, the numerical schemes developed for linear case were extended to solve numerically nonlinear MTFDEs, but adapted for each particular problem. Such work can be found in [16, 25], where it is approximated an equation from nervous conduction or in [24], where an equation which models vocal phonation is numerically solved. Taking into consideration that several numerical schemes were developed using the results about existence of solution for a class of MTFDEs, presented in Theorem 1 of [14], the main objective of the present work is to provide the proof of that theorem to complete the theoretical basis of such computational methods.

The outline of this article is resumed in two main sections. In the next section, we revisit the method of steps for a linear non-autonomous MTFDE with the form (1), a method used usually in delay differential equations (DDEs) which extend a known solution of equation in an interval to a larger interval. It is a way to increase our knowledge about the solutions of (1) as well as it provides us sufficient conditions for the existence of solution for this kind of MTFDE. In the third section, a theorem about the existence of solution from [14] is presented and proved.

2 Preliminaries

Similarly to [14], the main idea of the present work is to get a particular solution of Eq. (1) which satisfies the boundary conditions

$$x(t) = \begin{cases} \varphi_1(t), & \text{if } t \in [-1, 0], \\ f(t), & \text{if } t \in (k-1, k], \end{cases} \tag{2}$$

where φ_1 and f are smooth real-valued functions, defined on $[-1, 0]$ and $(k-1, k]$, respectively, $(1 < k \in \mathbb{N})$.

It is imposed that Eq. (1) is satisfied for almost all $t \in (0, k-1]$ (actually, we require that (1) is satisfied except possibly at the integer values of t). To avoid pathological cases (which we shall mention later), we also assume that our solution is continuous on $[-1, k]$ and has bounded variation. It follows that $x'(t)$ is continuous wherever (1) is satisfied on $(0, k-1)$. On $(1, k-2)$, one can differentiate (1) and conclude that $x''(t)$ is continuous wherever (1) is satisfied on $(1, k-2)$, and the process can be repeated. We can summarize by saying that the solution may have a discontinuity in the first derivative at $t = 0$ and/or $t = k-1$ and becomes progressively smoother on this sequence of internal subintervals.

In order to analyze and solve this BVP of (1) subject to (2), we consider first an initial value problem (IVP), with the conditions

$$x(t) = \varphi(t), \quad t \in [-1, 1], \tag{3}$$

where the function φ is defined by

$$\varphi(t) = \begin{cases} \varphi_1(t), \text{ if } t \in [-1, 0], \\ \varphi_2(t), \text{ if } t \in (0, 1]. \end{cases} \tag{4}$$

This reformulation provides a basis for both analytical and numerical construction of solutions using ideas based on Bellman's method of steps for solving delay differential equations. One solves the equation over successive intervals of unitary length. We need to assume the non-degeneracy condition that $\gamma(t) \neq 0$, for $t \geq 0$, so that Eq. (1) can be rewritten in the form

$$x(t+1) = a(t)x'(t) + b(t)x(t-1) + c(t)x(t), \quad t \geq 0 \tag{5}$$

where $a(t) = \frac{1}{\gamma(t)}$, $b(t) = -\frac{\beta(t)}{\gamma(t)}$ and $c(t) = -\frac{\alpha(t)}{\gamma(t)}$.

If x' is not defined for a particular value of t, then we shall use the value $x'(t^-)$ in (5). In principle, we can use formula (5) to construct a solution of Eq. (1) on an interval $[1, k]$, starting on $[-1, 1]$ using the initial functions given by (4).

So, for example, if $a, b, c \in C^1([0, 3])$, and supposing that all the appropriate derivatives of φ_i exist, we may obtain the following expressions for the solution in the first two intervals:

$$\begin{aligned} x(t) &= a\,(t-1)\,\varphi_2'(t-1) + b(t-1)\varphi_1(t-2) + c(t-1)\varphi_2(t-1), \qquad t \in (1, 2]; \\ x(t) &= a\,(t-1)\,a\,(t-2)\,\varphi_2''(t-2) + \big[a\,(t-1)\,\big(a'\,(t-2) + c(t-2)\big) \\ &\quad + c(t-1)a(t-2)\big]\,\varphi_2'(t-2) + \big[c'(t-2)a(t-1) + c\,(t-1)\,c\,(t-2) \\ &\quad + b\,(t-1)\big]\,\varphi_2(t-2) + [a\,(t-1)\,b\,(t-2)]\,\varphi_1'(t-3) \\ &\quad + \big[a\,(t-1)\,b'\,(t-2) + c\,(t-1)\,b\,(t-2)\big]\,\varphi_1(t-3), \qquad t \in (2, 3]. \end{aligned} \tag{6}$$

We remark that these formulae reduce to the corresponding formulae of Table 1 in [6], if we set $c(t) \equiv c$, $a(t) \equiv a$, $b(t) \equiv b$.

Continuing this process, we can extend the solution to any interval, provided that the initial function φ and the functions a, b, c are smooth enough functions and satisfy some simple relationships. In the next theorem, this result is formulated in more precise terms. In Sect. 2. of [14], the relationship between solutions of (1) subject to (2) and of (5) subject to (4) is detailed.

3 Existence Results

As we already remarked, the solution of the BVP becomes smoother as we move away from the ends of the interval. However, the solution of the IVP, constructed using the method of steps, becomes *less smooth* as time increases. The conclusions on smoothness for the solution of the non-autonomous IVP (5) subject to (4) constructed using the method of steps are summarized in Theorem 1.

Theorem 1. *Let x be the solution of problem (5),(4), where*

$$\begin{aligned}&\alpha(t),\quad \beta(t),\quad \gamma(t)\in C^{2L}([-1,2L+1]),\quad \gamma(t)\neq 0,\quad t\in[-1,2L+1],\\ &\varphi_1(t)\in C^{2L+1}([-1,0]),\quad \varphi_2(t)\in C^{2L+1}([0,1])\quad \textit{for some } L\in\mathbb{N}.\end{aligned} \tag{7}$$

Moreover, suppose that

$$\begin{aligned}\varphi_1^{(\ell)}(0^-) &= \varphi_2^{(\ell)}(0^+),\\ \varphi_2(1) &= a(0)\varphi_1'(0^-)+b(0)\varphi_1(-1)+c(0)\varphi_1(0);\\ \varphi_2^{(\ell)}(1^-) &= \tfrac{d^\ell}{dt^\ell}\left(a(t)\varphi_1'(t)+b(t)\varphi_1(t-1)+c(t)\varphi_1(t)\right)\Big|_{t=0^-},\ \ell=0,1,2,\ldots,2L+1.\end{aligned} \tag{8}$$

Then there exist functions $\delta_{i,l}$, $\epsilon_{i,l}$, $\bar{\delta}_{i,l}$, $\bar{\epsilon}_{i,l}\in C([-1,2L+1])$, $l=1,\ldots,L$, $i=0,1,\ldots,2l$, such that the following formulae are valid:

$$\begin{aligned}&x(t)=\sum_{i=0}^{2l-1}\delta_{i,l}(t)\varphi_1^{(i)}(t-2l)+\sum_{i=0}^{2l-1}\epsilon_{i,l}(t)\varphi_2^{(i)}(t-2l+1),\ \ t\in[2l-1,2l];\\ &x(t)=\sum_{i=0}^{2l}\bar{\epsilon}_{i,l}(t)\varphi_2^{(i)}(t-2l)+\sum_{i=0}^{2l-1}\bar{\delta}_{i,l}(t)\varphi_1^{(i)}(t-2l-1),\ \ t\in[2l,2l+1]\ l=1,2,\ldots.\end{aligned} \tag{9}$$

Moreover, the solution x, constructed according to the formulae (9), belongs to the class

$$C^{2L+1}([-1,1))\bigcap C^{2L}([-1,2))\bigcap\cdots\bigcap C^{1}([-1,2L+1)). \tag{10}$$

A detailed proof by induction is provided below.

Proof (Theorem 1). As usual, we begin by proving that formula (9) is true for $l=1$. If $t\in[1,2]$, we have

$$\begin{aligned}x(t)&=\sum_{i=0}^{1}\delta_{i,1}(t)\varphi_1^{(i)}(t-2)+\sum_{i=0}^{1}\epsilon_{i,1}(t)\varphi_2^{(i)}(t-1)\\ &=\delta_{0,1}(t)\varphi_1(t-2)+\delta_{1,1}(t)\varphi_1^{(1)}(t-2)+\epsilon_{0,1}(t)\varphi_2(t-1)+\epsilon_{1,1}(t)\varphi_2^{(1)}(t-1)\end{aligned}$$

$$\begin{cases}\delta_{0,1}(t)=0\\ \delta_{1,1}(t)=b(t-1)\\ \epsilon_{0,1}(t)=c(t-1)\\ \epsilon_{1,1}(t)=a(t-1)\end{cases} \tag{11}$$

by formula (2).

If $t \in [2, 3]$, we have

$$x(t) = \sum_{i=0}^{2} \bar{\epsilon}_{i,1}(t)\varphi_2^{(i)}(t-2) + \sum_{i=0}^{1} \bar{\delta}_{i,1}(t)\varphi_1^{(i)}(t-3)$$
$$= \bar{\epsilon}_{0,1}(t)\varphi_2(t-2) + \bar{\epsilon}_{1,1}(t)\varphi_2^{(1)}(t-2) + \bar{\epsilon}_{2,1}(t)\varphi_2^{(2)}(t-2) +$$
$$+\bar{\delta}_{0,1}(t)\varphi_1(t-3) + \bar{\delta}_{1,1}(t)\varphi_1^{(1)}(t-3)$$

with

$$\begin{cases} \bar{\delta}_{0,1}(t) = a(t-1)b'(t-2) + c(t-1)b(t-2) \\ \bar{\delta}_{1,1}(t) = a(t-1)b(t-2) \\ \bar{\epsilon}_{0,1}(t) = c'(t-2)a(t-1) + c(t-1)c(t-2) + b(t-1) \\ \bar{\epsilon}_{1,1}(t) = a(t-1)\left(a'(t-2) + c(t-2)\right) + c(t-1)a(t-2) \\ \bar{\epsilon}_{2,1}(t) = a(t-1)a(t-2) \end{cases} \tag{12}$$

by formula (2).

Now we shall assume that the assertion of the Theorem 1 is true for $l = 1, 2, \ldots, L$, where $L \in \mathbb{N}$, that is :

$$\begin{aligned} x(t) &= \textstyle\sum_{i=0}^{2l-1} \delta_{i,l}(t)\varphi_1^{(i)}(t-2l) + \sum_{i=0}^{2l-1} \epsilon_{i,l}(t)\varphi_2^{(i)}(t-2l+1), \quad t \in [2l-1, 2l]; \\ x(t) &= \textstyle\sum_{i=0}^{2l} \bar{\epsilon}_{i,l}(t)\varphi_2^{(i)}(t-2l) + \sum_{i=0}^{2l-1} \bar{\delta}_{i,l}(t)\varphi_1^{(i)}(t-2l-1), \quad t \in [2l, 2l+1]; \\ & \qquad l = 1, \ldots, L. \end{aligned} \tag{13}$$

Assuming that (13) is true, we will prove that the same equality holds true, when l is replaced by $L+1$. (For the sake of simplicity, we will write $l+1$ instead of $L+1$.) That is, we want to prove that

$$\begin{aligned} x(t) &= \textstyle\sum_{i=0}^{2l+1} \delta_{i,l+1}(t)\varphi_1^{(i)}(t-2l-2) + \sum_{i=0}^{2l+1} \epsilon_{i,l+1}(t)\varphi_2^{(i)}(t-2l-1), \quad t \in [2l+1, 2l+2]; \\ x(t) &= \textstyle\sum_{i=0}^{2l+2} \bar{\epsilon}_{i,l+1}(t)\varphi_2^{(i)}(t-2l-2) + \sum_{i=0}^{2l+1} \bar{\delta}_{i,l+1}(t)\varphi_1^{(i)}(t-2l-3), \quad t \in [2l+2, 2l+3]; \\ & \qquad l = 1, \ldots, L. \end{aligned} \tag{14}$$

Assuming that (13) is true, $\boxed{\text{for } l+1}$, and using Eq. (2), we obtain two separate cases **(A)** and **(B)**:

(A) Consider $t \in [2l+1, 2l+2]$. Notice that $t-1 \in [2l, 2l+1]$ and $t-2 \in [2l-1, 2l]$

$$x(t) = a(t-1)\left[\sum_{i=0}^{2l} \bar{\epsilon}_{i,l}(t-1)\varphi_2^{(i)}(t-2l-1) + \sum_{i=0}^{2l-1} \bar{\delta}_{i,l}(t-1)\varphi_1^{(i)}(t-2l-2)\right]'$$
$$+b(t-1)\left[\sum_{i=0}^{2l-1} \delta_{i,l}(t-2)\varphi_1^{(i)}(t-2l-2) + \sum_{i=0}^{2l-1} \epsilon_{i,l}(t-2)\varphi_2^{(i)}(t-2l-1)\right]$$

$$+c(t-1)\left[\sum_{i=0}^{2l}\bar{\epsilon}_{i,l}(t-1)\varphi_2^{(i)}(t-2l-1)+\sum_{i=0}^{2l-1}\bar{\delta}_{i,l}(t-1)\varphi_1^{(i)}(t-2l-2)\right]$$

$$= a(t-1)\left[\sum_{i=0}^{2l}\left(\bar{\epsilon}'_{i,l}(t-1)\varphi_2^{(i)}(t-2l-1)+\bar{\epsilon}_{i,l}(t-1)\varphi_2^{(i+1)}(t-2l-1)\right)\right.$$

$$\left.+\sum_{i=0}^{2l-1}\left(\bar{\delta}'_{i,l}(t-1)\varphi_1^{(i)}(t-2l-2)+\bar{\delta}_{i,l}(t-1)\varphi_1^{(i+1)}(t-2l-2)\right)\right]$$

$$+b(t-1)\left[\sum_{i=0}^{2l-1}\delta_{i,l}(t-2)\varphi_1^{(i)}(t-2l-2)+\sum_{i=0}^{2l-1}\epsilon_{i,l}(t-2)\varphi_2^{(i)}(t-2l-1)\right]$$

$$+c(t-1)\left[\sum_{i=0}^{2l}\bar{\epsilon}_{i,l}(t-1)\varphi_2^{(i)}(t-2l-1)+\sum_{i=0}^{2l-1}\bar{\delta}_{i,l}(t-1)\varphi_1^{(i)}(t-2l-2)\right].$$

Rearranging the sums, we obtain

$$\begin{aligned}x(t) &= \sum_{i=0}^{2l}a(t-1)\bar{\epsilon}'_{i,l}(t-1)\varphi_2^{(i)}(t-2l-1)+\sum_{i=0}^{2l}a(t-1)\bar{\epsilon}_{i,l}(t-1)\varphi_2^{(i+1)}(t-2l-1)\\ &+\sum_{i=0}^{2l-1}a(t-1)\bar{\delta}'_{i,l}(t-1)\varphi_1^{(i)}(t-2l-2)+\sum_{i=0}^{2l-1}a(t-1)\bar{\delta}_{i,l}(t-1)\varphi_1^{(i+1)}(t-2l-2)\\ &+\sum_{i=0}^{2l-1}b(t-1)\delta_{i,l}(t-2)\varphi_1^{(i)}(t-2l-2)+\sum_{i=0}^{2l-1}b(t-1)\epsilon_{i,l}(t-2)\varphi_2^{(i)}(t-2l-1)\\ &+\sum_{i=0}^{2l}c(t-1)\bar{\epsilon}_{i,l}(t-1)\varphi_2^{(i)}(t-2l-1)+\sum_{i=0}^{2l-1}c(t-1)\bar{\delta}_{i,l}(t-1)\varphi_1^{(i)}(t-2l-2)\end{aligned}$$

or

$$x(t)=\sum_{i=0}^{2l+1}\epsilon_{i,l+1}(t)\varphi_2^{(i)}(t-2l-1)+\sum_{i=0}^{2l}\delta_{i,l+1}(t)\varphi_1^{(i)}(t-2l-2) \tag{15}$$

with

$$\begin{cases}\epsilon_{i,l+1}(t) = \left[a(t-1)\left(\bar{\epsilon}'_{i,l}(t-1)+\bar{\epsilon}_{i-1,l}(t-1)\right)+b(t-1)\epsilon_{i,l}(t-2)+c(t-1)\bar{\epsilon}_{i,l}(t-1)\right]\\ \delta_{i,l+1}(t) = \left[a(t-1)\left(\bar{\delta}'_{i,l}(t-1)+\bar{\delta}_{i-1,l}(t-1)\right)+b(t-1)\delta_{i,l}(t-2)+c(t-1)\bar{\delta}_{i,l}(t-1)\right].\end{cases} \tag{16}$$

We consider the following restrictions:

$$\bar{\epsilon}_{-1,l}(t-1) = \bar{\epsilon}_{2l,l}(t-1) = \epsilon_{2l,l}(t-2) = \epsilon_{2l+1,l}(t-2) = \bar{\epsilon}_{2l+1,l}(t-1) = 0. \quad (17)$$

(B) Consider $t \in [2l+2, 2l+3]$. Notice that $t-1 \in [2l+1, 2l+2]$ and $t-2 \in [2l, 2l+1]$.

Using formulae (13), (15) and the mixed Eq. (5), we obtain

$$\begin{aligned}
x(t) &= a(t-1)\left[\sum_{i=0}^{2l+1} \epsilon_{i,l+1}(t-1)\varphi_2^{(i)}(t-2l-2) + \sum_{i=0}^{2l} \delta_{i,l+1}(t-1)\varphi_1^{(i)}(t-2l-3)\right]' \\
&\quad + b(t-1)\left[\sum_{i=0}^{2l} \bar{\epsilon}_{i,l}(t-2)\varphi_2^{(i)}(t-2l-2) + \sum_{i=0}^{2l-1} \bar{\delta}_{i,l}(t-2)\varphi_1^{(i)}(t-2l-3)\right] \\
&\quad + c(t-1)\left[\sum_{i=0}^{2l+1} \epsilon_{i,l+1}(t-1)\varphi_2^{(i)}(t-2l-2) + \sum_{i=0}^{2l} \delta_{i,l+1}(t-1)\varphi_1^{(i)}(t-2l-3)\right] \\
&= a(t-1)\left[\sum_{i=0}^{2l+1} \left(\epsilon'_{i,l+1}(t-1)\varphi_2^{(i)}(t-2l-2) + \epsilon_{i,l+1}(t-1)\varphi_2^{(i+1)}(t-2l-2)\right)\right. \\
&\quad \left. + \sum_{i=0}^{2l} \left(\delta'_{i,l+1}(t-1)\varphi_1^{(i)}(t-2l-3) + \delta_{i,l+1}(t-1)\varphi_1^{(i+1)}(t-2l-3)\right)\right] \\
&\quad + b(t-1)\left[\sum_{i=0}^{2l} \bar{\epsilon}_{i,l}(t-2)\varphi_2^{(i)}(t-2l-2) + \sum_{i=0}^{2l-1} \bar{\delta}_{i,l}(t-2)\varphi_1^{(i)}(t-2l-3)\right] \\
&\quad + c(t-1)\left[\sum_{i=0}^{2l+1} \epsilon_{i,l+1}(t-1)\varphi_2^{(i)}(t-2l-2) + \sum_{i=0}^{2l} \delta_{i,l+1}(t-1)\varphi_1^{(i)}(t-2l-3)\right].
\end{aligned}$$

Rearranging the sums, we obtain

$$\begin{aligned}
x(t) &= \sum_{i=0}^{2l+1} a(t-1)\epsilon'_{i,l+1}(t-1)\varphi_2^{(i)}(t-2l-2) + \sum_{i=0}^{2l+1} a(t-1)\epsilon_{i,l+1}(t-1)\varphi_2^{(i+1)}(t-2l-2) \\
&\quad + \sum_{i=0}^{2l} a(t-1)\delta'_{i,l+1}(t-1)\varphi_1^{(i)}(t-2l-3) + \sum_{i=0}^{2l} a(t-1)\delta_{i,l+1}(t-1)\varphi_1^{(i+1)}(t-2l-3) \\
&\quad + \sum_{i=0}^{2l} b(t-1)\bar{\epsilon}_{i,l}(t-2)\varphi_2^{(i)}(t-2l-2) + \sum_{i=0}^{2l-1} b(t-1)\bar{\delta}_{i,l}(t-2)\varphi_1^{(i)}(t-2l-3)) \\
&\quad + \sum_{i=0}^{2l+1} c(t-1)\epsilon_{i,l+1}(t-1)\varphi_2^{(i)}(t-2l-2) + \sum_{i=0}^{2l} c(t-1)\delta_{i,l+1}(t-1)\varphi_1^{(i)}(t-2l-3),
\end{aligned}$$

or rewritting

$$x(t) = \sum_{i=0}^{2l+2} \bar{\epsilon}_{i,l+1}(t)\varphi_2^{(i)}(t-2l-2) + \sum_{i=0}^{2l+1} \bar{\delta}_{i,l+1}(t)\varphi_1^{(i)}(t-2l-3), \qquad (18)$$

with the coefficients $\bar{\epsilon}_{i,l+1}(t)$ and $\bar{\delta}_{i,l+1}(t)$ given by

$$\begin{aligned}\bar{\epsilon}_{i,l+1}(t) &= a(t-1)\left(\epsilon'_{i,l+1}(t-1) + \epsilon_{i,l+1}(t-1)\right)\\ &\quad + b(t-1)\bar{\epsilon}_{i,l}(t-2) + c(t-1)\epsilon_{i,l+1}(t-1),\\ \bar{\delta}_{i,l+1}(t) &= a(t-1)\left(\delta'_{i,l+1}(t-1) + \delta_{i-1,l+1}(t-1)\right)\\ &\quad + b(t-1)\bar{\delta}_{i,l}(t-2) + c(t-1)\delta_{i,l+1}(t-1).\end{aligned}$$

The following restrictions are imposed:

$$\delta_{-1,l+1}(t-1) = \delta'_{2l+1,l+1}(t-1) = \bar{\delta}_{2l+1,l}(t-2) = \bar{\delta}_{2l,l}(t-2) = \delta_{2l+1,l+1}(t-1) = 0,$$
$$\epsilon_{-1,l+1}(t-1) = \epsilon_{2l+2,l+1}(t-1) = \bar{\epsilon}_{2l+1,l}(t-2) = \bar{\epsilon}_{2l+2,l}(t-2) = \epsilon_{2l+2,l+1}(t-1) = 0.$$

□

If the hypothesis of Theorem 1 is verified for some $L \in \mathbb{N}$, the solution x, computed using formulae (9), has at least $2L - l + 1$ continuous derivatives on each interval $[l, l+1)$, for $l \geq 1$. This means that when the solution x is given on $[-1, 0]$ and $(0, 1]$ by functions φ_1, φ_2 of class $C^{2L+1}([0, 1])$ that satisfy (8), its degree of smoothness decreases by one on each successive subinterval.

4 Final Remarks

In this article, some work presented earlier is revisited, more precisely the method of steps. Taking into consideration that some numerical methods were developed using the results of Theorem 1 presented in [14] about existence of solution for a class of MTFDEs is proved of that theorem. It was made to contribute to theoretical basis of the introduced computational methods to solve numerically MTFDEs of the type (1). With the same purpose, the numerical analysis of the method presented in [15], based on method of steps and FEM, is still ongoing. The idea is to generalize the numerical analysis done in [15] for a larger class of MTFDEs.

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Magnetohydrodynamic Flow of a Power-Law Fluid over a Stretching Sheet with a Power-Law Velocity

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Abstract The boundary-layer flow in a viscous non-Newtonian fluid containing over a nonlinear stretching sheet is analyzed. The stretching velocity is assumed to vary as a power function of the distance from the origin. The governing partial differential equation and auxiliary conditions are reduced to nonlinear ordinary differential equation with the appropriate corresponding conditions. The properties and nonexistence of the solutions to the boundary value problem are examined. The resulting nonlinear ordinary differential equation is solved numerically with a Chebyshev spectral method. On the base of our calculations, the effects of various parameters, namely, the power-law exponent, the MHD, and the nonlinear stretching parameter on the dimensionless velocity gradient at the wall, are discussed.

Keywords Boundary layer • Power-law fluid • Stretching sheet • Similarity method

AMS Subject Classifications: 34B40, 35G45

1 Introduction

The study of a boundary layer flow over a continuous solid surface due to motion with a constant speed in an otherwise quiescent viscous fluid was investigated by Sakiadis [13]. This type of problem is encountered in many sheeting manufacturing processes, such as plastic sheets. Ericson et al. [9] extended this problem to investigate the temperature distribution in the thermal boundary layer when the temperature of the sheet is kept at constant value. After that the examination of the velocity and the temperature distribution has been extended in various ways.

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Crane discussed the two-dimensional flow caused by a stretching of an elastic flat sheet which moves with a velocity varying linearly with the distance from the die [8]. Afzal et al. [1], Kuiken [11], and Banks [2] considered the case of stretching of sheet with a power-law velocity.

The linear stretching problem was investigated when the effect of a constant transverse magnetic field is included [4]. The boundary layer flow caused by a sheet stretching with a power-law velocity in the presence of a magnetic field was analyzed by Chiam [5].

All of the above approaches were made for Newtonian fluids. However, in many real situations, non-Newtonian fluids are encountered. The most frequently used model is the power-law Ostwald-de Waele model when a power-law relationship is given between the shear stress and the shear rate. The boundary layer over a power-law stretched sheet in a non-Newtonian power-law fluid was studied for permeable surface by Guedda et al. [10] and Yacob and Ishak [14]. The numerical study of the flow of an electrically conducting power-law fluid in the presence of a magnetic field for linearly stretching sheet was given by Cortell [7].

Our aim is to study the flow of a power-law fluid in the presence of a magnetic field over a sheet of stretching with power-law velocity. The properties and existence of similarity solution to laminar boundary layer flow of non-Newtonian power-law fluid over a continuous moving surface in the presence of transverse magnetic field is investigated. The resulting ordinary differential equations are then solved numerically. The influence of various fluid parameter is examined on the flow characteristics.

2 Problem Formulation

The steady laminar flow of a non-Newtonian electrically conducting incompressible fluid past a two-dimensional body is considered. The velocity components are represented by u and v in the coordinates along and normal to the body surface, x and y directions, respectively. The stretching velocity is $u_w = U_w x^m$ and the imposed external transverse magnetic field is denoted by $B(x) = B_0 x^{(m-1)/2}$, where $B_0 > 0$ and m are constants [6]. The continuity and momentum equations are given by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{K}{\rho}\frac{\partial}{\partial y}\left(\left|\frac{\partial u}{\partial y}\right|^{n-1}\frac{\partial u}{\partial y}\right) + u_w\frac{\partial u_w}{\partial y} - \sigma B^2 u, \tag{2}$$

where ρ denotes the density and σ the electric conductivity, and the nonlinear model describing the non-Newtonian fluid is

$$\tau_{xy} = K\left|\frac{\partial u}{\partial y}\right|^{n-1}\frac{\partial u}{\partial y}; \quad (3)$$

the viscosity function varies with the magnitude of the strain rate and depends on two fluid properties, K and n, the consistency coefficient, and the power-law exponent, respectively. The constitutive equation (3) represents the shear-thinning (pseudoplastic) fluids for $0 < n < 1$ and the shear-thickening (dilatant) fluids for $n > 1$. For $n = 1$, one recovers a Newtonian fluid. The deviation of n from a unity indicates the degree of deviation from Newtonian behavior.

The boundary conditions for impermeable surface are the following

(i) at the solid surface $y = 0$ neither slip nor mass transfer is taken: $u\,(x,0) = u_w(x)$, $v\,(x,0) = 0$,
(ii) outside the viscous boundary layer the streamwise velocity component is zero:

$$\lim_{y\to\infty} u\,(x,y) = 0. \quad (4)$$

We apply the similarity solution approach by introducing

$$u = \frac{\partial\psi}{\partial y}, \quad v = -\frac{\partial\psi}{\partial x}; \quad (5)$$

then the continuity equation (2) is automatically satisfied. Upon substitutions, the momentum equation (2) reduces to

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial y\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} = \alpha\frac{\partial}{\partial y}\left(\left|\frac{\partial^2\psi}{\partial y^2}\right|^{n-1}\frac{\partial^2\psi}{\partial y^2}\right) + u_w\frac{\partial u_w}{\partial x} - \sigma B^2\frac{\partial\psi}{\partial y}, \quad (6)$$

$\alpha = K/\rho$ and the boundary conditions are

$$\frac{\partial\psi}{\partial y}(x,0) = U_w x^m, \quad \frac{\partial\psi}{\partial x}(x,0) = 0, \quad \lim_{y\to\infty}\frac{\partial\psi}{\partial y}(x,0) = 0. \quad (7)$$

Applying similarity transformation

$$\psi\,(x,y) = bx^{\beta}f(\eta), \quad \eta = dyx^{-\delta} \quad (8)$$

for some parameters $b,\ d,\ \kappa,\ \delta$, (6) is reduced to the ordinary differential equation

$$\left(\left|f''\right|^{n-1}f''\right)' + \beta ff'' - mf'^2 - Mf' = 0, \quad \eta \in (0,\infty)\,, \quad (9)$$

where

$$\delta = \beta - m, \quad \beta = \frac{m(2n-1)+1}{n+1}, \tag{10}$$

and $M = \sigma B_0^2/(u_w \rho)$ denotes the magnetic parameter. The prime indicates differentiation with respect to η. The corresponding boundary conditions (7) become

$$f(0) = 0, \quad f'(0) = 1, \tag{11}$$

$$\lim_{\eta\to\infty} f'(\eta) = 0. \tag{12}$$

An important boundary layer characteristic is the skin-friction coefficient C_f, which is a nondimensional form of the wall shear stress

$$C_f = 2\mathrm{Re}_x^{-1/(n+1)} \left[\frac{m(2n-1)+1}{n(n+1)}\right]^{n/(n+1)} [-\gamma]^n, \tag{13}$$

where $\gamma = f''(0)$ and

$$\mathrm{Re}_x = \frac{u_w(x)^{2-n} x^n}{\rho} \tag{14}$$

is the local Reynolds number. C_f is directly related to $f''(0)$.

Equation (9) for a non-Newtonian fluid can be obtained as a special case ($n = 1$) and we have

$$f''' + \beta f f'' - m f'^2 - M f' = 0, \quad \eta \in (0, \infty), \tag{15}$$

with boundary conditions (11) and (12). Exact analytical solution to (15) and (11) and (12) for $m = 0$ of the form

$$f(\eta) = \frac{1}{\sqrt{1+M}} \left(1 - e^{-\sqrt{1+M}\eta}\right) \tag{16}$$

was given by Pavlov [12].

3 Properties and Nonexistence of Solutions

The existence of solutions can be established by a shooting method. This approach is used to find values of $f''(0) = \gamma$ for which f exists on $[0, \infty)$ such that $f'(\infty) = 0$. So, the boundary condition at infinity (12) is replaced by $f''(0) = \gamma$, where $\gamma \neq 0$. The initial value problem is written as (9) and

$$f(0) = 0, f'(0) = 1, \ f''(0) = \gamma. \tag{17}$$

Our aim is to derive conditions on the parameters involved in (9), (17) such that solution f_γ is global and satisfies boundary condition (12) for $n > 1$. The local solution f_γ satisfies the equality

$$\left|f_\gamma''(\eta)\right|^{n-1} f_\gamma''(\eta)+\beta f_\gamma'(\eta) f_\gamma(\eta)-M f_\gamma(\eta) = \left|f''(0)\right|^{n-1} f''(0)+\frac{3nm+1}{n+1}\int_0^\eta f_\gamma(s)^2 ds, \tag{18}$$

$0 \leq \eta < \eta_\gamma$, for all η_γ, where $(0, \eta_\gamma)$ is the maximal interval of existence. Since $\gamma \in \mathbf{R}$ is arbitrary, problem (9), (17) has infinitely many solutions.

First, we introduce the following definitions:

A function $f_\gamma(\eta)$ is called a solution to (9), (17) if

(i) $f_\gamma(\eta) \in C^2(0, \infty)$,
(ii) $\left|f_\gamma''\right|^{n-1} f_\gamma'' \in C^1(0, \infty)$,
(iii) $\lim_{\eta\to\infty} f_\gamma'(\eta) = 0$ and $\lim_{\eta\to\infty} f_\gamma''(\eta) = 0$.

Let us define the Lyapunov energy function as

$$E(\eta) = \frac{n}{n+1}\left|f_\gamma''\right|^{n+1} - \frac{m}{3} f_\gamma'^3 - \frac{M}{2} f_\gamma'^2,$$

which satisfies

$$E'(\eta) = -\frac{m(2n-1)+1}{n+1} f_\gamma f_\gamma''^2,$$

on $(0, \eta_\gamma)$ due to the differential equation of (9). Note, that

$$E(0) = \frac{n}{n+1}|\gamma|^{n+1} - \frac{m}{3} - \frac{M}{2}.$$

Theorem 1. *For any* $M > 0$, $m + M < 0$, $n > 1$, *and* $m(2n-1)+1 > 0$ *satisfying*

$$|\gamma|^{n+1} \leq \frac{n+1}{n}\left[\frac{m}{3} + \frac{M}{2}\right],$$

(i) solution f_γ *is positive and monotonic increasing on* $(0, \eta_\gamma)$ *and global;*
(ii) $\lim_{\eta\to\infty} f_\gamma''(\eta) = 0$ *and* $\lim_{\eta\to\infty} f_\gamma'(\eta) = 0$.

Proof. As $f(0) = 0$ and $f'(0) = 1$, one can assume that f_γ and f_γ' are positive on some interval $(0, \eta_0)$, for $0 < \eta_0 < \eta_\gamma$. Then, E is monotonic decreasing on $(0, \eta_0)$, i.e.,

$$E(\eta_0) < E(0). \tag{19}$$

Applying (1), we get $E(\eta_0) \leq 0$.

If $f'_\gamma(\eta_0) = 0$, then $E(\eta_0) = E(0) = 0$ and $E(\eta) = 0$ for all $0 \le \eta \le \eta_0$. Hence, $f''_\gamma \equiv 0$ on $(0, \eta_0)$ and $\gamma = 0$ a contradiction. Therefore, f_γ is strictly monotonic increasing.

Using function E, we show that f_γ is global. We have that

$$\frac{n}{n+1}\left|f''_\gamma\right|^{n+1} - \frac{m}{3}f'^3_\gamma - \frac{M}{2}f'^2_\gamma \le \frac{n}{n+1}|\gamma|^{n+1} - \frac{m}{3} - \frac{M}{2}, \tag{20}$$

therefore, both f'_γ and f''_γ are bounded. It implies that function f_γ is also bounded on $(0, \eta_\gamma)$ if η_γ is finite, which is absurd. Consequently, η_γ is infinite and f_γ is global.

Next, we show that

$$\lim_{\eta\to\infty} f''_\gamma(\eta) = 0, \tag{21}$$

which is the case if f''_γ is monotone on some interval $[\eta_0, \infty)$ since f'_γ and f''_γ are bounded. Assume that $\left|f''_\gamma\right|^{n-1} f''_\gamma$ is not monotone on any interval $[\eta_0, \infty)$. Then there exists a sequence $\{\eta_r\}$ tending to infinity as $r \to \infty$ such that $\left(\left|f''_\gamma\right|^{n-1} f''_\gamma\right)'(\eta_r) = 0$, and $\left(\left|f''_\gamma\right|^{n-1} f''_\gamma\right)(\eta_{2r})$ is a local maximum, $\left(\left|f''_\gamma\right|^{n-1} f''_\gamma\right)(\eta_{2r+1})$ is a local minimum. Applying $\eta = \eta_r$ to the differential equation, one gets

$$\frac{m(2n-1)+1}{n+1} f''_\gamma(\eta_r) = \frac{m f'^2_\gamma(\eta_r) + M f'_\gamma(\eta_r)}{f_\gamma(\eta_r)}. \tag{22}$$

As f'_γ is bounded and tends to zero as $r \to \infty$ then $f''_\gamma(\eta_r) \to 0$ as $r \to \infty$ and

$$\lim_{\eta\to\infty} f''_\gamma(\eta) = 0. \tag{23}$$

Since f''_γ goes to 0, this implies that $-\frac{m}{3}f'^3_\gamma - \frac{M}{2}f'^2_\gamma$ tends to $\lim_{\eta\to\infty} E(\eta)$ as $\eta \to \infty$.

It remains to prove that $\lim_{\eta\to\infty} f'_\gamma(\eta) = 0$. Let us assume that $\lim_{\eta\to\infty} f'_\gamma(\eta) = L$ with some $L > 0$. Next, applying identity (18) one gets

$$\left|f''_\gamma(\eta)\right|^{n-1} f''_\gamma(\eta) = -\frac{m(2n-1)+1}{n+1}L^2\eta + \frac{1+3nm}{n+1}L^2\eta + o(\eta), \tag{24}$$

$$\left|f''_\gamma(\eta)\right|^{n-1} f''_\gamma(\eta) = mL^2\eta + o(\eta) \tag{25}$$

as $\eta \to \infty$. From this, we deduce that $L = 0$. This implies that $\lim_{\eta\to\infty} E(\eta) = 0$. □

Moreover, the following nonexistence result will be established:

Theorem 2. *Problem (9)–(12) has no nonnegative solution for* $n > 1$, $M > 0$, $m + M < 0$, $m(2n - 1) + 1 < 0$, *and*

$$|\gamma|^{n+1} \geq \frac{n+1}{n}\left(\frac{m}{3} + \frac{M}{2}\right). \tag{26}$$

Proof. Let us assume that f is a nonnegative solution to (9)–(12). Then $E'(\eta) = -\beta f_\gamma f_\gamma''^2$ is nonnegative. Therefore, E is monotonic increasing and hence

$$E(0) \leq \lim_{\eta\to\infty} E(\eta), \tag{27}$$

$$\frac{n}{n+1}|\gamma|^{n+1} - \frac{m}{3} - \frac{M}{2} \leq 0, \tag{28}$$

which contradicts (26). □

4 Numerical Results and Discussion

The non-Newtonian MHD flow problem and the influence of the parameter values on the dimensionless velocity gradient at the wall $[-f''(0)]$ can also be investigated through numerical solutions. We solve the ordinary differential equation (9) under boundary conditions (11) and (12) using a Chebyshev spectral method, in which the method is suitable to provide very accurate results when the solution is smooth enough.

In our calculations the collocation method is used. During collocation the function values of the interpolating polynomial at the collocation points are determined [3]. The nth order Chebyshev polynomial of the first kind, $T_n(x)$ is applied. The spectral differentiation for Chebyshev polynomials is carried out by the matrix-vector multiplication method. For solving the boundary value problem on semi-infinite interval, we perform truncation and linear mapping. After discretization, the resulting system of nonlinear equations is solved with the Levenberg–Marquardt algorithm in Matlab for different values of the stretching parameter m, of the power-law exponent n, and of the magnetic parameter M. The values of $[-f''(0)]$ are calculated for different parameter values of n, m, and M. The demonstration of these values is exhibited in Figs. 1 and 2. The effect of the power-law exponent is shown in Fig. 1 for $m = 1$. It is observed that $[-f''(0)]$ increases monotonically with M. Moreover, it demonstrates that $[-f''(0)]$ decreases with increasing n. Figure 2 shows that the effect of m is opposite. Larger m provides larger values of $[-f''(0)]$.

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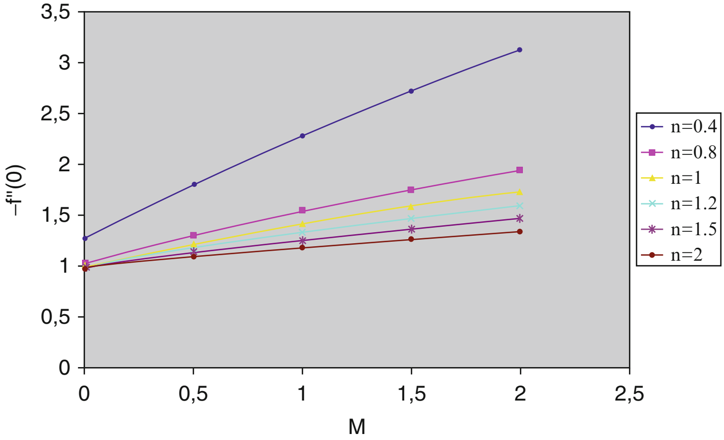

Fig. 1 The graph of $-f''(0)$ for $m = 1$ and for different values of n

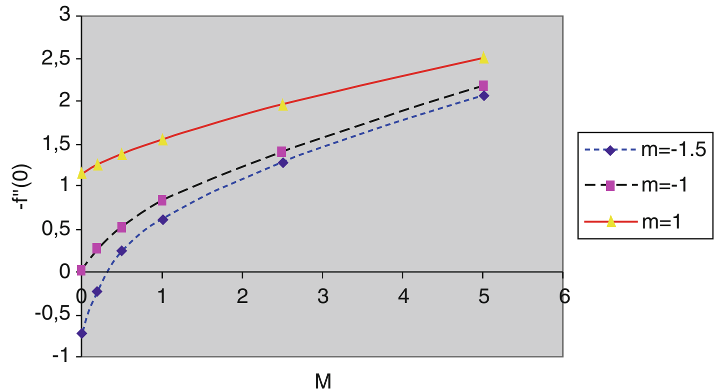

Fig. 2 The graph of $-f''(0)$ for $n = 1$ and for different values of m

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Existence of Mild Solutions for Impulsive Fractional Functional Differential Equations of Order $\alpha \in (1, 2)$

Ganga Ram Gautam and Jaydev Dabas

Abstract This paper investigates the existence result for fractional order functional differential equations subject to non-instantaneous impulsive condition by applying the classical fixed point technique. At last, an example involving partial derivatives is presented to verify the uniqueness result.

Keywords Fractional order differential equation • Functional differential equations • Impulsive conditions • Fixed point theorem

Mathematics Subject Classification (2000): 26A33, 34K05, 34A12, 26A33

1 Introduction

In this paper, we investigate the existence and uniqueness result of mild solutions for the following non-instantaneous impulsive fractional functional differential equation of the form

$$ {}^{C}D_t^{\alpha} y(t) = Ay(t) + f(t, y_{\rho(t,y_t)}),\ t \in (s_i, t_{i+1}] \subset J,\ i = 0, 1, \ldots, N, \tag{1} $$

$$ y(t) = g_i(t, y(t)),\ y'(t) = q_i(t, y(t)),\ t \in (t_i, s_i],\ i = 1, 2, \ldots, N, \tag{2} $$

$$ y(t) = \phi(t),\ y'(t) = \varphi(t),\ t \in [-d, 0], \tag{3} $$

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where ${}^C D_t^\alpha$ denotes the Caputo's fractional derivative of order $\alpha \in (1,2)$ and $A : D(A) \subset X \to X$ is the sectorial operator defined on a complex Banach space X. Functions $f : J \times PC_0 \to X;\ \rho : J \times PC_0 \to [-d, T]$ are continuous and satisfy some assumptions, where PC_0 is an abstract space defined in the next section. The map y_t is the element of PC_0 and defined as $y_t(\theta) = y(t+\theta),\ \theta \in [-d, 0]$. $J = [0, T]$ is operational interval such that $0 = t_0 = s_0 < t_1 \leq s_1 \leq t_2 < \cdots < t_N \leq s_N \leq t_{N+1} = T$ are prefixed numbers. Here y' denotes the derivative of y with respect to t and $g_i, q_i \in C((t_i, s_i] \times X; X)$ for all $i = 1, 2, \ldots, N$. The functions ϕ, φ belong to PC_0 respectively.

The impulsive differential equations have been appeared as in natural description evolution processes. The impulsive effects may be instantaneous or non-instantaneous which is shown in many disciplines. Instantaneous impulse is characterized by abrupt changes of the state at certain moments, but in case of non-instantaneous impulse, it starts abruptly at the fixed moments as the points t_i, and their action continues on the finite interval $[t_i, s_i]$. For the future development and recent update of theory for fractional functional differential equations, we refer the papers [1, 2, 4–6, 9, 10] for state-dependent delay, and for non-instantaneous impulse, one can see the papers [7, 8, 11, 12] and the references therein.

On the available of literature, we found that Hernandez et al. [11] used the first time non-instantaneous impulsive condition for abstract differential equations for order one and established the existence results. Kumar et al. [12] have studied the fractional order problem with non-instantaneous impulse, and by using the Banach fixed point theorem with condensing map, they established the existence and uniqueness results. Motivated by the work [11, 12], we have studied the problem considered in [8] for the order $\alpha \in (0,1)$ and established the existence results of mild solution of problem. Shu et al. [14] gave the definition of mild solution for fractional differential equations of order $\alpha \in (1,2)$ and then established the existence results of mild solutions using the Krasnoselskii's fixed point theorem and analytic operator theory.

Inspired by the work [11, 12, 14] and by the survey, we found that there is no literature on fractional functional differential equation with state-dependent delay subject to non-instantaneous impulsive condition of order $(1,2)$. This is the reason to investigate the problems (1)–(3) and establish the existence of uniqueness result. For further information, we have divided our work in four sections.

2 Preliminary

In this section, we have introduced some notations, basic definitions, and preliminary result, which were required to establish our main results. Let $(X, \|\cdot\|_X)$ be a complex Banach space of functions with the sup-norm $\|u\|_X = \sup_{t\in J}\{|u(t)| : u \in X\}$, and let $L(X)$ denote the space of bounded linear operators from X into X endowed with the natural norm of operators denoted by $\|\cdot\|_{L(X)}$.

As usual, $PC_0 = C([-d, 0], X)$ (with $[-d, 0] \subset$ R) is the space formed by all the continuous functions defined from $[-d, 0]$ into X, endowed with the norm

$$\|u(t)\|_{PC_0} = \sup_{t\in[-d,0]} \{|u(t)|_X\}.$$

In the case of impulsive conditions, we consider

$$PC_T = PC([-d, T]; X),\ 0 < T < \infty,$$

which is a Banach space of all such functions $u : [-d, T] \to X$, which are absolutely continuous everywhere except for a finite number of points $t_i \in (0, T),\ i = 1, 2, \ldots, N$, at which $u(t_i^+)$ and $u(t_i^-) = u(t_i)$ exists and endowed with the norm

$$\|u\|_{PC_T} = \sup_{t\in[-d,T]} \{\|u(t)\|_X, u \in PC_T\}.$$

For a function $u \in PC_T$ and $i \in \{0, 1, \ldots, N\}$, we introduce the function $\bar{u}_i \in A([t_i, t_{i+1}]; X)$ given by

$$\bar{u}_i(t) = \begin{cases} u(t), \text{ for } t \in (t_i, t_{i+1}], \\ u(t_i^+), \text{ for } t = t_i. \end{cases}$$

For further analysis, again consider

$$PC_T^1 = PC([-d, T]; X),\ 0 < T < \infty,$$

which is a Banach space of all such functions $u : [-d, T] \to X$, which are absolutely continuously differentiable everywhere except for a finite number of points $t_i \in (0, T),\ i = 1, 2, \ldots, N$, at which $u'(t_i^+)$ and $u'(t_i^-) = u'(t_i)$ exists and endowed with the norm

$$\|u\|_{PC_T^1} = \sup_{t\in[-d,T]} \left\{ \sum_{j=0}^{1} \|u^j(t)\|_X, u \in PC_T^1 \right\}.$$

For a function $u \in PC_T^1$ and $i \in \{0, 1, \ldots, N\}$, we introduce the function $\bar{u}_i \in C^1([t_i, t_{i+1}]; X)$ given by

$$\bar{u}_i(t) = \begin{cases} u'(t), \text{ for } t \in (t_i, t_{i+1}], \\ u'(t_i^+), \text{ for } t = t_i. \end{cases}$$

Definition 1. [13] Caputo's derivative of order $\alpha > 0$ with lower limit a, for a function $g : [a, \infty) \to \mathrm{R}$ such that $g \in C^n([a, \infty), X)$ is defined as

$$ {}_a^C D_t^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} g^{(n)}(s) ds = {}_a J_t^{n-\alpha} g^{(n)}(t), $$

where $n - 1 < \alpha < n, \ a \geq 0, \ n \in N$.

Definition 2. The Riemann–Liouville fractional integral operator of order $\alpha > 0$ with lower limit a, for a function $g \in L^1_{\mathrm{loc}}([a, \infty), X)$ is defined by

$$ {}_a J_t^0 g(t) = g(t), \ {}_a J_t^{\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} g(s) ds, \quad \alpha > 0, \ t > 0, $$

where $a \geq 0, \ n \in N$ and $\Gamma(\cdot)$ denotes the Gamma function.

Definition 3. Let $A : D(A) \subset X \to X$ be a closed and linear operator and $\alpha, \beta > 0$. We can say that A is the generator of (α, β) operator function if there exists $\omega \geq 0$ and a strongly continuous function $W_{\alpha,\beta} : \mathrm{R}^+ \to L(X)$ such that $\{\lambda^{\alpha} : \mathrm{Re}\lambda > \omega\} \subset \rho(A)$ and

$$ \lambda^{\alpha-\beta} (\lambda^{\alpha} I - A)^{-1} u = \int_0^{\infty} e^{-\lambda t} W_{\alpha,\beta}(t) u dt, \ \mathrm{Re}\lambda > \omega, \ u \in X. $$

Here $W_{\alpha,\beta}(t)$ is called the operator function generated by A.

Remark 1. The operator function $W_{\alpha,\beta}(t)$ is a general case of α-resolvent family and solution operator. In the case $\beta = 1$, operator function corresponds to solution operator $S_{\alpha}(t)$ by Definition 2.1 in [2], whereas in the case $\beta = \alpha$, operator function corresponds to α-resolvent family defined in [3] in Definition (2.3), and operator function corresponds to $K_{\alpha}(t)$ in [14] in the case $\beta = 2$.

The following result is based on Definition 2.1 in [11].

Definition 4. A function $y : [-d, T] \to X$ s.t. $y \in PC^1_T$ is called a mild solution of the problems (1)–(3) if $y(0) = \phi(0), y'(0) = \varphi(0), \ y(t) = g_j(t, y(t)), y'(t) = q_j(t, y(t))$ for $t \in (t_j, s_j]$ for each $j = 1, 2, \ldots, N$, and satisfying the following integral equation

$$ y(t) = \begin{cases} \phi(0) S_{\alpha}(t) + \varphi(0) K_{\alpha}(t) & \\ + \int_0^t T_{\alpha}(t) f(s, y_{\rho(s,y_s)}) ds, & t \in [0, t_1], \\ g_i(s_i, y(s_i)) S_{\alpha}(t - s_i) + q_i(s_i, y(s_i)) K_{\alpha}(t - s_i) & \\ + \int_{s_i}^t T_{\alpha}(t - s) f(s, y_{\rho(s,y_s)}) ds, & t \in [s_i, t_{i+1}], \end{cases} $$

for $i = 1, 2, \ldots, N$.

3 Main Results

In this section, we have established the existence result of solution for the problems (1)–(3). Let A be a sectorial operator and then strongly continuous functions $\|S_\alpha(t)\| \leq M; \|K_\alpha(t)\| \leq M; \|T_\alpha(t)\| \leq M$. Let us assume the function $\rho : [0,T] \times PC_0 \to [-d,T]$ is continuous. Now, we introduce the following assumption:

(H_1) The function f is continuous and $\exists$ positive constants L_{f1} such that

$$\|f(t,\psi) - f(t,\xi)\|_X \leq L_{f1}\|\psi - \xi\|_{PC_0}, \ \forall\ \psi, \xi \in PC_0.$$

(H_2) The functions g_i, q_i are continuous and $\exists$ positive constants L_{g_i}, L_{q_i} such that

$$\|g_i(t,x) - g_i(t,y)\|_X \leq L_{g_i}\|x-y\|_X;\ \|q_i(t,x) - q_i(t,y)\|_X \leq L_{q_i}\|x-y\|_X$$

for all $x, y \in X, t \in (t_i, s_i]$ and each $i = 1, 2, \ldots, N$.

Theorem 1. *Let the assumptions* (H_1) *and* (H_2) *hold and are constant:*

$$\Delta = \max\{MTL_{f_1}, L_{g_i}M + L_{q_i}M + MTL_{f1}\} < 1,$$

for $i = 1, \ldots, N$. *Then there exists a unique mild solution* $y(t)$ *of problems (1)–(3) on* J.

Proof. We convert problems (1)–(3) in to the fixed point problem. Consider $\mathscr{B} = \{y : y \in PC_T^1, y(0) = \phi(0), y'(0) = \varphi(0)\}$. Define an operator $\mathscr{P} : \mathscr{B} \to \mathscr{B}$ as

$$\mathscr{P}y(t) = \begin{cases} \phi(0)S_\alpha(t) + \varphi(0)K_\alpha(t) & \\ + \int_0^t T_\alpha(t-s)f(s, y_{\rho(s,y_s)})ds, & t \in [0,t_1], \\ g_i(s_i, y(s_i))S_\alpha(t-s_i) + q_i(s_i, y(s_i))K_\alpha(t-s_i) & \\ + \int_{s_i}^t T_\alpha(t-s)f(s, y_{\rho(s,y_s)})ds, & t \in [s_i, t_{i+1}]. \end{cases} \tag{4}$$

It is obvious that $\mathscr{P}$ is well defined. Now, we will express that the operator $\mathscr{P}$ has a unique fixed point. So let $y(t), y^*(t) \in \mathscr{B}$ and $t \in [0,t_1]$; we get

$$\begin{aligned} \|\mathscr{P}y - \mathscr{P}y^*\|_X &\leq \int_0^t \|T_\alpha(t-s)\|_{L(X}\|f(s, y_{\rho(s,y_s)}) - f(s, y^*_{\rho(s,y_s^*)})\|_X ds \\ &\leq TML_{f1}\|y - y^*\|_X. \end{aligned}$$

For $t \in [s_i, t_{i+1}]$, we have

$$\begin{aligned} \|\mathscr{P}y - \mathscr{P}y^*\|_X \leq\ & \|g_i(s_i, y(s_i)) - g_i(s_i, y^*(s_i))\|_X \|S_\alpha(t-s_i)\|_{L(X} \\ & + \|q_i(s_i, y(s_i)) - q_i(s_i, y^*(s_i))\|_X \|K_\alpha(t-s_i)\|_{L(X} \end{aligned}$$

$$+\int_{s_i}^{t} \|T_\alpha(t-s)\|_{L(X}\|f(s,y_{\rho(s,y_s)})-f(s,y^*_{\rho(s,y^*_s)})\|_X ds$$
$$\leq (L_{g_i}M + L_{q_i}M + TML_{f1})\|y-y^*\|_X.$$

For $t \in (t_j, s_j]$, we get

$$\|\mathscr{P}y - \mathscr{P}y^*\|_X \leq L_{g_j}\|y-y^*\|_X,\ j = 1,2,\ldots,N.$$

Gathering above results, we obtain

$$\begin{aligned}\|\mathscr{P}y - \mathscr{P}y^*\|_X &\leq \max\{MTL_{f_1}, L_{g_i}M + L_{q_i}M + MTL_{f1}\}\|y-y^*\|_X\\ &\leq \Delta\|y-y^*\|_X.\end{aligned}$$

Since $\Delta < 1$, which implies that $\mathscr{P}$ is a contraction map, there exists a unique fixed point which is the mild solution of problems (1)–(3) on J.

4 Example

In this section, we gave an example to illustrate our main result. Consider the following fractional order functional differential equation:

$$\frac{\partial^\alpha u(t,x)}{\partial t^\alpha} = \frac{\partial^2 u(t,x)}{\partial y^2} + \frac{u(t-\sigma(\|u\|),x)}{49}, (t,x) \in \cup_{i=1}^{N}[s_i, t_{i+1}] \times [0,\pi], \quad (5)$$

$$u(t,0) = u(t,\pi) = 0,\quad t \geq 0, \quad (6)$$

$$u(t,x) = \phi(t,x), u'(t,x) = \varphi(t,x), t \in [-d,0], x \in [0,\pi], \quad (7)$$

$$u(t,x) = G_i(t,y);\ u'(t,x) = H_i(t,y),\ t \in (t_i, s_i]. \quad (8)$$

where $\frac{\partial^\alpha}{\partial t^\alpha}$ denotes the partial Caputo's fractional derivative of order $\alpha \in (1,2), 0 = t_0 = s_0 < t_1 \leq s_1 < \cdots < t_N \leq sN < t_{N+1} = 1$ are prefixed numbers, and $\phi, \varphi \in PC_0$. Let $X = L^2[0,\pi]$ be a Banach space and define the operator $A : D(A) \subset X \to X$ by $Ay = y''$ with the domain $D(A) := \{y \in X :\ y, y'$ to be absolutely continuous, $y'' \in X, y(0) = 0 = y(\pi)\}$. Then

$$Ay = \sum_{n=1}^{\infty} n^2(y, y_n)y_n,\ y \in D(A),$$

where set $y_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx),\ n \in N$ is the space of eigenvectors of A in which element is orthogonal. It is clear that that the operator A stays the infinitesimal generator of an analytic semigroup operator $(T(t))_{t\geq 0}$ in Banach space X and is defined as

$$T(t)\omega = \sum_{n=1}^{\infty} e^{-n^2 t}(\omega, \omega_n)\omega_n, \text{ for all } \omega \in X, \text{ and every } t > 0.$$

The subordination opinion of solution operator implies that A stays the infinitesimal generator of $K(t), S(t)$. Since $K(t), S(t)$ are strongly continuous operators on interval $[0, \infty)$ by the theorem of uniformly boundedness, there exists a constant $M > 0$ such that $\|S(t)\| \leq M$, $\|K(t)\| \leq M$ for $t \in [0, 1]$. We have for $(t, \phi) \in [0, 1] \times PC_0$.

Setting $u(t)(x) = u(t, x)$, and

$$\rho(t, \phi) = t - \sigma(\|\phi(0)\|), \ (t, \phi) \in J \times PC_0,$$

we have

$$f(t, \phi) = \frac{\phi}{49}; \ g_i(t, y) = G_i(t, y); \ q_i(t, y) = H_i(t, y),$$

then by the above Eqs. (5)–(8) can be composed in the given abstract form as (1)–(3). Furthermore, we can see that for $(t, \phi), (t, \psi) \in J \times PC_0$, we may verify that

$$\|f(t, \phi) - f(t, \psi)\|_{L^2} \leq \left[\int_0^{\pi} \left\{\|\frac{\phi}{49} - \frac{\psi}{49}\|\right\}^2 dy\right]^{1/2} \leq \frac{\sqrt{\pi}}{49}\|\phi - \psi\|.$$

Hence function f satisfies (H_1). Similarly we can show that the functions g_i, q_i satisfy (H_2). All the conditions of Theorem 1 have been satisfied, so we can drive that the system (5)–(8) has a unique mild solution on [0, 1].

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Nonlinear Dynamical Systems in Modeling and Control of Infectious Disease

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Abstract This paper deals with nonlinear dynamical systems in the form of mathematical modeling to describe modeling of the dynamic behavior of biological and biomedical systems. Nonlinear ordinary differential equations have been studied to investigate the mysterious and complex mechanisms of the dynamics of infectious diseases in the human body. In particular, we study a nonlinear model of HIV immunology which describes the interactions between the human immune systems and the viruses. In this work, we propose a modification of the HIV model proposed by Joshi in Optim Control Appl Methods 23(4):199–213 (2002) by introducing state constraint to the dynamics. The aim is to obtain optimal immunotherapeutic strategy where the state constraint may play a crucial role. We treat our problem numerically and compare the results with existing literature to illustrate the significant effect of introducing state constraint to the dynamics of the model.

Keywords Mathematical model • Nonlinear ODEs • HIV immunology • Optimal control • State constraints • Numerical simulations

Mathematics Subject Classification (2000): 93A30, 49K15.

1 Introduction

Nonlinear phenomena characterize all aspects of global change dynamics, from the Earth's climate system to human physiology [20]. These nonlinear phenomena of rapid change in the human physiological systems can be captured and modeled by the nonlinear ordinary differential equations (NODEs) in the form of mathematical modeling. Since the human body is a highly nonlinear, robust, and an adaptive physiological control system, there is a close relationship between control theory

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and biology [16]. So nonlinearity plays an influential role in describing the mysterious and complex mechanisms of the dynamics of infectious diseases in the human body.

In recent years, mathematical models have become the most important tools in analyzing the dynamics of biological and biomedical systems. The processes in biology and medicine can be, in general, described by mathematical models where the nonlinear ordinary differential equations are the key ingredients. The spread of infectious diseases such as HIV [6], NiV [8], and flu [9] may be modeled as a nonlinear system of differential equations. In this paper, we study an HIV immunology model to analyze the nonlinear behavior of the disease dynamics. Immunotherapy is one of the most effective varieties of chemotherapy used for the treatment of HIV-positive patients which not only kills/halts the pathogen in the body but also helps in increasing the long-term internal resistance of our immune systems so that the body itself can fight against the viruses. This immunotherapeutic treatment in the form multidrug therapy from the early stage of the infections has shown remarkable milestone toward the evolution of AIDS treatment [18]. Optimal control technique is applied to obtain the better immunotherapeutic strategy, special feature of which is the introduction of state constraint. Some numerical simulations illustrate the results.

2 Nonlinear Mathematical Model

Mathematical models can provide better insights of the disease mechanisms which lead to design better prevention, therapy, and control programs. Numbers of mathematical models for different infectious diseases have been proposed and investigated by several authors over the years. We refer readers to [1–5, 7, 9] for more detailed discussions on some of the recent mathematical models of different infectious diseases. However, the cell-virus interactions in the human body are very complex, especially when these are the cases of HIV infections. The HIV model we now discuss here is a simple deterministic optimal control model first proposed by Kirschner and Webb [15] which describes the interactions between human immune systems and HIV virus in terms of a set of nonlinear ordinary differential equations (ODEs) given by

$$\begin{aligned} \frac{dT(t)}{dt} &= s_1 - \frac{s_2 V(t)}{\beta_1 + V(t)} - \mu T(t) - \lambda V(t) T(t) \\ \frac{dV(t)}{dt} &= \frac{\gamma V(t)}{\beta_2 + V(t)} - \alpha V(t) T(t) \end{aligned} \tag{1}$$

with the initial conditions

$$T(0) = T_0, \quad V(0) = V_0. \tag{2}$$

In the above model, $T(t)$ and $V(t)$ represent the uninfected $CD4^{+}T$ cells and concentrations of free infectious virus particles, respectively. μ is the natural death rate of $CD4^{+}T$ cells; λ is the infection rate by the free virus particles. γ represents the input rate of an external virus source; α is the loss rate of virus and β_1 and β_2 are half-saturation constants. $s_1 - \dfrac{s_2 V(t)}{\beta_1 + V(t)}$ represents the *source/proliferation* term of uninfected $CD4^{+}T$ cells and $\mu T(t)$ is the natural loss of uninfected $CD4^{+}T$; $\lambda V(t)T(t)$ is the loss by infection, $\dfrac{\gamma V(t)}{\beta_2 + V(t)}$ is the viral contribution to plasma, and $\alpha V(t)T(t)$ is the viral loss.

The model (1) was further studied, explored, and extended by Joshi in [14] in an optimal control problem introducing two control variables u_i for $i = 1, 2$. When modeling the immunotherapeutic treatment in a time interval $[0, T]$, the rate of immunotherapy at each instant is t. Taking into account the immunotherapy, the above two compartmental dynamic models (1) of HIV infections can be reformulated by the following nonlinear systems of ordinary differential equations:

$$\begin{aligned} \frac{dT(t)}{dt} &= s_1 - \frac{s_2 V(t)}{\beta_1 + V(t)} - \mu T(t) - \lambda V(t)T(t) - u_1(t)T(t) \\ \frac{dV(t)}{dt} &= \frac{\gamma(1 - u_2(t))V(t)}{\beta_2 + V(t)} - \alpha V(t)T(t) \end{aligned} \tag{3}$$

with the same initial conditions (2).

Here u_1 and u_2 act as the control variables representing the immune-boosting and viral-suppressing drugs, respectively, and the set of controls $(u_1(t), u_2(t)) \in U$ is Lebesgue measurable, where

$$U = \{(u_1(t), u_2(t)) : 0 \le a_i \le u_i(t) \le b_i \le 1 \text{ for } i = 1, 2, \text{ a.e. } t \in [0, T]\}.$$

$u = 0$ indicates no drugs at all and $u = 1$ indicates the maximum drug doses over time.

The aim is to find the optimal control strategy so that the number of uninfected $CD4^{+}T$ cell count at the end of treatment is maximized as much as possible while minimizing counts of the hazardous side effects of the antiretroviral drug doses as well as the systematic cost. The objective functional is chosen as

$$\text{Minimize } J(u_1, u_2) := \int_0^T -T(t) + B_1 u_1^2(t) + B_2 u_2^2(t)\, dt, \tag{4}$$

where B_1 and B_2 are the balancing parameters which determine the relative importance of the two factors in the objective functional.

One of the important aspects for the treatment of HIV infections is the regular monitoring of the $CD4^{+}T$ cell count in the blood. We observe from the literatures that the $CD4^{+}T$ cell count is very crucial for the treatment of HIV infections. The $CD4^{+}T$ cell count "less than 200/mm^3" indicates the severity of the disease [17].

Our intention here is to find a new solution of the model in [14] imposing some state constraints in the data. Our idea behind imposing the state constraints is to guarantee that the uninfected $CD4^{+}T$ cell count should not go below a certain level, for example, $200/\text{mm}^3$, during the entire treatment which can be ensured by increasing the internal immunity of the $CD4^{+}T$ cells with several drug administrations like antiretroviral therapy (ART) and highly active antiretroviral therapy (HAART). We recall that HAART is defined as treatment with at least three active antiretroviral drugs (ARVs) and so it is often called the drug "cocktail" or triple therapy. HAART affords us a potent way of suppressing viral replication in the blood while attempting to prevent the virus from rapidly developing resistance to the individual ARVs. Suppressing viral replication with HAART allows the body time to rebuild its immune system and replenish the destroyed $CD4^{+}T$ cells. Until today HAART is highly recommended for the immunotherapy of HIV-positive patients as it has been clearly shown to delay progression to AIDS and prolong life. See, for example, [13, 18] for some recent developments in HAART treatment and "functional cure" from HIV infections. Failure of HAART is a sustained and high rise in the viral load because when HAART is stopped, HIV becomes detectable in the blood once again. So we now modify the model proposed by Joshi [14] to construct a new problem. Our proposed modification in the above model is to introduce a state constraint in the state variable of virus concentration meaning that the number of free virus particle cannot pass a certain upper limit during the immunotherapeutic treatment. We take the state constraint

$$V(t) \leq \tilde{V}, \ \forall \ t \in [0, T], \tag{5}$$

where $\tilde{V}$ is an upper bound on the free virus particle taking values in $\mathbb{R}$.

3 Characterization of Optimal Control Problem

The model (3) along with the objective functional (4) and the state constraint (5) can be reformulated as the following state-constrained optimal control problem:

$$(\text{P}) \begin{cases} \text{Minimize } \displaystyle\int_0^T L(x(t), u(t))dt \\ \text{subject to} \\ \dot{x}(t) = f(x(t)) + g(x(t))u(t) \ \text{ for a.e. } t, \\ h(x(t)) \leq 0 \ \text{ for all } t, \\ u(t) \in [0, 1] \ \text{ for a.e. } t, \\ x(0) = x_0 \end{cases}$$

where

$$x(t) = (T(t), V(t)), \quad L(x,u) = -T(t) + B_1 u_1^2(t) + B_2 u_2^2(t),$$
$$f(x) = \Big(s_1 - \frac{s_2 V(t)}{\beta_1 + V(t)} - \mu T(t) - \lambda V(t) T(t), \frac{\gamma V(t)}{\beta_2 + V(t)} - \alpha V(t) T(t)\Big),$$

and

$$g(x) = \begin{pmatrix} T(t) & 0 \\ 0 & -\frac{\gamma V(t)}{\beta_2 + V(t)} \end{pmatrix}, \quad u(t) = (u_1(t), u_2(t)), \quad h(x(t)) = V(t) - \tilde{V}.$$

We define the Hamiltonian

$$H(x,u,p,\lambda) = p \cdot f(x) + p \cdot g(x) u - \lambda L(x,u).$$

In the absence of state constraint $h(x(t)) \leq 0$ for all t, the necessary conditions of optimality for optimal control problem (P) can be obtained by applying the well-known Pontryagin Maximum Principle [19] for optimal control problem. In vein of Vinter [22], the necessary conditions give closed forms for the controls (taking into account the control constraints) of our problem. It is worth mentioning that our cost is convex in u and the dynamics are linear in u. In such case the optimal solution of our model is guaranteed by Fleming and Rishel [11].

Suppose that (x^*, u^*) is the optimal solution of the above problem (P) without state constraint. The maximum principle in [22] asserts the existence of an absolutely continuous function p and a scalar $\lambda_0 \geq 0$ such that:

(i) $||p||_\infty + \lambda_0 > 0$,
(ii) $-\dot{p}(t) = p(t) \cdot f_x(x^*(t)) + p(t) \cdot g_x(x^*(t)) u^*(t) - \lambda L_x(x^*(t), u^*(t))$
(iii) $\forall\, u \in U, \;\; p(t) \cdot g(x^*(t)) u^*(t) - \lambda u^{*2} \leq p(t) \cdot g(x^*(t)) u(t) - \lambda u^2$ a.e.,

together with the transversality condition $p(T) = (0,0)$. Consider that $p(t) = (p_T, p_V)$. Then we deduce from (iii) an explicit characterization of the optimal control pair in normal form (i.e., $\lambda = 1$) given in terms of the multipliers p:

$$\begin{aligned}(u_1^*(t), u_2^*(t)) = \Big(&\min\Big\{\max\Big\{a_1, \frac{p_T(t) T(t)}{2B_1}\Big\}, b_1\Big\}, \\ &\min\Big\{\max\Big\{a_2, \frac{-p_V(t) V(t)}{2B_2(\beta_2 + V(t))}\Big\}, b_2\Big\}\Big).\end{aligned} \tag{6}$$

It is worth mentioning that the introduction of state constraint in the model makes the analytical solution quite complicated due to the presence of nonnegative Radon measure [22]. In such case, additional verification as well as validation (e.g., regularity) of minimizer for optimal solution is needed. However, for such discussions some literature (see, e.g., [12, 21]) can be of help for analytical treatments. An adapted theorem discussed in [9] (see also [5]) for the existence of minimizer may provide more information for such analysis. However, in this paper we only perform a numerical simulation of our state-constrained model to compare the dynamic behavior of the disease before and after the antiretroviral drug administration. We also compare the results for optimal immunotherapeutic strategy with and without state constraint omitting the detailed analytical treatment.

4 Numerical Results

We consider the model when the virus particles are assumed to be very active and the number of uninfected $CD4^{+}T$ cells counts is very low (i.e., at the late stage of the disease). So we take the initial values as $T(0) = 400$ and $V(0) = 2.0$ and all other parameters are as in Table 1. We perform all simulations for the fixed final time $T = 50$ days because of the short-term drug dosages for the HAART. Before proceeding to in-depth analysis, we would like to show readers the importance of immunotherapy treatment using HAART. For this purpose, we first solve the model when no immunotherapy is administered. In this case, we take the control variables $u_1 = u_2 = 0$ and we solve the problem by using the known nonlinear "ODE solver" written in "MATLAB" code. We then take $u_1 = 0.02$, $u_2 = 0.9$ and run the program using the same "ODE solver." The simulation results of these two cases are shown in Figs. 1 and 2. From Fig. 1, it is easy to observe that at the very beginning of the HIV infections, when any form of drugs as "immunotherapy" is not initiated for treatment, the number of uninfected $CD4^{+}T$ cells is decreasing quickly over time [see Fig. 1(upper one)], and at the same time the virus concentrations are increasing very fast [see Fig. 1(lower one)]. On the other hand, Fig. 2 shows that the immunotherapeutic treatment for HIV infections in the form of HAART as the $CD4^{+}T$ cell count is growing up immediately after drug initiations [see Fig. 2(upper one)] and the virus particles are decreasing almost to zero [see Fig. 2(lower one)].

Table 1 Definitions of the parameters and constants with their values [14]

Parameters and constants	Definition of parameters	Values
s_1	First source coefficient	2.0
s_2	Second source coefficient	0.002
λ	Infection rate of $CD4^{+}T$ cells	0.00025
γ	Input rate of an external virus source	30
α	Loss rate of virus	0.007
β_1	First half-saturation constant	14
β_2	Second half-saturation constant	1.0
T	Number of days	50
T_0	Initial $CD4^{+}T$ cells	400
V_0	Initial virus concentrations	2
$\tilde{V}$	Upper bound on virus concentrations	2.5

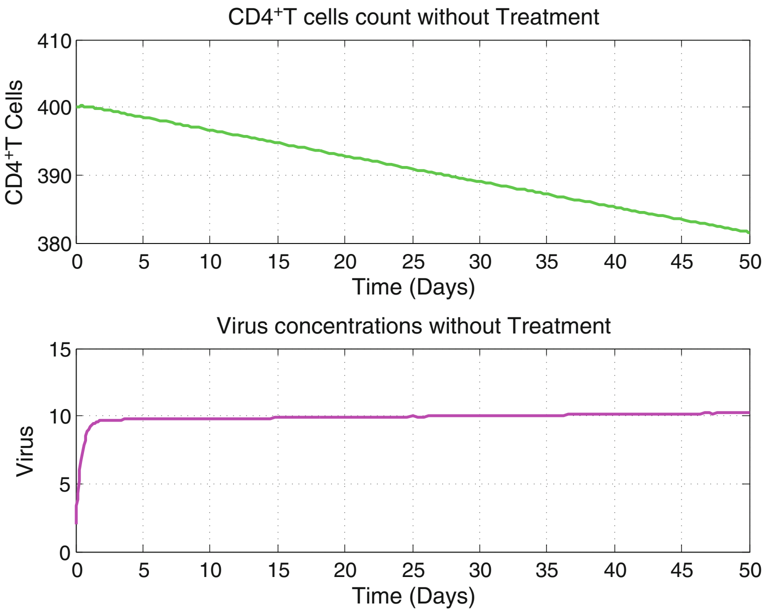

Fig. 1 Uninfected $CD4^+T$ cells decrease quickly like a straight line (*upper one*) and free virus particles increase (*lower one*) when no drugs are administered as a treatment measure

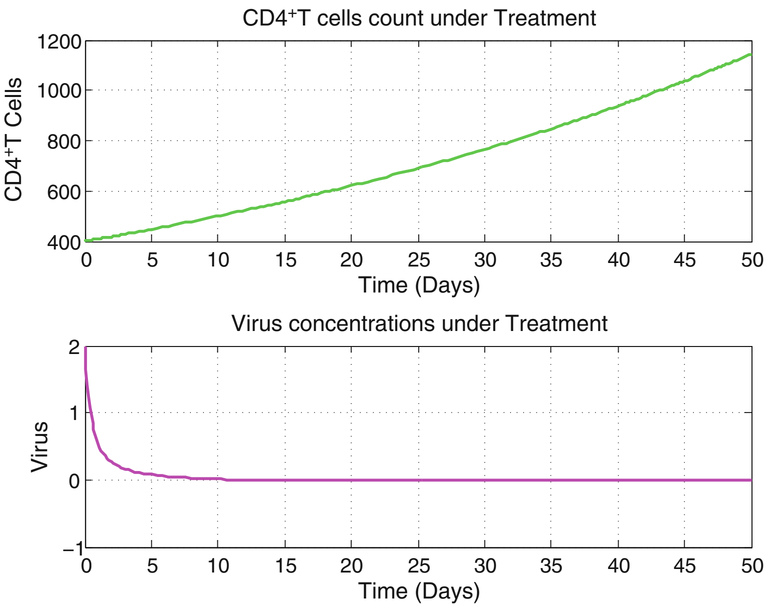

Fig. 2 Uninfected $CD4^+T$ cells are increasing dramatically (*upper one*) and free virus particles are decreasing to zero (*lower one*) when some antiretroviral drugs are administered as a treatment

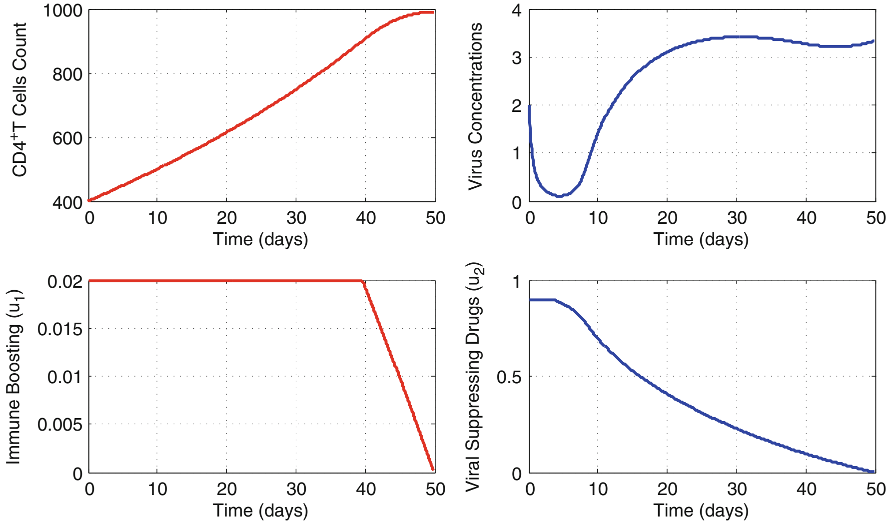

Fig. 3 Optimal state trajectories and optimal immunotherapeutic rates without state constraint

Now we solve the problem for the optimality systems when immunotherapy is effective as "immune-boosting"(i.e. $u_1 \in [0, 0.02]$) and "viral-suppressing drugs" (i.e., $u_2 \in [0, 0.9]$) during the whole treatment period. We solve the optimality systems numerically by using the known nonlinear optimal control solver "ICLOCS – version 0.1b" [10]. We first solve our model in the absence of state constraint. For a better comparison, we take all initial values and parameters same as in [14] with a fixed time interval [0, 50] and the results obtained in this case are presented in Fig. 3. Now, we turn to the case of state-constrained model. We take the upper bound of virus concentrations, i.e., $\tilde{V} = 2.5$ and all other values are same as before. The numerical simulation of this run is presented in Fig. 4. From a brief overview on the comparison of Figs. 3 and 4, we can see that the virus concentration in Fig. 3 is increasing almost after 7 days of the therapy administration until the end of final time, whereas our state-constrained model in Fig. 4 shows that this increasing tendency of virus concentration can be halted at a certain upper bound during the whole therapeutic process.

5 Conclusions

Immunotherapy is one of the most effective treatment strategies in the absence of effective HIV vaccine until today. Antiretroviral therapy (ART) or highly active antiretroviral therapy (HAART) aims to increase the internal immunity of HIV-positive people so that the body itself can fight against the virus. In this paper, we study a nonlinear mathematical model of HIV immunology, and a numerical

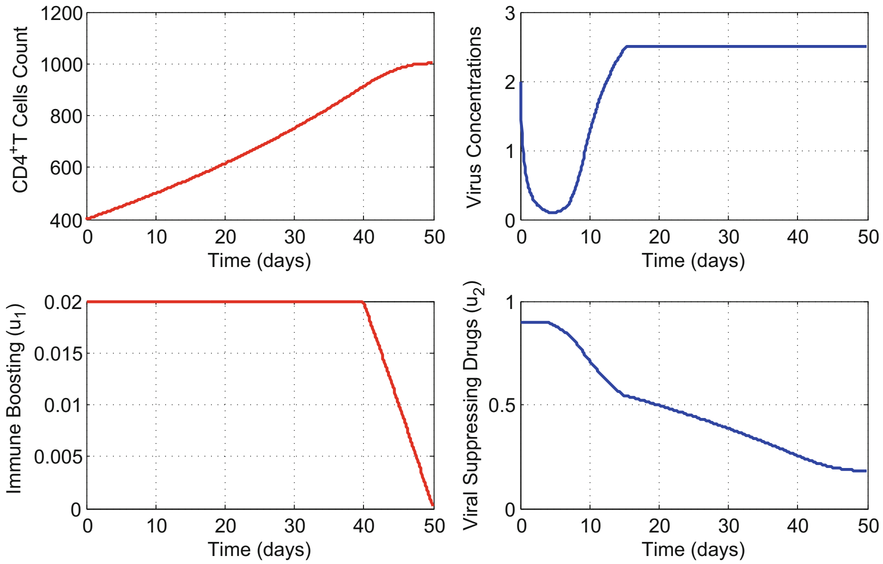

Fig. 4 Optimal state trajectories and optimal immunotherapeutic rates with state constraint

solution of the model for the optimal immunotherapy of antiretroviral treatment with a modification by introducing state constraint is presented. The numerical results showing a better immunotherapeutic strategy for state-constrained case are illustrated with simulations. Despite the challenge of analytical validations, this result may be of help in designing the combined antiretroviral therapy in such an efficient manner as to obtain maximum benefits from the immunotherapeutic treatment of HIV infections.

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Analysis of Difference Approximations to Delay Pseudo-Parabolic Equations

Gabil M. Amiraliyev, Mustafa Kudu, and Ilhame Amirali

Abstract This work deals with the one-dimensional initial-boundary Sobolev or pseudo-parabolic problem with delay. For solving this problem numerically, we construct fourth-order difference-differential scheme and obtain the error estimate for its solution. Further, for the time variable, we use the appropriate Runge–Kutta method for the realization of our differential-difference problem. Numerical results supporting the theory are presented.

Keywords Sobolev equation • Delay difference scheme • Error estimate

Mathematics Subject Classification: 65M15, 65M20, 65L05, 65L70

1 Introduction

We consider the initial-boundary value problem for pseudo-parabolic differential equation with delay in the domain $\overline{Q} = \overline{\Omega} \times [0, T]$; $\overline{\Omega} = [0, l]$, $Q = \Omega \times (0, T]$, $\Omega = (0, l)$:

$$\frac{\partial u(x,t)}{\partial t} - a(t)\frac{\partial^3 u(x,t)}{s\partial t \partial x^2} = b(t)\frac{\partial^2 u(x,t)}{\partial x^2} + c(t)\frac{\partial^2 u(x,t-r)}{\partial x^2} + d(t)u(x,t) + f(x,t), \ (x,t) \in Q, \tag{1}$$

$$u(x,t) = \varphi(x,t), \ (x,t) \in \overline{\Omega}\, x\, [-r, 0], \tag{2}$$

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$$u(0,t) = u(l,t) = 0,\ t \in (0,T], \tag{3}$$

where $a \geq \alpha > 0, b, c, d, f$ and φ are sufficiently smooth functions satisfying certain regularity conditions to be specified and $r > 0$ represents the delay parameter. Equations of this type arise in many areas of mechanics and physics. They are used to study heat conduction [1], homogeneous fluid flow in fissured rocks [2], shear in second-order fluids [3, 4], and other physical models. The important characteristic of these models is that they express the conservation of a certain quantity (mass, momentum, heat, etc.) in any sub-domain. For a discussion of existence and uniqueness results of pseudo-parabolic equations, see [5, 6, 8, 23]. Various finite difference schemes have been constructed to treat such problems [9–12, 20]. For example, in [13] two difference approximation schemes to a nonlinear pseudo-parabolic equation are developed. Each of these schemes possesses a unique solution which can be obtained by an iterative procedure. Further in [14] two difference streamline diffusion schemes for solving linear Sobolev equations with convection-dominated term are given. We can see other numerical methods of this type of equations in [15] (see also the references cited in them). In [17] a Crank–Nicolson–Galerkin approximation with extrapolated coefficients is presented for three cases for the nonlinear Sobolev equation along with a conjugate gradient iterative procedure which can be used efficiently to solve the different linear systems of algebraic equations arising at each step from the Galerkin method. In [28] the author studies a finite volume element approximation of pseudo-parabolic equations in three spatial dimensions. We also note that various approximate methods for delay parabolic equations were investigated in [3, 18, 19, 21, 22, 24–27]. In this study, we use the method of lines for the discretization in space variable for the problem (1), (2), and (3). The method of lines is a general technique for solving partial differential equations by typically using finite difference relationships for the spatial derivatives or the time derivative. Our aim is to get a fourth-order accurate difference-differential scheme and to establish the error estimate for its solution. Numerical results are also given at the end to demonstrate the efficiency of the method.

2 Construction of the Scheme

On the $\overline{\Omega}$, we introduce the uniform mesh

$$\omega_h = \{x_i = ih, \quad i = 1, 2, \ldots, N-1,\ \ h = l/N\}$$

and denote $g_{\bar{x}x,i} = (g_{i+1} - 2g_i + g_{i-1})/h^2$ for any mesh function g_i.

To construct the difference scheme, we will use the following relation which is valid for any $g(x) \in C^6\left(\overline{\Omega}\right)$:

$$\left[g''(x_{i+1}) + 10g''(x_i) + g''(x_{i-1})\right]/12 = g_{\bar{x}x,i} + \overline{R}_i, \tag{4}$$

where

$$\overline{R}_i = g^{(4)}(\xi_i)\, h^4/240, \quad \xi_i \in (x_{i-1}, x_{i+1}).$$

Let $x = x_i$ in (1)

$$\frac{\partial u(x_i,t)}{\partial t} - a(t)\frac{\partial^3 u(x_i,t)}{\partial t \partial x^2} = b(t)\frac{\partial^2 u(x_i,t)}{\partial x^2} + c(t)\frac{\partial^2 u(x_i,t-r)}{\partial x^2} + d(t)u(x_i,t)$$
$$+ f(x_i,t),\ x_i \in \omega_h,\ t \in (0,T]. \tag{5}$$

Using formula (4) in (5), we obtain

$$\left[u'_{i+1}(t) + 10u'_i(t) + u'_{i-1}(t)\right]/12 - a(t)u'_{\overline{x}x,i}(t) = b(t)u_{\overline{x}x,i}(t) + c(t)u_{\overline{x}x,i}(t-r)$$
$$+ d(t)\left[u_{i+1}(t) + 10u_i(t) + u_{i-1}(t)\right]/12 + \tilde{f}_i(t) + R_i(t),\ i = 1,2,\ldots,N-1, \tag{6}$$

$$u_i(t) = \varphi_i(t), \tag{7}$$

$$u_0(t) = u_N(t) = 0, \tag{8}$$

where

$$\tilde{f}_i(t) = \left[f_{i+1}(t) + 10f_i(t) + f_{i-1}(t)\right]/12,$$

$$R_i(t) = \frac{h^4}{240}\left[a(t)\frac{\partial^7 u(\xi_i,t)}{\partial t \partial x^6} + b(t)\frac{\partial^6 u(\xi_i,t)}{\partial x^6} + c(t)\frac{\partial^6 u(\xi_i,t-r)}{\partial x^6}\right], \quad \xi_i \in (x_{i-1}, x_{i+1}).$$

Taking into account the following relations

$$\left[u'_{i+1}(t) + 10u'_i(t) + u'_{i-1}(t)\right]/12 = u'_i(t) + u'_{\overline{x}x,i}(t)h^2/12,$$

$$d(t)\left[u_{i+1}(t) + 10u_i(t) + u_{i-1}(t)\right]/12 = d(t)u_i(t) + d(t)u_{\overline{x}x,i}(t)h^2/12,$$

and neglecting the remainder term R_i in (6), we propose the following difference-differential scheme for approximating (1), (2), and (3):

$$y'_i(t) - \left(a(t) - h^2/12\right)y'_{\overline{x}x,i}(t) = \left(b(t) + d(t)h^2/12\right)y_{\overline{x}x,i}(t) + c(t)y_{\overline{x}x,i}(t-r)$$
$$+ d(t)y_i(t) + \tilde{f}_i(t), \quad i = 1,2,\ldots,N-1, \quad t \in (0,T], \tag{9}$$

$$y_i(t) = \varphi_i(t), \quad i = 0,1,2,\ldots,N, \quad t \in (0,T], \tag{10}$$

$$y_0(t) = y_N(t) = 0, \quad t \in (0,T]. \tag{11}$$

3 The Error Estimate and Convergence

For the error function $z_i(t) = y_i(t) - u_i(t)$, from (6), (7), (8), (9), (10), and (11), we have the following difference-differential problem:

$$z_i'(t) - \left(a(t) - h^2/12\right) z'_{\bar{x}x,i}(t) = \left(b(t) + d(t)h^2/12\right) z_{\bar{x}x,i}(t) + c(t) z_{\bar{x}x,i}(t-r) + d(t) z_i(t) - R_i(t), \quad i = 1, 2, \ldots, N-1, \tag{12}$$

$$z_i(t) = 0, \quad t \in (0, T], \tag{13}$$

$$z_0(t) = z_N(t) = 0, \quad t \in (0, T]. \tag{14}$$

Theorem 3.1. Let the derivatives $\frac{\partial^7 u}{\partial t \partial x^6}$, $\frac{\partial^6 u}{\partial x^6}$ be bounded on the $\overline{Q}$ and $\alpha - h^2/12 \geq \alpha_* > 0$. Then the error of the problem (9), (10), and (11) satisfies

$$|y_i(t) - u_i(t)| \leq Ch^4, \quad i = 0, 1, \ldots, N, \quad t \in (0, T], \tag{15}$$

where C is a constant which is independent of h.

Proof. Let $Z(t) = (z_1(t), z_2(t), \ldots, z_{N-1}(t))^T$. Then Eqs. (12), (13), and (14) can be expressed in vector form as

$$Z'(t) + \left(a(t) - h^2/2\right) MZ'(t) = -\left(b(t) + d(t)h^2/2\right) MZ(t) - c(t) MZ(t-r) + d(t)Z(t) - R(t) \tag{13}$$

$$Z(0) = 0, \tag{17}$$

where

$$R(t) = (R_1(t), R_2(t), \ldots, R_{N-1}(t))^T, \quad M = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & 0 & \ldots & 0 \\ -1 & 2 & -1 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \ldots & \ldots & -1 & & 2 \end{pmatrix}.$$

The matrix M can be diagonalized as [7, 16]:

$$M = B^{-1} \text{diagonal}\left(\lambda_1, \lambda_2, \ldots \lambda_{N-1}\right) B$$

with

$$B = B^{-1} = (b_{ik})_{i,k=1}^{N-1} = \left((-1)^{i+k}\sqrt{2/N}\sin\left(\pi ik/N\right)\right)_{i,k=1}^{N-1},$$

$$\lambda_i = \frac{4}{h^2}\cos^2\left(\frac{\pi i}{2N}\right), \quad i = 1, \ldots, N-1.$$

Multiplying (16) on the left by B and denoting

$$BZ(t) = \Psi(t) = (\psi_1(t), \psi_2(t), \ldots, \psi_{N-1}(t))^T,$$

$$BR(t) = \Phi(t) = (\phi_1(t), \phi_2(t), \ldots, \phi_{N-1}(t))^T.$$

Equation (16) turning into the decomposed system as:

$$\psi_s'(t) + \left(a(t) - h^2/12\right)\lambda_s \psi_s'(t) = -\left(b(t) + d(t)h^2/12\right)\lambda_s \psi_s(t)$$

$$-c(t)\lambda_s \psi_s(t-r) + d(t)\psi_s(t) + \phi_s(t), \quad s = 1, 2, \ldots, N-1.$$

Therefore problems (16) and (17) reduce to

$$\psi_s'(t) + A_s(t)\psi_s(t) + B_s(t)\psi_s(t-r) = g_s(t), \tag{18}$$

$$\psi_s(0) = 0, \quad s = 1, 2, \ldots, N-1 \tag{19}$$

with

$$A_s(t) = \frac{\left(b(t) + d(t)h^2/12\right)\lambda_s - d(t)}{1 + \lambda_s\left(a(t) - h^2/12\right)}, \quad B_s(t) = \frac{c(t)\lambda_s}{1 + \lambda_s\left(a(t) - h^2/12\right)},$$
$$g_s(t) = \frac{\phi_s(t)}{1 + \lambda_s\left(a(t) - h^2/12\right)}.$$

It is not hard to show that the coefficients $A_s(t)$ and $B_s(t)$ are bounded independently of h:

$$|A_s(t)| \le \frac{\left|b(t)+d(t)h^2/12\right|\lambda_s}{1+\lambda_s\left(a(t)-h^2/12\right)} + \frac{|d(t)|}{1+\lambda_s\left(a(t)-h^2/12\right)}$$
$$\le \alpha_*^{-1}\left(\|b\|_\infty + \|d\|_\infty h^2/12\right)\lambda_s + \|d\|_\infty / \left(1 + \lambda_s \alpha_*\right).$$

Since $h \le \frac{l}{2}$ and $\lambda_s \ge \lambda_1 = \frac{4}{h^2}\cos^2\frac{\pi}{2N} \ge \frac{8}{l^2}$, then

$$|A_s(t)| \le c_0,$$

with

$$c_0 = \alpha_*^{-1}\left(\|b\|_\infty + \|d\|_\infty l^2/48\right) + \left(1 + 8\alpha_* l^{-2}\right)^{-1}\|d\|_\infty.$$

Similarly we have for $B_s(t)$

$|B_s(t)| \le c_1$ with $c_1 = \alpha_*^{-1}\|c\|_\infty$.

Further, (18) and (19) can be written

$$\psi_s(t) = \int_0^t g_s(t) e^{-\int_\tau^t A_s(\eta)\,\mathrm{d}\eta} \mathrm{d}\tau - \int_0^t B_s(t)\psi_s(\tau - r)\, e^{-\int_\tau^t A_s(\eta)\,\mathrm{d}\eta} \mathrm{d}\tau \quad (20)$$

Estimating that integral terms separately and taking into consideration that

$$\left| \int_0^t B_s(t)\psi_s(t) e^{\int_\tau^t A_s(\eta)\,d\eta} \mathrm{d}\tau \right| \le c_1 e^{c_0 T} \int_0^t |\psi_s(t-r)| \mathrm{d}\tau$$

$$\text{and} \quad \left| \int_0^t g_s(t) e^{\int_\tau^t A_s(\eta)\,\mathrm{d}\eta} \mathrm{d}\tau \right| \le e^{c_0 T}\|g_s\|_1,$$

after denoting $\delta_s(t) = |\psi_s(t)|$, from inequality (20), we have

$$\delta_s(t) \le C_0\|g_s\|_1 + C_1 \int_0^t |\delta_s(\tau - r)| \mathrm{d}\tau \quad (21)$$

with $C_0 = e^{c_0 T}, C_1 = c_1 e^{c_0 T}$. Using variable transformation $\tau - r = \xi$ in (21), we get

$$\delta_s(t) \le C_0\|g_s\|_1 \ \text{ for } \ 0 < t \le r$$

and

$$\delta_s(t) \le C_0\|g_s\|_1 + C_1 \int_0^{t-r} |\delta_s(\xi)| \mathrm{d}\xi \le C_0\|g_s\|_1 + C_1 \int_0^t |\delta_s(\xi)|\, \mathrm{d}\xi \ \text{ for } \ t > r.$$

From here, by virtue of Gronwall's inequality, we obtain

$$\delta_s(t) \le C_0\|g_s\|_1 e^{C_1 t}.$$

Thereby

$$\delta_s(t) \le C\|g_s\|_1 \quad (22)$$

with constant C independently of h.

The inequality (22) in turn implies that

$$|\psi_s(t)| \le C\lambda_s^{-1}\alpha_*^{-1}\|\phi_s\|_1. \quad (23)$$

Since

$$|\phi_s(t)| \le \sum_{k=1}^{N} |b_{sk}|\,|R_k| \le \sqrt{2/N}\sum_{k=1}^{N} |R_k| \le \sqrt{2/N}\,(N-1)\,Ch^4 \le \sqrt{N}Ch^4 \le Ch^{3.5},$$

the inequality (23) leads to

$$|\psi_s(t)| \le \alpha_*^{-1}\lambda_s^{-1}Ch^{3.5}.$$

Further, from

$$z_i(t) = \sum_{k=1}^{N-1} b_{ik}\psi_k,$$

we obtain

$$\begin{aligned} |z_i(t)| &\le \alpha_*^{-1}Ch^{3.5}\sum_{k=1}^{N-1}\lambda_k^{-1}\,|b_{ik}| \le Ch^{3.5}\sqrt{2/N}\,(N-1)\sum_{k=1}^{N-1} h^2/\left(4\cos^2\left(\tfrac{\pi k}{2N}\right)\right) \\ &\le Ch^4h^2\sum_{k=1}^{N-1} 1/\left(4\sin^2\left(\tfrac{\pi(N-k)}{2N}\right)\right). \end{aligned} \tag{24}$$

Taking into account the following inequality $\sin x > 2x/\pi$, $0 < x < \pi/2$, in (24), consequently we obtain

$$|z_i(t)| \le Ch^6\sum_{k=1}^{N-1} 1/\left(\frac{4}{\pi^2}\left(\frac{\pi\,(N-k)}{2N}\right)^2\right) = Ch^6N^2\sum_{k=1}^{N-1}\frac{1}{(N-k)^2} \le Ch^4,$$

i.e., (15) is proved.

4 Numerical Example

Consider the particular problem

$$\begin{aligned} &\frac{\partial u}{\partial t}(x,t) - 2\frac{\partial^3 u}{\partial t\partial x^2}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) + e^{-1}\frac{\partial^2 u}{\partial x^2}(x,t-1) + u(x,t) \\ &\quad = e^{-t}\sin h(x), \quad (x,t) \in [0,1]\times(0,2], \end{aligned}$$

$$u(x,t) = 100e^{-t}\left(\sin h(x) - x\sin h(1)\right),\ (x,t) \in [0,1]\times[-1,0],$$

$$u(0,t) = u(1,t) = 0,\ t \in (0,2].$$

Table 1 The results on [0, 1]x[0, 1]

(x, t)	Exact solution	R.K. approximation	Absolute error
(0.1,0.1)	−1.570197786	−1.570153231	4.455494E-5
(0.2,0.2)	−2.759469471	−2.759409134	6.033661E-5
(0.3,0.3)	−3.558895504	−3. 558807959	8.754480E-5
(0.4,0.4)	−3.976884921	−3.9766781183	1.037385E-4
(0.5,0.5)	−4.033749807	−4.033583955	1.658516E-4
(0.6,0.6)	−3.757555971	−3.75741264	1.433313E-4
(0.7,0.7)	−3.180983155	−3.180821732	1.614228E-4
(0.8,0.8)	−2.338980692	−2.338813886	1.668061E-4
(0.9,0.9)	−1.267047838	−1.266830828	2.170103E-4

Table 2 The results on [0, 1]x[1, 2]

(x, t)	Exact solution	R.K. approximation	Absolute error
(0.1,1.1)	−0.5776434849	−0.577438881	2.0460337E-4
(0.2,1.2)	−1.015152085	−1.014844225	3.0786018E-4
(0.3,1.3)	−1.309244489	−1.308933987	3.1050242E-4
(0.4,1.4)	−1.463014204	−1.462684133	3.3007124E-4
(0.5,1.5)	−1.483933616	−1.483571592	3.6202417E-4
(0.6,1.6)	−1.382327593	−1.381934256	3.9333711E-4
(0.7,1.7)	−1.170218303	−1.169809452	4.0885078E-4
(0.8,1.8)	−0.860462914	−0.860194326	2.6858776E-4
(0.9,1.9)	−0.466120867	−0.465889942	2.3092442E-4

The exact solution of this problem is

$$u(x,t) = 100e^{-t}\left(\sin h(x) - x\sin h(1)\right),\ (x,t) \in [0,1] \times [-1,0]\,.$$

To solve this problem numerically, we use the appropriate Runge–Kutta method. The spatial and time steps are both taken to be 0.1. The values for exact and numerical solutions and appropriate pointwise errors are shown in Tables 1 and 2.

It can be observed that the obtained results are essentially in agreement with the theoretical analysis described above.

5 Conclusion

In this paper, we have designed a fourth-order accurate difference-differential scheme to solve a time-delayed pseudo-parabolic partial differential equation in one dimension. An appropriate error estimate has provided. For the realization of our differential-difference problem, we use the fourth-order Runge–Kutta method. We have implemented the present method on standard test problem. It is observed from the results that the present method approximates the exact solution very well. The main lines for the analysis of the convergence carried out here can be used for the study of more complicated linear differential problems with second- and third-type boundary conditions.

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Oscillations of Delay and Difference Equations with Variable Coefficients and Arguments

I.P. Stavroulakis

Abstract Consider the first-order linear differential equation with several deviating arguments:

$$x'(t) + \sum_{i=1}^{m} p_i(t)x(\tau_i(t)) = 0 \left[x'(t) - \sum_{i=1}^{m} p_i(t)x(\sigma_i(t)) = 0 \right], t \geq t_0$$

and the discrete analogue difference equation

$$\Delta x(n) + \sum_{i=1}^{m} p_i(n)x(\tau_i(n)) = 0, n \geq 0 \left[\nabla x(n) - \sum_{i=1}^{m} p_i(n)x(\sigma_i(n)) = 0, n \geq 1 \right]$$

where the functions $p_i, \tau_i, \sigma_i \in C([t_{0,}\infty), \mathbb{R}^+)$ and $\tau_i(t)$ $[\sigma_i(t)]$ are retarded $(\tau_i(t) \leq t)$ [advanced $(\sigma_i(t) \geq t)$] arguments, for every $i = 1, 2, \ldots, m$, $\lim_{t\to\infty} \tau_i(t) = \infty$, and $(p_i(n))$, $1 \leq i \leq m$ are sequences of nonnegative real numbers, $\tau_i(n)$ $[\sigma_i(n)]$, $1 \leq i \leq m$ are retarded $(\tau_i(n) \leq n-1)$ [advanced $(\sigma_i(n) \geq n+1)$] arguments, $\lim\limits_{n\to\infty} \tau_i(n) = \infty$, and Δ $[\nabla]$ denotes the forward [backward] difference operator $\Delta x(n) = x(n+1) - x(n)$ $[\nabla x(n) = x(n) - x(n-1)]$. A survey on the oscillation of all solutions to these equations is presented in the case of several deviating arguments and especially when well-known oscillation conditions are not satisfied. Examples illustrating the results are given.

Keywords Oscillation • Retarded • Advanced differential equations • Difference equations

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Dedicated to the memory of George Roger Sell an excellent scientist and colleague.

1 Introduction

Consider the differential equation with several variable coefficients and arguments:

$$x'(t) + \sum_{i=1}^{m} p_i(t)x(\tau_i(t)) = 0 \left[x'(t) - \sum_{i=1}^{m} p_i(t)x(\sigma_i(t)) = 0 \right], t \geq t_0, \tag{1.1}$$

where the functions $p_i, \tau_i, \sigma_i \in C([t_{0,}\infty), \mathbb{R}^+)$ and $\tau_i(t)$ $[\sigma_i(t)]$ are retarded arguments ($\tau_i(t) \leq t$) [advanced arguments $\sigma_i(t) \geq t$] for every $i = 1, 2, \ldots, m$, and $\lim_{t\to\infty} \tau_i(t) = \infty$, and the discrete analogue difference equation

$$\Delta x(n) + \sum_{i=1}^{m} p_i(n)x(\tau_i(n)) = 0, n \geq 0 \left[\nabla x(n) - \sum_{i=1}^{m} p_i(n)x(\sigma_i(n)) = 0, n \geq 1 \right] \tag{1.2}$$

where $(p_i(n))$, $1 \leq i \leq m$ are sequences of nonnegative real numbers, $\tau_i(n)$ $[\sigma_i(n)]$, $1 \leq i \leq m$ are retarded $\tau_i(n) \leq n-1$ [advanced $\sigma_i(n) \geq n+1$] arguments, $\lim\limits_{n\to\infty} \tau_i(n) = \infty$, and Δ $[\nabla]$ denotes the forward [backward] difference operator $\Delta x(n) = x(n+1) - x(n)$ $[\nabla x(n) = x(n) - x(n-1)]$.

Let $T_0 \in [t_0, +\infty)$, $\tau(t) = \min\{\tau_i(t) : i = 1, \ldots, m\}$ and $\tau_{(-1)}(t) = \inf\{\tau(s) : s \geq t\}$. By a solution of the retarded Eq. (1.1), we understand a function $u \in C([t_0, +\infty); R)$, continuously differentiable on $[\tau_{(-1)}(T_0), +\infty)$ and that satisfies (1.1) for $t \geq \tau_{(-1)}(T_0)$. [Analogously for the advanced Eq. (1.1)]. Such a solution is called *oscillatory* if it has arbitrarily large zeros, and otherwise it is called *nonoscillatory*.

By a *solution* of the retarded difference Eq. (1.2), we mean a sequence of real numbers $(x(n))_{n\geq -w}$ which satisfies (1.2) for all $n \geq 0$. Here, $w = -\min\limits_{\substack{n\geq 0\\ 1\leq i\leq m}} \tau_i(n)$.

It is clear that, for each choice of real numbers $c_{-w}, c_{-w+1}, \ldots, c_{-1}, c_0$, there exists a unique solution $(x(n))_{n\geq -w}$ of (1.2) which satisfies the initial conditions $x(-w) = c_{-w}, x(-w+1) = c_{-w+1}, \ldots, x(-1) = c_{-1}, x(0) = c_0$.

By a solution of the advanced difference Eq. (1.2), we mean a sequence of real numbers $(x(n))_{n\geq 0}$ which satisfies (1.2) for all $n \geq 1$.

A solution $(x(n))_{n\geq -w}$ (or $(x(n))_{n\geq 0}$) of the difference Eq. (1.2) is called *oscillatory*, if the terms $x(n)$ of the sequence are neither eventually positive nor eventually negative. Otherwise, the solution is said to be *nonoscillatory*.

For the general theory of these equations, the reader is referred to [1, 9, 11, 13, 14, 18, 19, 25].

In this paper we present a survey on the oscillation of all solutions to these equations in the case of several variable coefficients and arguments and when well-known oscillation conditions are not satisfied.

2 Oscillation Criteria for Eq. (1.1)

For Eq. (1.1), the following results have been established.

In 1982, Ladas and Stavroulakis [17] (see also in 1984, Arino et al. [2]) studied the equation with several constant arguments of the form

$$x'(t) + \sum_{i=1}^{m} p_i(t)x(t-\tau_i) = 0 \left[x'(t) - \sum_{i=1}^{m} p_i(t)x(t+\tau_i) = 0 \right], t \geq t_0,) \quad (1.1')$$

under the assumption that

$$\liminf_{t\to\infty} \int_{t-\tau_i/2}^{t} p(s)ds > 0 \left[\liminf_{t\to\infty} \int_{t}^{t+\tau_i/2} p(s)ds > 0 \right], i = 1, 2, \ldots, m,$$

and proved that each one of the following conditions

$$\liminf_{t\to\infty} \int_{t-\tau_i}^{t} p_i(s)ds > \frac{1}{\mathrm{e}} \left[\liminf_{t\to\infty} \int_{t}^{t+\tau_i} p_i(s)ds > \frac{1}{\mathrm{e}} \right] \text{ for some } i, i = 1, 2, \ldots, m, \quad (2.1)$$

$$\liminf_{t\to\infty} \int_{t-\tau}^{t} \sum_{i=1}^{m} p_i(s)ds > \frac{1}{\mathrm{e}} \left[\liminf_{t\to\infty} \int_{t}^{t+\tau} \sum_{i=1}^{m} p_i(s)ds > \frac{1}{\mathrm{e}} \right],$$
$$\text{where } \tau = \min\{\tau_1, \tau_2, \ldots, \tau_m\}, \quad (2.2)$$

$$\left[\prod_{i=1}^{m} \left(\sum_{j=1}^{m} \liminf_{t\to\infty} \int_{t-\tau_j}^{t} p_i(s)ds \right) \right]^{\frac{1}{m}} > \frac{1}{\mathrm{e}} \left[\left[\prod_{i=1}^{m} \left(\sum_{j=1}^{m} \liminf_{t\to\infty} \int_{t}^{t+\tau_j} p_i(s)ds \right) \right]^{\frac{1}{m}} > \frac{1}{\mathrm{e}} \right] \quad (2.3)$$

or

$$\frac{1}{m}\sum_{i=1}^{m}\left(\liminf_{t\to\infty}\int_{t-\tau_i}^{t}p_i(s)ds\right)$$
$$+\frac{2}{m}\sum_{\substack{i<j\\ i,j=1}}^{m}\left[\left(\liminf_{t\to\infty}\int_{t-\tau_j}^{t}p_i(s)ds\right)\left(\liminf_{t\to\infty}\int_{t-\tau_i}^{t}p_i(s)ds\right)\right]^{\frac{1}{2}}>\frac{1}{\mathrm{e}} \tag{2.4}$$

$$\left[\frac{1}{m}\sum_{i=1}^{m}\left(\liminf_{t\to\infty}\int_{t}^{t+\tau_i}p_i(s)ds\right)\right.$$
$$\left.+\frac{2}{m}\sum_{\substack{i<j\\ i,j=1}}^{m}\left[\left(\liminf_{t\to\infty}\int_{t}^{t+\tau_j}p_i(s)ds\right)\left(\liminf_{t\to\infty}\int_{t}^{t+\tau_j}p_i(s)ds\right)\right]^{\frac{1}{2}}\right]>\frac{1}{\mathrm{e}} \tag{2.4}$$

implies that all solutions of Eq. (1.1′) oscillate. Later in 1996, Li [20] proved that the same conclusion holds if

$$\liminf_{t\to\infty}\sum_{i=1}^{m}\int_{t-\tau_i}^{t}p_i(s)ds>\frac{1}{\mathrm{e}}. \tag{2.5}$$

In 1984, Hunt and Yorke [15] considered the equation with variable coefficients of the form

$$x'(t)+\sum_{i=1}^{m}p_i(t)x(t-\tau_i(t))=0,\ t\geq t_0, \tag{1.1$''$}$$

under the assumption that there is a uniform upper bound τ_0 on the τ_i's and proved that if

$$\liminf_{t\to\infty}\sum_{i=1}^{m}\tau_i(t)p_i(t)>\frac{1}{\mathrm{e}}$$

then all solutions of Eq. (1.1″) oscillate.

In 1984, Fukagai and Kusano [10], for Eq. (1.1), established the following theorem.

Theorem 2.1 ([10], Theorem 1′). *Consider Eq.* (1.1) *and assume that there is a continuous non-decreasing function* $\tau^*(t)$ $[\sigma_*(t)]$ *such that* $\tau_i(t) \le \tau^*(t) \le t$ $[t \le \sigma_*(t) \le \sigma_i(t)]$ *for* $t \ge t_0$, $1 \le i \le m$. *If*

$$\liminf_{t\to\infty} \int_{\tau^*(t)}^{t} \sum_{i=1}^{m} p_i(s)ds > \frac{1}{\mathrm{e}} \left[\liminf_{t\to\infty} \int_{t}^{\sigma_*(t)} \sum_{i=1}^{m} p_i(s)ds > \frac{1}{\mathrm{e}}\right] \tag{2.6}$$

then all solutions of Eq. (1.1) *oscillate. If, on the other hand, there exists a continuous non-decreasing function* $\tau_*(t)$ $[\sigma^*(t)]$ *such that* $\tau_*(t) \le \tau_i(t)$ $[\sigma_i(t) \le \sigma^*(t)]$ *for* $t \ge t_0$, $1 \le i \le m$, $\lim_{t\to\infty} \tau_*(t) = \infty$ *and*

$$\int_{\tau_*(t)}^{t} \sum_{i=1}^{m} p_i(s)ds \le \frac{1}{\mathrm{e}} \left[\int_{t}^{\sigma^*(t)} \sum_{i=1}^{m} p_i(s)ds \le \frac{1}{\mathrm{e}}\right] \text{ for all sufficiently large } t,$$

then Eq. (1.1) *has a nonoscillatory solution.*

In 2000, Grammatikopoulos et al. [12] improved the above results, in the case of the retarded Eq. (1.1), as follows:

Theorem 2.2 ([12], Theorems 2.6). *Assume that the functions* τ_i *are non-decreasing for all* $i \in \{1, \ldots, m\}$*:*

$$\int_0^{\infty} \left|p_i(t) - p_j(t)\right| dt < +\infty, \quad i, j = 1, \ldots, m$$

and

$$\liminf_{t\to\infty} \int_{\tau_i(t)}^{t} p_i(s)ds > 0, \quad i = 1, \ldots, m.$$

If

$$\sum_{i=1}^{m} \left(\liminf_{t\to\infty} \int_{\tau_i(t)}^{t} p_i(s)ds\right) > \frac{1}{\mathrm{e}}, \tag{2.7}$$

then all solutions of Eq. (1.1) *oscillate.*

Observe that all the above-mentioned oscillation conditions (2.1)–(2.7) involve lim inf only. Moreover, it is an interesting problem to investigate Eq. (1.1) with non-monotone arguments and derive sufficient oscillation conditions, involving lim sup, which is the main objective in the following.

Theorem 2.3 ([16]). *Assume that there exist non-decreasing functions* $\mu_i \in C([t_0, +\infty))$ *such that*

$$\tau_i(t) \le \mu_i(t) \le t \quad (i = 1, \ldots, m), \tag{2.8}$$

and

$$\limsup_{t\to+\infty}\prod_{j=1}^{m}\left[\prod_{i=1}^{m}\int_{\mu_j(t)}^{t}p_i(s)\exp\left(\int_{\tau_i(s)}^{\mu_i(t)}\sum_{i=1}^{m}p_i(\xi)\right.\right.$$
$$\left.\left.\times\exp\left(\int_{\tau_i(\xi)}^{\xi}\sum_{i=1}^{m}p_i(u)du\right)d\xi\right)ds\right]^{\frac{1}{m}}>\frac{1}{m^m}. \tag{2.9}$$

Then all solutions of Eq. (1.1) *oscillate.*

In the case of monotone arguments, we have the following.

Theorem 2.4 ([16]). *Let τ_i be non-decreasing functions and*

$$\limsup_{t\to+\infty}\prod_{j=1}^{m}\left[\prod_{i=1}^{m}\int_{\tau_j(t)}^{t}p_i(s)\exp\left(\int_{\tau_i(s)}^{\tau_i(t)}\sum_{i=1}^{m}p_i(\xi)\right.\right.$$
$$\left.\left.\times\exp\left(\int_{\tau_i(\xi)}^{\xi}\sum_{i=1}^{m}p_i(u)du\right)d\xi\right)ds\right]^{\frac{1}{m}}>\frac{1}{m^m}. \tag{2.10}$$

Then all solutions of Eq. (1.1) *oscillate.*

Corollary 2.1 ([16]). *Let τ_i be non-decreasing functions and*

$$\limsup_{t\to+\infty}\prod_{j=1}^{m}\left(\prod_{i=1}^{m}\int_{\tau_j(t)}^{t}p_i(s)ds\right)^{\frac{1}{m}}>\frac{1}{m^m}, \tag{2.11}$$

Then all solutions of Eq. (1.1) *oscillate.*

Corollary 2.2 ([16]). *Let τ_i be non-decreasing functions, $p_i(t)\ \geq\ p(t)$ $(i = 1,\ldots,m)$ and*

$$\limsup_{t\to+\infty}\prod_{j=1}^{m}\int_{\tau_j(t)}^{t}p(s)\,ds>\frac{1}{m^m}, \tag{2.12}$$

Then all solutions of Eq. (1.1) *oscillate.*

Corollary 2.3 ([16]). *Let τ_i be non-decreasing functions, $p_i(t) \geq p = \text{const}$ and*

$$p^m \limsup_{t\to+\infty} \prod_{i=1}^{m} (t - \tau_i(t)) > \frac{1}{m^m}, \tag{2.13}$$

Then all solutions of Eq. (1.1) *oscillate.*

Remark 2.1. It should be pointed out that the condition (2.9) of Theorem 2.3 presents for the first time sufficient conditions (in terms of lim sup) for the oscillation of all solutions to Eq. (1.1) with several non-monotone arguments. They are also independent and essentially improve all the related oscillation conditions in the literature.

The following examples illustrate the significance of the results.

Example 2.1 (cf. [4, 16, 22]). We consider a generalization of an example presented in [4], where the equation

$$x'(t) + \frac{1}{\mathrm{e}} x(\tau(t)) = 0, t \geq 0,$$

with the retarded argument

$$\tau(t) := \begin{cases} t-1, & t \in [3n, 3n+1], \\ -3t + (12n+3), & t \in [3n+1, 3n+2], \\ 5t - (12n+13), & t \in [3n+2, 3n+3]. \end{cases}$$

was studied. Here we discuss the more general equation:

$$x'(t) + px(\tau(t)) = 0, t \geq 0, p > 0, \tag{2.14}$$

and illustrate how our methodology can be utilized to prove the existence of oscillatory solutions for some range of the parameter p. In this case, as in [4], one may choose the function

$$\sigma(t) = \begin{cases} t-1, & t \in [3n, 3n+1], \\ 3n, & t \in [3n+1, 3n+2.6], \\ 5t - (12n+13), & t \in [3n+2.6, 3n+3]. \end{cases}$$

Now note that, since $\tau(t) \leq t-1$,

$$\int_{\tau(t)}^{t} p du \geq \int_{t-1}^{t} p du = p.$$

The choice of $t_n = 3n + 3$ gives

$$
\begin{aligned}
C &= \limsup_{t\to+\infty} \int_{\sigma(t)}^{t} p \exp\left(\int_{\tau(s)}^{\sigma(t)} p \exp\left(\int_{\tau(\xi)}^{\xi} p du \right) d\xi \right) ds \\
&\geq \lim_{n\to+\infty} \int_{3n+2}^{3n+3} p \exp\left(\int_{5s-(12n+13)}^{3n+2} p \exp(p) d\xi \right) ds = \frac{1}{5} \left(e^{5pe^{p}} - 1 \right) e^{-p}.
\end{aligned}
$$

The inequality

$$
\frac{1}{5} \left(e^{5pe^{p}} - 1 \right) e^{-p} > 1
$$

is satisfied for (the numbers that follow are rounded to the third decimal place unless exact) $p \in [0.303, 0.358]$. Thus, for $p \in [0.303, 0.358]$, the condition (2.10) of Corollary 2.1 is satisfied and therefore all solutions to the above Eq. (2.14) oscillate. Observe, however, that when $p \in [0.303, 0.358]$ in (2.14), we find

$$
A = \limsup_{t\to\infty} \int_{\sigma(t)}^{t} p ds = p \cdot (2.6) < 1
$$

$$
\mathfrak{a} := \liminf_{t\to\infty} \int_{\tau(t)}^{t} p ds = p < \frac{1}{e}
$$

and

$$
\begin{aligned}
\int_{\sigma(3n+3)}^{3n+3} p \exp\left\{ \int_{\tau(s)}^{\sigma(3n+3)} p d\xi \right\} ds &= \int_{3n+2}^{3n+3} p \exp\left\{ \int_{5s-(12n+13)}^{3n+2} p d\xi \right\} ds \\
&= \frac{1}{5}(e^{5p} - 1) < 1.
\end{aligned}
$$

That is, none of the oscillation conditions (2.1)–(2.6) is satisfied.

Remark 2.2 ([16]). It is obvious that if for some $i_0 \in \{1, \ldots, m\}$ all solutions of the equation

$$
x'(t) + p_{i_0}(t)\, x(\tau_{i_0}(t)) = 0
$$

oscillate, then all solutions of Eq. (1.1) also oscillate.

Example 2.2 ([16]). Let $p, \Delta_1, \Delta_2 \in (0, +\infty)$ and consider the sequences $\{t_k\}_{k=1}^{\infty}$ such that $t_k \uparrow +\infty$ for $k \uparrow +\infty$, $t_k + 2\Delta < t_{k+1}$ $(k = 1, 2, \ldots)$, where $\Delta = \max\{\Delta_i,\ i = 1, 2\}$. Choose p, Δ_1 and Δ_2 such that

$$
p^2 \Delta_1 \Delta_2 > \frac{1}{4} \tag{2.15}
$$

and

$$p\,\Delta_i < 1 \quad (i = 1, 2). \tag{2.16}$$

Let $p(t) = p$ for $t \in [t_k, t_k + \Delta]$ $(k = 1, 2, \ldots)$ and $p(t) = 0$ for $t \in R_+ \setminus \bigcup_{k=1}^{\infty} [t_k, t_k + \Delta]$.

According to (2.15) it is obvious that the condition (2.13) is fulfilled, where $m = 2$ and $\tau_i(t) = t - \Delta_i$ $(i = 1, 2)$ a.e. and therefore all solutions to Eq. (1.1) are oscillatory. However, for the equations

$$x'(t) + p(t)\,x(t - \Delta_i) = 0 \quad (i = 1, 2)$$

by (2.16), we have

$$\limsup_{t \to +\infty} \int_{t-\Delta_i}^{t} p(s)\,ds < 1 \quad (i = 1, 2)$$

and

$$\liminf_{t \to +\infty} \int_{t-\Delta_i}^{t} p(s)\,ds = 0 \quad (i = 1, 2).$$

Remark 2.4 ([16]). In the above-mentioned Example 2.2, by a solution, we mean an absolutely continuous function which satisfies the corresponding equation almost everywhere.

Example 2.3 ([16]). Consider the equation:

$$x'(t) + p_1 x(\tau_1(t)) + p_2 x(\tau_2(t)) = 0,\ t \geq 0,\ p_1, p_2 > 0, \tag{2.17}$$

where

$$\tau_1(t) = \begin{cases} t - 1, & t \in [3n, 3n + 1], \\ -3t + (12n + 3), & t \in [3n + 1, 3n + 2], \\ 5t - (12n + 13), & t \in [3n + 2, 3n + 3], \end{cases}$$

$$\tau_2(t) = \begin{cases} t - 2, & t \in [3n, 3n + 1], \\ -t + 6n, & t \in [3n + 1, 3n + 2], \\ 3t - (6n + 8), & t \in [3n + 2, 3n + 3]. \end{cases}$$

We can take

$$\sigma_1(t) = \begin{cases} t-1, & t \in [3n, 3n+1], \\ 3n, & t \in [3n+1, 3n+2.6], \\ 5t-(12n+13), & t \in [3n+2.6, 3n+3], \end{cases}$$

$$\sigma_2(t) = \begin{cases} t-2, & t \in [3n, 3n+1], \\ 3n-1, & t \in [3n+1, 3n+2.\bar{3}], \\ 3t-(6n+8), & t \in [3n+2.\bar{3}, 3n+3]. \end{cases}$$

Note that, since $\tau_1(t) \le t-1$ and $\tau_2(t) \le t-2$, we have

$$\int_{\tau_1(t)}^t du \ge \int_{t-1}^t du = 1, \quad \int_{\tau_2(t)}^t du \ge \int_{t-2}^t du = 2.$$

Set $P = p_1 \exp(p_1+p_2) + p_2 \exp(2p_1+2p_2)$. The choice of $t_n = 3n+3$ gives

$$\begin{aligned}
&\limsup_{t\to+\infty} \prod_{j=1}^2 \left(\prod_{i=1}^2 \int_{\sigma_j(t)}^t p_i \exp\left(\int_{\tau_i(s)}^{\sigma_i(t)} \sum_{i=1}^2 p_i \exp\left(\int_{\tau_i(\xi)}^{\xi} (p_1+p_2) du \right) d\xi \right) ds \right)^{\frac{1}{2}} \\
&\ge \lim_{n\to+\infty} \prod_{j=1}^2 \left(\prod_{i=1}^2 \int_{\sigma_j(3n+3)}^{3n+3} p_i \exp\left(\int_{\tau_i(s)}^{\sigma_i(3n+3)} \sum_{i=1}^2 p_i \exp\left(\int_{\tau_i(\xi)}^{\xi} (p_1+p_2) du \right) d\xi \right) ds \right)^{\frac{1}{2}} \\
&\ge \lim_{n\to+\infty} \prod_{j=1}^2 \left(\int_{\sigma_j(3n+3)}^{3n+3} p_1 \exp\left(\int_{\tau_1(s)}^{3n+2} P d\xi \right) ds \right)^{\frac{1}{2}} \times \left(\int_{\sigma_j(3n+3)}^{3n+3} p_2 \exp\left(\int_{\tau_2(s)}^{3n+1} P d\xi \right) ds \right)^{\frac{1}{2}} \\
&= \lim_{n\to+\infty} \left(\int_{3n+2}^{3n+3} p_1 \exp\left(\int_{\tau_1(s)}^{3n+2} P d\xi \right) ds \right)^{\frac{1}{2}} \times \left(\int_{3n+2}^{3n+3} p_2 \exp\left(\int_{\tau_2(s)}^{3n+1} P d\xi \right) ds \right)^{\frac{1}{2}} \\
&\quad \times \left(\int_{3n+1}^{3n+3} p_1 \exp\left(\int_{\tau_1(s)}^{3n+2} P d\xi \right) ds \right)^{\frac{1}{2}} \times \left(\int_{3n+1}^{3n+3} p_2 \exp\left(\int_{\tau_2(s)}^{3n+1} P d\xi \right) ds \right)^{\frac{1}{2}} \\
&= \lim_{n\to+\infty} \left(\int_{3n+2}^{3n+3} p_1 \exp\left(\int_{5s-(12n+13)}^{3n+2} P d\xi \right) ds \right)^{\frac{1}{2}} \times \left(\int_{3n+2}^{3n+3} p_2 \exp\left(\int_{3s-(6n+8)}^{3n+1} P d\xi \right) ds \right)^{\frac{1}{2}} \\
&\quad \times \left(\int_{3n+1}^{3n+2} p_1 \exp\left(\int_{-3s+(12n+3)}^{3n+2} P d\xi \right) ds + \int_{3n+2}^{3n+3} p_1 \exp\left(\int_{5s-(12n+13)}^{3n+2} P d\xi \right) ds \right)^{\frac{1}{2}} \\
&\quad \times \left(\int_{3n+1}^{3n+2} p_2 \exp\left(\int_{-s+6n}^{3n+1} P d\xi \right) ds + \int_{3n+2}^{3n+3} p_2 \exp\left(\int_{3s-(6n+8)}^{3n+1} P d\xi \right) ds \right)^{\frac{1}{2}} \\
&=: D(p_1, p_2).
\end{aligned}$$

Let $p_1 = 0.1$, then, by direct computation, we get

$$D > \frac{1}{4},$$

if $p_2 \geq 0.158$. That is, when $p_1 = 0.1$ and $p_2 \geq 0.158$ in Eq. (2.17), the condition (2.9) of Theorem 2.3 is satisfied and therefore all solutions to this equation oscillate.

Note that since the delays are not monotone, Theorem 2.2 cannot be applied to this example. We now compare our result with Theorem 2.1. Note that

$$\tau_1(t), \tau_2(t) \leq \sigma_1(t), \text{ forevery } t > 0.$$

The choice $p_1 = 0.1, p_2 = 0.158$ gives

$$\liminf_{t\to\infty} \int_{\sigma_1(t)}^{t} (p_1 + p_2)ds = p_1 + p_2 = 0.258 < \frac{1}{\mathrm{e}},$$

that is, the condition (2.6) is not satisfied.

3 Oscillation Criteria for Eq. (1.2)

In this section we study the difference equation with several variable arguments

$$\Delta x(n) + \sum_{i=1}^{m} p_i(n)x(\tau_i(n)) = 0 \left[\nabla x(n) - \sum_{i=1}^{m} p_i(n)x(\sigma_i(n)) = 0\right] \tag{1.2}$$

If $\tau_i(n) = n - k_i$ and $\sigma_i(n) = n + k_i$ where $k_i > 0$, $1 \leq i \leq m$, then Eq. (1.2) reduces to the difference equation with several constant arguments of the form

$$\Delta x(n) + \sum_{i=1}^{m} p_i(n)x(n - k_i) = 0 \left[\nabla x(n) - \sum_{i=1}^{m} p_i(n)x(n + k_i) = 0\right] \tag{1.2$'$}$$

In 1989, Erbe and Zhang [8], in 1999, Tang and Yu [23] and in 2001 Tang and Zhang [24] proved that either one of the following conditions

$$\sum_{i=1}^{m} \left(\liminf_{n\to\infty} p_i(n)\right) \frac{(k_i + 1)^{k_i+1}}{(k_i)^{k_i}} > 1, \tag{3.1}$$

$$\liminf_{n\to\infty} \sum_{i=1}^{m} \left(\frac{k_i+1}{k_i}\right)^{k_i+1} \sum_{j=n+1}^{n+k_i} p_i(j) > 1, \tag{3.2}$$

or

$$\limsup_{n\to\infty} \sum_{i=1}^{m} \sum_{j=n}^{n+k_i} p_i(j) > 1, \tag{3.3}$$

implies that all solutions of the retarded difference Eq. (1.2)′ oscillate, while in 2002, Li and Zhu [21] proved that if

$$\liminf_{n\to\infty} \sum_{i=1}^{m} \left(\frac{k_i+1}{k_i}\right)^{k_i+1} \sum_{j=n-k_i}^{n-1} p_i(j) > 1, \tag{3.4}$$

then all solutions of the advanced difference Eq. (1.2)′ oscillate.

Set

$$\tau(n) = \max_{1\le i\le m} \tau_i(n), \; n \in \mathbb{N}_0, \tag{3.5}$$

$$\sigma(n) = \min_{1\le i\le m} \sigma_i(n), \; n \in \mathbb{N}. \tag{3.6}$$

In 2005, Yan et al. [26] and, in 2006, Berezansky and Braverman [3] proved that if

$$\liminf_{n\to\infty} \sum_{j=\tau(n)}^{n-1} \sum_{i=1}^{m} p_i(j) \left(\frac{n-\tau_i(j)+1}{n-\tau_i(j)}\right)^{n-\tau_i(j)+1} > 1, \tag{3.7}$$

or

$$\limsup_{n\to\infty} \sum_{i=1}^{m} p_i(n) > 0 \quad \text{and} \quad \liminf_{n\to\infty} \sum_{i=1}^{m} \sum_{j=\tau(n)}^{n-1} p_i(j) > \frac{1}{\mathrm{e}}, \tag{3.8}$$

then all solutions of the retarded difference Eq. (1.2) oscillate.

In 2014, Chatzarakis et al. [5] proved that if

$$\limsup_{n\to\infty} \sum_{i=1}^{m} \sum_{j=\tau(n)}^{n} p_i(j) > 1 \; \left[\limsup_{n\to\infty} \sum_{i=1}^{m} \sum_{j=n}^{\sigma(n)} p_i(j) > 1\right], \tag{3.9}$$

or $\limsup_{n\to\infty} \sum_{i=1}^{m} p_i(n) > 0$ and

$$\liminf_{n\to\infty}\sum_{i=1}^{m}\sum_{j=\tau_i(n)}^{n-1}p_i(j)>\frac{1}{\mathrm{e}}\quad\left[\liminf_{n\to\infty}\sum_{i=1}^{m}\sum_{j=n+1}^{\sigma_i(n)}p_i(j)>\frac{1}{\mathrm{e}}\right],\tag{3.10}$$

then all solutions of Eq. (1.2) oscillate.

Also in 2014, Chatzarakis et al. [6] established the following theorem.

Theorem 3.1 (see [6], Theorems 2.1 [3.1]). *Assume that the sequences $(\tau_i(n))$ $[(\sigma_i(n))]$, $1\le i\le m$ are increasing and*

$$\alpha=\min\{\alpha_i : 1\le i\le m\},\tag{3.11}$$

where

$$\alpha_i=\liminf_{n\to\infty}\sum_{j=\tau_i(n)}^{n-1}p_i(j)\quad\left[\alpha_i=\liminf_{n\to\infty}\sum_{j=n+1}^{\sigma_i(n)}p_i(j)\right].\tag{3.12}$$

If $0<\alpha\le 1/e$, *and*

$$\limsup_{n\to\infty}\sum_{i=1}^{m}\sum_{j=\tau(n)}^{n}p_i(j)\left[\limsup_{n\to\infty}\sum_{i=1}^{m}\sum_{j=n}^{\sigma(n)}p_i(j)\right]>1-\left(1-\sqrt{1-\alpha}\right)^2,\tag{3.13}$$

then all solutions of (1.2) *oscillate.*

If, additionally,

$$p_i(n)\ge 1-\sqrt{1-\alpha}\ \text{ for all large} n,\quad (1\le i\le m)\tag{3.14}$$

and

$$\limsup_{n\to\infty}\sum_{i=1}^{m}\sum_{j=\tau(n)}^{n}p_i(j)\left[\limsup_{n\to\infty}\sum_{i=1}^{m}\sum_{j=n}^{\sigma(n)}p_i(j)\right]>1-\alpha\left[\frac{1}{3\sqrt{1-\alpha}+\alpha-2}-1\right],\tag{3.15}$$

then all solutions of (1.2) *oscillate.*

In 2015 the above result was improved by Chatzarakis et al. [7] as follows.

Theorem 3.2 ([7]). *Assume that the sequences $(\tau_i(n))$ $[(\sigma_i(n))]$, $1\le i\le m$ are increasing; $(\tau(n))$ $[(\sigma(n))]$ is defined by* (3.5) [(3.6)] *and define α by* (3.11)*.*

If $0<\alpha\le 1/e$, *and*

$$\limsup_{n\to\infty}\sum_{i=1}^{m}\sum_{j=\tau(n)}^{n}p_i(j)\left[\limsup_{n\to\infty}\sum_{i=1}^{m}\sum_{j=n}^{\sigma(n)}p_i(j)\right]>1-\frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2},\tag{3.16}$$

then all solutions of (1.2) *oscillate.*

If, additionally,

$$p_i(n) \geq \frac{\alpha}{2} \text{ for all large} n, \quad (1 \leq i \leq m) \tag{3.17}$$

and

$$\limsup_{n\to\infty} \sum_{i=1}^{m} \sum_{j=\tau(n)}^{n} p_i(j) \left[\limsup_{n\to\infty} \sum_{i=1}^{m} \sum_{j=n}^{\sigma(n)} p_i(j) \right] > 1 - \left[2\frac{1-\sqrt{1-2\alpha-\alpha^2}}{2+\alpha} - \alpha \right], \tag{3.18}$$

then all solutions of (1.2) *oscillate.*

Remark 3.1 ([7]). It is easy to see that

$$2\frac{1-\sqrt{1-2\alpha-\alpha^2}}{2+\alpha} - \alpha > \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}$$
$$> \alpha\left[\frac{1}{3\sqrt{1-\alpha}+\alpha-2} - 1\right] > \left(1-\sqrt{1-\alpha}\right)^2.$$

Therefore, when (3.17) holds, then the condition (3.18) is weaker than conditions (3.16), (3.15) and (3.13).

Remark 3.2 ([7]). When $\alpha \to 0$, then all the above-mentioned conditions (3.18), (3.16), (3.15) and (3.13) reduce to

$$\limsup_{n\to\infty} \sum_{i=1}^{m} \sum_{j=\tau(n)}^{n} p_i(j) > 1, \quad \left[\limsup_{n\to\infty} \sum_{i=1}^{m} \sum_{j=n}^{\sigma(n)} p_i(j) > 1, \right]$$

that is, to the condition (3.9). However the improvement is clear when

$$\alpha \to \frac{1}{\mathrm{e}} \simeq 0.367879441$$

For illustrative purposes we give the values of the lower bound on the above conditions when $\alpha = 0.367879441$:

(3.13) :	0.957999636
(3.15) :	0.879366479
(3.16) :	0.863457014
(3.18) :	0.826495955

That is, the conditions (3.16) and (3.18) essentially improve (3.9), (3.13) and (3.15).

4 Examples

We illustrate the significance of the results by the following examples.

Example 4.1 ([7]). Consider the difference equation with three retarded arguments:

$$\Delta x(n) + p_1(n)x(n-1) + p_2(n)x(n-2) + p_3(n)x(n-3) = 0, \quad n \geq 0, \tag{4.1}$$

where

$$p_1(2n) = \frac{7}{100},\ p_1(2n+1) = \frac{4}{10},$$
$$p_2(3n) = p_2(3n+1) = \frac{5}{100},\ p_2(3n+2) = \frac{35}{100},$$
$$p_3(4n) = p_3(4n+1) = p_3(4n+2) = \frac{3}{100},\ p_3(4n+3) = \frac{98}{1000}.$$

Here $m = 3$, $\tau_1(n) = n-1$, $\tau_2(n) = n-2$, $\tau_3(n) = n-3$ and $\tau(n) = n-1$. It is easy to see that

$$\alpha_1 = \liminf_{n\to\infty} \sum_{j=n-1}^{n-1} p_1(j) = \frac{7}{100} = 0.07,$$

$$\alpha_2 = \liminf_{n\to\infty} \sum_{j=n-2}^{n-1} p_2(j) = 2 \cdot \frac{5}{100} = 0.1,$$

$$\alpha_3 = \liminf_{n\to\infty} \sum_{j=n-3}^{n-1} p_3(j) = 3 \cdot \frac{3}{100} = 0.09.$$

Thus

$$\alpha = \min\{\alpha_i : 1 \leq i \leq 3\} = \min\{0.07, 0.1, 0.09\} = 0.07 < \frac{1}{\mathrm{e}}.$$

Also,

$$\limsup_{n\to\infty} \sum_{i=1}^{3} \sum_{j=n-1}^{n} p_i(j) = \limsup_{n\to\infty} \left[\sum_{j=n-1}^{n} p_1(j) + \sum_{j=n-1}^{n} p_2(j) + \sum_{j=n-1}^{n} p_3(j) \right]$$
$$= \frac{7}{100} + \frac{4}{10} + \frac{5}{100} + \frac{35}{100} + \frac{3}{100} + \frac{98}{1000} = 0.998.$$

Observe that

$$0.998 > 1 - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2} \simeq 0.997358086,$$

that is, condition (3.16) of Theorem 3.2 is satisfied and therefore all solutions of Eq. (4.1) oscillate.

Observe, however, that

$$\begin{aligned} 0.998 &< 1, \\ 0.998 &< 1-\left(1-\sqrt{1-\alpha}\right)^2 \simeq 0.998730152, \end{aligned}$$

$$\begin{aligned} \liminf_{n\to\infty}\sum_{i=1}^{3}\sum_{j=\tau(n)}^{n-1} p_i(j) &= \liminf_{n\to\infty}\left[\sum_{j=n-1}^{n-1} p_1(j)+\sum_{j=n-1}^{n-1} p_2(j)+\sum_{j=n-1}^{n-1} p_3(j)\right] \\ &= \frac{7}{100}+\frac{5}{100}+\frac{3}{100} = 0.15 < \frac{1}{\mathrm{e}}, \end{aligned}$$

$$\begin{aligned} \liminf_{n\to\infty}\sum_{i=1}^{3}\sum_{j=n-k_i}^{n-1} p_i(j) &= \liminf_{n\to\infty}\left[\sum_{j=n-1}^{n-1} p_1(j)+\sum_{j=n-2}^{n-1} p_2(j)+\sum_{j=n-3}^{n-1} p_3(j)\right] \\ &= \frac{7}{100}+2\cdot\frac{5}{100}+3\cdot\frac{3}{100} = 0.26 < \frac{1}{\mathrm{e}}, \end{aligned}$$

$$\begin{aligned} &\liminf_{n\to\infty}\sum_{i=1}^{3}\left(\frac{k_i+1}{k_i}\right)^{k_i+1}\sum_{j=n+1}^{n+k_i} p_i(j) \\ &= \liminf_{n\to\infty}\left[\left(\frac{2}{1}\right)^2\sum_{j=n+1}^{n+1} p_1(j)+\left(\frac{3}{2}\right)^3\sum_{j=n+1}^{n+2} p_2(j)+\left(\frac{4}{3}\right)^4\sum_{j=n+1}^{n+3} p_2(j)\right] \\ &= 2^2\cdot\frac{7}{100}+\left(\frac{3}{2}\right)^3\cdot 2\cdot\frac{5}{100}+\left(\frac{4}{3}\right)^4\cdot 3\cdot\frac{3}{100} = 0.901944444 < 1, \end{aligned}$$

$$\begin{aligned} \sum_{i=1}^{3}\left(\liminf_{n\to\infty} p_i(n)\right)\frac{(k_i+1)^{k_i+1}}{(k_i)^{k_i}} &= \frac{7}{100}\cdot\frac{2^2}{1^1}+\frac{5}{100}\cdot\frac{3^3}{2^2}+\frac{3}{100}\cdot\frac{4^4}{3^3} \\ &= 0.901944444 < 1, \end{aligned}$$

$$\liminf_{n\to\infty}\sum_{j=\tau(n)}^{n-1}\sum_{i=1}^{3} p_i(j)\left(\frac{n-\tau_i(j)+1}{n-\tau_i(j)}\right)^{n-\tau_i(j)+1}$$

$$= \left(\frac{2}{1}\right)^2 \cdot \frac{7}{100} + \left(\frac{3}{2}\right)^3 \cdot \frac{5}{100} + \left(\frac{4}{3}\right)^4 \cdot \frac{3}{100} = 0.543564814 < 1,$$

and therefore none of the conditions (3.9), (3.13), (3.8), (3.10), (3.2), (3.1) and (3.7) is satisfied.

Example 4.2 ([7]). Consider the difference equation with two retarded arguments

$$\Delta x(n) + p_1(n)x(n-2) + p_2(n)x(n-1) = 0, \quad n \geq 0, \tag{4.2}$$

where

$$p_1(3n) = p_1(3n+1) = \frac{1}{10},\; p_1(3n+2) = \frac{1}{2}, \quad n \geq 0,$$
$$p_2(2n) = \frac{7}{100},\; p_2(2n+1) = \frac{3273}{10000}, \quad n \geq 0.$$

Here $m = 2$, $\tau_1(n) = n-2$, $\tau_2(n) = n-1$ and $\tau(n) = n-1$. It is easy to see that

$$\alpha_1 = \liminf_{n\to\infty} \sum_{j=n-2}^{n-1} p_1(j) = 2 \cdot \frac{1}{10} = 0.2,$$
$$\alpha_2 = \liminf_{n\to\infty} \sum_{j=n-1}^{n-1} p_2(j) = \frac{7}{100} = 0.07.$$

Thus

$$\alpha = \min\{\alpha_i : 1 \leq i \leq 2\} = \min\{0.2, 0.07\} = 0.07 < \frac{1}{\mathrm{e}}.$$

Furthermore, it is clear that

$$p_i(n) > \frac{\alpha}{2} = 0.035 \text{ for all large } n,\ (1 \leq i \leq 2)\,.$$
$$p_i(n) > 1 - \sqrt{1-\alpha} \simeq 0.035634923 \text{ for all large } n,\ (1 \leq i \leq 2)$$

Also,

$$\limsup_{n\to\infty} \sum_{i=1}^{2} \sum_{j=n-1}^{n} p_i(j) = \limsup_{n\to\infty} \left[\sum_{j=n-1}^{n} p_1(j) + \sum_{j=n-1}^{n} p_2(j) \right]$$
$$= \frac{1}{10} + \frac{1}{2} + \frac{7}{100} + \frac{3273}{10000} = 0.9973.$$

Observe that

$$0.9973 > 1 - \left[2\frac{1-\sqrt{1-2\alpha-\alpha^2}}{2+\alpha} - \alpha\right] \simeq 0.997262002,$$

that is, conditions (3.17) and (3.18) of Theorem 3.2 are satisfied and therefore all solutions of Eq. (4.2) oscillate.

Observe, however, that

$$\begin{aligned} 0.9973 &< 1, \\ \liminf_{n\to\infty}\sum_{i=1}^{3}\sum_{j=\tau(n)}^{n-1} p_i(j) &= \liminf_{n\to\infty}\left[\sum_{j=n-1}^{n-1} p_1(j) + \sum_{j=n-1}^{n-1} p_2(j)\right] \\ &= \frac{1}{10} + \frac{7}{100} = 0.17 < \frac{1}{\mathrm{e}}, \end{aligned}$$

$$\begin{aligned} \liminf_{n\to\infty}\sum_{i=1}^{2}\sum_{j=\tau_i(n)}^{n-1} p_i(j) &= \liminf_{n\to\infty}\left[\sum_{j=n-2}^{n-1} p_1(j)+\sum_{j=n-1}^{n-1} p_2(j)\right]=2\cdot\frac{1}{10}+\frac{7}{100}=0.27<\frac{1}{\mathrm{e}}, \\ 0.9973 &< 1-\left(1-\sqrt{1-\alpha}\right)^2 \simeq 0.998730152, \\ 0.9973 &< 1-\alpha\left[\frac{1}{3\sqrt{1-\alpha}+\alpha-2} - 1\right] \simeq 0.997317675, \end{aligned}$$

$$\begin{aligned} \liminf_{n\to\infty}\sum_{i=1}^{2}\left(\frac{k_i+1}{k_i}\right)^{k_i+1}\sum_{j=n-k_i}^{n-1} p_i(j) &= \liminf_{n\to\infty}\left[\left(\frac{3}{2}\right)^3\cdot 2\cdot\frac{1}{10}+2^2\cdot\frac{7}{100}\right]=0.955<1, \\ 0.9973 &< 1-\frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}\simeq 0.997358086, \end{aligned}$$

$$\begin{aligned} &\liminf_{n\to\infty}\sum_{j=\tau(n)}^{n-1}\sum_{i=1}^{2} p_i(j)\left(\frac{n-\tau_i(j)+1}{n-\tau_i(j)}\right)^{n-\tau_i(j)+1} \\ &= \left(\frac{3}{2}\right)^3\cdot\frac{1}{10} + \left(\frac{2}{1}\right)^2\cdot\frac{7}{100} = 0.61754<1, \end{aligned}$$

and therefore none of the conditions (3.9), (3.8), (3.10), (3.13), (3.15), (3.4), (3.16) and (3.7) is satisfied.

Example 4.3 ([7]). Consider the advanced difference equation

$$\nabla x(n) - p_1(n)x(n+2) - p_2(n)x(n+1) = 0, \quad n \geq 1 \tag{4.3}$$

where

$$p_1(3n) = p_1(3n+1) = \frac{1}{10}, \; p_1(3n+2) = \frac{1}{2}, \quad n \geq 1$$

$$p_2(2n) = \frac{8}{100}, \; p_2(2n+1) = \frac{3164}{10000}, \quad n \geq 1.$$

Here $m = 2$, $\sigma_1(n) = n+2$, $\sigma_2(n) = n+1$ and $\sigma(n) = n+1$. It is easy to see that

$$\alpha_1 = \liminf_{n\to\infty} \sum_{j=n+1}^{n+2} p_1(j) = 2 \cdot \frac{1}{10} = 0.2,$$

$$\alpha_2 = \liminf_{n\to\infty} \sum_{j=n+1}^{n+1} p_2(j) = \frac{8}{100} = 0.08.$$

Thus

$$\alpha = \min\{\alpha_i : 1 \leq i \leq 2\} = \min\{0.2, 0.08\} = 0.08 < \frac{1}{\mathrm{e}}.$$

Furthermore, it is clear that $p_i(n) > \frac{\alpha}{2} = 0.04$ for all large n, $(1 \leq i \leq 2)$.

$$p_i(n) > 1 - \sqrt{1-\alpha} \simeq 0.040833695 \text{ for all large } n, \; (1 \leq i \leq 2)$$

Also,

$$\limsup_{n\to\infty} \sum_{i=1}^{2} \sum_{j=n}^{\sigma(n)} p_i(j) = \limsup_{n\to\infty} \left[\sum_{j=n}^{n+1} p_1(j) + \sum_{j=n}^{n+1} p_2(j) \right]$$
$$= \frac{1}{10} + \frac{1}{2} + \frac{8}{100} + \frac{3164}{10000} = 0.9964.$$

Observe that

$$0.9964 > 1 - \left[2\frac{1-\sqrt{1-2\alpha-\alpha^2}}{2+\alpha} - \alpha \right] \simeq 0.996362477,$$

that is, conditions (3.17) and (3.18) of Theorem 3.2 are satisfied and therefore all solutions of Eq. (4.3) oscillate.

Observe, however, that

$$0.9964 < 1,$$

$$\liminf_{n\to\infty}\sum_{i=1}^{2}\sum_{j=n+1}^{n+k_i}p_i(j)=\liminf_{n\to\infty}\left[\sum_{j=n+1}^{n+2}p_1(j)+\sum_{j=n+1}^{n+1}p_2(j)\right]$$
$$=0.2+0.08=0.28<\frac{1}{e},$$
$$0.9964<1-\left(1-\sqrt{1-\alpha}\right)^2\simeq 0.998332609,$$
$$0.9964<1-\alpha\left[\frac{1}{3\sqrt{1-\alpha}+\alpha-2}-1\right]\simeq 0.996448991,$$
$$\liminf_{n\to\infty}\sum_{i=1}^{2}\left(\frac{k_i+1}{k_i}\right)^{k_i+1}\sum_{j=n-k_i}^{n-1}p_i(j)=\liminf_{n\to\infty}\left[\left(\frac{3}{2}\right)^3\cdot 2\cdot\frac{1}{10}+2^2\cdot\frac{8}{100}\right]$$
$$=0.995<1,$$
$$0.9964<1-\frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}\simeq 0.996508488,$$

and therefore none of the conditions (3.9), (3.10), (3.13), (3.15), (3.4) and (3.16) is satisfied.

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On Asymptotic Classification of Solutions to Nonlinear Regular and Singular Third- and Fourth-Order Differential Equations with Power Nonlinearity

I.V. Astashova

Abstract For the equation

$$y^{(n)} + p_0 \, |y|^k \operatorname{sign} y = 0,$$

in the cases $n = 3, 4, \quad p_0 > 0$ or $p_0 < 0$ for regular nonlinearity $k > 1$ and singular nonlinearity $0 < k < 1$ asymptotic classification of all solutions are given.

It is the first time when all results on this classification are represented together for regular and singular cases.

Keywords Nonlinear higher-order ordinary differential equation • Asymptotic behavior • Qualitative properties • Asymptotic classification of solutions

UDK 517.91

Mathematics Subject Classification (2000): 34C15, 34C10

1 Introduction

The first asymptotic classification of solutions to the Emden–Fowler equation of the second order appears in [12]. Generalizations of the equation of higher orders were investigated from different points of view later in the book [16] and in a great number of articles of different authors. In particular, sufficient conditions are given for the existence of some special types of solutions to these equations (see, e.g., [1, 3, 7, 9, 13–15, 18, 19]). See also [5] with its references. Qualitative properties

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of solutions to third- and fourth-order equations of this type were investigated in [1–6, 11, 20, 21]. Note that qualitative properties of similar linear equations of the third and fourth orders were investigated in [16, 17, 21].

The previous author's results on asymptotic classification of all possible solutions are seen in [5, 8, 10]. The purpose of this article is to represent together the asymptotic classification of all possible solutions to Emden–Fowler type third- and fourth-order equations both in regular and singular cases for comparison of their asymptotic behavior.

2 Regular Nonlinearity ($k > 1$)

Theorem 1. *Suppose $k > 1$ and $p_0 > 0$. Then all nontrivial non-extensible solutions to the equation*

$$y'''(x) + p_0\,|y|^{k-1}\,y(x) = 0 \tag{1}$$

are divided into the following five types according to their asymptotic behavior (see Fig. 1):

1–2. Defined on semiaxes $(b, +\infty)$ Kneser (up to the sign) solutions:

$$y(x) = \pm C_{3k}\,(x-b)^{-\frac{3}{k-1}},$$

where

$$C_{3k} = \left|\frac{3(k+2)(2k+1)}{p_0\,(k-1)^3}\right|^{\frac{1}{k-1}}. \tag{2}$$

3. Defined on semiaxes $(-\infty, b)$ oscillatory, in both directions, solutions having the form

$$y(x) = (b-x)^{-\frac{3}{k-1}}\,h\,(\log(b-x))$$

with some oscillatory periodic function h.

4–5. Defined on bounded intervals (b', b'') oscillatory near the right boundary and nonvanishing near the left one solutions satisfying

$$y(x) = \pm C_{3k}\,(x-b')^{-\frac{3}{k-1}}\,(1+o(1)) \quad \text{as} \quad x \to b'+0,$$

and, at their local extremum points x',

$$\left|y(x')\right| = \left|b''-x'\right|^{-\frac{3}{k-1}+o(1)} \quad \text{as } x' \to b''-0.$$

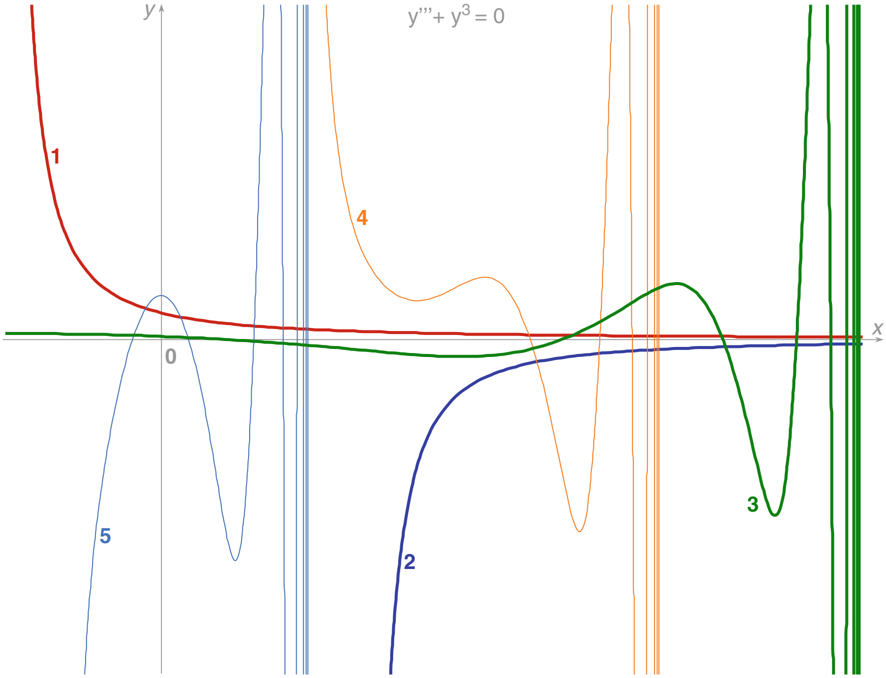

Fig. 1 Solutions to the equation $y''' + y^3 = 0$

Remark 1. The case $p_0 < 0$ can be reduced to the above one by the substitution $x \mapsto -x$.

Theorem 2. *Suppose* $k > 1$ *and* $p_0 > 0$*. Then all nontrivial non-extensible solutions to the equation*

$$y^{\mathrm{IV}}(x) + p_0\,|y|^{k-1}\,y(x) = 0 \tag{3}$$

are divided into the following three types according to their asymptotic behavior (see Fig. 2):

1. *Defined on semiaxes* $(-\infty, b)$ *oscillatory solutions. The distance between their neighboring zeros infinitely increases near the left boundaries of the domains and tends to zero near the right ones. The solutions and their derivatives satisfy the relations* $\lim\limits_{x\to-\infty} y^{(j)}(x) = 0$, $\overline{\lim\limits_{x\to b}}\,\left|y^{(j)}(x)\right| = \infty$ *for* $j = 0, 1, 2, 3$*. At the points of local extremum, the following estimates hold:*

$$C_1\,|x-b|^{-\frac{4}{k-1}} \le |y(x)| \le C_2\,|x-b|^{-\frac{4}{k-1}} \tag{4}$$

with the positive constants C_1 *and* C_2 *depending only on* k *and* p_0*.*

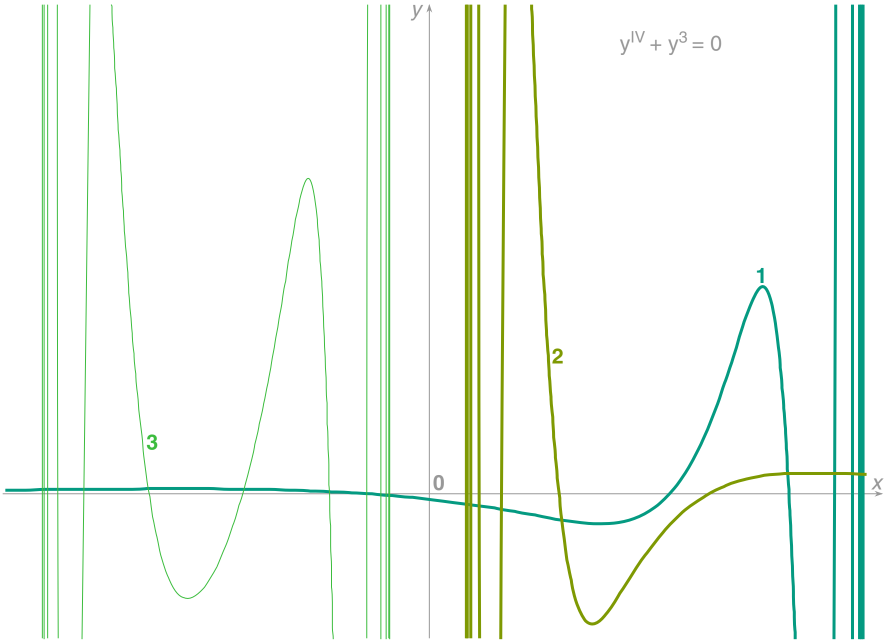

Fig. 2 Solutions to the equation $y^{\mathrm{IV}} + y^3 = 0$

2. *Defined on semiaxes* $(b, +\infty)$ *oscillatory solutions. The distance between their neighboring zeros tends to zero near the left boundaries of the domains and infinitely increases near the right ones. The solutions and their derivatives satisfy the relations* $\lim\limits_{x\to+\infty} y^{(j)}(x) = 0$, $\overline{\lim\limits_{x\to b}} \left|y^{(j)}(x)\right| = \infty$ *for* $j = 0, 1, 2, 3$. *At the points of local extremum, estimates* (4) *hold with the positive constants* C_1 *and* C_2 *depending only on* k *and* p_0.
3. *Defined on bounded intervals* (b', b'') *oscillatory solutions. All their derivatives* $y^{(j)}$, *with* $j = 0, 1, 2, 3, 4$, *satisfy*

$$\overline{\lim_{x\to b'}} \left|y^{(j)}(x)\right| = \overline{\lim_{x\to b''}} \left|y^{(j)}(x)\right| = \infty.$$

At the points of local extremum sufficiently close to any boundary of the domain, estimates (4) *hold, respectively, with* $b = b'$ *or* $b = b''$ *and the positive constants* C_1 *and* C_2 *depending only on* k *and* p_0.

Theorem 3. *Suppose* $k > 1$ *and* $p_0 < 0$. *Then all nontrivial non-extensible solutions to Eq.* (3) *are divided into the following 13 types according to their asymptotic behavior (see Fig. 3).*

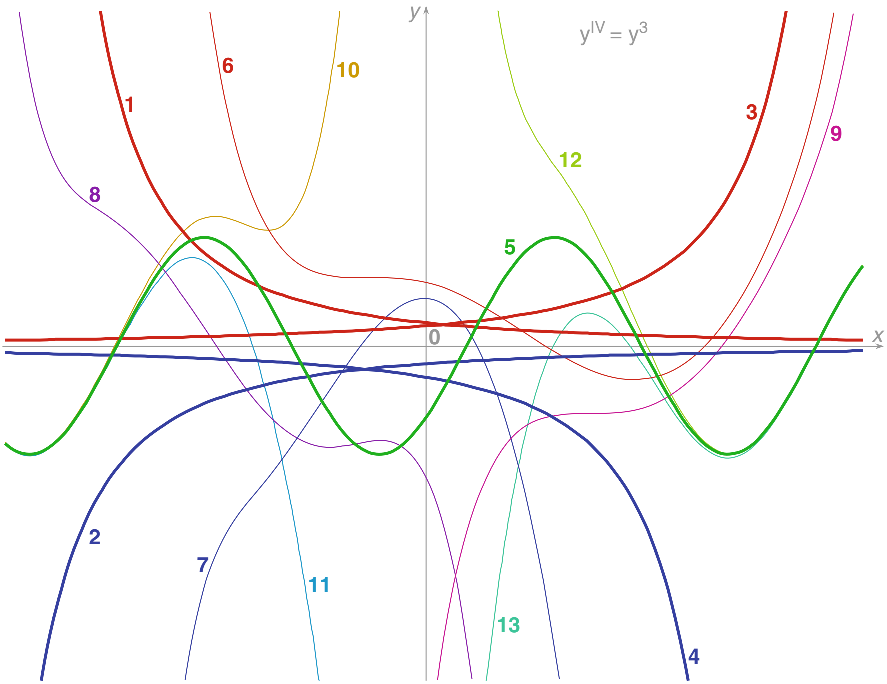

Fig. 3 Solutions to the equation $y^{\mathrm{IV}} = y^3$

1–2. Kneser (up to the sign) solutions on semiaxes $(b, +\infty)$:

$$y(x) = \pm C_{4k}\,(x-b)^{-\frac{4}{k-1}},$$

where

$$C_{4k} = \left(\frac{4(k+3)(2k+2)(3k+1)}{|p_0|\;(k-1)^4}\right)^{\frac{1}{k-1}}. \tag{5}$$

3–4. "Left" Kneser (up to the sign) solutions on semiaxes $(-\infty, b)$:

$$y(x) = \pm C_{4k}\,(b-x)^{-\frac{4}{k-1}}.$$

5. *Periodic oscillatory solutions on* $(-\infty, +\infty)$. *All of them can be received from one, say* $z(x)$, *by the relation*

$$y(x) = \lambda^4 z(\lambda^{k-1}x + x_0)$$

with arbitrary $\lambda > 0$ *and* x_0. *So, there exists such a solution with any maximum* $h > 0$ *and with any period* $T > 0$, *but not with any pair* (h, T).

6–9. *Defined on bounded intervals* (b', b'') *solutions with the power asymptotic behavior near the boundaries of the domain (with the independent signs* $\pm$*):*

$$y(x) \sim \pm C_{4k}(p(b'))\ (x - b')^{-\frac{4}{k-1}} \qquad x \to b' + 0,$$
$$y(x) \sim \pm C_{4k}(p(b''))\ (b'' - x)^{-\frac{4}{k-1}} \qquad x \to b'' - 0.$$

10–11. *Defined on semiaxes* $(-\infty, b)$ *solutions which oscillate near* $-\infty$ *and have the power asymptotic behavior near the right boundary of the domain:*

$$y(x) \sim \pm C_{4k}(p(b))\ (b - x)^{-\frac{4}{k-1}} \qquad x \to b - 0.$$

For each solution a finite limit of the absolute values of its local extrema exists as $x \to -\infty$*.*

12–13. *Defined on semiaxes* $(b, +\infty)$ *solutions which oscillate near* $+\infty$ *and have the power asymptotic behavior near the left boundary of the domain:*

$$y(x) \sim \pm C_{4k}(p(b))\ (x - b)^{-\frac{4}{k-1}} \qquad x \to b + 0.$$

For each solution a finite limit of the absolute values of its local extrema exists as $x \to +\infty$*.*

3 Singular Nonlinearity ($0 < k < 1$)

While studying the asymptotic behavior of solutions in the case of regular nonlinearity, $k > 1$, only maximally extended solutions are usually considered, because solutions can behave in a special way only near the boundaries of their domains. If $k < 1$, then special behavior can occur also near internal points of the domains. This is why a notion of *maximally unique (MU)* solutions is introduced.

Definition 1. A solution $u : (a,\, b) \to \mathbb{R}$ with $-\infty \le a < b \le +\infty$ to any ordinary differential equation is called a *MU solution* if the following two conditions hold:

(i) the equation has no other solution equal to u on some subinterval of $(a,\, b)$;
(ii) either there is no solution defined on another interval containing $(a,\, b)$ and equal to u on $(a,\, b)$, or there exist at least two such solutions not equal to each other at points arbitrary close to the boundary of $(a,\, b)$.

Theorem 4. *Suppose* $0 < k < 1$ *and* $p_0 > 0$*. Then all MU solutions to the equation*

$$y'''(x) = p_0\, |y|^{k-1}\, y(x) \tag{6}$$

are divided into the following five types according to their asymptotic behavior (see Fig. 4):

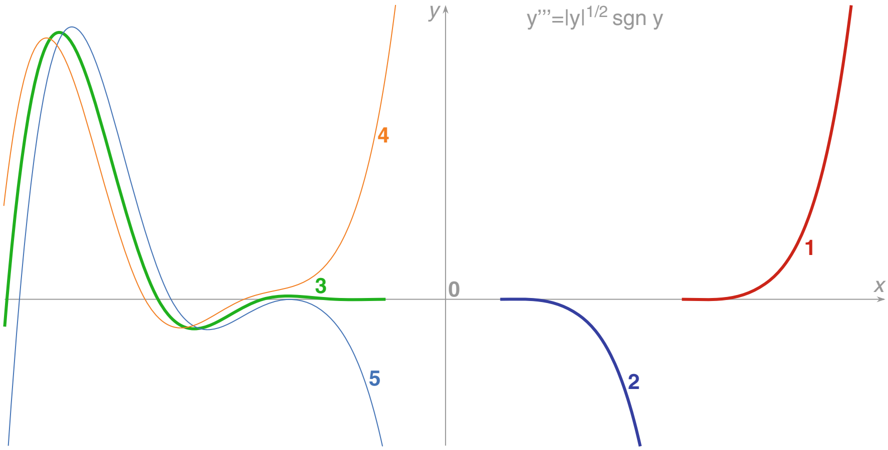

Fig. 4 MU solutions to the equation $y''' = |y|^{1/2}\text{sgn}\, y$

1–2. *Constant-sign solutions with power behavior on* $(b, +\infty)$:

$$y(x) = \pm C_{3k}\,(x-b)^{\frac{3}{1-k}},$$

where C_{3k} *is defined by* (2).

3. *Oscillatory, in both directions, solutions on* $(-\infty, b)$ *having the form*

$$y(x) = (b-x)^{\frac{3}{1-k}}\; h\,(\,\log(b-x)\,)$$

with some oscillatory periodic function h.

4–5. *Defined on* $(-\infty, +\infty)$ *solutions oscillating near* $-\infty$, *having asymptotically power behavior near* $+\infty$:

$$y(x) = \pm C_{3k}\, x^{\frac{3}{1-k}}(1 + o(1)) \quad \text{as } x \to +\infty,$$

and having no point x_0 *with* $y(x_0) = y'(x_0) = y''(x_0) = 0$. *At their local extremum points* x' *the satisfy*

$$\left|y(x')\right| = \left|x'\right|^{\frac{3}{1-k}+o(1)} \quad \text{as } x' \to -\infty.$$

Remark 2. The case $p_0 < 0$ can be reduced to the above one by the substitution $x \mapsto -x$.

Theorem 5. *Suppose* $0 < k < 1$ *and* $p_0 > 0$. *Then all MU solutions to Eq.* (3) *are divided into the following three types according to their asymptotic behavior (see Fig.* 5*):*

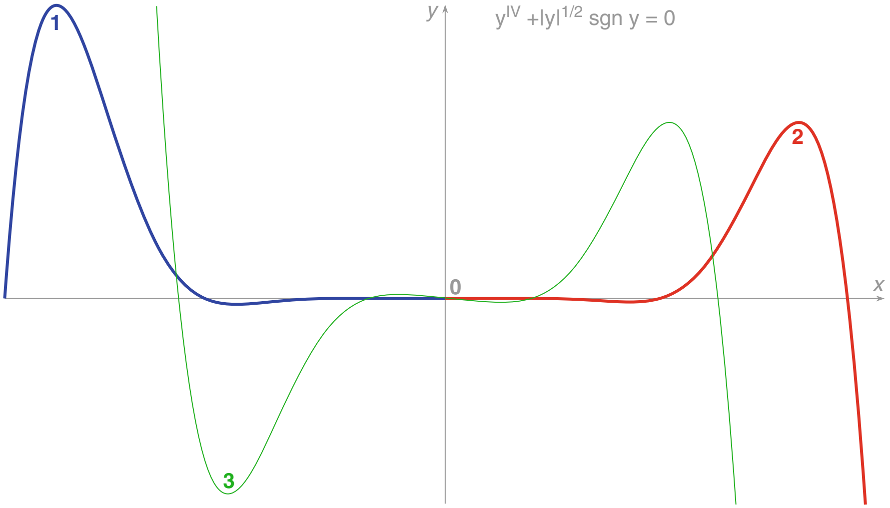

Fig. 5 MU solutions to the equation $y^{\mathrm{IV}} + |y|^{1/2}\operatorname{sgn} y = 0$

1. *Oscillatory solutions defined on* $(-\infty, b)$. *The distance between their neighboring zeros infinitely increases near* $-\infty$ *and tends to zero near* b. *The solutions and their derivatives satisfy the relations* $\lim\limits_{x\to b} y^{(j)}(x) = 0$, $\overline{\lim\limits_{x\to-\infty}} \left|y^{(j)}(x)\right| = \infty$ *for* $j = 0, 1, 2, 3$. *At the points of local extremum, the following estimates hold:*

$$C_1\,|x-b|^{-\frac{4}{k-1}} \le |y(x)| \le C_2\,|x-b|^{-\frac{4}{k-1}} \tag{7}$$

   with positive constants C_1 *and* C_2 *depending only on* k *and* p_0.
2. *Oscillatory solutions defined on* $(b, +\infty)$. *The distance between their neighboring zeros tends to zero near* b *and infinitely increases near* $+\infty$. *The solutions and their derivatives satisfy the relations* $\lim\limits_{x\to b} y^{(j)}(x) = 0$, $\overline{\lim\limits_{x\to+\infty}} \left|y^{(j)}(x)\right| = \infty$ *for* $j = 0, 1, 2, 3$. *At the points of local extremum, estimates* (7) *hold with positive constants* C_1 *and* C_2 *depending only on* k *and* p_0.
3. *Oscillatory solutions defined on* $(-\infty, +\infty)$. *All their derivatives* $y^{(j)}$ *with* $j = 0, 1, 2, 3, 4$ *satisfy*

$$\overline{\lim_{x\to-\infty}} \left|y^{(j)}(x)\right| = \overline{\lim_{x\to+\infty}} \left|y^{(j)}(x)\right| = \infty.$$

   At the points of local extremum, the estimates

$$C_1\,|x|^{-\frac{4}{k-1}} \le |y(x)| \le C_2\,|x|^{-\frac{4}{k-1}} \tag{8}$$

   hold near $-\infty$ *or* $+\infty$ *with positive constants* C_1 *and* C_2 *depending only on* k *and* p_0.

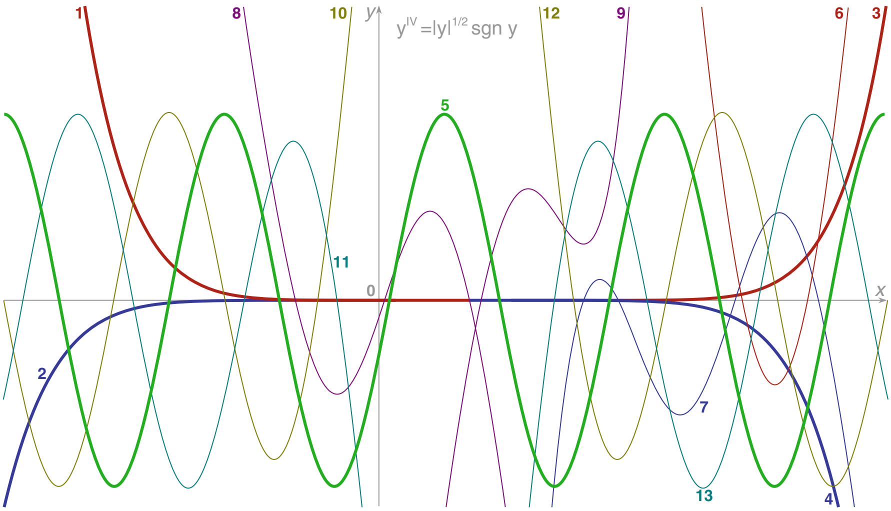

Fig. 6 MU solutions to the equation $y^{\mathrm{IV}} = |y|^{1/2}\mathrm{sgn}\, y$

Theorem 6. *Suppose* $0 < k < 1$ *and* $p_0 < 0$. *Then all MU solutions to Eq.* (3) *are divided into the following 13 types according to their asymptotic behavior (see Fig.* 6*):*

1–2. Defined on semiaxes $(-\infty, b)$ *solutions with the power asymptotic behavior near the boundaries of the domain (with the same signs* $\pm$*):*

$$y(x) \sim \pm C_{4k}\, |x|^{-\frac{4}{k-1}}, \qquad x \to -\infty,$$

$$y(x) \sim \pm C_{4k}\, (b-x)^{-\frac{4}{k-1}}, \qquad x \to b-0,$$

where

$$C_{4k} = \left(\frac{4(k+3)(2k+2)(3k+1)}{|p_0|\,(k-1)^4}\right)^{\frac{1}{k-1}}.$$

3–4. Defined on $(b, +\infty)$ *solutions with the power asymptotic behavior near the boundaries of the domain (with the same signs* $\pm$*):*

$$y(x) \sim \pm C_{4k}\, (x-b)^{-\frac{4}{k-1}}, \qquad x \to b+0,$$

$$y(x) \sim \pm C_{4k}\, x^{-\frac{4}{k-1}}, \qquad x \to +\infty.$$

5. Defined on the whole axis periodic oscillatory solutions. All of them can be received from one solution, say $z(x)$, *by the relation*

$$y(x) = \lambda^4 z(\lambda^{k-1}x + x_0)$$

with arbitrary $\lambda > 0$ *and* x_0. *So, there exists such a solution with any maximum* $h > 0$ *and with any period* $T > 0$, *but not with any pair* (h, T).

6–9. *Defined on* $(-\infty, +\infty)$ *solutions having the power asymptotic behavior near* $-\infty$ *and* $+\infty$ *(with all sign combinations admitted):*

$$y(x) \sim \pm C_{4k}\ |x|^{-\frac{4}{k-1}}, \qquad x \to \pm\infty.$$

10–11. *Defined on* $(-\infty, +\infty)$ *solutions which oscillate as* $x \to -\infty$ *and have the power asymptotic behavior near* $+\infty$*:*

$$y(x) \sim \pm C_{4k}\ x^{-\frac{4}{k-1}}, \qquad x \to +\infty.$$

Each solution has a finite limit of the absolute values of its local extrema as $x \to -\infty$.

12–13. *Defined on* $(-\infty, +\infty)$ *solutions which oscillate as* $x \to +\infty$ *and have the power asymptotic behavior near* $-\infty$*:*

$$y(x) \sim \pm C_{4k}\ |x|^{-\frac{4}{k-1}}, \qquad x \to -\infty.$$

Each solution has a finite limit of the absolute values of its local extrema as $x \to +\infty$.

4 Sketch of Proofs

To obtain the above results on asymptotic classification of all maximally extended solutions to the equation

$$y^{(n)} + p_0|y|^k \operatorname{sign} y = 0, \ \ p_0 \neq 0, \tag{9}$$

with $k > 1$ and all MU solutions to (9) with $0 < k < 1$, an auxiliary dynamical system is investigated on the $(n-1)$-dimensional sphere (see [4, 5, Chap. 5–7]; [8] for regular nonlinearity).

Note that if a function $y(x)$ is a solution to Eq. (9), the same is true for the function

$$z(x) = Ay(Bx + C), \tag{10}$$

where $A \neq 0$, $B > 0$, and C are any constants satisfying

$$|A|^{k-1} = B^n. \tag{11}$$

Any nontrivial solution $y(x)$ to Eq. (9) generates in $\mathbb{R}^n \setminus \{0\}$ the curve $\big(y(x), y'(x), y''(x), \dots, y^{n-1}(x)\big)$. We can define an equivalence relation on $\mathbb{R}^n \setminus \{0\}$ such that all solutions obtained from $y(x)$ by (10)–(11) generate equivalent curves, i.e., curves passing through equivalent points (may be for different x). We assume the points $(y_0, y_1, y_2, \dots, y_{n-1})$ and $(z_0, z_1, z_2, \dots, z_{n-1})$ in $\mathbb{R}^n \setminus \{0\}$ to be equivalent if and only if there exists a positive constant λ such that

$$z_j = \lambda^{n+j(k-1)} y_j, \qquad j = 0, 1, 2, \dots, n-1.$$

The quotient space obtained is homeomorphic to the $(n-1)$-dimensional sphere:

$$S^{n-1} = \big\{y \in \mathbb{R}^n : \quad y_0^2 + y_1^2 + y_2^2 + \cdots + y_{n-1}^2 = 1\big\}$$

having exactly one representative of each equivalence class since the equation

$$\lambda^{2n} y_0^2 + \lambda^{2(n+2(k-1))} y_1^2 + \cdots + \lambda^{2(n+(n-1)(k-1))} y_{n-1}^2 = 1$$

has exactly one positive root λ for any $(y_0, y_1, y_2, \dots, y_{n-1}) \in \mathbb{R}^n \setminus \{0\}$.

Now, equivalent curves in $\mathbb{R}^n \setminus \{0\}$ generate the same curves in the quotient space. The last ones are trajectories of an appropriate dynamical system, which can be described, in different charts covering the quotient space, by different formulas using different independent variables.

For example, on the chart that covers the points corresponding to positive values of solutions and has the coordinate functions

$$u_j = y^{(j)} y^{-\beta_j} \ \text{ with } \beta_j = 1 + \frac{j(k-1)}{n}, \ \ j = 1, \dots, n-1,$$

the dynamical system can be written as

$$\begin{cases} \dfrac{du_1}{dt} = u_2 - \beta_1 u_1^2, \\ \dfrac{du_j}{dt} = u_{j+1} - \beta_j u_1 u_j, \ \ j = 2, \dots, n-2, \\ \dfrac{du_{n-1}}{dt} = -p_0 - \beta_{n-1} u_1 u_{n-1} \end{cases} \tag{12}$$

with the independent variable

$$t = \int_{x_0}^{x} y(\xi)^{\frac{k-1}{n}} \, d\xi.$$

Qualitative properties of the trajectories of the dynamical system on the sphere do not depend essentially on whether k in (9) is greater or less than 1. However, the properties of the related solutions to Eq. (9) differ according to the case, regular or singular, considered.

Globally, the dynamical system can have some fixed points, which depends on the sign of p_0 and the parity of n. They correspond to the solutions to Eq. (9) with power-law behavior, which can be defined by explicit formulas, namely,

$$y(x) = \pm C\,|x - x^*|^{-n/(k-1)}$$

with arbitrary x^* and C defined by (2), (5), or similar formulas for $n > 4$.

In the regular case, these solutions, if maximally extended, have a vertical asymptote at one of their domain boundaries (which is finite) and tend to zero near another one (which is infinite). In the singular case, the related MU solutions vanish with all their $n-1$ lower-order derivatives at one of their domain boundaries (which is finite) and infinitely grow in absolute value near another one (which is infinite).

The dynamical system on the sphere can also have nonconstant periodical trajectories. They correspond to oscillatory solutions to (9) that can be written with the help of some periodic functions, but can be nonperiodic themselves. Their extrema and the lengths of their constant-sign intervals behave in different ways according to the sign of p_0, the parity of n, and regular or singular case considered.

Investigation of stability of the fixed points and periodical trajectories gives information on the rest of the solutions to Eq. (9), which appear to have, near the boundaries of their domains, asymptotically the same behavior as that of the solutions mentioned before.

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Some Properties of a Generalized Solution for 3-D Flow of a Compressible Viscous Micropolar Fluid Model with Spherical Symmetry

Ivan Dražić and Nermina Mujaković

Abstract We consider the nonstationary 3-D flow of a compressible viscous and heat-conducting micropolar fluid bounded with two concentric spheres that present solid thermoinsulated walls. We assume that the fluid is perfect and polytropic in the thermodynamical sense, as well as that the initial density and temperature are strictly positive. We take sufficiently smooth spherically symmetric initial functions and analyze the corresponding problem with homogeneous boundary data.

In this work we give the overview of the current progress in mathematical analysis of the described problem with particular emphasis on the existence theorems and the large time behavior of the solution.

Keywords Micropolar fluid • Spherical symmetry • Generalized solution

AMS Subject Classifications: 35Q35, 76N10

1 Introduction

The micropolar fluid is a type of fluid which exhibits microrotational effects, as well as microrotational inertia which enables us to consider some physical phenomena that cannot be treated by the classical Navier–Stokes equations. It is important to emphasize that it has been shown experimentally that the inclusion of the phenomena at the microlevel significantly improves the mathematical model of the fluid flow [10].

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The first considerations that go in the direction of studying micro phenomena, and which are known to science today, appear in the works of Cosserat brothers created at the beginning of the last century. However, due to its complexity, this theory remained neglected for many years until in the 1960s. A.C. Eringen introduced the concept of the micropolar fluid. The model of micropolar fluids in the last two decades has become an important area of interest for mathematicians and engineers especially in the modeling of liquid crystals with rigid molecules, magnetic fluids, clouds with dust, muddy fluids, some biological fluids, etc.

Here we analyze the compressible flow of an isotropic, viscous, and heat-conducting micropolar fluid, which is in the thermodynamical sense perfect and polytropic. The model for this kind of flow in the one-dimensional case was first described by Mujaković in [7]. In her later works, she analyzed the one-dimensional model in relation to existence, regularity, and stabilization for different kinds of problems with homogeneous and nonhomogeneous boundary conditions. A significant number of results related to this one-dimensional model have been systematized in the fifth and sixth chapters of the book [11], but for recent progress in this area, we refer to [8] and [4] and the references cited therein.

In this work we analyze the motion of the described fluid between two concentric spheres, which enables us to consider the spherically symmetric solution to the governing system if we assume that the initial functions are spherically symmetric and smooth enough.

The paper is organized as follows. In the next section, we will describe the governing three-dimensional system and derive its spherically symmetric form in the Lagrangian description. Then we will give an overview of the current progress in mathematical analysis of this problem. We will introduce the generalized solution to the problem together with the existence and uniqueness theorems. Finally, we will mention some recent results concerning the large time behavior of the solution.

2 The Mathematical Model

The mathematical model of the described fluid is stated, for example, in the book of Lukaszewicz [6] and reads

$$\dot{\rho} + \rho \operatorname{div} \mathbf{v} = 0, \tag{1}$$

$$\rho \dot{\mathbf{v}} = \operatorname{div} \mathbf{T} + \rho \mathbf{f}, \tag{2}$$

$$\rho j_I \dot{\boldsymbol{\omega}} = \operatorname{div} \mathbf{C} + \mathbf{T}_x + \rho \mathbf{g}, \tag{3}$$

$$\rho \dot{E} = \mathbf{T} : \nabla \mathbf{v} + \mathbf{C} : \nabla \boldsymbol{\omega} - \mathbf{T}_x \cdot \boldsymbol{\omega} - \operatorname{div} \mathbf{q} + \rho \delta, \tag{4}$$

$$\mathbf{T} = (-p + \lambda \operatorname{div} \mathbf{v})\mathbf{I} + 2\mu \operatorname{sym} \nabla \mathbf{v} + 2\mu_r \operatorname{skw} \nabla \mathbf{v} - 2\mu_r \boldsymbol{\omega}_{skw}, \tag{5}$$

$$\mathbf{C} = c_0(\operatorname{div} \boldsymbol{\omega})\mathbf{I} + 2c_d \operatorname{sym} \nabla \boldsymbol{\omega} + 2c_a \operatorname{skw} \nabla \boldsymbol{\omega}, \tag{6}$$

$$\mathbf{q} = -k\nabla\theta, \tag{7}$$

$$p = R\rho\theta, \tag{8}$$

$$E = c_v\theta. \tag{9}$$

Here ρ, $\mathbf{v}$, $\boldsymbol{\omega}$, E, and θ are, respectively, mass density, velocity, microrotation velocity, internal energy density, and absolute temperature. $\mathbf{T}$ is the stress tensor, $\mathbf{C}$ is the couple stress tensor, $\mathbf{q}$ is the heat flux density vector, $\mathbf{f}$ is the body force density, $\mathbf{g}$ is body couple density, and δ is the body heat density. p denotes pressure and the positive constant j_I is micro-inertia density. λ and μ are coefficients of the viscosity, and μ_r, c_0, c_d, and c_a are coefficients of the micro-viscosity. By the constant k ($k \geq 0$), we denote the heat conduction coefficient, the positive constant R is the specific gas constant, and the positive constant c_v denotes the specific heat for a constant volume.

Equations (1)–(4) are, respectively, local forms of conservation laws for the mass, momentum, momentum moment, and energy. Equations (5)–(6) are constitutive equations for the micropolar continuum. Equation (7) is the Fourier law, and Eqs. (8)–(9) present the assumptions that our fluid is perfect and polytropic. Coefficients of viscosity and coefficients of micro-viscosity are related through the Clausius–Duhamel inequalities, as follows:

$$\mu \geq 0, \qquad 3\lambda + 2\mu \geq 0, \qquad \mu_r \geq 0. \tag{10}$$

$$c_d \geq 0, \quad 3c_0 + 2c_d \geq 0, \quad |c_d - c_a| \leq c_d + c_a. \tag{11}$$

Vector $\mathbf{T}_x$ in Eqs. (3) and (4) is an axial vector with the Cartesian components $(\mathbf{T}_x)_i = \varepsilon_{ijk}\mathbf{T}_{jk}$, where ε_{ijk} is the Levi-Civita alternating tensor[1] and sym $\mathbf{T}$ and skw $\mathbf{T}$ are the symmetric and skew-symmetric parts of the tensor $\mathbf{T}$. The differential (dot) operator in Eqs. (1)–(4) denotes the material derivative defined by

$$\dot{\mathbf{a}} = \frac{\partial \mathbf{a}}{\partial t} + (\nabla \mathbf{a}) \cdot \mathbf{v}. \tag{12}$$

The colon operator in Eq. (4) is the scalar product of tensors defined by

$$\mathbf{A} : \mathbf{B} = \mathrm{tr}(\mathbf{A}^T\mathbf{B}). \tag{13}$$

We take the following homogeneous boundary conditions:

$$\mathbf{v}|_{\partial\Omega} = 0, \quad \boldsymbol{\omega}|_{\partial\Omega} = 0, \quad \left.\frac{\partial\theta}{\partial\boldsymbol{\nu}}\right|_{\partial\Omega} = 0, \tag{14}$$

[1] We assume the Einstein notation for summation.

where $\Omega \subset \mathbf{R}^3$ is the spatial domain of our problem and the vector $\boldsymbol{\nu}$ is the exterior unit normal vector. These boundary conditions mean that we analyze the flow of the fluid through a chamber with solid thermo-insulated walls. In our case it will be the flow between two concentric spheres and we have

$$\Omega = \left\{ \mathbf{x} = (x_1, x_2, x_3) : \mathbf{x} \in \mathbf{R}^3,\ a < |\mathbf{x}| < b,\ |\mathbf{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2} \right\}. \tag{15}$$

We introduce the spherically symmetric initial conditions:

$$\rho_0(\mathbf{x}) = \rho_0(r),\ \mathbf{v}_0(\mathbf{x}) = \frac{\mathbf{x}}{r} v_0(r),\ \boldsymbol{\omega}_0(\mathbf{x}) = \frac{\mathbf{x}}{r}\omega_0(r),\ \theta_0(\mathbf{x}) = \theta_0(r), \tag{16}$$

where ρ_0, v_0, ω_0, and θ_0 are known real functions defined on $]a,b[$[2], $\mathbf{x} = (x_1, x_2, x_3) \in \mathbf{R}^3$, $r = |\mathbf{x}|$ and assume that ρ, $\mathbf{v}$, $\boldsymbol{\omega}$, and θ are spherically symmetric too:

$$v_i(\mathbf{x}, t) = \frac{x_i}{r} v(r,t),\ \omega_i(\mathbf{x}, t) = \frac{x_i}{r}\omega(r,t), i = 1,2,3, \tag{17}$$

$$\rho(\mathbf{x}, t) = \rho(r,t),\ \theta(\mathbf{x}, t) = \theta(r,t). \tag{18}$$

Using the assumptions (17) and (18), the spatial domain (15) becomes a one-dimensional domain $]a,b[$. The governing system now takes the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial r}(v\rho) + \frac{2\rho}{r} v = 0, \tag{19}$$

$$\rho\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r}\right) = -R\frac{\partial}{\partial r}(\rho\theta) + (\lambda + 2\mu)\frac{\partial}{\partial r}\left(\frac{\partial v}{\partial r} + 2\frac{v}{r}\right), \tag{20}$$

$$\rho j_I\left(\frac{\partial \omega}{\partial t} + v\frac{\partial \omega}{\partial r}\right) = -4\mu_r\omega + (c_0 + 2c_d)\frac{\partial}{\partial r}\left(\frac{\partial \omega}{\partial r} + 2\frac{\omega}{r}\right), \tag{21}$$

$$\begin{aligned}\rho c_v\left(\frac{\partial \theta}{\partial t} + v\frac{\partial \theta}{\partial r}\right) &= k\left(\frac{\partial^2\theta}{\partial r^2} + \frac{2}{r}\frac{\partial \theta}{\partial r}\right) - R\rho\theta\left(\frac{\partial v}{\partial r} + 2\frac{v}{r}\right)\\ &\quad + (\lambda + 2\mu)\left(\frac{\partial v}{\partial r} + 2\frac{v}{r}\right)^2 - 4\mu\frac{v}{r}\left(2\frac{\partial v}{\partial r} + \frac{v}{r}\right)\\ &\quad + (c_0 + 2c_d)\left(\frac{\partial \omega}{\partial r} + 2\frac{\omega}{r}\right)^2 - 4c_d\frac{\omega}{r}\left(2\frac{\partial \omega}{\partial r} + \frac{\omega}{r}\right) + 4\mu_r\omega^2,\end{aligned} \tag{22}$$

$$\rho(r,0) = \rho_0(r),\ v(r,0) = v_0(r),\ \omega(r,0) = \omega_0(r),\ \theta(r,0) = \theta_0(r), \tag{23}$$

$$v(a,t) = v(b,t) = 0,\ \omega(a,t) = \omega(b,t) = 0,\ \frac{\partial \theta}{\partial r}(a,t) = \frac{\partial \theta}{\partial r}(b,t) = 0, \tag{24}$$

for $r \in]a,b[$ and $t \in]0,T[$.

[2] a and b are the radii of boundary spheres from (15).

In the mathematical analysis of compressible fluids, it is convenient to use Lagrangian description. The Eulerian coordinates (r, t) are connected to the Lagrangian coordinates (ξ, t) by the relation

$$r(\xi, t) = r_0(\xi) + \int_0^t \tilde{v}(\xi, t)d\tau, \quad r_0(\xi) = r(\xi, 0) = \xi, \tag{25}$$

where $\tilde{v}(\xi, t)$ is defined by $\tilde{v}(\xi, t) = v\,(r(\xi, t), t)$.

We introduce the new function η and the constant L by

$$\eta(\xi) = \int_a^\xi s^2\rho_0(s)ds, \quad \eta(b) = \int_a^b s^2\rho_0(s)ds = L \tag{26}$$

and define the new coordinate, $x = L^{-1}\eta(\xi)$. With this new coordinate the spatial domain becomes $]0, 1[$, and we finally get the following initial-boundary problem:

$$\frac{\partial\rho}{\partial t} = -\frac{1}{L}\rho^2\frac{\partial}{\partial x}\left(r^2 v\right), \tag{27}$$

$$\frac{\partial v}{\partial t} = -\frac{R}{L}r^2\frac{\partial}{\partial x}(\rho\theta) + \frac{\lambda + 2\mu}{L^2}r^2\frac{\partial}{\partial x}\left(\rho\frac{\partial}{\partial x}\left(r^2 v\right)\right), \tag{28}$$

$$\rho\frac{\partial\omega}{\partial t} = -\frac{4\mu_r}{j_I}\omega + \frac{c_0 + 2c_d}{j_I L^2}r^2\rho\frac{\partial}{\partial x}\left(\rho\frac{\partial}{\partial x}\left(r^2\omega\right)\right), \tag{29}$$

$$\begin{aligned}\rho\frac{\partial\theta}{\partial t} &= \frac{k}{c_v L^2}\rho\frac{\partial}{\partial x}\left(r^4\rho\frac{\partial\theta}{\partial x}\right) - \frac{R}{c_v L}\rho^2\theta\frac{\partial}{\partial x}\left(r^2 v\right) + \frac{\lambda + 2\mu}{c_v L^2}\left[\rho\frac{\partial}{\partial x}\left(r^2 v\right)\right]^2 \\ &\quad -\frac{4\mu}{c_v L}\rho\frac{\partial}{\partial x}\left(r v^2\right) + \frac{c_0+2c_d}{c_v L^2}\left[\rho\frac{\partial}{\partial x}\left(r^2\omega\right)\right]^2 - \frac{4c_d}{c_v L}\rho\frac{\partial}{\partial x}\left(r\omega^2\right) + \frac{4\mu_r}{c_v}\omega^2,\end{aligned} \tag{30}$$

$$\rho(x, 0) = \rho_0(x), \quad v(x, 0) = v_0(x), \quad \omega(x, 0) = \omega_0(x), \quad \theta(x, 0) = \theta_0(x), \tag{31}$$

$$v(0, t) = v(1, t) = 0, \quad \omega(0, t) = \omega(1, t) = 0, \quad \frac{\partial\theta}{\partial x}(0, t) = \frac{\partial\theta}{\partial x}(1, t) = 0, \tag{32}$$

considered on the domain $Q_T =]0, 1[\times]0, T[$.

The function $r(x, t)$ is defined by

$$r(x, t) = r_0(x) + \int_0^t v(x, \tau)d\tau, \ (x, t) \in Q_T. \tag{33}$$

where

$$r_0(x) = \left(a^3 + 3L\int_0^x \frac{1}{\rho_0(y)}dy\right)^{\frac{1}{3}}, \ x \in]0, 1[\tag{34}$$

and $a > 0$ is a radius of smaller boundary sphere.

3 Properties of the Solution

In this section we consider the properties of the so-called generalized solution to the problem (27)–(32).

Definition 1. A generalized solution to the problem (27)–(32) in the domain Q_T is a function

$$(x,t) \mapsto (\rho, v, \omega, \theta)(x,t), \quad (x,t) \in Q_T, \tag{35}$$

where

$$\rho \in \mathrm{L}^\infty(0,T;\mathrm{H}^1(]0,1[)) \cap \mathrm{H}^1(Q_T)\ ,\ \inf_{Q_T} \rho > 0\ , \tag{36}$$

$$v, \omega, \theta \in \mathrm{L}^\infty(0,T;\mathrm{H}^1(]0,1[)) \cap \mathrm{H}^1(Q_T) \cap \mathrm{L}^2(0,T;\mathrm{H}^2(]0,1[)), \tag{37}$$

that satisfies the Eqs. (27)–(30) a.e. in Q_T and conditions (31)–(32) in the sense of traces.

Let us mention that by using the embedding and interpolation theorems, one can conclude that our generalized solution could be treated as a strong solution. In fact, we have

$$\rho \in \mathrm{L}^\infty(0,T;\mathrm{C}([0,1])) \cap \mathrm{C}([0,T],\mathrm{L}^2(]0,1[))\ , \tag{38}$$

$$v, \omega, \theta \in \mathrm{L}^2(0,T;\mathrm{C}^1([0,1])) \cap \mathrm{C}([0,T],\mathrm{H}^1(]0,1[)), \tag{39}$$

$$v, \omega, \theta \in \mathrm{C}(\overline{Q}_T). \tag{40}$$

We first analyzed the existence of the generalized solution to the problem (27)–(32). Using the Faedo–Galerkin method, we proved in [1] the existence locally in time. After that we analyzed the uniqueness of the solution in [9], and finally based on extension principle, we proved in [2] the global existence theorem for the problem (27)–(32). These results are summarized in the following theorem.

Theorem 1. *Let the functions* $\rho_0, \theta_0 \in \mathrm{H}^1(]0,1[)$, $v_0, \omega_0 \in \mathrm{H}_0^1(]0,1[)$ *satisfy the conditions*

$$\rho_0(x) \geq m, \quad \theta_0(x) \geq m \quad \text{for } x \in]0,1[\tag{41}$$

where $m \in \mathbf{R}^+$. *Then for any* $T \in \mathbf{R}^+$, *there exists a unique generalized solution to the problem (27)–(32) on the domain* Q_T *having the property*

$$\theta > 0 \ \ in\ \overline{Q}_T. \tag{42}$$

In the second stage of our research, we analyzed the large time behavior of the solution to the problem (27)–(32). Theorem 1 ensures the existence on the arbitrary but finite time interval $]0, T[$. Because of that we were not able to analyze the behavior of the solution when $t \to \infty$ and had to prove that the solution exists on the time interval $]0, \infty[$ as well, which is the purpose of next theorem obtained in [3].

Theorem 2. *Let the initial functions* ρ_0, v_0, ω_0, *and* θ_0 *satisfy the same conditions as in Theorem 1. Then the problem (27)–(32) has a solution on the domain* $Q_\infty =]0, 1[\times]0, \infty[$ *with the properties*

$$\rho \in \mathrm{L}^\infty(0, \infty; \mathrm{H}^1(]0, 1[)), \tag{43}$$

$$\frac{\partial \rho}{\partial t} \in \mathrm{L}^\infty(0, \infty; \mathrm{L}^2(]0, 1[)) \cap \mathrm{L}^2(Q_\infty), \tag{44}$$

$$\frac{\partial \rho}{\partial x} \in \mathrm{L}^2(0, \infty; \mathrm{L}^2(]0, 1[)), \tag{45}$$

$$v, \omega \in \mathrm{L}^\infty(0, \infty; \mathrm{H}^1(]0, 1[)) \cap \mathrm{H}^1(Q_\infty) \cap \mathrm{L}^2(0, \infty; \mathrm{H}^2(]0, 1[)), \tag{46}$$

$$\theta \in \mathrm{L}^\infty(0, \infty; \mathrm{H}^1(]0, 1[)), \tag{47}$$

$$\frac{\partial \theta}{\partial x} \in \mathrm{L}^2(0, \infty; \mathrm{H}^1(]0, 1[)), \tag{48}$$

$$\frac{\partial \theta}{\partial t} \in \mathrm{L}^2(Q_\infty). \tag{49}$$

In the following theorem, which is also proved in [3], we proved the stabilization of the solution when $t \to \infty$.

Theorem 3. *Let* $(\rho, v, \omega, \theta)$ *be a generalized solution to the problem (27)–(32) in the domain* Q_∞. *Then we have the convergence*

$$(\rho, v, \omega, \theta) \to (\rho^*, 0, 0, \theta^*) \tag{50}$$

in the space $\left(\mathrm{H}^1(]0, 1[)\right)^4$ *when* $t \to \infty$, *where*

$$\rho^* = \int_0^1 \frac{1}{\rho_0(x)} dx, \tag{51}$$

$$\theta^* = \frac{1}{c_v} \int_0^1 \left(c_v \theta_0(x) + \frac{1}{2} v_0^2 + \frac{j_I}{2} \omega_0^2 \right) dx. \tag{52}$$

For the function r, we have the convergence

$$r \to r^* \tag{53}$$

in the space $\mathrm{H}^2(]0,1[)$ *when* $t \to \infty$ *where*

$$r^*(x) = \left(a^3 + 3xu^*\right)^{\frac{1}{3}}, \quad x \in [0,1], \tag{54}$$

and the constant a is the same as in (34).

The proof of Theorem 2 is based on a series of uniform-in-time a priory estimates and the proof of Theorem 3 on the results Theorem 2 and application of Friedrichs and Poincare inequalities.

Recently, in [5], Huang and Lian deduced the nature of the convergences (50) and (53). They showed that the solution $(\rho, v, \omega, \theta)$ decays to a constant state with exponential rate. Their result is stated in the next theorem.

Theorem 4. *Let* $(\rho, v, \omega, \theta)$ *be a generalized solution to the problem (27)–(32) in the domain* Q_∞*. Then there exist constants* $C_1 > 0$ *and* $\gamma_1 = \gamma_1(C) > 0$ *such that for any fixed* $\gamma \in]0, \gamma_1]$ *and for any* $t > 0$*, we have*

$$\begin{aligned} e^{\gamma t} & \left(\|\rho(t) - \rho^*\|_{H^1}^2 + \|v(t)\|_{H^1}^2 + \|\omega(t)\|_{H^1}^2 + \|\theta(t) - \theta^*\|_{H^1}^2\right) \\ & + \int_0^t e^{\gamma s} \left(\|\rho_x\|_{H^1}^2 + \|v_x\|_{H^1}^2 + \|\omega_x\|_{H^1}^2 + \|\theta_x\|_{H^1}^2 \right. \\ & \left. + \|v_t\|^2 + \|\omega_t\|^2 + \|\theta_t\|^2\right)(s) ds \le C_1, \end{aligned} \tag{55}$$

where ρ^* *and* θ^* *are defined by (51) and (52).*

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Nonoscillatory Solutions of the Four-Dimensional Neutral Difference System

Jana Krejčová

Abstract We study nonoscillatory solutions of four-dimensional nonlinear neutral difference systems. We state asymptotic properties of solutions, and we establish sufficient conditions for the system to have weak property B and property B.

Keywords Difference systems • Weak property B • Non-oscillatory solutions • Asymptotic properties

Mathematics Subject Classification (2000): 39A10

1 Introduction

In this paper, we study asymptotic behavior of solutions of a four-dimensional system:

$$\begin{aligned} \Delta\left(x_n + p_n x_{n-\sigma}\right) &= A_n f_1\left(y_n\right) \\ \Delta y_n &= B_n f_2\left(z_n\right) \\ \Delta z_n &= C_n f_3\left(w_n\right) \\ \Delta w_n &= D_n f_4\left(x_{\gamma_n}\right), \end{aligned} \qquad \text{(S)}$$

where $n \in \mathbb{N}_0 = \{n_0, n_0 + 1, \ldots\}$, n_0 is a positive integer, σ is a nonnegative integer, and $\{A_n\}, \{B_n\}, \{C_n\}, \{D_n\}$ are positive real sequences defined for $n \in \mathbb{N}_0$. Δ is the forward difference operator given by $\Delta x_n = x_{n+1} - x_n$.

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The sequence $\gamma : \mathbb{N} \to \mathbb{N}$ satisfies

$$\lim_{n\to\infty} \gamma_n = \infty. \tag{H1}$$

The most common form of this sequence is $\gamma = n \pm \tau$, where $\tau \in \mathbb{N}$. The sequence $\{p_n\}$ is a sequence of the real numbers and it satisfies $0 \le p_n < 1$. Functions $f_i : \mathbb{R} \to \mathbb{R}$ for $i = 1, .., 4$ satisfy

$$\frac{f_i(u)}{u} \ge 1,\ u \in \mathbb{R}\backslash 0. \tag{H2}$$

This property implies that u and $f_i(u)$ have the same sign for $i = 1, .., 4$ and $u \in \mathbb{R}\backslash 0$. Throughout our paper we assume that the system (S) is in the canonical form, which means that the following conditions hold

$$\sum_{n=n_0}^{\infty} A_n = \infty, \quad \sum_{n=n_0}^{\infty} B_n = \infty, \quad \sum_{n=n_0}^{\infty} C_n = \infty. \tag{H3}$$

By a solution of the system (S), we mean a vector sequence (x, y, z, w) which satisfies the system (S) for $n \in \mathbb{N}_0$. We investigate nonoscillatory solutions in this paper. Therefore, the first important thing is to divide solutions into oscillatory and nonoscillatory.

The component x is said to be nonoscillatory if it is either eventually positive or eventually negative. The non-oscillation of the components y, z, w is defined by the same way. A solution of the system (S) is said to be *nonoscillatory* if all of its components x, y, z, w are nonoscillatory. Otherwise, a solution is said to be *oscillatory*.

Another important property is the boundedness. A solution of the system (S) is said to be *bounded* if all of its components x, y, z, w are bounded. Otherwise, a solution is said to be *unbounded*.

Definition 1. The system (S) has **weak property B** if every nonoscillatory solution of (S) satisfies

$$x_n z_n > 0 \quad \text{and } y_n w_n > 0 \quad \text{for large } n. \tag{1}$$

Definition 2. The system (S) has **property B** if any nonoscillatory solution of (S) satisfies either

$$\lim_{n\to\infty} |x_n| = \lim_{n\to\infty} |y_n| = \lim_{n\to\infty} |z_n| = \lim_{n\to\infty} |w_n| = \infty, \tag{2}$$

or

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n = \lim_{n\to\infty} w_n = 0. \tag{3}$$

Solutions satisfying (1) and $x_n y_n > 0$ are called *strongly monotone solutions*, while solutions satisfying (1) and $x_n y_n < 0$ are called *Kneser solutions*. Hence, weak property B means that any nonoscillatory solution is either Kneser or strongly monotone solutions and property B means that these solutions are either unbounded or vanishing at infinity in all their components. Property B is defined in accordance with those for the higher-order differential equations or for the system of differential equations; see [7] and references therein.

In the last few years, great attention has been paid to the study of neutral difference equations. The system (S) is a prototype of even-order neutral systems and can be easily rewritten as a fourth-order nonlinear neutral difference equation. Equations with quasi-differences have been widely studied in the literature; see, for example, [2–6, 8, 9]. Oscillatory properties of solutions of the fourth-order difference equations are investigated in [6]. Their approach is based on studying the considered equation as a four-dimensional difference system, where $\{D_n\}$ is a negative real sequence. Asymptotic properties of neutral type difference equations can be found in [8]. The problem of boundedness of solutions of (S) with $\gamma_n = n-\tau$ has been investigated in the recent paper [1].

The aim of this paper is to investigate asymptotic behavior of nonoscillatory solutions of (S). We are motivated by the paper [2], where asymptotic properties of (S) with $\{p_n\} = \{0\}$ have been investigated. We extend results from [1] and [2]. We give sufficient conditions that (S) has weak property B and property B. This completes the results from [6], where they study property A.

2 Preliminaries

First, we point out some basic properties of (S) which we use to prove the main results of the paper.

Throughout our article, we use the notation

$$s_n = x_n + p_n x_{n-\sigma}, \tag{4}$$

where $n \in \mathbb{N}_0$.

Lemma 1. *Let $\{x_n\}$ and $\{p_n\}$ be real sequences, where $n \in \mathbb{N}_0$ and p_n satisfy $0 \leq p_n < 1$. Let $\{s_n\}$ be the sequence defined by (4). Then $\{x_n\}$ is bounded if and only if $\{s_n\}$ is bounded.*

Moreover, if s is positively increasing for large n, then for large n

$$x_n = s_n - p_n x_{n-\sigma} \geq s_n - p_n s_{n-\sigma} \geq s_{n-\sigma}(1 - p_n). \tag{5}$$

Proof. By (4) and the fact $p_n \geq 0$, we get the equivalency between the boundedness of x and s.

The second statement follows from the fact $x_n \leq s_n$ and $s_{n-\sigma} \leq s_n$.

Lemma 2 ([6, Lemma 1]). *Assume* $\lim p_n = P$, $0 < P < 1$. *If x is bounded and*

$$\lim s_n = S \in \mathbb{R},$$

then x is convergent and

$$\lim x_n = \frac{S}{1+P}.$$

In particular, if s tends to zero as $n \to \infty$, then x tends to zero as $n \to \infty$, too.

If the system (S) has a solution (x, y, z, w), then it has the solution $(-x, -y, -z, -w)$ as well. Thus, throughout the paper, we can focus on solutions whose first component is eventually positive for large n.

The following lemma describes the possible types of nonoscillatory solutions.

Lemma 3. *Any nonoscillatory solution (x, y, z, w) of the system (S) with eventually positive x is one of the following types:*

$$\begin{aligned}
&\textit{type } (a)\ x_n > 0\ y_n > 0\ z_n > 0\ w_n > 0 \textit{ for large } n,\\
&\textit{type } (b)\ x_n > 0\ y_n > 0\ z_n > 0\ w_n < 0 \textit{ for large } n,\\
&\textit{type } (c)\ x_n > 0\ y_n < 0\ z_n > 0\ w_n < 0 \textit{ for large } n.
\end{aligned}$$

Proof. Let (x, y, z, w) be a nonoscillatory solution of (S) such that $x_n > 0$ for large n. There are eight possible types of these solutions. We prove that solutions of the following types do not exist:

$$\begin{aligned}
&\text{type (i)} \quad x_n > 0\ y_n > 0\ z_n < 0\ w_n > 0 \text{ for large n},\\
&\text{type (ii)} \quad x_n > 0\ y_n < 0\ z_n < 0\ w_n > 0 \text{ for large n},\\
&\text{type (iii)} \ x_n > 0\ y_n < 0\ z_n < 0\ w_n < 0 \text{ for large n},\\
&\text{type (iv)} \ x_n > 0\ y_n > 0\ z_n < 0\ w_n < 0 \text{ for large n},\\
&\text{type (v)} \quad x_n > 0\ y_n < 0\ z_n > 0\ w_n > 0 \text{ for large n}.
\end{aligned}$$

First, assume that there exist $n_1 \in \mathbb{N}_0$ and a solution such that $z_n < 0$, $w_n > 0$ for $n \geq n_1 \geq n_0$. From the fourth equation of (S), we have $\Delta w_n > 0$ and this implies that there exists $k > 0$ such that $w_n \geq k$ for large n. Using (H2) we have $f_3(w_n) \geq w_n \geq k$. By the summation of the third equation of (S), we have

$$z_n - z_{n_0} = \sum_{i=n_0}^{n-1} C_i f_3(w_i) \geq k \sum_{i=n_0}^{n-1} C_i.$$

Passing $n \to \infty$, we get a contradiction with the fact that $z_n < 0$. This excludes solutions of types (i) and (ii).

Let us suppose that there exists solution such that $x_n > 0$, $y_n < 0$, $z_n < 0$ for large n. By (4) s is positive and the first equation of (S) implies that s is decreasing for large n. Therefore, s is bounded. Since y is eventually negatively decreasing, there exists $h < 0$ such that $y_n \leq h$. Using (H2) we have $f_1(y_n) \leq y_n \leq h$. Using the summation of the first equation of (S) and passing $n \to \infty$, we get a contradiction with the boundedness of s. Therefore, a solution of type (iii) cannot exist.

Assume that there exists a solution of type (iv). Therefore, we have $z_n < 0$ and z is decreasing for all large n. This implies that there exists $l < 0$ such that $z_n \leq l$ for large n. From (H2) we get $f_2(z_n) \leq z_n \leq l$. By the summation of the second equation of (S) and passing $n \to \infty$, we get a contradiction with the positivity of y.

Finally, assume that there exists a solution of type (v). Therefore, we have $z_n > 0$ and z is increasing for all large n. This implies that there exists $g > 0$ such that $z_n \geq g$ for large n. From (H2) we get $f_2(z_n) \geq z_n \geq g$. Using the summation of the second equation of (S), we get a contradiction with negativity of y. Thus, solutions of type (v) cannot exist.

By Definition 1, the system (S) has weak property B if there exist only solutions of type (a) and (c). Solutions of type (a) are called strongly monotone and solutions of type (c) are called Kneser solutions. We have to determine some asymptotic properties of these solutions for the purpose of investigation property B. Properties of strongly monotone solutions are summarized in the following lemma.

Lemma 4. *Any solution of type (a) satisfies*

$$\lim_{n\to\infty} x_n = \infty, \lim_{n\to\infty} y_n = \infty, \lim_{n\to\infty} z_n = \infty. \tag{6}$$

Proof. Let (x, y, z, w) be a solution of type (a). Because y is positive and increasing, there exists $k > 0$ such that $y_n \geq k$ for large n. By the summation of the first equation of (S), we get

$$s_n - s_{n_0} = \sum_{i=n_0}^{n-1} A_i f_1(y_i) \geq \sum_{i=n_0}^{n-1} A_i y_i \geq k \sum_{i=n_0}^{n-1} A_i.$$

Passing $n \to \infty$ we get $s_n \to \infty$. Lemma 1 implies that s is unbounded if and only if x is unbounded. Therefore $\lim_{n\to\infty} x_n = \infty$.

To prove the other statements, we use similar arguments. Because z is eventually positively increasing, there exists $h > 0$ such that $z_n \geq h$ for large n. Using the summation of the second equation of (S), we get $y_n \to \infty$ for $n \to \infty$.

Finally, w is eventually positively increasing; thus, there exists $l > 0$ such that $w_n \geq l$ for large n. By the summation of the third equation of (S), we obtain that $z_n \to \infty$ for $n \to \infty$.

The following lemma summarizes properties of Kneser solutions.

Lemma 5. *Any solution of type (c) satisfies*

$$\lim_{n\to\infty} y_n = 0, \lim_{n\to\infty} z_n = 0, \lim_{n\to\infty} w_n = 0. \tag{7}$$

Proof. Assume that the solution (x, y, z, w) is of type (c). Since y is negative and increasing, there exists $\lim_{n\to\infty} y_n = k$, $k \leq 0$. If $k < 0$, then from the summation of the first equation of (S), we get $s_n \to -\infty$ as $n \to \infty$, which is a contradiction with the boundedness of s. Therefore $\lim_{n\to\infty} y_n = 0$.

Since z is eventually positively decreasing, there exists $\lim_{n\to\infty} z_n = h \geq 0$. Suppose $h > 0$. By the summation of the second equation of (S), we obtain a contradiction with the negativity of y. Thus, $\lim_{n\to\infty} z_n = 0$.

Similarly, since w is eventually negatively increasing, there exists $\lim_{n\to\infty} w_n = l \leq 0$. Suppose $l < 0$. By the summation of the third equation of (S), we obtain a contradiction with the positivity of z. Therefore $\lim_{n\to\infty} w_n = 0$.

Now, we can continue to state sufficient conditions for the system (S) to have weak property B and property B.

3 Property B

The first theorem gives the simple criterion that system (S) has property B.

Theorem 1. *If*

$$\sum_{n=n_0}^{\infty} D_n = \infty \tag{8}$$

holds, then the system (S) has property B.

Proof. By Lemma 3, there are only three possible types of nonoscillatory solutions. Assume that (x, y, z, w) is a type (b) solution. Since x is positive, then s is positive and from the first equation of (S) we get that s is increasing. Therefore, by Lemma 1 there exists a real constant $k > 0$ such that $x_n \geq k$ for large n. By the summation of the fourth equation of (S), we get

$$w_n - w_{n_0} = \sum_{i=n_0}^{n-1} D_i f_4\left(x_{\gamma_i}\right) \geq \sum_{i=n_0}^{n-1} D_i x_{\gamma_i} \geq k \sum_{i=n_0}^{n-1} D_i. \tag{9}$$

Passing $n \to \infty$ we get the contradiction with the negativity of w. Thus, the system (S) does not have solutions of type (b).

If (x, y, z, w) is a solution of type (a), then using the same argument as in the previous and by (9), we get $w_n \to \infty$ for $n \to \infty$. From this fact and Lemma 4, we get that all solutions of type (a) satisfy (2).

If (x, y, z, w) is a solution of type (c), then there exists $\lim_{n\to\infty} x_n = h$, $h \geq 0$. If we assume $h > 0$, then by the summation of the fourth equation of (S), we get a contradiction with the negativity of w. Therefore, $\lim_{n\to\infty} x_n = 0$. From that fact and Lemma 5, we get that all solutions of type (c) satisfy (3).

Now, we get the assertion by Definition 2.

In view of Theorem 1, in the sequel, we assume $\sum_{n=n_0}^{\infty} D_n < \infty$.

In the following, we state sufficient conditions for the system (S) to have weak property B and property B. To ensure these properties, we have to exclude solutions of type (b). The following theorem gives a condition for the nonexistence of solutions of type (b).

Theorem 2. *If*

$$\sum_{i=n_0}^{\infty} D_i \left(\sum_{j=n_0}^{i-1} C_j \right) = \infty \tag{10}$$

holds, then the system (S) has weak property B.

Proof. By Definition 1, weak property B means that there exist only solutions of types (a) and (c). Assume that (x, y, z, w) is a type (b) solution. Since x is positive and s is positive and increasing, by Lemma 1 there exists $k > 0$ such that $x_n \geq k$ for large n. By the summation of the fourth equation of (S), we get

$$w_\infty - w_n = \sum_{i=n}^{\infty} D_i f_4 \left(x_{\gamma_i} \right) \geq \sum_{i=n}^{\infty} D_i x_{\gamma_i} \geq k \sum_{i=n}^{\infty} D_i, \tag{11}$$

$$-w_n \geq k \sum_{i=n}^{\infty} D_i.$$

Using the summation of the third equation of (S), we have

$$z_n - z_{n_0} = \sum_{i=n_0}^{n-1} C_i f_3 \left(w_i \right) \leq \sum_{i=n_0}^{n-1} C_i w_i,$$

$$-z_n + z_{n_0} \geq \sum_{i=n_0}^{n-1} C_i \left(-w_i \right) \geq k \sum_{i=n_0}^{n-1} C_i \left(\sum_{j=i}^{\infty} D_j \right).$$

Passing $n \to \infty$ and using the change of summation

$$\sum_{i=n_0}^{\infty} C_i \left(\sum_{j=i}^{\infty} D_j \right) = \sum_{i=n_0}^{\infty} D_i \left(\sum_{j=n_0}^{i-1} C_j \right) = \infty,$$

we get the contradiction with the boundedness of z. Thus, solutions of type (b) do not exist and (S) has weak property B.

Theorem 3. *Assume* $\lim p_n = P$, $0 < P < 1$ *and*

$$\sum_{i=n_0}^{\infty} D_i \left(\sum_{j=n_0}^{\gamma_i-\sigma-1} A_j \left(\sum_{k=n_0}^{j-1} B_k \left(\sum_{l=n_0}^{k-1} C_l \right) \right) \right) = \infty. \tag{12}$$

In addition, if (10) *holds, then the system (S) has property B.*

Proof. By Theorem 2, the system (S) has only solutions of types (a) and (c). Let (x, y, z, w) be a solution of type (a). Thus, w is positively increasing, and there exists a constant $t > 0$ such that $w_n \geq t$ for large n. From the third equation of (S), we get

$$z_n \geq \sum_{i=n_0}^{n-1} C_i f_3(w_i) \geq \sum_{i=n_0}^{n-1} C_i w_i \geq t \sum_{i=n_0}^{n-1} C_i.$$

Substituting this into the second equation of (S), we obtain

$$y_n \geq \sum_{i=n_0}^{n-1} B_i f_2(z_i) \geq \sum_{i=n_0}^{n-1} B_i z_i \geq t \sum_{i=n_0}^{n-1} B_i \left(\sum_{j=n_0}^{i-1} C_j \right).$$

Using the first equation of (S), we have

$$s_n \geq \sum_{i=n_0}^{n-1} A_i f_1(y_i) \geq t \sum_{i=n_0}^{n-1} A_i \left(\sum_{j=n_0}^{i-1} B_j \left(\sum_{k=n_0}^{j-1} C_k \right) \right). \tag{13}$$

Since s is positively increasing, by Lemma 1 the inequality (5) holds. Taking into account $\lim(1 - p_n) = 1 - P > 0$, there exists $p > 0$ such that $1 - p_n \geq p$; therefore,

$$x_n \geq s_{n-\sigma}(1 - p_n) \geq p\, s_{n-\sigma}.$$

Substituting (13) into the fourth equation of (S), we get

$$w_n \geq \sum_{i=n_0}^{n-1} D_i x_{\gamma_i} \geq p\, t \sum_{i=n_0}^{n-1} D_i \left(\sum_{j=n_0}^{\gamma_i-\sigma-1} A_j \left(\sum_{k=n_0}^{j-1} B_k \left(\sum_{l=n_0}^{k-1} C_l \right) \right) \right).$$

Passing $n \to \infty$ we get $w_n \to \infty$. From here and Lemma 4, solution of type (a) satisfies (2).

Now assume that (x, y, z, w) is a solution of type (c). By Lemma 5, we have to prove that $\lim_{n\to\infty} x_n = 0$. Because s is positive and decreasing, it is bounded.

Thus, there exists $\lim_{n\to\infty} s_n = S$, $S \geq 0$. By Lemma 2, there exists $\lim_{n\to\infty} x_n = S/(1+P)$. Put $h = S/(1+P)$. Assume $S > 0$. Then $h > 0$. Using Lemma 5, (H2), and the summation from n to infinity of the fourth equation of (S), we obtain

$$-w_n \geq h \sum_{i=n}^{\infty} D_i.$$

Substituting and using the summation of the third equation of (S), we have

$$-z_n + z_{n_0} \geq \sum_{j=n_0}^{n-1} C_j(-w_j) \geq h \sum_{j=n_0}^{n-1} C_j \left(\sum_{i=j}^{\infty} D_i \right).$$

Passing $n \to \infty$ and using the change of summation, we get that (10) implies $\sum_{j=n_0}^{\infty} C_j \left(\sum_{i=j}^{\infty} D_i \right) = \infty$. This leads to the contradiction with the boundedness of z. Therefore, $h = 0$.

Now, we get the assertion by Definition 2.

Example 1. Consider the system:

$$\begin{aligned} \Delta\left(x_n + P x_{n-\sigma}\right) &= y_n \\ \Delta y_n &= z_n \\ \Delta z_n &= w_n \\ \Delta w_n &= n^{(-2)} f_4\left(x_{n+\tau}\right), \end{aligned} \tag{14}$$

where $0 < P < 1$, $\tau \in \mathbb{N}$.

The system is in the canonical form. We apply conditions from Theorem 3:

$$\sum_{i=n_0}^{\infty} i^{(-2)} \left(\sum_{j=n_0}^{i+\tau-\sigma-1} \left(\sum_{k=n_0}^{j-1} \left(\sum_{l=n_0}^{k-1} 1 \right) \right) \right) = \infty,$$

$$\sum_{i=n_0}^{\infty} i^{(-2)} \left(\sum_{j=n_0}^{i-1} 1 \right) = \infty.$$

They are all satisfied. Thus, the system (14) has property B.

4 Concluding Remarks

Results of this paper generalized results in [2] and make a motion to study the system (S), for example, when (S) is not in the canonical form.

Another interesting problem is to study (S) with $-1 < p_n < 0$. In this case, the problem is existence of unbounded solution of type $x_n > 0$, $y_n < 0$, $z_n < 0$, $w_n < 0$ for large n.

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Comparison Theorems for Weighted Focal Points of Conjoined Bases of Hamiltonian Difference Systems

Julia Elyseeva

Abstract In this paper we prove comparison theorems for the number of weighted focal points of conjoined bases of Hamiltonian difference systems. The notion of a weighted focal point introduced by O. Došlý and J. Elyseeva (Appl. Math. Lett. (43) 2015, 114–119) plays an important role in spectral theory for discrete Hamiltonian eigenvalue problems with nonlinear dependence on the spectral parameter. We present new relations between the numbers of weighted focal points of conjoined bases of two Hamiltonian systems and derive corollaries to these relations generalizing comparison results for the classical number of focal points. The consideration is based on the comparative index theory for symplectic difference systems.

Keywords Hamiltonian difference systems • Weighted focal point • Discrete Sturmian theory • Comparative index

Mathematics Subject Classification (2000): 39A21, 39A22

1 Introduction

We consider the discrete Hamiltonian systems [3, 15]

$$\Delta x_{k+1} = A_k x_{k+1} + B_k u_k, \quad \Delta u_k = C_k x_{k+1} - A_k^T u_k, \quad \det (I - A_k) \neq 0, \tag{1}$$

$$\Delta \hat{x}_{k+1} = \hat{A}_k \hat{x}_{k+1} + \hat{B}_k \hat{u}_k, \quad \Delta \hat{u}_k = \hat{C}_k \hat{x}_{k+1} - \hat{A}_k^T \hat{u}_k, \quad \det \left(I - \hat{A}_k\right) \neq 0, \tag{2}$$

$$k = 0, 1, \dots, N,$$

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associated with the Hamiltonians

$$\mathscr{H}_k = \mathscr{H}_k^T, \quad \mathscr{H}_k = \begin{pmatrix} -C_k & A_k^T \\ A_k & B_k \end{pmatrix}, \quad \hat{\mathscr{H}}_k = \hat{\mathscr{H}}_k^T, \quad \hat{\mathscr{H}}_k = \begin{pmatrix} -\hat{C}_k & \hat{A}_k^T \\ \hat{A}_k & \hat{B}_k \end{pmatrix}, \tag{3}$$

where $A_k,\ B_k,\ C_k,\ \hat{A}_k,\ \hat{B}_k,\ \hat{C}_k \in \mathbb{R}^{n\times n}$. System (1) is the most important special case of the discrete symplectic systems

$$y_{k+1} = \mathscr{S}_k y_k, \quad y_k = \begin{pmatrix} x_k \\ u_k \end{pmatrix} \in \mathbb{R}^{2n}, \quad k = 0, 1, \dots, N. \tag{4}$$

The matrix $\mathscr{S}_k$ with the blocks $\mathscr{A}_k,\ \mathscr{B}_k,\ \mathscr{C}_k,\ \mathscr{D}_k \in \mathbb{R}^{n\times n}$ is symplectic, i.e.,

$$\mathscr{S}_k = \begin{pmatrix} \mathscr{A}_k & \mathscr{B}_k \\ \mathscr{C}_k & \mathscr{D}_k \end{pmatrix}, \quad \mathscr{S}_k^T J \mathscr{S}_k = J, \quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad k = 0, 1, \dots, N, \tag{5}$$

I being the $n \times n$ identity matrix. For Hamiltonian system (1), this matrix has the form

$$\mathscr{S}_k = \begin{pmatrix} (I - A_k)^{-1} & (I - A_k)^{-1} B_k \\ C_k (I - A_k)^{-1} & C_k (I - A_k)^{-1} B_k + I - A_k^T \end{pmatrix}. \tag{6}$$

In this paper, we derive comparison results for the number of *weighted* focal points of conjoined bases of (1) and (2). This new notion was introduced in [8] for discrete eigenvalue problems for the Hamiltonian difference systems with nonlinear dependence on spectral parameter. The notion of a weighted focal point coincides with the classical notion of focal points [3, 4, 16] in the case when the symmetric matrix B_k in (1) is nonnegative definite, i.e., $B_k \geq 0$. In the general case, $\operatorname{ind}(B_{k)} \neq 0$ (here $\operatorname{ind} A$ is the number of negative eigenvalues of $A = A^T$); the number of weighted focal points is closely related to the notions of *weighted nodes* (for $n = 1$) and *relative oscillation numbers* (for $n > 1$) introduced for the Wronskians of solutions of the scalar and matrix difference Sturm–Liouville equations (see [1, 2, 14]). For example, for the discrete Sturm-Liouville equation

$$\Delta(r_k \Delta x_k) - q_k x_{k+1} = 0,\ r_k \neq 0,\ k = 0, \dots, N-1 \tag{7}$$

(which can be rewritten in form (1)) the number of (forward) weighted focal points is defined as follows (see [8, Example 2.2])

$$\#(x_k) = \begin{cases} 1, & r_k > 0,\ x_k \neq 0,\ x_k x_{k+1} \leq 0, \\ -1, & r_k < 0,\ x_{k+1} \neq 0,\ x_k x_{k+1} \leq 0\ , \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

Then, for the case $r_k > 0$, the quantity $\#(x_k) \geq 0$ coincides with the multiplicity of a generalized zero of x_k and takes the values from the set $\{0, 1\}$. For arbitrary $r_k \neq 0$, we have by (8) that any strictly positive (negative) solution x_k of Eq. (7) is weighted nonoscillatory, i.e., $\#(x_k) = 0,\ k \geq 0$. Note that for any other nontrivial solution $\hat{x}_k$ of (7), in this case we have the estimate $|\#(\hat{x}, 0, N)| \leq 1$, $\#(\hat{x}, 0, N) = \sum_{k=0}^{N} \#(\hat{x}_k)$ (see the next section), and then for the total number $l(\hat{x}, 0, N)$ of generalized zeros in $(0, N+1]$, we derive $\sum_{k=0}^{N} \operatorname{ind}(r_k) - 1 \leq l(\hat{x}, 0, N) \leq \sum_{k=0}^{N} \operatorname{ind}(r_k) + 1$, where $\operatorname{ind}(r_k) = 1$ for $r_k < 0$ and $\operatorname{ind}(r_k) = 0,\ r_k > 0$. For example, this estimate holds for any nontrivial solution of the Fibonacci sequence $x_{k+2} = x_{k+1} + x_k$ rewritten in form (7) with $r_k = (-1)^k$. So we see that the notion of a weighted focal point can be useful in the investigation of the oscillatory behavior of conjoined bases of (1) with respect to the behavior of $\sum \operatorname{ind}(B_k)$.

In this paper, we present analogs of classical comparison theorems [5–7, 11] for weighted focal points (see Sect. 3). So we prove an analog of [11, Theorem 2.1] presenting relations between the number of weighted focal points of conjoined bases of (1) and (2) (see Theorem 1). Then we derive corollaries to Theorem 1 based on the modified majorant condition $\mathscr{H}_k \geq \hat{\mathscr{H}}_k$ for the discrete Hamiltonians (3). Note that this condition coincides with the classical one for discrete Hamiltonian systems [5, 7] if and only if $\operatorname{ind}(\hat{B}_k) = \operatorname{ind}(B_k)$ (see Corollary 2 in the next section). In the general case, the classical majorant condition is not assumed to be satisfied in the results of Sect. 3.

2 Number of Weighted Focal Points

We will use the following notation. For a matrix A, we denote by A^T, A^{-1}, $A^{\dagger}$, $\operatorname{rank} A$ and $\operatorname{ind} A$, respectively, its transpose, inverse, Moore–Penrose pseudoinverse, rank (i.e., the dimension of its image), and index (i.e., the number of its negative eigenvalues). We use the notation $Sp(2m)$ for the group of symplectic matrices of the dimension $2m$, and we also use the notation ΔA_k for the forward difference operator $A_{k+1} - A_k$. By I and 0, we denote the identity and zero matrices of appropriate dimensions.

Recall now some basic concepts of the oscillation theory of symplectic difference systems (4) (see [4]).

A $2n \times n$ matrix solution $Y_k = \binom{X_k}{U_k}$ of (4) with $n \times n$ matrices $X_k,\ U_k$ is said to be a *conjoined basis* if

$$\operatorname{rank} Y_k = n \quad \text{and} \quad X_k^T U_k = U_k^T X_k \tag{9}$$

and the conjoined basis Y_k of (4) with the initial condition $Y_M = \binom{0}{I}$ at $k = M$ is said to be *the principal solution* at M.

The concept of multiplicity of a focal point of a conjoined basis Y_k of (4) was introduced in [16] as follows. Let

$$M_k = (I - X_{k+1}X_{k+1}^{\dagger})\mathscr{B}_k,\ T_k = I - M_k^{\dagger}M_k,\ P_k = T_k^T X_k X_{k+1}^{\dagger}\mathscr{B}_k T_k,$$

then obviously $M_kT_k = 0$, and it can be shown (see [16]) that the matrix P_k is symmetric. The *multiplicity* of a *forward focal point* of the conjoined basis Y_k in the interval $(k, k+1]$ is defined as the number

$$m(Y_k) := m_1(Y_k) + m_2(Y_k), \quad m_1(Y_k) := \operatorname{rank} M_k, \quad m_2(Y_k) := \operatorname{ind} P_k. \tag{10}$$

Based on the previous definition, we introduce the number of weighted focal points as follows (see [8]).

Definition 1. A conjoined basis Y_k of Hamiltonian difference system (1) has a *weighted (forward) focal point in* $(k, k+1]$ if $m(Y_k) - \operatorname{ind}(B_k) \neq 0$. In this case, the number of weighted (forward) focal points in $(k, k+1]$ is defined as

$$\#_k = \#(Y_k) := m(Y_k) - \operatorname{ind}(B_k), \tag{11}$$

where $m(Y_k)$ is given by (10).

Note that we have the estimate for $\#(Y_k)$ (see [8])

$$|\#(Y_k)| \leq \operatorname{rank} B_k \leq n.$$

Another important estimate for the number of weighted focal points of conjoined bases Y, $\hat{Y}$ of (1) follows from the separation result (see [10, Corollary 3.1] and [9]) $|l(Y, 0, N) - l(\hat{Y}, 0, N)| \leq \operatorname{rank} w(Y, \hat{Y}) \leq n$, $l(Y, 0, N) = \sum_{k=0}^{N} m(Y_k)$, where the Wronskian given by

$$w(Y, \hat{Y}) = Y^T J \hat{Y} \tag{12}$$

is constant for conjoined bases of Hamiltonian system (1) (see [4]). Using (11), we have the same estimate for weighted focal points of conjoined bases of (1)

$$|\#(Y, 0, N) - \#(\hat{Y}, 0, N)| \leq \operatorname{rank} w(Y, \hat{Y}) \leq n, \ \#(Y, 0, N) = \sum_{k=0}^{N} \#(Y_k). \tag{13}$$

The main results of this paper are based on the comparative index theory established in [10, 11]. According to [10], we define the comparative index for $2n \times n$ matrices Y, $\hat{Y}$ with condition (9) using the notation

$$\begin{cases} \mathscr{M} = (I - XX^{\dagger})\hat{X}, \ X = [I\,0]Y, \ \hat{X} = [I\,0]\hat{Y}, \\ \mathscr{T} = I - \mathscr{M}^{\dagger}\mathscr{M}, \ \mathscr{D} = \mathscr{D}^T = \mathscr{T} w^T(Y, \hat{Y}) X^{\dagger} \hat{X} \mathscr{T}, \end{cases}$$

where $w(Y, \hat{Y})$ is the Wronskian given by (12). The comparative index is defined by $\mu(Y, \hat{Y}) = \mu_1(Y, \hat{Y}) + \mu_2(Y, \hat{Y})$, where $\mu_1(Y, \hat{Y}) = \operatorname{rank} \mathscr{M}$ and $\mu_2(Y, \hat{Y}) =$

ind$\mathscr{D}$. According to [11, p. 449, Property 7] for the comparative index, we have the estimate

$$0 \le \mu(Y, \hat{Y}) \le \operatorname{rank} w(Y, \hat{Y}) \le n. \tag{14}$$

Comparison theorems for symplectic difference systems derived in [11] are based on the following notion of the comparative index for a pair of symplectic matrices. Here, we use the notation

$$\langle \mathscr{S} \rangle = \begin{pmatrix} \mathscr{X} \\ \mathscr{U} \end{pmatrix}, \; \mathscr{X} = \begin{pmatrix} I & 0 \\ \mathscr{A} & \mathscr{B} \end{pmatrix}, \; \mathscr{U} = \begin{pmatrix} 0 & -I \\ \mathscr{C} & \mathscr{D} \end{pmatrix}, \quad \mathscr{S} = \begin{pmatrix} \mathscr{A} & \mathscr{B} \\ \mathscr{C} & \mathscr{D} \end{pmatrix}$$

for arbitrary symplectic matrix $\mathscr{S}$ separated into $n \times n$ blocks $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$, $\mathscr{D}$ according to (5). In [11, Lemma 2.3], we proved that $4n \times 2n$ matrices $\langle \mathscr{S} \rangle$, $\langle \hat{\mathscr{S}} \rangle$ associated with $\mathscr{S}$, $\hat{\mathscr{S}} \in Sp(2n)$ obey (9) (with n replaced by $2n$) and then the comparative index for the pair $\langle \mathscr{S} \rangle$, $\langle \hat{\mathscr{S}} \rangle$ is well defined. For the comparative index of the symplectic coefficient matrices $\mathscr{S}_k$, $\hat{\mathscr{S}}_k$ associated with (1) and (2) by (6), we derive the following result.

Lemma 1. *Let $\mathscr{S}_k$, $\hat{\mathscr{S}}_k$ be the symplectic coefficient matrices associated with* (1) *and* (2) *via* (6)*, then*

$$\mu(\langle \mathscr{S}_k \rangle, \langle \hat{\mathscr{S}}_k \rangle) = \operatorname{ind}(\mathscr{H}_k - \hat{\mathscr{H}}_k) + \operatorname{ind}(\hat{B}_k) - \operatorname{ind}(B_k), \tag{15}$$

where $\mathscr{H}_k$, $\hat{\mathscr{H}}_k$ are the discrete Hamiltonians given by (3)*.*

Proof. Using (6) it is easy to verify by direct computations that

$$L \langle \mathscr{S}_k \rangle P = \begin{pmatrix} I & 0 \\ A_k & B_k \\ C_k & -A_k^T \\ 0 & I \end{pmatrix} = \mathfrak{N}^T \begin{pmatrix} I \\ -\mathscr{H}_k \end{pmatrix}, \; L = \operatorname{diag}\left(\begin{pmatrix} 0 & I \\ -I & I \end{pmatrix}, \begin{pmatrix} I & I \\ -I & 0 \end{pmatrix} \right),$$

$$\mathfrak{N} = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & I & 0 \\ 0 & -I & 0 & 0 \end{pmatrix},$$

where L, $\mathfrak{N} \in Sp(4n)$ and $P = \begin{pmatrix} I - A_k & -B_k \\ 0 & I \end{pmatrix} \in \mathbb{R}^{2n \times 2n}$. Since $\det P \neq 0$ and L is symplectic block diagonal, we can apply [10, p. 448, Properties 1,2] evaluating

$$\mu(\langle \mathscr{S}_k \rangle, \langle \hat{\mathscr{S}}_k \rangle) = \mu(L \langle \mathscr{S}_k \rangle P, L \langle \hat{\mathscr{S}}_k \rangle \hat{P}) = \mu\left(\mathfrak{N}^T \begin{pmatrix} I \\ -\mathscr{H}_k \end{pmatrix}, \mathfrak{N}^T \begin{pmatrix} I \\ -\hat{\mathscr{H}}_k \end{pmatrix} \right),$$

where $\hat{P}$ associated with (2) is defined by analogy with P.

Next, applying [10, Theorem 2.2] with $W = \mathfrak{N}$, $Z[0\,I]^T = \binom{I}{-\mathscr{H}_k}$, $\hat{Z}[0\,I]^T = \binom{I}{-\hat{\mathscr{H}}_k}$, we derive

$$\begin{aligned}\mu\left(\mathfrak{N}^T\binom{I}{-\mathscr{H}_k}, \mathfrak{N}^T\binom{I}{-\hat{\mathscr{H}}_k}\right) &= \mu\left(\binom{I}{-\mathscr{H}_k}, \binom{I}{-\hat{\mathscr{H}}_k}\right)\\ &+\mu\left(\binom{I}{-\hat{\mathscr{H}}_k}, \mathfrak{N}\binom{0}{I}\right) - \mu\left(\binom{I}{-\mathscr{H}_k}, \mathfrak{N}\binom{0}{I}\right),\end{aligned} \tag{16}$$

where we have used [10, p. 449, Property 9]. By the definition of the comparative index, the addends in the right-hand side of (16) coincide with the respective addends in (15). The proof is completed. □

Corollary 1. *The condition*

$$\mu(\langle \mathscr{S}_k \rangle, \langle \hat{\mathscr{S}}_k \rangle) = 0 \tag{17}$$

holds if and only if

$$\operatorname{ind}(\mathscr{H}_k - \hat{\mathscr{H}}_k) = \operatorname{ind}(B_k) - \operatorname{ind}(\hat{B}_k) \tag{18}$$

Corollary 2. *The condition*

$$\mathscr{H}_k \geq \hat{\mathscr{H}}_k \tag{19}$$

and (17) are simultaneously satisfied if and only if $\operatorname{ind}\hat{B}_k = \operatorname{ind} B_k$.

3 Main Results

In this section, we derive relations between the number of weighted focal points of conjoined bases of (1) and (2) (compare with [11, Theorem 2.1]).

Theorem 1. *Let Y_k, $\hat{Y}_k$ be conjoined bases of (1) and (2) and Z_k, $\hat{Z}_k$ be symplectic fundamental matrices of these systems such that the conditions $Y_k = Z_k[0\,I]^T$, $\hat{Y}_k = \hat{Z}_k[0\,I]^T$ hold. Then for the numbers $\#(Y_k)$, $\#(\hat{Y}_k)$ of weighted focal points of Y_k, $\hat{Y}_k$ in $(k, k+1]$, we have the relation*

$$\#(\hat{Y}_k) - \#(Y_k) + \Delta\mu(Y_k, \hat{Y}_k) = \operatorname{ind}(\mathscr{H}_k - \hat{\mathscr{H}}_k) - \mu(\langle \hat{Z}_{k+1}^{-1} Z_{k+1}\rangle, \langle \hat{Z}_k^{-1} Z_k\rangle), \tag{20}$$

and hence

$$\begin{aligned}&\#(\hat{Y}, M, N) - \#(Y, M, N) + \mu(Y_{N+1}, \hat{Y}_{N+1}) - \mu(Y_M, \hat{Y}_M)\\ &\quad = \sum_{k=M}^{N} (\operatorname{ind}(\mathscr{H}_k - \hat{\mathscr{H}}_k) - \mu(\langle \hat{Z}_{k+1}^{-1} Z_{k+1}\rangle, \langle \hat{Z}_k^{-1} Z_k\rangle)),\end{aligned} \tag{21}$$

where $\#(\hat{Y}, M, N)$, $\#(Y, M, N)$ are the numbers of (forward) weighted focal points in $(M, N+1]$.

Proof. For the proof we use [11, Theorem 2.1] and [7, Theorem 3.3]. Under the notation of Theorem 1, we have the following relation between the classical number $m(Y_k)$, $m(\hat{Y}_k)$ of focal points of conjoined bases Y_k, $\hat{Y}_k$ of symplectic difference systems with the coefficient matrices $\mathscr{S}_k$, $\hat{\mathscr{S}}_k$

$$m(\hat{Y}_k) - m(Y_k) + \Delta\mu(Y_k, \hat{Y}_k) = \mu(\langle \mathscr{S}_k \rangle, \langle \hat{\mathscr{S}}_k \rangle) - \mu(\langle \hat{Z}_{k+1}^{-1} Z_{k+1} \rangle, \langle \hat{Z}_k^{-1} Z_k \rangle). \tag{22}$$

Substituting into (22) representation (15) for $\mu(\langle \mathscr{S}_k \rangle, \langle \hat{\mathscr{S}}_k \rangle)$ and using Definition 1 of the number of weighted focal points, we derive (20). Summing (20) from $k = 0$ to $k = N$, we prove (21). The proof is completed. □

Corollary 3. *For the left-hand side of* (20)*, we have the estimate*

$$-\mathrm{ind}(\hat{\mathscr{H}}_k - \mathscr{H}_k) \le \#(\hat{Y}_k) - \#(Y_k) + \Delta\mu(Y_k, \hat{Y}_k) \le \mathrm{ind}(\mathscr{H}_k - \hat{\mathscr{H}}_k) \tag{23}$$

and then

$$\begin{aligned} -\sum_{k=M}^{N} \mathrm{ind}(\hat{\mathscr{H}}_k - \mathscr{H}_k) - \mu(Y_{N+1}, \hat{Y}_{N+1}) &\le \#(\hat{Y}, M, N) - \#(Y, M, N) \\ &\le \sum_{k=M}^{N} \mathrm{ind}(\mathscr{H}_k - \hat{\mathscr{H}}_k) + \mu(Y_M, \hat{Y}_M) \end{aligned} \tag{24}$$

Proof. The right inequality in (23) is the direct consequence of (20) because $\mu(\langle \hat{Z}_{k+1}^{-1} Z_{k+1} \rangle, \langle \hat{Z}_k^{-1} Z_k \rangle) \ge 0$. The left inequality is based on the equality (see [11, Theorem 2.1])

$$\begin{aligned} &\mu(\langle \mathscr{S}_k \rangle, \langle \hat{\mathscr{S}}_k \rangle) - \mu(\langle \hat{Z}_{k+1}^{-1} Z_{k+1} \rangle, \langle \hat{Z}_k^{-1} Z_k \rangle) \\ &\quad = \mu(\langle \hat{Z}_k^{-1} Z_k \rangle, \langle \hat{Z}_{k+1}^{-1} Z_{k+1} \rangle) - \mu(\langle \hat{\mathscr{S}}_k \rangle, \langle \mathscr{S}_k \rangle), \end{aligned} \tag{25}$$

then, using (25) by analogy with the proof of Theorem 1, we derive

$$\mathrm{ind}(\mathscr{H}_k - \hat{\mathscr{H}}_k) - \mu(\langle \hat{Z}_{k+1}^{-1} Z_{k+1} \rangle, \langle \hat{Z}_k^{-1} Z_k \rangle) = \mu(\langle \hat{Z}_k^{-1} Z_k \rangle, \langle \hat{Z}_{k+1}^{-1} Z_{k+1} \rangle) - \mathrm{ind}(\hat{\mathscr{H}}_k - \mathscr{H}_k). \tag{26}$$

The left inequality in (23) follows from (26) because $\mu(\langle \hat{Z}_k^{-1} Z_k \rangle, \langle \hat{Z}_{k+1}^{-1} Z_{k+1} \rangle) \ge 0$. Summing (23) and using that any comparative index is nonnegative, we derive (24). □

So we see that inequality (24) generalizes estimate (13) to the case when Y, $\hat{Y}$ are conjoined bases of (1) and (2) (note that $\mu(Y, \hat{Y}) \le n$ by (14)).

Corollary 4. *Assume* (19)*; then for any conjoined bases* Y_k, $\hat{Y}_k$ *of* (1) *and* (2)*, we have*

$$-\sum_{k=M}^{N} \mathrm{rank}(\hat{\mathscr{H}}_k - \mathscr{H}_k) - n \le \#(\hat{Y}, M, N) - \#(Y, M, N) \le n, \tag{27}$$

in particular, if $\hat{Y}_k$ *is the principal solution of* (2) *at* M*, then*

$$-\sum_{k=M}^{N} \operatorname{rank}(\hat{\mathscr{H}}_k - \mathscr{H}_k) - n \le \#(\hat{Y}, M, N) - \#(Y, M, N) \le 0, \tag{28}$$

Remark 1. (i) Note that the classical majorant condition (17) (see [7, formula (2.13)]) for systems (1) and (2) is not assumed in the results of this section. In particular, under assumption (19) we have $\mu(\langle \mathscr{S}_k \rangle, \langle \hat{\mathscr{S}}_k \rangle) = \operatorname{ind}(\hat{B}_k) - \operatorname{ind}(B_k) \ge 0$. If we assume (19) and $\#(Y, M, N) = 0$, we have by (28) that

$$-\sum_{k=M}^{N} \operatorname{rank}(\hat{\mathscr{H}}_k - \mathscr{H}_k) - n \le \#(\hat{Y}, M, N) \le 0 \tag{29}$$

for the principal solution $\hat{Y}_k$ of (2) at M, i.e., the number of weighted focal points of this solution is non-positive.

(ii) If we assume majorant condition (17), then for the classical number of focal points of $Y, \hat{Y}$ in $(M, N+1]$, we have by Corollary 1 and (24) that $l(\hat{Y}, M, N) - l(Y, M, N) \le n$ and $l(\hat{Y}, M, N) - l(Y, M, N) \le 0$ when $\hat{Y}_k$ is the principal solution of (2) at M (see [6, Theorems 1.2,1.3]). In particular, $l(Y, M, N) = 0$ implies $l(\hat{Y}, M, N) = 0$ for the principal solution of (2) at M (compare with (29)).

(iii) As it was pointed out in [8, Remark 2.8(ii)], the notion of a weighted focal point is important in the development of the relative oscillation theory for Hamiltonian eigenvalue problems. Relative oscillation theory developed for *linear* symplectic spectral problems in [12, 13] measures the difference between the spectra of *two* different eigenvalue problems, rather than measuring the spectrum of one single problem only. Similar results, incorporating the new notion of weighted focal points, we hope to prove for discrete eigenvalue problems with the different Hamiltonians $\mathscr{H}_k(\lambda)$, $\hat{\mathscr{H}}_k(\lambda)$. Theorem 1 is the first step in this direction.

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Interpolation Method of Shalva Mikeladze with Following Applications

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Abstract The paper deals with the numerical method of Shalva Mikeladze, the accuracy of which depends on the number of interpolation points. The method called the method without saturation is devoted to the numerical solution of ordinary differential equations. It is constructed on the basis of an interpolation formula to solve numerically linear and nonlinear ordinary differential equations of any order and systems of such equations. Using its different versions, it is possible to solve boundary value, eigenvalue, and Cauchy problems (Mikeladze, Soobsh AN GSSR 45(2):284–296, 1967 and Mikeladze, Soobsh AN GSSR 47(2):263–268, 1967). This method in combination with the method of lines can also be applied to solve boundary value problems for partial differential equations of elliptic type (Makarov, Karalashvili, Soobsh AN GSSR 131(1):33–36, 1988). As a model, the Dirichlet problem for a Poisson equation in the symmetric rectangle is considered. This application created a semi-discrete difference scheme with matrices of central symmetry having certain properties.

Keywords Method without saturation • Interpolation points • Boundary value problem • Centro-symmetric matrices

Mathematics Subject Classification (2000): 35J25, 65N40

1 Introduction

In the 60s of the last century, several works of Shalva Mikeladze [5, 6] were published, where he proposed a new numerical method of solving ordinary differential equations. The method was based on his general interpolation formula for solving numerically any order linear and nonlinear ordinary differential equations and systems of such equations

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$$y^{(k)}\left(a+ht_\beta\right)=H_{n,k}^{t_\alpha,t_\beta}+h^{n-k}\sum_{\nu=1}^{m}A_{\nu,k}^{t_\alpha,t_\beta}y^{(n)}\left(a+ht_\nu\right)+R_{m,k}^{t_\alpha,t_\beta}, \quad (1)$$

where

$$H_{n,k}^{t_\alpha,t_\beta}=\sum_{\lambda=0}^{n-k-1}\left(t_\beta-t_\alpha\right)^\lambda\frac{h^\lambda}{\lambda!}y^{(k+\lambda)}\left(a+ht_\alpha\right),\quad A_{\nu,k}^{t_\alpha,t_\beta}=\frac{1}{(n-k-1)!}$$
$$\times\sum_{\mu=\nu}^{m}\left[\prod_{\substack{s=1\\ s\neq\nu}}^{\mu}\left(t_\nu-t_s\right)\right]^{-1}\int_{t_\alpha}^{t_\beta}\left(t_\beta-t\right)^{n-k-1}P_{\mu-1}(t)\,dt, \quad (2)$$

$$R_{m,k}^{t_\alpha,t_\beta}=\frac{h^{m+n-k}}{(n-k-1)!}$$
$$\times\int_{t_\alpha}^{t_\beta}\left(t_\beta-t\right)^{n-k-1}P_m(t)\,y^{(n)}\left(a+ht,a+ht_1,\ldots,a+ht_m\right)dt, \quad (3)$$

$h=\frac{(b-a)}{m+1}$, $(b-a)$ is the length of integration segment, $k=\overline{0,n-1}$, $\beta=\overline{1,\mathrm{m}+1}$.

Formula (1) in different versions can be used to solve boundary value, eigenvalue, and Cauchy problems.

The same formula can also be obtained by means of the Lagrange interpolation formula [2], in which

$$A_{\nu,k}^{t_\alpha,t_\beta}=\frac{1}{(n-k-1)!}\int_{t_\alpha}^{t_\beta}\left(t_\beta-t\right)^{n-k-1}l_\nu^m(t)\,dt, \quad (4)$$

where $l_j^{(m)}(t)=\dfrac{(t-1)(t-2)\ldots(t-m)}{(t-j)(j-1)!(-1)^{m-j}(m-j)!}$, $j=\overline{1,m}$ are Lagrange fundamental polynomials.

Because of the uniqueness of interpolation polynomials (2) and (4), they are identical. In the case of equidistant location of interpolation points on the integration segment $t_\beta=\beta$, $t_\alpha=0$, formula (2) can be simplified for easy calculations.

2 Semi-Discrete Scheme for the Dirichlet Problem with a Poisson Equation

One of the methods of solving partial differential equations is the well-known method of lines [1]. According to this method, in the two-dimensional case, the discretization of a differential operator is carried out by one independent variable, and the initial problem is reduced to a system of ordinary differential equations.

Here, we use the method of lines which is modified with the aid of Mikeladze's formula. This modified method establishes the relation between the function $u(x, y)$ and its second-order partial derivatives by y variable with a constant step h

$$u(x, -b + ih) = H_{2,0}^{0,i} + h^2 \sum_{\nu=1}^{m} A_{\nu,0}^{0,i} \frac{\partial^2 u(x, -b + \nu h)}{\partial y^2} + R_{m,0}^{0,i}(x), \tag{5}$$

where

$$H_{2,0}^{0,i} = \sum_{\lambda=0}^{1} \frac{(ih)^{\lambda}}{\lambda!} \frac{\partial^{\lambda} u\,(x, -b)}{\partial y^2}, \quad A_{\nu,0}^{0,i} = \int_0^i (i - t) \cdot l_{\nu}^{(m)}\,(t)\, dt \tag{6}$$

$$R_{m,0}^{0,i}(x) = h^{m+2} \int_0^i (i - t) \times P_m\,(t)\, \frac{\partial^2 u\,(x, -b + ht, -b + h, \ldots, -b + mh)}{\partial y^2} dt. \tag{7}$$

Let us assume that at any region Ω, the function $u(x, y)$ has $m + 2$ partial derivatives with respect to the variable y. Using (5) we obtain

$$u\,(x, y + (i + 1)\,h) - 2u\,(x, y + ih) + u\,(x, y + (i - 1)\,h) = h^2 \sum_{\nu=1}^{m} \Delta_i^2 A_{\nu,0}^{0,i-1} \frac{\partial^2 u\,(x, y + \nu h)}{\partial y^2} + \Delta_i^2 R_{m,0}^{0,i-1}. \tag{8}$$

It is required to find a function, which is a solution of the following boundary value problem

$$\frac{\partial^2 u\,(x, y)}{\partial x^2} + \frac{\partial^2 u\,(x, y)}{\partial y^2} = -f(x, y), \quad (x, y) \in \Omega u(x, y) = 0, \quad (x, y) \in \Gamma. \tag{9}$$

Let us cover the region $\Omega = \{(x, y)\,|\, -a \le x \le a; -b \le y \le b\}$ by the lines $y = y_i = -b + ih$, $i = \overline{1, m}$, which are parallel to the x-axis with the interval $h = 2b/(m + 1)$. Denote $u_i\,(x) = u\,(x, y_i)$ and $f_i\,(x) = f\,(x, y_i)$. Each triple of functions $u_{i-1}\,(x)\,,\ u_i\,(x)\,,\ u_{i+1}\,(x)$ satisfies equality (8) which we rewrite as follows

$$-\sum_{\nu=1}^{m} (A_{\nu,0}^{0,i+1} - 2A_{\nu,0}^{0,i} + A_{\nu,0}^{0,i-1}) \frac{\partial^2 u\,(x, y)}{\partial y^2}\,\Big|_{y=y_\nu} + h^{-2}[u_{i+1}(x) - 2u_i(x) + u_{i-1}(x)] = h^{-2} \Delta \nabla_i R_{m,0}^{0,i}.$$

But by virtue of Eq. (9), we have the relation $\frac{\partial^2 u\,(x,y)}{\partial y^2}\big|_{y=y_\nu} = -f_\nu\,(x) - u_\nu''(x)$. Substituting this expression into the preceding equality, we obtain the following system of ordinary differential equations

$$\sum_{\nu=1}^{m}(A_{\nu,0}^{0,i+1} - 2A_{\nu,0}^{0,i} + A_{\nu,0}^{0,i-1})[u_\nu''(x) + f_\nu\,(x)]$$
$$+ h^{-2}[u_{i+1}(x) - 2u_i(x) + u_{i-1}(x)] = h^{-2}\Delta\nabla_i R_{m,0}^{0,i}, \tag{10}$$
$$x \in (-a,a)\,, u_0\,(x) = u_{m+1}\,(x) = 0, u_i\,(\pm a) = 0, \quad i = \overline{1,m},$$

where

$$\Delta\nabla_i R_{m,0}^{0,i} = R_{m,0}^{0,i+1}(x) - 2R_{m,0}^{0,i}(x) + R_{m,0}^{0,i-1}(x), \tag{11}$$

Neglecting the remainders and denoting the approximate solutions by $v_i(x)$, we get the semi-discrete scheme of the modified method of lines

$$\sum_{\nu=1}^{m} a_{i\nu}^{(m)}[v_\nu''(x) + f_\nu\,(x)] + h^{-2}[v_{i+1}(x) - 2v_i(x) + v_{i-1}(x)] = 0, \tag{12}$$

where

$$a_{i\nu}^{(m)} = A_{\nu,0}^{0,i+1} - 2A_{\nu,0}^{0,i} + A_{\nu,0}^{0,i-1} = \Delta\nabla_i A_{\nu,0}^{0,i} \tag{13}$$
$$v_0\,(x) = v_{m+1}\,(x) = 0, \quad x \in (-a,a)\,, \quad v_i\,(\pm a) = 0, \quad i = \overline{1,m}.$$

Let us write (10) and (12) in the vector-matrix form

$$A(U'' + F) + h^{-2}MU = h^{-2}MR, \quad U(\pm a) = 0 \tag{14}$$
$$A(V'' + F) + h^{-2}MV = 0, \quad V(\pm a) = 0, \tag{15}$$

where

$$A = A_m = [a_{ij}^{(m)}]_{i,j=1}^m, \quad M = M_m = [1,-2,1]_{i,j=1}^m,$$
$$U = (u_1(x), u_2(x), \ldots, u_m(x))^T,$$
$$V = (v_1(x), v_2(x), \ldots, v_m(x))^T, \quad F = (f_1(x), f_2(x), \ldots, f_m(x))^T,$$
$$R = (R_{m,0}^{0,1}(x), R_{m,0}^{0,2}(x), \ldots, R_{m,0}^{0,m}(x))^T.$$

In the case of equidistant lines, $a_{ij}^{(m)} = \Delta\nabla_i A_{\nu,0}^{0,i}$ can be transformed to the following form, convenient for calculations

$$a_{ij}^{(m)} = \int_{i-1}^{i} \int_{z}^{z+1} l_j^{(m)}(t)\, dtdz \tag{16}$$

$$a_{ij}^{(m)} = 2\int_{0}^{1} \int_{z}^{2z} l_j^{(m)}(t+i-1)\, dtdz \tag{17}$$

$$a_{ij}^{(m)} = \int_{0}^{1} (1-x)\left[l_j^{(m)}(x+i) + l_j^{(m)}(-x+i)\right] dx. \tag{18}$$

As to the matrices in semi-discrete scheme (15), we may say that $M_m = [1, -2, 1]_1^m$ is a three-diagonal non-singular double-symmetric matrix (symmetric and centrosymmetric), corresponding to the second difference derivative with the well-studied properties. As to the matrix A_m, it is a centrosymmetric matrix with certain properties, which behaves as a unit matrix. If we multiply matrix-vector Eq. (15) from the left by M^{-1} and solve the eigenvalue problem for the matrix $B = M^{-1}A$, then the system of differential Eq. (12) splits into m equations with only one unknown function in each equation and can be solved by the well-known methods.

3 Order of Approximation

Let us first check whether (12) tends to the initial equation when step $h = 2b/(m+1)$ tends to zero:

$$\lim_{h\to 0} \frac{v_{i+1}(x) - 2v_i(x) + v_{i-1}(x)}{h^2} = -\lim_{h\to 0} \sum_{\nu=1}^{m} a_{i\nu}^{(m)} [v_\nu''(x) + f_\nu(x)]\,.$$

Expanding the right side of this equality into a Taylor series at the point $y_i = -b + ih$, we obtain

$$-\frac{\partial^2 v(x,y)}{\partial y^2}\Big|_{y=y_\nu} = \left[f(x,y_\nu) + \frac{\partial^2 v(x,y_\nu)}{\partial x^2}\right] \sum_{\nu=1}^{m} a_{i\nu}^{(m)}$$
$$+ \lim_{h\to 0} \sum_{j=1}^{m-1} \frac{h^j}{j!} \left[\frac{\partial^j f(x,y_i)}{\partial y^j} + \frac{\partial^{j+2} v(x,y_\nu)}{\partial x^2 \partial y^j}\right] \times \sum_{\nu=1}^{m} (\nu - i)^j a_{i\nu}^{(m)}.$$

Lemma 3.1. *For the numbers $A_{\nu,k}^{t_\alpha,t_\beta}$, the following relation is fulfilled*

$$\sum_{\nu=1}^{m} (t_\nu - t_\alpha)^\mu A_{\nu,k}^{t_\alpha,t_\beta} = \frac{(t_\beta - t_\alpha)^{\mu+n-k}}{(\mu+n-k)!}\mu!, \quad \mu = \overline{1, m-1}. \tag{19}$$

For the considered problem (9), *formula* (19) ($t_\nu = \nu$, $t_\alpha = 0$, $n = 2$, $k = 0$) *takes the form*

$$\sum_{\nu=1}^{m} \nu^{\mu} A_{\nu,0}^{0,i} = \frac{i^{\mu+2}\mu!}{(\mu+2)!}, \quad \mu = \overline{0, m-1} \tag{20}$$

Since $a_{i\nu}^{(m)} = \Delta_i^2 A_{\nu,0}^{0,i}$, *by* (21) *we have*

$$\begin{aligned} \sum_{\nu=1}^{m} \nu^{\mu} a_{i\nu}^{(m)} &= \sum_{\nu=1}^{m} \nu^{\mu} \Delta_i^2 A_{\nu,0}^{0,i} \\ &= \Delta_i^2 \frac{(i-1)^{\mu+2}\mu!}{(\mu+2)!} = \frac{\mu!}{(\mu+2)!}\left[(i-1)^{\mu+2} - 2i^{\mu+2} + (i+1)^{\mu+2}\right] \end{aligned} \tag{21}$$

Using Lemma 3.1 and the properties of binomial coefficients, after some further simplifications, we come to

$$\begin{aligned} -\frac{\partial^2 v(x, y_i)}{\partial y^2} &= f(x, y_i) + \frac{\partial^2 v(x, y_i)}{\partial x^2} \\ &+ 2\lim_{h\to 0} \sum_{j=1}^{\left[\frac{m+1}{2}\right]} \frac{h^{2j}}{(2j+2)!}\left[\frac{\partial^{2j} f(x, y_i)}{\partial y^{2j}} + \frac{\partial^{2j+2} v(x, y_i)}{\partial x^2 \partial y^{2j}}\right]. \end{aligned} \tag{22}$$

Since this limit equals zero, we obtain the initial Eq. (9).

Using the above notations $v_i(x) = v(x, y_i)$, $v_i(x) + f_i(x) = -\frac{\partial^2 v(x,y)}{\partial y^2}\big|_{y=y_i}$ in (12) and expanding the unknown function and its second derivative by y into a Taylor series at the point $y = y_0 = -b$, we have

$$\begin{aligned} v(x, y_i) &= \sum_{\lambda=1}^{m+1} \frac{(ih)^{\lambda}}{\lambda!} \frac{\partial^{\lambda} v(x, -b)}{\partial y^{\lambda}} + O\left(h^{m+2}\right) \\ \frac{\partial^2 v(x, y_\nu)}{\partial y^2} &= \sum_{\lambda=0}^{m-1} \frac{(\nu h)^{\lambda}}{\lambda!} \frac{\partial^{\lambda+2} v(x, -b)}{\partial y^{\lambda+2}} + O(h^m). \end{aligned} \tag{23}$$

Let us substitute (23) into (12), then

$$\begin{aligned} &\sum_{\nu=1}^{m} a_{i\nu}^{(m)} \left\{-\sum_{\lambda=0}^{m-1} \frac{(\nu h)^{\lambda}}{\lambda!} \frac{\partial^{\lambda+2} v(x, -b)}{\partial y^{\lambda+2}} + O(h^m)\right\} \\ &\quad + h^{-2}\left\{\sum_{\lambda=0}^{m+1} \frac{h^{\lambda}}{\lambda!} \frac{\partial^{\lambda} v(x, -b)}{\partial y^{\lambda}}\left[(i+1)^{\lambda} - 2i^{\lambda} + (i-1)^{\lambda}\right]\right\} + O(h^m) \end{aligned}$$

$$= \sum_{\lambda=0}^{m-1} \frac{h^\lambda}{\lambda!} \frac{\partial^{\lambda+2} v(x,-b)}{\partial y^{\lambda+2}} \left(-\sum_{\nu=1}^{m} \nu^\lambda a_{i\nu}^{(m)} \right)$$
$$+ \sum_{\lambda=0}^{m-1} \frac{h^\lambda}{(\lambda+2)!} \frac{\partial^{\lambda+2} v(x,-b)}{\partial y^{\lambda+2}} \Delta_i^2 (i-1)^{\lambda+2} + O(h^m).$$

By (21) we have

$$= \sum_{\lambda=0}^{m-1} \frac{h^\lambda}{\lambda!} \frac{\partial^{\lambda+2} v(x,-b)}{\partial y^{\lambda+2}} \left\{ -\Delta_i^2 \frac{(i-1)^{\lambda+2} \lambda!}{(\lambda+2)!} + \Delta_i^2 \frac{(i-1)^{\lambda+2}}{(\lambda+1)(\lambda+2)} \right\} + O(h^m).$$

Lemma 3.2. *If the solution of problem (9) $u(x,y) \in C_{m+2}\left(\overline{\Omega}\right)$, then the approximation error of the modified method of lines is of m-th order.*

4 Convergence

Let $Z = U - V$ be the error of the modified method of lines. Using this notation and subtracting (12) from (10), for the error Z we obtain the following system of equations

$$\sum_{\nu=1}^{m} a_{i\nu}^{(m)} z_\nu''(x) + h^{-2}[z_{i+1}(x) - 2z_i(x) + z_{i-1}(x)] = h^{-2} \Delta \nabla_i R_{m,0}^{0,i}, \tag{24}$$
$$z_0(x) = z_{m+1}(x) = 0, \quad x \in (-a,a), \quad z_i(\pm a) = 0, \quad i = \overline{1,m},$$

where $R_{m,0}^{0,i}$, $i = \overline{1,m}$, is given by expression (7) and the matrix-vector form of this system is

$$AZ''(x) + h^{-2} MZ(x) = h^{-2} MR, \quad Z(\pm a) = 0, \tag{25}$$
$$Z = (z_1(x), z_2(x), \ldots, z_m(x)), \quad R = (R_{m,0}^{0,1}(x), R_{m,0}^{0,2}(x), \ldots, R_{m,0}^{0,m}(x))^T.$$

Let us multiply (25) by Z from the right using the scalar product formula (26) for the semi-discrete schemes

$$(U,V)_{\Omega \times \widehat{\omega}} = \int_{-a}^{a} (U,V)_{\widehat{\omega}}\, dx = \int_{-a}^{a} \sum_{i=1}^{m} h u_i(x) v_i(x)\, dx \tag{26}$$

$$\int_{-a}^{a} (AZ'', Z)\, dx + h^{-2} \int_{-a}^{a} (MZ, Z)\, dx = h^{-2} \int_{-a}^{a} (MR, Z)\, dx. \tag{27}$$

Let us use Green's first formula to transform the first addend of (27) as follows

$$\begin{aligned}\int_{-a}^{a}\left(AZ'',Z\right)dx &= \int_{-a}^{a}\sum_{i=1}^{m}h\left(\sum_{j=1}^{m}a_{ij}^{(m)}z_j''(x)\right)\cdot z_i(x)\,dx\\ &= \int_{-a}^{a}\sum_{i=1}^{m}h\left[\sum_{j=1}^{m}a_{ij}^{(m)}(-z_j'(x))z_i'(x)+z_i(x)\,z_j'(x)\,|_{-a}^{a}\right]dx\\ &= -\int_{-a}^{a}\sum_{i=1}^{m}h\left(\sum_{j=1}^{m}a_{ij}^{(m)}z_j'(x)\right)z_i'(x)\,dx = -\int_{-a}^{a}\left(AZ',Z'\right)dx.\end{aligned} \tag{28}$$

It is known that the matrix $(-M)$ is positive definite. Besides, by direct calculations we see that $(AZ,Z)\geq\gamma^2\,(Z,Z)$ when $m\leq 5$. Taking into account (28) and a fact that matrices A and $(-M)$ are positive definite, from (27) we obtain

$$h^{-2}\int_{-a}^{a}(-MZ,Z)\,dx\leq h^{-2}\int_{-a}^{a}(-MR,Z)\,dx. \tag{29}$$

Let us consider

$$\begin{aligned}h^{-2}\int_{-a}^{a}(-MZ,Z)\,dx &= -\int_{-a}^{a}\left(Z_{\overline{y}y},Z\right)dx\\ &= \int_{-a}^{a}\left(Z_{\overline{y}},Z_{\overline{y}}\right)dx = \left\|Z_{\overline{y}}\right\|^2_{\Omega\times\widehat{\omega}}.\end{aligned} \tag{30}$$

Similarly, for the right-hand side of inequality (29), we have

$$h^{-2}\int_{-a}^{a}(-MR,Z)\,dx = \int_{-a}^{a}\left(R_{\overline{y}},Z_{\overline{y}}\right)dx. \tag{31}$$

By virtue of the Cauchy–Buniakowski inequality, we obtain

$$\begin{aligned}\int_{-a}^{a}\left(R_{\overline{y}},Z_{\overline{y}}\right)dx &\leq \left(\int_{-a}^{a}\left(R_{\overline{y}},R_{\overline{y}}\right)^2dx\right)^{\frac{1}{2}}\cdot\left(\int_{-a}^{a}\left(Z_{\overline{y}},Z_{\overline{y}}\right)^2dx\right)^{\frac{1}{2}}\\ &= \left\|R_{\overline{y}}\right\|_{\Omega\times\widehat{\omega}}\cdot\left\|Z_{\overline{y}}\right\|_{\Omega\times\widehat{\omega}}.\end{aligned} \tag{32}$$

Using (31) and (32), we get $h^{-2}\int_{-a}^{a}(-MR,Z)\,dx \leq \left\|R_{\overline{y}}\right\|_{\Omega\times\widehat{\omega}}\cdot\left\|Z_{\overline{y}}\right\|_{\Omega\times\widehat{\omega}}$. Then (29) with (30) and last relation will give the following inequality: $\left\|Z_{\overline{y}}\right\|_{\Omega\times\widehat{\omega}}\leq\left\|R_{\overline{y}}\right\|_{\Omega\times\widehat{\omega}}$. Thus, the convergence rate estimate in the difference-continuous norm W_2^1 reduces to the estimate of the first difference derivative norm of remainder (7) [4].

Theorem 4.1. *If the solution of problem (9) $u(x,y) \in W_2^1(\Omega)$, then for the error of the semi-discrete scheme of lines (15) in the difference-continuous norm W_2^1 the following estimate holds:*

$$\left\|Z_{\overline{y}}\right\|_{\Omega\times\widehat{\omega}} \le \left\|R_{\overline{y}}\right\|_{\Omega\times\widehat{\omega}} \le C\left(\frac{4e^2b}{m+1}\right)^{m-2} |u|_{W_2^{m+2}(\Omega)}\,,\ m\le 5,\ C=\frac{16b^3e^5}{\pi}\sqrt{\frac{2}{3\pi}}.$$

Appendix: Examples of Centrosymmetric Matrices

Elements of inverse of the three-diagonal $M_m = [1,-2,1]_1^m$ matrix $M^{-1} = \left[m_{ij}^{-1}\right]_{i,j=1}^m$ are:

$$m_{ij}^{-1} = -\frac{1}{m+1}\begin{cases} i\,(m+1-j)\,, & i<j \\ i\,(m+1-i)\,, & i=j \\ (m+1-i)\,j, & i>j \end{cases}$$

$A_1 = I_1$ and $A_2 = I_2$ are unit matrices. In general, A_m behaves as a unit matrix [3]. It is easy to notice for $m = 3$, $m = 4$, and for $m = 5$ that

$$A_3 = \frac{1}{12}\begin{bmatrix} 13 & -2 & 1 \\ 1 & 10 & 1 \\ 1 & -2 & 13 \end{bmatrix} = I_3 + \frac{1}{12}\begin{bmatrix} 1 & -2 & 1 \\ 1 & -2 & 1 \\ 1 & -2 & 1 \end{bmatrix},$$

$$A_3^{-1} = \frac{1}{12}\begin{bmatrix} 11 & 2 & -1 \\ -1 & 14 & -1 \\ -1 & 2 & 11 \end{bmatrix} = I_3 - \frac{1}{12}\begin{bmatrix} 1 & -2 & 1 \\ 1 & -2 & 1 \\ 1 & -2 & 1 \end{bmatrix}.$$

$$A_4 = \frac{1}{12}\begin{bmatrix} 14 & -5 & 4 & -1 \\ 1 & 10 & 1 & 0 \\ 0 & 1 & 10 & 0 \\ -1 & 4 & -5 & 14 \end{bmatrix} = I_4 + \frac{1}{12}\begin{bmatrix} 2 & -5 & 4 & -1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ -1 & 4 & -5 & 2 \end{bmatrix},$$

$$A_4^{-1} = \frac{1}{12}\begin{bmatrix} 10 & 5 & -4 & 1 \\ -1 & 14 & -1 & 0 \\ 0 & -1 & 14 & -1 \\ 1 & -4 & 5 & 10 \end{bmatrix} = I_4 - \frac{1}{12}\begin{bmatrix} 2 & -5 & 4 & -1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ -1 & 4 & -5 & 2 \end{bmatrix}.$$

$$A_5 = \frac{1}{720}\begin{bmatrix} 897 & -528 & 582 & -288 & 57 \\ 57 & 612 & 42 & 12 & -3 \\ -3 & 72 & 582 & 72 & -3 \\ -3 & 12 & 42 & 612 & 57 \\ 57 & -288 & 582 & -528 & 897 \end{bmatrix}$$

$$= I_5 + \frac{1}{720}\begin{bmatrix} 177 & -528 & 582 & -288 & 57 \\ 57 & -108 & 42 & 12 & -3 \\ -3 & 72 & -138 & 72 & -3 \\ -3 & 12 & 42 & -108 & 57 \\ 57 & -288 & 582 & -528 & 177 \end{bmatrix}$$

$$A_5^{-1} = \frac{1}{720}\begin{bmatrix} 548 & 508 & -552 & 268 & -52 \\ -52 & 808 & -12 & -32 & 8 \\ 8 & -92 & 888 & -92 & 8 \\ 8 & -32 & -12 & 808 & -52 \\ -52 & 268 & -552 & 508 & 548 \end{bmatrix}$$

$$= I_5 - \frac{1}{720}\begin{bmatrix} 172 & -508 & 552 & -268 & 52 \\ 52 & -88 & 12 & 32 & -8 \\ -8 & 92 & -168 & 92 & -8 \\ -8 & 32 & 12 & -88 & 52 \\ 52 & -268 & 552 & -508 & 172 \end{bmatrix}$$

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Parabolic Obstacle Problem with Measurable Coefficients in Morrey-Type Spaces

Lubomira G. Softova

Abstract We consider obstacle problem related to linear divergence form parabolic system with measurable coefficients in domain with irregular boundary. Supposing that the data of the problem and the obstacle belong to Morrey-type space, we get Calderón–Zygmund type estimate for the gradient of the solution.

Keywords Parabolic systems • Obstacle • Morrey-type spaces • Measurable • Coefficients • Small BMO • Reifenberg-flat domain

Mathematics Subject Classification (2000): Primary 35K87, secondary 35B65, 35R05, 46E30

1 Introduction

The obstacle problem for partial differential equations arises naturally in the classical elasticity theory as one of the simplest unilateral problems in the study of mechanics of elastic membranes. Roughly speaking, it aims to find the equilibrium position of an elastic membrane, the boundary of which is keeping fixed and which is constrained to stay above a prescribed obstacle. More generally, the obstacle problems provide a basic analytic tool in the study of variational inequalities and free boundary problems for PDEs. They are involved in various geometric and potential theory problems such as capacities of sets or minimal surfaces. We refer the reader to the classical texts [9, 10, 13, 14] for more details.

Our work is motivated by the recent papers [1, 2, 19], where the authors developed the Calderón-Zygmund theory for nonlinear elliptic and parabolic problems with irregular obstacles. To the difference of [1, 2, 19], we deal with differential operators with rough coefficients having quite arbitrary discontinuities in one direction. This situation is closely related to the equilibrium equations of linearly

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elastic laminates and composite materials. Even if there have been recently a lot of works in this direction, most of the obtained results consider single equations without obstacles; see [3–5, 7, 8, 12, 16, 17].

Another difference with the cited results consists of the fact that we derive estimate for the gradient of the solution in the framework of the generalized Morrey-type spaces under possibly more general assumptions on the weight function extending in such a way the results obtained in [6, 12]. Regarding the considered non-smooth domain, we suppose that it is flat in the sense of Reifenberg [18]. Roughly speaking, this means that the boundary is well approximated by hyperplanes at each point and at each scale. This is a sort of "minimal regularity" of the boundary that guarantees the validity of the main results of the geometric analysis. For instance, C^1 or Lipschitz domains with small Lipschitz constant belong to that category. The class of Reifenberg-flat domains goes beyond these common examples and contains domains with rough fractal boundaries such as the von Koch snowflake. In addition, domain which is flat in the sense of Reifenberg is also Jones' flat and possesses the extension properties (see [18, 20]).

Turning back to our problem, let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$ and $Q = \Omega \times (0,T], T > 0$ be a cylinder. Denote by ∂Q the usual parabolic boundary $\{\Omega \times \{t=0\}\} \cup \{\partial\Omega \times [0,T]\}$. Hereafter, we adopt the standard summation convention on the repeated indexes, with $1 \leq \alpha, \beta \leq n$, and $1 \leq i,j \leq m$ where $m \geq 1$. The letter c will denote a positive constant that varies from one appearance to another and can be calculated explicitly in terms of known quantities as λ, Λ, m, n, p, and $|Q|$.

This announce extends some recent results obtained in [6] in collaboration with S.-S. Byun. We study *obstacle problem* related to the system

$$u_t^i - D_\alpha\big(A_{ij}^{\alpha\beta}(x,t)D_\beta u^j\big) = -D_\alpha f_i^\alpha(x,t) \quad \text{in } Q\,. \tag{1}$$

The obstacle is given by vector function $\psi = (\psi^1, \ldots, \psi^m) : Q \to \mathbb{R}^m$ with the same kind of regularity as the weak solution of (1):

$$\begin{cases} \psi \in L^2(0,T;H^1(\Omega,\mathbb{R}^m)),\ \psi_t \in L^2(0,T;H^{-1}(\Omega,\mathbb{R}^m), \\ \psi^i \leq 0 \text{ a.e.on } \partial Q,\ i=1,\ldots,m. \end{cases} \tag{2}$$

Further, we define *admissible set* $\mathscr{A}$ consisting of vector functions:

$$\begin{cases} \mathbf{u} \in C^0(0,T;L^2(\Omega;\mathbb{R}^m)) \cap L^2(0,T;H_0^1(\Omega,\mathbb{R}^m)) \\ u^i(\cdot,0) = 0 \text{ a.e.in } \Omega. \quad u^i \geq \psi^i \text{ a.e.in } Q\,. \end{cases}$$

The function $\mathbf{u} \in \mathscr{A}$ is called *weak solution* to the obstacle problem related to (1)–(2) if for all $\phi \in \mathscr{A}$ with $\phi_t \in L^2(0,T;H^{-1}(\Omega,\mathbb{R}^m))$ the variational inequality holds

$$\int_0^T \langle \phi_t^i, \phi^i - u^i \rangle \, dt + \int_Q A_{ij}^{\alpha\beta}(x,t) D_\beta u^j \cdot D_\alpha \left(\phi^i - u^i \right) \, dxdt$$
$$\geq \int_Q f_i^\alpha(x,t) \cdot D_\alpha \left(\phi^i - u^i \right) \, dxdt \,. \tag{3}$$

We assume that the coefficients $A_{ij}^{\alpha\beta} : Q \to \mathbb{R}^{mn \times mn}$ are uniformly elliptic and uniformly bounded, namely, there exist positive constants λ and Λ such that

$$\|A_{ij}^{\alpha\beta}\|_{L^\infty(Q)} \leq \Lambda \quad \text{and} \quad \lambda |\xi|^2 \leq A_{ij}^{\alpha\beta}(x,t) \xi_\alpha^i \xi_\beta^j, \text{ a.e. in } Q \tag{4}$$

for all matrices $\xi \in \mathbb{M}^{m \times n}$.

According to the classical theory and some recent results [1, 2, 9], if $\mathbf{F} \in L^2(Q, \mathbb{R}^{m \times n})$, there exists a unique weak solution $\mathbf{u} \in \mathscr{A}$ of (3) satisfying

$$\|D\mathbf{u}\|_{L^2(Q)} \leq c \left(\|\mathbf{F}\|_{L^2(Q)} + \|\psi_t\|_{L^2(Q)} + \|D\psi\|_{L^2(Q)} \right) . \tag{5}$$

2 Generalized Parabolic Morrey-type Spaces

Let us describe the spaces that we are going to use. We consider *parabolic cylinders* $\mathscr{I}_r(y,\tau) = \mathscr{B}_r(y) \times (\tau - r^2, \tau + r^2)$ with respect to the classical parabolic metric and *cylinders* $\mathscr{C}_r$ in which we isolate the variable x^1

$$\mathscr{C}_r(y,\tau) = \left\{ (x^1, x', t) \in \mathbb{R}^{n+1} : |x^1 - y^1| < r, \ |x' - y'| < r, |t - \tau| < r^2 \right\} .$$

For some fixed $x^1 \in (y^1 - r, y^1 + r)$, we consider the x^1-*slice* of $\mathscr{C}_r$:

$$\mathscr{C}_r^{x^1}(y', \tau) = \left\{ (x', t) \in \mathbb{R}^{n-1} \times \mathbb{R} : (x^1, x', t) \in \mathscr{C}_r(y,\tau) \right\}.$$

Taking now the τ-slice of $\mathscr{C}_r$, we get the *cube*:

$$\mathscr{C}_r^\tau(y) = \left\{ (x^1, x') \in \mathbb{R}^n : \ |x^1 - y^1| < r, |x' - y'| < r \right\} .$$

We call *weight* positive and measurable function $\varphi : \mathbb{R}^{n+1} \times \mathbb{R}_+ \to \mathbb{R}_+$.

Definition 1. Let Q be a cylinder in $\mathbb{R}^{n+1}$. A function $f \in L^q(Q), 1 < q < \infty$, belongs to the *generalized Morrey-type space* $L^{q,\varphi}(Q)$ if the following norm is finite

$$\|f\|_{L^{q,\varphi}(Q)} = \sup_{\substack{(y,\tau) \in Q \\ r > 0}} \left(\frac{1}{\varphi(\mathscr{I}_r(y,\tau))} \int_{Q_r} |f(x,t)|^q \, dxdt \right)^{\frac{1}{q}} .$$

These spaces are widely studied under various conditions on φ (see [11, 12, 15] and the references therein). Let $f \in L^1_{\rm loc}(\mathbb{R}^{n+1})$ and

$$\mathscr{M}f(y,\tau) = \sup_{r>0} \frac{1}{|\mathscr{I}_r(y,\tau)|} \int_{\mathscr{I}_r(y,\tau)} |f(x,t)|\, dxdt$$

be the *Hardy-Littlewood maximal operator.* It is well known that $\mathscr{M}$ is bounded in the Lebesgue and Morrey spaces. Moreover, it is bounded also in $L^{p,\varphi}$ for various φ, [11, 12, 15]. Here we need the following result.

Lemma 1 (Maximal inequality, [12]). *Assume that there is a positive constants κ such that for any fixed $(y,\tau) \in \mathbb{R}^{n+1}$ and any $r > 0$ holds true*

$$\sup_{r<s<\infty} \frac{\operatorname*{essinf}_{s<\sigma<\infty} \varphi(\mathscr{I}_\sigma(y,\tau))\sigma^{n+2}}{s^{n+2}} \le \kappa \frac{\varphi(\mathscr{I}_r(y,\tau))}{r^{n+2}} . \tag{6}$$

Then there is a constant $c_q > 0$ such that for any $q \in (1,\infty)$

$$\|f\|_{L^{q,\varphi}(\mathbb{R}^{n+1})} \le \|Mf\|_{L^{q,\varphi}(\mathbb{R}^{n+1})} \le c_q \|f\|_{L^{q,\varphi}(\mathbb{R}^{n+1})} \quad \forall\, f \in L^{q,\varphi}(\mathbb{R}^{n+1}) .$$

Imposing in addition a *monotonicity condition* on φ, precisely

$$\varphi(\mathscr{I}_r(y,\tau)) \le \varphi(\mathscr{I}_s(z,\xi)) \quad \text{for all} \quad \mathscr{I}_r(y,\tau) \subset \mathscr{I}_s(z,\xi) \tag{7}$$

we get the estimate

$$\sup_{\substack{(y,\tau)\in Q\\ r>0}} \frac{|\mathscr{I}_r(y,\tau) \cap Q|}{\varphi\,(\mathscr{I}_r(y,\tau))} \le \kappa_1, \tag{8}$$

with $\kappa_1 > 0$ depending on n, φ, and Q [5, 12]. Then the Hölder inequality implies

$$\|f\|^2_{L^2(Q)} \le c(n,p,|Q|,\varphi)\|f\|_{L^{p,\varphi}(Q)} . \tag{9}$$

3 Statement of the Problem and Main Result

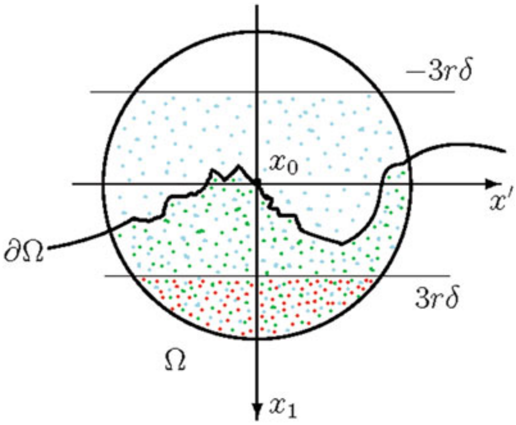
Definition 2. We say that $(A^{\alpha\beta}_{ij}, \Omega)$ are (δ, R)-vanishing of codimension 1, if:

i) For every fixed $(y,\tau) \in Q$ and $r \in (0, \frac{1}{3}R]$, there exists a new coordinate system (x,t) centered in $(y,\tau) \equiv (0,0)$ such that

$$\frac{1}{|\mathscr{C}_{3r}(0,0)|} \int_{\mathscr{C}_{3r}(0,0)} |A^{\alpha\beta}_{ij}(x,t) - \overline{A}^{\alpha\beta}_{ij\ \mathscr{C}^{x^1}_{3r}(0,0)}(x^1)|^2\, dxdt \le \delta^2 \tag{10}$$

where $\overline{A}^{\alpha\beta}_{ij\ \mathscr{C}^{x^1}_{3r}(0,0)}(x^1)$ is the integral average of $A^{\alpha\beta}_{ij}$ over $\mathscr{C}^{x^1}_{3r}(0,0)$.

ii) For every fixed $(y, \tau) \in Q$ and $r \in (0, \frac{1}{3}R]$ such that $\text{dist}(y, \partial\Omega) = \text{dist}(y, x_0) \leq \sqrt{2}r$ with $x_0 \in \partial\Omega$, there exists a coordinate system (x, t) in which $(x_0, \tau) \equiv (0, 0)$ is the origin and Ω verifies the *Reifenberg condition* illustrated on the graphic:



$$\Omega \cap \{x \in \mathscr{C}_{3r}^{\tau}(0) : x^1 > 3r\delta\} \subset \Omega \cap \mathscr{C}_{3r}^{\tau}(0) \subset \Omega \cap \{x \in \mathscr{C}_{3r}^{\tau}(0) : x^1 > -3r\delta\}\,.$$

The part **i)** means that the coefficients have *small mean oscillation* (small BMO) with respect to (x', t), while in x^1 they are only *measurable* and could have arbitrary jump. The second part of the definition asserts that Ω is (δ, R)-*Reifenberg flat domain* (see [18, 20]). Moreover, it implies (cf. [16, 17]) that there is a constant $0 < \alpha = \alpha(\delta, n, \partial\Omega) < \frac{1}{2}$ such that

$$\alpha\,|\mathscr{C}_{3r}^{\tau}(x_0)| \;\leq\; |\mathscr{C}_{3r}^{\tau}(x_0) \cap \Omega| \;\leq\; (1-\alpha)|\mathscr{C}_{3r}^{\tau}(x_0)|\,.$$

We prove the following result (see [6] for details).

Theorem 1. *For any given* $p \in (2, \infty)$ *and weight* φ *satisfying* (6)–(7)*, suppose that* $|\mathbf{F}|^2, |\psi_t|^2, |D\psi|^2 \in L^{\frac{p}{2},\varphi}(Q)$*. Let* $\mathbf{u} \in \mathscr{A}$ *be a solution of* (3)–(4)*. Then there exists a small constant* $\delta = \delta(\lambda, \Lambda, m, n, p, \varphi) > 0$ *such that if* $\left(A_{ij}^{\alpha\beta}, \Omega\right)$ *are* (δ, R)*-vanishing of codimension 1, then* $|D\mathbf{u}|^2 \in L^{\frac{p}{2},\varphi}(Q)$ *and*

$$\left\| |D\mathbf{u}|^2 \right\|_{L^{\frac{p}{2},\varphi}(Q)} \leq c\left(\left\| |\mathbf{F}|^2 \right\|_{L^{\frac{p}{2},\varphi}(Q)} + \left\| |\psi_t|^2 \right\|_{L^{\frac{p}{2},\varphi}(Q)} + \left\| |D\psi|^2 \right\|_{L^{\frac{p}{2},\varphi}(Q)} \right). \quad (11)$$

Let us note that (5) and (9) ensure the existence of unique weak solution of (3).

4 Auxiliary Results

In this section, we give several preliminary results that we need in order to prove the main theorem (see [5–7] for details). Our approach is based on the Vitali covering lemma and estimates of the upper level sets of the maximal function of the gradient $\mathscr{M}(|D\mathbf{u}|^2)$. Fix $(y_0, \tau_0) \in Q$, take a cylinder $\mathscr{I}_r(y_0, \tau_0)$, and consider $Q_r = \mathscr{I}_r(y_0, \tau_0) \cap Q$. For any solution $\mathbf{u}$ of (3), we define the upper level sets

$$
\begin{aligned}
\mathfrak{C} =& \{(x,t) \in Q_r : \mathscr{M}(|D\mathbf{u}|^2) > N^2\}, \\
\mathfrak{D} =& \{(x,t) \in Q_r : \mathscr{M}(|D\mathbf{u}|^2) > 1\} \cup \{(x,t) \in Q_r : \mathscr{M}(|\mathbf{F}|^2) > \delta^2\} \\
& \cup \{(x,t) \in Q_r : \mathscr{M}(|\psi_t|^2 + |D\psi|^2) > \delta^2\}, \qquad (12)\\
\mathfrak{C} & \subset \mathfrak{D} \subset Q_r \quad \text{for } N > 1 .
\end{aligned}
$$

For any $(y, \tau) \in \mathfrak{C}$ and for each $\rho > 0$, we define the measure function

$$
\Theta(\rho) = \frac{|\mathfrak{C} \cap \mathscr{C}_\rho(y,\tau)|}{|\mathscr{C}_\rho(y,\tau)|} \in C^0(0,\infty), \qquad \lim_{\rho \to 0+} \Theta(\rho) = 1, \qquad \lim_{\rho \to +\infty} \Theta(\rho) = 0 .
$$

Lemma 2. *Suppose that there exists $\varepsilon \in (0,1)$ for which $\Theta(1) < \varepsilon$. Then the following estimate holds $|\mathfrak{C}| \le \varepsilon \left(\frac{10\sqrt{2}}{1-\delta}\right)^{n+2} |\mathfrak{D}|$.*

Moreover, for those ρ for which $\Theta(\rho) \ge \varepsilon$ holds the inclusion $Q_r \cap \mathfrak{C}_\rho(y,\tau) \subset \mathfrak{D}$.

The estimate follows by the Vitali covering lemma applied to a set of mutually disjoint cylinders $\{\mathscr{C}_{\rho_i}(y_i, \tau_i)\}$ such that $(y_i, \tau_i) \in \mathfrak{C}$ and $\Theta(\rho_i) = \varepsilon$ (see [6, Lemma 4.1]). The inclusion can be proved as in [6, Lemma 5.1] using localizable solutions studied in [19].

The next result comes from the measure theory and has been proven in various functional spaces (see [5, 6, 12]).

Lemma 3. *Let $h \in L^1(Q)$ be a nonnegative function, $q \in (1,\infty)$ and $\zeta > 0, \theta > 1$ be constants. Then $h \in L^{q,\varphi}(Q)$ if and only if*

$$
\mathscr{S} := \sup_{\substack{(y,\tau)\in Q \\ r>0}} \sum_{k \ge 1} \frac{\theta^{kq} |\{(x,t) \in Q_r : h(x,t) > \zeta \theta^k\}|}{\varphi(\mathscr{I}_r(y,\tau))} < \infty .
$$

Moreover, there exists $c = c(\theta, \zeta, q, \varphi, Q)$ such that $\frac{1}{c}\mathscr{S} \le \|h\|^q_{L^{q,\varphi}(Q)} \le c(1 + \mathscr{S})$.

Fixing $\varepsilon > 0$ as in Lemma 2, we obtain a power decay estimate for the upper level sets of the maximal function of $D\mathbf{u}$.

Lemma 4. *Suppose $\Theta(1) < \varepsilon$ for some $\varepsilon \in (0,1)$, then for each $(y,\tau) \in Q_r$ and each positive integer k, we have*

$$\begin{aligned}\left|\left\{(x,t) \in Q_r\colon \mathscr{M}(|D\mathbf{u}|^2) > N^{2k}\right\}\right| &\le \epsilon_1^k \left|\left\{(x,t) \in Q_r\colon \mathscr{M}(|D\mathbf{u}|^2) > 1\right\}\right| \\ &+ \sum_{i=1}^{k} \epsilon_1^i \left|\left\{(x,t) \in Q_r\colon \mathscr{M}(|\mathbf{F}|^2) > \delta^2 N^{2(k-i)}\right\}\right) \\ &+ \sum_{i=1}^{k} \epsilon_1^i \left|\left\{(x,t) \in Q_r\colon \mathscr{M}(|\psi_t|^2 + |D\psi|^2) > \delta^2 N^{2(k-i)}\right\}\right|\end{aligned} \tag{13}$$

with $\epsilon_1 = \varepsilon \left(\frac{10\sqrt{2}}{1-\delta}\right)^{n+2}$.

The Lemma 2 ensures the validity of (13) for $k = 1$. Further, the proof follows by induction.

To get (11), we use the invariance of the problem under scaling and normalization. This property follows by straightforward calculations.

Proof (Theorem 1). By suitable change of the functions, we make the norms of $\mathbf{F}$ and ψ small enough. Precisely, taking

$$K \equiv \||\mathbf{F}|^2\|_{L^{\frac{p}{2},\varphi}(Q)} + \||\psi_t|^2\|_{L^{\frac{p}{2},\varphi}(Q)} + \||D\psi|^2\|_{L^{\frac{p}{2},\varphi}(Q)}$$

we define

$$\tilde{\mathbf{u}} = \frac{\delta \mathbf{u}(x,t)}{\sqrt{K}}, \quad \widetilde{\mathbf{F}} = \frac{\delta \mathbf{F}(x,t)}{\sqrt{K}}, \quad \tilde{\psi} = \frac{\delta \psi(x,t)}{\sqrt{K}}\,. \tag{14}$$

Consider now upper level sets $\mathfrak{C}$ and $\mathfrak{D}$ defined for $\tilde{\mathbf{u}}$. For each $(y,\tau) \in \mathfrak{C}$, we have

$$\begin{aligned}\Theta(1) \le c|\mathscr{C}| \le c\int_{Q_r} \mathscr{M}(|D\tilde{\mathbf{u}}|^2)\,dxdt &\le c\int_{Q} \left(|\widetilde{\mathbf{F}}|^2 + |\tilde{\psi}_t|^2 + |D\tilde{\psi}|^2\right) dxdt \\ &\le c\left(\||\widetilde{\mathbf{F}}|^2\|_{L^{\frac{p}{2},\varphi}(Q)} + \||\tilde{\psi}_t|^2\|_{L^{\frac{p}{2},\varphi}(Q)} + \||D\tilde{\psi}|^2\|_{L^{\frac{p}{2},\varphi}(Q)}\right) \le c\delta^2 < \varepsilon\,.\end{aligned}$$

Now applying Lemma 3 with $h = \mathscr{M}(|D\tilde{\mathbf{u}}|^2), \theta = N^2, \lambda = 1, q = \frac{p}{2}$, and Lemma 4, we get

$$\begin{aligned}\Sigma &\equiv \sum_{k=1}^{\infty} \frac{N^{2k\frac{p}{2}} \left|\left\{(x,t) \in Q_r\colon \mathscr{M}(|D\tilde{\mathbf{u}}|^2) > N^{2k}\right\}\right|}{\varphi(\mathscr{I}_r(y,\tau))} \\ &\le \sum_{k=1}^{\infty} \frac{N^{kp}\epsilon_1^k \left|\left\{(x,t) \in Q_r : \mathscr{M}(|D\tilde{\mathbf{u}}|^2) > 1\right\}\right|}{\varphi(\mathscr{I}_r(y,\tau))}\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=1}^{\infty} N^{kp} \sum_{i=1}^{k} \frac{\epsilon_1^i \left|\left\{(x,t)\in Q_r : \mathscr{M}(|\widetilde{\mathbf{F}}|^2) > \delta^2 N^{2(k-i)}\right\}\right|}{\varphi(\mathscr{I}_r(y,\tau))} \\
&+\sum_{k=1}^{\infty} N^{kp} \sum_{i=1}^{k} \frac{\epsilon_1^i \left|\left\{(x,t)\in Q_r : \mathscr{M}(|\tilde{\psi}_t|^2 + |D\tilde{\psi}|^2) > \delta^2 N^{2(k-i)}\right\}\right|}{\varphi(\mathscr{I}_r(y,\tau))} \\
\leq &\sum_{k=1}^{\infty} (N^p \epsilon_1)^k \frac{|Q_r|}{\varphi(\mathscr{I}_r(y,\tau))} \\
&+\sum_{i=1}^{\infty} (N^p \epsilon_1)^i \underbrace{\sum_{k=i}^{\infty} N^{(k-i)p} \frac{\left|\left\{(x,t)\in Q_r : \mathscr{M}(|\widetilde{\mathbf{F}}|^2) > \delta^2 N^{2(k-i)}\right\}\right|}{\varphi(\mathscr{I}_r(y,\tau))}}_{\Sigma'} \\
&+\sum_{i=1}^{\infty} (N^p \epsilon_1)^i \underbrace{\sum_{k=i}^{\infty} N^{(k-i)p} \frac{\left|\left\{(x,t)\in Q_r : \mathscr{M}(|\tilde{\psi}_t|^2 + |D\tilde{\psi}|^2) > \delta^2 N^{2(k-i)}\right\}\right|}{\varphi(\mathscr{I}_r(y,\tau))}}_{\Sigma''} \\
\leq &\sum_{k=1}^{\infty} (N^p \epsilon_1)^k \left(\kappa_1 + \Sigma' + \Sigma''\right)
\end{aligned}
$$

where we have used (8) in the last step. By the auxiliary Lemmas, we get

$$
\begin{aligned}
\Sigma' \leq & \frac{c}{\varphi(\mathscr{I}_r(y,\tau))}\left(|Q_r| + \int_{Q_r} \frac{|\widetilde{\mathbf{F}}(x,t)|^p}{\delta^p}\, dxdt\right) \\
\Sigma'' \leq & \frac{c}{\varphi(\mathscr{I}_r(y,\tau))}\left(|Q_r| + \int_{Q_r} \frac{|\tilde{\psi}_t(x,t)|^p + |D\tilde{\psi}(x,t)|^p}{\delta^p}\, dxdt\right).
\end{aligned}
$$

Unifying the above estimates, applying again (8) and (14), and taking the supremum of Σ over $(y_0, \tau_0) \in Q$ and $r > 0$, we get

$$
\begin{aligned}
\mathscr{S} &\leq c \sum_{k=1}^{\infty} (N^p \epsilon_1)^k \left[1 + \frac{1}{\delta^p}\|\widetilde{\mathbf{F}}\|^p_{L^{p,\varphi}(Q)} + \frac{1}{\delta^p}\left(\|\tilde{\psi}_t\|^p_{L^{p,\varphi}(Q)} + \|D\tilde{\psi}\|^p_{L^{p,\varphi}(Q)}\right)\right] \\
&\leq c \sum_{k=1}^{\infty} (N^p \epsilon_1)^k\,.
\end{aligned}
$$

Taking ε small enough such that $N^p \epsilon_1 \leq N^p c_1 \varepsilon < 1$, we get $\mathscr{S} < \infty$ and in view of Lemma 2 and (14), we get (11) through the maximal inequality.

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Chebyshev Spectral Approximation for Diffusion Equations with Distributed Order in Time

Maria Luísa Morgado and Magda Rebelo

Abstract In this work we provide a numerical method for the diffusion equation with distributed order in time. The basic idea is to expand the unknown function in Chebyshev polynomials for the time variable t and reduce the problem to the solution of a system of algebraic equations, which may then be solved by any standard numerical technique. We apply the method to the forward and backward problems. Some numerical experiments are provided in order to show the performance and accuracy of the proposed method.

Keywords Fractional differential equation • Caputo derivative • Diffusion equation • Chebyshev polynomials • Distributed order equation

Mathematics Subject Classification (2000): 26A33, 41A50

1 Introduction

In the last decades, lots of attention has been devoted to the time fractional diffusion equation (TFDE), namely, the one in the Caputo sense:

$$\frac{\partial^\alpha u(t,x)}{\partial t^\alpha} = D\frac{\partial^2 u(t,x)}{\partial x^2} + f(t,x), \quad 0 < t \le a,\ 0 < x < b, \tag{1}$$

where $0 < \alpha < 1$ and the fractional Caputo derivative is defined by [2]

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$$\frac{\partial^\alpha u(t,x)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{\partial^n u(s,x)}{\partial s^n}\, ds, \tag{2}$$

where n is the smallest integer greater than or equal to α. The TFDE has been found in a broad variety of engineering, biological, finance, and physical processes where anomalous diffusion (AD) occurs (see, e.g., [6, 8, 10]). More recently, a general equation has attracted the scientific community, the distributed-order time fractional diffusion equation, given by

$$\int_0^1 c(\alpha) \frac{\partial^\alpha u(t,x)}{\partial t^\alpha}\, d\alpha = \frac{\partial^2 u(t,x)}{\partial x^2} + f(t,x), \quad 0 < t \le a,\ 0 < x < b, \tag{3}$$

where the function $c(\alpha)$ acting as weight for the order of differentiation is such that [7, 9] $c(\alpha) \ge 0$ and $\int_0^1 c(\alpha)\, d\alpha = C > 0$. Obviously, if $c(\beta) = \frac{1}{D}$ if $\beta = \alpha$ and 0 otherwise, then (3) reduces to (1).

Here, we will be interested in the numerical approximation of this type of distributed-order equations with boundary conditions of Dirichlet type:

$$u(t,0) = \phi_0(t),\ u(t,b) = \phi_b(t), \quad 0 < t < a, \tag{4}$$

and we will distinguish the following two problems: a forward problem (FDODE) where (3) and (4) is subject to an initial condition

$$u(0,x) = g_0(x), \quad 0 < x < b, \tag{5}$$

and a backward problem (BDODE), the case where (5) is replaced with the terminal condition

$$u(a,x) = g_a(x), \quad 0 < x < b. \tag{6}$$

Numerical methods are crucial for this kind of fractional differential equations, since only in a very few special cases, the analytical solutions can be found. While the methods developed for TFDEs are already relatively wide, the same cannot be said for the distributed-order diffusion equation case, since, to the best of our knowledge, only a few works have been reported. In [4] an implicit finite-difference method has been derived for the one-dimensional distributed-order diffusion equation; in [11] the same idea has been followed for the numerical approximation of nonlinear reaction-diffusion equations with distributed order in time. In [12] a numerical scheme has been developed for the solution of a distributed-order diffusion equation containing also a fractional derivative in space. In [5], a finite-difference method was presented for the two-dimensional distributed-order diffusion equation, together with an extrapolation technique to improve the convergence orders in time. In all these papers, only finite-difference approximations have been considered for the fractional time derivative, which may become heavy from the computational point of view, due to the nonlocal property of fractional differential operators. Moreover, in all of these works, only forward problems have been investigated.

Here we will follow an alternative approach: we consider a Chebyshev polynomial approximation of the fractional derivatives. The paper is organized in the following way: we start with a section devoted to some preliminary results that will be used in the forthcoming sections. In Sect. 3 we describe the numerical method and we end with some numerical examples and some conclusions.

2 Preliminaries

In this section we present some auxiliary results that will be used in the derivation of the numerical scheme. For the approximation of the integral term, we will use Gaussian quadrature. N-point Gaussian quadrature rules are a special class of quadrature formulas that yield the exact value of a definite integral for integral functions that are polynomials of degree less than or equal to $(2N-1)$.This can be achieved by suitable choices of the points x_i and weights ω_i, $i = 1, \ldots, N$. These rules are conventionally given in the interval $[-1, 1]$ and may be given by $\displaystyle\int_{-1}^{1} f(x)\, dx \cong \sum_{i=1}^{N} \omega_i f(x_i)$. Obviously other intervals can be considered by using proper variable substitutions. In our case, since we are dealing with the interval $[0, 1]$, it is easy to see that

$$\int_0^1 f(x)\, dx = \frac{1}{2}\int_{-1}^{1} f\left(\frac{t+1}{2}\right) dt \cong \frac{1}{2}\sum_{j=1}^{N} \omega_j f\left(\frac{t_j+1}{2}\right).$$

Lemma 1 ([1]). *If* $f \in C^{(2N)}([0,1])$, *then with* $\gamma_N = \dfrac{(N!)^4}{(2N+1)[(2N)!]^3} \approx \dfrac{\Pi}{4^N}$, *we have:*

$$\int_0^1 f(x)\, dx - \frac{1}{2}\sum_{j=1}^{N} \omega_j f\left(\frac{t_j+1}{2}\right) = \gamma_N \frac{f^{(2N)}(c_N)}{(2N)!}, \quad 0 \le c_N \le 1.$$

As we have mentioned in the Sect. 1, we will use Chebyshev polynomials to approximate the fractional derivatives. Chebyshev polynomials of degree n, $T_n(z)$ are defined in the interval $[-1, 1]$.

In order to use them in the interval $[0, a]$, we introduce the change of variable $z = 2t/a - 1$ and obtain the so-called shifted Chebyshev polynomials $T_{a,n}(t) = T_n\left(\dfrac{2t}{a} - 1\right)$. These shifted Chebyshev polynomials can also be obtained from the following expression (see [3]):

$$T_{a,n}(t) = n\sum_{k=0}^{n}(-1)^{n-k}\frac{2^{2k}(n+k-1)!}{(2k)!(n-k)!a^k}t^k, \quad n = 1, 2, \ldots,$$

where

$$T_{a,i}(0) = (-1)^i \text{ and } T_{a,i}(a) = 1, \tag{7}$$

and satisfy the following orthogonality relation:

$$\int_0^a T_{a,j}(t)T_{a,k}(t)\omega_a(t)dt = \delta_{kj}h_k,$$

where $\omega_a(t) = \dfrac{1}{\sqrt{at-t^2}}$ and $h_0 = \pi$, $h_k = \frac{\pi}{2}$, $k = 1, 2, \ldots$.
A function $y(t)$ belonging to the space of square integrable functions on $[0, a]$ may be expressed as

$$y(t) = \sum_{i=0}^{\infty} c_i T_{a,i}(t), \tag{8}$$

where the coefficients c_i are given by

$$c_i = \frac{< y(t), T_{a,i}(t) >}{\|T_{a,i}\|^2} = \frac{1}{h_i}\int_0^a y(t)T_{a,i}(t)\omega_a(t)\, dt, \quad i = 0, 1, 2, \ldots.$$

For computational purposes, only the first $(m+1)$ terms in (8) are considered:

$$y_m(t) = \sum_{i=0}^{m} c_i T_{a,i}(t), \quad t \in [0, a], \tag{9}$$

and the following result holds:

Theorem 1 ([3]). *Let $y(t)$ be a square integrable function on $[0, a]$. Then, given $m \in \mathbf{N}$, $y(t)$ may be approximated by $y_m(t)$, defined by (9), and for $\alpha > 0$, we have*

$$D^\alpha y_m(t) = \sum_{i=\lceil\alpha\rceil}^{m}\sum_{k=\lceil\alpha\rceil}^{i} c_i w_{i,k}^{(\alpha)} t^{k-\alpha}, \quad w_{i,k}^{(\alpha)} = \frac{(-1)^{i-k}2^{2k}i(i+k-1)!\Gamma(k+1)}{(i-k)!(2k)!\Gamma(k+1-\alpha)a^k} \tag{10}$$

and the error $|E(m)| = |D^\alpha y(t) - D^\alpha y_m(t)| \le \sum_{i=m+1}^{\infty} c_i \left(\sum_{k=\lceil\alpha\rceil}^{i}\sum_{j=0}^{k-\lceil\alpha\rceil}\theta_{i,j,k}\right),$

where, for $h_0 = 2$, $h_j = 1,\ j = 1, 2, \dots$,

$$\theta_{i,j,k} = \frac{(-1)^{i-k} 2i(i+k-1)!\Gamma\left(k-\alpha+\frac{1}{2}\right)}{h_j \Gamma\left(k+\frac{1}{2}\right)(i-k)!\Gamma(k-\alpha-j+1)\Gamma(k+j-\alpha+1)a^{\alpha}}.$$

3 Numerical Method

In the derivation of the numerical method, we proceed as in the classical (integer order) case. Let

$$u(t,x) \approx u_m(t,x) = \sum_{i=0}^{m} v_i(x) T_{a,i}(t), \tag{11}$$

Using a Gaussian quadrature formula with n points, (9) and (10), we obtain

$$\frac{1}{2}\sum_{j=1}^{n} \omega_j c\left(\frac{\beta_j+1}{2}\right) \sum_{i=\lceil \frac{\beta_j+1}{2} \rceil}^{m} \sum_{k=0}^{i-\lceil \frac{\beta_j+1}{2} \rceil} v_i(x) w_{i,k}^{\left(\frac{\beta_j+1}{2}\right)} t^{i-k-\frac{\beta_j+1}{2}} = \sum_{i=0}^{m} \ddot{v}_i(x) T_{a,i}(t) + f(x,t). \tag{12}$$

Note that in this case $\frac{\beta_j+1}{2} \in [0,1]$, $j=1,\dots,n$, and then $\left\lceil \frac{\beta_j+1}{2} \right\rceil = 1$, $j=1,\dots,n$.

Now, we collocate Eq. (12) at m points t_p. For collocation points, we use the roots of the shifted Chebyshev polynomial of degree m, $T_{a,m}(t)$:

$$\frac{1}{2}\sum_{j=1}^{n} \omega_j c\left(\frac{\beta_j+1}{2}\right) \sum_{i=1}^{m} \sum_{k=0}^{i-1} v_i(x) w_{i,k}^{\left(\frac{\beta_j+1}{2}\right)} t_p^{i-k-\frac{\beta_j+1}{2}} = \sum_{i=0}^{m} \ddot{v}_i(x) T_{L,i}(t_p) + f(x,t_p),$$

$$p = 0, \dots, m-1. \tag{13}$$

We obtain in this way m ordinary differential equations on the $(m+1)$ unknowns $v_i(x),\ i = 0, \dots, m.$

Using the fact that $T_{a,i}(0) = (-1)^i$ and taking the initial condition (5) into account, we obtain the extra equation:

$$\sum_{i=0}^{m} (-1)^i v_i(x) = g_0(x). \tag{14}$$

Alternatively, since $T_{a,i}(a) = 1$, from the terminal condition (6), we obtain

$$\sum_{i=0}^{m} v_i(x) = g_a(x) \tag{15}$$

On the other hand, by substituting (11) on the boundary conditions (4), we obtain

$$\sum_{i=0}^{m} v_i(0)T_{a,i}(t) = \phi_0(t), \tag{16}$$

$$\sum_{i=0}^{m} v_i(b)T_{a,i}(t) = \phi_b(t). \tag{17}$$

At the collocation points $t_p, \ p = 0, \ldots, m-1$, (16) and (17) are as follows:

$$\sum_{i=0}^{m} v_i(0)T_{a,i}(t_p) = \phi_0(t_p), \quad p = 0, \ldots, m-1, \tag{18}$$

$$\sum_{i=0}^{m} v_i(b)T_{a,i}(t_p) = \phi_b(t_p), \quad p = 0, \ldots, m-1. \tag{19}$$

Therefore, in order to obtain the functions $\{v_i\}_{i=0}^{m}$ that define the approximate solution of the forward problem (3), (4), (5), we must solve the system of differential equations (13)–(14), with boundary conditions (18) and (19).
In order to obtain an approximate solution of the backward problem (3), (4), (6), we must solve the system of differential equations (13)–(15), with boundary conditions (18) and (19).

4 Numerical Results

In this section, we apply the proposed method to solve some examples for which the analytical solution is known. We define the absolute error at the point (t, x) by

$$e_m(t, x) = |u(t, x) - u_m(t, x)|, \quad (t, x) \in [0, a] \times [0, b].$$

Example 1. Forward problem:

$$c(\alpha) = \Gamma\left(\frac{5}{2} - \alpha\right)$$

$$f(t, x) = \frac{\sqrt{t}(x-1)^2\left(3\sqrt{\pi}(t-1)(x-1)^2x^2 - 8t(5x(3x-2)+1)\log(t)\right)}{4\log(t)},$$

with analytical solution given by $u(x, t) = t^{3/2}x^2(1-x)^4, \quad (t, x) \in [0, 1] \times [0, 1]$.

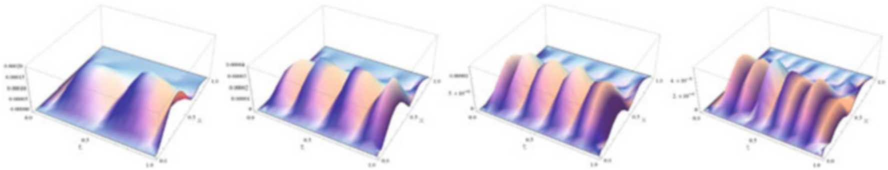

Fig. 1 Example 1: pointwise absolute error at the points $(t, x) \in [0, 1] \times [0, 1]$ for several values of m. From *left to right*: $m = 3, m = 5, m = 7, m = 9$

In order to approximate the integral that defines the distributed-order derivative, we will use a 3-point Gaussian quadrature formula.

In Fig. 1 the domain pointwise absolute errors are displayed. We see that the pointwise error goes up to the order of 2×10^{-4}, 4×10^{-5}, 1×10^{-5}, *and* 4×10^{-6} if we consider on the series expansion of u, (11), $m = 3$, $m = 5$, $m = 7$, and $m = 9$, respectively. This shows that the numerical solutions are in good agreement with the exact solutions, and we have more accuracy if we consider more terms on the series approximation (11) of u.

As a second example we consider, a backward problem which is defined by

Example 2. Backward problem

$$c(\alpha) = \Gamma\left(\frac{7}{2} - \alpha\right)$$

$$f(t, x) = \frac{t^{3/2}\left(15\sqrt{\pi}(t-1)x(x-1)^2 + 16t(2-3x)\log(t)\right)}{8\log(t)},$$

with analytical solution given by $u(t, x) = t^{5/2}(1-x)^2 x, \quad (t, x) \in [0, 1] \times [0, 1]$.

In these backward problems, the unknown solution $u(x, t)$ has to be determined from the boundary measurements $\phi_0(t)$ and $\phi_b(t)$ and terminal time measurement $g_a(x)$, which normally contain noises in practical problems. Thus, in order to test the proposed method, first we apply the method, with several values of m, to the second example without noise on the data and then we apply the method with some noise on the boundary and terminal data.

The comparison results between $u(0, x)$ and $u_m(0, x)$ are displayed in Table 1, for several values of $x \in [0, 1]$ and $m = 1, 3, 5, 7$ and 9. From the results in Table 1, it can be observed that the error is smaller for the biggest value of m that we consider. Thus, the overall errors can be made smaller by adding new terms from the series (11) that approximate $u(t, x)$.

Now, we consider Example 2 with several levels of noise, $\delta = 10^{-i}$, $i = 2, \ldots, 5$, on the boundary and terminal data: $\bar{g_a}(x) = g_a(x) + \delta$, $\bar{\phi_0}(t) = \phi_0(t) + \delta$, $\bar{\phi_b}(t) = \phi_b(t) + \delta$. In Fig. 2 we show the absolute error at points $(x, 0)$, $x \in [0, 1]$ obtained for the approximation (11) with $m = 5$ and several levels of noise. It can be observed that the noise has influence on the numerical results.

Table 1 Example 2: the absolute errors related with the approximate solutions $u_m(x,t)$, $m = 1, 3, 5, 7$ and $m = 9$, at the points $(0,x)$, $x \in \{0.2, 0.4, 0.5, 0.6, 0.7, 0.9\}$

x	$e_1(0,x)\|$	$e_3(0,x)\|$	$e_5(0,x)$	$e_7(0,x)$	$e_9(0,x)$
0.2	1.261×10^{-1}	3.614×10^{-3}	5.725×10^{-6}	8.529×10^{-6}	4.397×10^{-6}
0.4	1.583×10^{-1}	5.447×10^{-3}	2.407×10^{-5}	5.669×10^{-6}	3.122×10^{-6}
0.5	1.466×10^{-1}	5.545×10^{-3}	4.123×10^{-5}	1.809×10^{-6}	1.048×10^{-6}
0.6	1.221×10^{-1}	5.114×10^{-3}	5.335×10^{-5}	2.030×10^{-6}	1.064×10^{-6}
0.7	8.987×10^{-2}	4.223×10^{-3}	5.660×10^{-5}	4.689×10^{-6}	2.514×10^{-6}
0.9	5.522×10^{-2}	2.979×10^{-3}	4.870×10^{-5}	5.350×10^{-6}	2.825×10^{-6}

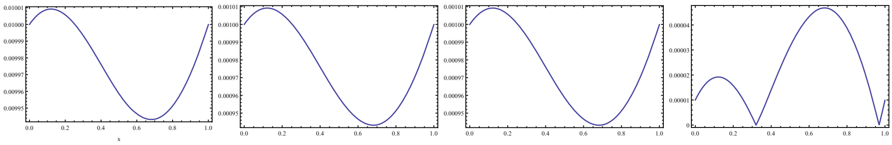

Fig. 2 Absolute error for Example 2 using $m = 5$ and different noise levels δ. From *left to right*: $\delta = 10^{-2}$, $\delta = 10^{-3}$, $\delta = 10^{-4}$, $\delta = 10^{-5}$

5 Conclusions

In this work we have presented an alternative method (than finite-difference methods) for the numerical approximation of time distributed-order diffusion equations that is able to deal with both initial (or forward) and terminal (or backward) problems. The numerical results presented for examples with known analytical solutions illustrate the accuracy of the proposed method. In the future we intend to provide a full comparison with the finite-difference methods and analyze the convergence of the scheme.

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On Polarization Dynamics in Ferroelectric Materials

M. Driss Aouragh, M. Hadda and M. Tilioua

Abstract We consider a mathematical model describing polarization dynamics in ferroelectric material. The model consists of a Maxwell system for electromagnetic field coupled with a second-order time-dependent equation for the evolution of polarization. We study the long-time behaviour of weak solutions and prove that all points of the ω-limit set of any trajectories are solutions of the stationary model.

Keywords Ferroelectrics • Polarization dynamics • Maxwell system • Global existence • Uniqueness • Long-time behaviour

AMS Subject Classifications: 35L10, 35K05

1 Introduction

In this work we are dealing with long-time behaviour of weak solutions of a mathematical system arising in the theory of ferroelectric materials. We shall consider the model considered in [2]. It is given by full Maxwell system for electromagnetic field coupled with a second-order time-dependent equation for the evolution of polarization. To describe the model equations, we consider $\Omega \subset \mathbb{R}^3$ a bounded and regular open set of $\mathbb{R}^3$. The generic point of $\mathbb{R}^3$ is denoted by x. We assume that a ferroelectric material occupies the domain Ω. The polarization field

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of the ferroelectric material is denoted by $P(t,x)$. Its evolution is governed by the following second-order time-dependent equation; see [2]

$$\begin{cases} \partial_t^2 P + \alpha\partial_t P + \operatorname{curl}^2 P + kP = \beta E \ \text{ in } \mathbb{R}^+ \times \Omega, \\ \operatorname{curl} P \times \nu + \delta\nu \times ((\partial_t P + \alpha P) \times \nu) = 0 \text{ on } \mathbb{R}^+ \times \partial\Omega, \\ P(t=0) = P_0 \text{ and } \partial_t P(t=0) = P_1 \text{ in } \Omega, \end{cases} \tag{1}$$

coupled with

$$\begin{cases} \partial_t H - \operatorname{curl} E = 0 \ \text{ in } \mathbb{R}^+ \times \Omega, \\ \partial_t (E+P) + \sigma E + \operatorname{curl} H = 0 \ \text{ in } \mathbb{R}^+ \times \Omega, \\ H \times \nu + \beta\nu \times (E \times \nu) = 0 \ \text{ on } \mathbb{R}^+ \times \partial\Omega, \\ H(0,x) = H_0(x) \ \text{ and } E(0,x) = E_0(x) \text{ in } \Omega, \end{cases} \tag{2}$$

where (E,H) represents the electromagnetic field, "$\times$" is the usual vector product, curl denotes the rotational operator and $\operatorname{curl}^2 P = \operatorname{curl}(\operatorname{curl} P) = \nabla\times(\nabla\times P)$. Here ν denotes the outward normal on the boundary of Ω, and α, β, σ and δ are positive constants. The Silver-Müller boundary condition considered in (2) is a first-order approximation to the so-called transparent boundary conditions, i.e. no energy loss is observed on the boundary. Silver-Müller condition allows for reflections back into Ω. It can be found in the literature under other names as well such as Leontovich or impedance boundary condition. For more information about the Silver-Müller boundary condition, we highly recommend the book of Müller [9].

Throughout, we make use of the following notation. For Ω an open bounded domain of $\mathbb{R}^3$, we denote by $\mathbb{L}^2(\Omega) = (L^2(\Omega))^3$ and $\mathbb{H}^1(\Omega) = (H^1(\Omega))^3$ the classical Hilbert spaces equipped with the usual norm denoted by $|\cdot|_2$ and $|\cdot|_{\mathbb{H}^1(\Omega)}$. We also consider the following classical space used in the theory of Maxwell equations $\mathscr{H}(\operatorname{curl}, \Omega) = \{u \in \mathbb{L}^2(\Omega), \operatorname{curl} u \in \mathbb{L}^2(\Omega)\}$.

We define the energy

$$\mathscr{E}(t) = |\partial_t P(t)|_2^2 + |\operatorname{curl} P(t)|_2^2 + k|P(t)|_2^2 + \alpha|\sqrt{\delta}P(t)\times\nu|_2^2 + |E(t)|_2^2 + |H(t)|_2^2 \tag{3}$$

and the initial energy

$$\mathscr{E}_0 = |P_1|_2^2 + |\operatorname{curl} P_0|_2^2 + k|P_0|_2^2 + \alpha|\sqrt{\delta}P_0 \times \nu|_2^2 + |E_0|_2^2 + |H_0|_2^2 \tag{4}$$

We have the following energy estimate:

Lemma 1. *If (E,H,P) is a regular solution of the problem (1) and (2), then we have the following energy estimate:*

$$\mathscr{E}(t) + 2\int_0^t (\alpha|\partial_t P|_2^2 + \sigma|E|_2^2 + |\sqrt{\beta}E\times\nu|_2^2 + |\sqrt{\delta}\partial_t P \times \nu|_2^2)\mathrm{d}s \le \mathscr{E}_0. \tag{5}$$

Proof. To obtain the energy inequality, we formally take the inner product of (1) by P, the first equation of (2) by H and the second of 2 by E, summing up and integrating over Ω the resulting equations and using the divergence theorem.

Before discussing problem (1) and (2), let us first review some previous results on ferroelectric systems. Greenberg et al. [7] considered particular solutions of ferroelectric system (1) and (2) with transverse magnetic symmetry and the boundary condition $P \times \nu = 0$. They supposed that the ferroelectric material occupies a cylinder with generators parallel to the x_3-axis and a uniform, simply connected cross section ω and considered only solutions which are independent of the variable x_3 and have, with some abuse of notation, the special form, $E = r_0 e u_3, H = \beta(h_1 u_1 + h_2 u_2)$, and $P = r_0 p u_3$ in $\omega \subset \mathbb{R}^2$, where $\beta = \beta_0 r_0$ and (u_1, u_2, u_3) is an orthonormal basis of $\mathbb{R}^3$. Reducing so the coupled full systems of Maxwell's equations (1) and (2) to scalar wave equations, they were able to study the asymptotic behaviour with respect to the time variable and prove that the reduced (scalar) ferroelectric system tends to a steady state in which the scalar polarization is governed by a non-linear scalar equation that has multiple solutions. Next results concern dimensional reduction for thin ferroelectric materials [1] and the limiting behaviour when the thickness of the medium tends to zero is obtained. In the framework of time harmonic dependency of the solutions, the work [8] discusses the model equations of ferroelectric media introduced in [7]. By classical methods, among other results existence and uniqueness of the solutions for frequencies which are far from 0 are proved, and the regularity of the solutions when the polarization satisfies the boundary condition $P \times \nu = 0$ is obtained. Finally, in a periodic setting, the work [6] addresses global existence of weak solutions for Landau–Lifshitz–Maxwell equations. In fact the model considered in [6] generalizes (1)–(2) in the sense that it is coupled with Landau–Lifshitz equation for magnetization field.

The rest of the paper is divided as follows. In the next section, we give a global existence and uniqueness result for the model (1) and (2). The purpose of Sect. 3 is to characterize the long-time behaviour of the solutions. We conclude the paper in Sect. 4 by giving some comments.

2 Global Existence of Weak Solutions

We first state the definition of weak solutions to problem (1) and (2).

Definition 1. We say that (E, H, P) is a weak solution to the problem (1) and (2) if (1) and (2) are satisfied in the sense of distributions and

$$\begin{aligned} &E, H \in L^\infty(\mathbb{R}^+; \mathbb{L}^2(\Omega)) \;\text{ and }\; E \in L^2(\mathbb{R}^+; \mathbb{L}^2(\Omega)) \\ &\partial_t P, \operatorname{curl} P \in L^\infty(\mathbb{R}^+; \mathbb{L}^2(\Omega)) \;\text{ and }\; \partial_t P \in L^2(\mathbb{R}^+; \mathbb{L}^2(\Omega)). \end{aligned} \tag{6}$$

Moreover, for all $t \geq 0$, the energy inequality (5) holds true.

Following the lines of the proof given in [2] (see also [4]), we may prove, by using classical results of the semigroup theory [10] and its application to semilinear equations, the following results dealing with the Silver-Müller boundary conditions. For more details, we refer to [3].

Theorem 1. *Let* $(H_0, E_0, P_0, P_1) \in \mathbb{L}^2(\Omega) \times \mathbb{L}^2(\Omega) \times \mathscr{H}(\mathrm{curl}\,, \Omega) \times \mathbb{L}^2(\Omega)$ *such that* $P_0 \times \nu \in \mathbb{L}^2(\partial\Omega)$*. Then there exists a unique weak solution* $H, E \in L^\infty(\mathbb{R}^+; \mathbb{L}^2(\Omega))$ *and* $P \in L^\infty(\mathbb{R}^+; \mathscr{H}(\mathrm{curl}\,, \Omega))$ *to the problem* (1) *and* (2)*. The tangential traces* $H \times \nu, E \times \nu, \partial_t P \times \nu$ *belong to* $L^2(\mathbb{R}^+; \mathbb{L}^2(\partial\Omega))$ *and* $P \times \nu \in L^\infty(\mathbb{R}^+; \mathbb{L}^2(\partial\Omega))$*.*

We have the following time regularity result:

Proposition 1. *Let* (H, E, P) *be a weak solution of* (1) *and* (2)*. We assume that*

$$H_0, E_0, P_0, P_1, \mathrm{curl}\, P_0 \in \mathscr{H}(\mathrm{curl}\,, \Omega)$$

We assume moreover that $P_0 \times \nu, P_1 \times \nu \in \mathbb{L}^2(\partial\Omega)$*. Then*

$$\begin{array}{l} \partial_t H, \partial_t E, \partial_t^2 P \in L^\infty(\mathbb{R}^+; \mathbb{L}^2(\Omega)) \\ H, E, P, \partial_t P \in L^\infty(\mathbb{R}^+; \mathscr{H}(\mathrm{curl}\,, \Omega)). \end{array} \tag{7}$$

Lemma 2. *There exists a constant* $C > 0$ *such that, if* (H, E, P) *is a global solution of* (1) *and* (2)*, we have*

$$\begin{cases} |E|^2_{L^\infty(\mathbb{R}+;\mathbb{L}^2(\Omega))} + |H|^2_{L^\infty(\mathbb{R}+;\mathbb{L}^2(\Omega))} \le C \\ |P|^2_{L^\infty(\mathbb{R}+;\mathbb{L}^2(\Omega))} + |E|^2_{L^2(\mathbb{R}+;\mathbb{L}^2(\Omega))} + |\partial_t P|^2_{L^2(\mathbb{R}+;\mathbb{L}^2(\Omega))} \le C \\ |\partial_t P|^2_{L^\infty(\mathbb{R}+;\mathbb{L}^2(\Omega))} + |\mathrm{curl}\, P|_{L^\infty(\mathbb{R}+;\mathbb{L}^2(\Omega))} \le C. \end{cases} \tag{8}$$

Moreover we have

$$\begin{cases} |E \times \nu|^2_{L^2(\mathbb{R}+;\mathbb{L}^2(\partial\Omega))} \le C, \\ |H \times \nu|^2_{L^2(\mathbb{R}+;\mathbb{L}^2(\partial\Omega))} \le C, \\ |P \times \nu|^2_{L^\infty(\mathbb{R}+;\mathbb{L}^2(\partial\Omega))} \le C, \\ |\partial_t P \times \nu|^2_{L^2(\mathbb{R}+;\mathbb{L}^2(\partial\Omega))} \le C. \end{cases} \tag{9}$$

In similar way, we get the following estimates for the time partial derivatives of the solution:

Lemma 3. *There exists a constant* $C > 0$ *such that, if* (H, E, P) *is a global solution of* (1)–(2)*, we have*

$$\begin{cases} |\partial_t^2 P|^2_{L^\infty(\mathbb{R}+;\mathbb{L}^2(\partial\Omega))} + |\operatorname{curl}\partial_t P|^2_{L^\infty(\mathbb{R}+;\mathbb{L}^2(\partial\Omega))} \leq C, \\ |\partial_t E|^2_{L^\infty(\mathbb{R}+;\mathbb{L}^2(\partial\Omega))} + |\partial_t H|^2_{L^\infty(\mathbb{R}+;\mathbb{L}^2(\partial\Omega))} + |\operatorname{curl} H|^2_{L^\infty(\mathbb{R}+;\mathbb{L}^2(\partial\Omega))} \leq C \\ |\operatorname{curl} P|^2_{L^\infty(\mathbb{R}+;\mathbb{L}^2(\partial\Omega))} + |\operatorname{curl}^2 P|^2_{L^\infty(\mathbb{R}+;\mathbb{L}^2(\partial\Omega))} \leq C \\ |\operatorname{curl}^2 \partial_t P|^2_{L^\infty(\mathbb{R}+;\mathbb{L}^2(\partial\Omega))} \leq C \\ |\operatorname{curl} E|^2_{L^\infty(\mathbb{R}+;\mathbb{L}^2(\partial\Omega))} \leq C. \end{cases} \tag{10}$$

3 The Limit as t Goes to $+\infty$

We investigate the long-time behaviour of the solutions of (1). More precisely, we study the ω-limit set of the trajectories and characterize the ω-limit points as solutions of a suitable stationary problem. We proceed as in Carbou–Fabrie [5].

Let P be a weak solution of (1). We call ω-limit set of the trajectory P the following set

$$\omega(P)=\{p \in \mathscr{H}(\operatorname{curl}, \Omega), \exists\, t_n, \lim_{n\to+\infty} t_n = +\infty, P(t_n,.) \rightharpoonup p \text{ in } \mathscr{H}(\operatorname{curl}, \Omega) \text{ weakly}\}$$

Consider a weak solution P of (1). From the energy estimate (5), the ω-limit set $\omega(P)$ is non-empty. We denote p a point of this set. Hence, there exists a sequence $(t_n)_{n\geq 1}$ with $\lim_{n\to\infty} t_n = +\infty$ such that $P(t_n,.)$ tends to p in $\mathscr{H}(\operatorname{curl}, \Omega)$ weakly, in $\mathbb{L}^2(\Omega)$ strongly and a.e. in Ω.

Let $a > 0$ fixed. For $s \in (-a, a)$ and $x \in \Omega$, we define for n large enough

$$p_n(s,x) = P(t_n + s, x).$$

We consider a function $\rho \in \mathscr{C}_0^\infty((-a,a))$ such that

$$\begin{cases} \rho_a(s) = 0 \text{ out of } [-a,a]; \quad \rho_a(s) = 1 \text{ on } [-a+1, a-1] \\ 0 \leq \rho_a \leq 1; \quad |\rho_a'(s)| \leq 2. \end{cases} \tag{11}$$

We set

$$P_a^n(x) = \frac{1}{2a}\int_{-a}^{a} P(t_n+s,x)\rho_a(s)\mathrm{d}s$$
$$H_a^n(x) = \frac{1}{2a}\int_{-a}^{a} H(t_n+s,x)\rho_a(s)\mathrm{d}s$$

and

$$E_a^n(x) = \frac{1}{2a}\int_{-a}^{a} E(t_n+s,x)\rho_a(s)\mathrm{d}s.$$

We have the following convergence result:

Lemma 4. *The sequence* $(p_n)_{n\geq 1}$ *satisfies the following convergences*

$$\begin{aligned} p_n &\to p \text{ in } \mathbb{L}^2((-a,a)\times\Omega) \text{ strongly,}\\ p_n &\rightharpoonup p \text{ in } \mathbb{L}^2((-a,a);\mathscr{H}(\mathrm{curl}\,,\Omega)) \text{ weakly.} \end{aligned} \tag{12}$$

By the estimates on P (Lemma 2), P_a^n is bounded in $\mathscr{H}(\mathrm{curl}\,,\Omega)$ uniformly with respect to n and a, extracting a subsequence there exists a subsequence such that $P_a^n \rightharpoonup P_a$ in $\mathscr{H}(\mathrm{curl}\,,\Omega)$ weak.

Lemma 2 shows also that E and H are bounded in $L^\infty(\mathbb{R}^+;\mathbb{L}^2(\Omega))$. Then H_a^n and E_a^n are bounded in $\mathbb{L}^2(\Omega)$ independently of a and n. So by extracting a subsequence, we may assume that $(E_a^n,H_a^n)_{n\geq 1}$ converges in $\mathbb{L}^2(\Omega)$ weakly to (E_a,H_a) as n tends to $+\infty$.

Passing to the limit for polarization. In the weak formulation of (1), we take as test function $\frac{1}{2a}\rho_a(t-t_n)\Psi(x)$ where Ψ is a function lying in $\mathscr{D}(\bar{\Omega})$. Letting $s=t-t_n$, we obtain

$$\begin{aligned} &\frac{1}{2a}\int_{-a}^{a}\int_\Omega (\partial_t^2 p_n(s,x)+a\partial_t p_n(s,x)+kp_n(s,x))\cdot\Psi(x)\rho_a(s)\,\mathrm{d}x\mathrm{d}s\\ &-\beta\int_\Omega E_a^n(x)\Psi(x)\mathrm{d}x+\frac{1}{2a}\int_{-a}^{a}\int_\Omega \mathrm{curl}\,p_n(s,x)\cdot\mathrm{curl}\,(\Psi(x)\rho_a(s))\,\mathrm{d}x\mathrm{d}s\\ &-\frac{\delta}{2a}\int_{-a}^{a}\int_{\partial\Omega}\partial_t p_n\times\nu\cdot\Psi(x)\rho_a(s)\times\nu\,\mathrm{d}\sigma\mathrm{d}s\\ &-\frac{\delta}{2a}\int_{-a}^{a}\int_{\partial\Omega}\alpha p_n\times\nu\cdot\Psi(x)\rho_a(s)\times\nu\,\mathrm{d}\sigma\mathrm{d}s=0. \end{aligned} \tag{13}$$

Now for a fixed value of the parameter a, we take the limit of the previous equation when n tends to $+\infty$. We then pass to the limit as a tends to $+\infty$ to get

$$\begin{aligned} &\int_\Omega kP_\infty\cdot\Psi(x)\,\mathrm{d}x+\int_\Omega \mathrm{curl}\,P_\infty(x)\cdot\mathrm{curl}\,\Psi(x)\,\mathrm{d}x\\ &\quad-\beta\int_\Omega E_\infty(x)\Psi(x)\mathrm{d}x-\delta\int_{\partial\Omega}\alpha P_\infty\times\nu\cdot\Psi(x)\times\nu\,\mathrm{d}\sigma=0. \end{aligned} \tag{14}$$

It remains to derive the equation satisfied by E_∞.

Passing to the limit for electromagnetic field. We first recall the equation verified by H_a^n and E_a^n. We write the weak formulation of the second equation of (2) by considering the test function $\Psi(t,x)=\frac{1}{2a}\rho_a(t-t_n)\zeta(x)$ with $\zeta\in\mathscr{D}(\mathbb{R}^3)$. Letting n tends to $+\infty$ in the weak formulation of (2), we get

$$\alpha_a+\int_\Omega H_a(x)\mathrm{curl}\,\zeta(x)\mathrm{d}x+\sigma\int_\Omega E_a(x)\zeta(x)\mathrm{d}x=0,$$

where α_a tends to 0 as a goes to $+\infty$.

Now, as a goes to $+\infty$, we obtain

$$\int_{\Omega} H_{\infty}(x) \cdot \operatorname{curl} \zeta(x) \mathrm{d}x = \sigma \int_{\Omega} E_{\infty}(x) \zeta(x) \mathrm{d}x. \tag{15}$$

In the same way, we pass to the limit in the first equation of (2).
Gathering all convergence results obtained, we have

Theorem 2. *If P is a weak solution of* (1)*, then each point P_∞ in $\omega(P)$ is a weak solution of the steady-state system*

$$\begin{cases} P_{\infty} \in \mathscr{H}(\operatorname{curl}, \Omega); \\ \operatorname{curl}^2 P_{\infty} + kP_{\infty} = \beta E_{\infty} \ in\ \Omega; \\ \operatorname{curl} P_{\infty} \times \nu + \delta\alpha\nu \times P_{\infty} \times \nu = 0 \ on\ \partial\Omega \end{cases} \tag{16}$$

coupled to

$$\begin{cases} \operatorname{curl} E_{\infty} = 0 \ in\ \Omega, \\ \sigma E_{\infty} + \operatorname{curl} H_{\infty} = 0 \ in\ \Omega, \\ H_{\infty} \times \nu + \beta\nu \times (E_{\infty} \times \nu) = 0 \ on\ \partial\Omega. \end{cases} \tag{17}$$

4 Concluding Remarks

In this paper, a ferroelectric system with Silver-Müller boundary condition is investigated. Global existence and uniqueness result is given, and the long-time behaviour of the solutions is studied. The calculations performed in this paper can be generalized to the model that couples magnetization field M with (E, H, P) [6]. More precisely, for the coupled system [with the same initial and boundary conditions as in (1) and (2)]

$$\begin{cases} \partial_t M - \alpha_1 M \times \partial_t M = -(1+\alpha_1^2) M \times (\triangle M + H) \ \text{ in } \mathbb{R}^+ \times \Omega, \\ \partial_t^2 P + \alpha \partial_t P + \operatorname{curl}^2 P + kP = \beta E \ \text{ in } \mathbb{R}^+ \times \Omega, \\ \partial_t (H + M) - \operatorname{curl} E = 0 \ \text{ in } \mathbb{R}^+ \times \Omega, \\ \partial_t (E + P) + \sigma E + \operatorname{curl} H = 0 \ \text{ in } \mathbb{R}^+ \times \Omega, \\ M(t=0) = M_0 \text{ in } \Omega, \quad \partial_\nu M = 0 \text{ on } \mathbb{R}^+ \times \partial\Omega \end{cases} \tag{18}$$

one obtains by similar calculations, as t goes to $+\infty$, the following limit problem

$$\begin{cases} M_{\infty} \times (\triangle M_{\infty} + H_{\infty}) = 0 \ \text{ in } \Omega; \ \partial_\nu M_{\infty} = 0 \text{ on } \partial\Omega \\ \operatorname{div}(H_{\infty} + M_{\infty}) = 0 \ \text{ in } \Omega, \end{cases} \tag{19}$$

coupled to (16) and (17). Note that in (18), the assumed initial data have to satisfy the compatibility condition $\operatorname{div}(H_0 + M_0) = 0$ in Ω.

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On a Discrete Number Operator Associated with the 5D Discrete Fourier Transform

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Abstract We construct an explicit form of a difference analogue of the quantum number operator in terms of the raising and lowering operators that govern eigenvectors of the 5D discrete (finite) Fourier transform. Eigenvalues of this difference operator are represented by distinct non-negative numbers so that it can be used to systematically classify, in complete analogy with the case of the continuous classical Fourier transform, eigenvectors of the 5D discrete Fourier transform, thus resolving the ambiguity caused by the well-known degeneracy of the eigenvalues of the discrete Fourier transform.

Keywords Discrete Fourier transform • Raising and lowering operators • 5D eigenvectors

Mathematics Subject Classification (2000): 39A10, 39A12, 42A38

1 Introduction

We are to begin by recalling first a few well-known facts about the classical Fourier transform (FT) and its finite analogue, discrete Fourier transform (DFT). It is known that the Hermite functions

$$\psi_n(x) := c_n^{-1} H_n(x) \exp\left(-x^2/2\right), \qquad c_n = \sqrt{\sqrt{\pi}\, 2^n\, n!}\,, \tag{1}$$

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where $H_n(x)$ are the classical Hermite polynomials, represent an important explicit example of an orthonormal and complete system in the Hilbert space $L^2(\mathbb{R}, dx)$ of square-integrable functions on the full real line $x \in \mathbb{R}$. It is further well known that the functions $\psi_n(x)$ possess the simple transformation property with respect to the Fourier transform: *they are eigenfunctions of the Fourier transform, associated with the eigenvalues* i^n,

$$(\mathscr{F}\,\psi_n)\,(x) \equiv \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\,\mathrm{i}\,xy}\,\psi_n(y)\,dy = \mathrm{i}^n\,\psi_n(x)\,. \tag{2}$$

The question is then can be posed whether there is a way of deriving the eigenfunctions (1) of the Fourier transform, which does not presuppose a knowledge of the analytic formula (2) to be proved.

Since mutually commuting operators have the same set of eigenfunctions, one may solve this problem by defining such a self-adjoint differential operator with simple spectrum of distinct eigenvalues that commutes with the FT operator $\mathscr{F}$. Then the eigenfunctions of that differential operator can be found by solving a corresponding to this case differential equation, and they will be at same time the eigenfunctions of the $\mathscr{F}$. So in this way, one reduces a problem of finding eigenfunctions of the FT operator $\mathscr{F}$ to one of solving some differential equation.

To illustrate how to find such differential operator, let us start with the first-order differential operator $\frac{d}{dx}$ and evaluate its action on the Fourier integral transform:

$$\frac{d}{dx}\int_{\mathbb{R}} e^{\,\mathrm{i}\,xy} f(y)\,dy = \,\mathrm{i}\int_{\mathbb{R}} e^{\,\mathrm{i}\,xy}\,y f(y)\,dy\,, \tag{3}$$

where $f(x) \in L^2(\mathbb{R}, dx)$. Consequently, from the right side of (3), one deduces that the next step should be to evaluate

$$x\int_{\mathbb{R}} e^{\,\mathrm{i}\,xy} f(y)\,dy = -\,\mathrm{i}\int_{\mathbb{R}} \left(\frac{d\,e^{\,\mathrm{i}\,xy}}{dy}\right) f(y)\,dy = \,\mathrm{i}\int_{\mathbb{R}} e^{\,\mathrm{i}\,xy}\,\frac{df(y)}{dy}\,dy\,, \tag{4}$$

upon integrating by parts the middle term in (4). From (3) and (4), it thus follows that

$$\left(x \pm \frac{d}{dx}\right)\int_{\mathbb{R}} e^{\,\mathrm{i}\,xy} f(y)\,dy = \pm\,\mathrm{i}\int_{\mathbb{R}} e^{\,\mathrm{i}\,xy}\left(y \pm \frac{d}{dy}\right) f(y)\,dy\,. \tag{5}$$

In the operator form, these identities can be written as intertwining relations

$$\mathbf{a}\,\mathscr{F} = \mathrm{i}\,\mathscr{F}\,\mathbf{a}\,, \qquad \mathbf{a}^{\dagger}\,\mathscr{F} = -\,\mathrm{i}\,\mathscr{F}\mathbf{a}^{\dagger}\,, \tag{6}$$

where

$$\mathbf{a} := \frac{1}{\sqrt{2}}\left(x + \frac{d}{dx}\right)\,, \qquad \mathbf{a}^{\dagger} := \frac{1}{\sqrt{2}}\left(x - \frac{d}{dx}\right)\,, \tag{7}$$

are the lowering and raising first-order differential operators, which obey the standard Heisenberg commutation relation

$$\left[\,\mathbf{a}\,,\,\mathbf{a}^{\dagger}\,\right] := \mathbf{a}\,\mathbf{a}^{\dagger} - \mathbf{a}^{\dagger}\,\mathbf{a} \equiv \left[\,\frac{d}{dx}\,,\,x\,\right] = \mathbf{I}\,. \tag{8}$$

The final step for finding the desired differential operator is actually revealed by the intertwining relations (6) because one readily concludes that

$$\mathbf{a}^{\dagger}\mathbf{a}\,\mathscr{F} = \mathrm{i}\,\mathbf{a}^{\dagger}\mathscr{F}\mathbf{a} = \mathscr{F}\mathbf{a}^{\dagger}\mathbf{a} \tag{9}$$

on account of both identities in (6). Consequently, the self-adjoint second-order differential number operator $\mathbf{N} := \mathbf{a}^{\dagger}\mathbf{a}$ does commute with the FT operator $\mathscr{F}$, and it only remains to resolve the eigenproblem $\mathbf{N}f_n(x) = \lambda_n f_n(x)$ for this operator $\mathbf{N}$. It is not difficult to show then that the eigenfunctions of the number operator $\mathbf{N}$ are the Hermite functions $\psi_n(x)$ (up to the arbitrariness in the choice of a normalization constant factor), whereas the corresponding eigenvalues are $\lambda_n = n$, $n = 0, 1, 2, \ldots$.

Turning to the discrete Fourier transform $\Phi^{(N)}$, we recall that it is based on N points and represented by the $N \times N$ unitary symmetric matrix with elements

$$\Phi^{(N)}_{m,n} = \frac{1}{\sqrt{N}}\,\exp\left(\frac{2\pi\mathrm{i}}{N}\,m\,n\right) \equiv \frac{1}{\sqrt{N}}\,q^{m\,n}\,, \tag{10}$$

where $q := e^{\frac{2\pi\mathrm{i}}{N}}$ and $m, n \in \{0, 1, \ldots, N-1\}$. Given a vector $\vec{v}$ with components $\{v_k\}_{k=0}^{N-1}$, one can compute another vector $\vec{u}$ with components

$$u_m = \sum_{n=0}^{N-1} \Phi^{(N)}_{m,n}\,v_n\,, \tag{11}$$

referred to as *the discrete (finite) Fourier transform* of the vector $\vec{v}$. Those vectors $\vec{f}_k$, which are solutions of the standard equations

$$\sum_{n=0}^{N-1} \Phi^{(N)}_{m,n}\left(\vec{f}_k\right)_n = \lambda_k\left(\vec{f}_k\right)_m, \qquad k \in \{0, 1, \ldots, N-1\}\,, \tag{12}$$

then represent eigenvectors of the DFT operator $\Phi^{(N)}$, associated with the eigenvalues λ_k. Since the fourth power of $\Phi^{(N)}$ is the unit matrix, the only four distinct eigenvalues among λ_ks are ± 1 and $\pm\,\mathrm{i}$.

Although there exists a plethora of discussion in the literature on eigenvectors of the DFT (see, e.g. [1–11] and the relevant references quoted there), the problem of deriving eigenvectors of DFT analytically still remains to be solved. Recently, Atakishiyeva and Atakishiyev [12] have proposed a strategy for resolving this problem by constructing a *self-adjoint difference operator* $\mathscr{N}^{(N)}$ *(with distinct non-negative eigenvalues) in terms of the difference raising and lowering operators, which are defined by the intertwining relations*

$$\mathbf{b_N}\,\Phi^{(N)} = \mathrm{i}\,\Phi^{(N)}\,\mathbf{b_N}, \qquad \mathbf{b_N^T}\,\Phi^{(N)} = -\,\mathrm{i}\,\Phi^{(N)}\,\mathbf{b_N^T}. \tag{13}$$

The ability to solve a difference equation for eigenvectors of this discrete number operator $\mathscr{N}^{(\mathbf{N})}$, which commutes with the DFT operator $\Phi^{(N)}$, then enables one to define an analytical form of the desired set of eigenvectors for the latter operator.

An important aspect to observe at this point is that although the idea of making use an analogy with the continuous case for deriving eigenvectors of the DFT is not new (see, e.g. [3, 4]), it seems, however, that there never was consistent attempt to find out how symmetry properties of the continuous Fourier transform might best be transferred to the discrete case.

The limited aim of this presentation is to restrict our attention to the 5D DFT and give a detailed account of how one can solve the eigenproblem for the discrete number operator $\mathscr{N}^{(\mathbf{5})}$ by using the difference raising and lowering operators that govern eigenvectors of the 5D discrete Fourier transform $\Phi^{(5)}$.

The motivation for selecting this special dimension $N = 5$ of the general discrete Fourier transform $\Phi^{(N)}$ is twofold. First, this dimension is large enough to contain a multiple eigenvalue, and therefore one has to handle the same degeneracy problem as in the more general case. Second, this dimension is small enough in order to have calculational advantages that appear in the process of resolving the eigenproblem for the discrete number operator $\mathscr{N}^{(\mathbf{5})}$. We hope that this study will deepen our understanding of the case with an arbitrary ND discrete Fourier transform and help us to provide some rigorous proofs, still needed for general values of N.

2 5D Raising and Lowering Difference Operators

We recall that the *5D discrete (finite) Fourier transform* (DFT) is traditionally represented by a 5×5 unitary symmetric matrix $\Phi^{(5)}$ with elements defined as in (10) with $N = 5$ (see, e.g. [2, 6]). So the matrix form of $\Phi^{(5)}$ is

$$\left(\Phi^{(5)}_{m,n}\right) = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & q & q^2 & q^3 & q^4 \\ 1 & q^2 & q^4 & q & q^3 \\ 1 & q^3 & q & q^4 & q^2 \\ 1 & q^4 & q^3 & q^2 & q \end{pmatrix}, \tag{14}$$

where $q = \exp(2\pi\mathrm{i}/5)$ is the 5th root of unity and indices $m, n \in \{0, 1, 2, 3, 4\}$. In [12] it was shown how to construct the *difference lowering* $\mathbf{b_5}$ *and raising* $\mathbf{b_5^T}$ *operators for eigenvectors of the DFT operator* $\Phi^{(5)}$, which satisfy ‘proper’ intertwining relations with the $\Phi^{(5)}$ of the form (13) for $N = 5$. Let us draw attention here to those intertwining relations, which evidently imply that if a vector $\vec{f}_k$ is the eigenvector of the DFT operator $\Phi^{(5)}$, associated with the eigenvalue i^{k}, $0 \le k \le 3$, then the vectors $\mathbf{b}_5^{\mathbf{T}} \vec{f}_k$ and $\mathbf{b}_5 \vec{f}_k$ are also the eigenvectors of the same operator $\Phi^{(5)}$, associated with the eigenvalues $\mathrm{i}^{\mathrm{k}+1}$ and $\mathrm{i}^{\mathrm{k}-1}$, respectively. But note carefully that this does not necessarily mean that those vectors $\mathbf{b}_5^{\mathbf{T}} \vec{f}_k$ and $\mathbf{b}_5 \vec{f}_k$ essentially coincide (to within constant factors) with the eigenvectors $\vec{f}_{k+1}$ and $\vec{f}_{k-1}$ of the DFT operator $\Phi^{(5)}$, respectively, as it does happen to be the case with the Fourier transform operator $\mathscr{F}$. Later we detail the action of the operators $\mathbf{b}_5$ and $\mathbf{b}_5^{\mathbf{T}}$ on eigenvectors of the DFT operator $\Phi^{(5)}$. The operators $\mathbf{b}_5$ and $\mathbf{b}_5^{\mathbf{T}}$ are explicitly given as

$$\mathbf{b}_5 := c\Big(2\,\mathbf{S} + \mathbf{T}^{(+)} - \mathbf{T}^{(-)}\Big), \quad \mathbf{b}_5^{\mathbf{T}} := c\Big(2\,\mathbf{S} - \mathbf{T}^{(+)} + \mathbf{T}^{(-)}\Big), \quad c = \frac{1}{4}\sqrt{\frac{5}{\pi}}, \tag{15}$$

where the operator $\mathbf{S}$ represents the diagonal matrix with elements $S_{kl} := sin(k\theta)\delta_{kl}$, $\theta := 2\pi/5$, $0 \le k, l \le 4$ and a pair of the shift operators $\mathbf{T}^{(\pm)}$ are defined as $T_{kl}^{(\pm)} := \delta_{k\pm 1,l}$ with $\delta_{-1,l} \equiv \delta_{4,l}$ and $\delta_{5,l} \equiv \delta_{0,l}$. We also display the matrix form of the lowering and raising operators $\mathbf{b}_5$ and $\mathbf{b}_5^{\mathbf{T}}$, respectively:

$$\Big((\mathbf{b}_5)_{m,m'}\Big) = c \begin{pmatrix} 0 & 1 & 0 & 0 & -1 \\ -1 & 2\sin\theta & 1 & 0 & 0 \\ 0 & -1 & 2\sin 2\theta & 1 & 0 \\ 0 & 0 & -1 & -2\sin 2\theta & 1 \\ 1 & 0 & 0 & -1 & -2\sin\theta \end{pmatrix}, \tag{16}$$

$$\Big((\mathbf{b}_5^{\mathbf{T}})_{m,m'}\Big) = c \begin{pmatrix} 0 & -1 & 0 & 0 & 1 \\ 1 & 2\sin\theta & -1 & 0 & 0 \\ 0 & 1 & 2\sin 2\theta & -1 & 0 \\ 0 & 0 & 1 & -2\sin 2\theta & -1 \\ -1 & 0 & 0 & 1 & -2\sin\theta \end{pmatrix}. \tag{17}$$

It is not hard to show that the determinants of both matrices $\mathbf{b}_5$ and $\mathbf{b}_5^{\mathbf{T}}$ are equal to 0; therefore, they are not invertible. Observe also that both of these matrices are of 'almost' tridiagonal form: they have ± 1 elements in the upper-right and lower-left corners but otherwise are tridiagonal. Since those ± 1 elements can be regarded as cyclic extensions of the subdiagonal and the superdiagonal elements, these types of matrices are referred to as *extended tridiagonal* matrices in [7–9]. Moreover, another confirmation of the 'cyclic' properties of the operators $\mathbf{b}_5$ and $\mathbf{b}_5^{\mathbf{T}}$ is revealed by the identities

$$\Big(\mathbf{b}_5\Big)^5 + 5\,c^4\tau\,\mathbf{b}_5 = 0\,, \qquad \Big(\mathbf{b}_5^{\mathbf{T}}\Big)^5 + 5\,c^4\tau\,\mathbf{b}_5^{\mathbf{T}} = 0 \tag{18}$$

where τ is the golden ratio, $\tau := (\sqrt{5}+1)/2 = -2\cos 2\theta$. This particular irrational number τ is known to turn out frequently in geometry, particularly in figures with pentagonal symmetry (see, e.g. [13, 14]); so it is not surprising that it appears here as well. Since the successive powers of τ obey the Fibonacci recurrence $\tau^{n+1} = \tau^n + \tau^{n-1}$, $n \geq 0$, this characteristic property of the golden ratio allows any polynomial in τ to be reduced to a linear expression in τ. In the sequel, it proves therefore convenient to parametrize the operators $\mathbf{b_5}$ and $\mathbf{b_5^T}$ in terms of the golden ratio τ and its conjugate $\tau^{-1} := (\sqrt{5}-1)/2 = 2\cos\theta = \tau - 1$. Taking into account that $2\sin\theta = \kappa\tau^{1/2}$ and $2\sin 2\theta = \kappa\tau^{-1/2}$, where $\kappa := (5)^{1/4}$, one rewrites matrices (16) and (17) as

$$\left((\mathbf{b_5})_{m,m'}\right) = c\begin{pmatrix} 0 & 1 & 0 & 0 & -1 \\ -1 & \kappa\tau^{1/2} & 1 & 0 & 0 \\ 0 & -1 & \kappa\tau^{-1/2} & 1 & 0 \\ 0 & 0 & -1 & -\kappa\tau^{-1/2} & 1 \\ 1 & 0 & 0 & -1 & -\kappa\tau^{1/2} \end{pmatrix}, \tag{19}$$

$$\left((\mathbf{b_5^T})_{m,m'}\right) = c\begin{pmatrix} 0 & -1 & 0 & 0 & 1 \\ 1 & \kappa\tau^{1/2} & -1 & 0 & 0 \\ 0 & 1 & \kappa\tau^{-1/2} & -1 & 0 \\ 0 & 0 & 1 & -\kappa\tau^{-1/2} & -1 \\ -1 & 0 & 0 & 1 & -\kappa\tau^{1/2} \end{pmatrix}. \tag{20}$$

From the definition (15) of the lowering $\mathbf{b_5}$ and raising $\mathbf{b_5^T}$ operators, it follows that their commutator $\mathscr{K} := \left[\mathbf{b_5}, \mathbf{b_5^T}\right]_- \equiv \mathbf{b_5}\,\mathbf{b_5^T} - \mathbf{b_5^T}\,\mathbf{b_5}$ is equal to

$$\mathscr{K} = 4\,c^2\left[\mathbf{T}^{(+)} - \mathbf{T}^{(-)},\, \mathbf{S}\right]_- . \tag{21}$$

Its explicit matrix form in terms of the golden ratio τ is

$$\left((\mathscr{K})_{m,m'}\right) = 2\,\kappa\sqrt{\tau}\,c^2\begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & \tau-2 & 0 & 0 \\ 0 & \tau-2 & 0 & 2(1-\tau) & 0 \\ 0 & 0 & 2(1-\tau) & 0 & \tau-2 \\ 1 & 0 & 0 & \tau-2 & 0 \end{pmatrix}. \tag{22}$$

To compare (22) with the continuous case, recall that the lowering and raising first-order differential operators $\mathbf{a}$ and $\mathbf{a}^\dagger$, associated with the Fourier transform operator $\mathscr{F}$, obey the Heisenberg commutation relation (8).

It is worthwhile to close this section by emphasizing that it was intuitively understood much earlier that probably the *extended tridiagonal* type matrices lie at the core of the adequate description of eigenvectors of the general ND discrete

Fourier transform [3, 9]. But it seems that this particular band structure was imprecisely attributed to those operators with distinct eigenvalues, which *commute* with the associated DFT operators and can be therefore used for the unambiguous classification of the eigenvectors of the latter ones. Only recently it has become clear that it is the lowering $\mathbf{b_N}$ and raising $\mathbf{b_N^T}$ operators for the eigenvectors of the DFT operator $\Phi^{(N)}$ that are of the extended tridiagonal type [12]. Defined by the standard intertwining relations (13), these operators $\mathbf{b_N}$ and $\mathbf{b_N^T}$ do not commute with the $\Phi^{(N)}$. Nevertheless, from the same defining identities for the $\mathbf{b_N}$ and $\mathbf{b_N^T}$, it follows at once that their product $\mathbf{b_N^T}\,\mathbf{b_N}$ does commute with the DFT operator $\Phi^{(N)}$. Moreover, although the operator $\mathscr{N}^{(\mathbf{N})} := \mathbf{b_N^T}\,\mathbf{b_N}$ is not of the extended tridiagonal type, it turns out to be quite sufficient for finding explicit forms of all mutually orthogonal eigenvectors the DFT operator $\Phi^{(N)}$ in a systematic and unambiguous way. As we shall see in the next section, this briefly outlined above algebraic approach [12] to solving the eigenproblem for the operator $\mathscr{N}^{(\mathbf{N})}$ can be effectively employed in the particular case of the 5D DFT.

3 Eigenvalues and Eigenvectors of the Discrete Number Operator

Let us study in detail a discrete number operator $\mathscr{N}^{(\mathbf{5})} := \mathbf{b_5^T}\,\mathbf{b_5}$, whose matrix elements are defined as

$$\left(\mathscr{N}^{(\mathbf{5})}\right)_{m,m'} = c^2 \begin{pmatrix} 2 & -\kappa\tau^{1/2} & -1 & -1 & -\kappa\tau^{1/2} \\ -\kappa\tau^{1/2} & 4+\tau & \kappa\tau^{-3/2} & -1 & -1 \\ -1 & \kappa\tau^{-3/2} & 5-\tau & 2\kappa\tau^{-1/2} & -1 \\ -1 & -1 & 2\kappa\tau^{-1/2} & 5-\tau & \kappa\tau^{-3/2} \\ -\kappa\tau^{1/2} & -1 & -1 & \kappa\tau^{-3/2} & 4+\tau \end{pmatrix}. \tag{23}$$

As a product of a matrix and its transpose, the defining matrix in (23) is symmetric and all of its eigenvalues are non-negative. Moreover, since the determinant of the matrix (23) is equal to zero, at least one of the eigenvalues should have zero value as well; but this lowest eigenvalue turns out to be unique, and all eigenvalues of the matrix (23) are actually distinct.

Before entering into further details about explicit forms of the eigenvalues and eigenvectors of the operator $\mathscr{N}^{(\mathbf{5})}$, we may recall first the following important facts, associated with this eigenproblem.

In this particular case under study, when (23) is just a 5×5 matrix, one can use some computer program in order to evaluate the eigenvalues and eigenvectors of the discrete number operator $\mathscr{N}^{(\mathbf{5})}$. For instance, this is what one gets by using *Mathematica*:

Eigenvalues of the $\mathscr{N}^{(\mathbf{5})}$ are $c^2\,\nu_k$, $0 \le k \le 4$, where ν_ks, arranged in the descending order, are given by

$$\nu_4 = \frac{1}{2}\Big(15 - \sqrt{5}\Big) + \sqrt{5 - 2\sqrt{5}} = \kappa\Big[\kappa(3\tau - 2) + \tau^{-3/2}\Big], \tag{24}$$

$$\nu_3 = \frac{1}{2}\Big(5 + \sqrt{5}\Big) + \sqrt{5 + 2\sqrt{5}} = \kappa\Big(\kappa + \tau^{1/2}\Big)\tau, \tag{25}$$

$$\nu_2 = \frac{1}{2}\Big(15 - \sqrt{5}\Big) - \sqrt{5 - 2\sqrt{5}} = \kappa\Big[\kappa(3\tau - 2) - \tau^{-3/2}\Big], \tag{26}$$

$$\nu_1 = \frac{1}{2}\Big(5 + \sqrt{5}\Big) - \sqrt{5 + 2\sqrt{5}} = \kappa\Big(\kappa - \tau^{1/2}\Big)\tau, \quad \nu_0 = 0; \tag{27}$$

Eigenvectors $\vec{y}_k$ of $\mathcal{N}^{(\mathbf{5})}$, associated with these eigenvalues ν_k, $0 \le k \le 4$, have the following components:

$$\Big(\vec{y}_4\Big)_{k=0}^{4} = \Big\{0,\ \kappa + \tau^{1/2},\ \tau^{-1/2},\ -\tau^{-1/2},\ -\kappa - \tau^{1/2}\Big\}, \tag{28}$$

$$\Big(\vec{y}_3\Big)_{k=0}^{4} = \Big\{2(1-\tau),\ 1,\ 1,\ 1,\ 1\Big\}, \tag{29}$$

$$\Big(\vec{y}_2\Big)_{k=0}^{4} = \Big\{2, -\Big(\tau + 2\kappa\tau^{1/2}\Big), 2\kappa\tau^{1/2} + 3\tau - 2,\ III, II\Big\}, \tag{30}$$

$$\Big(\vec{y}_1\Big)_{k=0}^{4} = \Big\{0,\ \tau^{1/2} - \kappa,\ \tau^{-1/2},\ -\tau^{-1/2},\ \kappa - \tau^{1/2}\Big\}, \tag{31}$$

$$\Big(\vec{y}_0\Big)_{k=0}^{4} = \Big\{2\tau + \kappa\tau^{1/2},\ 1 + \kappa\tau^{-1/2},\ 1,\ 1,\ 1 + \kappa\tau^{-1/2}\Big\}. \tag{32}$$

Since the discrete number operator $\mathcal{N}^{(\mathbf{5})}$ commutes with the DFT operator $\Phi^{(5)}$, the above eigenvectors of the $\mathcal{N}^{(\mathbf{5})}$ are at the same time eigenvectors of the $\Phi^{(5)}$: two of them, $\vec{y}_0$ and $\vec{y}_2$, are associated with the same eigenvalue $\mathrm{i}^0 = 1$ of the $\Phi^{(5)}$, while the eigenvectors $\vec{y}_4$, $\vec{y}_3$ and $\vec{y}_1$ correspond to the eigenvalues i, $\mathrm{i}^2 = -1$ and $\mathrm{i}^3 = -\mathrm{i}$ of the $\Phi^{(5)}$, respectively. Obviously, these multiplicities corresponding to the eigenvalues i^k, $0 \le k \le 3$, of the 5D DFT operator $\Phi^{(5)}$, are the particular $N = 5$ cases of the general explicit expressions for the multiplicities $m_k(\mathrm{i}^k)$ of the eigenvalues of the ND DFT [1, 10],

$$m_0(1) = \left[\frac{N}{4}\right] + 1, \qquad m_1(\mathrm{i}) = \left[\frac{\mathrm{N}+1}{4}\right],$$
$$m_2(-1) = \left[\frac{N+2}{4}\right], \qquad m_3(-\mathrm{i}) = \left[\frac{\mathrm{N}+3}{4}\right] - 1, \tag{33}$$

where the symbol $[X]$ stands for the greatest integer in X.

It is important also to realize that the eigenvectors $\vec{y}_k$, $0 \le k \le 4$, are distinct from those eigenvectors of the DFT operator $\Phi^{(5)}$, which have appeared before in the literature. For instance, in [10] Matveev evaluated explicit forms of the eigenvectors of the ND DFT operator $\Phi^{(N)}$ for the values of N from $N = 2$ to $N = 8$ by combining the technique of spectral projectors for the operator $\Phi^{(N)}$ with the Gramm–Schmidt orthogonalization algorithm. In particular, Matveev's eigenvectors $\vec{v}_n$, $1 \le n \le 5$, of the operator $\Phi^{(5)}$ (see the very end of page 644 in [10]) are interrelated with the eigenvectors $\vec{y}_k$, $0 \le k \le 4$, in (28)–(32) in the following way: the eigenvectors $\vec{v}_1$ and $\vec{v}_2$ are some linear combinations of the $\vec{y}_0$ and $\vec{y}_2$, whereas the eigenvectors $\vec{v}_3$, $\vec{v}_4$ and $\vec{v}_5$ coincide with the $\vec{y}_3$, $\vec{y}_4$ and $\vec{y}_1$, respectively, up to normalization by constant factors:

$$\vec{v}_3 = \vec{y}_3 , \qquad \vec{v}_4 = \frac{\kappa}{2}\vec{y}_4 , \qquad \vec{v}_5 = -\frac{\kappa}{2}\vec{y}_1 .$$

Finally, the incentive for making those extended comments, given above, is just to emphasize that this set of eigenvectors of the discrete number operator $\mathcal{N}^{\mathbf{(5)}}$, produced by *Mathematica*, is still ambiguous until a rule is given for ordering those eigenvectors and associated eigenvalues of the operator $\mathcal{N}^{\mathbf{(5)}}$.

We return now to a study of the eigenvectors and eigenvalues of the operator $\mathcal{N}^{\mathbf{(5)}}$ in a systematic algebraic way. Since the lowest eigenvalue of the $\mathcal{N}^{\mathbf{(5)}}$ is 0, its lowest eigenvector $\vec{f}_0$ is defined as

$$\mathcal{N}^{\mathbf{(5)}} \vec{f}_0 = 0 . \tag{34}$$

Moreover, an explicit form of the same eigenvector $\vec{f}_0$ can be found from the simpler equation

$$\mathbf{b_5} \vec{f}_0 = 0 . \tag{35}$$

Since the symmetric matrix $(\mathcal{N}^{\mathbf{(5)}})_{m,m'}$ in (23) clearly exhibits additional symmetry among the entries of the all antidiagonals, it is evident that all eigenvectors of the operator $\mathcal{N}^{\mathbf{(5)}}$ must be either 'even' or 'odd' with respect to that particular reflection symmetry about the subantidiagonal $\{(\mathcal{N}^{\mathbf{(5)}})_{5-k,k}\}_{k=1}^{4}$ of the matrix $(\mathcal{N}^{\mathbf{(5)}})_{m,m'}$, that is,

$$\left(\vec{f}_n\right)_k = (-1)^n \left(\vec{f}_n\right)_{5-k} , \quad 0 \le n, k \le 4 . \tag{36}$$

This means that we should look for a lowest eigenvector $\vec{f}_0$, whose componentwise structure is of the form

$$\left(\vec{f}_0\right)_{k=0}^{4} = \left(\alpha, \beta, \gamma, \gamma, \beta\right) . \tag{37}$$

Substituting (37) into (35) and employing an explicit form of the matrix (19), one obtains only two linearly independent equations for the three unknowns α, β and γ

$$\alpha \, = \, \beta \, \kappa \, \tau^{1/2} + \gamma \, , \qquad \beta \, = \, \gamma \, (1 + \kappa \, \tau^{-1/2}) \, . \tag{38}$$

Taking into account that $\kappa^2 = 2\tau - 1$, from (38) it follows that $\alpha \, = \, \gamma \, (2\tau + \kappa \, \tau^{1/2})$ and the lowest eigenvector $\vec{f}_0$ with the components

$$\left(\vec{f}_0\right)_{k=0}^{4} = \gamma \left(2\tau + \kappa \, \tau^{1/2}, 1 + \kappa \, \tau^{-1/2}, 1, 1, 1 + \kappa \, \tau^{-1/2}\right) \tag{39}$$

is thus determined by the Eq. (35) up to normalization by a constant γ. Notice that $\vec{f}_0 \, = \, \gamma \, \vec{y}_0$; thus, the $\vec{f}_0$ is also the eigenvector of the DFT operator $\Phi^{(5)}$ corresponding to the eigenvalue $\mathrm{i}^0 \, = \, 1$. Also, to normalize the lowest eigenvector $\vec{f}_0$ to have length one, it suffices to choose the normalization constant γ in (39) as $\gamma = \gamma_0 \equiv (\nu_2 \nu_3)^{-1/2}$, and we shall employ in what follows the same notation $\vec{f}_0$ for the *unit-length* lowest eigenvector with the components as in (39), but $\gamma = \gamma_0$.

In order to find next eigenvectors of the number operator $\mathcal{N}^{(\mathbf{5})}$, we first define 4 vectors of the form

$$\vec{f}_k := \frac{d_k}{c^k} \left(\mathbf{b_5^T}\right)^k \vec{f}_0 \, , \qquad 1 \le k \le 4 \, , \tag{40}$$

where $\vec{f}_0$ is the lowest eigenvector of the $\mathcal{N}^{(\mathbf{5})}$ and d_ks are some normalization scalar factors. Since $\vec{f}_0$ is the eigenvector of the DFT operator $\Phi^{(5)}$ also and it corresponds to the eigenvalue $\mathrm{i}^0 \, = \, 1$, from the second intertwining relation in (13) for $N \, = \, 5$, it follows at once that all vectors $\vec{f}_k$, $1 \, \le \, k \, \le \, 4$, are, in effect, the eigenvectors of the $\Phi^{(5)}$,

$$\begin{aligned} \Phi^{(5)} \vec{f}_k &= \frac{d_k}{c^k} \, \Phi^{(5)} \left(\mathbf{b_5^T}\right)^k \vec{f}_0 = \mathrm{i} \, \frac{d_k}{c^k} \, \mathbf{b_5^T} \, \Phi^{(5)} \left(\mathbf{b_5^T}\right)^{k-1} \vec{f}_0 = \ldots \\ &= \mathrm{i}^k \, \frac{d_k}{c^k} \left(\mathbf{b_5^T}\right)^k \Phi^{(5)} \vec{f}_0 = \mathrm{i}^k \, \frac{d_k}{c^k} \left(\mathbf{b_5^T}\right)^k \vec{f}_0 = \mathrm{i}^k \vec{f}_k \, , \end{aligned} \tag{41}$$

corresponding to the eigenvalues i^k, respectively. Moreover, it actually turns out that all vectors $\vec{f}_k$, $\; 0 \le k \le 4$, are at the same time the eigenvectors of the number operator $\mathcal{N}^{(\mathbf{5})}$. Indeed, since the operators $\Phi^{(5)}$ and $\mathcal{N}^{(\mathbf{5})}$ commute, one checks easily that

$$\Phi^{(5)} \, \mathcal{N}^{(\mathbf{5})} \vec{f}_k \, = \, \mathcal{N}^{(\mathbf{5})} \, \Phi^{(5)} \vec{f}_k \, = \, \mathrm{i}^k \, \mathcal{N}^{(\mathbf{5})} \vec{f}_k \tag{42}$$

is valid for all integer values of k between 0 and 4. This means that for any integer $k \, \in \, [0, 4]$, both vectors $\vec{f}_k$ and $\mathcal{N}^{(\mathbf{5})} \vec{f}_k$ are associated with the same eigenvalues i^k of the 5D DFT operator $\Phi^{(5)}$. Consequently, three vectors $\vec{f}_k$, $1 \, \le \, k \, \le \, 3$, do represent the eigenvectors of the number operator $\mathscr{N}^{(\mathbf{5})}$ because the corresponding multiplicities $m(\mathrm{i}^k) = 1$ for those values of $1 \leq k \leq 3$. As for the last vector $\vec{f}_4$, one readily verifies that

$$\mathscr{N}^{(\mathbf{5})}\vec{f}_4 \simeq \mathbf{b}_\mathbf{5}^\mathbf{T}\,\mathbf{b}_\mathbf{5}\left(\mathbf{b}_\mathbf{5}^\mathbf{T}\right)^4\vec{f}_0 \simeq \mathbf{b}_\mathbf{5}^\mathbf{T}\mathbf{b}_\mathbf{5}\,\mathbf{b}_\mathbf{5}^\mathbf{T}\vec{f}_3 \simeq \mathbf{b}_\mathbf{5}^\mathbf{T}\left(\mathscr{N}^{(\mathbf{5})} + \mathscr{K}\right)\vec{f}_3 \simeq \mathbf{b}_\mathbf{5}^\mathbf{T}\vec{f}_3 \simeq \vec{f}_4\,, \quad (43)$$

where the symbol $\mathbf{A} \simeq \mathbf{B}$ indicates that $\mathbf{A}$ is equal to $\mathbf{B}$ multiplied by a non-zero scalar constant factor and we employed the fact that the vector $\vec{f}_3$ is an eigenvector of the operator $\mathscr{K}$ also, for the same reason as it happens to be true in the case (42) of the operator $\mathscr{N}^{(\mathbf{5})}$ (despite the non-commutativity of the operators $\mathscr{K}$ and $\mathscr{N}^{(\mathbf{5})}$).

Since the vector $\vec{f}_0$ has been already defined by (34) as the lowest eigenvector of the operator $\mathscr{N}^{(\mathbf{5})}$, one does conclude that the five orthonormal vectors $\vec{f}_k$, $0 \leq k \leq 4$, explicitly given as

$$\vec{f}_0 = \frac{1}{\sqrt{\nu_2\nu_3}}\,\vec{y}_0, \qquad \vec{f}_1 = \frac{1}{2}\sqrt{\frac{\tau}{\nu_3}}\,\vec{y}_4, \qquad \vec{f}_2 = \frac{\sqrt{\tau}}{2\kappa}\,\vec{y}_3,$$
$$\vec{f}_3 = \frac{1}{2}\sqrt{\frac{\tau}{\nu_1}}\,\vec{y}_1, \qquad \vec{f}_4 = \frac{1}{2\sqrt{\nu_2\nu_3}}\,\vec{y}_2, \qquad (44)$$

do represent the desired set of the eigenvectors for the number operator $\mathscr{N}^{(\mathbf{5})}$,

$$\mathscr{N}^{(\mathbf{5})}\vec{f}_k = \lambda_k\vec{f}_k, \qquad 0 \leq k \leq 4, \qquad (45)$$

associated with the eigenvalues

$$\lambda_0 = 0, \quad \lambda_1 = c^2\nu_4, \quad \lambda_2 = c^2\nu_3, \quad \lambda_3 = c^2\nu_1, \quad \lambda_4 = c^2\nu_2, \qquad (46)$$

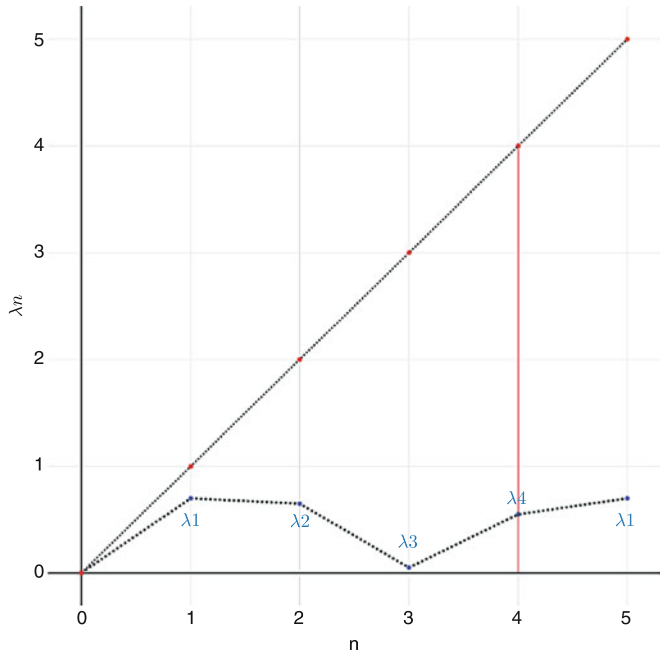
respectively.

The explicit analytical form of the spectrum of the discrete number operator $\mathscr{N}^{(\mathbf{5})}$ can be thus represented as

$$\lambda_k = c^2\Big[5(1-\delta_{k0}) + 4\Big((\tau-1)\sin k\theta + \cos k\theta\Big)\sin 2k\theta\Big], \qquad (47)$$

where $\theta = 2\pi/5$ and $0 \leq k \leq 4$.

Our first graph compares the eigenvalues λ_k, $0 \leq k \leq 4$, and the first 5 eigenvalues of the quantum number operator $\mathbf{N} = \mathbf{a}^\dagger\mathbf{a}$.



The next step is to clarify how these eigenvectors (44) transform under the action of the operators $\mathbf{b}_5^{\mathbf{T}}$ and $\mathbf{b}_5$ and then to compare them with the behaviour of their continuous counterparts $\psi_n(x)$, which satisfy the well-known relations

$$\mathbf{a}\,\psi_n(x) \;=\; \sqrt{n}\,\psi_{n-1}(x), \qquad \mathbf{a}^\dagger\,\psi_n(x) \;=\; \sqrt{n+1}\,\psi_{n+1}(x). \tag{48}$$

But observe first that from (40), it follows at once that

$$\vec{f}_{k+1} = \frac{d_{k+1}}{c\,d_k}\,\mathbf{b}_5^{\mathbf{T}}\,\vec{f}_k\,, \qquad 0 \le k \le 3\,, \qquad d_0 = 1\,. \tag{49}$$

Therefore, the relations (49) can be explicitly written for each appropriate value of the index k as

$$\mathbf{b}_5^{\mathbf{T}}\,\vec{f}_0 = \frac{4\,\kappa\,c^2}{\sqrt{\tau\lambda_4}}\vec{f}_1 \equiv \eta\,\vec{f}_1, \qquad \mathbf{b}_5^{\mathbf{T}}\,\vec{f}_k = \sqrt{\lambda_{k+1}}\vec{f}_{k+1}, \qquad 1 \le k \le 3, \tag{50}$$

where $\eta := 4\kappa/\sqrt{\tau\nu_2\nu_4} \equiv 4/\sqrt{5\tau+21}$. It remains only to evaluate the last identity

$$\mathbf{b}_5^{\mathbf{T}}\vec{f}_4 = \frac{1}{4c^4\kappa^2}(\mathbf{b}_5^{\mathbf{T}})^5\vec{f}_0 = -\frac{5\tau}{4\kappa^2}\mathbf{b}_5^{\mathbf{T}}\vec{f}_0 = -\sqrt{(1-\eta^2)\,\lambda_1}\,\vec{f}_1, \tag{51}$$

which is a direct consequence of the second 'cyclic' identity in (18) and the relation

$$\vec{f}_4 = \frac{1}{4c^4\kappa^2}(\mathbf{b}_5^{\mathbf{T}})^4\vec{f}_0, \tag{52}$$

readily obtained by the successive use of all entries in the chain of relations (49). The formulas (50) and (51) are thus discrete analogues of those, which are collected in the second identity of the continuous case (48).

As for the action of the lowering difference operator $\mathbf{b_5}$, the situation here is slightly different. The point is that already at the first step one evaluates that

$$\mathbf{b_5}\vec{f}_1 = \delta\, \mathbf{b_5}\mathbf{b}_5^{\mathbf{T}}\vec{f}_0 = \delta\left(\mathscr{K} + \mathscr{N}^{(5)}\right)\vec{f}_0 = \delta\, \mathscr{K}\vec{f}_0 = \delta\left(\alpha\vec{f}_0 + \beta\vec{f}_4\right), \tag{53}$$

where $\delta := \left(\eta\sqrt{\lambda_1}\right)^{-1}$ and coefficients α and β can be explicitly defined by the following easy algebra:

$$(\vec{f}_0, \mathbf{b_5}\vec{f}_1) = \delta\alpha = (\mathbf{b}_5^{\mathbf{T}}\vec{f}_0, \vec{f}_1) = \delta^{-1}(\vec{f}_1, \vec{f}_1) = \delta^{-1} = \eta\sqrt{\lambda_1}, \tag{54}$$

$$(\vec{f}_4, \mathbf{b_5}\vec{f}_1) = \delta\beta = (\mathbf{b}_5^{\mathbf{T}}\vec{f}_4, \vec{f}_1) = -\sqrt{(1-\eta^2)\lambda_1}\,(\vec{f}_1, \vec{f}_1) = -\sqrt{(1-\eta^2)\lambda_1}\,. \tag{55}$$

From (54) and (55), one thus concludes that $\alpha = \eta^2\lambda_1,\ \beta = -\eta\sqrt{1-\eta^2}\,\lambda_1$, and the relation (53) now explicitly reads

$$\mathbf{b_5}\vec{f}_1 = \sqrt{\lambda_1}\left[\eta\vec{f}_0 - \sqrt{1-\eta^2}\vec{f}_4\right]. \tag{56}$$

The evaluation of the action of the lowering operator $\mathbf{b_5}$ on the remaining three eigenvectors $\vec{f}_n$, $n = 2, 3, 4$, requires less efforts for the following reason. As we have already remarked in the process of deriving formula (43), three vectors $\vec{f}_1, \vec{f}_2$ and $\vec{f}_3$, associated with the eigenvalues i^k, $k = 1, 2, 3$, with multiplicities 1, are actually common eigenvectors of the number operator $\mathscr{N}^{(5)}$ and the operator $\mathscr{K}$ (although these operators do not commute). Moreover, the explicit form of corresponding eigenvalues of the operator $\mathscr{K}$,

$$\mathscr{K}\vec{f}_n = (\lambda_{n+1} - \lambda_n)\vec{f}_n\,, \qquad n = 1, 2, 3, \tag{57}$$

is a direct consequence of the evident intertwining relation $\mathscr{N}^{(5)}\mathbf{b}_5^{\mathbf{T}} = \mathbf{b}_5^{\mathbf{T}}\left(\mathscr{N}^{(5)} + \mathscr{K}\right)$. Therefore, for $2 \le k \le 4$ one readily derives that

$$\mathbf{b_5}\vec{f}_k = \frac{1}{\sqrt{\lambda_k}}\,\mathbf{b_5}\,\mathbf{b}_5^{\mathbf{T}}\vec{f}_{k-1} = \frac{1}{\sqrt{\lambda_k}}\left(\mathscr{K} + \mathscr{N}^{(5)}\right)\vec{f}_{k-1} = \sqrt{\lambda_k}\vec{f}_{k-1}\,. \tag{58}$$

We are now in a position to write down all matrix elements of the raising and lowering operators $\mathbf{b}_5^{\mathbf{T}}$ and $\mathbf{b_5}$ in the basis, built over the eigenvectors $\vec{f}_n$, $0 \le n \le 4$. In particular, using (50) and (51), one readily evaluates that

$$\left(\left(\vec{f}_k\,,\mathbf{b}_5^{\mathbf{T}}\vec{f}_l\right)\right) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \eta\sqrt{\lambda_1} & 0 & 0 & 0 & -\sqrt{(1-\eta^2)\lambda_1} \\ 0 & \sqrt{\lambda_2} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\lambda_3} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{\lambda_4} & 0 \end{pmatrix}. \tag{59}$$

In a like manner, from (45), (56) and (58), it follows at once that

$$\left(\left(\vec{f}_k\,,\mathbf{b}_5\vec{f}_l\right)\right) = \begin{pmatrix} 0 & \eta\sqrt{\lambda_1} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\lambda_2} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{\lambda_3} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\lambda_4} \\ 0 & -\sqrt{(1-\eta^2)\lambda_1} & 0 & 0 & 0 \end{pmatrix}. \tag{60}$$

To close this section, we point out here that as a consistency check, one may verify that these matrix realizations (59) and (60) of the raising and lowering operators $\mathbf{b}_5^{\mathbf{T}}$ and $\mathbf{b}_5$ in the $\vec{f}_n$-basis, respectively, do possess the same 'cyclic' properties as indicated in identities (18). Indeed, a direct computation of the 5th power of the matrix (60) shows that

$$\left(\left(\vec{f}_k\,,\mathbf{b}_5\vec{f}_l\right)\right)^5 + \left[(1-\eta^2)\lambda_1\lambda_2\lambda_3\lambda_4\right]^{1/2}\left(\left(\vec{f}_k\,,\mathbf{b}_5\vec{f}_l\right)\right) = 0, \tag{61}$$

where

$$\left[(1-\eta^2)\lambda_1\lambda_2\lambda_3\lambda_4\right]^{1/2} = c^4\left[(1-\eta^2)\nu_1\nu_2\nu_3\nu_4\right]^{1/2} = 5\,c^4\tau, \tag{62}$$

upon using definitions of the eigenvalues ν_k, $1 \le k \le 4$, given in (24)–(27). Finally, since the operator $\mathbf{b}_5^{\mathbf{T}}$ is the matrix transpose of $\mathbf{b}_5$, the former one has the same 'cyclic' property as the latter.

4 Eigenvectors of the $\mathscr{N}^{(5)}$ Versus the Hermite Functions $\psi_n(x)$

From the outset the ND discrete Fourier transform $\Phi^{(N)}$ was conceived as a finite (discrete) analogue of the Fourier transform $\mathscr{F}$, and the N eigenvectors $\vec{f}_k$, $0 \le k \le N-1$, of the former transform operator were therefore required to converge to the corresponding Hermite functions $\psi_n(x)$ in the limit as $N \to \infty$. So the question is: How the eigenvectors $\vec{f}_k$, $0 \le k \le N-1$, of a ND DFT $\Phi^{(N)}$ with a fixed integer value of N can be related to the Hermite functions $\psi_n(x)$, $0 \le n < \infty$? Note first that a ND discrete Fourier transform is actually a discrete (finite) image of the N-dimensional subspace of the infinite-dimensional Hilbert space $L^2(\mathbb{R}, dx)$, spanned by the first N basis functions $\psi_n(x)$, $0 \le n \le N-1$, in this space, rather than of the whole Hilbert space $L^2(\mathbb{R}, dx)$ itself. Consequently, one should find out how the eigenvectors $\vec{f}_k$ match the first N Hermite functions $\psi_n(x)$, $0 \leq n \leq N-1$. In this regard, it is important to take into account the following fundamental properties of the Hermite functions $\psi_n(x)$: $\psi_n(-x) = (-1)^n \psi_n(x)$, and each function $\psi_n(x)$ has exactly n alternations in its sign. As we have already remarked above [see formula (36)], the eigenvectors $\vec{f}_k$ do exhibit the same type of the reflection symmetry as $\psi_n(-x) = (-1)^n \psi_n(x)$ in the continuous case, so that it remains only to verify that each eigenvector $\vec{f}_k$ has the same number of alternations in its components $\left(\vec{f}_k\right)_l$, $0 \leq l \leq N-1$, as a Hermite function $\psi_n(x)$, associated with it. But the careful examination of the eigenvectors $\vec{f}_n$ under study, explicitly defined by relations (28)–(32) and (44), indicates that their components are not appropriately structured in order to enable one to match them with the first 5 Hermite functions $\psi_n(x)$. It has been then realized that one actually needs to rearrange components of the eigenvectors $\vec{f}_k$ and introduce another set of *centred vectors* $\vec{f}_k^{(c)}$ with the components $\left(\vec{f}_k^{(c)}\right)_{l=-2}^{2}$, defined as

$$\left(\vec{f}_k^{(c)}\right)_{l-2} = \left((\mathbf{U})_{l,m}\right)\left(\vec{f}_k\right)_m, \qquad 0 \leq k, l, m \leq 4, \tag{63}$$

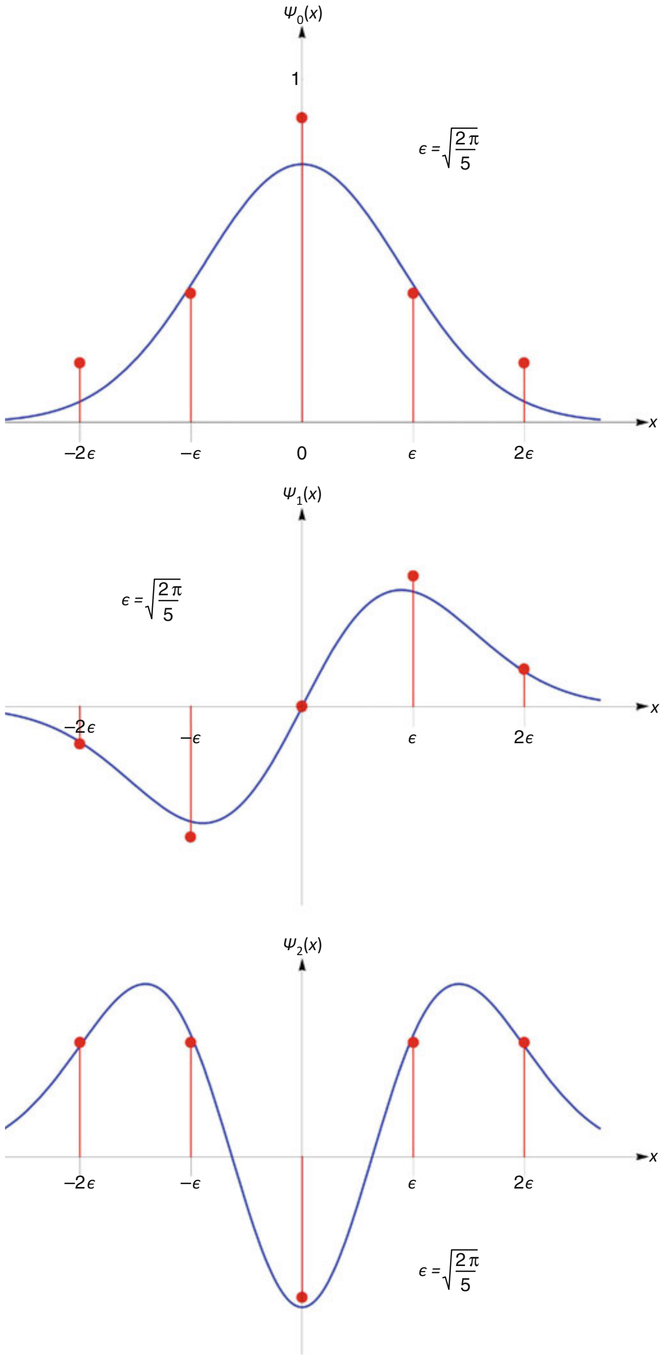
where $\mathbf{U}$ is the unitary operator, $\mathbf{U}\,\mathbf{U}^{\mathbf{T}} = \mathbf{U}^{\mathbf{T}}\,\mathbf{U} = \mathbf{I}$, with the matrix elements

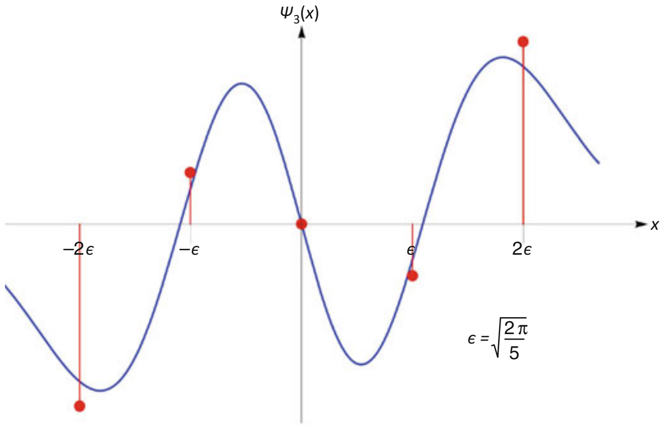
$$\left((\mathbf{U})_{m,m'}\right) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}. \tag{64}$$

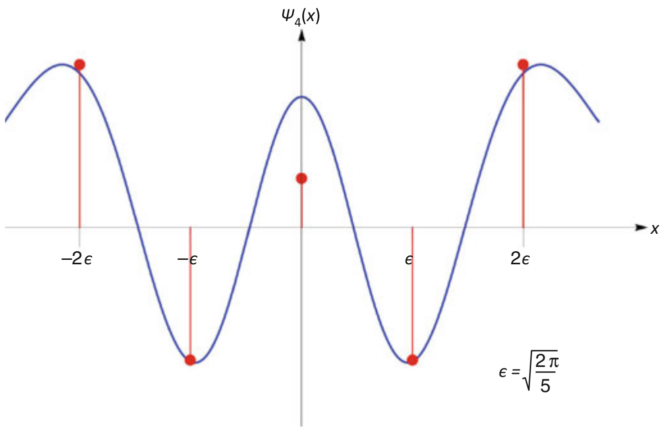
The explicit componentwise forms of these centred eigenvectors $\vec{f}_k^{(c)}$, $0 \leq k \leq 4$, are

$$\begin{aligned}
\vec{f}_0^{(c)} &= \frac{1}{\sqrt{\nu_2\,\nu_3}}\left\{1,\ 1+\kappa\tau^{-1/2},\ 2\tau+\kappa\tau^{1/2},\ II,\ I\right\},\\
\vec{f}_1^{(c)} &= \frac{1}{2}\sqrt{\frac{\tau}{\nu_3}}\left\{-\ \tau^{-1/2},\ -\kappa-\tau^{1/2},\ 0,\ -II,\ -I\right\},\\
\vec{f}_2^{(c)} &= \frac{\sqrt{\tau}}{2\kappa}\left\{1, 1, 2(1-\tau), 1, 1\right\},\\
\vec{f}_3^{(c)} &= \frac{1}{2}\sqrt{\frac{\tau}{\nu_1}}\left\{-\ \tau^{-1/2},\ \kappa-\tau^{1/2},\ 0, -II, -I\right\},\\
\vec{f}_4^{(c)} &= \frac{1}{2\sqrt{\nu_2\,\nu_3}}\left\{2\kappa\tau^{1/2}+3\tau-2,\ -\left(\tau+2\kappa\tau^{1/2}\right),\ 2,\ II,\ I\right\}.
\end{aligned} \tag{65}$$

In next 5 graphs, we compare the centred vectors $\vec{f}_n^{(c)}$ and the first Hermite functions $\psi_n(x)$, $0 \leq n \leq 4$, respectively.


Ψ0(x)
1
ϵ = √(2π/5)
x
−2ϵ
−ϵ
0
ϵ
2ϵ
Ψ1(x)
ϵ = √(2π/5)
x
−2ϵ
−ϵ
ϵ
2ϵ
Ψ2(x)
x
−2ϵ
−ϵ
ϵ
2ϵ
ϵ = √(2π/5)





It is to be emphasized that thus introduced centred vectors $\vec{f}_n^{(c)}$ are actually eigenvectors of the *centred discrete number operator* $\mathscr{N}^{(\mathbf{5;c})}$,

$$\mathscr{N}^{(\mathbf{5;c})} := \mathbf{U}\,\mathscr{N}^{(\mathbf{5})}\,\mathbf{U}^{\mathbf{T}}\,, \tag{66}$$

associated with the same eigenvalues λ_n as in (45), that is, $\mathscr{N}^{(\mathbf{5;c})}\vec{f}_n^{(c)} = \lambda_n \vec{f}_n^{(c)}$. Matrix elements of this operator $\mathscr{N}^{(\mathbf{5;c})} := (\mathbf{b}_\mathbf{5}^{(\mathbf{c})})^{\mathbf{T}}\mathbf{b}_\mathbf{5}^{(\mathbf{c})}$ are explicitly given as (*cf* (23))

$$\left(\mathscr{N}^{(\mathbf{5;c})}\right)_{m,m'} = c^2 \begin{pmatrix} 5-\tau & \kappa\tau^{-\frac{3}{2}} & -1 & -1 & 2\kappa\tau^{-\frac{1}{2}} \\ \kappa\tau^{-\frac{3}{2}} & 4+\tau & -\kappa\tau^{\frac{1}{2}} & -1 & -1 \\ -1 & -\kappa\tau^{\frac{1}{2}} & 2 & -\kappa\tau^{\frac{1}{2}} & -1 \\ -1 & -1 & -\kappa\tau^{\frac{1}{2}} & 4+\tau & \kappa\tau^{-\frac{3}{2}} \\ 2\kappa\tau^{-\frac{1}{2}} & -1 & -1 & \kappa\tau^{-\frac{3}{2}} & 5-\tau \end{pmatrix}, \tag{67}$$

where the centred lowering and raising operators $\mathbf{b}_5^{(\mathbf{c})}$ and $(\mathbf{b}_5^{(\mathbf{c})})^{\mathbf{T}}$ are defined through the operators $\mathbf{b}_5$ and $\mathbf{b}_5^{\mathbf{T}}$, respectively, by the same similarity transformation as in (66).

Furthermore, the centred vectors $\vec{f}_n^{(c)}$ turn out to be also eigenvectors of the *centred discrete Fourier transform operator* $\Phi^{(5;c)}$,

$$\Phi^{(5;c)} := \mathbf{U}\,\Phi^{(5)}\,\mathbf{U}^{\mathbf{T}}\,, \tag{68}$$

corresponding to the respective eigenvalues i^n. The eigenproblem for the centred DFT operator $\Phi^{(5;c)}$ can be thus written in the matrix form as

$$\sum_{n=-2}^{2} \Phi_{m,n}^{(5;c)} \left(\vec{f}_k^{(c)}\right)_n = \lambda_k \left(\vec{f}_k^{(c)}\right)_m, \qquad 0 \le k \le 4, \qquad -2 \le m \le 2, \tag{69}$$

and associated matrix for this eigenproblem has elements (*cf* (14))

$$\left(\Phi_{m,n}^{(5;c)}\right) = \frac{1}{\sqrt{5}} \begin{pmatrix} q^4 & q^2 & 1 & q^3 & q \\ q^2 & q & 1 & q^4 & q^3 \\ 1 & 1 & 1 & 1 & 1 \\ q^3 & q^4 & 1 & q & q^2 \\ q & q^3 & 1 & q^2 & q^4 \end{pmatrix}. \tag{70}$$

A word of explanation regarding the present results (66)–(70) is in order at this point, but let us recall first the following. Square matrices $\mathbf{A}$ and $\mathbf{B}$ are said to be *similar*, if there is a non-singular matrix $\mathbf{C}$ (which is referred to as a *transforming matrix* of $\mathbf{B}$ to $\mathbf{A}$) such that $\mathbf{A} = \mathbf{C}\mathbf{B}\mathbf{C}^{-1}$. In the case when the transforming matrix $\mathbf{C}$ is a unitary matrix $\mathbf{U}$, $\mathbf{U}\,\mathbf{U}^{\dagger} = \mathbf{I}$, then $\mathbf{B}$ is *unitarily similar* to $\mathbf{A}$ (see, e.g. pages 130 and 175 in [15]). Taking that into account, note that in the process of establishing how the eigenvectors $\vec{f}_n$ are related to the Hermite functions $\psi_n(x)$, we actually arrived at another form (69) of the eigenproblem for the 5D DFT operator, which appear to be different from the traditional one (12). This form of DFT is also well known but less frequently used. However, some authors have even suggested that *it is better not to use the standard Fourier matrix that represents a discretization of the Fourier transform but rather to use a 'centred' version of it* [8]. But from (68) it is clear that $\Phi^{(5)}$ and $\Phi^{(5;c)}$ are actually unitarily similar, with the transforming matix (64), and they represent therefore the same linear transformation after a change of basis. Nonetheless, it is true that there is considerable merit in working with the non-standard 5D DFT operator $\Phi^{(5;c)}$ because it explicitly displays all those symmetry properties in the eigenproblem (69), which are so characteristic of the continuous Fourier transform. As a direct consequence of this appealing feature of (69), the matrix form of $\Phi^{(5;c)}$ clearly exhibits its remarkable symmetry: the matrix $\Phi^{(5;c)}$ is a *centrosymmetric matrix* [3], meaning that it is reproduced by the similarity transformation with the transforming matrix $\mathbf{J}$,

$$\Phi^{(5;c)} = \mathbf{J}\,\Phi^{(5;c)}\,\mathbf{J}\,, \tag{71}$$

where $\mathbf{J}$ is the 5×5 matrix with ones on the antidiagonal,

$$\left(J_{m,n}\right) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Finally, the operator $\mathscr{N}^{(5;c)}$, which commutes with $\Phi^{(5;c)}$, is also of the centrosymmetric type, as is readily seen from (67).

5 Concluding Remarks

To summarize, we have constructed an explicit form of a difference analogue of the quantum number operator in terms of the raising and lowering operators that govern eigenvectors of the 5D discrete (finite) Fourier transform. The main algebraic properties of this operator have been examined in detail. The eigenvalues of this discrete number operator are represented by distinct non-negative numbers so that this operator has been used to systematically classify, in complete analogy with the case of the continuous classical Fourier transform, eigenvectors of the 5D discrete Fourier transform, thus resolving the ambiguity caused by the well-known degeneracy of the eigenvalues of the discrete Fourier transform. We hope that this particular knowledge will help us to extend our results to the case of an arbitrary ND discrete Fourier transform.

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Finite Difference Formulation for the Model of a Compressible Viscous and Heat-Conducting Micropolar Fluid with Spherical Symmetry

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Abstract We are dealing with the nonstationary 3D flow of a compressible viscous heat-conducting micropolar fluid, which is in the thermodynamical sense perfect and polytropic. It is assumed that the domain is a subset of $\mathbf{R}^3$ and that the fluid is bounded with two concentric spheres. The homogeneous boundary conditions for velocity, microrotation, heat flux, and spherical symmetry of the initial data are proposed. By using the assumption of the spherical symmetry, the problem reduces to the one-dimensional problem. The finite difference formulation of the considered problem is obtained by defining the finite difference approximate equation system. The corresponding approximate solutions converge to the generalized solution of our problem globally in time, which means that the defined numerical scheme is convergent. Numerical experiments are performed by applying the proposed finite difference formulation. We compare the numerical results obtained by using the finite difference and the Faedo–Galerkin approach and analyze the properties of the numerical solutions.

Keywords Micropolar fluid flow • Spherical symmetry • Finite difference approximations • Strong and weak convergence

Mathematics Subject Classification (2010): 35Q35, 76M20, 65M06, 76N99

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1 Introduction

The theory of micropolar fluids, established by Eringen [4], provides a mathematical foundation for studying the model of a fluid, which takes into account the interactions between the micromotion effects of fluid particles and the macromotion. The micropolar fluid equation system is actually an extension of the classical Navier–Stokes equations with the additional variable called microrotation, describing the angular momentum of the particles.

In this paper, we focus on the compressible flow of the isotropic, viscous, and heat-conducting micropolar fluid, which is in the thermodynamical sense perfect and polytropic. The model for this type of flow was first considered by Mujaković in [6] where she developed one-dimensional model. In [6, 7], she considered the model with homogeneous and nonhomogeneous boundary conditions and proved the existence and the uniqueness of the generalized solutions. The model in the three-dimensional case, which we consider in this work, was introduced in [1]. It is assumed that the fluid occupies the domain $\Omega \subset \mathrm{R}^3$, bounded with two concentric spheres with radii a and b, $b > a > 0$; that the initial data are spherically symmetric; and that the homogeneous boundary conditions for velocity, microrotation, and heat flux are valid. Taking into account the spherical symmetry, the problem reduces to the one-dimensional problem, which we consider here in the Lagrangian description. The local existence and the uniqueness of the generalized solution were proved in [1, 8] by using the Faedo–Galerkin method. Additionally, the global existence and the stabilization of the generalized solution for the same model were established in [2, 3].

We consider here the finite difference formulation of the described problem, which is based on the approximate equation system obtained by using the finite difference approach. It is proved in [9] that the sequence of the corresponding approximate solutions converges to the generalized solution of our problem, which means that the defined numerical scheme is convergent. In this work, we analyze some properties of numerical solutions obtained with the proposed finite difference scheme. Furthermore, we compute the numerical results by using the Faedo–Galerkin method [1] and compare it with the results obtained with the finite difference scheme.

The paper is organized as follows. In the second section, we introduce the mathematical formulation of our problem. In the third section, we define the corresponding finite difference formulation. In the fourth section, we present and analyze some properties of the numerical solutions.

2 Mathematical Model

In this work, we consider the three-dimensional flow of the compressible viscous and heat-conducting micropolar fluid, being thermodynamically perfect and polytropic. In the Eulerian description, the starting domain is $\{\mathbf{x} \mid \mathbf{x} \in \mathrm{R}^3, a < |\mathbf{x}| < b\}$, where $|\mathbf{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ and $b > a > 0$. The homogeneous boundary conditions for velocity, microrotation, heat flux, and spherical symmetry of the initial data are proposed. This spherically symmetric problem is transformed in [1] to the one-dimensional problem in Lagrangian coordinates in the domain $\langle 0, 1\rangle$ and described by the following system of equations:

$$\frac{\partial \rho}{\partial t} = -\frac{1}{L}\rho^2 \frac{\partial}{\partial x}\left(r^2 v\right), \tag{1}$$

$$\frac{\partial v}{\partial t} = -\frac{R}{L} r^2 \frac{\partial}{\partial x}(\rho\theta) + \frac{\lambda + 2\mu}{L^2} r^2 \frac{\partial}{\partial x}\left(\rho \frac{\partial}{\partial x}\left(r^2 v\right)\right), \tag{2}$$

$$\rho\frac{\partial \omega}{\partial t} = -\frac{4\mu_r}{j_I}\omega + \frac{c_0 + 2c_d}{j_I L^2} r^2 \rho \frac{\partial}{\partial x}\left(\rho \frac{\partial}{\partial x}\left(r^2 \omega\right)\right), \tag{3}$$

$$\begin{aligned} \rho\frac{\partial \theta}{\partial t} = {} & \frac{K}{c_v L^2}\rho \frac{\partial}{\partial x}\left(r^4 \rho \frac{\partial \theta}{\partial x}\right) - \frac{R}{c_v L}\rho^2 \theta \frac{\partial}{\partial x}\left(r^2 v\right) + \frac{\lambda + 2\mu}{c_v L^2}\left(\rho \frac{\partial}{\partial x}\left(r^2 v\right)\right)^2 \\ & - \frac{4\mu}{c_v L}\rho \frac{\partial}{\partial x}\left(r v^2\right) + \frac{c_0 + 2c_d}{c_v L^2}\left(\rho \frac{\partial}{\partial x}\left(r^2 \omega\right)\right)^2 - \frac{4c_d}{c_v L}\rho \frac{\partial}{\partial x}\left(r\omega^2\right) + \frac{4\mu_r}{c_v}\omega^2. \end{aligned} \tag{4}$$

Here ρ, v, w, and θ denote, respectively, the mass density, velocity, microrotation velocity ,and temperature in the Lagrangian description and $L = \int_a^b s^2 \rho_0(s)\, ds$, ($a$ and b are radii of the starting domain). $\mu, \mu_r, \lambda, c_0, c_d, K, c_v, j_I,$ and R are the physical constants describing fluid properties for which the following relations should be valid: $\mu, \mu_r, c_d, j_I \geq 0$, $3\lambda + 2\mu \geq 0$, $3c_0 + 2c_d \geq 0$, $c_v, R, K > 0$. The system is considered in the domain $Q_T = \langle 0, 1\rangle \times \langle 0, T\rangle$, where $T > 0$ is arbitrary. Equations (1)–(4) are, respectively, local forms of the conservation laws for the mass, momentum, momentum moment, and energy. We take the homogeneous boundary conditions:

$$v(0,t) = v(1,t) = 0,\ \omega(0,t) = \omega(1,t) = 0,\ \partial_x\theta(0,t) = \partial_x\theta(1,t) = 0, \tag{5}$$

for $t \in \langle 0, T\rangle$ and the nonhomogeneous initial conditions:

$$\rho(x,0) = \rho_0(x),\ \ v(x,0) = v_0(x),\ \ \omega(x,0) = \omega_0(x),\ \ \theta(x,0) = \theta_0(x), \tag{6}$$

for $x \in \langle 0, 1\rangle$ and for the given functions ρ_0, v_0, ω_0, and θ_0. The function r is defined by:

$$r(x,t) = r_0(x) + \int_0^t v(x,\tau)\,d\tau, \quad (x,t) \in Q_T, \tag{7}$$

where:

$$r_0(x) = \left(a^3 + 3L\int_0^x \frac{1}{\rho_0(y)}\,dy\right)^{1/3}, \quad x \in \langle 0,1\rangle. \tag{8}$$

We assume that the initial functions satisfy:

$$\rho_0(x) \geq m,\ \theta_0(x) \geq m, \quad x \in \langle 0,1\rangle, \tag{9}$$

for some constant $m \in \mathrm{R}^+$ and that

$$\rho_0, \theta_0 \in H^1(\langle 0,1\rangle) \quad \text{and} \quad v_0, \omega_0 \in H_0^1(\langle 0,1\rangle). \tag{10}$$

Under the stated assumptions (9) and (10), in previous papers [1, 2, 8], it is proven that the problem (1)–(5) has unique solution $(\rho, v, \omega, \theta)$ in the domain Q_T, for any $T > 0$, with the following properties:

$$\rho \in L^\infty(0,T;H^1(\langle 0,1\rangle)) \cap H^1(Q_T), \tag{11}$$

$$v, \omega, \theta \in L^\infty(0,T;H^1(\langle 0,1\rangle)) \cap H^1(Q_T) \cap L^2(0,T;H^2(\langle 0,1\rangle)), \tag{12}$$

$$\rho > 0, \quad \theta > 0 \quad \text{on} \quad \bar{Q}_T. \tag{13}$$

For the function r, it holds $r \in L^\infty(0,T;H^2(\langle 0,1\rangle)) \cap H^2(Q_T) \cap C(\bar{Q}_T)$, and $r \geq a$ in $\bar{Q}_T$. These results were obtained by using the Faedo–Galerkin method for a local existence theorem [1] and the extension principle for a global existence theorem [2].

3 Finite Difference Formulation

In this section, we introduce the finite difference formulation for the considered problem (1)–(6). More precisely, we define the finite difference scheme resulting with the system of ordinary differential equations.

Let h be an increment in x such that $Nh = 1$ for $N \in \mathbf{Z}^+$. The staggered grid points are denoted with $x_k = kh, k \in \{0,1,\ldots,N\}$, and $x_j = jh, j \in \left\{\frac{1}{2},\ldots,N-\frac{1}{2}\right\}$. For each integer N, we construct the following time-dependent functions:

$$\rho_j(t), \theta_j(t),\ j = \tfrac{1}{2},\ldots,N-\tfrac{1}{2}, \quad \text{and} \quad v_k(t), \omega_k(t),\ k = 0,1,\ldots,N, \tag{14}$$

that form a discrete approximation to the solution at defined grid points:

$$\rho(x_j,t),\theta(x_j,t),\ j=\tfrac{1}{2},\dots,N-\tfrac{1}{2},\quad \text{and}\quad v(x_k,t),\omega(x_k,t),\ k=0,1,\dots,N.$$

We define the operator δ with $\delta g_l = \dfrac{g_{l+\frac{1}{2}}-g_{l-\frac{1}{2}}}{h}$, where $l=j$ or $l=k$. For $k\in\{1,\dots,N\}$ and $j\in\{\frac{1}{2},\dots,N-\frac{1}{2}\}$, the functions ρ_k, θ_k and v_j, ω_j are defined by $\rho_k=\rho_{k-\frac{1}{2}}$, $\theta_k=\theta_{k-\frac{1}{2}}$, and $v_j=v_{j+\frac{1}{2}}$, $\omega_j=\omega_{j+\frac{1}{2}}$. In accordance with the given initial conditions (6), we introduce the discrete initial conditions as:

$$(\rho_j,\theta_j)(0)=\left(\frac{1}{h}\int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h}\rho_0(x)dx,\ \frac{1}{h}\int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h}\theta_0(x)dx\right),\ j\in\left\{\tfrac{1}{2},\dots,N-\tfrac{1}{2}\right\},\tag{15}$$

$$(v_k,\omega_k)(0)=\left(\frac{1}{h}\int_{(k-1)h}^{kh}v_0(x)dx,\ \frac{1}{h}\int_{(k-1)h}^{kh}\omega_0(x)dx\right),\ k\in\{1,\dots,N-1\}.\tag{16}$$

and $v_0(0)=v_N(0)=0$, $\omega_0(0)=\omega_N(0)=0$, $\delta\theta_0(0)=\delta\theta_N(0)=0$. Because of (8), we take:

$$r_k(0)=\left(a^3+3L\sum_{j=\frac{1}{2}}^{k-\frac{1}{2}}\frac{h}{\rho_j(0)}\right)^{1/3},\ k=1,\dots,N\tag{17}$$

and $r_0(0)=a$. The functions $\rho_j(t),v_k(t),\omega_k(t),\theta_j(t)$, $j=\frac{1}{2},\dots,N-\frac{1}{2}$, $k=1,\dots,N-1$, and $r_k(t)$, $k=0,\dots,N$ are determined by using an appropriate spatial discretization of the equation system (1)–(4) and (7):

$$\dot\rho_j = -\frac{1}{L}\rho_j^2\delta(r^2v)_j,\quad j=\tfrac{1}{2},\dots,N-\tfrac{1}{2},\tag{18}$$

$$\dot v_k = -\frac{R}{L}r_k^2\,\delta(\rho\theta)_k+\frac{\lambda+2\mu}{L^2}r_k^2\,\delta(\rho\delta(r^2v))_k,\quad k=1,\dots,N-1,\tag{19}$$

$$\rho_k\dot\omega_k = -\frac{4\mu_r}{j_I}\omega_k+\frac{c_0+2c_d}{j_IL^2}r_k^2\,\rho_k\delta(\rho\delta(r^2\omega))_k,\quad k=1,\dots,N-1,\tag{20}$$

$$\begin{aligned}\rho_j\dot\theta_j &= \frac{K}{c_vL^2}\rho_j\delta(r^4\rho\delta\theta)_j-\frac{R}{c_vL}\rho_j^2\theta_j\delta(r^2v)_j\\ &\quad+\frac{\lambda+2\mu}{c_vL^2}\left(\rho_j(\delta(r^2v)_j)\right)^2-\frac{4\mu}{c_vL}\rho_j\delta(rv^2)_j+\\ &\quad+\frac{c_0+2c_d}{c_vL^2}\left(\rho_j(\delta(r^2\omega)_j)\right)^2-\frac{4c_d}{c_vL}\rho_j\delta(r\omega^2)_j+\frac{4\mu_r}{c_v}\omega_j^2,\\ &\quad j=\tfrac{1}{2},\dots,N-\tfrac{1}{2},\end{aligned}\tag{21}$$

$$\dot r_k = v_k,\quad k=0,\dots,N.\tag{22}$$

In accordance with the boundary conditions (5), we take

$$v_0(t) = v_N(t) = 0,\ \omega_0(t) = \omega_N(t) = 0,\ \delta\theta_0(t) = \delta\theta_N(t) = 0. \tag{23}$$

It is clear that (18)–(22), supplemented with the discretized equations (because, there are 3 equations included in (23)), represent the time-dependent system of ordinary differential equations with the initial conditions (15)–(17) for $5N - 1$ unknown functions. By using the solutions of this system, the approximate solutions of the considered problem can be defined, for example, as linear splines. It is proved in [9] that these approximate solutions converge to the generalized global solution of our problem.

4 Numerical Examples

The obtained system of equations (18)–(22) has the form $\dot{\mathbf{u}}(t) = \mathbf{F}(\mathbf{u}(t))$. For solving it numerically, the second-order strong stability-preserving Runge–Kutta method is used [5]:

$$\mathbf{u}^{(1)} = \mathbf{u}^n + \Delta t\mathbf{F}(\mathbf{u}^n)$$
$$\mathbf{u}^{n+1} = \frac{1}{2}\mathbf{u}^n + \frac{1}{2}\mathbf{u}^{(1)} + \frac{1}{2}\Delta t\mathbf{F}(\mathbf{u}^{(1)}).$$

Here $\mathbf{u}^n$ denotes the numerical solution at time moment $t^n = n\Delta t$ for the chosen time step Δt. For stability reasons of obtained numerical scheme, we choose $\Delta t = \mathscr{O}(h^2)$. In this way, the positivity of the density and the temperature are preserved.

Test Example 1 We take the following initial conditions: $\rho_0(x) = \left|x^2 - \frac{1}{4}\right| + 1$, $v_0(x) = 0$, $\omega_0(x) = 4(x^2 - x^4)$, and $\theta_0(x) = 0.1$ and parameters $\mu = \mu_r = K = c_0 = c_d = 0.01$, $R = c_v = 1$, $j_I = 1$, $a = 1$, and $L = 1$. Numerical parameters are set to $N = 16$, $t = 10^{-3}$.

In Fig. 1, we present the numerical results obtained with the proposed finite difference scheme at different time moments. In shown figures, it is nicely visible that, for larger t, the stabilization of the solution arises, which is in accordance with the result that was proved in [3] and which states that the solution of our problem converges to the stationary constant solution of the form $(\rho^*, 0, 0, \theta^*)$ in the space $(H_1(\langle 0, 1\rangle))^4$ (when $t \to \infty$), where $\rho^* = \left(\int_0^1 \frac{1}{\rho_0(x)}dx\right)^{-1}$, $\theta^* = \frac{1}{c_v}\int_0^1 \left(\frac{1}{2}|v_0(x)|^2 + \frac{j_I}{2}|\omega_0(x)|^2 + c_v|\theta_0(x)|\right)dx$.

Test Example 2—Influence of the Microrotation to the Solution Now, the initial conditions are taken as follows: $\rho_0(x) = 1$, $v_0(x) = 0$, $\omega_0(x) = \sin(\pi x)$, and $\theta_0(x) = 0.1$. The fixed parameters in this test are $K = c_0 = c_d = 0.01$, $R = c_v = 1$, $j_I = 1$, $a = 1$, and $L = 1$. Numerical parameters are set to $N = 16$, $t = 10^{-3}$.

In order to investigate the influence of the microrotation to the solution, in first case, we fix the viscosity parameter $\mu = 0.01$ and variate the micropolar viscosity coefficient μ_r. The obtained results for velocity and microrotation are presented in Fig. 2. In the second case, for fixed $\mu_r = 0.01$, we variate the viscosity parameter μ. The corresponding numerical results are presented in Fig. 3. In both cases, the influence of the microrotation and the corresponding parameters to the solution is clearly visible.

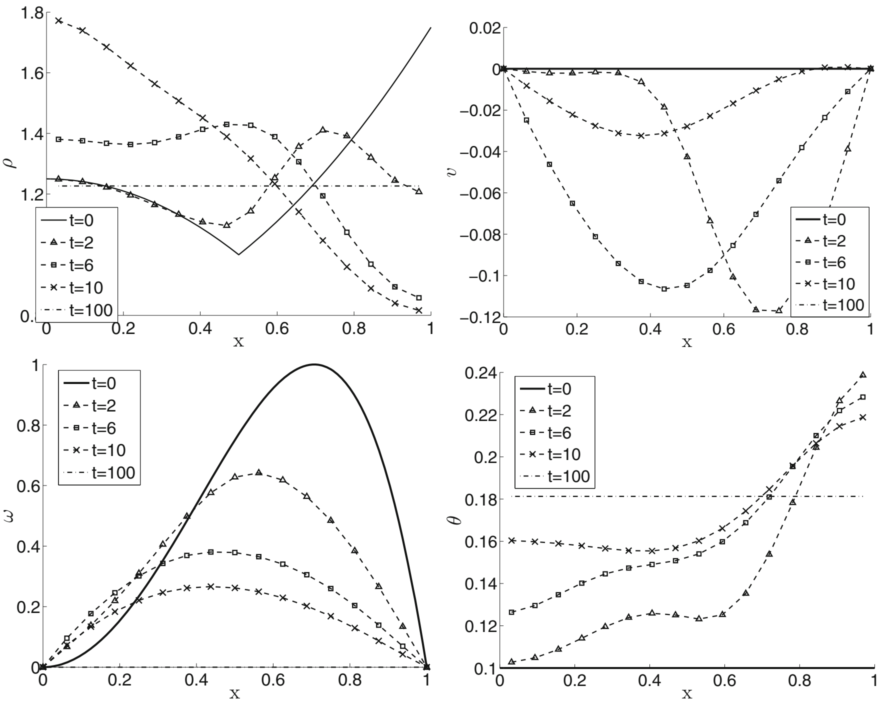

Fig. 1 Numerical results obtained at different time moments

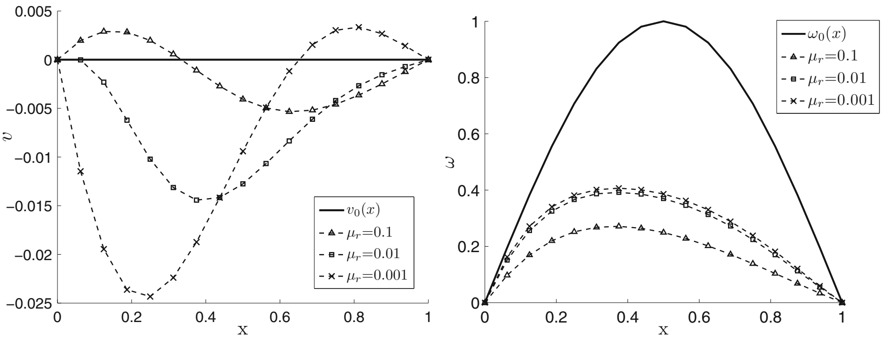

Fig. 2 Comparison of the numerical results obtained for fixed viscosity $\mu = 0.01$ and different microrotation viscosity μ_r

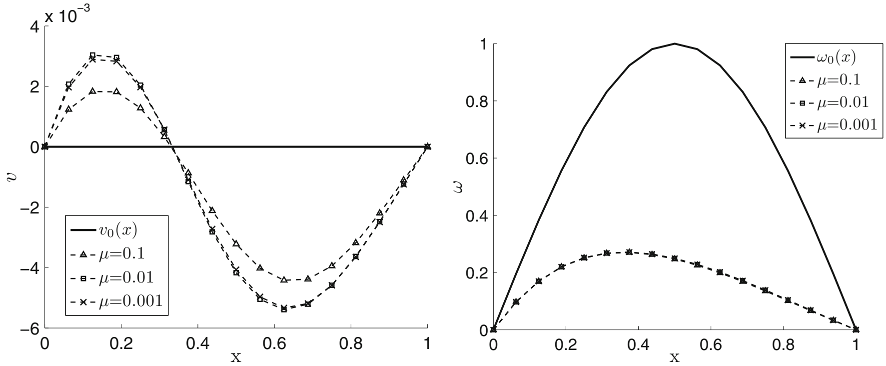

Fig. 3 Comparison of the numerical results obtained for different viscosity μ and fixed microrotation viscosity $\mu_r = 0.01$

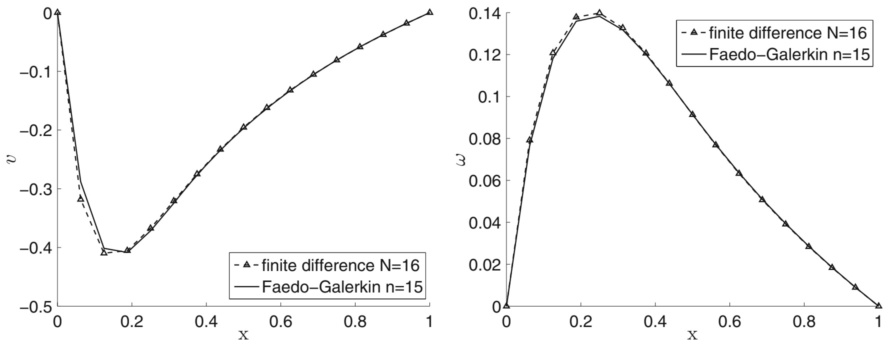

Fig. 4 Comparison of the numerical results obtained by finite difference and Faedo–Galerkin method

Test Example 3—Comparison with the Faedo–Galerkin Method In order to compare the numerical results obtained with the finite difference scheme and with the Faedo–Galerkin method (see [1]), we take the smooth initial functions: $\rho_0(x) = 1.0$, $v_0(x) = \sin(\pi x)$, $\omega_0(x) = \sin(2\pi x)$, and $\theta_0(x) = 2 + \cos(\pi x)$. The parameters are set to $\mu = \mu_r = K = c_0 = c_d = 0.01$, $R = c_v = 1$, $j_I = 1$, $a = 1$, and $L = 1$. As before, numerical parameters are set to $N = 16$, $t = 10^{-3}$ for the finite difference scheme. To obtain the numerical approximations with the Faedo–Galerkin method, we take $n = 15$ terms in the expression for approximating functions. From the numerical results shown in Fig. 4, where the comparison of both approaches is given, one can conclude that the results coincide quite well.

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An Optimal Control Problem in Mathematical and Computer Models of the Information Warfare

Nugzar Kereselidze

Abstract This article deals with the application of mathematical and computer methods in sociology, specifically information warfare. Pursuing such research can sometimes be met with new and interesting problems, worthy of serious attention of mathematicians. Given ChilKer tasks, it is possible to represent that case. ChilKer task refers to the boundary value problem for a system of ordinary differential equations and optimal control problem for which the right-side boundary conditions are given at different points of time for different coordinates of the unknown vector function. For ChilKer type the boundary value problem of system of ordinary differential equations proposed a method for finding solutions.

Keywords Information warfare • ChilKer task • Boundary value problem • Optimal control problem • Controllable

Mathematics Subject Classification (2010): 34B60, 49J15, 90C99

1 Introduction

The term "information warfare" has become strongly established in the scientific and other spheres of human activity. And despite the fact that the exact definition of this term is still does not exist, it is perceived in the intuitive sense. This situation, however, does not prevent the fact that in this area research is conducted; scientific magazines' are published, created by the introduction of technologies

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of information warfare, developing national doctrine on information security, to prevent and reflect information or cyber warfare. We have to explore information confrontation with the aid of mathematical and computer models since 2009. Sokhumi State University professor *Temur Chilachava* offered me to deal with confrontation issues in the information warfare. We are engaged in mathematical models of one of the directions of information warfare—information confrontation. Therefore, the report will use the term "information warfare" in the narrow sense [2, 3]. Under "information warfare" we mean usage of mass media (printed or electronic press, the Internet) by two countries or the union of two countries or two strong economic structures (consortium's) to conduct purposeful disinformation or propaganda. The union of international organizations (*UN, OSCE, EU, NATO, WTO*, and others) appears as the third side in this process, the effort of which is to neutralize the tension between the two rival countries, sides and to achieve the termination of information warfare.

The aims of information warfare can be:

- Infliction of losses to the image of the antagonist country—creating the image of an enemy.
- Discredit of the management of the antagonist country.
- Demoralization of the personnel of the armed forces and the civilians of the antagonist country.
- Creation of public opinion, inside and outside of the country, for justification and argumentation of possible military operations.
- Opposition to the geopolitical ambitions of the antagonist country etc.

Of course, not all of the information warfare transformed into fighting, "hot" war. However, it is safe to assert that every "hot" war was preceded by information warfare. Information warfare is something similar to the artillery preparation before approach. However, if the artillery preparation is completed as soon as the attack begins, the information warfare is not completed and continues along with a "hot" war.

Professor *T. Chilachava* proposed to consider models of information confrontation between two subjects which are "reconciled"—by peacekeeping third party. This approach was clearly a pioneer in the mathematical modeling of the information confrontation. This approach has been seen by scientists working in the field of mathematical modeling. In particular, the Indian scientists, Professors *Bimal Kumar Mishra* and *Aheksha Prajapati* in their work [8], used our proposed model and determined its singular points. On the other hand, the interesting work on mathematical modeling of *Samarski A.'s* and *Mikhailov A.'s* [10] advertising campaign and *Pugacheva E.'s* and *Solovenko K.'s* [9] conflict situation should be noted.

2 Theoretical and Experimental Methods

We will consider the mathematical model of IW, taking into account the IT levels of the parties [5, 6]:

$$\begin{aligned}\frac{dx(t)}{dt} &= \alpha_1 x(t)\left(1-\frac{x(t)}{I_1}\right)-\beta_1 z(t),\\ \frac{dy(t)}{dt} &= \alpha_2 y(t)\left(1-\frac{y(t)}{I_2}\right)-\beta_2 z(t),\\ \frac{dz(t)}{dt} &= \gamma\,(x(t)+y(t))\left(1-\frac{z(t)}{I_3}\right)\end{aligned} \tag{1}$$

$$x(0)=x_0\le I_1,\qquad y(0)=y_0\le I_0,\qquad z(0)=z_0\le I_3 \tag{2}$$

Where considered $x(t)$, $y(t)$, functions are amounts of information (as a rule—disinformation "black" PR) disseminated by the respective antagonistic parties to achieve information superiority in information warfare; $z(t)$ is the amount of information of the peacekeeping party in a time point t, containing appeals to the antagonistic parties to finish distributions of misinformation and to stop information warfare; in the model, α_1 ,α_2—are indexes of the aggressiveness of the relevant sides; β_1,β_2—are indexes of peacekeeping readiness of the parties; γ is an index of peacekeeping activity of the third party; and I_1, I_2, I_3 are kinds of "equilibrium" amounts of information of the respective parties, a certain level of their own IT development: financial or other opportunities to use foreign IT. We consider the process of information confrontation described by the mathematical model (1), (2)—at a certain large segment of time $[0, T]$.

Initially, we have found the solution of the *Cauchy problem* (1), (2), examined it, and determined it for different values of model parameters, developing this way the whole process of information confrontation. In general, the solution of the Cauchy problem (1), (2) depends on the parameters of the model:

$$\begin{aligned}&x(t,\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma,x_0,y_0,z_0,I_1,I_2,I_3);\\ &y(t,\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma,x_0,y_0,z_0,I_1,I_2,I_3);\\ &z(t,\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma,x_0,y_0,z_0,I_1,I_2,I_3).\end{aligned} \tag{3}$$

Analytical solution of the *Cauchy problem* was obtained in [2, 3]; when the right sides of (1) are linear relationships, there are no I_1, I_2, I_3, peculiar "equilibrium" volumes of information of the relevant parties, and—$\alpha=\alpha_1=\alpha_2$, $\beta=\beta_1=\beta_2$, and (4) is transformed to:

$$x(t,\alpha,\beta,\gamma,x_0,y_0,z_0);\quad y(t,\alpha,\beta,\gamma,x_0,y_0,z_0);\quad z(t,\alpha,\beta,\gamma,x_0,y_0,z_0) \tag{4}$$

The correlation between quantities $\alpha, \beta, x_0, y_0, z_0$ for which (4) vanishes, equal to zero, i.e., information warfare completed or not, has been established.

Further, we will be interested in a question of completion of information warfare. We will consider such points $t^*, t^{**} \in [0, T]$ on which information warfare and conditions:

$$x(t^*) = 0, \quad y(t^{**}) = 0 \tag{5}$$

are fair.

Thus, the task of completing the information warfare actually turns into a boundary value problem for a system of ordinary differential equations: (1), (2), (5). Note that, we do not describe boundary conditions for the third function in (5), as they are not relevant for fixing complete information warfare.

We cannot speak uniquely about the existence of solutions of the boundary value problem (1), (2), (5). At each set of fixed parameters $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma, x_0, y_0, z_0, I_1, I_2, I_3$, the solution of the boundary value problem either exists or is not present. Therefore, naturally there is a task using possibilities of variation of the parameters connected with the peacekeeping side to solve a boundary value problem. That is, naturally there is a problem of the control of information warfare, with the aid of peacekeeping activity. Parameters, which the peacekeeping side can control, are its peacekeeping activity—γ , level of development of information technologies—I_3, and its initial value—z_0 , with which it enters into the information warfare.

Let us touch controllability, which corresponds to the boundary value problem (1), (2), (5). Using computer simulations by controlling parameters, specifically increasing them, in works [5, 6], let us install an opportunity to translate the process of information confrontation with position (2) to the state (5), i.e., problem (1), (2), (5) is controllable.

But if controllability of systems is possible, why not to search for the optimal control? So, we naturally come to the optimal control problem [7]. We believe that $u_1(t)$ and $u_2(t)$ are control functions. In the system (1), we will assume transformation of the boundary value problem (1), (2), (5) into the optimal control problem. Thus, the purpose of the control is to transfer the system from position (2) to state (5) for the minimum expenditure for peace side. We assume that costs of the third party by one of the peacekeeping activities and the level of information technology are equal to one conventional unit. So, we have:

$$J = \int_0^{\max(t^*, t^{**})} (u_1(t)^2 + u_2(t)^2) dt \to \inf \tag{6}$$

$$\begin{aligned}
\frac{dx(t)}{dt} &= \alpha_1 x(t) \left(1 - \frac{x(t)}{I_1}\right) - \beta_1 z(t), \\
\frac{dy(t)}{dt} &= \alpha_2 y(t) \left(1 - \frac{y(t)}{I_2}\right) - \beta_2 z(t), \\
\frac{dz(t)}{dt} &= u_2(t) \left(x(t) + y(t)\right) \left(1 - \frac{z(t)}{u_1(t)}\right)
\end{aligned} \tag{7}$$

$$x(0) = x_0, \quad y(0) = y_0, \quad z(0) = z_0. \tag{8}$$

$$x(t^*) = 0, \quad y(t^{**}) = 0 \tag{9}$$

Note, that the points $t^*, t^{**} \in [0, T]$ usually are different and free. *The optimal control problem* (6)–(9) differs from the usual control tasks. Namely, a pair $(x(t), y(t))$ which characterizes the condition of information confrontation should not be reset to zero at the same time. There may be cases: $x(t^*) = 0$, $y(t^*) \neq 0$, and $x(t^{**}) \neq 0$, $y(t^{**}) = 0$. This state of things is consistent with the social processes. If the process is described by a certain vector function, the completion of the process can mean output zeros on the coordinates of these vector functions at different time. That is a characteristic feature of the optimal control problem (6)–(9) which, for brevity, we shall call the *ChilKer task*.

We believe that $x_0, y_0 > 0; z_0 \geq 0$ $\Delta = [0, T]$; $t^*, t^{**} \in \Delta$; or $t^* = t^{**}$, or $t^* > t^{**}$, or $t^* < t^{**}$; $x(t), y(t)z(t) \in PC^1(\Delta)$ function with a piecewise continuous derivative and absolutely continuous functions and $u_1(t), u_2(t) \in PC(\Delta)$ with piecewise continuous functions.

For all the meaning of time t, we have $x(t) \leq I_1, y(t) \leq I_2, z(t) \leq I_3$ and $0 \leq u_1(t) \leq I_1, 0 \leq u_2(t) \leq Q$, where Q is some positive real number. Five functions $x(t), y(t), z(t), u_1(t), u_2(t)$ are called admissible controlled process or admissible process, if (8) is performed at the points of continuity of the control parameters and boundary conditions (8), (9) are satisfied.

$||x(t), y(t), z(t)|| = \max(\max_{t \in [0,T]} |x(t)|, \max_{t \in [0,T]} |y(t)|, \max_{t \in [0,T]} |z(t)|)$. In accordance with this standard, a strong minimum is considered.

Note that we do not know in advance which of the relations $t^* = t^{**}$, $t^* > t^{**}$, and $t^* < t^{**}$ is performed and t^* and t^{**} are not fixed in the Δ. These circumstances do not allow us to proceed directly to the solution of the problem (6)–(9). Thus, the problem ChilKer should be presented by two subtasks. In fact, we must decompose the ChilKer problem by the domain of definition and get two tasks of the classical type—*Lagrange problem in the form of Pontryagin*. Case $t^* = t^{**}$ will not be considered, because it is already a well-known Lagrange problem in the Pontryagin form. For the information warfare problem, it was considered in [4]. In general, it is successfully applied to physical type problems, in particular to variational problems of mechanics. Therefore, we consider the case when $t^* < t^{**}$, the second case—$t^* > t^{**}$ is treated similarly.

The functional (6) divide the interval of integration $[0, \max(t^*, t^{**)}]$ into two subinterval of integration: $[0, \min(t^*, t^{**)}]$ and $[\min(t^*, t^{**}), \max(t^*, t^{**})]$:

Select the task A as a ChilKer subtask—a problem of optimal control:

$$A = \int_0^{\min(t^*, t^{**})} (u_1(t)^2 + u_2(t)^2) dt \to \inf \tag{10}$$

$$\frac{dx(t)}{dt} = \alpha_1 x(t)\left(1 - \frac{x(t)}{I_1}\right) - \beta_1 z(t),$$
$$\frac{dy(t)}{dt} = \alpha_2 y(t)\left(1 - \frac{y(t)}{I_2}\right) - \beta_2 z(t), \tag{11}$$
$$\frac{dz(t)}{dt} = u_2(t)\,(x(t) + y(t))\left(1 - \frac{z(t)}{u_1(t)}\right)$$

$$x(0) = x_0, \quad y(0) = y_0, \quad z(0) = z_0. \tag{12}$$

$$x(t^*) = 0; y(t^*) = y_A, \text{where } 0 < y_A \le I_2; z(t^*) = z_A, \text{where } 0 \le z_A \le I_3 \tag{13}$$

Task A is a *Lagrange task in Pontryagin form* and it is possible to apply the necessary and sufficient conditions of existence of the solution [1].

Let:

$$x_A(t), y_A(t), z_A(t), u_{1A}(t), u_{2A}(t), t_A^* \tag{14}$$

be the solutions of task A (10)–(13).

Task B will be in the following form (B is optimal control problem):

$$B = \int_{\min(t^*,t^{**})}^{\max(t^*,t^{**})} (u_1(t)^2 + u_2(t)^2)dt \to \inf \tag{15}$$

$$\frac{dx(t)}{dt} = \alpha_1 x(t)\left(1 - \frac{x(t)}{I_1}\right) - \beta_1 z(t),$$
$$\frac{dy(t)}{dt} = \alpha_2 y(t)\left(1 - \frac{y(t)}{I_2}\right) - \beta_2 z(t), \tag{16}$$
$$\frac{dz(t)}{dt} = u_2(t)\,(x(t) + y(t))\left(1 - \frac{z(t)}{u_1(t)}\right)$$

$$x(t^*) = x_A(t^*), \quad y(t^*) = y_A(t^*), \quad z(t^*) = z_A(t^*). \tag{17}$$

$$x(t) \le 0. \tag{18}$$

$$x(t^{**}) = x_B \le 0; \quad y(t^{**}) = 0; \quad z(t^{**}) = z_B, \text{where } 0 \le z_B \le I_3. \tag{19}$$

Tasks A and B are broadly similar, but there are differences. Here it is essential to add conditions (18) to the B problem and it makes sense—having reached the termination of information warfare of the first party, i.e., $x(t^*) = 0$, it should not renew it again. Therefore, on the segment $[t^*, t^{**}]$, the first side does not resume information warfare, because of the fact that there is inequality $x(t) \leq 0$. Denote solutions of problem B—(15)–(19) by:

$$x_B(t), y_B(t), z_B(t), u_{1B}(t), u_{2B}(t), t_B^{**}. \tag{20}$$

Now with the help of solutions (14) of problems A (10)–(13) and (20) of problems B (15)–(19), we will have:

$$\begin{aligned} S &= (x_A(t), y_A(t), z_A(t), u_{1A}(t), u_{2A}(t)) \quad \textit{if} \quad 0 \leq t \leq t_A^* = t^*; \\ S &= (x_B(t), y_B(t), z_B(t), u_{1B}(t), u_{2B}(t)) \quad \textit{if} \quad t_A^* \leq t \leq t_B^{**} = t^{**}. \end{aligned} \tag{21}$$

where $S = (x(t), y(t), z(t), u_1(t), u_2(t))$.

It is easy to notice that the solution (21) is admissible controlled process of the problem ChilKer (6)–(9). Thus, we have the following :

Theorem 1. *If (14) and (20) are, respectively, solutions of problems A and B, resulting from decomposition of ChilKer problem, then (21) is an admissible controlled process for ChilKer (6)–(9) problem.*

Thus, we can conclude that the functions $x(t), y(t), z(t)$ of (21) are solutions of ChilKer problem, (1), (2), (5) for the ODE, and transfer the system from the state of (2) to (5), when $\gamma = u_2(t)$ and $I_3 = u_1(t)$

We have three points:

$$P_0 - (0, x_0, y_0, z_0) \tag{22}$$

$$P_A - (t^*, x(t^*) = 0, y_A, z_A) \tag{23}$$

$$P_B - (t^{**}, x_B, y(t^{**}) = 0, z_B). \tag{24}$$

Note that, among all admissible control processes, ChilKer problem (6)–(9) passing through the points P_0, P_A, P_B solution (21)—is the optimal for the functional J (6). So, we have the following theorem:

Theorem 2. *For the solution of ChilKer problem (6)–(9) to which are added conditions of passing $x(t), y(t), z(t)$ functions through the points (22)–(24), it is necessary and sufficient existence of tasks A (14) and B (20) solutions, which are resulted by decomposition of ChilKer problem.*

3 Conclusions

For the boundary value problem of type ChilKer system of ordinary differential equations, (1), (2), (5) proposed a method for finding solutions. The system of ODE (1) defines the control parameters and the optimal control problem is generated ChilKer type (6)–(9). The ChilKer type optimal control problem is divided into two suboptimal control, i.e., decomposition of ChilKer problem occurs on the domain and the corresponding solutions are (14), (20) for the tasks A (10)–(13) and B (15)–(19). It is alleged that “stapling” solutions of A and B (21) is a valid process control for optimal control problems such as ChilKer (6)–(9) and one of the solutions to the boundary value problem of type ChilKer system of ordinary differential equations (1), (2), (5). The solution (21) is the best among all the solutions of the optimal control type ChilKer (6)–(9) passing through the points (22)–(24).

4 Gratitude

On this topic, the problems (1), (2), (5) and (6)–(9) useful exchange of views were held with *V. Kiguradze*, academic National Academy of Sciences of Georgia; *A. Fursikov* chairman of Department of General Control Problems of Mechanics and Mathematics Faculty of Lomonosov Moscow State University, Russian Federation; *T. Chilachava*, professor of Sokhumi State University, Tbilisi, Georgia; *S. Israylov*, professor from Grozny, Chechnya, Russian Federation; and *F. Dvalishvili*, professor of the University of Georgia, Tbilisi, Georgia. To all of them, I express my deep gratitude. Special thanks to *L. Karalashvili*, professor University of Georgia, Tbilisi, Georgia.

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Forward Attractors in Discrete Time Nonautonomous Dynamical Systems

Peter E. Kloeden, Thomas Lorenz, and Meihua Yang

Abstract Two recent developments provide insights on the appropriate definition of a forward attractor in nonautonomous dynamical systems. One is the construction of the component subsets of a forward attractor defined as a time-dependent family of compact subsets and conditions that ensure that the so constructed family forward attracts bounded subsets. Such a family is Lyapunov asymptotically stable, but often does not exist even in simple examples of dissipative nonautonomous systems. The other development is the recent discovery that the forward omega-limit set is asymptotically positively invariant. This makes this set, which Vishik proposed as the forward attractor and called the uniform attractor, a more useful concept of forward attractor since it now provides more information about the dynamics in current time as it approaches the omega-limit set. These developments are discussed in this chapter in the context of discrete time nonautonomous dynamical systems that are formulated as processes.

Keywords Nonautonomous difference equations • Process • Two-parameter semigroup • Pullback attractor • Forward attractor • Omega-limit points

Mathematics Subject Classification (2000): Primary 34B45, 37B55; Secondary 37C70, 37L30

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1 Introduction

The nature of time in a nonautonomous dynamical system is very different from that in autonomous systems, which depends on the time that has elapsed since starting rather on the actual time. Consequently, limiting objects may not exist in actual time as in autonomous systems. This is most apparent in the shortcomings in the definition of a forward attractor in a nonautonomous systems [2, 10]. The usual definition is an intuitive and natural counterpart of that of a pullback attractor: an invariant family of compact subsets that attracts bounded sets in the sense of forward rather than pullback convergence [1–3, 8]. Moreover, this definition requires the subsets to exist for all time, also in the distant past, even though forward attraction is really only about what happens in the distant future.

Vishik [3, 15] proposed using the forward omega-limit set as the forward attractor, which he called the *uniform attractor*. This set indicates just where the forward limit points are to be found. It is generally not invariant and provides little information about the dynamics in actual time on the approach to the limit set. In addition, the uniform qualifier refers to the skew product-like systems considered by Vishik rather to more general nonautonomous dynamical systems.

Some recent developments, which clarify the situation, are discussed here. Firstly, it was shown in [7] that the subsets of a forward attractor can be constructed by a pullback argument within a positively invariant family of subsets just like those of a pullback attractor. Nothing is assumed here about what is happening outside the family of subsets. Additional conditions must be satisfied to ensure that forward attraction holds. These are, however, not always satisfied in even simple examples. Consequently, forward attractors in this sense may often not exist. The second development is the proof that the forward omega-limit set is, in fact, asymptotically positively invariant [9]. This is weaker than most other invariance concepts, but nevertheless provides more information about the approach of the dynamics to the omega-limit set. As such it makes Vishik's definition of a uniform attractor much more useful.

The forward attractor in the sense of an invariant family nonempty compact subsets that forward attracts all families of nonempty bounded subsets is a Lyapunov asymptotically stable family of sets. It is a special case that provides much more information about the dynamics of the nonautonomous system in current as well in the limiting future, but does not exist in many systems.

2 Nonautonomous Attractors

A nonautonomous difference equation on $\mathbb{R}^d$ has the form:

$$x_{n+1} = f_n(x_n) \tag{1}$$

with mappings $f_n : \mathbb{R}^d \to \mathbb{R}^d$, which are assumed to be continuous here and may vary with time $n \in \mathbb{Z}$.

Define $\mathbb{Z}^2_{\geq} := \{(n, n_0) \in \mathbb{Z}^2 : n \geq n_0\}$. The nonautonomous difference equation (1) generates a *solution mapping* $\phi : \mathbb{Z}^2_{\geq} \times \mathbb{R}^d \to \mathbb{R}^d$ through iterated composition, i.e., $\phi(n, n_0, x_0) := f_{n-1} \circ \cdots \circ f_{n_0}(x_0)$ for all $n > n_0$ with $n_0 \in \mathbb{Z}$ and each $x_0 \in X$ with the initial value $\phi(n_0, n_0, x_0) := x_0$.

The solution mappings of nonautonomous difference equations like (1) generate an abstract discrete time nonautonomous dynamical system formulated as a 2-parameter semigroup or process [4, 5] on the state space $\mathbb{R}^d$ and time set $\mathbb{Z}$.

Definition 1. A *(discrete time) process* on the state space $\mathbb{R}^d$ is a mapping $\phi : \mathbb{Z}^2_{\geq} \times \mathbb{R}^d \to \mathbb{R}^d$ which satisfies the following initial value, 2-parameter evolution and continuity properties:

i) $\phi(n_0, n_0, x_0) = x_0$ for all $n_0 \in \mathbb{Z}$ and $x_0 \in \mathbb{R}^d$,
ii) $\phi(n_2, n_0, x_0) = \phi(n_2, n_1, \phi(n_1, n_0, x_0))$ for all $n_0 \leq n_1 \leq n_2$ in $\mathbb{Z}$ and $x_0 \in \mathbb{R}^d$,
iii) the mapping $x_0 \mapsto \phi(n, n_0, x_0)$ of $\mathbb{R}^d$ into itself is continuous for all $(n, n_0) \in \mathbb{Z}^2_{\geq}$.

The general nonautonomous case differs crucially from the autonomous one in that the starting time n_0 is just as important as the time that has elapsed since starting, i.e., $n - n_0$. This has some profound consequences in terms of definitions and the interpretation of dynamical behavior, so many of the concepts that have been developed and extensively investigated for autonomous dynamical systems, in general, and autonomous difference equations, in particular, are either too restrictive or no longer valid or meaningful in the nonautonomous context. The following definitions and results are taken from [2, 8, 11, 13, 14].

Definition 2. A family $\mathscr{A} = \{A_n : n \in \mathbb{Z}\}$ of nonempty subsets of $\mathbb{R}^d$ is ϕ*-invariant* if:

$$\phi(n, n_0, A_{n_0}) = A_n, \quad \text{for all } (n, n_0) \in \mathbb{Z}^2_{\geq},$$

or, equivalently, if $f_n(A_n) = A_{n+1}$ for all $n \in \mathbb{Z}$. It is said to be uniformly bounded if there exists a bounded subset B of $\mathbb{R}^d$ such that $A_n \subset B$ for all $n \in \mathbb{Z}$.

Forward and pullback convergences can be used to define *two distinct types* of nonautonomous attractors for a process ϕ.

Definition 3. A ϕ-invariant family $\mathscr{A} = \{A_n : n \in \mathbb{Z}\}$ of nonempty compact subsets of $\mathbb{R}^d$ is called a *forward attractor* if it forward attracts families $\mathscr{D} = \{D_n : n \in \mathbb{Z}\}$ of bounded subsets of $\mathbb{R}^d$, i.e.:

$$\mathrm{dist}_{\mathbb{R}^d}(\phi(n, n_0, D_{n_0}),\ A_n) \to 0 \quad \text{as } n \to \infty \quad (n_0 \text{ fixed}) \tag{2}$$

and a *pullback attractor* if it pullback attracts families $\mathscr{D} = \{D_n : n \in \mathbb{Z}\}$ of bounded subsets of $\mathbb{R}^d$, i.e.:

$$\mathrm{dist}_{\mathbb{R}^d}(\phi(n, n_0, D_{n_0}),\ A_n) \to 0 \quad \text{as } n_0 \to -\infty \quad (n \text{ fixed}). \tag{3}$$

Here:

$$\mathrm{dist}_{\mathbb{R}^d}(x,B) := \inf_{b\in B} \|x-b\|, \qquad \mathrm{dist}_{\mathbb{R}^d}(A,B) := \sup_{a\in A} \mathrm{dist}_{\mathbb{R}^d}(a,B)$$

for nonempty subsets A, B of $\mathbb{R}^d$.

Remark 1. A forward attractor is, in fact, a Lyapunov asymptotically stable ϕ-invariant family of sets. The required attractivity is given by the forward convergence (2), while the Lyapunov stability part follows from forward attraction combined with the continuity in initial conditions uniformly on a bounded time interval. Lyapunov stability can be interpreted as continuity in initial conditions uniformly over all future time. Analogously, pullback convergence (3) implies what can be considered as continuity in initial conditions uniformly over all past initial times.

Note that each uniformly bounded ϕ-invariant family is characterized by the bounded entire solutions.

Proposition 1. *A uniformly bounded family $\mathscr{A} = \{A_n : n \in \mathbb{Z}\}$ is ϕ-invariant if and only if, for every pair $n_0 \in \mathbb{Z}$ and $x_0 \in A_{n_0}$, there exists a bounded entire solution χ such that $\chi_{n_0} = x_0$ and $\chi_n \in A_n$ for all $n \in \mathbb{Z}$.*

The existence of a pullback attractor follows from that of a pullback absorbing family in the following generalization of the theorem for autonomous global attractors. The proof is simpler if the pullback absorbing family is assumed to be ϕ-positive invariant.

Definition 4. A family $\mathscr{B} = \{B_n : n \in \mathbb{Z}\}$ of nonempty compact subsets of $\mathbb{R}^d$ is called *pullback absorbing* if, for every family $\mathscr{D} = \{D_n : n \in \mathbb{Z}\}$ of bounded subsets of X and $n \in \mathbb{Z}$, there exists an $N(n,\mathscr{D}) \in \mathbb{N}$ such that:

$$\phi\left(n, n_0, D_{n_0}\right) \subseteq B_n \quad \text{for all } n_0 \leq n - N(n,\mathscr{D}).$$

It is said to be *ϕ-positively invariant* if $\phi\left(n, n_0, B_{n_0}\right) \subseteq B_n$ for all $(n,n_0) \in \mathbb{Z}^2_{\geq}$.

The assumption about a ϕ-positively invariant pullback absorbing family is not a serious restriction, since one can always be constructed given a general pullback absorbing family [11].

Theorem 1. *Suppose that a process ϕ has a ϕ-positively invariant pullback absorbing family $\mathscr{B} = \{B_n : n \in \mathbb{Z}\}$.*

Then there exists a global pullback attractor $\mathscr{A} = \{A_n : n \in \mathbb{Z}\}$ with component sets determined by:

$$A_n = \bigcap_{j\geq 0} \phi\left(n, n-j, B_{n-j}\right) \quad \text{for all } n \in \mathbb{Z}. \tag{4}$$

Moreover, if $\mathscr{A}$ is uniformly bounded, then it is unique.

The situation is somewhat more complicated for forward attractors than for pullback attractors due to some peculiarities of forward attractors [14], e.g., they need not be unique. For each $r \geq 0$ the process generated by:

$$x_{n+1} = f_n(x_n) := \begin{cases} x_n, & n \leq 0, \\ \frac{1}{2}\, x_n, & n > 0 \end{cases} \tag{5}$$

has a forward attractor $\mathscr{A}^{(r)}$ with component subsets:

$$A_n^{(r)} = \begin{cases} r\,[-1, 1], & n \leq 0, \\ \\ \frac{1}{2^n}\, r\,[-1, 1], & n > 0. \end{cases} \tag{6}$$

These forward attractors are not pullback attractors.

It is often asserted in the literature that there is *no counterpart* of Theorem 1 for nonautonomous forward attractors. In fact, such construction (4) in continuous time was shown by Kloeden & Lorenz [7] to hold within any positively invariant family but provides only a candidate for a forward attractor; other conditions must also hold.

3 Construction of Forward Attractors

The following important property of forward attractors holds for essentially the same reasons as in continuous time [7].

Proposition 2. *A uniformly bounded forward attractor $\mathscr{A} = \{A_n : n \in \mathbb{Z}\}$ in $\mathbb{R}^d$ has a ϕ-positively invariant family $\mathscr{B} = \{B_n : n \in \mathbb{Z}\}$ of nonempty compact subsets with $A_n \subset B_n$ for $n \in \mathbb{Z}$, which is forward absorbing.*

A key observation for the construction of a forward attractor is provided by the next theorem [7].

Theorem 2. *Suppose that a process ϕ on $\mathbb{R}^d$ has a ϕ-positively invariant family $\mathscr{B} = \{B_n : n \in \mathbb{Z}\}$ of nonempty compact subsets of X.*

Then ϕ has a maximal ϕ-invariant family $\mathscr{A} = \{A_n : n \in \mathbb{Z}\}$ in $\mathscr{B}$ of nonempty compact subsets determined by:

$$A_n = \bigcap_{n_0 \leq n} \phi\,(n, n_0, B_{n_0}) \quad \textit{for each} \;\; n \in \mathbb{Z}. \tag{7}$$

In view of Proposition 2, the component sets of *any* uniformly bounded forward attractor can be constructed in this way. Note that nothing is assumed here about the dynamics *outside* of the family $\mathscr{B}$.

3.1 A Condition Ensuring Forward Convergence

The ϕ-invariant family $\mathscr{A} = \{A_n, n \in \mathbb{Z}\}$ constructed in Theorem 1 need *not* be a forward attractor, even when the ϕ-positively invariant family $\mathscr{B}$ is a forward absorbing family, e.g., consider any example of a pullback attractor that is not a forward attractor.

Another important observation, if somewhat obvious, is that there should be no ω-limit points from inside the family $\mathscr{B}$ that are not ω-limit points from inside the family $\mathscr{A}$. For each $n_0 \in \mathbb{Z}$, the forward ω-limit set with respect to $\mathscr{B}$ is defined by:

$$\omega_{\mathscr{B}}(n_0) := \bigcap_{m \geq n_0} \overline{\bigcup_{n \geq m} \phi(n, n_0, B_{n_0})}.$$

The set $\omega_{\mathscr{B}}(n_0)$ is nonempty and compact as the intersection of nonempty nested compact subsets and:

$$\lim_{n \to \infty} \text{dist}_X \left(\phi(n, n_0, B_{n_0}),\ \omega_{\mathscr{B}}(n_0)\right) = 0 \quad (\text{fixed } n_0).$$

Since $A_{n_0} \subset B_{n_0}$ and $A_n = \phi(n, n_0, A_{n_0}) \subset \phi(n, n_0, B_{n_0})$,

$$\lim_{n \to \infty} \text{dist}_X \left(A_n,\ \omega_{\mathscr{B}}(n_0)\right) = 0 \quad (\text{fixed } n_0). \tag{8}$$

Moreover, $\omega_{\mathscr{B}}(n_0) \subset \omega_{\mathscr{B}}(n_0') \subset B$ for $n_0 \leq n_0'$, where the final inclusion is from the uniform boundedness of $\mathscr{B}$, since $\phi(n, n_0, B_{n_0}) \subset B_n$ for each $n \geq n_0$. Hence the set:

$$\omega_{\mathscr{B}}^{\infty} := \overline{\bigcup_{n_0 \in \mathbb{Z}} \omega_{\mathscr{B}}(n_0)}$$

is nonempty and compact. It is often called the *uniform attractor* of the non-autonomous system, even though it is not invariant [3, 15].

From (8) it is clear that:

$$\lim_{n \to \infty} \text{dist}_X \left(A_n,\ \omega_{\mathscr{B}}^{\infty}\right) = 0. \tag{9}$$

The ω-limit points for dynamics starting inside the family of sets $\mathscr{A}$ are defined by:

$$\omega_{\mathscr{A}}^{\infty} := \bigcap_{n_0 \in \mathbb{Z}} \overline{\bigcup_{n \geq n_0} A_n} = \bigcap_{n_0 \in \mathbb{Z}} \overline{\bigcup_{n \geq n_0} \phi(n, n_0, A_{n_0})} \subset B,$$

which is nonempty and compact as a family of nested compact sets. Obviously, $\omega_{\mathscr{A}}^{\infty} \subset \omega_{\mathscr{B}}^{\infty} \subset B$. The example below shows that the inclusions may be strict.

The following result was proved in [7].

Theorem 3. *$\mathscr{A}$ is forward attracting from within $\mathscr{B}$ if and only if $\omega^{\infty}_{\mathscr{A}} = \omega^{\infty}_{\mathscr{B}}$.*

$\mathscr{A}$ will then be a forward attractor, if $\mathscr{B}$ is forward absorbing.

3.2 Example

Consider the piecewise autonomous equation:

$$x_{n+1} = \frac{\lambda_n\, x_n}{1 + |x_n|}, \qquad \lambda_n := \begin{cases} \lambda, & n \geq 0, \\ \lambda^{-1}, & n < 0 \end{cases} \tag{10}$$

for some $\lambda > 1$, which corresponds to a switch between the two autonomous problems at $n = 0$. Its pullback attractor $\mathscr{A}$ of the resulting nonautonomous system has component sets $A_n \equiv \{0\}$ for all $n \in \mathbb{Z}$ corresponding to the zero entire solution. Note that the trivial fixed point $\bar{x} = 0$ seems to be "asymptotically stable" for $n < 0$ and then "unstable" for $n \geq 0$. Moreover, the interval $[x_-, x_+]$ with $x_\pm = \pm(\lambda - 1)$ is like a global attractor for the whole equation on $\mathbb{Z}$, but it is not really one since it is not invariant or minimal for $n < 0$ (Fig. 1).

A similar situation occurs in the nonautonomous difference equation:

$$x_{n+1} = f_n(x_n) := \frac{\lambda_n\, x_n}{1 + |x_n|}, \tag{11}$$

where $\{\lambda_n\}_{n\in\mathbb{Z}}$ be a monotonically increasing sequence with $\lim_{n\to\pm\infty} \lambda_n = \lambda^{\pm 1}$ for $\lambda > 1$, which is asymptotically autonomous in both directions with the limiting systems as (10).

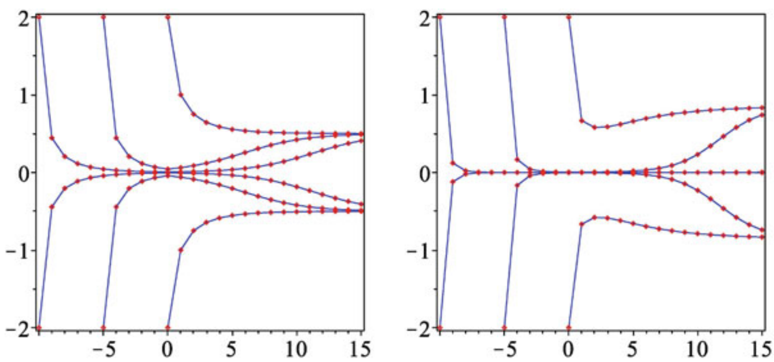

Fig. 1 Trajectories of the piecewise autonomous equation (10) with $\lambda = 1.5$ (*left*) and the asymptotically autonomous equation (11) with $\lambda_n = 1 + \frac{0.9\,n}{1+|n|}$ (*right*)

The nonautonomous difference equations (10) and (11) are asymptotically autonomous in both directions, but the pullback attractor does not reflect the full limiting dynamics, in particular in the forward time direction. It clearly does not satisfy the assumptions of Theorem 3.

4 What Really Is a Forward Attractor?

The definition of a forward attractor in Definition 3 as ϕ-invariant family $\mathscr{A} = \{A_n, n \in \mathbb{Z}\}$ of nonempty compact subsets of $\mathbb{R}^d$ that forward attracts families of bounded sets was introduced as an obvious counterpart to a pullback attractor.

Forward attraction (2) is, however, very different conceptually from pullback attraction (3) in that it is about what happens in the distant future and not in actual time, i.e., current time. Pullback convergence, seemingly contradictorily, is in this sense the natural generalization of convergence in autonomous systems, which depend only on the elapsed time since starting, so limit sets exist, in fact, in actual time. Moreover, as seen in system (5) above, forward attractors need not be unique.

Another curious feature of forward attractors in the sense of Definition 3 is that they require the entire past history of the system to be known. Indeed, its construction in Theorem 2 is based on pullback convergence, although forward convergence is about the distant future and should be independent of the past. In fact, forward convergence should not even require the system to be defined in the past.

The future limiting dynamics in (10) and (11) is contained in the set $[x_-, x_+]$ with $x_\pm = \pm(\lambda - 1)$, which corresponds to the omega-limit set $\omega_{\mathscr{B}}$ in the previous section. It is an example of what Vishik calls a uniform attractor.

Vishik [15] defined the *uniform attractor* to be a compact set which attracts the forward dynamics of the system and minimal in the sense that it is contained in all sets with this property. Nothing is said about invariance in this definition.

When the positively invariant family $\mathscr{B}$ in the previous section is forward absorbing, then $\omega_{\mathscr{B}}$ is a uniform attractor. The set $[x_-, x_+]$ for the piecewise autonomous system (10) is invariant only for positive time, but for both systems (10) and (11), it is positively invariant for all time. These are simple examples; in general, the future dynamics can be much more complicated, in fact, even arbitrary.

Nevertheless, when $\mathscr{B}$ consists of a single compact set B, the uniform attractor $\omega_{\mathscr{B}}$ is asymptotically positively invariant, see [6, 12]. The following result is taken from Kloeden & Yang [9].

Proposition 3. *Suppose that $\mathscr{B}$ consists of a single compact set B that is forward absorbing and positively invariant. Then, under the above assumptions, $\omega_{\mathscr{B}}$ is asymptotically positively invariant, i.e., for any monotonic decreasing sequence*

$\varepsilon_p \to 0$ as $p \to \infty$, there exists a monotonic increasing sequence $N_p \to \infty$ as $p \to \infty$ such that for each $n_0 \geq N_p$:

$$\phi(n, n_0, \omega_{\mathscr{B}}) \subset B_{\varepsilon_p}(\omega_{\mathscr{B}}) \qquad \textit{for all } n \geq n_0, \tag{12}$$

where $B_{\varepsilon_p}(\omega_{\mathscr{B}}) := \{x \in \mathbb{R}^d : \mathrm{dist}_{\mathbb{R}^d}(x, \omega_{\mathscr{B}}) < \varepsilon_p\}$.

Note that Proposition 3 does not require the process ϕ and the family $\mathscr{B}$ to be defined in the past. In particular, it applies when the system is defined, say, only for times $n \in \mathbb{Z}^+$ or even for $n \geq n_0$ for some $n_0 \in \mathbb{Z}^+$. Since the dynamics is assumed to be forward dissipative, it may be easier then to find a single absorbing set.

5 Definition of a Forward Attractor

The qualifier "uniform" in Vishik's definition of a uniform attractor comes from uniformity assumptions on the skew product flows where the term was introduced. The dynamics need not be, in general, uniform. What remains is the minimal compact attracting set. Simple examples show that this set need not be invariant, although it may be positively invariant in some cases. In general, from Proposition 3, it is asymptotically positively invariant. This suggests the following definition.

Definition 5. The "*forward attractor*" of a nonautonomous dynamical system is an asymptotically positively invariant compact attracting set that is contained in all other sets with these properties.

This is an improvement on Vishik's original definition, because it provides more information about what happens in actual time on the approach to the limiting object in the infinite future.

A forward attractor in the sense of Definition 3, i.e., a ϕ-invariant family $\mathscr{A} = \{A_n, n \in \mathbb{Z}\}$ of nonempty compact subsets of $\mathbb{R}^d$ that forward attracts all families $\mathscr{D}$ of nonempty bounded subsets of $\mathbb{R}^d$ (in the sense of (2)) could then be called a *Lyapunov asymptotically stable ϕ-invariant family* of sets, see Remark 1.

The corresponding omega-limit set $\omega_{\mathscr{B}}$ is then the forward attractor in the sense of Definition 5. For the difference equation (5), which has uncountably many forward attractors in the sense of Definition 3, it is simply the set $\{0\}$.

The existence of a Lyapunov asymptotically stable family of sets is a very strong property and not typical in many nonautonomous systems. The forward attractor in the sense of Definition 5 better captures the forward limiting behavior of such systems.

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Asymptotic Stability Regions for Certain Two Parametric Full-Term Linear Difference Equation

Petr Tomášek

Abstract We introduce an efficient form of necessary and sufficient conditions for asymptotic stability of the k-th order linear difference equation $y(n+k)$ $+a\sum_{j=1}^{k-1}(-1)^j y(n+k-j)+by(n)=0$, where $a,b\in\mathbb{R}$. The asymptotic stability region in (a,b) plane for this equation will be constructed and discussed with respect to some related linear difference equations.

Keywords Difference equation • Stability • The Schur-Cohn criterion

Mathematics Subject Classification: 39A06, 39A10, 39A30.

1 Introduction

The aim of the paper is to introduce and discuss asymptotic stability conditions for linear difference equation with constant parameters

$$y(n+k)+a\sum_{j=1}^{k-1}(-1)^j y(n+k-j)+by(n)=0, \qquad n=0,1,2,\dots \tag{1}$$

where $a,b\in\mathbb{R}$, $k\in\mathbb{N}$. We are going to utilize a certain technique based on the Schur–Cohn criterion, which for particular cases of linear higher-order difference equations enables us to obtain an efficient form of necessary and sufficient conditions for asymptotic stability of such equations. First the technique was applied to a few-term linear difference equations, where the already known

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results were confirmed and reformulated to a more convenient form and then a new significant result was obtained in [2], where asymptotic stability of difference equation

$$y_{n+k} + \alpha y_{n+k-1} + \beta y_{n+1} + \gamma y_n = 0, \qquad n = 0, 1, 2, \ldots,$$

with real parameters α, β and γ was studied. Moreover, previous results were mainly formulated as a parametric expression of stability region's boundary, which naturally followed from proof procedures based on the boundary locus technique. Moreover there was necessary to solve an auxiliary nonlinear equation (considering a fixed order of linear difference equation). On the other hand, conditions obtained by the new approach are of a more compact and explicit form. In this connection we recall some works dealing with asymptotic stability of scalar higher-order few-term linear difference equations:

Analysed difference equation ($n = 0, 1, 2, \ldots$)	Paper
$y_{n+k} - y_{n+k-1} + \gamma y_n = 0$	Levin, May [11]
$y_{n+k} + \alpha y_{n+k-1} + \gamma y_n = 0$	Kuruklis [10]
$y_{n+k} + \alpha y_{n+m} + \gamma y_n = 0$	Dannan [5], Cheng, Huang [4],
$k > m > 0$	Kipnis, Nigmatullin [9]
$y_{n+k} + \alpha y_{n+k-1} + \beta y_{n+1} + \gamma y_n = 0$	Čermák, Jánský, Kundrát [2]
$y_{n+k} + \alpha y_{n+k-2} + \gamma y_n = 0$	Ren [12], Čermák, Tomášek [1]
$y_{n+k} + \alpha y_{n+k-2} + \beta y_{n+2} + \gamma y_n = 0$	Čermák, Jánský, Tomášek [3]

Analysis of necessary and sufficient conditions for equations with more parameters or more terms turns out to be a very difficult problem. On the other hand, a special full-term higher-order equation with two parameters gives a very simple form of such conditions. Asymptotic stability analysis of equation

$$y(n+k) + a \sum_{j=1}^{k-1} y(n+k-j) + by(n) = 0, \qquad n = 0, 1, 2, \ldots \tag{2}$$

was done in [7] and alternative proof procedure was realized in [13]. The asymptotic stability conditions for (2) can be captured by the next assertion.

Proposition 1. *Consider Eq.* (2) *with* $a, b \in \mathbb{R}$, $k \geq 2$. *Then* (2) *is asymptotically stable if and only if*

$$a - 1 < b < 1, \qquad -1 + (1-k)a < b. \tag{3}$$

The above mentioned papers inspired the author to analyse asymptotic stability of another special two-parametric full-term linear equation (1): In Sect. 2 we introduce, after some preliminaries, the necessary and sufficient conditions for (1) to be asymptotically stable. In Sect. 3 we discuss the result with respect to its geometrical interpretation and in a connection with asymptotic stability conditions for linear difference equation (2).

2 Asymptotic Stability Conditions

First we introduce a sufficient condition for (1) to be asymptotically stable, which is just an appropriate reformulation of the well known Cohn stability condition for linear difference equation (1).

Proposition 2. *Consider Eq.* (1) *with* $a, b \in \mathbb{R}$, $k \in \mathbb{N}$. *If the condition*

$$|b| < 1 + (1-k)|a|$$

is satisfied then (1) *is asymptotically stable.*

At the end of the paper is the Cohn condition compared with the necessary and sufficient one introduced below.

In the sequel we recall the notion of inner matrices, which determinants play a key role in our proof procedure. Let $M = \Delta_\ell$ be an $\ell \times \ell$ matrix. We construct $(\ell-2)\times(\ell-2)$ matrix $\Delta_{\ell-2}$ from Δ_ℓ by deleting its first and its last column and row. Repeating this procedure we obtain a set of matrices $\{\Delta_1, \Delta_3, \dots, \Delta_\ell\}$ in the case ℓ odd and a set of matrices $\{\Delta_2, \Delta_4, \dots, \Delta_\ell\}$ in the case ℓ even. The appropriate set of matrices (with respect to the parity of ℓ) is called the *inners* of the matrix M. Illustrating this we introduce

$$M_5 = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} & m_{15} \\ m_{21} & m_{22} & m_{23} & m_{24} & m_{25} \\ m_{31} & m_{32} & m_{33} & m_{34} & m_{35} \\ m_{41} & m_{42} & m_{43} & m_{44} & m_{45} \\ m_{51} & m_{52} & m_{53} & m_{54} & m_{55} \end{bmatrix}, \quad M_6 = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} & m_{15} & m_{16} \\ m_{21} & m_{22} & m_{23} & m_{24} & m_{25} & m_{26} \\ m_{31} & m_{32} & m_{33} & m_{34} & m_{35} & m_{36} \\ m_{41} & m_{42} & m_{43} & m_{44} & m_{45} & m_{46} \\ m_{51} & m_{52} & m_{53} & m_{54} & m_{55} & m_{56} \\ m_{61} & m_{62} & m_{63} & m_{64} & m_{65} & m_{66} \end{bmatrix}.$$

The concentric rectangles determine the inner matrices Δ_1, Δ_3 and $\Delta_5 = M_5$ within matrix M_5 and the inner matrices Δ_2, Δ_4 and $\Delta_6 = M_6$ within matrix M_6. Much more about inners and their interesting properties and usage can be found in [8]. One useful property of square matrices is the following one: a square matrix M is said to be *positive innerwise* if all its inners have a positive determinant.

Theorem 1. *Consider Eq.* (1) *with* $a, b \in \mathbb{R}$, $k \in \mathbb{N}$. *Then* (1) *is asymptotically stable if and only if*

$$-1 < b < 1 - a, \qquad b < 1 + (k-1)a \qquad \text{for } k \text{ odd;} \tag{4}$$

$$a - 1 < b < 1, \qquad -1 + (1-k)a < b \qquad \text{for } k \text{ even.} \tag{5}$$

Proof. It is well known that (1) is asymptotically stable if and only if polynomial

$$P(\lambda) = \lambda^k + a \sum_{j=1}^{k-1} (-1)^j \lambda^{k-j} + b \tag{6}$$

has all its roots inside the unit disk in the complex plane. This property is guaranteed (in the if and only if sense) by the Schur–Cohn criterion (see [6]), which reformulation for polynomial $P(\lambda)$ given by (6) can be captured by

Proposition 3. *Polynomial* (6) *has all its roots inside the unit disk if and only if the sequel three conditions simultaneously hold:*

1.

$$P(1) > 0, \quad \text{i.e.} \quad \begin{array}{l} 1 + b > 0 \quad \text{for} \quad k \quad \text{odd} \\ 1 - a + b > 0 \quad \text{for} \quad k \quad \text{even} \end{array}$$

2.

$$(-1)^k P(-1) > 0, \quad \text{i.e.} \quad \begin{array}{l} 1 + (k-1)a - b > 0 \quad \text{for} \quad k \quad \text{odd} \\ 1 + (k-1)a + b > 0 \quad \text{for} \quad k \quad \text{even} \end{array}$$

3. $(k-1) \times (k-1)$ *matrices given by polynomial coefficients*

$$B^{\pm}_{k-1} = \begin{bmatrix} 1 & 0 & \cdots & & 0 \\ -a & \ddots & \ddots & & \vdots \\ a & \ddots & \ddots & \ddots & \\ -a & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \\ \frac{a}{(-1)^k} & \cdots & -a & a & -a & 1 \end{bmatrix} \pm \begin{bmatrix} 0 & \cdots & & 0 & b \\ \vdots & & ⋰ & ⋰ & \frac{a}{(-1)^{k-1}} \\ & ⋰ & ⋰ & ⋰ & \vdots \\ & ⋰ & ⋰ & ⋰ & a \\ 0 & ⋰ & ⋰ & ⋰ & -a \\ b & \frac{a}{(-1)^{k-1}} & \cdots & a & -a & a \end{bmatrix}$$

are positive innerwise.

The first two conditions are easy to verify, hence our aim is to simplify the third one. We are supposed to verify that determinants of all inner central matrices of B^{+}_{k-1} and of B^{-}_{k-1} are positive. First we introduce matrices D, I, J for efficient formulation of the proof steps. Let I_s be $s \times s$ identity matrix and let D_s and J_s be $s \times s$ matrices given by

$$D_s = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ -a & \ddots & & & & \vdots \\ a & \ddots & \ddots & & & \vdots \\ -a & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \frac{a}{(-1)^{s+1}} & \cdots & -a & a & -a & 0 \end{bmatrix}, \quad J_s = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ \vdots & {\cdot}^{\cdot^{\cdot}} & {\cdot}^{\cdot^{\cdot}} & 0 \\ 0 & {\cdot}^{\cdot^{\cdot}} & {\cdot}^{\cdot^{\cdot}} & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix}.$$

Matrix J_s is usually referred as the skew identity matrix. Using this notion we can express

$$B_{k-1}^{\pm} = \begin{cases} D_{k-1} + I_{k-1} \mp D_{k-1}J_{k-1} \pm bJ_{k-1} & \text{for } k \text{ odd,} \\ D_{k-1} + I_{k-1} \pm D_{k-1}J_{k-1} \pm bJ_{k-1} & \text{for } k \text{ even} \end{cases}$$

and all their inner matrices as

$$\Delta_s^{\pm} = \begin{cases} D_s + I_s \mp D_sJ_s \pm bJ_s & s = 2, 4, 6, \ldots, k-1 \text{ for } k \text{ odd,} \\ D_s + I_s \pm D_sJ_s \pm bJ_s & s = 1, 3, 5, \ldots, k-1 \text{ for } k \text{ even.} \end{cases}$$

In the case of k odd we realize the next operations within matrix B_{k-1}^{+} (which preserve determinants of all its inners) to get matrix $\tilde{B}_{k-1}^{+}$:

1. addition of the $(k-i)$-th column to the i-th column for $i = 1, 2, \ldots, (k-1)/2$
2. subtraction of the i-th row from the i-th one, $i = 1, 2, \ldots, (k-1)/2$

and similarly within matrix B_{k-1}^{-} to get matrix $\tilde{B}_{k-1}^{-}$:

1. subtraction of the $(k-i)$-th column from the i-th column, $i = 1, 2, \ldots, (k-1)/2$
2. addition of the i-th row to the $(k-i)$-th one for $i = 1, 2, \ldots, (k-1)/2$

Now any inner matrix of above obtained $\tilde{B}_{k-1}^{+}$ and $\tilde{B}_{k-1}^{-}$ can be expressed as

$$\tilde{\Delta}_s^{+} = \left[\begin{array}{c|c} (1+b)I_{s/2} & -D_{s/2}J_{s/2} + bJ_{s/2} \\ \hline O_{s/2} & (1+a-b)I_{s/2} + 2D_{s/2} + 2D_{s/2}^T \end{array}\right],$$

$$\tilde{\Delta}_s^{-} = \left[\begin{array}{c|c} (1+b)I_{s/2} & D_{s/2}J_{s/2} - bJ_{s/2} \\ \hline O_{s/2} & (1-a-b)I_{s/2} \end{array}\right],$$

where O_v means $v \times v$ zero matrix. The corresponding determinants can be captured as

$$\det \tilde{\Delta}_s^{+} = (1+b)^{s/2}(1-a-b)^{s/2-1}(1+(s-1)a-b), \tag{7}$$

$$\det \tilde{\Delta}_s^{-} = (1+b)^{s/2}(1-a-b)^{s/2} \tag{8}$$

for $s = 2, 4, 6, \ldots, k-1$.

In the case of k even we realize the next operations within matrix B^+_{k-1} (which preserve determinants of all its inners) to get matrix $\tilde{B}^+_{k-1}$:

1. subtraction of the $(k-i)$-th column from the i-th column, $i = 1, 2, \ldots, (k-2)/2$
2. addition of the i-th row to the $(k-i)$-th one for $i = 1, 2, \ldots, (k-2)/2$

and similarly within matrix B^-_{k-1} to get matrix $\tilde{B}^-_{k-1}$:

1. addition of the $(k-i)$-th column to the i-th column for $i = 1, 2, \ldots, (k-2)/2$
2. subtraction of the i-th row from the i-th one, $i = 1, 2, \ldots, (k-2)/2$

Now we can express any inner matrix of $\tilde{B}^+_{k-1}$ and $\tilde{B}^-_{k-1}$ as

$$\Delta^+_s = \left[\begin{array}{c|c|c} (1-b)I_{(s-1)/2} & o^T_{(s-1)/2} & D_{(s-1)/2}J_{(s-1)/2} + bJ_{(s-1)/2} \\ \hline o_{(s-1)/2} & 1+b & \omega_{(s-1)/2} \\ \hline O_{(s-1)/2} & 2\omega^T_{(s-1)/2} & (1+a+b)I_{(s-1)/2} + 2D_{(s-1)/2} + 2D^T_{(s-1)/2} \end{array}\right],$$

$$\Delta^-_s = \left[\begin{array}{c|c|c} (1-b)I_{(s-1)/2} & o^T_{(s-1)/2} & -D_{(s-1)/2}J_{(s-1)/2} - bJ_{(s-1)/2} \\ \hline o_{(s-1)/2} & 1-b & -\omega_{(s-1)/2} \\ \hline O_{(s-1)/2} & o^T_{(s-1)/2} & (1-a+b)I_{(s-1)/2} \end{array}\right],$$

where o_v is $1 \times v$ zero matrix and ω_v is a $1 \times v$ matrix, which is given by $\omega_v = [-a, a, -a, a, \ldots, (-1)^v a]$. The corresponding determinants then are

$$\det \tilde{\Delta}^+_s = (1-b)^{(s-1)/2}(1-a+b)^{(s-1)/2}(1+(s-1)a+b), \tag{9}$$

$$\det \tilde{\Delta}^-_s = (1-b)^{(s+1)/2}(1-a+b)^{(s-1)/2} \tag{10}$$

for $s = 1, 3, 5, \ldots, k-1$.

Considering simultaneous validity of all three points of Proposition 3 we can formulate Theorem 1. Indeed, in the case of k odd relations $1+b > 0$, $1+(k-1)a-b > 0$ and positivity of all $\det \tilde{\Delta}^+_s$ and $\det \tilde{\Delta}^-_s$, $s = 2, 4, 6, \ldots, k-1$ given by (7) and (8), respectively, give condition (4). In the case of k even relations $1-a+b > 0$, $1+(k-1)a+b > 0$ and positivity of all $\det \tilde{\Delta}^+_s$ and $\det \tilde{\Delta}^-_s$, $s = 1, 3, 5, \ldots, k-1$ given by (9) and (10), respectively, give condition (5). The theorem is proved. □

3 Discussion and Observations

For a better insight into asymptotic stability dependency on equation parameters a, b and k there can be constructed stability regions in (a, b) plane. Figure 1 depicts asymptotic stability region for (1) in the case of k odd whereas Fig. 2 captures the situation in the case of k even. As we can see, the asymptotic stability region has in the both cases a triangle shape (grey colour highlighted area). A part of the triangle situated in the right half-plane remains preserved for any admissible value of k,

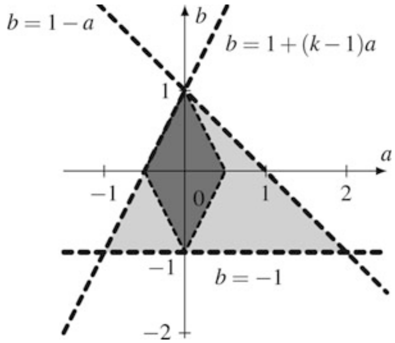

Fig. 1 Asymptotic stability region of (1) in (a, b) plane for k odd

whereas a part in the left half-plane depends on a value of parameter k. The greater value of k is the smaller area remains for asymptotic stability, since the left side of the triangle rotates towards the b axis round the vertex $(0, 1)$ in the case of k odd and round the vertex $(0, -1)$ in the case of k even. The Cohn asymptotic stability condition (the sufficient one) forms a diamond in (a, b) plane which is in Figs. 1 and 2 highlighted by a dark-grey colour. As we can observe from the both figures, the higher order k of difference equation we consider the higher ratio of asymptotic stability region area given by Theorem 1 to the diamond area given by Proposition 2 becomes.

Of course Proposition 3 can be used directly for checking whether some particular linear difference equation is asymptotically stable or not. But validity of the third condition requires a number of determinants computing and checking their positiveness. The form of necessary and sufficient conditions introduced in Theorem 1 gives a realistic idea about the asymptotic stability of a class of higher-order linear difference equations given by (1).

In the sequel we compare the obtained result with the conditions obtained for Eq. (2), which served as a motivation for the presented analysis. If we consider difference equation (2), we can observe that the necessary and sufficient asymptotic stability condition (3) for (2) (considering any $k \geq 2$) corresponds with the k-even branch of Theorem 1. Thence the asymptotic stability region (as well as the Cohn stability domain) for Eq. (2) corresponds to situation depicted in Fig. 2, but considering any integer order $k \geq 2$.

As it was stated in several papers, a formulation of some closed form of necessary and sufficient asymptotic stability conditions for higher-order linear difference equations is a very difficult problem. This fact is documented by a relatively small number of particular cases of such equations, which have been successfully analysed in the past. It still remains a challenge to obtain some efficient closed form of necessary and sufficient conditions for a more general higher-order linear difference equations, e.g. in the sense of considering more then two or three parameters or considering some special configuration of a few-term difference equations (see the survey in Sect. 1).

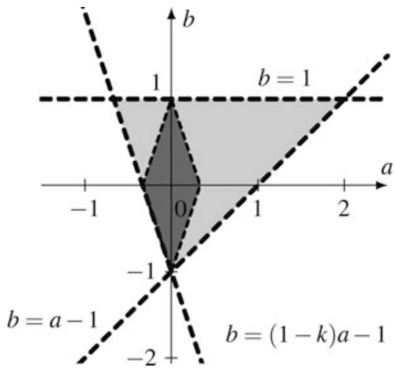

Fig. 2 Asymptotic stability region of (1) in (a,b) plane for k even

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Limit Point Criteria for Second-Order Sturm–Liouville Equations on Time Scales

Petr Zemánek

Abstract Necessary and sufficient conditions for the classification of the second-order Sturm–Liouville equation on time scales being in the limit point case are established. They unify and extend some of the criteria known for the second-order Sturm–Liouville differential and difference equations.

Keywords Sturm-Liouville equation • time scale • limit point case • criteria

Mathematics Subject Classification (2000): Primary: 34N05 Secondary: 34B20, 34B24, 39A12

1 Introduction

In this paper we focus on the limit point case of the second-order Sturm–Liouville dynamic equation

$$-[p(t)\,y^{\Delta}(t,\lambda)]^{\Delta} + q(t)\,y^{\sigma}(t,\lambda) = \lambda\,w(t)\,y^{\sigma}(t,\lambda), \quad t \in [a,\infty)_{\mathbb{T}}. \qquad (\mathrm{E}_{\lambda})$$

Here $\lambda \in \mathbb{C}$ and $[a,\infty)_{\mathbb{T}} := [a,\infty) \cap \mathbb{T}$, where $\mathbb{T}$ denotes a time scale (i.e., any nonempty closed subset of $\mathbb{R}$), which is bounded from below with $a := \min \mathbb{T}$ and unbounded from above. The coefficients $p(\cdot)$, $q(\cdot)$, and $w(\cdot)$ are real-valued piecewise rd-continuous functions on $[a,\infty)_{\mathbb{T}}$ such that

$$\inf_{t\in[a,b]_{\mathbb{T}}} |p(t)| > 0 \quad \text{for all } b \in (a,\infty)_{\mathbb{T}} \quad \text{and} \quad w(t) > 0 \quad \text{for all } t \in [a,\infty)_{\mathbb{T}}. \quad (1)$$

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Observe that $p(\cdot)$ is allowed to change its sign. Let us emphasize that the first condition in (1) cannot be replaced by the weaker assumption $p(t) \neq 0$ on $[a,\infty)_{\mathbb{T}}$ (see [4, Remark 2.2]). We also note that (E_λ) includes several equations of particular interest, especially the second-order Sturm–Liouville differential and difference equations.

The present results belong to the Weyl–Titchmarsh theory, whose history goes back to the seminal paper [13] devoted to Eq. (E_λ) in the case $\mathbb{T} = \mathbb{R}$ with $p(t) \equiv 1 \equiv w(t)$. One of the basic questions of this theory concerns the number of linearly independent solutions of Eq. (E_λ), which are square-integrable with respect to the weight $w(\cdot)$. It can be shown that there exists at least one square-integrable solution for every $\lambda \in \mathbb{C} \setminus \mathbb{R}$, which leads to the dichotomy of Eq. (E_λ) as being in the *limit point case* (i.e., at least one solution is not square-integrable) or in the *limit circle case* (i.e., all solutions are square-integrable) (see Sect. 2 for more details). The knowledge of the number of linearly independent square-integrable solutions plays a crucial role in the spectral theory of operators or linear relations associated with the underlying equation. In particular, it is closely connected with the deficiency indices of the operator or linear relation (see, e.g., [6, 14]). Therefore, it is very useful to develop some criteria for this classification. Some of them can be found in [5, 12, 15]. In this paper we derive necessary and sufficient conditions for the limit point case. They unify and extend the results derived in [1, Theorems 8, 18, and 19] and [14, Theorems 7.4.1 and 7.4.2].

The paper is organized as follows. In the next section we recall basic results concerning the limit point and limit circle classification of Eq. (E_λ). The main results are established in Sect. 3.

2 Preliminaries

Fundamental results of the time scale calculus can be found in [2]. For brevity we write only $y^{\sigma 2}(t)$ instead of $[y^\sigma(t)]^2 = y^2(\sigma(t)) = [y^2(t)]^\sigma$, where $\sigma(\cdot)$ is the time scale forward jump operator. By a *solution* of Eq. (E_λ), we mean a function $y(\cdot,\lambda) : [a,\infty)_{\mathbb{T}} \to \mathbb{C}$ such that $y(\cdot,\lambda), p(\cdot)\, y^\Delta(\cdot,\lambda) \in \mathrm{C}^1_{\mathrm{prd}}$, i.e., they are piecewise rd-continuously delta-differentiable on $[a,\infty)_{\mathbb{T}}$, and it satisfies (E_λ) for all $t \in [a,\infty)_{\mathbb{T}}$ (see [8, p. 4]). In addition, if $\lambda \in \mathbb{R}$, then we may consider only real-valued solutions.

The Weyl–Titchmarsh theory has been extended in many directions during the last 100 years. Recently, its generalization for the symplectic dynamic systems

$$z^\Delta(t,\lambda) = \mathbb{S}(t,\lambda)\, z(t,\lambda), \qquad \mathbb{S}(t,\lambda) = \mathscr{S}(t) + \lambda \mathscr{V}(t), \tag{S_λ}$$

where $\mathbb{S}(\cdot,\lambda) : [a,\infty)_{\mathbb{T}} \to \mathbb{C}^{2n\times 2n}$ is a piecewise rd-continuous function satisfying for every $\lambda \in \mathbb{C}$ the symplectic-type identity

$$\mathbb{S}^*(t,\lambda)\,\mathscr{J} + \mathscr{J}\,\mathbb{S}(t,\bar{\lambda}) + \mu(t)\,\mathbb{S}^*(t,\lambda)\,\mathscr{J}\,\mathbb{S}(t,\bar{\lambda}) = 0, \qquad \mathscr{J} := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

was established by the author and his collaborator in [8–10], compare with [11]. Here $\bar{\lambda}$ stands for the complex conjugate of λ and $\mathbb{S}^*(t,\lambda) = \overline{[\mathbb{S}(t,\lambda)]}^{\top}$. We note that system ($\mathrm{S}_\lambda$) reduces to the linear Hamiltonian differential system in the case $\mathbb{T} = \mathbb{R}$. The following lemma shows the relationship between system (S_λ) and Eq. (E_λ). It guarantees the existence and uniqueness of the solution of any initial value problem associated with (E_λ) (see, e.g., [8, Theorem 3.4]). Moreover, it implies that the Weyl–Titchmarsh theory for Eq. (E_λ) can be derived directly from the corresponding results for system (S_λ). This approach was used in [7] (see also the references therein and compare with [15]).

Lemma 1. *Equation* (E_λ) *is equivalent with system* (S_λ)*, where*

$$z(t,\lambda) = \begin{pmatrix} y(t,\lambda) \\ p(t)\, y^{\Delta}(t,\lambda) \end{pmatrix}, \quad \mathscr{S}(t) = \begin{pmatrix} 0 & 1/p(t) \\ q(t) & \mu(t)\, q(t)/p(t) \end{pmatrix},$$

$$\mathscr{V}(t) = -\begin{pmatrix} 0 & 0 \\ w(t) & \mu(t)\, w(t)/p(t) \end{pmatrix}.$$

Proof. The proof follows by straightforward calculations. □

We denote by $\mathscr{L}^2_w$ and $\mathscr{N}(\lambda)$ the linear spaces consisting of all functions and solutions of Eq. (E_λ), respectively, which are square-integrable with respect to the weight $w(\cdot)$, i.e.,

$$\mathscr{L}^2_w := \left\{ y : [a,\infty)_{\mathbb{T}} \to \mathbb{C},\ \int_a^\infty w(t) |y^\sigma(t)|^2\, \Delta t < \infty \right\},$$

$$\mathscr{N}(\lambda) := \left\{ y(\cdot,\lambda) \in \mathscr{L}^2_w,\ y(\cdot,\lambda) \text{solves } (E_\lambda) \right\}.$$

Moreover, we put $n(\lambda) := \dim \mathscr{N}(\lambda)$, i.e., $n(\lambda)$ is the number of linearly independent square-integrable solutions of (E_λ). According to [7, Theorem 3.10] it holds $1 \le n(\lambda) \le 2$ for any $\lambda \in \mathbb{C} \setminus \mathbb{R}$. This estimate is obtained as a consequence of a construction of the so-called Weyl circles, which are nested and converge to a circle ($n(\lambda) = 2$) or a point ($n(\lambda) = 1$) (see, e.g., [9, Sects. 3 and 4]). Motivated by this geometric background, for a given $\lambda \in \mathbb{C}$ Eq. (E_λ) is said to be in the *limit circle case* if $n(\lambda) = 2$, while it is in the *limit point case* if $n(\lambda) \le 1$. Although this classification of (E_λ) is introduced for every particular $\lambda \in \mathbb{C}$, the following theorem yields the so-called invariance of the limit circle case. It is a simple consequence of Lemma 1 and [9, Theorem 6.1], because $\operatorname{tr} \mathscr{V}(t) = -\mu(t)\, w(t)/p(t) = 0$ for every right-dense point $t \in [a,\infty)_{\mathbb{T}}$ as discussed in [9, Remark 6.2(ii)] (see also [15, Theorem 3.2], [5, Theorem 3.1], and [10]).

Theorem 1. *If there exists* $\lambda_0 \in \mathbb{C}$ *such that* $n(\lambda_0) = 2$*, then* $n(\lambda) = 2$ *all* $\lambda \in \mathbb{C}$*.*

Consequently we obtain the following result, which is known as the Weyl alternative (see [9, Sect. 6]). It shows that the classification of Eq. (E_λ) as being in the limit point or limit circle case does not depend on λ.

Theorem 2. *Equation* (E_λ) *is either*

(i) *in the limit circle case for every* $\lambda \in \mathbb{C}$, *i.e.,* $n(\lambda) \equiv 2$, *or*
(ii) *in the limit point case for every* $\lambda \in \mathbb{C}$, *i.e.,* $n(\lambda) \leq 1$. *In this case,* $n(\lambda) = 1$ *for all* $\lambda \in \mathbb{C} \setminus \mathbb{R}$ *and* $n(\lambda) \in \{0, 1\}$ *for all* $\lambda \in \mathbb{R}$.

We conclude this section with a sufficient condition for the existence of an eventually positive increasing solution of equation (E_0), which shall be useful in the development of a limit point criterion (see Theorem 4). In the case $\mathbb{T} = \mathbb{R}$ it reduces to [14, Lemma 7.4.1].

Lemma 2. *In addition to* (1) *let us assume that there exists* $b \in [a, \infty)_{\mathbb{T}}$ *such that* $p(t) > 0$ *and* $q(t) \geq 0$ *on* $[b, \infty)_{\mathbb{T}}$. *Then equation* (E_0) *possesses a positive increasing solution on* $(b, \infty)_{\mathbb{T}}$.

Proof. We show that a positive increasing solution $y(\cdot)$ on $(b, \infty)_{\mathbb{T}}$ can be obtained as the solution of equation (E_0) determined by the initial conditions

$$y(b) = 0 \quad \text{and} \quad p(b)\, y^{\Delta}(b) = 1. \tag{2}$$

Let $y(\cdot)$ satisfy (E_0) and (2). Then there exists $c \in (b, \infty)_{\mathbb{T}}$ such that $y(t) > 0$ on $(b, c)_{\mathbb{T}}$. At first we show that $y(t) > 0$ on $(b, \infty)_{\mathbb{T}}$. Let $d \in [c, \infty)_{\mathbb{T}}$ be the first point such that $y(d) \leq 0$. Then, by the First Mean Value Theorem, we obtain

$$0 \geq y(d) - y(b) = \int_b^d y^{\Delta}(t)\, \Delta t = \int_b^d \frac{1}{p(t)} [p(t)\, y^{\Delta}(t)]\, \Delta t = K \int_b^d \frac{1}{p(t)} \Delta t,$$

i.e., $K \leq 0$, where $\inf\{p(t)\, y^{\Delta}(t),\ t \in [b, d)_{\mathbb{T}}\} \leq K \leq \sup\{p(t)\, y^{\Delta}(t),\ t \in [b, d)_{\mathbb{T}}\}$ (see [3, Theorem 5.41]). On the other hand,

$$p(t)\, y^{\Delta}(t) = p(b)\, y^{\Delta}(b) + \int_b^t q(\tau)\, y^{\sigma}(\tau)\, \Delta\tau, \quad t \in [b, \infty)_{\mathbb{T}}.$$

But the nonnegativity of $q(\tau)\, y^{\sigma}(\tau)$ on $(b, d)_{\mathbb{T}}$ implies $p(t)\, y^{\Delta}(t) \geq 1$ for $t \in [b, d)_{\mathbb{T}}$, which yields a contradiction with $K \leq 0$. Hence $y(t) > 0$ for all $t \in (b, \infty)_{\mathbb{T}}$. Therefore, $p(t) y^{\Delta}(t) \geq 1$ for all $t \in [b, \infty)_{\mathbb{T}}$ and consequently $y^{\Delta}(t) > 0$ on $(b, \infty)_{\mathbb{T}}$, i.e., $y(\cdot)$ is positively increasing on $(b, \infty)_{\mathbb{T}}$, which completes the proof. □

3 Main Results

The first result yields a necessary and sufficient condition for Eq. (E_λ) being in the limit point case. If $w(t) \equiv 1$, we obtain [1, Theorems 8 and 18] in the case $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$, respectively.

Theorem 3. *Equation* (E_λ) *is in the limit point case for all* $\lambda \in \mathbb{C}$ *if and only if there exists a function* $y : [a,\infty)_{\mathbb{T}} \to \mathbb{R}$ *such that* $y(\cdot), p(\cdot)\, y^{\Delta}(\cdot) \in \mathrm{C}^1_{\mathrm{prd}}$ *and*

$$\int_a^\infty w(t)\, y^{\sigma 2}(t) \times \left(1 + \int_a^{\sigma(t)} \frac{1}{w(s)} \Big[-\big(p(s)\, y^{\Delta}(s)\big)^{\Delta} + q(s)\, y^{\sigma}(s)\Big]^2 \Delta s\right)^{-1} \Delta t = \infty. \tag{3}$$

Proof. According to Theorem 2, Eq. (E_λ) is in the limit point case if and only equation (E_0) has a solution, which does not belong to $\mathscr{L}^2_w$. Therefore, it suffices to restrict our attention only to equation (E_0).

If (E_0) is in the limit point case, then there exists its solution such that $y(\cdot) \notin \mathscr{L}^2_w$. In this case $y(\cdot), p(\cdot)\, y^{\Delta}(\cdot) \in \mathrm{C}^1_{\mathrm{prd}}$ and identity (3) is satisfied, because the second integral is identically zero. Hence, the necessity is established.

On the other hand, let us assume there exists $y(\cdot)$ such that $y(\cdot), p(\cdot)\, y^{\Delta}(\cdot) \in \mathrm{C}^1_{\mathrm{prd}}$ and identity (3) holds. Moreover, we suppose that all solutions of equation (E_0) belong to $\mathscr{L}^2_w$. Let $u(\cdot)$ and $v(\cdot)$ be normalized solutions of (E_0), i.e.,

$$W[u(t), v(t)] := u(t)\, [p(t)\, v^{\Delta}(t)] - v(t)\, [p(t)\, u^{\Delta}(t)] \equiv 1.$$

If we denote

$$f(t) := -[p(t)\, y^{\Delta}(t)]^{\Delta} + q(t)\, y^{\sigma}(t),$$

then the Variation of Constants Formula (see [2, Theorems 4.24 and 4.33]) yields

$$y(t) = u(t)\left[A + \int_a^t v^{\sigma}(s) f(s)\, \Delta s\right] - v(t)\left[B + \int_a^t u^{\sigma}(s) f(s)\, \Delta s\right]$$

for some $A, B \in \mathbb{R}$. Hence

$$|y(t)| \le D^{1/2}(t)\left[C + 2\int_a^t \big(D^{\sigma}(s)\big)^{1/2} |f(s)|\, \Delta s\right],$$

where $C := |A| + |B|$ and $D(t) := u^2(t) + v^2(t)$. Upon applying the Cauchy–Schwarz inequality (see [2, Theorem 6.15]) and the inequality of arithmetic and geometric means, we obtain

$$\begin{aligned} |y(t)| &\le D^{1/2}(t)\left[C + 2\Big(\int_a^t w(s)\, D^{\sigma}(s)\, \Delta s\Big)^{1/2} \Big(\int_a^t \frac{1}{w(s)} f^2(s)\, \Delta s\Big)^{1/2}\right] \\ &\le D^{1/2}(t)\left[C + 2\int_a^t w(s)\, D^{\sigma}(s)\, \Delta s\right]^{1/2} \left[C + 2\int_a^t \frac{1}{w(s)} f^2(s)\, \Delta s\right]^{1/2} \end{aligned}$$

and consequently we have

$$y^2(t)\left[C+2\int_a^t \frac{1}{w(s)} f^2(s)\,\Delta s\right]^{-1} \le D(t)\left[C+2\int_a^t w(s)\,D^\sigma(s)\,\Delta s\right] \tag{4}$$

Since $u, v \in \mathscr{L}_w^2$ it follows $\int_a^\infty w(t)\,D^\sigma(t)\,\Delta t =: K < \infty$. Hence, inequality (4) implies

$$\begin{aligned}\int_a^\infty w(t)\,y^{\sigma 2}(t)&\left(C+2\int_a^{\sigma(t)} \frac{1}{w(s)}\Big[-\big(p(s)\,y^\Delta(s)\big)^\Delta + q(s)\,y^\sigma(s)\Big]^2\Delta s\right)^{-1}\Delta t\\ &\le \int_a^\infty w(t)\,D^\sigma(t)\left[C+2\int_a^{\sigma(t)} w(s)\,D^\sigma(s)\,\Delta s\right]\Delta t \qquad (5)\\ &\le \int_a^\infty w(t)\,D^\sigma(t)\left[C+2\int_a^{\infty} w(s)\,D^\sigma(s)\,\Delta s\right]\Delta t \le K(C+2K) < \infty,\end{aligned}$$

which contradicts (3), because the integrals on the left-hand sides of (3) and (5) converge or diverge simultaneously by [3, Theorem 5.53]. Therefore, equation (E_0) is in the limit point case and the proof is complete. □

Several limit point criteria can be obtained as a consequence of Theorem 3. For example, the choice $y(t) \equiv 1$ yields the following sufficient conditions, which unify and extend [1, Theorems 1 and 19]. The second part of the corollary generalizes [14, Theorem 7.4.2]. Observe that these criteria do not include the coefficient $p(\cdot)$.

Corollary 1. *Equation* (E_λ) *is in the limit point case if*

$$\int_a^\infty w(t)\left(1+\int_a^{\sigma(t)} \frac{q^2(s)}{w(s)}\Delta s\right)^{-1}\Delta t = \infty. \tag{6}$$

In particular, identity (6) *is satisfied when*

$$\int_a^\infty \frac{q^2(t)}{w(t)}\Delta t < \infty \quad \text{and} \quad \int_a^\infty w(t)\,\Delta t = \infty. \tag{7}$$

Now we apply the main result in several illustrative examples.

Example 1.

(i) Let $\mathbb{T}$ be arbitrary and consider Eq. (E_λ) with $q(t) \equiv 0$, i.e.,

$$-[p(t)\,y^\Delta(t,\lambda)]^\Delta = \lambda\,w(t)\,y^\sigma(t,\lambda). \tag{8}$$

If $w(t)$ satisfies the second condition in (7), then Corollary 1 implies that Eq. (8) is in the limit point case for any $p(\cdot)$, e.g., put $[a,\infty)_\mathbb{T} = [0,\infty)_\mathbb{Z}$ and $p(t) = (-1)^t$. Note that this fact can be easily verified, because $y(t) \equiv 1$ is a solution of Eq. (8) with $\lambda = 0$.

(ii) Let $\mathbb{T}$ be arbitrary and consider Eq. (E_λ) with the coefficients $p(t) = t + \alpha$, $q(t) = 1/[\sigma(t) + \beta]$, and $w(t) \equiv 1$, i.e.,

$$-[(t+\alpha)\, y^{\Delta}(t,\lambda)]^{\Delta} + \frac{1}{\sigma(t)+\beta}\, y^{\sigma}(t,\lambda) = \lambda\, y^{\sigma}(t,\lambda), \tag{9}$$

where the constants $\alpha, \beta \in \mathbb{R}$ are such that $p(\cdot)$ and $q(\cdot)$ satisfy the basic assumptions, i.e., $-\alpha \notin \mathbb{T}$, $\beta \neq -\sigma(t)$ for all $t \in \mathbb{T}$, and $\beta \neq -t$ for all $t \in \mathbb{T}$, which are left-dense and right-scattered at the same time. In particular, Eq. (9) reduces to

$$-[(t+\alpha)\, y^{\Delta}(t,\lambda)]^{\Delta} + \frac{1}{rt+\beta}\, y^{\sigma}(t,\lambda) = \lambda\, y^{\sigma}(t,\lambda),$$

on the time scale $\mathbb{T} = r^{\mathbb{N}_0} = \{1, r, r^2, \dots\}$, where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $r > 1$ is fixed, and $\alpha \neq -r^k$, $\beta \neq -r^{k+1}$ for all $k \in \mathbb{N}_0$ (see [2, Exercise 4.4]).

We show that Eq. (9) is in the limit point case. The second condition in (7) is trivially satisfied. Since $\sigma(t) \geq t$, it follows

$$\int_b^\infty \frac{q^2(t)}{w(t)}\Delta t \leq \int_b^\infty \frac{q^2(t)}{w(t)}\Delta t = \int_b^\infty \frac{\Delta t}{[\sigma(t)+\beta]^2} \leq \int_b^\infty \frac{\Delta t}{t\sigma(t)+2\sigma(t)\,\beta}, \tag{10}$$

where $b \in [a,\infty)_{\mathbb{T}}$ is such that $b > 0$ and $t\sigma(t) + 2\sigma(t)\,\beta > 0$ on $[b.\infty)_{\mathbb{T}}$. The integral on the right-hand side of (10) is convergent by [3, Theorem 5.53], because $\lim_{t\to\infty}[t\sigma(t) + 2\sigma(t)\,\beta]/[t\sigma(t)] = 1$ and $\int_b^\infty 1/[t\sigma(t)]\,\Delta t = 1/b < \infty$. Therefore, the first condition in (7) is also satisfied and Corollary 1 implies the limit point case. Note that this conclusion can be verified directly, because $y(t) = t + \beta \notin \mathscr{L}^2_w$ solves Eq. (9) with $\lambda = 0$.

Finally, based on Lemma 2, we obtain yet another limit point criterion.

Theorem 4. *In addition to* (1) *let us assume that there exist* $b \in [a,\infty)_{\mathbb{T}}$ *and* $M \in \mathbb{R}$ *such that* $p(t) > 0$ *on* $[b,\infty)_{\mathbb{T}}$,

$$q(t) \geq Mw(t) \quad \text{for all} t \in [b,\infty)_{\mathbb{T}}, \quad \text{and} \quad \int_a^\infty w(t)\,\Delta t = \infty. \tag{11}$$

Then, Eq. (E_λ) *is in the limit point case for all* $\lambda \in \mathbb{C}$.

Proof. Similarly as in the proof of Theorem 3, it suffices to show that equation (E_M) has a solution, which does not belong to $\mathscr{L}^2_w$. The first condition in (11) implies that equation (E_M) satisfies the assumptions of Lemma 2. Hence, (E_M) possesses a positive increasing solution $y(\cdot, M)$ on $(b,\infty)_{\mathbb{T}}$. Let us assume that equation (E_M) is in the limit circle case. Then, $y(\cdot, M) \in \mathscr{L}^2_w$ and for any $d \in (b,\infty)_{\mathbb{T}}$ we have

$$\int_d^\infty w(t)\, y^{\sigma 2}(t, M)\,\Delta t \geq y^{\sigma 2}(d, M) \int_d^\infty w(t)\,\Delta t = \infty.$$

But this contradicts the assumption $y(\cdot, M) \in \mathscr{L}^2_w$ and the proof is complete. □

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Continuous Dependence of the Minimum of a Functional on Perturbations in Optimal Control Problems with Distributed and Concentrated Delays

Phridon Dvalishvili and Tamaz Tadumadze

Abstract Continuity of the minimum of a general functional is proved with respect to perturbations of the initial data and right-hand side of the equation with variable distributed and concentrated delays. Under the initial data, we understand the collection of initial moment, of variable delays, and initial function. Perturbations of the right-hand side of the equation are small in the integral sense.

Keywords Continuity of functional minimum • Optimal control • Delay differential equation • Perturbation

Mathematics Subject Classification (2000): 34K35; 34K27

1 Statement of the Problem: Formulation of the Main Result

Let $\mathbb{R}^n_x$ be an n-dimensional vector space of points $x = (x^1, \dots, x^n)^T$. Let $a < t_{01} < t_{02} < t_{11} < t_{12}$ be given numbers and let $O \subset \mathbb{R}^n_x$ be an open set. Let D be the set of continuously differentiable scalar functions (delay functions) $\tau(t), t \in \mathbb{R}^1_t$, satisfying the conditions

$$\tau(t) < t,\ \dot{\tau}(t) > 0, t \in \mathbb{R}^1_t,\ \ \inf\{\tau(a) : \tau(t) \in D\} := \hat{\tau} > -\infty.$$

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We denote by Φ and Ω, respectively, the sets of continuous initial functions $\varphi : [\hat{\tau}, t_{02}] \to O$ and measurable control functions $u : I \to U$, where $I = [a, t_{12}]$ and $U \subset \mathbb{R}^r_u$ is a given compact set. Furthermore, $f(t, x_1, x_2, x_3, u), (t, x_1, x_2, x_3, u) \in I \times O^3 \times U$ is an n-dimensional *Carathéodory* function satisfying the standard conditions: for any compact set $K \subset O$, there exist $m_{f,K}(t), L_{f,K}(t) \in L_1(I, [0, \infty))$, such that

$$|f(t, x_1, x_2, x_3, u)| \le m_{f,K}(t) \ \forall (t, x_1, x_2, x_3, u) \in I \times K^3 \times U$$

and

$$|f(t, x_1', x_2', x_3', u) - f(t, x_1'', x_2'', x_3'', u)| \le L_{f,K}(t) \sum_{i=1}^{3} |x_i' - x_i''|$$

$$\forall (t, x_1', x_2', x_3'), (t, x_1'', x_2'', x_3'') \in I \times K^3, \quad \forall u \in U.$$

Let Q be the set of continuous scalar functions $q(t, x), (t, x) \in [t_{11}, t_{12}] \times O$.

To each initial $\mu = (t_0, \theta(t), \tau(t), \varphi(t)) \in \Lambda = [t_{01}, t_{02}] \times D^2 \times \Phi$ and control $u(t) \in \Omega$, we assign the differential equation with distributed delay on the interval $[\theta(t), t]$ and with the concentrated delay $\tau(t)$:

$$\dot{x}(t) = \int_{\theta(t)}^{t} f(t, x(t), x(\tau(t)), x(s), u(t)) ds, \tag{1}$$

$$x(t) = \varphi(t), \quad t \in [\hat{\tau}, t_0]. \tag{2}$$

Definition 1.1. Let $\mu = (t_0, \theta(t), \tau(t), \varphi(t)) \in \Lambda$ and $u(t) \in \Omega$ be the initial data and the control function. A function $x(t; \mu, u) \in O, t \in [\hat{\tau}, t_1]$, where $t_1 \in [t_{11}, t_{12}]$, is called a solution of the Eq. (1) with the initial condition (2) or a solution corresponding to the initial data μ and the control $u(t)$ defined on the interval $[\hat{\tau}, t_1]$, if it satisfies the condition (2), is absolutely continuous on the interval $[t_0, t_1]$ and satisfies the Eq. (1) almost everywhere (a.e.) on $[t_0, t_1]$.

For given $\mu_0 = (t_{00}, \theta_0(t), \tau_0(t), \varphi_0(t))$, let us consider the optimal control problem

$$\dot{x}(t) = \int_{\theta_0(t)}^{t} f(t, x(t), x(\tau_0(t)), x(s), u(t)) ds, \quad t \in [t_{00}, t_1], \tag{3}$$

$$x(t) = \varphi_0(t), \quad t \in [\hat{\tau}, t_{00}], \tag{4}$$

$$J(w) = q_0(t_1, x(t_1; \mu_0, u)) \to \min, \tag{5}$$

where

$$w = (t_1, u(t)) \in W = \{w = (t_1, u(t)) : t_1 \in [t_{11}, t_{12}], u(t) \in \Omega\}$$

and $q_0(t, x) \in Q$ is a given function.

Here and in what follows, it is assumed that *the solution* $x(t;\mu_0,u)$ *is defined on the interval* $[\hat{\tau},t_{12}]$ *for every* $u(t)\in\Omega$.

Definition 1.2. An element $w_0=(t_{10},u_0(t))\in W$ is called an optimal element or a solution of the problem (3)–(5) if

$$J(w_0)=\inf\{J(w):w\in W\}:=J_0.$$

(3)–(5) is called the optimal control problem with distributed and concentrated delays.

Theorem 1.1. *Let the following conditions hold:*

(a) *there exists a compact set* $K_0\subset O$ *such that*

$$x(t;\mu_0,u)\in K_0,t\in[\hat{\tau},t_{12}],\ \forall u(t)\in\Omega.$$

(b) *the set*

$$\Big\{f(t,x_1,x_2,x_3,u):u\in U\Big\}$$

is convex for every fixed $(t,x_1,x_2,x_3)\in[a,t_{12}]\times O^3$.

Then, there exists an optimal element w_0 *for the initial problems (3)–(5); there exists a number* $\delta>0$ *such that for every*

$$\mu_\delta=(t_{0\delta},\theta_\delta(t),\tau_\delta(t),\varphi_\delta(t))\in V_{\mu_0,\delta}=V_{t_{00},\delta}\times V_{\theta_0,\delta}\times V_{\tau_0,\delta}\times V_{\varphi_0,\delta}$$

and

$$g_\delta(t,x_1,x_2,x_3)\in V_\delta(K_1),\ q_\delta(t,x)\in V_{q_0,\delta}(K_1)$$

the perturbed optimal control problem

$$\dot{x}(t)=\int_{\theta_\delta(t)}^{t}\Big[f(t,x(t),x(\tau_\delta(t)),x(s),u(t))$$
$$+g_\delta(t,x(t),x(\tau_\delta(t)),x(s))\Big]ds,\quad t\in[t_{0\delta},t_1], \tag{6}$$

$$x(t)=\varphi_\delta(t),\quad t\in[\hat{\tau},t_{0\delta}], \tag{7}$$

$$J(w;\delta)=q_\delta(t_1,x(t_1;\mu_\delta,u))\to\min \tag{8}$$

has a solution $w_{0\delta} = (t_{1\delta}, u_{0\delta}(t))$. *Moreover, for any sequence* $\delta_i \in (0, \delta)$, $i = 1, 2, \ldots$ *with* $\delta_i \to 0$, *we have*

$$\lim_{i\to\infty} J(w_{0\delta_i}; \delta_i) = J_0.$$

Here

$$V_{t_{00},\delta} = \{t_0 \in [t_{01}, t_{02}] : \ |t_{00} - t_0| < \delta\},$$
$$V_{\theta_0,\delta} = \{\theta(t) \in D : \ \max_{t\in I} |\theta_0(t) - \theta(t)| < \delta\},$$
$$V_{\tau_0,\delta} = \{\tau(t) \in D : \ \max_{t\in I} |\tau_0(t) - \tau(t)| < \delta\},$$
$$V_{\varphi_0,\delta} = \{\varphi(t) \in \Phi : \ \max_{t\in[\hat{\tau}, t_{02}]} |\varphi_0(t) - \varphi(t)| < \delta\};$$

$K_1 \subset O$ *is a compact set containing a certain neighborhood of the set* K_0 *and*

$$V_\delta(K_1) = \Big\{g_\delta(t, x_1, x_2, x_3) \in G : \Big| \int_{t'}^{t''} g_\delta(t, x_1, x_2, x_3)dt \Big| < \delta, \forall t', t'' \in I,$$
$$\forall x_i \in K_1, i = 1, 2, 3, \ \int_I [m_{g_\delta,K_1}(t) + L_{g_\delta,K_1}(t)]dt < \alpha\Big\}, \tag{9}$$

where G *is the set of* Carathéodory *functions* $g(t, x_1, x_2, x_3) : I \times O^3 \to \mathbb{R}^n_x$ *satisfying the standard conditions and* $\alpha > 0$ *is a fixed number independent of* g_δ;

$$V_{q_0,\delta}(K_1) = \Big\{q(t, x) \in Q : \max_{(t,x)\in I\times K_1} |q_0(t, x) - q(t, x)| < \delta\Big\}.$$

Some Comments Let

$$f(t, x_1, x_2, x_3, u) = A(t, x_1, x_2, x_3) + B(t, x_1, x_2, x_3)u$$

and U be a convex set. Then the condition (b) of Theorem 1.1 is fulfilled.

Let

$$f(t, x_1, x_2, x_3, u) = A(t)x_1 + B(t)x_2 + C(t)x_3 + D(t)u$$

and U be a convex set. Then the conditions (a), (b) of Theorem 1.1 are fulfilled. Perturbations $g_\delta \in V_\delta(K_1)$ are small in the integral sense [see (9)]. In Sect. 3, Theorem 1.1 is proved by the scheme proposed in [6]. Finally, we note that various small values are as a rule ignored in the numerical solution of optimal control problems; therefore, it is important to establish the connection between the initial and the perturbed problems. Theorems about well-posedness for various classes of optimal control problems which contain ordinary and functional differential equations are given in [1–7].

2 Auxiliary Assertions

To each initial data $\mu = (t_0, \theta(t), \tau(t), \varphi(t)) \in \Lambda$ and control $u(t) \in \Omega$, we assign the functional differential equation

$$\dot{y}(t) = \int_{\theta(t)}^{t} f(t, x(t), h(t_0, \varphi, y)(\tau(t)), h(t_0, \varphi, y)(s), u(t))ds \tag{10}$$

with the initial condition

$$y(t_0) = \varphi(t_0), \tag{11}$$

where $h(\cdot)$ is the operator given by the formula

$$h(t_0, \varphi, y)(t) = \begin{cases} \varphi(t), & \text{for } t \in [\hat{\tau}, t_0), \\ y(t), & \text{for } t \in [t_0, b]. \end{cases}$$

Definition 2.1. An absolutely continuous function $y(t) = y(t; \mu, u) \in O, t \in [r_1, r_2] \subset I$ is called solution of the Eq. (10) with the initial condition (11) or the solution corresponding to the initial data $\mu \in \Lambda$ and control $u(t) \in \Omega$, defined on $[r_1, r_2]$, if $t_0 \in [r_1, r_2], y(t_0) = \varphi(t_0)$, and satisfying the Eq. (10) a.e. on the interval $[r_1, r_2]$.

Remark 2.1. Let $y(t; \mu, u), t \in [r_1, r_2]$ be a solution of the Eq. (10) with the initial condition (11). Then the function

$$x(t; \mu, u) = h(t_0, \varphi, y(\cdot; \mu, u))(t), \quad t \in [\hat{\tau}, r_2]$$

is a solution of the Eq. (1) with the initial condition (2). It is clear that

$$y(t; \mu_0, u) = x(t; \mu_0, u), \quad t \in [t_{00}, t_{12}], \quad \forall u(t) \in \Omega.$$

Theorem 2.1 ([8]). *Let the condition* (a) *hold, i.e.,* $y(t; \mu_0, u) \in K_0, t \in [t_{00}, t_{12}]$. *Then there exists a number* $\delta > 0$ *such that for any*

$$\mu_\delta = (t_{0\delta}, \theta_\delta(t), \tau_\delta(t), \varphi_\delta(t)) \in V_{\mu_0,\delta}, \;\; g_\delta(t, x_1, x_2, x_3) \in V_\delta(K_1)$$

and $u(t) \in \Omega$ *the perturbed equation*

$$\dot{y}(t) = \int_{\theta_\delta(t)}^{t} [f(t, y(t), h(t_{0\delta}, \varphi_\delta, y)(\tau_\delta(t)), h(t_{0\delta}, \varphi_\delta, y)(s), u(t))$$
$$+ g_\delta(t, y(t), h(t_{0\delta}, \varphi_\delta, y)(\tau_\delta(t)), h(t_{0\delta}, \varphi_\delta, y)(s))]ds$$

with the perturbed initial condition

$$y(t_{0\delta}) = \varphi_\delta(t_{0\delta})$$

has a solution $y(t;\mu_\delta, g_\delta, u)$ *defined on* $[t_{00}-\delta, t_{12}] \subset (a, t_{12}]$. *Also,* $y(t;\mu_\delta, g_\delta, u) \in K_1, \forall t \in [t_{00}-\delta, t_{12}], \forall u(t) \in \Omega$ *and*

$$\lim_{i\to\infty} |y(t;\mu_{\delta_i}, g_{\delta_i}, u) - y(t;\mu_0, 0, u)| = 0$$

uniformly for $t \in [t_{00}-\delta, t_{12}]$ *and* $u(t) \in \Omega$, *where* $\delta_i \in (0,\delta), i = 1, 2, \dots$ *with* $\delta_i \to 0$.

Due to the uniqueness, the solution $y(t;\mu_0, 0, u)$ is the continuation of the solution $y(t;\mu_0, u)$ on the interval $[t_{00}-\delta, t_{12}]$.

Theorem 2.2 ([8]). *Let* $x(t), y(t), z(t) \in K_0, t \in [\hat{\tau}, t_{12}]$ *be fixed continuous functions and* $\theta(t), \tau(t) \in D$ *be fixed delay functions. Moreover, let* $g_{\delta_i} \in V_{\delta_i}(K_0)$ *with* $\delta_i \to 0$. *Then*

$$\lim_{i\to\infty} \max_{t',t''\in I} \left| \int_{t'}^{t''} \left[\int_{\theta(t)}^{t} g_{\delta_i}(t, x(t), y(\tau(t)), z(s))ds \right] dt \right| = 0.$$

3 Proof of Theorem 1.1

It is not difficult to see that the convexity of the set

$$P_f(t;x(\cdot), \theta_0, \tau_0) = \left\{ \int_{\theta_0(t)}^{t} f(t, x(t), x(\tau_0(t)), x(s), u)ds : u \in U \right\} \tag{12}$$

follows from the condition (b) for every fixed continuous function $x(t) \in O, t \in [\hat{\tau}, t_{12}]$.

The convexity of the set (12) and the condition (a) guarantee the existence of a solution w_0 for the initial optimal control problem (3)–(5), [6].

Let $\delta > 0$ be so small (see Theorem 2.1) that from $\varphi_\delta \in V_{\varphi_0,\delta}$, it follows that $\varphi_\delta(t) \in K_1, t \in [\hat{\tau}, t_{12}]$. Since $y(t;\mu_\delta, g_\delta, u) \in K_1, t \in [t_{00}-\delta, t_{12}]$, we have

$$x(t;\mu_\delta, g_\delta, u) = h(t_{0\delta}, \varphi_\delta, y(\cdot;\mu_\delta, g_\delta, u))(t) \in K_1, \quad t \in [\hat{\tau}, t_{12}] \tag{13}$$

(see Remark 2.1). The set $P_{f+g_\delta}(t;x(\cdot), \theta_\delta, \tau_\delta)$ is convex. Consequently, the perturbed optimal control problem (6)–(8) has a solution $w_{0\delta}$.

Let $\delta_i \in (0, \delta), i = 1, 2, \ldots$ with $\delta_i \to 0$. By virtue of Theorem 2.1, we conclude that

$$\lim_{i\to\infty} |x(t; \mu_{\delta_i}, g_{\delta_i}, u_0) - x(t; \mu_0, 0, u_0)| = 0 \;\; \text{uniformlyfor } t \in [\hat{\tau}, t_{12}]$$

[see (13)]. The element w_0 is an admissible element for the perturbed problem (6)–(8), i.e.,

$$J(w_{0,\delta_i}; \delta_i) \le J(w_0; \delta_i), \quad i = 1, 2, \ldots.$$

From this it follows that

$$\underline{J} = \lim_{i\to\infty} \inf J(w_{0,\delta_i}; \delta_i) \le \overline{J} = \lim_{i\to\infty} \sup J(w_{0,\delta_i}; \delta_i) \le J_0.$$

To prove the theorem, it suffices to show that $\underline{J} = J_0$.

Let $\underline{J} < J_0$. From the sequence δ_i, $i = 1, 2, \ldots$, we extract a subsequence, which will again be denoted by δ_i, $i = 1, 2, \ldots$, such that

$$\lim_{i\to\infty} J(w_{0,\delta_i}; \delta_i) = \lim_{i\to\infty} q_{\delta_i}(t_{1\delta_i}, x(t_{1\delta_i}; \mu_{\delta_i}, u_{0\delta_i})) = \underline{J}, \quad \lim_{i\to\infty} t_{1\delta_i} = \hat{t}_{10} \in [t_{11}, t_{12}].$$

It is clear that

$$x(t; \mu_{\delta_i}, g_{\delta_i}, u_{0\delta_i}) = h(t_{0\delta_i}, \varphi_{\delta_i}, y_i)(t), \quad t \in [\hat{\tau}, t_{12}],$$

where $y_i(t) := y(t; \mu_{\delta_i}, g_{\delta_i}, u_{0\delta_i})$, $t \in [t_{00} - \delta, t_{12}]$ satisfies the integral equation

$$y_i(t) = \varphi_{\delta_i}(t_{0\delta_i}) + \Theta_{1i}(t) + \Theta_{2i}(t), \tag{14}$$

where

$$\Theta_{1i}(t) = \int_{t_{0\delta_i}}^{t} \left\{ \int_{\theta_{\delta_i}(\xi)}^{\xi} f(\xi, y_i(\xi), h(t_{0\delta_i}, \varphi_{\delta_i}, y_i)(\tau_{\delta_i}(\xi)), h(t_{0\delta_i}, \varphi_{\delta_i}, y_i)(s), \right.$$
$$\left. u_{0\delta_i}(\xi))ds \right\} d\xi,$$
$$\Theta_{2i}(t) = \int_{t_{0\delta_i}}^{t} \left\{ \int_{\theta_{\delta_i}(\xi)}^{\xi} g_{\delta_i}(\xi, y_i(\xi), h(t_{0\delta_i}, \varphi_{\delta_i}, y_i)(\tau_{\delta_i}(\xi)), h(t_{0\delta_i}, \varphi_{\delta_i}, y_i)(s)) \Big] ds \right\} d\xi.$$

From Theorem 2.1 we get

$$\lim_{i\to\infty} |y(t; \mu_{\delta_i}, g_{\delta_i}, u_{0\delta_i}) - y(t; \mu_0, 0, u_{0\delta_i})| = 0 \tag{15}$$

The sequence $\{y(t; \mu_0, 0, u_{0\delta_i})\}$ is equicontinuously and uniformly bounded. Therefore, we can choose from it a subsequence which we denote again by $\{y(t; \mu_0, 0, u_{0\delta_i})\}$ and which uniformly converges on $[t_{00}, t_{12}]$ to some function $\hat{y}(t)$. Thus,

$$\lim_{i\to\infty} y(t; \mu_{\delta_i}, g_{\delta_i}, u_{0\delta_i}) = \hat{y}(t)$$

[see (15)]. We rewrite the Eq. (14) in the form

$$y_i(t) = \varphi_{\delta_i}(t_{0\delta_i}) + \Psi_{1i}(t) + \Psi_{2i}(t) + \Pi_{1i}(t) + \Pi_{2i}(t), \tag{16}$$

where

$$\Psi_{1i}(t) = \int_{t_{00}}^{t} \Big\{ \int_{\theta_0(\xi)}^{\xi} f(\xi, \hat{y}(\xi), h(t_{00}, \varphi_0, \hat{y})(\tau_0(\xi)), h(t_{00}, \varphi_0, \hat{y})(s), u_{0\delta_i}(\xi))ds \Big\} d\xi,$$

$$\Psi_{2i}(t) = \int_{t_{00}}^{t} \Big\{ \int_{\theta_0(\xi)}^{\xi} g_{\delta_i}(\xi, \hat{y}(\xi), h(t_{00}, \varphi_0, \hat{y})(\tau_0(\xi)), h(t_{00}, \varphi_0, \hat{y})(s))ds \Big\} d\xi,$$

$$\Pi_{1i}(t) = \Theta_{1i}(t) - \Psi_{1i}(t), \Pi_{2i}(t) = \Theta_{2i}(t) - \Psi_{2i}(t).$$

According to Theorem 2.2, we have

$$\lim_{i\to\infty} \Psi_{2i}(t) = 0 \text{ uniformlyfor } t \in [t_{00} - \delta, t_{12}].$$

By a simple transformation, we can prove

$$\lim_{i\to\infty} \Pi_{1i}(t) = 0, \ \lim_{i\to\infty} \Pi_{2i}(t) = 0 \text{ uniformlyfor } t \in [t_{00} - \delta, t_{12}].$$

Using the convexity and compactness of the set $P(\xi, \hat{y}(\cdot), \theta_0, \tau_0)$, $\xi \in [t_{00} - \delta, t_{12}]$, we prove in the well-known manner [5] that from the sequence

$$F_i(\xi) = \int_{\theta_0(\xi)}^{\xi} f(\xi, \hat{y}(\xi), h(t_{00}, \varphi_0, \hat{y})(\tau_0(\xi)), h(t_{00}, \varphi_0, \hat{y})(s), u_{0\delta_i}(\xi))ds$$

we can choose a subsequence weakly converging to some function

$$F(\xi) \in P(\xi, \hat{y}(\cdot), \theta_0, \tau_0), \quad \xi \in [t_{00} - \delta, t_{12}].$$

Moreover, there exists a measurable function $\hat{u}(t) \in \Omega$ (see [5]) such that for any $\xi \in [t_{00} - \delta, t_{12}]$

$$F(\xi) = \int_{\theta_0(\xi)}^{\xi} f(\xi, \hat{y}(\xi), h(t_{00}, \varphi_0, \hat{y})(\tau_0(\xi)), h(t_{00}, \varphi_0, \hat{y})(s), \hat{u}(\xi))ds.$$

Passing to the limit in (15), we obtain

$$\hat{y}(t) = \varphi_0(t_{00}) + \int_{t_{00}}^{t} F(\xi)d\xi = \varphi_0(t_{00})$$
$$+ \int_{t_{00}}^{t} \Big[\int_{\theta_0(\xi)}^{\xi} f(\xi, \hat{y}(\xi), h(t_{00}, \varphi_0, \hat{y})(\tau_0(\xi)), h(t_{00}, \varphi_0, \hat{y})(s), \hat{u}(\xi))ds \Big] d\xi,$$
$$t \in [t_{00} - \delta, t_{12}].$$

Now let us define a function $\hat{x}(t), t \in [\hat{\tau}, t_{12}]$, in the following manner

$$\hat{x}(t) = \begin{cases} \varphi_0(t), & t \in [\hat{\tau}, t_{00}], \\ \hat{y}(t), & t \in [t_{00}, t_{12}]. \end{cases}$$

The function $\hat{x}(t)$ satisfies the equation

$$\dot{x}(t) = \int_{\theta_0(t)}^{t} f(t, x(t), x(\tau_0(t)), x(s), \hat{u}(t))ds, \quad t \in [t_{00}, \hat{t}_{10}],$$

and the initial condition

$$x(t) = \varphi_0(t), \quad t \in [\hat{\tau}, t_{00}],$$

i.e., $(\hat{t}_{10}, \hat{u}(t)) \in W$. Moreover, $\underline{J} = q_0(\hat{t}_{10}, \hat{x}(\hat{t}_{10}))$. So, the assumption $\underline{J} < J_0$ contradicts the optimality of the element w_0. Thus, $\underline{J} = \overline{J} = J_0$. The theorem is proved.

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Existence of Positive Solutions for a System of Fractional Boundary Value Problems

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Abstract We study the existence and nonexistence of positive solutions of a system of nonlinear Riemann–Liouville fractional differential equations with integral boundary conditions which contain some positive constants.

Keywords Riemann-Liouville fractional differential equations • Integral boundary conditions • Positive solutions

Mathematics Subject Classification (2000): 34A08, 45G15

1 Introduction

Fractional differential equations describe many phenomena in various fields of engineering and scientific disciplines such as physics, biophysics, chemistry, biology (such as blood flow phenomena), economics, control theory, signal and image processing, aerodynamics, viscoelasticity, electromagnetics, and so on (see [6, 12, 14, 15]). For some recent developments on the topic, see [1–5, 7, 11, 16] and the references therein.

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We consider the system of nonlinear ordinary fractional differential equations

$$\begin{cases} D_{0+}^{\alpha}u(t) + a(t)f(v(t)) = 0, & t \in (0,1), \\ D_{0+}^{\beta}v(t) + b(t)g(u(t)) = 0, & t \in (0,1), \end{cases} \tag{S}$$

with the uncoupled integral boundary conditions

$$\begin{cases} u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \ \ u(1) = \int_0^1 u(s)dH(s) + a_0, \\ v(0) = v'(0) = \cdots = v^{(m-2)}(0) = 0, \ \ v(1) = \int_0^1 v(s)dK(s) + b_0, \end{cases} \tag{BC}$$

where $n-1 < \alpha \le n$, $m-1 < \beta \le m$, $n, m \in \mathbf{N}$, $n, m \ge 3$, D_{0+}^{α} and D_{0+}^{β} denote the Riemann–Liouville derivatives of orders α and β, respectively, the integrals from (BC) are Riemann–Stieltjes integrals, and a_0 and b_0 are positive constants.

Under some assumptions on the functions f and g, by using the Schauder fixed point theorem, we shall prove the existence of positive solutions of problem (S)–(BC). By a positive solution of (S)–(BC), we mean a pair of functions $(u, v) \in C([0,1]; \mathbf{R}_+) \times C([0,1]; \mathbf{R}_+)$ satisfying (S) and (BC) with $u(t) > 0$, $v(t) > 0$ for all $t \in (0,1]$. We shall also give sufficient conditions for the nonexistence of positive solutions for this problem. Some systems of fractional equations with parameters subject to integral boundary conditions were studied in [8] and [10] by using the Guo-Krasnoselskii fixed point theorem. We also mention the paper [9], where the authors investigated the existence and multiplicity of positive solutions for the system $D_{0+}^{\alpha}u(t)+f(t,v(t)) = 0$, $t \in (0,1)$, $D_{0+}^{\beta}v(t)+g(t,u(t)) = 0$, $t \in (0,1)$, with the integral boundary conditions (BC) with $a_0 = b_0 = 0$ [denoted by $(\widetilde{BC})$] by using some theorems from the fixed point index theory and the Guo-Krasnoselskii fixed point theorem. In [9], the nonlinearities f and g may be nonsingular or singular in $t = 0$ and/or $t = 1$. A system of fractional differential equations with parameters where the functions f and g are sign-changing with boundary conditions $(\widetilde{BC})$ is investigated in [13].

In Sect. 2, we present some auxiliary results which investigate a Riemann–Liouville fractional equation subject to integral boundary conditions. In Sect. 3, we prove our main results, and an example which supports the obtained results is finally presented in Sect. 4.

2 Auxiliary Results

In this section we present some auxiliary results from [9] related to a Riemann–Liouville fractional equation with integral boundary conditions.

We consider the fractional differential equation

$$D_{0+}^{\alpha}u(t) + y(t) = 0, \quad t \in (0,1), \tag{1}$$

with the integral boundary conditions

$$u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \;\; u(1) = \int_0^1 u(s)dH(s), \tag{2}$$

where $n-1 < \alpha \leq n$, $n \in \mathbf{N}$, $n \geq 3$, and $H : [0,1] \to \mathbf{R}$ is a function of bounded variation.

Lemma 1 ([9]). *If $H : [0,1] \to \mathbf{R}$ is a function of bounded variation, $\Delta_1 = 1 - \int_0^1 s^{\alpha-1} dH(s) \neq 0$ and $y \in C((0,1)) \cap L^1(0,1)$, then the solution of problem (1)–(2) is given by $u(t) = \int_0^1 G_1(t,s)y(s)\,ds$, where*

$$G_1(t,s) = g_1(t,s) + \frac{t^{\alpha-1}}{\Delta_1}\int_0^1 g_1(\tau,s)\,dH(\tau), \;\; (t,s) \in [0,1] \times [0,1], \tag{3}$$

and

$$g_1(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1}, \;\; 0 \leq s \leq t \leq 1, \\ t^{\alpha-1}(1-s)^{\alpha-1}, \;\; 0 \leq t \leq s \leq 1. \end{cases} \tag{4}$$

Lemma 2 ([9]). *If $H : [0,1] \to \mathbf{R}$ is a nondecreasing function and $\Delta_1 > 0$, then the Green's function G_1 of the problem (1)–(2) is continuous on $[0,1] \times [0,1]$ and satisfies $G_1(t,s) \geq 0$ for all $(t,s) \in [0,1] \times [0,1]$. Moreover, if $y \in C((0,1)) \cap L^1(0,1)$ satisfies $y(t) \geq 0$ for all $t \in (0,1)$, then the unique solution u of problem (1)–(2) satisfies $u(t) \geq 0$ for all $t \in [0,1]$.*

Lemma 3 ([9]). *Assume that $H : [0,1] \to \mathbf{R}$ is a nondecreasing function and $\Delta_1 > 0$. Then the Green's function G_1 of the problem (1)–(2) satisfies the inequalities*

a) $G_1(t,s) \leq J_1(s), \;\; \forall\, (t,s) \in [0,1] \times [0,1]$, *where*

$$J_1(s) = g_1(\theta_1(s),s) + \frac{1}{\Delta_1}\int_0^1 g_1(\tau,s)\,dH(\tau), \;\; \forall\, s \in [0,1].$$

b) *For every $c \in (0,1/2)$, we have*

$$\min_{t \in [c,1-c]} G_1(t,s) \geq \gamma_1 J_1(s) \geq \gamma_1 G_1(t',s), \;\; \forall\, t',\, s \in [0,1],$$

where $\gamma_1 = c^{\alpha-1}$ *and* $\theta_1(s) = \begin{cases} \dfrac{s}{1-(1-s)^{\frac{\alpha-1}{\alpha-2}}}, & s \in (0,1], \\ \frac{\alpha-2}{\alpha-1}, & s = 0, \end{cases}$ *if* $n-1 < \alpha \leq n$, $n \geq 3$.

Lemma 4 ([9]). *Assume that $H : [0,1] \to \mathbf{R}$ is a nondecreasing function and $\Delta_1 > 0$, $c \in (0,1/2)$ and $y \in C((0,1)) \cap L^1(0,1)$, $y(t) \geq 0$ for all $t \in (0,1)$. Then the solution $u(t)$, $t \in [0,1]$ of problem (1)–(2) satisfies the inequality $\inf_{t\in[c,1-c]} u(t) \geq \gamma_1 \sup_{t'\in[0,1]} u(t')$.*

We can also formulate similar results as Lemmas 1–4 above for the fractional differential equation

$$D_{0+}^{\beta} v(t) + h(t) = 0, \quad 0 < t < 1, \tag{5}$$

with the integral boundary conditions

$$v(0) = v'(0) = \cdots = v^{(m-2)}(0) = 0, \quad v(1) = \int_0^1 v(s)\, dK(s), \tag{6}$$

where $m-1 < \beta \leq m$, $m \in \mathbf{N}$, $m \geq 3$, $K : [0,1] \to \mathbf{R}$ is a function of bounded variation and $h \in C((0,1)) \cap L^1(0,1)$. We denote by Δ_2, γ_2, g_2, θ_2, G_2, and J_2 the corresponding constants and functions for the problem (5)–(6) defined in a similar manner as Δ_1, γ_1, g_1, θ_1, G_1, and J_1, respectively.

3 Main Results

We present first the assumptions that we shall use in the sequel.

$(J1)$ $H, K : [0,1] \to \mathbf{R}$ are nondecreasing functions, $\Delta_1 = 1 - \int_0^1 s^{\alpha-1}\, dH(s) > 0$, $\Delta_2 = 1 - \int_0^1 s^{\beta-1}\, dK(s) > 0$.

$(J2)$ The functions $a, b : [0,1] \to [0,\infty)$ are continuous and there exist $t_1, t_2 \in (0,1)$ such that $a(t_1) > 0$, $b(t_2) > 0$.

$(J3)$ $f, g : [0,\infty) \to [0,\infty)$ are continuous functions, and there exists $c_0 > 0$ such that $f(u) < \frac{c_0}{L}$, $g(u) < \frac{c_0}{L}$ for all $u \in [0,c_0]$, where $L = \max\{\int_0^1 a(s)J_1(s)\, ds, \int_0^1 b(s)J_2(s)\, ds\}$ and J_1, J_2 are defined in Sect. 2 (Lemma 3).

$(J4)$ $f, g : [0,\infty) \to [0,\infty)$ are continuous functions and satisfy the conditions

$$\lim_{u\to\infty} \frac{f(u)}{u} = \infty, \quad \lim_{u\to\infty} \frac{g(u)}{u} = \infty.$$

By $(J2)$, we deduce that $\int_0^1 a(s)J_1(s)\, ds > 0$ and $\int_0^1 b(s)J_2(s)\, ds > 0$, that is, the constant L from $(J3)$ is positive.

Our first theorem is the following existence result for problem (S)–(BC).

Theorem 1. *Assume that assumptions $(J1)$–$(J3)$ hold. Then problem (S)–(BC) has at least one positive solution for $a_0 > 0$ and $b_0 > 0$ sufficiently small.*

Proof. We consider the problems

$$\begin{cases} D_{0+}^{\alpha} h(t) = 0, \quad t \in (0,1), \\ h(0) = h'(0) = \cdots = h^{(n-2)}(0) = 0, \quad h(1) = \int_0^1 h(s)dH(s) + 1, \end{cases} \tag{7}$$

$$\begin{cases} D_{0+}^{\beta} k(t) = 0, \quad t \in (0,1), \\ k(0) = k'(0) = \cdots = k^{(m-2)}(0) = 0, \quad k(1) = \int_0^1 k(s)dK(s) + 1. \end{cases} \tag{8}$$

The above problems (7) and (8) have the solutions

$$h(t) = \frac{t^{\alpha-1}}{\Delta_1}, \quad k(t) = \frac{t^{\beta-1}}{\Delta_2}, \quad t \in [0,1], \tag{9}$$

where Δ_1 and Δ_2 are defined in $(J1)$. By assumption $(J1)$ we obtain $h(t) > 0$ and $k(t) > 0$ for all $t \in (0,1]$.

We define the functions $x(t)$ and $y(t)$, for all $t \in [0,1]$, by $x(t) = u(t) - a_0 h(t)$ and $y(t) = v(t) - b_0 k(t)$, for $t \in [0,1]$, where (u,v) is a solution of (S)–(BC). Then (S)–(BC) can be equivalently written as

$$\begin{cases} D_{0+}^{\alpha} x(t) + a(t) f(y(t) + b_0 k(t)) = 0, \quad t \in (0,1), \\ D_{0+}^{\beta} y(t) + b(t) g(x(t) + a_0 h(t)) = 0, \quad t \in (0,1), \end{cases} \tag{10}$$

with the boundary conditions

$$\begin{cases} x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0, \quad x(1) = \int_0^1 x(s)dH(s), \\ y(0) = y'(0) = \cdots = y^{(m-2)}(0) = 0, \quad y(1) = \int_0^1 y(s)dK(s). \end{cases} \tag{11}$$

Using the Green's functions G_1 and G_2 from Sect. 2 (Lemma 1), a pair (x,y) is a solution of problem (10)–(11) if and only if (x,y) is a solution for the nonlinear integral equations

$$\begin{cases} x(t) = \int_0^1 G_1(t,s) a(s) f\left(\int_0^1 G_2(s,\tau) b(\tau) g(x(\tau) + a_0 h(\tau))\, d\tau + b_0 k(s) \right) ds, \\ y(t) = \int_0^1 G_2(t,s) b(s) g(x(s) + a_0 h(s))\, ds, \quad t \in [0,1], \end{cases} \tag{12}$$

where $h(t),\ k(t)$, for $t \in [0,1]$, are given by (9).

We consider the Banach space $X = C([0,1])$ with the supremum norm $\|\cdot\|$ and define the set $E = \{x \in C([0,1]),\ \ 0 \le x(t) \le c_0,\ \ \forall\, t \in [0,1]\} \subset X$.

We also define the operator $\mathcal{S} : E \to X$ by

$$(\mathcal{S}x)(t) = \int_0^1 G_1(t,s) a(s) f\left(\int_0^1 G_2(s,\tau) b(\tau) g(x(\tau) + a_0 h(\tau)) d\tau + b_0 k(s) \right) ds,$$

for all $t \in [0,1]$ and $x \in E$. For sufficiently small $a_0 > 0$ and $b_0 > 0$, by $(J3)$, we deduce $f(y(t) + b_0k(t)) \le \frac{c_0}{L}$ and $g(x(t) + a_0h(t)) \le \frac{c_0}{L}$, for all $t \in [0,1]$, $x, y \in E$.

Then, by using Lemma 2, we obtain $(\mathcal{S}x)(t) \ge 0$ for all $t \in [0,1]$ and $x \in E$. By Lemma 3, for all $x \in E$, we have

$$\int_0^1 G_2(s,\tau)b(\tau)g(x(\tau) + a_0h(\tau))\,d\tau \le \frac{c_0}{L}\int_0^1 J_2(\tau)b(\tau)\,d\tau \le c_0, \quad \forall\, s \in [0,1],$$

and

$$\begin{aligned}(\mathcal{S}x)(t) &\le \int_0^1 J_1(s)a(s)f\left(\int_0^1 G_2(s,\tau)b(\tau)g(x(\tau) + a_0h(\tau))d\tau + b_0k(s)\right) ds \\ &\le \frac{c_0}{L}\int_0^1 J_1(s)a(s)\,ds \le c_0, \quad \forall\, t \in [0,1].\end{aligned}$$

Therefore, $\mathcal{S}(E) \subset E$.

Using standard arguments, we deduce that $\mathcal{S}$ is completely continuous. By using the Schauder fixed point theorem, we conclude that $\mathcal{S}$ has a fixed point $x \in E$. This element together with y given by (12) represents a solution for (10)–(11). This shows that our problem (S)–(BC) has a positive solution $u = x + a_0h$, $v = y + b_0k$ for sufficiently small a_0 and b_0. □

In what follows, we present sufficient conditions for the nonexistence of positive solutions of (S)–(BC).

Theorem 2. *Assume that assumptions* $(J1)$, $(J2)$, *and* $(J4)$ *hold. Then problem* (S)–(BC) *has no positive solution for* a_0 *and* b_0 *sufficiently large.*

Proof. We suppose that (u,v) is a positive solution of (S)–(BC). Then $x = u - a_0h$, $y = v - b_0k$ is a solution for (10)–(11), where h and k are the solutions of problems (7) and (8) [given by (9)]. By $(J2)$ there exists $c \in (0,1/2)$ such that $t_1, t_2 \in (c, 1-c)$, and then $\int_c^{1-c} a(s)J_1(s)\,ds > 0$, $\int_c^{1-c} b(s)J_2(s)\,ds > 0$. Now by using Lemma 2, we have $x(t) \ge 0$, $y(t) \ge 0$ for all $t \in [0,1]$, and by Lemma 4 we obtain $\inf_{t\in[c,1-c]} x(t) \ge \gamma_1\|x\|$ and $\inf_{t\in[c,1-c]} y(t) \ge \gamma_2\|y\|$.

Using now (9), we deduce that $\inf_{t\in[c,1-c]} h(t) = h(c) = \gamma_1\|h\|$ and $\inf_{t\in[c,1-c]} k(t) = k(c) = \gamma_2\|k\|$.

Therefore, we obtain $\inf_{t\in[c,1-c]}(x(t) + a_0h(t)) \ge \gamma_1\|x + a_0h\|$ and $\inf_{t\in[c,1-c]}(y(t) + b_0k(t)) \ge \gamma_2\|y + b_0k\|$.

We now consider

$$R = \left(\min\left\{\gamma_1\gamma_2\int_c^{1-c} a(s)J_1(s)\,ds,\ \gamma_1\gamma_2\int_c^{1-c} b(s)J_2(s)\,ds\right\}\right)^{-1} > 0.$$

By using $(J4)$, for R defined above, we conclude that there exists $M > 0$ such that $f(u) > 2Ru$, $g(u) > 2Ru$ for all $u \ge M$. We consider $a_0 > 0$ and $b_0 > 0$ sufficiently large such that $\inf_{t\in[c,1-c]}(x(t)+a_0h(t)) \geq M$ and $\inf_{t\in[c,1-c]}(y(t)+b_0k(t)) \geq M$. By $(J2)$, (10), (11), and the above inequalities, we deduce that $\|x\| > 0$ and $\|y\| > 0$.

Now by using Lemma 3 and the above considerations, we have

$$\begin{aligned} y(c) &\geq \gamma_2 \int_c^{1-c} J_2(s)b(s)g(x(s) + a_0h(s))\, ds \\ &\geq 2R\gamma_2 \int_c^{1-c} J_2(s)b(s) \inf_{\tau\in[c,1-c]}(x(\tau) + a_0h(\tau))\, ds \\ &\geq 2R\gamma_1\gamma_2 \int_c^{1-c} J_2(s)b(s)\|x + a_0h\|\, ds \geq 2\|x + a_0h\| \geq 2\|x\|. \end{aligned}$$

Therefore, we obtain

$$\|x\| \leq y(c)/2 \leq \|y\|/2. \tag{13}$$

In a similar manner, we deduce $x(c) \geq 2\|y + b_0k\| \geq 2\|y\|$. So, we obtain

$$\|y\| \leq x(c)/2 \leq \|x\|/2. \tag{14}$$

By (13) and (14), we conclude that $\|x\| \leq \|y\|/2 \leq \|x\|/4$, which is a contradiction, because $\|x\| > 0$. Then, for a_0 and b_0 sufficiently large, our problem (S)–(BC) has no positive solution. □

4 An Example

We consider $a(t) = 1$, $b(t) = 1$ for all $t \in [0,1]$, $\alpha = \frac{5}{2}$ $(n = 3)$, $\beta = \frac{10}{3}$ $(m = 4)$, $H(t) = \begin{cases} 0, & t \in [0,1/4), \\ 3, & t \in [1/4,3/4), \\ 7/2, & t \in [3/4,1] \end{cases}$ and $K(t) = t^4$ for all $t \in [0,1]$. Then $\int_0^1 u(s)\,dH(s) = 3u\left(\frac{1}{4}\right) + \frac{1}{2}u\left(\frac{3}{4}\right)$ and $\int_0^1 v(s)\,dK(s) = 4\int_0^1 s^3v(s)\,ds$. We also consider the functions $f, g : [0,\infty) \to [0,\infty)$, $f(x) = \frac{\tilde{a}x^{\alpha_0}}{x^{\beta_0}+\tilde{c}}$, $g(x) = \frac{\tilde{b}x^{\gamma_0}}{x^{\delta_0}+\tilde{d}}$, for all $x \in [0,\infty)$, with $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} > 0$, $\alpha_0, \beta_0, \gamma_0, \delta_0 > 0$, $\alpha_0 > \beta_0 + 1$, $\gamma_0 > \delta_0 + 1$. We have $\lim_{x\to\infty} f(x)/x = \lim_{x\to\infty} g(x)/x = \infty$.

Therefore, we consider the system of fractional differential equations

$$\begin{cases} D_{0+}^{5/2}u(t) + \dfrac{\tilde{a}v^{\alpha_0}(t)}{v^{\beta_0}(t) + \tilde{c}} = 0, & t \in (0,1), \\ D_{0+}^{10/3}v(t) + \dfrac{\tilde{b}u^{\gamma_0}(t)}{u^{\delta_0}(t) + \tilde{d}} = 0, & t \in (0,1), \end{cases} \tag{S_0}$$

with the boundary conditions

$$\begin{cases} u(0) = u'(0) = 0, \;\; u(1) = 3u\left(\frac{1}{4}\right) + \frac{1}{2}u\left(\frac{3}{4}\right) + a_0, \\ v(0) = v'(0) = v''(0) = 0, \;\; v(1) = 4\int_0^1 s^3 v(s)\, ds + b_0. \end{cases} \qquad (BC_0)$$

We obtain $\Delta_1 = 1 - \int_0^1 s^{3/2}\, dH(s) = 1 - 3\left(\frac{1}{4}\right)^{3/2} - \frac{1}{2}\left(\frac{3}{4}\right)^{3/2} = \frac{10-3\sqrt{3}}{16} \approx 0.3002 > 0$, $\Delta_2 = 1 - \int_0^1 s^{7/3}\, dK(s) = 1 - 4\int_0^1 s^{16/3}\, ds = \frac{7}{19} \approx 0.3684 > 0$. Then we deduce that assumptions $(J1)$, $(J2)$, and $(J4)$ are satisfied. In addition, after some computations, we obtain $\widetilde{A} = \int_0^1 J_1(s)\, ds \approx 0.42677595$, $\widetilde{B} = \int_0^1 J_2(s)\, ds \approx 0.04007233$, and then $L = \widetilde{A}$. We choose $c_0 = 1$, and if we select $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, $\tilde{d}$ satisfying the conditions $\tilde{a} < \frac{1+\tilde{c}}{L} = \frac{1+\tilde{c}}{\widetilde{A}}$, $\tilde{b} < \frac{1+\tilde{d}}{L} = \frac{1+\tilde{d}}{\widetilde{A}}$, then we conclude that $f(x) \le \frac{\tilde{a}}{1+\tilde{c}} < \frac{1}{L}$, $g(x) \le \frac{\tilde{b}}{1+\tilde{d}} < \frac{1}{L}$ for all $x \in [0, 1]$. For example, if $\tilde{c} = \tilde{d} = 1$, then for $\tilde{a} \le 4.68$ and $\tilde{b} \le 4.68$, the above conditions for f and g are satisfied. So, assumption $(J3)$ is also satisfied. By Theorems 1 and 2, we deduce that problem (S_0)–(BC_0) has at least one positive solution for sufficiently small $a_0 > 0$ and $b_0 > 0$ and no positive solution for sufficiently large a_0 and b_0.

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Reid's Construction of Minimal Principal Solution at Infinity for Linear Hamiltonian Systems

Peter Šepitka and Roman Šimon Hilscher

Abstract Recently the authors introduced a theory of principal solutions at infinity for nonoscillatory linear Hamiltonian systems in the absence of the complete controllability assumption. In this theory the so-called minimal principal solution at infinity plays a distinguished role (the minimality refers to the rank of the first component of the solution). In this paper we show that the minimal principal solution at infinity can be obtained by a suitable generalization of the Reid construction of the principal solution known in the controllable case. Our new result points to some applications of the minimal principal solution at infinity, e.g., in the spectral theory of linear Hamiltonian systems.

Keywords Linear Hamiltonian system • Principal solution at infinity • Antiprincipal solution at infinity • Minimal principal solution at infinity • Controllability • Moore–Penrose pseudoinverse

Mathematics Subject Classification (2000): 34C10

1 Introduction

In this paper we consider the linear Hamiltonian system

$$x' = A(t)\,x + B(t)\,u, \quad u' = C(t)\,x - A^T(t)\,u, \qquad t \in [a, \infty), \tag{H}$$

where $A, B, C : [a, \infty) \to \mathbb{R}^{n\times n}$ are piecewise continuous matrix-valued functions such that $B(t)$ and $C(t)$ are symmetric, and the Legendre condition

$$B(t) \geq 0 \quad \text{for all} \;\; t \in [a, \infty) \tag{1}$$

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holds. Here $n \in \mathbb{N}$ is a given dimension and $a \in \mathbb{R}$ is a fixed number. We will not impose any controllability (normality) assumption on system (H), i.e., vector solutions (x, u) of (H) may have $x(t) \equiv 0$ and $u(t) \neq 0$ on nondegenerate subintervals of $[a, \infty)$ or even throughout $[a, \infty)$. Similarly, matrix solutions (X, U) of (H) may have $X(t)$ singular on subintervals of $[a, \infty)$ with positive length. Such possibly abnormal linear Hamiltonian systems were originally studied in [12, 13, 21] and more recently in [6–8, 10, 11, 18, 19, 22]. We recall that in the abnormal case, the nonoscillation of system (H) means that every conjoined basis (X, U) of (H) has the kernel of $X(t)$ eventually constant, say on the interval $[\alpha, \infty)$ for some $\alpha \geq a$. In addition, in [15, Remark 5.4] we showed that $X(t)$ satisfies the estimate of its rank

$$n - d_\infty \leq n - d[\alpha, \infty) \leq \operatorname{rank} X(t) \leq n, \quad t \in [\alpha, \infty). \tag{2}$$

Here $d[\alpha, \infty)$ is the *order of abnormality* of system (H) on $[\alpha, \infty)$, i.e., it is the dimension of the space of solutions (x, u) of (H) with $x(t) \equiv 0$ on $[\alpha, \infty)$ (see also [13, Sect. VII.3]), and

$$d_\infty := \lim_{t \to \infty} d[t, \infty) = \max\{d[t, \infty),\ t \in [a, \infty)\}, \quad 0 \leq d_\infty \leq n.$$

In [15, 16] we initiated a general theory of principal solutions at infinity for such an abnormal system (H). By [16, Definition 7.1] a conjoined basis $(\hat{X}, \hat{U})$ of (H) is a *principal solution at infinity* if the kernel of $\hat{X}(t)$ is constant on $[\alpha, \infty)$ and

$$\lim_{t \to \infty} \hat{S}^\dagger(t) = 0, \quad \hat{S}(t) := \int_\alpha^t \hat{X}^\dagger(s)\, B(s)\, \hat{X}^{\dagger T}(s)\, \mathrm{d}s. \tag{3}$$

The dagger in (3) denotes the Moore–Penrose pseudoinverse [2, 3]. Based on (2) we then say $(\hat{X}, \hat{U})$ is a *minimal principal solution at infinity*, denoted by $(\hat{X}_{\min}, \hat{U}_{\min})$, if the rank of $\hat{X}_{\min}(t)$ is the smallest possible number, i.e., if $\operatorname{rank} \hat{X}_{\min}(t) = n - d_\infty$ on $[\alpha, \infty)$. By [15, Theorems 7.2 and 7.6], the nonoscillation of (H) is equivalent with the existence of a minimal principal solution at infinity, which is in this case unique up to a right nonsingular multiple. Moreover, in [16, Sect. 7], the solution $(\hat{X}_{\min}, \hat{U}_{\min})$ was utilized in order to construct all other principal solutions $(\hat{X}, \hat{U})$ at infinity with $\operatorname{rank} \hat{X}(t) = r$ on $[\alpha, \infty)$ for any integer r between $n - d_\infty$ and n. Most recently in [17, Theorem 6.5], we showed that the solution $(\hat{X}_{\min}, \hat{U}_{\min})$ is the smallest solution of (H) at infinity in the sense of the limit

$$\lim_{t \to \infty} X^\dagger(t)\, \hat{X}(t) = 0,$$

where (X, U) is any (antiprincipal) solution of (H) at infinity.

In the present paper we derive another important property of the minimal principal solution $(\hat{X}_{\min}, \hat{U}_{\min})$ at infinity, namely, its Reid construction. In the controllable case (i.e., when $d_\infty = 0$), the principal solution $(\hat{X}, \hat{U})$ at infinity has $\hat{X}(t)$ eventually invertible, and it can be constructed by the pointwise limit (the so-called Reid construction)

$$\big(\hat{X}(t), \hat{U}(t)\big) = \lim_{\tau\to\infty} \big(X_\tau(t), U_\tau(t)\big), \quad t \in [a, \infty), \tag{4}$$

where (X_τ, U_τ) is the conjoined basis of (H) given by the initial conditions $X_\tau(\tau) = 0$ and $U_\tau(\tau)$ invertible (see [4, p. 44] or [13, Theorem VII.3.4]). In our main result (Theorem 1), we derive the property in (4) for the minimal principal solution $(\hat{X}_{\min}, \hat{U}_{\min})$ at infinity of an abnormal system (H). This result completes the information about the solution $(\hat{X}_{\min}, \hat{U}_{\min})$, and it can be used in further applications, where the property in (4) was earlier applied (see Remark 6).

We note that already in [21, Definition, p. 40], Stokes uses (4) as the definition of a principal solution $(\hat{X}, \hat{U})$ at infinity for an abnormal system (H). The construction in [21, Theorem 1, p. 40] uses properties of symmetric solutions of the associated Riccati matrix equation on $[a, \infty)$, hence properties of conjoined bases (X, U) of (H) with $X(t)$ invertible. The approach in the present paper is direct and more general without using the Riccati equation. Moreover, our results in Theorem 1 and Remark 5 imply that a principal solution $(\hat{X}, \hat{U})$ in [21, Definition, p. 40] is in fact the minimal principal solution $(\hat{X}_{\min}, \hat{U}_{\min})$ at infinity in the sense of (3).

Our recent results in [17] also reveal more information about the conjoined basis (X_τ, U_τ) used in the Reid construction in (4). In particular, we prove that (X_τ, U_τ) is an *antiprincipal solution at infinity*, according to [17, Definition 5.1]. This is defined as a conjoined basis (X, U) of (H) such that for some $\alpha \geq a$ the kernel of $X(t)$ is constant on $[\alpha, \infty)$, $d[\alpha, \infty) = d_\infty$, and

$$\lim_{t\to\infty} S^\dagger(t) = T, \quad S(t) := \int_\alpha^t X^\dagger(s)\, B(s)\, X^{\dagger T}(s)\, \mathrm{d}s, \quad \operatorname{rank} T = n - d_\infty. \tag{5}$$

In general, by [17, Corollary 4.11], the matrix T in (5) satisfies $0 \leq \operatorname{rank} T \leq n - d_\infty$. Therefore, upon comparing (3) and (5), we can see that antiprincipal solutions of (H) at infinity are defined by the maximal possible rank of their associated matrix T in (5). This is in full agreement with the controllable case (see [1, 5]).

2 Main Result

In this section we state and prove the main result of this paper. First we recall some notions and results about linear Hamiltonian systems and their solutions (see [4, 9, 13]). A matrix solution (X, U) is a *conjoined basis* of (H) if $X^T(t)\, U(t)$ is symmetric and $\operatorname{rank}(X^T(t),\ U^T(t)) = n$ for some and hence for all $t \in [a, \infty)$. Every conjoined basis (X, U) of (H) can be completed by another conjoined basis $(\bar{X}, \bar{U})$ to form a symplectic fundamental matrix of (H). In this case the two conjoined bases (X, U) and $(\bar{X}, \bar{U})$ are *normalized*, i.e., their Wronskian $X^T(t)\, \bar{U}(t) - U^T(t)\, \bar{X}(t) \equiv I$ on $[a, \infty)$. For a conjoined basis (X, U) of (H), we define its associated orthogonal projectors onto the subspaces $\operatorname{Im} X^T(t)$ and $\operatorname{Im} X(t)$ by

$$P(t) := X^\dagger(t)\, X(t), \quad R(t) := X(t)\, X^\dagger(t), \quad t \in [a, \infty). \tag{6}$$

By the *kernel* of (X, U), we mean the kernel of its first component X. If (X, U) has constant kernel on $[\alpha, \infty)$, then $P(t)$ is constant on $[\alpha, \infty)$, and we set

$$P := P(t) \quad \text{on } [\alpha, \infty). \tag{7}$$

In this case we have $r := \operatorname{rank} X(t) = \operatorname{rank} P = \operatorname{rank} R(t)$ on $[\alpha, \infty)$, and we say that (X, U) has *rank* r. The matrix-valued function $X^\dagger$ is then piecewise continuously differentiable on $[\alpha, \infty)$; in particular $X^\dagger$ is continuous on $[\alpha, \infty)$, so that the function $S(t)$ in (5) is well defined for any such a conjoined basis (X, U). It follows that under (1) the matrix $S(t)$ is symmetric, nonnegative definite, piecewise continuously differentiable on $[\alpha, \infty)$, and the set $\operatorname{Im} S(t)$ is nondecreasing and hence eventually constant with $\operatorname{Im} S(t) \subseteq \operatorname{Im} P$ (see [15, Theorem 4.2]). Therefore, the orthogonal projector $P_{\mathscr{S}}(t)$ onto the set $\operatorname{Im} S(t)$ is eventually constant, and we write

$$P_{\mathscr{S}}(t) := S(t)\, S^\dagger(t) = S^\dagger(t)\, S(t), \quad P_{\mathscr{S}\infty} := P_{\mathscr{S}}(t) \quad \text{for } t \to \infty. \tag{8}$$

In addition, we have the inclusions $\operatorname{Im} S(t) = \operatorname{Im} P_{\mathscr{S}}(t) \subseteq \operatorname{Im} P_{\mathscr{S}\infty} \subseteq \operatorname{Im} P$ on $[\alpha, \infty)$.

Remark 1. Let (X, U) be a conjoined basis of (H) with constant kernel on $[\alpha, \infty)$. In this paper we will utilize a special conjoined basis $(\bar{X}, \bar{U})$ satisfying

$$X^T(t)\, \bar{U}(t) - U^T(t)\, \bar{X}(t) \equiv I, \quad X^\dagger(\alpha)\, \bar{X}(\alpha) = 0. \tag{9}$$

This means that the two conjoined bases (X, U) and $(\bar{X}, \bar{U})$ are normalized. The existence of $(\bar{X}, \bar{U})$ is proven in [15, Theorem 4.4].

Following (2) we say that a conjoined basis (X, U) of (H) is a *minimal conjoined basis* on $[\alpha, \infty)$ if (X, U) has constant kernel on $[\alpha, \infty)$ and $\operatorname{rank} X(t) = n - d[\alpha, \infty)$. Similarly, (X, U) is a *maximal conjoined basis* on $[\alpha, \infty)$ if $X(t)$ is invertible on $[\alpha, \infty)$. Therefore, a minimal/maximal principal solution $(\hat{X}_{\min}, \hat{U}_{\min})$ of (H) at infinity is defined as a minimal/maximal conjoined basis on $[\alpha, \infty)$ for some $\alpha \geq a$, for which condition (3) is satisfied. Analogous terminology is used for minimal/maximal antiprincipal solutions of (H) at infinity. It is easy to see that minimal conjoined bases (X, U) of (H) are characterized by the property that their associated projectors P and $P_{\mathscr{S}\infty}$ in (7) and (8) satisfy the equality $P = P_{\mathscr{S}\infty}$ (see [15, Remark 5.14]). Minimal conjoined bases are extremely important, since they serve as a tool for the construction of the minimal principal solution $(\hat{X}_{\min}, \hat{U}_{\min})$ at infinity. More precisely, by the proof of [15, Theorem 7.2],

$$\begin{pmatrix} \hat{X}_{\min}(t) \\ \hat{U}_{\min}(t) \end{pmatrix} = \begin{pmatrix} X(t) & \bar{X}(t) \\ U(t) & \bar{U}(t) \end{pmatrix} \begin{pmatrix} I \\ -T \end{pmatrix}, \quad t \in [a, \infty), \tag{10}$$

where (X, U) is a minimal conjoined basis on $[\alpha, \infty)$, $(\bar{X}, \bar{U})$ is the associated conjoined basis from (9), and $T := \lim_{t\to\infty} S^{\dagger}(t)$.

By [16, Remark 7.11] and [17, Theorem 5.5], the property of being a principal or an antiprincipal solution of (H) at infinity is invariant under the translation of the initial point α to the right. Therefore, without loss of generality we will assume further on in this paper that the point α is such that the abnormality of system (H) on $[\alpha, \infty)$ is maximal, i.e.,

$$d[\alpha, \infty) = d_{\infty}. \tag{11}$$

In the following theorem we present the main result of this paper.

Theorem 1. *Assume* (1) *and* (11) *and suppose that* (X, U) *is a minimal conjoined basis of* (H) *on* $[\alpha, \infty)$*. Let* $(\bar{X}, \bar{U})$ *be the associated conjoined basis of* (H) *satisfying* (9)*. Then there exists* $\beta \geq \alpha$ *such that* $\bar{X}(t)$ *is invertible for all* $t \geq \beta$*, and the solution* (X_τ, U_τ) *of* (H) *given by the initial conditions*

$$X_\tau(\tau) = 0, \quad U_\tau(\tau) = -\bar{X}^{T-1}(\tau), \tag{12}$$

is a conjoined basis satisfying

$$\big(\hat{X}_{\min}(t), \hat{U}_{\min}(t)\big) = \lim_{\tau\to\infty} \big(X_\tau(t), U_\tau(t)\big), \quad t \in [a, \infty). \tag{13}$$

Proof. We start by deriving some additional properties of the conjoined basis $(\bar{X}, \bar{U})$. By [15, Theorem 4.4] and [14, Theorem 2.2.11], the functions $\bar{X}(t)$, $\bar{X}(t)\, P$, and $\bar{U}(t)\, P$ are uniquely determined by (X, U) on $[\alpha, \infty)$, and on this interval

$$\bar{X}(t)\, P = X(t)\, S(t), \quad \bar{U}(t)\, P = U(t)\, S(t) + X^{\dagger T}(t) + U(t)\, (I - P)\, \bar{X}^T(t)\, X^{\dagger T}(t), \tag{14}$$

$$\operatorname{Ker} \bar{X}(t) = \operatorname{Im}\, [P - P_{\mathscr{S}}(t)], \quad S^{\dagger}(t) = \bar{X}^{\dagger}(t)\, X(t)\, P_{\mathscr{S}}(t). \tag{15}$$

Since (X, U) is a minimal conjoined basis on $[\alpha, \infty)$, we have $P = P_{\mathscr{S}\infty}$, and with

$$\beta := \inf\{t \geq \alpha,\ \operatorname{rank} S(t) = n - d_\infty\} \tag{16}$$

it follows that $P_{\mathscr{S}}(t) = P_{\mathscr{S}\infty}$ on (β, ∞). From the first equality in (15), we then obtain that $\operatorname{Ker} \bar{X}(t) = \{0\}$, so that $\bar{X}(t)$ is invertible for all $t > \beta$. Moreover, from the second condition in (15), we then get

$$S^{\dagger}(t) = \bar{X}^{-1}(t)\, X(t), \quad t \in (\beta, \infty). \tag{17}$$

Fix now $\tau > \beta$ and define on $[a, \infty)$ the solution (X_τ, U_τ) of (H) by the initial conditions (12). It is easy to see that (X_τ, U_τ) is a conjoined basis of (H). Consider the matrices M_τ and N_τ such that

$$\begin{pmatrix} X_\tau(t) \\ U_\tau(t) \end{pmatrix} = \begin{pmatrix} X(t) & \bar{X}(t) \\ U(t) & \bar{U}(t) \end{pmatrix} \begin{pmatrix} M_\tau \\ N_\tau \end{pmatrix}, \quad t \in [a, \infty). \tag{18}$$

Since by (9) the fundamental matrix in (18) is symplectic, it follows that

$$M_\tau = \bar{U}^T(t)\, X_\tau(t) - \bar{X}^T(t)\, U_\tau(t), \quad N_\tau = X^T(t)\, U_\tau(t) - U^T(t)\, X_\tau(t) \tag{19}$$

on $[a, \infty)$. Evaluating (19) at $t = \tau$ and using (12) yields that

$$M_\tau = I, \quad N_\tau = -X^T(\tau)\, \bar{X}^{T-1}(\tau) \overset{(17)}{=} -[S^\dagger(\tau)]^T = -S^\dagger(\tau). \tag{20}$$

This shows that the limit of (M_τ, N_τ) as $\tau \to \infty$ indeed exists, and by (20) it is equal to $(I, -T)$. Hence, from (18) we get that for each $t \in [a, \infty)$, the limit of $(X_\tau(t), U_\tau(t))$ as $\tau \to \infty$ also exists, and by (10) it is equal to $(\hat{X}_{\min}(t), \hat{U}_{\min}(t))$. Therefore, (13) holds and the proof is complete. □

Remark 2. We note that by [15, Remark 5.3], the point β in (16) also satisfies

$$\beta = \inf\{t \geq \alpha,\ (\hat{X}_\alpha, \hat{U}_\alpha) \text{ has constant kernel on } [t, \infty)\},$$

where $(\hat{X}_\alpha, \hat{U}_\alpha)$ is the principal solution of (H) at the point α, i.e., it is the solution given by the initial conditions $\hat{X}_\alpha(\alpha) = 0$ and $\hat{U}_\alpha(\alpha) = I$.

Remark 3. The proof of Theorem 1 gives also the answer to the question, when the limit in (13) exists and what is its value depending on the chosen initial conditions in (12). In this respect we have the following result. Assume that (X_τ, U_τ) is a solution of (H) given by the initial conditions $X_\tau(\tau) = 0$ and $U_\tau(\tau) = K(\tau)$, where $K(\tau)$ is invertible for all $\tau \geq \gamma$ for some $\gamma > \beta$. Then for $t \in [a, \infty)$ the limit of $\big(X_\tau(t), U_\tau(t)\big)$ as $\tau \to \infty$ exists if and only if $K(\tau) = -\bar{X}^{T-1}(\tau)\, E(\tau)$ for $\tau \geq \gamma$, where $E(\tau)$ is an invertible matrix such that the limit $E := \lim_{\tau\to\infty} E(\tau)$ exists. In this case

$$\lim_{\tau\to\infty} \big(X_\tau(t), U_\tau(t)\big) = \big(\hat{X}_{\min}(t)\, E, \hat{U}_{\min}(t)\, E\big), \quad t \in [a, \infty),$$

where $(\hat{X}_{\min}, \hat{U}_{\min})$ is the minimal principal solution of (H) at infinity from (10). This statement follows from the proof of Theorem 1, in which the matrices M_τ and N_τ are given by $M_\tau = E(\tau)$ and $N_\tau = -S^\dagger(\tau)\, E(\tau)$.

Remark 4. It follows by [17, Proposition 5.15] that for $\tau > \beta$ the conjoined basis (X_τ, U_τ) used in the Reid construction of $(\hat{X}_{\min}, \hat{U}_{\min})$ in Theorem 1 is a (minimal) antiprincipal solution of (H) at infinity.

Formula (17) implies an interesting property of the conjoined basis $(\bar{X}, \bar{U})$.

Proposition 1. *Let* (X, U) *be a minimal conjoined basis of* (H) *on* $[\alpha, \infty)$*. Then the associated conjoined basis* $(\bar{X}, \bar{U})$ *from Remark 1 is a maximal antiprincipal solution of* (H) *at infinity.*

Proof. Let $\beta \geq \alpha$ be such that $\bar{X}(t)$ is invertible on $[\beta, \infty)$. Let $\bar{S}(t)$ be defined as in (5) through $\bar{X}(t)$ on $[\beta, \infty)$. By [17, Theorem 5.3 and Remark 5.4], it is enough to show that the matrix $\bar{S}(t)$ has a limit as $t \to \infty$. From (17) we obtain that $S^{\dagger}(t)\, X^{\dagger}(t) = \bar{X}^{-1}(t)\, R(t)$ on $[\beta, \infty)$, while from [15, Theorem 4.2(ii)], we get $R(t)\, B(t) = B(t) = B(t)\, R(t)$. Hence, for $t \in [\beta, \infty)$ we have

$$\begin{aligned}\bar{S}(t) &= \int_{\beta}^{t} \bar{X}^{-1}(s)\, B(s)\, \bar{X}^{T-1}(s)\, \mathrm{d}s = \int_{\beta}^{t} S^{\dagger}(s)\, X^{\dagger}(s)\, B(s)\, X^{\dagger T}(s)\, S^{\dagger}(s)\, \mathrm{d}s \\ &= \int_{\beta}^{t} S^{\dagger}(s)\, S'(s)\, S^{\dagger}(s)\, \mathrm{d}s = -\int_{\beta}^{t} [S^{\dagger}(s)]'\, \mathrm{d}s = S^{\dagger}(\beta) - S^{\dagger}(t).\end{aligned}$$

This shows that $\lim_{t\to\infty} \bar{S}(t) = S^{\dagger}(\beta) - T$, i.e., this limit exists, and by [17, Remark 5.4] the conjoined basis $(\bar{X}, \bar{U})$ is a (maximal) antiprincipal solution of (H) at infinity. □

In the following remark we prove that the approach in [21, Theorem 1, p. 40] is a special case of our construction in Theorem 1 and Remark 3. Moreover, in view of Proposition 1, we may conclude that the choice of the initial conditions in (12) with $(\bar{X}, \bar{U})$ being a maximal antiprincipal solution of (H) at infinity is natural and the only possible in order to guarantee the existence of the limit in (13).

Remark 5. The comments in Remarks 3 and 4 lead to an explanation of the construction of a principal solution of (H) at infinity by Stokes in [21, Theorem 1, p. 40]. Let $(\tilde{X}, \tilde{U})$ be a conjoined basis of (H) such that $\tilde{X}(t)$ is invertible for all $t \geq \gamma$ for some $\gamma \geq \alpha$ as in [21, pp. 39–40]. For $\tau \geq \gamma$, let (X_τ, U_τ) be the solution of (H) given by the initial conditions $X_\tau(\tau) = 0$ and $U_\tau(\tau) = \tilde{X}^{T-1}(\tau)$. Note that Stokes uses the notation (Y, Z) and (Y_c, Z_c) instead of our $(\tilde{X}, \tilde{U})$ and (X_τ, U_τ). Then by applying Remark 3 with $K(\tau) := \tilde{X}^{T-1}(\tau)$ and $E(\tau) := -\bar{X}^{T}(\tau)\, \tilde{X}^{T-1}(\tau)$, we shall prove that

$$\left.\begin{array}{c}\lim_{\tau\to\infty}(X_\tau, U_\tau) \text{ exists if and only if} \\ (\tilde{X}, \tilde{U}) \text{ is a (maximal) antiprincipal solution at infinity.}\end{array}\right\} \tag{21}$$

And in this case the limit in (21) is the minimal principal solution at infinity. Assume first that $(\tilde{X}, \tilde{U})$ is a (maximal) antiprincipal solution at infinity. Then by [16, Theorem 6.3 and Remark 6.4] (with the maximal genus $\mathscr{G}_{\max}$, since $\tilde{X}(t)$ and $\bar{X}(t)$ are invertible for large t)

$$\lim_{\tau\to\infty}(-M_\tau^T) = \lim_{\tau\to\infty} \tilde{X}^{-1}(\tau)\, \bar{X}(\tau) = L \quad \text{with} \quad \operatorname{rank} L = \operatorname{rank} \bar{T} + d_\infty, \tag{22}$$

where $\bar{T}$ is the matrix in (5) associated with $(\bar{X}, \bar{U})$. From Proposition 1 we know that $(\bar{X}, \bar{U})$ is a (maximal) antiprincipal solution at infinity, i.e., $\operatorname{rank} \bar{T} = n - d_\infty$. Hence, the matrix L in (22) is invertible. Therefore, the limit in (21) exists and, by Remark 3 with $E := -L^T$ being invertible, it is equal to the minimal principal solution of (H) at infinity. Conversely, assume that the limit in (21) exists, i.e.,

$$E := \lim_{\tau\to\infty} E(\tau) = -\lim_{\tau\to\infty} \bar{X}^T(\tau)\, \tilde{X}^{T-1}(\tau) \tag{23}$$

exists. From the fact that $(\bar{X}, \bar{U})$ is a maximal antiprincipal solution at infinity (by Proposition 1) and from [16, Theorem 6.3 and Remark 6.4], it follows that the limit

$$F := \lim_{\tau\to\infty} \bar{X}^{-1}(\tau)\, \tilde{X}(\tau) = -\lim_{\tau\to\infty} E^{T-1}(\tau) \quad \text{with} \quad \operatorname{rank} F = \operatorname{rank} \tilde{T} + d_\infty \tag{24}$$

also exists, where $\tilde{T}$ is defined in (5) through $\tilde{X}(t)$ on $[\beta, \infty)$. From (23) and (24) we then conclude that E and F are invertible with $E = -F^{T-1}$, which in turn implies by Remark 3 that the limit in (21) is the minimal principal solution of (H) at infinity. Finally, the second condition in (24) implies that $\operatorname{rank} \tilde{T} = \operatorname{rank} F - d_\infty = n - d_\infty$, so that $(\tilde{X}, \tilde{U})$ is a maximal antiprincipal solution at infinity.

3 Comments and Concluding Remarks

In this section we make some additional observations and comments to the main result of this paper and future research directions.

Remark 6. The construction of the minimal principal solution of (H) at infinity in Theorem 1 can be utilized in applications in the same way as for the controllable system (H). For example, we expect that this will be possible in the Friedrichs extension for symmetric operators associated with the linear Hamiltonian system (H) (see [20, Theorem 3.1]) and its proof. This topic is under our investigation.

Remark 7. In this paper we show that the minimal principal solution of (H) at infinity can be represented by the Reid construction in (13). It is an open problem whether other principal solutions of (H) at infinity (i.e., principal solutions at infinity with rank strictly bigger than $n - d_\infty$) have a similar representation. We will address this problem in our future work.

In the last part of this paper, we comment the construction in Theorem 1 with respect to the used initial data. Since the representation of (X_τ, U_τ) in (18) with $M_\tau = I$ and $N_\tau = -S^\dagger(\tau)$ from (20) does not depend on a particular choice of $(\bar{X}, \bar{U})$, the construction of $(\hat{X}_{\min}, \hat{U}_{\min})$ in (13) also does not depend on the choice of $(\bar{X}, \bar{U})$ for a given minimal conjoined basis (X, U). In the next result we will show that the above construction of $(\hat{X}_{\min}, \hat{U}_{\min})$ does not depend on the choice of the minimal conjoined basis (X, U). More precisely, we will show that starting with a different minimal conjoined basis (X, U) leads to a right nonsingular multiple of $(\hat{X}_{\min}, \hat{U}_{\min})$ in (13). For this purpose we first derive an auxiliary lemma.

Lemma 1. *Let* (X_1, U_1) *and* (X_2, U_2) *be minimal conjoined bases of* (H) *on* $[\alpha, \infty)$*, and let* $(\bar{X}_1, \bar{U}_1)$ *and* $(\bar{X}_2, \bar{U}_2)$ *be their associated conjoined bases from Remark 1, which satisfy* $X_i^T(t)\,\bar{U}_i(t) - U_i^T(t)\,\bar{X}_i(t) \equiv I$ *and* $X_i^\dagger(\alpha)\,\bar{X}_i(\alpha) = 0$ *for* $i \in \{1, 2\}$*. Then there exists a constant invertible matrix K such that*

$$\bar{X}_2(t) = \bar{X}_1(t)\,K, \quad t \in [\alpha, \infty). \tag{25}$$

Proof. Let $P_1, R_1(t)$ and $P_2, R_2(t)$ be the orthogonal projectors in (7), (6) associated with X_1 and X_2, respectively, First we note that $\operatorname{Im} X_1(\alpha) = \operatorname{Im} X_2(\alpha)$, by [15, Remark 5.16], so that $R_1(\alpha) = R_2(\alpha)$. Next we represent (X_1, U_1) in terms of (X_2, U_2) and $(\bar{X}_2, \bar{U}_2)$ and both (X_2, U_2) and $(\bar{X}_2, \bar{U}_2)$ in terms of (X_1, U_1) and $(\bar{X}_1, \bar{U}_1)$. Thus, we have for $i \in \{1, 2\}$

$$\begin{pmatrix} X_{3-i}(t) \\ U_{3-i}(t) \end{pmatrix} = \begin{pmatrix} X_i(t) & \bar{X}_i(t) \\ U_i(t) & \bar{U}_i(t) \end{pmatrix} \begin{pmatrix} M_i \\ N_i \end{pmatrix}, \quad \begin{pmatrix} \bar{X}_2(t) \\ \bar{U}_2(t) \end{pmatrix} = \begin{pmatrix} X_1(t) & \bar{X}_1(t) \\ U_1(t) & \bar{U}_1(t) \end{pmatrix} \begin{pmatrix} \bar{M}_1 \\ \bar{N}_1 \end{pmatrix} \tag{26}$$

on $[a, \infty)$, where according to [15, Theorem 4.6], the matrices M_1 and M_2 are invertible with $M_2 = M_1^{-1}$ and $N_2 = -N_1^T$. Since the fundamental matrices in (26) are symplectic, we get from (26) at $t = \alpha$

$$\bar{M}_1 = \bar{U}_1^T(\alpha)\,\bar{X}_2(\alpha) - \bar{X}_1^T(\alpha)\,\bar{U}_2(\alpha), \tag{27}$$

$$\bar{N}_1 = X_1^T(\alpha)\,\bar{U}_2(\alpha) - U_1^T(\alpha)\,\bar{X}_2(\alpha) = M_2^T = M_1^{T-1}. \tag{28}$$

Therefore, $\bar{N}_1$ is invertible. Since by (14) at $t = \alpha$ we have $P_1\bar{U}_1^T(\alpha) = X_1^\dagger(\alpha)$ and $P_1\bar{X}_1^T(\alpha) = S_1(\alpha)\,X_1^T(\alpha) = 0$ and $R_2(\alpha)\,\bar{X}_2(\alpha) = X_2(\alpha)\,X_2^\dagger(\alpha)\,\bar{X}_2(\alpha) = 0$, it follows from (27) that

$$P_1\bar{M}_1 = X_1^\dagger(\alpha)\,\bar{X}_2(\alpha) = X_1^\dagger(\alpha)\,R_1(\alpha)\,\bar{X}_2(\alpha) = X_1^\dagger(\alpha)\,R_2(\alpha)\,\bar{X}_2(\alpha) = 0.$$

Therefore, again by (26) we get for $t \in [\alpha, \infty)$

$$\bar{X}_2(t) = X_1(t)\,\bar{M}_1 + \bar{X}_1(t)\,\bar{N}_1 = X_1(t)\,P_1\bar{M}_1 + \bar{X}_1(t)\,\bar{N}_1 = \bar{X}_1(t)\,\bar{N}_1.$$

This shows that (25) holds with $K := \bar{N}_1$, which is by (28) invertible. □

Assume now that in addition to (X, U) in Theorem 1, we start with another minimal conjoined basis (X_*, U_*) of (H) on $[\alpha, \infty)$. Let $(\bar{X}_*, \bar{U}_*)$ be the associated conjoined basis from Remark 1. If we represent (X_*, U_*) in terms of (X, U) by

$$\begin{pmatrix} X_*(t) \\ U_*(t) \end{pmatrix} = \begin{pmatrix} X(t) & \bar{X}(t) \\ U(t) & \bar{U}(t) \end{pmatrix} \begin{pmatrix} M \\ N \end{pmatrix}, \quad t \in [a, \infty),$$

where M is invertible (see [15, Theorem 4.6] again), then by Lemma 1 we have $\bar{X}_*(t) = \bar{X}(t)\, K$ on $[\alpha, \infty)$ with invertible $K := M^{T-1}$. Following (12) we construct for $\tau \geq \beta$ the conjoined basis $(X_{*\tau}, U_{*\tau})$ by the initial conditions $X_{*\tau}(\tau) = 0$ and $U_{*\tau}(\tau) = -\bar{X}_*^{T-1}(\tau) = -\bar{X}^{T-1}(\tau)\, M$. Hence, we have $X_{*\tau}(t) = X_\tau(t)\, M$ and $U_{*\tau}(t) = U_\tau(t)\, M$ on $[a, \infty)$, which implies that for $t \in [a, \infty)$

$$\lim_{\tau\to\infty} \big(X_{*\tau}(t), U_{*\tau}(t)\big) = \lim_{\tau\to\infty} \big(X_\tau(t)\, M, U_\tau(t)\, M\big) = \big(\hat{X}_{\min}(t)\, M, \hat{U}_{\min}(t)\, M\big).$$

Finally, let us mention that the construction of $(\hat{X}_{\min}, \hat{U}_{\min})$ in (13) does not also depend on the choice of the point α, which defines the interval $[\alpha, \infty)$ on which (X, U) is a minimal conjoined basis. More precisely, similarly to the above we get that moving the point α to the right yields a constant right nonsingular multiple in the representation (13). The analysis of this problem goes however beyond the scope and length of this paper, so that the details are omitted.

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On Numerical Regularity of Trisection-Based Algorithms in 3D

Sergey Korotov, Ángel Plaza, José P. Suárez, and Pilar Abad

Abstract The longest-edge (LE-) trisection of the given tetrahedron is obtained by joining two equally spaced points on its longest edge with the opposite vertices, and, thus, splitting the tetrahedron into three sub-tetrahedra. On the base such LE-trisections we introduce and numerically test the refinement algorithms for tetrahedral meshes. Computations conducted show that the quality of meshes generated by these algorithms does not seem to degenerate.

Keywords Longest-edge trisection • Tetrahedral mesh • Mesh adaptivity • Mesh regularity

1 Introduction

Theoretical results regarding the regularity properties of tetrahedral meshes generated are still very limited in comparison to the vast amount of empirical studies and tests performed for this type of meshes in the mesh generation community. The complexity of methods to subdivide tetrahedra and an existence of many ways to performs such subdivisions are only two possible reasons. For example, Delaunay-type refinement in 3D is much more complicated than that one in 2D, and no mathematical guarantee exists for the shape quality of subtetrahedra produced [2]. Concerning the longest-edge (LE-) based refinements, whether the shapes of subtetrahedra produced by repeated LE-bisections degenerate or not is still an open problem [13]. However, many practical implementations have shown that the

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LE-bisection is a good choice for qualitative mesh refinements [9, 22, 25]. Various shape measures, currently used as effective quantitative means for evaluating the quality of meshes, can be found in [8] (see also [5, 6, 15]).

There are currently two main approaches for subdividing a single tetrahedron: octasection, sometimes called the 3D red refinement (see, e.g., [10, 28]), and bisection (also called the green refinement) (see also [12] for a combination of these techniques). The octasection may be preferable to bisection in the cases where the initial mesh is relatively coarse and a high ratio of element explosion is needed. However, the octasection-based algorithms are, in general, quite complicated to implement, and the processing time required may result in a poor efficiency. Even for adaptive processes, where it is aimed to produce meshes that optimally control a measure of the error, those methods based on tetrahedral octasection may be prohibitive, because they rarely perform satisfactory in a real time.

Typically, the main goal in the mesh generation is a generation of well-shaped mesh elements and at the same time providing with an optimal distribution of points within the mesh. However, no formal definition of optimality can be tracked as the mesh is refined [14].

Some algorithms based on the LE-bisections were developed to overcome with simplicity and efficiency the problem of mesh refinement and obtaining qualitative tetrahedral meshes. In [25], Rivara and Levin introduced a pure three-dimensional LE-refinement scheme. Carey et al. in [20, 21] also developed a LE-based algorithm for evolution problems. See also [9, 26] for many numerical tests for conforming version of LE-bisections. All mentioned above algorithms guarantee the construction of good-quality tetrahedral meshes. Empirical experimentation was provided showing that the solid angles decrease slowly when the algorithm is iteratively applied and that even a quality improvement behavior holds, similarly to the two-dimensional case. Their algorithms have been successfully used in practice for the finite element method (FEM) and multigrid applications even though no formal proof exists on the nondegeneracy property. It should be noted that there are other similar algorithms for local tetrahedral refinement, like those by Bänsch [3], Kossaczký [11], Maubach [17], or Mukherjee et al. [1]. They are however not purely LE-based algorithms.

Less attention has been given to LE-subdivisions based on the insertion of two equidistant points on the longest edge of tetrahedra, which is also known as the 3-tetrahedra longest edge (3T-LE) or simply the LE-trisection [23] (see Fig. 1 for an illustration).

In this paper we extend to three dimensions the study of the nondegeneracy property of the LE-trisection presented earlier for the two-dimensional case in [23]. We introduce an algorithm for the refinement of tetrahedral meshes based on the LE-trisection. The proposed algorithm can be viewed as an extension to three dimensions of the recently published algorithm in 2D [24]. As for the LE-bisection, the algorithm is based on equally division of the longest edge of the tetrahedron but now in three parts. The methodology used to assert that the proposed refinement scheme does not seem degenerate is similar to that in [9, 22, 25].

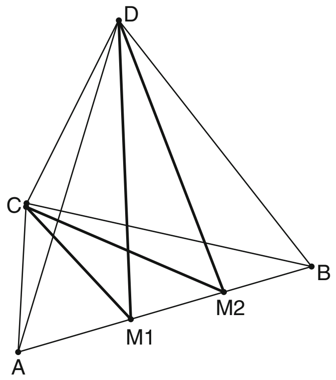

Fig. 1 Longest-edge trisection of a tetrahedron

We conduct a number of numerical experiments that allow us to state the conjecture that the nondegeneracy property of the LE-trisection holds in 3D when the iterative refinement is applied to unstructured tetrahedral meshes.

The paper is organized as follows. In Sect. 2 we present the nondegeneracy study of the LE-trisection algorithm for tetrahedral meshes. An empirical study is carried out together with the evaluation of the mesh quality by means of several shape measures. Section 3 introduces and discusses a new algorithm for the refinement of unstructured tetrahedral meshes based on the LE-trisection. The algorithm is then tested through a local refinement scenario. Finally, in the last section some final remarks are given, and some guidelines are presented in order to motivate further research on the refinements based on the LE-trisections.

2 Nondegeneracy of the LE-Trisection Algorithm

In this section, we present the results of numerical experiments, where we monitor whether the iterative application of the LE-trisection algorithm to tetrahedral meshes generates nondegenerated elements. In each of the tests presented, we iteratively applied the algorithm until getting a reasonably high number (approximately, a million and a half) of sub-tetrahedra in the final meshes.

To judge on the quality of the meshes, various shape measures are proposed in the literature. Normally, a shape measure is a continuous function [8, 15, 19, 25], which has to be invariant under translation, rotation, reflection, and uniform scaling of the tetrahedron and which takes the maximum value for the regular tetrahedron and the minimum ones—for degenerate tetrahedra. In our experiments we will use the following four shape measures:

1. In [16] Liu and Joe introduced the quality indicator for a tetrahedron t as $\eta(t) = \frac{12\ (3\ \text{volume}(t))^{2/3}}{\sum_{i=1}^{6} l_i^2}$, where $l_1, \ldots, l_6$ are the lengths of six edges of t.

2. In [27] Whitehead introduced the *relative thickness* of a tetrahedron t as $\frac{\rho(t)}{\delta(t)}$, where $\rho(t)$ stands for the distance from the centroid of a tetrahedron t to its boundary, and $\delta(t)$ is the diameter of t.
3. The ratio $\frac{r(t)}{R(t)}$, where $r(t)$ is the radius of the inscribed sphere and $R(t)$ the radius of the circumscribed sphere around a tetrahedron t.
4. The solid angle at a vertex of a tetrahedron is related to the sum of the three dihedral angles associated with the edges incident on that vertex. The solid angle is a representative measure for the quality of the tetrahedral meshes.

Let t be some given tetrahedron to which the LE-trisection refinement is applied. Then, after trisecting the initial mesh $\tau_1 = \{t = t_1^1\}$, a new mesh $\tau_2 = \{t_i^2\}$ is obtained. The successive application of the LE-trisection scheme to any tetrahedron and its successors yields an infinite sequence of tetrahedral meshes denoted as $\tau_1, \tau_2, \tau_3, \ldots$.

In what follows, we will focus on the Liu–Joe shape measure as in [18, 22]. We give experimental evidence showing the convergence of the standard shape measure η to a fixed positive value. This indicates that the LE-trisection algorithm does not produce degenerating meshes, which can be further used as an argument to mathematically investigate the validity of the nondegeneracy property for such an algorithm. Namely, we show that for any tetrahedron $t_i^n \in \tau_n$, $n \geq 1$, it holds

$$\eta\left(t_i^n\right) \geq c\eta(t),$$

where c is a positive constant. To effectively approximate such a constant c, we define $\eta\left(\tau_j\right) = \min_i\{\eta\left(t_i^j\right)\}$ and compute values

$$c_j = \frac{\eta\left(\tau_j\right)}{\eta\left(\tau_1\right)},$$

where $j = 1, \ldots, 13$. In the final meshes we have $1,594,323$ tetrahedra. The value c_{13} is taken then as an estimation of the constant of interest c.

We will study the refinements made for the input tetrahedra that are, in some sense, representative for the overall mesh quality [4, 19]. Several known tetrahedral shapes have been satisfactory used to test various meshing algorithms (see e.g. [16, 18, 22, 25]). We give descriptive names to the five different element types commonly used in this respect. The *round* tetrahedron is that one which has no bad angles of any kind (the main example is the regular tetrahedron). The *needle* (or *thin*) is such a tetrahedron which has one small solid angle. The *wedge-like* element has small but not large dihedrals and no large angles of any kind. The *sliver* is a tetrahedron which has small and large dihedrals, but no large solid angle. The *cap-like* tetrahedron has a large (nearly flat) solid angle. For the cap tetrahedron, the circumscribed sphere's radius is hence much larger than the longest edge. Table 1 reports on the exact coordinates of vertices of five tetrahedra used in our tests.

Table 1 The xyz coordinates of each tetrahedron used in the numerical experiments

Round	Needle	Wedge	Sliver	Cap
0.00 0.00 0.00	0.00 0.00 00.00	0.00 0.00 0.00	0.00 0.00 0.00	0.00 0.00 0.00
1.73 1.00 2.82	1.73 1.00 56.56	0.08 0.50 0.14	1.73 -1.81 0.22	1.73 1.00 0.14
1.73 3.00 0.00	1.73 3.00 00.00	1.73 3.00 0.00	1.73 3.00 0.00	1.73 3.00 0.00
3.46 0.00 0.00	3.46 0.00 00.00	3.46 0.00 0.00	3.46 0.00 0.00	3.46 0.00 0.00

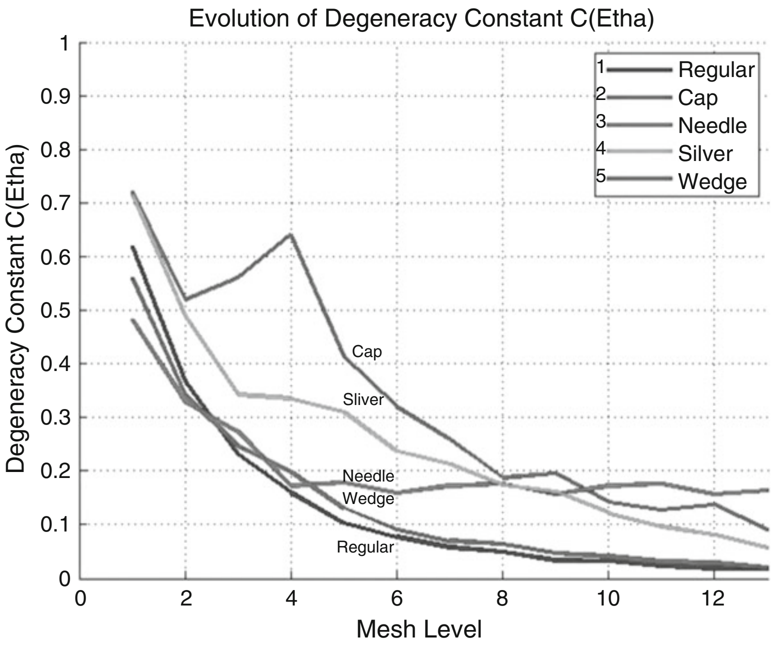

Fig. 2 Dynamics of $c_j = \frac{\eta(\tau_j)}{\eta(\tau_1)}$, $j = 1, \ldots, 13$, for five selected tetrahedra

In Fig. 2 we plot the evolution of $c_j = \frac{\eta(\tau_j)}{\eta(\tau_1)}$ for $j = 1, \ldots, 13$. In addition, Table 2 reports values c_j for each tetrahedron considered. The results obtained confirm that c_j seems to converge asymptotically to a certain positive value. It is worth noting that the most unfavorable case for the degeneracy constant is attained for the regular tetrahedron, where $c_{13} = 0.0174$. On the other hand, the best case for the degeneracy constant corresponds to the worst initial tetrahedron, of the needle type. This is in complete agreement with many previous works, where regular and needle tetrahedra present the worst and the best cases, respectively, for the mesh quality evolution.

We deduce that the estimated value for the nondegeneracy constant in the LE-trisection algorithm approaches $c = 0.0174$ which is held for the regular tetrahedron (see the third column in Table 2).

Table 2 Values of $c_j = \frac{\eta(\tau_j)}{\eta(\tau_1)}$, $j = 1, \ldots, 13$, for five selected tetrahedra. Approximated constant $c_{13} = 0.0174$ (regular tetrahedron) is the minimum value reached among each tetrahedron type at refinement level 13

Level	Number of tets	Regular	Needle	Wedge	Sliver	Cap
1	3	0.6181	0.4818	0.5606	0.7141	0.7206
2	9	0.3671	0.3291	0.3418	0.4888	0.5194
3	27	0.2308	0.2727	0.2459	0.3421	0.5598
4	81	0.1602	0.1727	0.1971	0.3355	0.6404
5	243	0.1023	0.1782	0.1302	0.3104	0.4132
6	729	0.0769	0.1582	0.0910	0.2368	0.3207
7	2187	0.0582	0.1727	0.0705	0.2122	0.2582
8	6561	0.0494	0.1764	0.0633	0.1729	0.1869
9	19,683	0.0330	0.1564	0.0470	0.1615	0.1948
10	59,049	0.0328	0.1727	0.0410	0.1211	0.1422
11	177,147	0.0240	0.1764	0.0307	0.0966	0.1269
12	531,441	0.0189	0.1564	0.0283	0.0807	0.1363
13	1,594,323	**0.0174**	0.1636	0.0199	0.0573	0.0900

Note that this value of 0.0174 is the minimum from those obtained at refinement level 13 for each tetrahedron considered in the experiment. For that reason, and for the purpose of this work the constant $c = 0.0174$ is the experimental nondegeneracy constant of the LE trisection method.

To complete the study, Figs. 3, 4, 5, and 6 present the evolution of the four shape measures in average, η, relative thickness, ratio, and solid angle, where the function values have been normalized between 0 and 1. It is interesting to note as the LE-trisection refinement improves in average the overall quality of the mesh for the cases of needle, sliver, cap, and wedge (not showed in the figures) elements. And for the regular tetrahedron, the evolving mesh is not deteriorating through the refinements.

3 A Conforming Refinement Algorithm Based on LE-Trisections

3.1 Description of the Algorithm

The 3D-LE-Refinement algorithm performing the conforming refinement by the LE-trisections can be described in two steps. First, some target elements τ_0 belonging to the input mesh τ are refined by the LE-trisections. This leads to the appearance of three new tetrahedra for each subdivided element from τ_0. The second step of the algorithm consists in making the conformity (i.e., absence of hanging nodes) of the overall mesh by performing appropriate post-refinement of

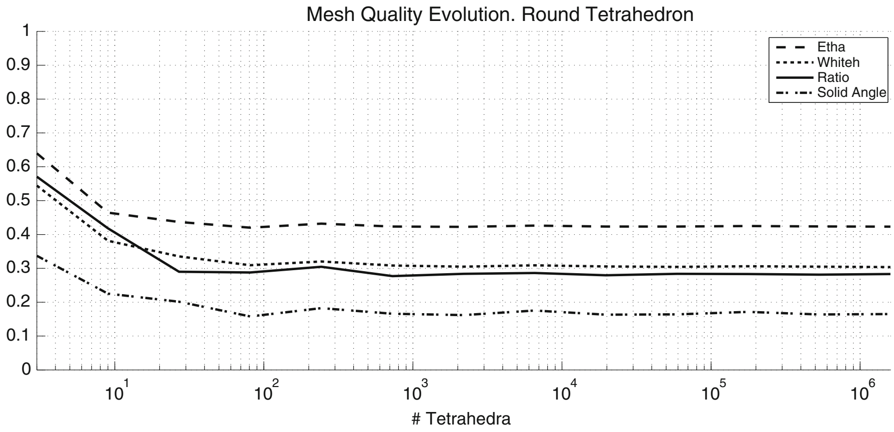

Fig. 3 LE-trisection of the round-type tetrahedron

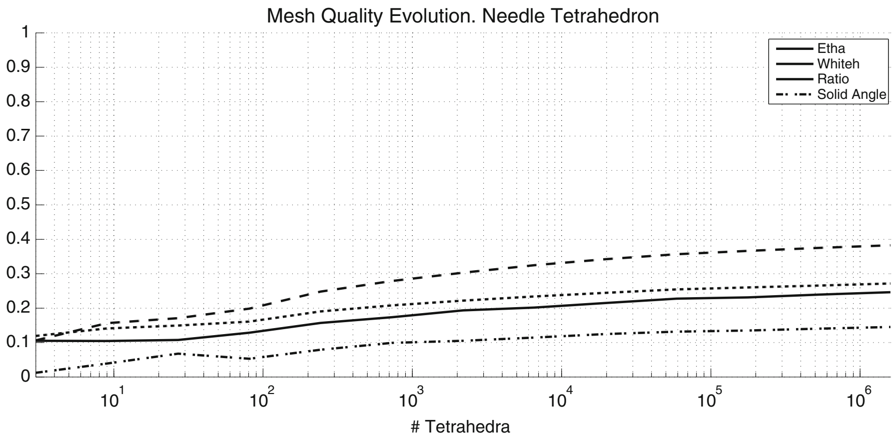

Fig. 4 LE-trisection of the needle-type tetrahedron

some elements in the mesh (3D conformity step). The only neighboring information used in the algorithm is the tetrahedra attached to a given edge. Those neighboring tetrahedra have to be also subdivided in order to assure the conformity of the final mesh.

The algorithm is described as follows:

Algorithm 3D-LE-Refinement(τ, τ_0)
/* Input: τ – tetrahedral mesh, τ_0 – list of tets to be refined
/* Output: new mesh τ
Perform **LE-Trisection** of each tet in τ_0
3D-Conformity(τ, τ_0)
End

A separate sub-algorithm is devoted to providing the mesh conformity:

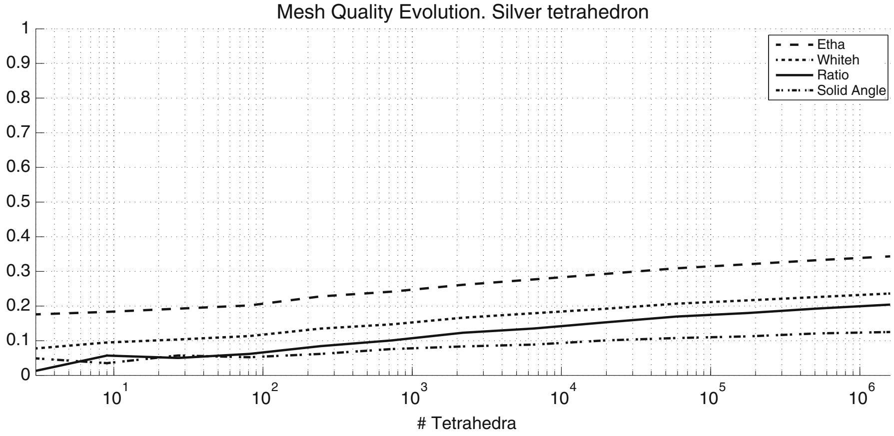

Fig. 5 LE-trisection of the sliver-type tetrahedron

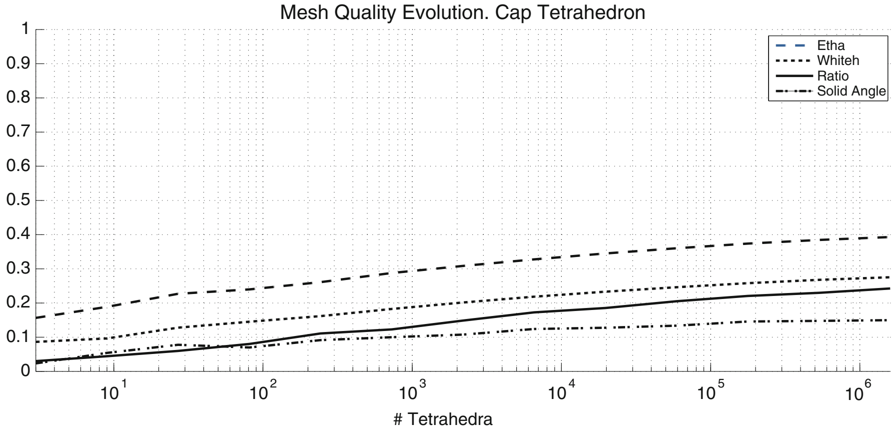

Fig. 6 LE-trisection of the cap-type tetrahedron

Algorithm 3D-Conformity(τ, τ_0)
/* Input: τ – tetrahedral mesh, τ_0 – list of tets to be refined
/* Output: new mesh τ
/* Let L be the set of non-conforming edges in τ
L=Longest-Edges of each tet in τ_0
/* *The conformity is ensured*
While L is not empy **do**
    Let e be the non-conforming edge $\in L$
    S= find all tetrahedra that share the edge e
    For each tetrahedron $t \in S$ **do**
        Perform **LE-Trisection** of t
        Let e be the new non-conforming edge in t
        $L = L \cup e$

End For
End While
End

In Fig. 7, two steps of the above algorithm are illustrated. A simple input mesh consisting of five tetrahedra is considered (see Fig. 7a: $t_1 = (ABCD)$, $t_2 = (ABDE)$, $t_3 = (AFEB)$, $t_4 = (AFEH)$, and $t_5 = (AFGH)$). The elements chosen to be refined are t_1 and t_3 (see Fig. 7b), where the LE-trisection is already applied to the tetrahedra t_1 and t_3. At this point, the conformity process is triggered through the 3D conformity algorithm. Hanging nodes are present at edges AB and AF, and so, the algorithm finds all tetrahedra that share those edges (Fig. 7c). As a result, the following tetrahedra have to be subdivided: $(ABDE), (ABXE), (AFEH), and (AEGH)$ (see Fig. 7d).

The proposed algorithm can be viewed as a three-dimensional analogue of the recently proposed algorithm in [24] for triangulations. Some benefits of the 3D-LE-Refinement algorithm are:

1. The required processing time is linear in the number of new elements generated. The storage requirement for the algorithm is proportional to the number of tetrahedra targeted for the refinement.
2. A simple data structure is employed based on a list of nodes and a list of coordinates. This data structure contributes to a better clarity of the algorithm, thereby facilitating coding. Regarding running efficiency it should be noted that the only neighboring information required is the number of tetrahedra attached to an edge.
3. Any predefined or complicated configurations of tetrahedra depending on the number of points in edges are not needed for the refining process. This simplifies both, code and programming. Only the edge length calculations are needed for the subdivision.
4. The longest edges are more likely to be split than the shorter ones. This leads to a more uniform distribution of edge lengths of tetrahedra, thus improving the overall aspect ratio of the mesh.

3.2 Numerical Test

Here, we describe the numerical test for studying how the algorithm performs in a local refinement scenario. The input to the **3D-LE-Refinement** algorithm is the initial mesh of a cubic domain, which consists of 12 tetrahedra.

The local refinement is applied to tetrahedra which are interior to the cube. At each refinement level, one selects for the initial refinement those tetrahedra which share the vertex located at the center of the domain. The refinement is repeated until the volume covered by the selected tetrahedra is less than a prescribed value. Figure 8 shows the final mesh.

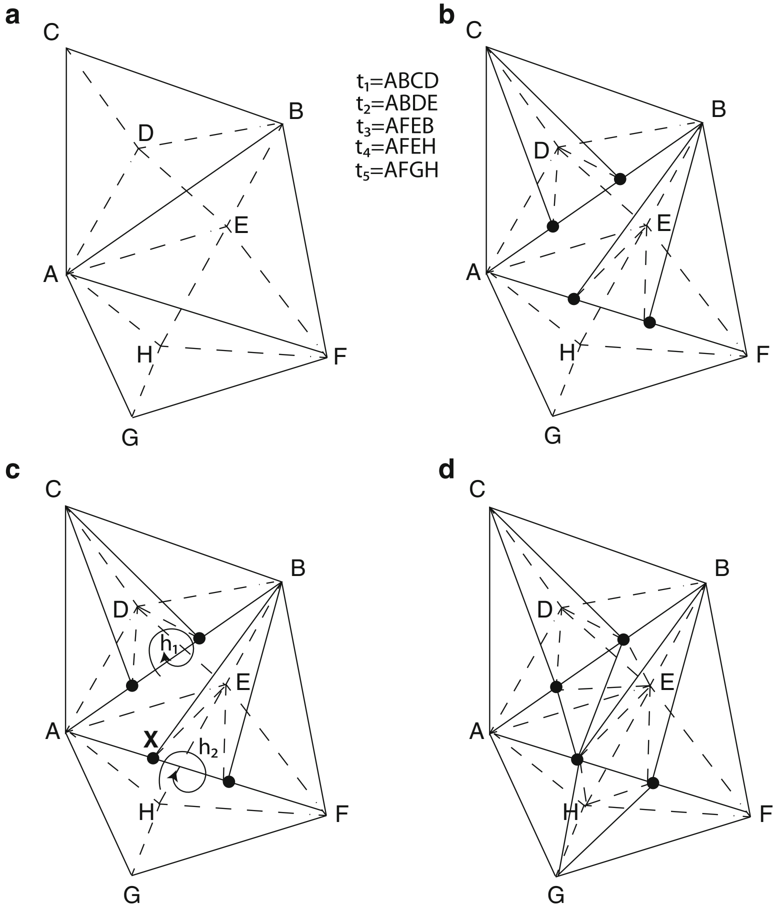

Fig. 7 (**a**) The initial mesh with five tets $t_1, \ldots, t_5$. (**b**) The tetrahedra t_1 and t_3 divided by the LE-trisection. (**c**) Tets sharing the nonconforming sides AB and AF are computed. (**d**) The conformity is assured by subdividing the sharing tetrahedra $(ABDE)$, $(ABXE)$, $(AFEH)$, $(AEGH)$

The subject of special interest in refinement methods is the types and the number of different shapes of mesh elements produced. For example, in [25], Rivara and Levin studied the LE-bisection algorithm performing the classification of tetrahedra generated through refinement. This is of help to assess the quality of elements generated in the refinement process. To this end we group all the elements in four classes in relation to the quality measure η: class A consisting of tetrahedra with $\eta \in [0.0, 0.2]$, class B containing tetrahedra with $\eta \in (0.2, 0.4]$, class C having tetrahedra with $\eta \in (0.4, 0.6]$, and the class D being a set of tetrahedra with $\eta \in (0.6, 1.0]$.

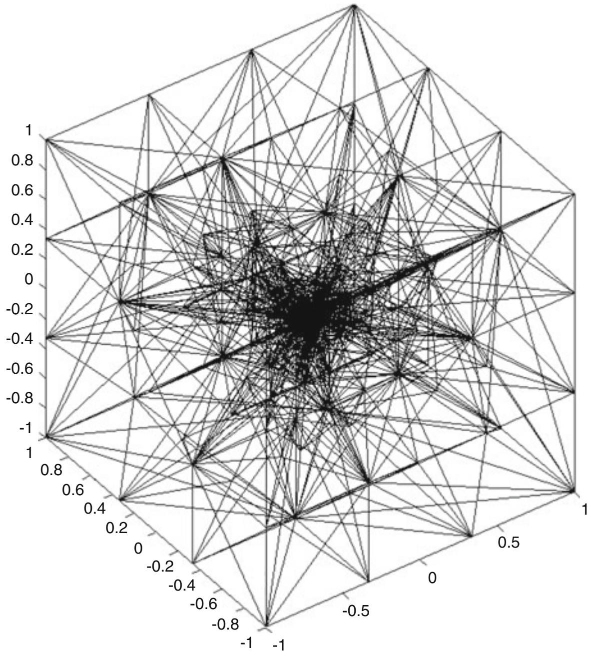

Fig. 8 The final mesh in the test of Sect. 3

Table 3 Classification of tetrahedra during 600 refinement steps

Ref. level	Number of tets	Class A (%)	Class B (%)	Class C (%)	Class D (%)
0	12	0,00	0,00	0,00	100,00
10	2316	0,00	35,23	23,32	41,45
200	50,196	0,00	35,79	23,74	40,47
400	100,596	0,00	35,75	23,77	40,47
600	150,996	24,90	26,82	17,89	30,39

For practical reasons, we consider the tetrahedra from the classes C and D as acceptable for numerical computation, while those elements from the classes A and B are perceived as poorly shaped and potentially causing various numerical difficulties in computations.

In Table 3 we report on values for the quality of the meshes obtained during the local refinements in our test. Percentage values for tetrahedra of the classes A, B, C, and D are showed for a sequence of refinement levels 10, 200, 400, and 600. It is interesting to note that the algorithm does not generate bad elements of class A during the first 400 refinement levels. It should be noted that usually in the adaptive mesh refinement, one can get an approximate solution with an acceptable error using rarely more than 10 or 20 refinement steps [7].

Although the refinement on the domain is probably destroying the quality of the mesh due to appearance of some “badly behaving” tetrahedra, it can be noted that at the finest mesh with 150,966 elements, the percentage of good elements (of classes C and D) is still superior to those of a poorer quality (from classes A and B). This result is in a good agreement with the nondegeneracy property studies from Sect. 2.

4 Final Remarks

There is no mathematical result guaranteeing the nondegeneracy property of the LE-trisection of tetrahedra, while this has been recently obtained for triangulations in [23]. As already seen for the LE-bisection of tetrahedra in [25], we demonstrate in this paper that the LE-trisection also possesses the nondegeneracy property. For example, the algorithm principally subdivides longest edges in tetrahedra, causing so to avoid distorted elements as those appearing when subdividing shortest edges. Furthermore, a reasonably element explosion is yielded as an element is subdivided into three tetrahedra. This permits the application to large meshes without degrading the algorithm efficiency.

We also introduced an algorithm which is a 3D analogue of the recently invented algorithm for the LE-trisection of triangles (see [24]). The required processing time is linear in the number of new element generated. The algorithm avoids any predefined or complicated configuration of tetrahedra that might depend on points and edges. This simplifies both, code and programming. In addition, only edge length calculation is needed for subdivision. As in the family of LE-refinement schemes, longest edges are more likely to be split than shorter ones. This leads to a more uniform distribution of edge length of tetrahedra, thereby improving the aspect ratio of the mesh. From the numerical experiments with the 3D algorithm presented here, the meshes obtained generally lead to well-behaved (regular) tetrahedra.

Some open questions arising from the algorithm are:

(1) To confirm our conjecture by a formal proof of the nondegeneracy of the refining process. It should be noted that at this moment this is still an open problem even for the LE-bisection algorithm in 3D.
(2) To check the validity of the algorithm in adaptive mesh refinement scenarios, where it is needed to track the features of the simulation as the computation progresses.
(3) To probably improve the quality of the generated meshes by combining the longest-edge refinement, for example, with the Delaunay strategy or with some other mesh enrichment techniques.

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A Nonclassical and Nonautonomous Diffusion Equation Containing Infinite Delays

Tomás Caraballo, Antonio M. Márquez-Durán, and Felipe Rivero

Dedicated to George R. Sell, in Memoriam

Abstract We first study the well-posedness of a nonclassical and nonautonomous diffusion equation containing unbounded delays. Then, we prove the existence and uniqueness of local solutions, and finally we prove the global in time existence of solutions as well as the continuous dependence on the initial values.

Keywords Delay equation • Pullback attractor • Nonautonomous problem • Evolution process • Nonclassical diffusion equation

Mathematics Subject Classification (2000): 35R15, 35B41, 37B55, 47J35

1 Introduction

In this paper we are interested in the following nonclassical diffusion equation with infinite delays, where the latter are written in an abstract functional formulation which allows us to consider several kinds of memory terms in a unified way:

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$$
\begin{cases}
\dfrac{\partial u}{\partial t} - \gamma(t)\Delta \dfrac{\partial u}{\partial t} - \Delta u = g(u) + f(t, u_t) \text{ in } (\tau, +\infty) \times \Omega, \\
u = 0 \text{ on } (\tau, +\infty) \times \partial\Omega \\
u(t, x) = \phi(t - \tau, x), t \in (-\infty, \tau], x \in \Omega
\end{cases}
\tag{1}
$$

where $\tau \in \mathbb{R}$ is the initial time, $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain, $\gamma : \mathbb{R} \to (0, +\infty)$ is a continuous bounded function with $0 < \gamma_0 \leq \gamma(t) \leq \gamma_1 < \infty$, and the nonlinearity g is a function satisfying the following growth conditions:

$$
g \in C^1(\mathbb{R}), \qquad \limsup_{|a| \to +\infty} \frac{g(a)}{a} \leq 0 \tag{2}
$$

$$
|g(a) - g(b)| \leq c|a - b|(1 + |a|^{\rho - 1} + |b|^{\rho - 1}), \tag{3}
$$

with $1 < \rho < \frac{n+2}{n-2}$. The time-dependent delay term $f(t, u_t)$ represents, for instance, the influence of an external force with some kind of delay, memory, or hereditary characteristics, although it can also model some kind of feedback control. Here, u_t denotes the past of the solution up to time t, which is also called a segment of the solution; in other words, given a function $u : (-\infty, +\infty) \times \Omega \to \mathbb{R}$, for each $t \in \mathbb{R}$, we can define the mapping $u_t : (-\infty, 0] \times \Omega \to \mathbb{R}$ by

$$
u_t(\theta, x) = u(t + \theta, x), \text{ for } \theta \in (-\infty, 0], x \in \Omega.
$$

As we have already mentioned, this abstract formulation allows to consider several types of delay terms in a unified formulation. For instance, terms like

$$
F_1(u(t - \sigma(t))), \int_{-\infty}^{0} F_2(t, \theta, u(t + \theta))\, d\theta, \tag{4}
$$

where F_i $(i = 1, 2)$ are suitable functions, and $\sigma : \mathbb{R} \to [0, +\infty)$, can be described by the following corresponding f_i defined as

$$
f_1(t, \phi) = F_1(\phi(-\sigma(t))), \; f_2(t, \phi) = \int_{-\infty}^{0} F_2(t, \theta, \phi(\theta))\, d\theta, \tag{5}
$$

where $\phi : (-\infty, 0] \to X$ (X denotes certain Banach or Hilbert space concerning the spatial variable). Then, when we replace ϕ by u_t in (5), we obtain (4).

Nonclassical parabolic equations are used to model physical phenomena such as non-Newtonian flow, soil mechanics, heat conduction, etc. (see [1–3, 11–13, 15, 17, 18] and references therein). The asymptotic behavior of the model without the delay term and with constant coefficients is studied in [19], where the well-posedness of the problem and the existence of the global attractor in $H_0^1(\Omega)$ and in $H^2(\Omega)$, depending on the regularity of the initial data, are shown. However, there are many real situations in which the model can be better described if some terms containing delays appear in the equations.

The introduction of a time dependence in coefficient $\gamma(t)$ represents the variability of viscosity in time due, for example, to external environment temperatures. This time dependence provides the system with a nonautonomous nature.

In [16], Rivero studied the existence of the pullback attractor and its continuity under nonautonomous perturbations without delay, showing the existence of a concrete structure under some assumptions on the nonlinearity. In [4], the authors analyzed the case with bounded delay for the first time, establishing the well-posedness of the problem when $\gamma(t) \equiv \gamma$ is constant. Also, it was proved in [4] the stability of the stationary solutions under some appropriate hypotheses on the delay term. In [10], Hu and Wang studied this equation with a specific variable delay term with bounded derivative, showing the existence of the pullback attractor in H_0^1 and H^2 without nonlinearity nor variable coefficients, and recently, it has been analyzed in [5] a version of our model in which the delay is bounded or finite. Our aim in this paper is to carry out a program in which the delays do not need to be bounded; in order words, the whole history of the problem has influence in the future behavior of the system. This requires of nontrivial different technicalities in our analysis arising mainly from the weighted Banach space which is necessary to consider in order to set up an appropriate framework for the problem.

The content of this paper is the following: in Sect. 2 we introduce a phase space which will be useful for the abstract framework and prove the existence and uniqueness of local solution for (1). Section 3 is devoted to the study of the global existence of solutions and the continuous dependence of solutions with respect to the initial values. The existence of stationary solutions of our problem, the asymptotic behavior of such stationary solutions, and the existence of attracting sets are being investigated in a subsequent paper in preparation right now.

2 Existence of Solution

We consider the following usual spaces $H = L^2(\Omega)$ with inner product $(\cdot,\cdot)$ and associate norm $|\cdot|$ and $V = H_0^1(\Omega)$ with scalar product $((\cdot,\cdot)) = (A^{1/2}u, A^{1/2}v)$, for $u, v \in V$, and associate norm $\|\cdot\|$, where $Au = -\Delta u$ for any $u \in D(A)$ with $D(A) = \{u \in V : Au \in H\} = H_0^1(\Omega) \cap H^2(\Omega)$.

One possibility to deal with infinite delays, and which we will use here, is to consider, for any $\delta > 0$, the following space, as has been considered previously in the literature (see, e.g., [8, 9, 14]) :

$$C_\delta(V) = \left\{\varphi \in C((-\infty,0]; H_0^1(\Omega)) \;:\; \exists \lim_{s\to-\infty} e^{\delta s}\varphi(s) \in H_0^1(\Omega)\right\},$$

which is a Banach space with the norm

$$\|\varphi\|_\delta := \sup_{s\in(-\infty,0]} e^{\delta s}\|\varphi(s)\|.$$

In order to state the problem in the correct framework, let us first establish some initial assumptions on some terms of the equation.

For the delay term, we assume that $f : \mathbb{R} \times C_\delta(V) \to V$ satisfies:

(f1) is continuous in t,

(f2) is locally Lipschitz in $C_\delta(V)$ uniformly in time, that is, there exists a nondecreasing function $L_f : \mathbb{R} \to \mathbb{R}$, such that for all $R > 0$ if $\|\xi\|_\delta, \|\eta\|_\delta \leq R$, then

$$\|f(t,\xi) - f(t,\eta)\| \leq L_f(R)\|\xi - \eta\|_\delta,$$

for all $t \in \mathbb{R}$, and

(f3) there exist a constant $C_f > 0$ and a nonnegative function $\psi \in L^1(\tau, T)$, for all $T > \tau$, such that, for any $\xi \in C_\delta(V)$,

$$\|f(t,\xi)\|^2 \leq C_f\|\xi\|_\delta^2 + \psi(t), \quad \text{for all} \quad t > \tau.$$

Finally, we suppose that $\phi \in C_\delta(V)$.

Proceeding as in [16], we can define operators $B(t) = (I + \gamma(t)A)^{-1}$ and $\tilde{A}(t) = AB(t)$, where $A = -\Delta$ with Dirichlet boundary conditions and the functions $\tilde{g}(t,u) = B(t)g(u)$ and $\tilde{f}(t,\phi) = B(t)f(t,\phi)$, for all $t \in \mathbb{R}$, and all $\phi \in C_\delta(V)$.

Then, we can write problem (1) as

$$\frac{du}{dt} = h(t, u_t), \tag{6}$$

with $h : \mathbb{R} \times C_\delta(V) \to V$ defined as $h(t,\phi) = \tilde{A}(t)\phi(0) + \tilde{g}(t,\phi(0)) + \tilde{f}(t,\phi)$, for all $t \in \mathbb{R}$, and $\phi \in C_\delta(V)$.

The domain of the operator $\tilde{A}(t)$ does not depend on time. In fact, if we define our problem in $H_0^1(\Omega)$, then $D(\tilde{A}(t)) = H_0^1(\Omega)$. This operator is uniformly bounded in time and

$$\tilde{A}(t) = \frac{1}{\gamma(t)}\left[I - (1 + \gamma(t)A)^{-1}\right], \tag{7}$$

for any $t \in \mathbb{R}$. Also, for any $\alpha > 0$ and $x \in D(A^\alpha)$, $A^\alpha\tilde{A}(t)x = \tilde{A}(t)A^\alpha x$.

Thanks to the continuity of the function $\mathbb{R} \ni t \mapsto B(t) \in \mathscr{L}(H_0^1(\Omega))$, we obtain the following estimate (see [16]):

$$\|\tilde{A}(t) - \tilde{A}(s)\|_{\mathscr{L}(H_0^1(\Omega))} \leq C|\gamma(t) - \gamma(s)|,$$

for a constant $C \in \mathbb{R}$.

We can now state and prove the existence of solution to our problem.

Theorem 1. *For each $\phi \in C_\delta(V)$ and under assumptions (2), (3), and (f1)–(f2), there exists $\epsilon > 0$ such that there is a unique solution of problem (1) defined in the interval $(-\infty, \tau + \epsilon)$. In other words, there exists a function $u \in C((-\infty, \tau + \epsilon); H_0^1(\Omega))$ with $u(t, \tau; \phi) = \phi(t - \tau)$ for all $t \in (-\infty, \tau]$ which satisfies*

$$u(t, \tau; \phi) = \phi(0) + \int_\tau^t h(r, u_r)dr,$$

for all $t \in [\tau, \tau + \epsilon)$.

Proof. The proof is based on the contraction mapping theorem. To this end, for the given initial datum $\phi \in C_\delta(V)$, and for a positive T to be determined later on, we define the following space

$$X_\phi^T = \Big\{u \in C\left((-\infty, T); H_0^1(\Omega)\right) : u(t) = \phi(t - \tau) \text{ for all } \quad t \in (-\infty, \tau], \\ \text{and } \|u\|_{X_\phi^T} \le 2\|\phi\|_\delta \Big\}, \tag{8}$$

where $\|u\|_{X_\phi^T} = \sup_{\sigma \in (-\infty, T)} \|u(\sigma)\|$.

This space X_ϕ^T is a complete metric space (since it is a closed subset of a Banach space).

Now we consider the operator $\Phi : X_\phi^T \to X_\phi^T$ given by

$$\Phi(u)(t) = \begin{cases} \phi(t - \tau), & t \in (-\infty, \tau] \\ \phi(0) + \displaystyle\int_\tau^t h(r, u_r)dr, & t \in (\tau, T). \end{cases}$$

Let us first check that Φ is well defined, i.e., $\Phi(u) \in X_\phi^T$ for all $u \in X_\phi^T$.

Given $u \in X_\phi^T$, the mapping $\Phi(u)(\cdot) : (-\infty, T) \to H_0^1(\Omega)$ is continuous thanks to the continuity of ϕ and the mapping $t \in (\tau, T) \to h(t, u_t)$. Moreover, for all $t \ge \tau$ we have that

$$\begin{aligned} \|\Phi(u)(t)\| &= \left\|\phi(0) + \int_\tau^t h(r, u_r)dr\right\| \\ &\le \|\phi(0)\| + \int_s^t \|h(r, u_r)\|dr \\ &\le \|\phi(0)\| + \int_s^t \|\tilde{A}(r)\|_{\mathscr{L}(H_0^1)}\|u(r)\|dr \\ &\quad + \int_s^t \|B(r)g(u(r))\|dr + \int_s^t \|B(r)f(r, u_r)\|dr. \end{aligned} \tag{9}$$

But, $\|\tilde{A}(r)\|_{\mathscr{L}(H_0^1)} \leq a$, for all $r \geq s$, and proceeding as in [16], we can prove that $B(t) \circ g$ is locally Lipschitz in $H_0^1(\Omega)$ uniformly in t. Indeed, by (3),

$$\begin{aligned} \|g(u)-g(v)\|_{L^{\frac{2n}{n+2}}} &\leq c\left[\int_\Omega \left[|u-v|(1+|u|^{\rho-1}+|v|^{\rho-1})\right]^{\frac{2n}{n+2}}\right]^{\frac{n+2}{2n}} \\ &\leq \tilde{c}\|u-v\|_{L^{\frac{2n}{n-2}}}\left(1+\|u\|^{\rho-1}_{L^{\frac{n(\rho-1)}{2}}}+\|v\|^{\rho-1}_{L^{\frac{n(\rho-1)}{2}}})\right). \end{aligned} \tag{10}$$

Since

$$\frac{2n(\rho-1)}{4} \leq \frac{2n}{n-2},$$

we have that

$$H_0^1(\Omega) \subset L^{\frac{n(\rho-1)}{2}}.$$

Due to the following composition of operators,

$$H_0^1(\Omega) \xrightarrow{g} H^{-1}(\Omega) \xrightarrow{B(t)} H_0^1(\Omega),$$

function $B(t) \circ g$ is locally Lipschitz in $H_0^1(\Omega)$.

Therefore, for $R = 2\|\phi\|_\delta$ there exists $L_g(R)$ such that

$$\begin{aligned} \|B(r)g(u(r))\| &\leq \|B(r)\left(g(u(r))-g(0)\right)+B(r)g(0)\| \\ &\leq L_g(R)\|u(r)\|+b_0|g(0)|, \end{aligned} \tag{11}$$

where $b_0 > 0$ is a constant such that $\|B(t)\|_{\mathscr{L}(H_0^1)} \leq b_0$, for all t.

On the other hand, we have

$$\begin{aligned} \|u_r\|_\delta &= \sup_{s\in(-\infty,0]} e^{\delta s}\|u_r(s)\| \leq \sup_{s\in(-\infty,0]} \|u(r+s)\| \leq \sup_{\sigma\in(-\infty,T)} \|u(\sigma)\| \\ &= \|u\|_{X_\phi^T} \leq 2\|\phi\|_\delta, \end{aligned} \tag{12}$$

for all $\tau \leq r \leq t < T$.

And as f is locally Lipschitz,

$$\begin{aligned} \|B(r)f(r,u_r)\| &\leq \|B(r)\left(f(r,u_r)-f(r,0)\right)+B(r)f(r,0)\| \\ &\leq b_0L_f(R)\|u_r\|_\delta+b_0\|f(r,0)\|. \end{aligned} \tag{13}$$

Then, for all $t \geq \tau$ we have that

$$\|\Phi(u)(t)\| \leq \|\phi(0)\| + a\int_\tau^t \|u(r)\|dr + L_g(R)\int_\tau^t \|u(r)\|dr + b_0|g(0)|(t-\tau) + b_0L_f(R)\int_\tau^t \|u_r\|_\delta dr + b_0\int_\tau^t \|f(r,0)\|dr.$$

Therefore,

$$\|\Phi(u)(t)\| \leq \|\phi(0)\| + \big(a + L_g(R) + b_0L_f(R)\big)\int_\tau^t \|u_r\|_\delta dr + b_0|g(0)|(t-\tau) + b_0\int_\tau^t \|f(r,0)\|dr.$$

Then, we obtain that

$$\|\Phi(u)(t)\| \leq \|\phi\|_\delta + R(t-\tau)\big(a + L_g(R) + b_0L_f(R)\big) + b_0|g(0)|(t-\tau) + b_0\int_\tau^t \|f(r,0)\|dr, \quad \forall t \in (\tau, T).$$

If we write $T = \tau + \epsilon$, we then have

$$\|\Phi(u)(t)\| \leq \|\phi\|_\delta + R\,\epsilon\big(a + L_g(R) + b_0L_f(R)\big) + b_0|g(0)|\epsilon + b_0\int_\tau^{\tau+\epsilon} \|f(r,0)\|dr, \quad \forall t \in (\tau, \tau+\epsilon),$$

and considering $\epsilon > 0$ small enough, we can ensure that

$$\|\Phi(u)(t)\| \leq 2\|\phi\|_\delta, \quad \forall t \in (\tau, \tau+\epsilon).$$

We also have the same conclusion for $t \in (-\infty, \tau]$, in fact $\|\Phi(u)(t)\| \leq \|\phi\|_\delta$, for all $t \in (-\infty, \tau]$, and therefore, we can conclude that the operator Φ is well defined.

Now, by using the contraction mapping theorem, we prove the existence of a fixed point for $\Phi(\cdot)$, which will be the solution of our problem. To this end, we need to prove that Φ is a contracting mapping. Let us take $u, v \in X_\phi^T$. We have

$$\begin{aligned}\|\Phi(u)(t) - \Phi(v)(t)\| &\leq \int_s^t \|h(r,u_r) - h(r,v_r)\|dr \\ &\leq \int_s^t \|\tilde{A}(r)\|_{\mathscr{L}(H_0^1)}\|u(r) - v(r)\|dr \\ &\quad + \int_s^t \|B(r)\,(g(u(r)) - g(v(r)) + f(r,u_r) - f(r,v_r))\,\|dr.\end{aligned}$$

Using the uniform bound in time for $\tilde{A}(t)$ and $B(t)$, the fact that $B(t) \circ g$ is locally Lipschitz in $H_0^1(\Omega)$, and $(f2)$, taking into account that $\|u(t)\|, \|v(t)\| \leq R$ for all $t \in [s, T)$ and (12), we obtain

$$\begin{aligned}\|\Phi(u)(t) - \Phi(v)(t)\| &\leq K_1 \int_\tau^t \|u(r) - v(r)\| dr + K_2 \int_\tau^t \|f(r, u_r) - f(r, v_r)\| dr \\ &\leq K_1 \int_\tau^t \|u(r) - v(r)\| dr \\ &\quad + K(R) \int_\tau^t \sup_{\theta \in (-\infty, 0]} \|u(r+\theta) - v(r+\theta)\| dr.\end{aligned}$$

Taking supremum in $[\tau, T)$ with $T = \tau + \epsilon$

$$\|\Phi(u) - \Phi(v)\|_{X_\phi^T} \leq K_1 \epsilon \|u - v\|_{X_\phi^T} + K(R)\epsilon \left(\sup_{r \in [\tau, T)} \sup_{\theta \in (-\infty, 0]} \|u(r+\theta) - v(r+\theta)\| \right),$$

but, if $u, v \in X_\phi^T$,

$$\sup_{r \in [\tau, T)} \sup_{\theta \in (-\infty, 0]} \|u(r+\theta) - v(r+\theta)\| = \sup_{r \in (-\infty, T)} \|u(r) - v(r)\| = \|u - v\|_{X_\phi^T}.$$

Therefore, for $\epsilon > 0$ small enough, Φ is well defined and is a contraction in X_ϕ. The proof is therefore complete.

3 Global Solution and Absorbing Sets

In this section we will prove that the local solution, whose existence has been proved in Theorem 1, is in fact a global one, i.e., it is defined in the whole future and not only in a small time interval. However, we will deduce this result after obtaining some a priori estimates which will be also useful in our future investigation of global asymptotic behavior to deduce the existence of absorbing sets for the evolution process generated by our model.

For any $\varphi \in H_0^1(\Omega)$, taking into account (2) and arguing as in [7], for each $\rho > 0$, there is a constant $K_\rho > 0$ such that

$$\begin{aligned}\int_\Omega g(u)u &\leq \rho|u|^2 + K_\rho, \\ \int_\Omega G(u) &\leq \rho|u|^2 + K_\rho\end{aligned} \tag{14}$$

for all $u \in L^2(\Omega)$, where $G(r) = \int_0^r g(\theta) d\theta$.

Let $L_b(\varphi)$ be the following energy functional

$$L_b(\varphi) = \frac{1}{2}\left(|\varphi|^2 + b\|\varphi\|^2\right) - b\int_\Omega G(\varphi), \tag{15}$$

with $b \geq 0$. It is easy to prove that for $\rho = \frac{\lambda_1}{6}$,

$$L_b(\varphi) \geq \frac{b}{3}\|\varphi\|^2 - bK_{\frac{\lambda_1}{6}} \tag{16}$$

and for any $\rho > 0$,

$$L_b(\varphi) \leq \frac{1 + b(\lambda_1 + 2\rho)}{2\lambda_1}\|\varphi\|^2 + bK_\rho, \tag{17}$$

with λ_1 the first eigenvalue of A.

Taking a solution $u(t, \tau; \phi)$ of (1) and for $b > 0$,

$$\begin{aligned}
\frac{d}{dt}L_b(u) &= (u, \frac{du}{dt}) + b((u, \frac{du}{dt})) - b\int_\Omega g(u)\frac{du}{dt} \\
&= -\gamma(t)((u, \frac{du}{dt})) - \|u\|^2 + (u, g(u)) + (u, f(t, u_t)) \\
&\quad + b\left[-|\frac{du}{dt}|^2 - \gamma(t)\|\frac{du}{dt}\|^2 + (g(u), \frac{du}{dt}) + (f(t, u_t), \frac{du}{dt})\right] \\
&\quad - b(g(u), \frac{du}{dt}) \\
&\leq -\left(1 - \frac{\gamma_1\varepsilon_1}{2} - \frac{2\rho + \varepsilon_2}{2\lambda_1}\right)\|u\|^2 + \frac{\varepsilon_2 + 1}{2\varepsilon_2}|f(t, u_t)|^2 \\
&\quad + \gamma(t)\left(\frac{1}{2\varepsilon_1} - b\right)\|\frac{du}{dt}\|^2 + K_\delta,
\end{aligned}$$

for $\varepsilon_1, \varepsilon_2, \rho > 0$. Taking $\varepsilon_1 = \frac{1}{4\gamma_1}$, $\varepsilon_2 = \frac{\lambda_1}{4}$, $\rho = \frac{\lambda_1}{8}$ and $b \geq \frac{1}{2\varepsilon_1} = 2\gamma_1$, we obtain

$$\begin{aligned}
\frac{d}{dt}L_b(u) &\leq -\frac{1}{2}\|u\|^2 + \left(\frac{\lambda_1 + 4}{2\lambda_1}\right)|f(t, u_t)|^2 + K_{\frac{\lambda_1}{8}} \\
&\leq -\left(\frac{\lambda_1}{1 + b(\lambda_1 + 2\tilde{\rho})}\right)L_b(u) + \left(\frac{\lambda_1 + 4}{2\lambda_1}\right)|f(t, u_t)|^2 \\
&\quad + \left(\frac{\lambda_1}{1 + b(\lambda_1 + 2\tilde{\rho})}\right)K_{\tilde{\rho}} + K_{\frac{\lambda_1}{8}},
\end{aligned}$$

where $\tilde{\rho}$ is a fixed positive constant.

Denoting $C_b = \left(\frac{\lambda_1}{1+b(\lambda_1+2\tilde{\rho})}\right)$, $C_{\lambda_1} = \left(\frac{\lambda_1+4}{2\lambda_1}\right)$ and $\tilde{K}_b = C_b K_{\tilde{\rho}} + K_{\frac{\lambda_1}{8}}$, we have that

$$\frac{d}{dt}\left(e^{C_b t}L_b(u)\right) = C_b e^{C_b t}L_b(u) + e^{C_b t}\frac{d}{dt}L_b(u)$$
$$\leq e^{C_b t}(C_{\lambda_1}|f(t,u_t)|^2 + \tilde{K}_b).$$

Integrating between τ and t, $t \geq \tau$, and using hypothesis (f3),

$$\begin{aligned} e^{C_b t}L_b(u(t)) &\leq e^{C_b\tau}L_b(\phi(0)) + C_{\lambda_1}\int_\tau^t e^{C_b r}|f(r,u_r)|^2 dr + \frac{\tilde{K}_b}{C_b}\left(e^{C_b t} - e^{C_b\tau}\right) \\ &\leq e^{C_b\tau}L_b(\phi(0)) + \frac{\tilde{K}_b}{C_b}\left(e^{C_b t} - e^{C_b\tau}\right) \\ &\quad + \lambda_1^{-1}C_{\lambda_1}\left(\int_\tau^t e^{C_b r}\|f(r,u_r)\|^2 dr\right) \\ &\leq e^{C_b\tau}L_b(\phi(0)) + \lambda_1^{-1}C_{\lambda_1}C_f\int_\tau^t e^{C_b r}\|u_r\|_\delta^2 dr + \lambda_1^{-1}C_{\lambda_1}\int_\tau^t e^{C_b r}\psi(r)dr \\ &\quad + \frac{\tilde{K}_b}{C_b}\left(e^{C_b t} - e^{C_b\tau}\right). \end{aligned}$$

Taking into account (16) and (17), we obtain

$$\begin{aligned} \frac{b}{3}e^{C_b t}\|u(t)\|^2 &\leq e^{C_b\tau}(\tilde{C}_b\|\phi(0)\|^2 + bK_{\rho_2}) + \lambda_1^{-1}C_{\lambda_1}C_f\int_\tau^t e^{C_b r}\|u_r\|_\delta^2 dr \\ &\quad + \lambda_1^{-1}C_{\lambda_1}\int_\tau^t e^{C_b r}\psi(r)dr + e^{C_b t}\left(\frac{\tilde{K}_b}{C_b} + bK_{\frac{\lambda_1}{6}}\right), \end{aligned}$$

where $\rho_2 > 0$ is chosen and

$$\tilde{C}_b = \frac{1 + b(\lambda_1 + 2\rho_2)}{2\lambda_1}.$$

Consequently, if $t \geq \tau$, we have

$$\begin{aligned} e^{C_b t}\|u(t)\|_\delta^2 \leq \max\Bigg\{ &\sup_{\theta\in(-\infty,\tau-t]} e^{C_b t}e^{2\delta\theta}\|\phi(t+\theta-\tau)\|^2, \\ &\sup_{\theta\in[\tau-t,0]} \frac{3}{b}e^{C_b\tau}e^{(2\delta-C_b)\theta}(\tilde{C}_b\|\phi(0)\|^2 + bK_{\rho_2}) \end{aligned}$$

$$+\frac{3}{b}\lambda_1^{-1}C_{\lambda_1}C_f e^{(2\delta-C_b)\theta}\int_\tau^{t+\theta} e^{C_b r}\|u_r\|_\delta^2 dr$$
$$+\frac{3}{b}\lambda_1^{-1}C_{\lambda_1}e^{(2\delta-C_b)\theta}\int_\tau^{t+\theta} e^{C_b r}\psi(r)dr$$
$$+\frac{3}{b}e^{(2\delta-C_b)\theta}e^{C_b t}\left(\frac{\tilde{K}_b}{C_b}+2K_{\frac{\lambda_1}{6}}\right)\Bigg\},$$

but, on the one hand,

$$\begin{aligned}\sup_{\theta\in(-\infty,\tau-t]} e^{\delta\theta}\|\phi(t+\theta-\tau)\| &= \sup_{\theta\in(-\infty,0]} e^{\delta(\theta-(t-\tau))}\|\phi(\theta)\|\\ &= e^{-\delta(t-\tau)}\|\phi\|_\delta\\ &\le \|\phi\|_\delta,\end{aligned}$$

and, $\|\phi(0)\| \le \|\phi\|_\delta$.

On the other hand, taking $2\delta > C_b$, we have

$$\sup_{\theta\in[\tau-t,0]}\left(e^{(2\delta-C_b)\theta}\int_\tau^{t+\theta} e^{C_b r}\|u_r\|_\delta^2 dr + e^{(2\delta-C_b)\theta}\int_\tau^{t+\theta} e^{C_b r}\psi(r)dr\right)$$
$$\le \int_\tau^t e^{C_b r}\|u_r\|_\delta^2 dr + \int_\tau^t e^{C_b r}\psi(r)dr.$$

Therefore,

$$\frac{b}{3}e^{C_b t}\|u_t\|_\delta^2 \le e^{C_b\tau}\left(\tilde{C}_b\|\phi\|_\delta^2 + bK_{\rho_2}\right) + \lambda_1^{-1}C_{\lambda_1}\int_\tau^t e^{C_b r}\psi(r)dr$$
$$+ e^{C_b t}\left(\frac{\tilde{K}_b}{C_b}+bK_{\frac{\lambda_1}{6}}\right) + \lambda_1^{-1}C_{\lambda_1}C_f\int_\tau^t e^{C_b r}\|u_r\|_\delta^2 dr.$$

Assuming that

$$\frac{3}{b}\lambda_1^{-1}C_{\lambda_1}C_f < C_b \tag{18}$$

and calling $\beta = \frac{3}{b}\lambda_1^{-1}C_{\lambda_1}C_f$ (it means $\beta < C_b$) and

$$\alpha(t) = \frac{3}{b}e^{C_b\tau}\left(\tilde{C}_b\|\phi\|_\delta^2 + bK_{\rho_2}\right)+\frac{3}{b}\lambda_1^{-1}C_{\lambda_1}\int_\tau^t e^{C_b r}\psi(r)dr+\frac{3}{b}e^{C_b t}\left(\frac{\tilde{K}_b}{C_b}+bK_{\frac{\lambda_1}{6}}\right),$$

by the Gronwall lemma, we obtain that

$$e^{C_b t}\|u_t\|_\delta^2 \le \alpha(t) + \beta \int_\tau^t \alpha(r) e^{\beta(t-r)} dr.$$

Now,

$$\begin{aligned}\beta \int_\tau^t \alpha(r) e^{\beta(t-r)} dr \le\ & \frac{3}{b} e^{C_b \tau} e^{\beta(t-\tau)} \left(\tilde{C}_b \|\phi\|_\delta^2 + bK_{\rho_2}\right) \\ & + \frac{3}{b} \frac{\beta}{C_b - \beta} \left(\frac{\tilde{K}_b}{C_b} + bK_{\frac{\lambda_1}{6}} \right) e^{C_b t} \\ & + \frac{3}{b} \beta \lambda_1^{-1} C_{\lambda_1} e^{\beta t} \int_\tau^t e^{(C_b - \beta) r} \psi(r) dr.\end{aligned}$$

Then,

$$\begin{aligned}\|u_t\|_\delta^2 \le\ & e^{-C_b t}\alpha(t) + \frac{3}{b} e^{(C_b-\beta)(\tau - t)} \left(\tilde{C}_b \|\phi\|_\delta^2 + bK_{\rho_2}\right) \\ & + \frac{3}{b} \frac{\beta}{C_b - \beta} \left(\frac{\tilde{K}_b}{C_b} + bK_{\frac{\lambda_1}{6}} \right) \\ & + \frac{3}{b} \beta \lambda_1^{-1} C_{\lambda_1} e^{-(C_b-\beta) t} \int_\tau^t e^{(C_b - \beta) r} \psi(r) dr.\end{aligned}$$

Assuming that there exists a $\eta_0 \ge 0$ such that for any $\eta \in [0, \eta_0]$,

$$\int_{-\infty}^t e^{\eta r} \psi(r) dr < +\infty, \tag{19}$$

we have

$$\|u_t\|_\delta^2 \xrightarrow{\tau \to -\infty} l(t), \tag{20}$$

where

$$\begin{aligned}l(t) =\ & \frac{3}{b} \left(\frac{\tilde{K}_b}{C_b} + bK_{\frac{\lambda_1}{6}} \right) \left(1 + \frac{\beta}{C_b - \beta} \right) \\ & + \frac{3}{b} \lambda_1^{-1} C_{\lambda_1} \left(\int_{-\infty}^t e^{C_b r} \psi(r) dr + \beta e^{-(C_b-\beta)t} \int_{-\infty}^t e^{(C_b-\beta) r} \psi(r) dr \right).\end{aligned}$$

Then, we have the global existence of any solution $u(t, \tau; \phi)$ of (1), i.e., for each $\phi \in C_\delta(V)$, $u(\cdot, \tau; \phi) \in C((-\infty, +\infty), H_0^1(\Omega))$ in Theorem 1.

Remark 1. We would like to emphasize that once we justify that the solutions of our problem generates a nonautonomous dynamical system, this also ensures the existence of a family of closed subsets $\{\overline{B}_{C_\delta(V)}(0, l^{1/2}(t)) : t \in \mathbb{R}\}$ in which pullback attracts bounded subsets of $C_\delta(V)$. This will be investigated in the forthcoming paper [6].

Now we prove a result on the continuous dependence on the initial data which will be very useful for our future investigations on this field, in particular, when we analyze the existence of pullback attractors for the evolution process generated by our model.

Proposition 1. *Under the assumptions of Theorem 1, any solution $u(t, \tau; \phi)$ of (1) is continuous with respect to the initial condition $\phi \in C_\delta(V)$. More precisely, if u^i, for $i = 1, 2$, are the corresponding solutions to the initial data $\phi^i \in C_\delta(V)$, $i = 1$, 2, the following estimate holds:*

$$\max_{r\in[\tau,t]} \|u^1(r) - u^2(r)\| \tag{21}$$
$$\leq (\|\phi^1(0) - \phi^2(0)\| + \frac{a + L_g(R) + b_0 L_f(R)}{\delta} \|\phi^1 - \phi^2\|_\delta) e^{(a+L_g(R)+b_0L_f(R))(t-\tau)},$$

for all $t \in [\tau, T)$, where $R \geq 0$ is given by

$$R = \max(2\|\phi^1\|_\delta, 2\|\phi^2\|_\delta).$$

Proof. Let u^i, for $i = 1, 2$, be the corresponding solutions to the initial data $\phi^i \in C_\delta(V)$, $i = 1, 2$, in the interval $(-\infty, T)$, for a fixed $T > \tau$. Then we have that

$$u^1(t) - u^2(t) = \phi^1(0) - \phi^2(0) + \int_\tau^t (h(r, u_r^1) - h(r, u_r^2))dr, \quad t \in (\tau, T).$$

Now, taking into account (f2), (8), (12), that $\|\tilde{A}(r)\|_{\mathscr{L}(H_0^1)} \leq a$, and function $B(t) \circ g$ is locally Lipschitz in $H_0^1(\Omega)$, for $R = \max(2\|\phi^1\|_\delta, 2\|\phi^2\|_\delta)$, it is not difficult to deduce that

$$\|u^1(t) - u^2(t)\| \leq \|\phi^1(0) - \phi^2(0)\| \tag{22}$$
$$+ (a + L_g(R) + b_0 L_f(R)) \int_\tau^t \|u_r^1 - u_r^2\|_\delta dr, \quad t \in (\tau, T).$$

As for $r \in [\tau, t]$ one has

$$\begin{aligned}
\|u_r^1 - u_r^2\|_\delta &= \sup_{\theta \le 0} e^{\gamma\theta} \|u^1(r+\theta) - u^2(r+\theta)\| \\
&= \max\left\{ \sup_{\theta \in (-\infty, \tau - r]} e^{\delta\theta} \|\phi^1(r+\theta-\tau) - \phi^2(r+\theta-\tau)\|, \right. \\
&\qquad \left. \sup_{\theta \in [\tau - r, 0]} e^{\delta\theta} \|u^1(r+\theta) - u^2(r+\theta)\| \right\} \\
&\le \max\left\{ e^{\delta(\tau - r)} \|\phi^1 - \phi^2\|_\gamma, \max_{\theta \in [\tau, r]} \|u^1(\theta) - u^2(\theta)\| \right\},
\end{aligned}$$

we conclude from (22) that for all $t \in (\tau, T)$,

$$\begin{aligned}
\|u^1(t) - u^2(t)\| &\le \|u^1(\tau) - u^2(\tau)\| \\
&\quad + (a + L_g(R) + b_0 L_f(R)) \|\phi^1 - \phi^2\|_\delta \int_\tau^t e^{\delta(\tau - r)} dr \\
&\quad + (a + L_g(R) + b_0 L_f(R)) \int_\tau^t \max_{\theta \in [\tau, r]} \|u^1(\theta) - u^2(\theta)\| dr.
\end{aligned}$$

If we now substitute t by $r \in [\tau, t]$ and consider the maximum when varying this r, from the above, we can conclude that

$$\begin{aligned}
\max_{r \in [\tau, t]} \|u^1(r) - u^2(r)\| &\le \|u^1(\tau) - u^2(\tau)\| + \frac{a + L_g(R) + b_0 L_f(R)}{\delta} \|\phi^1 - \phi^2\|_\delta \\
&\quad + (a + L_g(R) + b_0 L_f(R)) \int_\tau^t \max_{r \in [\tau, \theta]} \|u^1(r) - u^2(r)\| d\theta.
\end{aligned}$$

Hence, by the Gronwall lemma, we obtain (21).

4 Conclusions and Future Directions

In this paper we have investigated the existence, uniqueness, and continuous dependence with respect to the initial values of solutions of a nonclassical and nonautonomous reaction diffusion equation with unbounded delays. This is a preliminary step in order to study the asymptotic behavior of the solutions of the problem. In fact, we are working on this direction in order to prove the existence and exponential stability of the stationary (steady-state) solutions of our equation, as well as the global asymptotic behavior within the framework of pullback attractor. These results will be reported in [6].

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On a Weak Solvability of a System of Thermoviscoelasticity of Oldroyd's Type

Victor Zvyagin and Vladimir Orlov

Abstract We study the solvability in the weak sense of the initial-boundary value problem for an Oldroyd's type model of motion of a viscoelastic continuum.

Keywords Thermoviscoelastic continuum • A priori estimates • Successive approximations • Weak solution

Mathematics Subject Classification (2000): Primary 80A17; Secondary 35Q35

1 Introduction

In $Q_T = \{(t,x) : [0,T] \times \Omega\}$, where $\Omega \subset \mathbf{R}^n$, $n = 2, 3$, is a bounded open domain with the boundary $\partial\Omega \in C^2$, the following problem is considered:

$$\partial u/\partial t + \sum_{i=1}^{n} u_i \partial u/\partial x_i + \nabla p = \operatorname{Div} \sigma + f; \ \operatorname{Div} u = 0; \tag{1}$$

$$u|_{\partial\Omega} = 0, \ u|_{t=0} = u_0; \tag{2}$$

$$\sigma + \lambda(\partial\sigma/\partial t + \sum_{i=1}^{n} u_i \partial\sigma/\partial x_i) = 2\eta(\theta)E(u) + 2\kappa(\partial E(u)/\partial t + \sum_{i=1}^{n} u_i \partial E(u)/\partial x_i); \tag{3}$$

$$\sigma|_{t=0} = \sigma_0; \tag{4}$$

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$$\partial\theta/\partial t + \sum_{i=1}^{n} u_i \partial\theta/\partial x_i - \chi\Delta\theta = g + \sigma : E(u); \tag{5}$$

$$\theta|_{t=0} = \theta_0,\ \theta|_{\partial\Omega} = 0. \tag{6}$$

The unknowns are $u(t,x) = (u_1(t,x),\dots,u_n(t,x))$, $p(x,t)$, and $\theta(x,t)$ as temperature, i.e., the velocity, the pressure, the temperature, and the stress deviator $\sigma(t,x) = \{\sigma_{ij}(t,x)\}_{i,j=1}^n$, respectively, $E(u) = \{E_{ij}(u)\}_{i,j=1}^n$, $E_{ij}(u) = \frac{1}{2}(\partial u_i/\partial x_j + \partial u_j/\partial x_i)$ is the strain velocity tensor.

Next, $f(x,t)$ and $g(x,t)$ are given body force and heat generation per unit volume, respectively; $u_0(x)$, $\theta_0(x)$, and $\sigma_0(x)$ are given initial velocity, temperature, and stress deviator; $\eta(\theta) > 0$ is the viscosity of the continuum, λ is relaxation time, and κ is the retardation time ($0 < \lambda < \kappa$).

In the present work, we study the solvability in the weak sense of the initial-boundary value problem for an Oldroyd's type model of motion of a viscoelastic continuum with the constitutive law (3) (see [1, 2]).

In such a model stresses after the instant stopping of motion damped as $\exp(-\lambda t)$, and the velocities after instant stress relief damped as $\exp(-\lambda t)$.

2 Statement of the Problem and the Main Result

We shall use the standard Sobolev spaces $L_p(\Omega)$, $W_p^l(\Omega)$, $L_p(Q_T)$, $W_p^{k,m}(Q_T)$. By $p = 2$ the norms in these spaces will be denoted by $|\cdot|_0$, $|\cdot|_l$, $\|\cdot\|_0$, $\|\cdot\|_{k,m}$, respectively, $(\cdot,\cdot)$ is the scalar products in L_2 spaces. $H_p^\beta(\Omega)$ is the space of Bessel potentials, $H_p^1(\Omega) = W_p^1(\Omega)$.

Next, $\overset{\circ}{W}{}_p^m(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W_p^l(\Omega)$ ($m > 0$) norm, $W_{p,0}^m(\Omega) = W_p^m(\Omega) \cap \overset{\circ}{W}{}_p^1(\Omega)$, $W_p^{-m}(\Omega) = (\overset{\circ}{W}{}_{p'}^m(\Omega))'$ (conjugate space), $m > 0$, $p' = p/(p-1)$; $\mathcal{V} = \{u : u \in C_0^\infty(\Omega), \operatorname{div} u = 0\}$; H and V are the closure of $\mathcal{V}$ in $L_2(\Omega)$ and $W_2^1(\Omega)$ norms, respectively.

$\mathbf{R}^{n\times n}$ is the space of matrices of the order $n \times n$, and $\mathbf{R}_s^{n\times n}$ is its subspace of symmetric matrices.

It will be convenient to us setting

$$\sigma = \tau + 2\mu_1 E(u),\ \mu_1 = 2\kappa\lambda^{-1},\ \mu_2(\theta) = (\eta(\theta) - \kappa)\lambda^{-1} \tag{7}$$

to rewrite problem (1)–(6) in the form

$$\partial u/\partial t + \sum_{i=1}^{n} u_i \partial u/\partial x_i - \operatorname{Div}\tau - \mu_1\Delta u + \nabla p = f; \tag{8}$$

$$\operatorname{div} u = 0; \tag{9}$$

$$\lambda^{-1}\tau + (\partial\tau/\partial t + \sum_{i=1}^{n} u_i \partial\tau/\partial x_i) = 2\mu_2(\theta)E(u); \tag{10}$$

$$\partial\theta/\partial t + \sum_{i=1}^{n} u_i \partial\theta/\partial x_i - \chi\Delta\theta = g + \tau : E(u) + 2\mu_1 E(u) : E(u); \tag{11}$$

$$u|_{\partial\Omega} = 0, \; u|_{t=0} = u_0; \tag{12}$$

$$\tau|_{t=0} = \sigma_0 - 2\mu_1 E(u_0) \equiv \tau_0; \tag{13}$$

$$\theta|_{t=0} = \theta_0, \; \theta|_{\partial\Omega} = 0. \tag{14}$$

Introduce the spaces

$$W_1 = L_2(0,T;V) \cap C_\omega([0,T];H) \cap W_1^1(0,T;V');$$

$$W_2 = L_2(0,T;L_2(\Omega,\mathbf{R}_s^{n\times n})) \cap C_\omega([0,T];W_p^{-1}(\Omega,\mathbf{R}_s^{n\times n}))$$

$$\cap W_2^1(0,T;W_p^{-2}(\Omega,\mathbf{R}_s^{n\times n}));$$

$$W_3 = L_p(0,T;\overset{\circ}{W}{}_p^1(\Omega)) \cap W_1^1(0,T;W_p^{-1}(\Omega)) \cap C_\omega([0,T];W_p^{1-1/p}(\Omega)).$$

Here $\langle a, b\rangle$ is the action of functional $a \in E'$ from E' upon $b \in E$.

Definition 1. A triple of functions (u,τ,θ), $u \in W_1$, $\tau \in W_2$, $\theta \in W_3(u,\sigma)$,

$$u \in L_2(0,T;V) \bigcap C_w([0,T];H), \; \frac{du}{dt} \in L_1(0,T;V^*),$$

$$\sigma \in L_2(0,T;L_2(\Omega,\mathbf{R}_S^{N\times N})) \bigcap C_w([0,T];H^{-1}(\Omega,\mathbf{R}_S^{N\times N}))$$

is a weak solution to problem (8)–(14) if it satisfies identities

$$\frac{d}{dt}(u,\varphi) - \sum_{i=1}^{n}(u_i, \partial\varphi/\partial x_i) + (\nabla\tau,\nabla\varphi) + 2\mu_1(\nabla u,\nabla\varphi) = \langle f,\varphi\rangle; \tag{15}$$

$$\frac{d}{dt}(\tau,\Phi) - \sum_{i=1}^{n}(u_i\tau, \partial\Phi/\partial x_i) + \lambda^{-1}(\tau,\Phi) = 2\lambda^{-1}(\mu_2(\theta)E(u),\Phi); \tag{16}$$

$$\frac{d}{dt}(\theta,\psi) - \sum_{i=1}^{n}(u_i\theta, \partial\psi/\partial x_i) + \chi(\nabla\theta,\nabla\psi)$$
$$= \langle g,\psi\rangle + (\tau : E,\psi) + \mu_1(E(u) : E(u),\psi) \tag{17}$$

for any $\varphi \in V$, $\Phi \in H_0^1(\Omega,\mathbf{R}_s^{n\times n})$, $\psi \in C_0^\infty(\Omega)$ and conditions (12), (13), (14).

Formally identities (15), (16), (17) are obtained from (8)–(10) by multiplying (8), (10), and (11) by test functions in L_2 and integrating by parts.

Hereafter we deal with problem (8)–(14). The main result:

Theorem 1. *Let* $1 < p < 4/3$ *by* $n = 2$ *and* $1 < p < 5/4$ *by* $n = 3$. *Let*

(a) $f \in L_2(0,T;V')$, $u_0 \in H$,
(b) $\sigma_0 - \eta(\theta_0)\frac{2\kappa}{\lambda}E(u_0) \in L_2(\Omega, R_s^{n\times n})$, $\sigma_0 \in W_2^{-1}(\Omega, R_s^{n\times n})$, $\theta_0 \in W_p^{1-2/p}(\Omega)$,
(c) $g \in L_1(0,T;L_1(\Omega))$,
(d) $\eta(s) \in C(-\infty,+\infty)$ *and for some* $d > 0$

$$0 < \kappa/\lambda \le \eta(s) \le d, \ s \in (-\infty,+\infty). \tag{18}$$

Then there exists a weak solution to problem (8)–(14).

3 Auxiliary Problems

For the proof of Theorem 1, we consider the pair of auxiliary problems. The first one is

$$\partial u/\partial t + \sum_{i=1}^{n} u_i \partial u/\partial x_i - \operatorname{Div}\tau - \mu_1 \Delta u + \nabla p = f, \tag{19}$$

$$\operatorname{div} u = 0, \ u|_{t=0} = u_0, \ u|_{\partial\Omega} = 0; \tag{20}$$

$$\tau + \lambda(\partial\tau/\partial t + \sum_{i=1}^{n} u_i \partial\tau/\partial x_i) = 2\mu_2(\overline{\theta})E(u), \tag{21}$$

$$\tau|_{t=0} = \tau_0 =: \sigma_0 - 2\mu_1 E(u_0) \tag{22}$$

by fixed $\overline{\theta} \in W_3$.

A weak solution to (19)–(22) is the pair (u,τ), $u \in W_1$, $\tau \in W_2$, which solves identities (15)–(16) by $\theta = \overline{\theta}$.

The second one is

$$\partial\theta/\partial t + \sum_{i=1}^{n} u_i \partial\theta/\partial x_i - \kappa\Delta\theta = g + \mathscr{E}(u) : \tau + 2\mu_1 E(u) : E(u); \tag{23}$$

$$\theta|_{t=0} = \theta_0, \ \theta|_{\partial\Omega} = 0 \tag{24}$$

by fixed $\tau \in W_2$ and $u \in W_1$.

A weak solution to (23)–(24) is $\theta \in W_3$, which solves identities (17) and (24) by any $\psi \in C_0^\infty(\Omega)$.

3.1 Solvability of Problem (19)–(22)

Theorem 2. *Let conditions of Theorems 1 are fulfilled. Then problem (19)–(22) has a weak solution and the estimate holds:*

$$\|u\|_{L_\infty(0,T;H)} + \|u\|_{L_2(0,T;V)} + \|\tau\|_{L_\infty(0,T;L_2(\Omega,R_s^{n\times n}))} + \|\partial u/\partial t\|_{L_1(0,T;V')} \quad (25)$$

$$+ \|\partial\tau/\partial t\|_{L_2(0,T;W_2^{-2}(\Omega,R_s^{n\times n}))} \le M_1(|u_0|_0, |\tau_0|_0, \|f\|_{L_2(0,T;V')}), \quad (26)$$

where M_1 does not depend of $\overline{\theta}$.

Theorem 2 is proved with the help of the approximation-topological method (see [4, 6]), based on a priori estimates and the Leray–Schauder degree theory (see [3, 5]).

For this we consider the following auxiliary problem:

$$\frac{d}{dt}(\tau,\Phi) + \frac{1}{\lambda}(\tau,\Phi) - \xi\sum_{i=1}^{n}\Big(\frac{u_i\tau}{1+\delta(\frac{|\tau|^2}{2\mu}+|u|^2)}, \frac{\partial\Phi}{\partial x_i}\Big) + 2\xi(\mu_2(\bar{\theta})u, Div\,\Phi)$$

$$+ \frac{\varepsilon}{\lambda}(\nabla\tau,\nabla\Phi) = 0, \quad (27)$$

$$\frac{d}{dt}(u,\varphi) - \xi\sum_{i=0}^{n}\Big(\frac{u_i u}{1+\delta(\frac{|\tau|^2}{2\mu_2}+|u|^2)}, \frac{\partial\varphi}{\partial x_i}\Big) + \mu_1(\nabla u,\nabla\varphi)$$

$$+ \xi(\tau,\nabla\varphi) = \langle f,\varphi\rangle \quad (28)$$

for all $\varphi \in V,\ \Phi \in H_0^1$ a.e. in $(0, T)$;

$$u|_{t=0} = u_0,\ \tau|_{t=0} = \tau_0 \quad (29)$$

The numbers $\delta > 0,\ 0 \le \xi \le 1, 0 < \varepsilon \le 1$ are parameters.

Then problem (27)–(29) is equivalent to the operator equation

$$\tilde{A}_\varepsilon(u,\tau) + \xi Q(u,\tau) = (f, 0, u_0, \tau_0) \quad (30)$$

The linear operator is generated by the linear part of problem (27)–(29), while Q by its nonlinear part. The invertibility of $\tilde{A}_\varepsilon$ allows us rewrite in the form

$$(u,\tau) - \xi\tilde{A}_\varepsilon^{-1}Q(u,\tau) = \tilde{A}^{-1}(f, 0, u_0, \tau_0) \quad (31)$$

in $W \times W_M$, with compact operator $\tilde{A}_\varepsilon Q$.

Here

$$W = \{u \in L_2(0, T; V),\ \frac{du}{dt} \in L_2(0, T; V^*)\}$$

$$W_M = \{\tau \in L_2(0, T; H_0^1(\Omega, \mathbf{R}_S^{n\times n})),\ \frac{d\tau}{dt} \in L_2(0, T; H^{-1}(\Omega, \mathbf{R}_S^{n\times n}))\}$$

with natural intersection norms.

The application of the Leray–Schauder degree theory yields the existence of solution $(u_{\delta,\varepsilon}, \tau_{\delta,\varepsilon})$ of Eq. (31).

The passage to the limit by $\delta \to 0, \varepsilon \to 0$ gives the solution (u, τ) of (27)–(29).

3.2 Solvability of Problem (23)–(24)

Theorem 3. *Let $u \in W_1$, $\theta_0 \in W_p^{1-2/p}(\Omega)$, $\tau \in W_2$, $g \in L_1(0, T; L_1(\Omega))$, $1 < p < 4/3$ by $n = 2$ and $1 < p < 5/4$ by $n = 3$. Then problem (23)–(24) has a weak solution and the estimate holds:*

$$\|\partial\theta/\partial t\|_{L_1(0,T;W_p^{-1}(\Omega))} + \|\theta\|_{L_p(0,T;W_p^1(\Omega))} + \sup_t \|\theta(t,\cdot)\|_{W_p^{1-2/p}(\Omega)} \tag{32}$$

$$\leq M\left(\|g\|_{L_1(0,T;L_1(\Omega))} + \|u\|_{0,1}^2 + \|\tau\|_0^2 + \|\theta_0\|_{W_p^{1-2/p}(\Omega)}\right). \tag{33}$$

The main difficulties in the problem of a weak solvability to the linear parabolic equation (23) consists in the membership of second and third summands to $L_1(0, T; L_1(\Omega))$. In order to get the weak solvability, we regularize equation (23) by means of application of operator $A^{-1/2}$ and reduce the problem to the solvability of equation

$$\partial(A^{-1/2}\theta(t,x))/\partial t + A^{-1/2}\partial(u_i\theta)/\partial x_i + A^{1/2}\theta(t,x) \tag{34}$$

$$= A^{-1/2}g(t,x) + A^{-1/2}(\sigma : E(u)) + \mu_1 A^{-1/2}E(u) : E(u)).$$

in the functional class $W(p) = \{\theta : A^{-1/2}\theta(t,x) \in W_1^1(0, T; L_p(\Omega)),\ A^{1/2}\theta(t,x) \in L_p(0, T; L_p(\Omega))\}$, by p from Theorem 3.

Here $A^{-1/2}$ is the fractional power of acting in $L_p(\Omega)$, $1 < p < +\infty$ operator $Au = -\chi\Delta u$, $\chi > 0$ with domain $D(A) = W_{p,0}^2(\Omega)$. Operator $A^{-1/2}$ is the integral operator of potential type and maps $L_1(0, T; L_1(\Omega))$ into by $L_1(0, T; W_p^{\varepsilon}(\Omega))$ suitable $\varepsilon > 0$.

This provides sufficient smoothness for the solvability of Eq. (34) and necessary a priori estimates which guarantee the convergence of iterative process in the proof of Theorem 1.

4 Proof of Theorem 1

Consider the sequence of problems

$$\partial u^{n+1}/\partial t + \sum_{i=1}^{n} u_i^{n+1} \partial u^{n+1}/\partial x_i - \mu_1 \text{Div}\, E(u^{n+1}) + \nabla p^{n+1}$$

$$= \text{Div}\, \tau^{n+1} + f, \ \text{div}\, u^{n+1} = 0; \tag{35}$$

$$u^{n+1}|_{\partial\Omega} = 0, \ u^{n+1}|_{t=0} = u_0; \tag{36}$$

$$d\tau^{n+1}/dt + \frac{1}{\lambda}\tau^{n+1} - 2\mu_2(\theta^n) E(u^{n+1}) = 0; \tag{37}$$

$$\tau^{n+1}|_{t=0} = \tau_0; \tag{38}$$

$$\partial \theta^{n+1}/\partial t + \sum_{i=1}^{n} u_i^{n+1} \partial \theta^{n+1}/\partial x_i - \kappa \Delta \theta^{n+1}$$

$$= g + \tau^{n+1} : E(u^{n+1}) + 2\mu_1 E(u^{n+1}) : E(u^{n+1}); \tag{39}$$

$$\theta|_{t=0} = \theta_0, \ \theta|_{\partial\Omega} = 0. \tag{40}$$

Let u^n, τ^n, θ^n be known. Substituting θ^n in (35)–(38), we find weak solutions u^{n+1}, τ^{n+1}. Then substituting u^{n+1} and τ^{n+1} in (39)–(40), we find weak solution θ^{n+1}. Thus, knowing (u^n, τ^n, θ^n), we find $(u^{n+1}, \tau^{n+1}, \theta^{n+1})$. As u^0, τ^0, and θ^0 we take u_0, τ_0, and θ_0, respectively.

Under conditions of Theorems 2 and 3 the successive approximations (u^n, τ^n, θ^n), $n = 0, 1, \ldots$ are well defined, and from these theorems, the estimates follow

$$\|u^n\|_{W_1} \le M; \ \|\tau^n\|_{W_2} \le M; \ \|\theta^n\|_{W_3} \le M. \tag{41}$$

Using estimates (41) we can assume that take place convergence

$$u^n \to u_* \text{ weakly in } L_2(0, T; V); \tag{42}$$

$$u^n \to u_* \ * -\text{weakly in } L_\infty(0, T; H); \tag{43}$$

$$\tau^n \to \tau_* \text{ weakly in } L_2(0, T; H_0^1(\Omega)); \tag{44}$$

$$\tau^n \to \tau_* \text{ weakly in } L_2(0, T; L_2(\Omega)); \tag{45}$$

$$\theta^n \to \theta_* \text{ strongly in } L_p(Q_T). \tag{46}$$

By this $u_* \in W_1$, $\tau_* \in W_2$, $\theta_* \in W_3$, and initial-boundary conditions (2), (4), (6) are fulfilled.

These convergence are not enough to pass to the limit. We need the better properties for the sequence u^n.

Lemma 1. *The sequence u^n strongly converges to u^* in $L_2(0, T; V)$.*

Proof. By the standard way, we deduce from (35 the equality

$$\frac{1}{2}|u^{n+1}(t,x)|_0^2 + 2\mu_1 \int_0^T |\mathscr{E}(u^{n+1})(t,x)|_0^2 dt \tag{47}$$

$$= -\int_0^T (\tau^{n+1}, \nabla u^{n+1}) dt + \int_0^T \langle f, u^{n+1}\rangle \, dt + \frac{1}{2}|u_0|_0^2.$$

It is convenient to write (47 in the form

$$\|u^{n+1}\|_{\mathscr{N}}^2 = \int_0^T \langle f, u^{n+1}\rangle \, dt + \int_0^T (\tau^{n+1}, \nabla u^{n+1}) dt + \frac{1}{2}|u_0|^2. \tag{48}$$

Here we have introduced the Hilbert space $\mathscr{N} = H \times L_2(0, T; V)$ with the scalar product

$$(z, w)_{\mathscr{N}} = \frac{1}{2}(z_1, w_1)_H + \mu_1(\mathscr{E}(z_2), \mathscr{E}(w_2))_{L_2(0,T;L_2(\Omega))}, \; z = (z_1, z_2), w = (w_1, w_2).$$

The convergence (42)–(46) implies the weak convergence of u^n to u_* in $\mathscr{N}$.

The same convergence (42)–(46) implies the convergence of the right-hand side of (48) to $\int_0^T \langle f, u^*\rangle \, dt + \int_0^T (\tau^*, \nabla u^*) dt + \frac{1}{2}|u_0|^2$.

Thus, we have the weak convergence of u^n in $\mathscr{N}$ and the convergence of $\|u^{n+1}\|_{\mathscr{N}} \to \|u_*\|_{\mathscr{N}}$. It is known fact that this convergence implies strong convergence of u^n to u_* in $\mathscr{N}$.

□

Let us show that (u_*, τ_*, θ_*) (23)–(24) is a weak solution to problem (8)–(14). Since $u_* \in W_1$, $\tau_* \in W_2$, $\theta_* \in W_3$, and conditions (2), (4), (6) are fulfilled, it is sufficient to prove that (u_*, τ_*, θ_*) solves identities (15), (16), (17).

Since (u^{n+1}, τ^{n+1}) is a weak solution to problem (35)–(38), then

$$d(u^{n+1}, \varphi)/dt - \sum_{i=1}^{n} \left((u^{n+1})_i u^{n+1}, \partial\varphi/\partial x_i\right) + (\nabla\tau^{n+1}, \nabla\varphi) \tag{49}$$

$$+ \mu_1(\nabla u^{n+1}, \nabla\varphi) = \langle f, \varphi\rangle, \;\; \varphi \in V. \tag{50}$$

$$d(\tau^{n+1}, \Phi)/dt + \frac{1}{\lambda}(\tau^{n+1}, \Phi) - 2(\mu_2(\theta^n)\mathscr{E}(u^{n+1}), \Phi) = 0. \tag{51}$$

Since θ^{n+1} is a weak solution to this problem, then

$$d(\theta^{n+1}, \psi)/dt - \sum_{i=1}^{n} \big(u_i^{n+1}\theta^{n+1}, \partial\psi/\partial x_i\big) + \chi(\nabla\theta, \nabla\varphi) \tag{52}$$

$$= (g, \psi) + (\tau^{n+1} : E(u^{n+1}), \psi) + 2\mu_1(E(u^{n+1}) : E(u^{n+1}), \psi). \tag{53}$$

The convergence (42)–(46) and Lemma 1 allow us to pass to the limit in (35)–(40) and obtain identities (15), (16), (17) for (u_*, τ_*, θ_*) that mean that (u_*, τ_*, θ_*) is a weak solution to problem (8)–(14). Theorem 2 is proved.

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Well-Posedness and Spectral Analysis of Hyperbolic Volterra Equations of Convolution Type

N.A. Rautian and V.V. Vlasov

Abstract We study the correct solvability of abstract integrodifferential equations in Hilbert space generalizing integrodifferential equations arising in the theory of viscoelasticity. The equations under considerations are the abstract hyperbolic equations perturbed by the terms containing Volterra integral operators. We establish the correct solvability in the weighted Sobolev spaces of vector-valued functions on the positive semiaxis. We also provide the spectral analysis of operator-valued functions which are the symbols of these equations.

Keywords Integrodifferential equations • Sobolev space • Gurtin-Pipkin heat equation • Spectra • Operator-function

Mathematics Subject Classification (2010): Primary 34D05, Secondary 34C23

1 Introduction

Numerous problems arise in the research of integrodifferential equation applications. Let us point to some problems in studying the reduction of integrodifferential equations in mechanics and the physics which are considered in the previous chapters.

The first class of problems are the problems arising in the theory of viscoelasticity. Integral terms like convolution describe in this case long-term memory; thus, functions of a kernel of convolution are defined as a result of experiment. Curves often received as a result of experiment are approximated in practice by the sum of finite number of exponentials or series of exponents. A rather complete description of the problems arising in the theory of viscoelasticity is given in the recent monograph [1] (see also [5, 11] and references therein).

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The second class of problems are the problems of acoustics of emulsions. The mixture of two liquids with various characteristics (density, viscosity, compressibility coefficient) is called an emulsion. It is possible to prove by using the homogenization theory that the equation describing an average value of sound pressure for one-dimensional distribution of a sound wave has the abstract form coinciding with a form of the equation, considered in this work (see [15, 16]).

The third class of problems are the problems of homogenization in multiphase media where one of the phases is an elastic (or viscoelastic) media and the other is a viscous (compressible or incompressible) liquid (see [15, 16]).

This paper is devoted to researching integrodifferential equations with unbounded operator coefficients in a Hilbert space. Most of the equations under consideration are abstract hyperbolic equations perturbed by terms containing Volterra integral operators. These equations are the abstract forms of the integrodifferential equations arising in the theory of viscoelasticity (see [1, 5]) and the Gurtin-Pipkin integrodifferential equations (see [3, 7, 9] for more details), which describe heat propagation in media; it also arises in homogenization problems in porous media (Darcy's law) (see [15, 21]).

Due to the fact that we study not only concrete partial neutral integrodifferential equation but a wide class of integrodifferential equations, it is natural and convenient to consider integrodifferential equations with unbounded operator coefficients (abstract integrodifferential equations), which can be realized as integrodifferential partial differential equations with respect to spatial variables when necessary. For the self-adjoint positive operator A considered in what follows, we can take, in particular, the operator $Ay = -y''$, where $x \in (0, \pi)$, $y(0) = y(\pi) = 0$, or the operator $Ay = -\Delta y$ satisfying the Dirichlet conditions on a bounded domain with sufficiently smooth boundary or more general elliptic self-adjoint operator of the order $2m$ in bounded domain with sufficiently smooth boundary. At present, there is an extensive literature on abstract integrodifferential equations. We restrict ourselves by citing monographs [1, 18, 20] (see also references therein). The works most closely related to these questions are [2, 10, 13, 14, 16, 17, 19].

2 Statement of the Problem

Let H be a separable Hilbert space, and let A be a self-adjoint positive operator with bounded inverse acting on H.

Consider the following problem for a second-order integrodifferential equation on $\mathbb{R}_+ = (0, \infty)$:

$$\frac{d^2u(t)}{dt^2} + Au(t) + Bu(t) - \int_0^t K(t-s)Au(s)ds - \int_0^t Q(t-s)Bu(s)ds = f(t), \quad t \in \mathbb{R}_+, \tag{1}$$

$$u(+0) = \varphi_0, \quad u^{(1)}(+0) = \varphi_1, \tag{2}$$

where A is a positive self-adjoint operator acting on the separable Hilbert space H, $A^* = A \geqslant \kappa_0$ ($\kappa_0 > 0$), having the compact inverse operator; I is the identity operator in the separable Hilbert space H; and operator B is symmetric on the $Dom\,(A)$, nonnegative, $(Bx, y) = (x, By)$, $(Bx, x) \geqslant 0$ for arbitrary $x, y \in Dom\,(A)$ and satisfying the inequality $\|Bx\| \leqslant \kappa\,\|Ax\|$, $0 < \kappa < 1$, $x \in Dom\,(A)$.

We suppose that the kernels $K(t)$ and $Q(t)$ can be represented in the following form:

$$K(t) = \sum_{j=1}^{\infty} c_j e^{-\gamma_j t}, \quad Q(t) = \sum_{j=1}^{\infty} d_j e^{-\gamma_j t}, \tag{3}$$

where the coefficients $c_j > 0$, $d_j \geqslant 0$, $\gamma_{j+1} > \gamma_j > 0$, $j \in \mathbb{N}$, $\gamma_j \to +\infty$ ($j \to +\infty$), and moreover, we suppose that

$$\sum_{j=1}^{\infty} \frac{c_j}{\gamma_j} < \infty, \quad \sum_{j=1}^{\infty} \frac{d_j}{\gamma_j} < \infty. \tag{4}$$

The conditions (4) mean that $K(t), Q(t) \in L_1(\mathbb{R}_+)$. If, in addition to (4), conditions

$$K(0) = \sum_{j=1}^{\infty} c_j < +\infty, \quad Q(0) = \sum_{j=1}^{\infty} d_j < +\infty \tag{5}$$

hold, then the kernels $K(t)$ and $Q(t)$ belong to the space $W_1^1(\mathbb{R}_+)$.

Equation (1) can be regarded as an abstract form of dynamical viscoelastic integrodifferential equation where operators A and B are generated by the following differential expressions:

$$A = -\rho^{-1}\mu\,(\Delta u + \mathrm{grad}(\mathrm{div} u))\,, \quad B = -\rho^{-1}\lambda \cdot \mathrm{grad}(\mathrm{div} u),$$

where $u = \mathbf{u}(x, t) \in \mathbb{R}^3$ is displacement vector of viscoelastic hereditary isotropic media that fill the bounded domain $\Omega \subset \mathbb{R}^3$ with smooth boundary, $\partial\Omega$; ρ is a constant density, $\rho > 0$; Lame parameters λ, μ are the positive constants; and $K(t)$, $Q(t)$ are the relaxation functions characterizing hereditary properties of media. On the domain boundary $\partial\Omega$, the Dirichlet condition

$$u|_{\partial\Omega} = 0. \tag{6}$$

is satisfied. The Hilbert space H can be realized as the space of three-dimensional vector functions $L_2(\Omega)$.

Let us denote

$$A_0 = A + B.$$

Due to the known result (theorem [4], p.361), operator A_0 is a self-adjoint positive operator. We convert the domain $Dom(A_0^\beta)$ of the operator A_0^β, $\beta > 0$ into a Hilbert space H_β by endowing $Dom(A_0^\beta)$ with the norm $\|\cdot\|_\beta = \|A_0^\beta \cdot\|$, which is equivalent to the graph norm of the operator A_0^β.

We note that operator-function $L(\lambda)$ is the symbol of Eq. (1), and it has the following form:

$$L(\lambda) = \lambda^2 I + A + B - \hat{K}(\lambda)A - \hat{Q}(\lambda)B, \tag{7}$$

where $\hat{K}(\lambda)$ and $\hat{Q}(\lambda)$ are the Laplace transforms of the kernels $K(t)$ and $Q(t)$, having the representations

$$\hat{K}(\lambda) = \sum_{j=1}^{\infty} \frac{c_j}{(\lambda + \gamma_j)}, \quad \hat{Q}(\lambda) = \sum_{j=1}^{\infty} \frac{d_j}{(\lambda + \gamma_j)}. \tag{8}$$

2.1 Correct Solvability

By $W_{2,\gamma}^n \left(\mathbb{R}_+, A_0^n\right)$ we denote the Sobolev space of vector functions on the half-axis $\mathbb{R}_+ = (0, \infty)$ taking values in H endowed with the norm

$$\|u\|_{W_{2,\gamma}^n(\mathbb{R}_+,A_0^n)} \equiv \left(\int_0^\infty e^{-2\gamma t}\left(\left\|u^{(n)}(t)\right\|_H^2 + \|A_0^n u(t)\|_H^2\right) dt\right)^{1/2}, \quad \gamma \geq 0.$$

For more information about the spaces $W_{2,\gamma}^n \left(\mathbb{R}_+, A_0^n\right)$, see monograph [6, Chap. 1]. For $n = 0$ we set $W_{2,\gamma}^0 \left(\mathbb{R}_+, A_0^0\right) \equiv L_{2,\gamma}\left(\mathbb{R}_+, H\right)$.

Definition 1. We say that vector function u is a strong solution of the problem (1), (2), if it belongs to the space $W_{2,\gamma}^2(\mathbb{R}_+, A_0)$ for some $\gamma \geqslant 0$, satisfies (1) almost everywhere on the half-axis $\mathbb{R}_+$, and satisfies the initial conditions (2).

The following theorem gives conditions for problem (1), (2) to be well solvable.

Theorem 1. *If condition (5) holds, $f'(t) \in L_{2,\gamma_0}(\mathbb{R}_+, H)$ for certain $\gamma_0 \geqslant 0$ and $f(0) = 0$, $\varphi_0 \in H_1$, $\varphi_1 \in H_{1/2}$, then there exists $\gamma_1 \geqslant \gamma_0$, that for any $\gamma > \gamma_1$ problem (1), (2) is uniquely solvable in the space $W_{2,\gamma}^2\left(\mathbb{R}_+, A_0\right)$, and its solution satisfies the inequality*

$$\|u\|_{W_{2,\gamma}^2(\mathbb{R}_+,A_0)} \leqslant d\left(\left\|f'(t)\right\|_{L_{2,\gamma}(\mathbb{R}_+,H)} + \|A_0\varphi_0\|_H + \left\|A_0^{1/2}\varphi_1\right\|_H\right), \tag{9}$$

where the constant d does not depend on the vector function f and the vectors φ_0 and φ_1.

2.2 Spectral Analysis

We proceed to study the structure of the spectrum of the operator-function $L(\lambda)$ in the case where conditions (4), (5) and conditions

$$\sup_{k\in\mathbb{N}} \gamma_k^2(\gamma_{k+1}-\gamma_k) = +\infty. \tag{10}$$

$$\lim_{k\to\infty} \frac{\gamma_k-\gamma_{k-1}}{\gamma_k} = 0. \tag{11}$$

hold.

Moreover we suppose that

$$\sum_{k=1}^{\infty} \frac{c_k}{\gamma_k} < 1, \quad \sum_{k=1}^{\infty} \frac{d_k}{\gamma_k} < 1. \tag{12}$$

Theorem 2. *If conditions* (12), (5), (10), *and* (11) *hold, then the spectrum of the operator-function* $L(\lambda)$ *is contained in the union of the intervals* $\Delta_k = (-\gamma_k, \tilde{p}_k) \subset (-\gamma_k, -\gamma_{k-1})$, $k \in \mathbb{N}$ *(*$\gamma_0 = 0$*) and the strip* $\{\lambda \in \mathbb{C} | \alpha_1 \leqslant \operatorname{Re}\lambda \leqslant \alpha_2\}$, *where* $\tilde{p}_k = \max\{p_k(\tau'), p_k(\tau'')\}$, $p_k(\tau)$ *are the real roots of the equation*

$$\Phi_\tau(p) := \tau \sum_{k=1}^{\infty} c_k(p+\gamma_k)^{-1} + (1-\tau)\sum_{k=1}^{\infty} d_k(p+\gamma_k)^{-1} = 1, \quad (0 \le \tau \le 1).$$

belonging to the intervals $(-\gamma_k, -\gamma_{k-1})$, $k \in \mathbb{N}$ *(*$\gamma_0 = 0$*)*, $\tau' := \left\| A^{-1/2}A_0A^{-1/2} \right\|^{-1}$, $\tau'' := \left\| A_0^{-1/2}AA_0^{-1/2} \right\|$, $(0 < \tau' < \tau'' \leqslant 1)$ *and*

$$\alpha_1 = -\frac{1}{2}\sup_{\|f\|=1}\sum_{k=1}^{\infty}\frac{((c_kA+d_kB)f,f)}{((A+B)f,f)}, \quad \alpha_2 = -\frac{1}{2}\inf_{\|f\|=1}\sum_{k=1}^{\infty}\frac{((c_kA+d_kB)f,f)}{\left((A+B+\gamma_k^2I)f,f\right)}.$$

Remark. According to Lemma 2.1 of [12], the operator $A^{-1/2}BA^{-1/2}$ has a bounded closure on the space H. It follows that so does the operator $A^{-1/2}A_0A^{-1/2} = I + A^{-1/2}BA^{-1/2}$ H. In turn, Lemma 2.1 of [12] mentioned above and the self-adjointness of $A_0 = A + B$ imply that the operator $A_0^{-1/2}AA_0^{-1/2}$ has a bounded closure on H as well. Thus, the quantities τ' and τ'' in the statement of Theorem 2 are well defined.

Remark. The constants α_1 and α_2 in the statement of Theorem 2 can be estimated as

$$\alpha_1 \geqslant -\frac{1}{2}\left\| A_0^{-1/2}\left(\sum_{k=1}^{\infty} c_kA + \sum_{k=1}^{\infty} d_kB\right)A_0^{-1/2}\right\|,$$

$$\alpha_2 < -\frac{1}{2}\left\|(c_1A+d_1B)^{-1/2}\left(A_0+\gamma_1^2I\right)(c_1A+d_1B)^{-1/2}\right\|^{-1}, \quad c_1 > d_1$$

Theorem 3. *The nonreal spectrum of the operator-function $L(\lambda)$ is symmetric with respect to the real axis and consists of eigenvalues of finite algebraic multiplicity; moreover, for any $\varepsilon > 0$ the eigenvalues are isolated, i.e., have no points of accumulation, in the domain $\Omega_\varepsilon := \mathbb{C} \setminus \{\lambda : \alpha_1 \leq \operatorname{Re}\lambda \leq \alpha_2, |\operatorname{Im}\lambda| < \varepsilon\}$.*

Equation (1) is related to applications: if $B \equiv 0$ then it is an abstract form of the Gurtin-Pipkin integrodifferential equation modeling the finite speed of heat propagation in media with memory. That integrodifferential equation is deduced in [3].

Equations of the above type are currently investigated by many authors (see [1, 3, 10, 14, 16–19] and references therein). We impose the following assumptions:

1. The operator B is identically zero.
2. The real-valued function $K(t) = \sum_{k=1}^{\infty} c_k e^{-\gamma_k t}$ satisfies the assumptions (5), (12).

Let $\{a_j^2\}_{j=1}^{\infty}$ be eigenvalues of the operator A ($Ae_j = a_j^2 e_j$), numbered according to the increasing order: $0 < a_1^2 < a_2^2 < \cdots < a_n^2 < \ldots$; $a_n \to +\infty$ ($n \to +\infty$). The corresponding eigenvectors $\{e_j\}_{j=1}^{\infty}$ form an orthonormal basis of the space H.

Now we consider the structure of the spectrum of the operator-valued function $L_1(\lambda)$:

$$L_1(\lambda) = \lambda^2 I + A - \hat{K}(\lambda)A,$$

where $\hat{K}(\lambda)$ is the Laplace transform of the function K.

In the considered case, (1) can be decomposed into a countable set of scalar integrodifferential equations

$$u_n^{(2)}(t) + a_n^2 u_n(t) - \int_0^t \sum_{k=1}^{\infty} c_k e^{-\gamma_k(t-s)} a_n^2 u_n(s)\, ds = f_n(t),\ t > 0 \tag{13}$$

where $u_n(t) = (u(t), e_n)$ and $f_n(t) = (f(t), e_n)$, $n = 1, 2, \ldots$. Those equations are projections (1) onto the one-dimensional spaces spanned by vectors $\{e_n\}$.

Using the Laplace transform, we naturally arrive at the countable set of meromorphic functions $l_n(\lambda) = \lambda^2 + a_n^2 - a_n^2\left(\sum_{k=1}^{\infty} \frac{c_k}{\lambda+\gamma_k}\right)$, $n = 1, 2, \ldots$ which are symbols of the integrodifferential equations given by (13).

The spectrum of the operator-valued function $L_1(\lambda)$ is described as follows.

Theorem 4. *If conditions (12), (5), (10) hold, then the spectrum of the operator-function $L_1(\lambda)$ coincides with the closure of the union of the sets of zeros for the functions $\{l_n(\lambda)\}_{n=1}^{\infty}$. The zeros of the meromorphic function $l_n(\lambda)$ form a countable set of real roots $\{\lambda_{n,k}\}_{n=1}^{\infty}$, satisfying the inequalities*

$$\begin{gathered} -\gamma_k < \lambda_{n,k} < x_k < -\gamma_{k-1} \quad [\gamma_0 = 0], \\ \lambda_{n,k} = x_k + \underline{O}\left(\frac{1}{a_n^2}\right) \quad k \in \mathbb{N}\,(k > 1),\ (a_n \to +\infty) \end{gathered} \tag{14}$$

where x_k *are the real zeros of the function* $g(\lambda) = 1 - \sum_{k=1}^{\infty} \frac{c_k}{\lambda + \gamma_k}$, *and a pair of complex-conjugate roots* $\left\{\lambda_n^{\pm}\right\}_1^{\infty}$, $\lambda_n^{+} = \overline{\lambda_n^{-}}$ *such that*

$$\lambda_n^{\pm} = -\frac{1}{2}\sum_{k=1}^{\infty} c_k \pm i a_n + \underline{O}\left(\frac{1}{a_n^2}\right), \quad (a_n \to +\infty). \tag{15}$$

Thus, the spectrum $\sigma(L_1)$ *of the operator-valued function* $L_1(\lambda)$ *is representable as*

$$\sigma(L_1) \equiv \overline{\left(\cup_{k=1}^{\infty} \cup_{n=1}^{\infty} \{\lambda_{n_k}\}\right) \cup \left(\cup_{n=1}^{\infty} \lambda_n^{\pm}\right)},$$

where $\lim_{n\to\infty} \lambda_{nk} = x_k$, $k = 1, 2, \ldots$.

The proof of Theorem 4 is given in [14] (see also [17]). Picture of the spectrum of operator-function $L_1(\lambda)$ is given in Fig. 1.

Remark. The spectrum of the operator-valued function $L_1(\lambda)$ is located in the left-hand semi-plane $\{\lambda : \Re\lambda < 0\}$, if $\sum_{j=1}^{\infty} \frac{c_j}{\gamma_j} < 1$. If $\sum_{j=1}^{\infty} \frac{c_j}{\gamma_j} > 1$, then the accumulation point x_1 of the poles is located in the right-hand semi-plane $\{\lambda : \Re\lambda > 0\}$; this corresponds to the instability case.

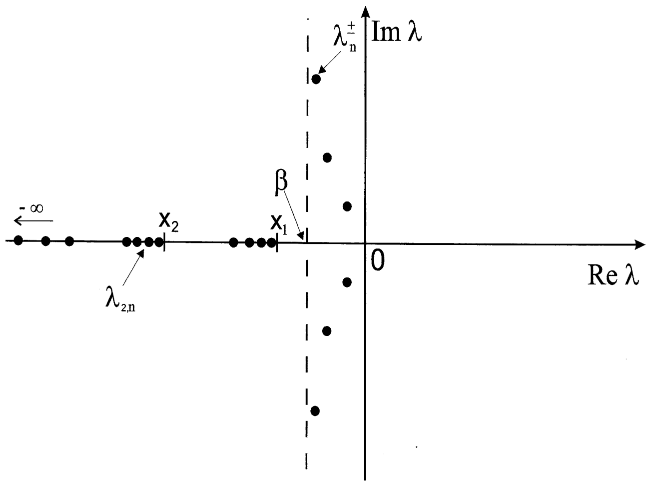

Fig. 1 Spectral structure in case $K(t) \in W_1^1(\mathbb{R}_+)$, $\beta = -\frac{1}{2}K(0)$

3 Conclusion and Remarks

We emphasize that our method of the proof of the theorem on the correct solvability of the initial boundary value problem for an abstract integrodifferential equation differs substantially from the approach used by Pandolfi in [10] and Miller in [8]. Moreover, L. Pandolfi studied solvability in function space on a finite time interval $(0, T)$, whereas we study solvability in the weighted Sobolev spaces $W^2_{2,\gamma}(\mathbb{R}_+, A^2)$ on the positive semiaxis $\mathbb{R}_+$.

Our proof of the solvability Theorem 1 essentially uses the Hilbert structure of the space $W^2_{2,\gamma}(\mathbb{R}_+, A^2_0)$, $L_{2,\gamma}(\mathbb{R}_+, H)$ and Paley-Wiener theorem, while in [8, 10], considerations are performed in Banach function spaces consisting of smooth functions on a finite time interval $(0, T)$.

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Difference Equations and Boundary Value Problems

Alexander V. Vasilyev and Vladimir B. Vasilyev

Abstract We study multidimensional difference equations with a continual variable in the Sobolev–Slobodetskii spaces. Using ideas and methods of the theory of boundary value problems for elliptic pseudo-differential equations, we suggest to consider certain boundary value problems for such difference equations. Special boundary conditions permit to prove unique solvability for these boundary value problems in appropriate Sobolev–Slobodetskii spaces.

Keywords Difference equation • Symbol • Factorization • Index • Boundary value problem • Solvability

Mathematics Subject Classification (2010): Primary 39A14; Secondary 35J40

1 Introduction

We consider a general difference equation of the type

$$\sum_{|k|=0}^{\infty} a_k(x)u(x+\alpha_k) = v(x), \quad x \in D, \tag{1.1}$$

where $D \subset \mathbb{R}^m$ is a canonical domain like $\mathbb{R}^m, \mathbb{R}^m_\pm = \{x \in \mathbb{R}^m : x = (x_1, \cdots, x_m), \pm x_m > 0\}, C^a_+ = \{x \in \mathbb{R}^m : x_m > a|x'|, x' = (x_1, \cdots, x_{m-1}), a > 0\}$, k is a multi-index, $|k| = k_1 + \cdots + k_m$, $\{\alpha_k\} \subset D$. Equations of a such type have a long history [4, 5, 8] and in general there is no algorithm for solving the Eq. (1.1). If so then any assertion on a solvability of such equations is very important and required. One can

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add that Eq. (1.1) appear in very distinct branches of a science like mathematical biology, technical problems, etc. Also such equations have arisen in studies of the second author [10, 11] related to boundary value problems in a plane corner. One-dimensional case for such equations was considered in [12].

Here we will start from the equation

$$\sum_{|k|=0}^{\infty} a_k u(x+\alpha_k) = v(x), \quad x \in \mathbb{R}^m_+, \tag{1.2}$$

with constant coefficients because further we will try to use a local principle [6] to obtain some results on Fredholm properties of the general Eq. (1.1). We use methods of the theory of boundary value problems for elliptic pseudo-differential equations [1, 9]. For our case of a half-space, these methods are based on the theory of one-dimensional singular integral equations and classical Riemann boundary value problem [2, 3, 7].

2 Spaces, Operators, and Symbols

2.1 Spaces

Let $S(\mathbb{R}^m)$ be the Schwartz class of infinitely differentiable rapidly decreasing at infinity functions and $S'(\mathbb{R}^m)$ be the space of distributions over the space $S(\mathbb{R}^m)$. If $u \in S(\mathbb{R}^m)$, then its Fourier transform is defined by the formula

$$\tilde{u}(\xi) = \int_{\mathbb{R}^m} e^{-ix\cdot\xi} u(x)dx.$$

Definition 2.1. A Sobolev–Slobodetskii space $H^s(\mathbb{R}^m)$, $s \in \mathbb{R}$, consists of functions (distributions) with a finite norm

$$||u||_s = \left(\int_{\mathbb{R}^m} \tilde{u}(\xi)(1+|\xi|)^{2s} d\xi \right)^{1/2}.$$

Let us note $H^0(\mathbb{R}^m) = L_2(\mathbb{R}^m)$.

The space $S(\mathbb{R}^m)$ is a dense subspace in the $H^s(\mathbb{R}^m)$ [1]. The space $H^s(\mathbb{R}^m_+)$ consists of functions from the space $H^s(\mathbb{R}^m)$ which support belongs to $\overline{\mathbb{R}^m_+}$ with induced norm. Also we need the space $H^s_0(\mathbb{R}^m_+)$ which consists of distributions from $S'(\mathbb{R}^m_+)$ admitting a continuation in the whole space $H^s(\mathbb{R}^m)$. A norm in the space $H^s_0(\mathbb{R}^m_+)$ is defined by the formula

$$||u||^+_s = \inf ||lu||_s,$$

where *infimum* is taken from all continuations l.

2.2 Operators

Here we consider difference operators with constant coefficients only of the type

$$\mathcal{D} : u(x) \longmapsto \sum_{|k|=0}^{\infty} a_k u(x + \alpha_k), \tag{2.1}$$

where $\{a_k\} and \{\alpha_k\}$ are given sequences in $\mathbb{R}^m$, and

$$\sum_{|k|=0}^{\infty} |a_k| < +\infty. \tag{2.2}$$

Definition 2.2. An operator $\mathcal{D}$ of the type (2.1) with coefficients a_k satisfying (2.2) is called difference operator with constant coefficients.

Lemma 2.3. *Every operator* $\mathcal{D} : H^s(\mathbb{R}^m) \to H^s(\mathbb{R}^m)$ *with constant coefficients is a linear bounded operator* $\forall\ s \in \mathbb{R}$.

2.3 Symbols

Definition 2.4. The function

$$\sigma_{\mathcal{D}}(\xi) = \sum_{|k|=0}^{\infty} a_k e^{-i\alpha_k \cdot \xi} \tag{2.3}$$

is called a symbol of the operator $\mathcal{D}$. The symbol $\sigma_{\mathcal{D}}(\xi)$ is called an **elliptic symbol** if $\sigma_{\mathcal{D}}(\xi) \neq 0, \forall\ \xi \in \mathbb{R}^m$.

Evidently under condition (2.2) $\sigma_{\mathcal{D}} \in L_\infty(\mathbb{R}^m)$, but everywhere below we suppose that $\sigma_{\mathcal{D}} \in C(\dot{\mathbb{R}}^m)$ taking into account that $\dot{\mathbb{R}}^m$ is a compactification of $\mathbb{R}^m$.

3 Equations and Factorization

3.1 Equations

We are interested in studying solvability of the Eq. (1.2). It can be written in the operator form

$$(\mathcal{D}u)(x) = v(x), \quad x \in \mathbb{R}^m_+, \tag{3.1}$$

assuming that v is a given function in $\mathbb{R}^m_+$, $v \in H^s_0(\mathbb{R}^m_+)$, the unknown function u is defined in $\mathbb{R}^m_+$, $u \in H^s(\mathbb{R}^m_+)$, and $\{\alpha_k\} \subset \mathbb{R}^m_+$.

By notation, $u_+(x) = u(x)$, lv is an arbitrary continuation of v on $\mathbb{R}^m_+$. Then we put

$$u_-(x) = (lv)(x) - (\mathcal{D}u_+)(x),$$

and see that $u_-(x) = 0, \forall\ x \in \mathbb{R}^m_+$, to explain this notation. Further we rewrite the last equation

$$(\mathcal{D}u_+)(x) + u_-(x) = (lv)(x)$$

and apply the Fourier transform

$$\sigma_{\mathcal{D}}(\xi)\tilde{u}_+(\xi) + \tilde{u}_-(\xi) = \widetilde{lv}(\xi). \tag{3.2}$$

To solve the Eq. (3.2) with an elliptic symbol $\sigma_{\mathcal{D}}(\xi)$, we need to introduce a concept of a factorization. Everywhere below we write $\sigma(\xi)$ instead of $\sigma_{\mathcal{D}}(\xi)$ for a brevity.

3.2 *Factorization*

Let us denote $\xi = (\xi', \xi_m), \xi' = (\xi_1, \cdots, \xi_{m-1})$.

Definition 3.1. Let $\sigma(\xi)$ be an elliptic symbol. Factorization of elliptic symbol $\sigma(\xi)$ is called its representation in the form

$$\sigma(\xi) = \sigma_+(\xi)\sigma_-(\xi),$$

where factors $\sigma_\pm(\xi)$ admit an analytical continuation in upper and lower complex planes $\mathbb{C}_\pm$ on the last variable ξ_m for almost all $\xi' \in \mathbb{R}^{m-1}$ and $\sigma_\pm(\xi) \in L_\infty(\mathbb{R}^m)$.

Definition 3.2. Index of factorization for the elliptic symbol $\sigma(\xi)$ is called an integer

$$æ = \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} d \arg \sigma(\cdot, \xi_m).$$

Remark 3.3. The index æ is not really depended on ξ' because it is homotopic invariant.

Remark 3.4. It is a principal fact the index of factorization does not correlate with an order of operator. For our case the order of the operator $\mathcal{D}$ is zero in a sense of Eskin's book [1], but the index may be an arbitrary integer. It is essential the index is a **topological barrier** for a solvability.

Proposition 3.5. *If æ = 0, then for any elliptic symbol $\sigma(\xi)$, a factorization*

$$\sigma(\xi) = \sigma_+(\xi)\sigma_-(\xi)$$

exists, and it is unique up to a constant.

This is classical result, see details in [1–3, 7].

4 Solvability and Boundary Value Problems

4.1 Solvability

Everywhere below we will denote $\tilde{H}(D)$ the Fourier image of the space $H(D)$.

Theorem 4.1. *If $|s| < 1/2$, æ $= 0$, then the Eq. (3.1) has a unique solution $u \in H^s(\mathbb{R}^m_+)$ for arbitrary right-hand side $v \in H^s_0(\mathbb{R}^m_+)$.*

Proof. is a very simple. It is based on properties of the Hilbert transform

$$(H_{\xi'}u)(\xi', \xi_m) = \frac{1}{\pi i} v.p. \int\limits_{-\infty}^{+\infty} \frac{u(\xi', \eta_m)d\eta_m}{\xi_m - \eta_m}$$

which is a linear bounded operator $H^s(\mathbb{R}^m) \to H^s(\mathbb{R}^m)$ for $|s| < 1/2$ [1]. This operator generates two projectors on some spaces consisting of boundary values of analytical functions in $\mathbb{C}_\pm$ on the last variable ξ_m [1–3, 7]

$$\Pi_\pm = 1/2(I \pm H_{\xi'}),$$

so that the representation

$$f = f_+ + f_- \equiv \Pi_+ f + \Pi_- f$$

is unique for arbitrary $f \in H^s(\mathbb{R}^m)$, $|s| < 1/2$. Further after factorization we write the equality (3.2) in the form

$$\sigma_+(\xi)\tilde{u}_+(\xi) + \sigma_-^{-1}(\xi)\tilde{u}_-(\xi) = \sigma_-^{-1}(\xi)\widetilde{lv}(\xi),$$

and else

$$\sigma_+(\xi)\tilde{u}_+(\xi) - (\Pi_+(\sigma_-^{-1} \cdot \widetilde{lv}))(\xi) = (\Pi_-(\sigma_-^{-1} \cdot \widetilde{lv}))(\xi) - \sigma_-^{-1}(\xi)\tilde{u}_-(\xi).$$

So the left-hand side belongs to the space $\widetilde{H}^s(\mathbb{R}^m_+)$ and the left-hand side belongs to the space $\widetilde{H}^s(\mathbb{R}^m_-)$, and these should be zero. Hence

$$\tilde{u}_+(\xi) = \sigma_+^{-1}(\xi)(\Pi_+(\sigma_-^{-1} \cdot \widetilde{lv}))(\xi).$$

It completes the proof. $\triangle$

4.2 General Solution

Let æ $\in \mathbb{Z}$. First we introduce a function

$$\omega(\xi', \xi_m) = \left(\frac{\xi_m - i|\xi'| - i}{\xi_m + i|\xi'| + i}\right)^{æ},$$

which belongs to $C(\dot{\mathbb{R}}^m)$.

Evidently the functions $z \pm i|\xi'|$ for fixed $\xi' \in \mathbb{R}^{m-1}$ are analytical functions in complex half planes $\mathbb{C}_\pm$. Moreover

$$\frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} d \arg \frac{\xi_m - i|\xi'| - i}{\xi_m + i|\xi'| + i} = 1.$$

According to the index property [1–3, 7], a function

$$\omega^{-1}(\xi', \xi_m)\sigma(\xi', \xi_m)$$

has a vanishing index, and it can be factorized

$$\omega^{-1}(\xi', \xi_m)\sigma(\xi', \xi_m) = \sigma_+(\xi', \xi_m)\sigma_-(\xi', \xi_m),$$

so we have

$$\sigma(\xi', \xi_m) = \omega(\xi', \xi_m)\sigma_+(\xi', \xi_m)\sigma_-(\xi', \xi_m),$$

where

$$\sigma_\pm(\xi', \xi_m) = \exp(\Psi^\pm(\xi', \xi_m)), \quad \Psi^\pm(\xi', \xi_m) = \frac{1}{2\pi i} \lim_{\tau \to 0+} \int\limits_{-\infty}^{+\infty} \frac{\ln(\omega^{-1}\sigma)(\xi, \eta_m) d\eta_m}{\xi_m \pm i\tau - \eta_m}.$$

Now the Eq. (3.2), we rewrite in the form

$$(\xi_m + i|\xi'| + i)^{-æ}\sigma_+(\xi)\tilde{u}_+(\xi) + (\xi_m - i|\xi'| - i)^{-æ}\sigma_-^{-1}(\xi)\tilde{u}_-(\xi)$$
$$= (\xi_m - i|\xi'| - i)^{-æ}\sigma_-^{-1}(\xi)\widetilde{lv}(\xi). \qquad (4.1)$$

Let us note the right-hand side of the Eq. (4.1) belongs to the space $\widetilde{H}^{s+æ}(\mathbb{R}^m)$. If $|æ + s| < 1/2$, we go to Sect. 4.1.

4.2.1 Positive Case

If $s + æ > 1/2$, we choose a minimal $n \in \mathbb{N}$ so that $0 < s + æ - n < 1/2$. Further we use a decomposition formula for operators $\Pi_\pm$ [1] for $\tilde{f} \in \widetilde{H}^{s+æ}(\mathbb{R}^m)$

$$\Pi_\pm \tilde{f} = \sum_{k=1}^{n} \frac{\Pi' \Lambda_\pm^{k-1}\tilde{f}}{\Lambda_\pm^k} + \frac{1}{\Lambda_\pm^n}\Pi_\pm \Lambda_\pm^n \tilde{f}, \qquad (4.2)$$

where

$$\Lambda_\pm(\xi', \xi_m) = \xi_m \pm |\xi'| \pm i, \quad (\Pi'\tilde{f})(\xi') = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \tilde{f}(\xi', \xi_m)d\xi_m.$$

We rewrite the Eq. (4.1)

$$\sigma_+(\xi)\tilde{w}_+(\xi) + \sigma_-^{-1}(\xi)\tilde{w}_-(\xi) = \tilde{h}(\xi),$$

where $\tilde{w}_\pm(\xi) = (\xi_m \pm i|\xi'| \pm i)^{-æ}\tilde{u}_\pm(\xi), \tilde{h}(\xi) = (\xi_m - i|\xi'| - i)^{-æ}\sigma_-^{-1}(\xi)\widetilde{lv}(\xi)$.

Obviously $\tilde{w}_\pm \in \widetilde{H}^{s+æ}(\mathbb{R}^m_\pm), \tilde{h} \in \widetilde{H}^{s+æ}(\mathbb{R}^m)$. We set $s+æ-n = \alpha, 0 < \alpha < 1/2$. Since $s + æ = n + \alpha > \alpha$ then $\tilde{h} \in \widetilde{H}^{s+æ}(\mathbb{R}^m) \Longrightarrow h \in \widetilde{H}^{\alpha}(\mathbb{R}^m)$. According to Theorem 4.1, we have a solution of the last equation $\tilde{w}_+ \in \widetilde{H}^{\alpha}(\mathbb{R}^m_+)$ in the form

$$\tilde{w}_+(\xi) = \sigma_+^{-1}(\xi)(\Pi_+\tilde{h})(\xi).$$

Thus

$$\tilde{u}_+(\xi) = (\xi_m + i|\xi'| + i)^{æ}\sigma_+^{-1}(\xi)(\Pi_+\tilde{h})(\xi),$$

so that $\tilde{u}_+ \in \widetilde{H}^{\alpha-æ}(\mathbb{R}^m_+)$. Now we apply the formula (4.2) to the expression $\Pi_+\tilde{h}$ and obtain the following representation

$$\tilde{u}_+(\xi) = \sum_{k=1}^{n} \frac{\tilde{c}_k(\xi')}{\sigma_+(\xi)\Lambda_+^{k-æ}(\xi', \xi_m)} + \frac{1}{\sigma_+(\xi)\Lambda_+^{n-æ}(\xi', \xi_m)}(\Pi_\pm\Lambda_+^n\tilde{h})(\xi', \xi_m), \quad (4.3)$$

where $\tilde{c}_k = (\Pi' \Lambda_+^{k-1})\tilde{h}$. It is not hard concluding $\tilde{c}_k \in \widetilde{H}^{s_k}(\mathbb{R}^{m-1})$, $s_k = s + æ - k + 1/2$. So we have the following

Proposition 4.2. *If $s + æ > 1/2$, then for the solution of the Eq. (3.1), the representation (4.3) is valid.*

Note. One can prove that the functions $\tilde{c}_k \in \widetilde{H}^{s_k}(\mathbb{R}^{m-1}) and s_k = s + æ - k + 1/2$ are defined uniquely.

4.2.2 Negative Case

If $s + æ < -1/2$, we choose a polynomial $Q_n(\xi)$ without real zeroes so that $-1/2 < s + æ + n < 0$, and use the equality

$$\sigma_+(\xi)\tilde{w}_+(\xi) + \sigma_-^{-1}(\xi)\tilde{w}_-(\xi) = \tilde{h}(\xi)$$

from Sect. 4.2.1 once again. Since $\tilde{h} \in \widetilde{H}^{s+æ}(\mathbb{R}^m)$, we represent

$$\tilde{h} = Q\Pi_+(Q^{-1}\tilde{h}) + Q\Pi_-(Q^{-1}\tilde{h})$$

because $Q^{-1}\tilde{h} \in \widetilde{H}^{s+æ+n}(\mathbb{R}^m)$. Further we work with the equality

$$\sigma_+(\xi)\tilde{w}_+(\xi) + \sigma_-^{-1}(\xi)\tilde{w}_-(\xi) = Q\Pi_+(Q^{-1}\tilde{h}) + Q\Pi_-(Q^{-1}\tilde{h})$$

or in other words

$$\sigma_+(\xi)\tilde{w}_+(\xi) - Q\Pi_+(Q^{-1}\tilde{h}) = Q\Pi_-(Q^{-1}\tilde{h}) - \sigma_-^{-1}(\xi)\tilde{w}_-(\xi)$$

So the left-hand side belongs to the space $\widetilde{H}^{s+æ}(\mathbb{R}^m_+)$, and the left-hand side belongs to the space $\widetilde{H}^{s+æ}(\mathbb{R}^m_-)$ so it is distribution supported on $\mathbb{R}^{m-1}$. Its general form in Fourier images is [1]

$$\sum_{j=1}^{n} \tilde{c}_j(\xi')\xi_m^{j-1}.$$

Thus we have the formula ($\tilde{g}_+ = \Pi_+(Q^{-1}\tilde{h})$

$$(\xi_m + i|\xi'| + i)^{-æ}\sigma_+(\xi)\tilde{u}_+(\xi) - Q_n(\xi)g_+(\xi) = \sum_{j=1}^{n} \tilde{c}_j(\xi')\xi_m^{j-1}.$$

and a lot of solutions

$$\tilde{u}_+(\xi) = (\xi_m + i|\xi'| + i)^{æ}\sigma_+^{-1}(\xi)Q_n(\xi)g_+(\xi) + (\xi_m + i|\xi'| + i)^{æ}\sigma_+^{-1}(\xi)\sum_{j=1}^{n} \tilde{c}_j(\xi')\xi_m^{j-1}.$$

It is left to verify that functions $\tilde{C}_j(\xi) = (\xi_m + i|\xi'| + i)^{æ}\sigma_+^{-1}(\xi)\tilde{c}_j(\xi')\xi_m^j$ belong to $\widetilde{H}^s(\mathbb{R}^m)$. We have

$$||C_j||_s^2 = \int\limits_{\mathbb{R}^m} |\tilde{c}_j(\xi')|^2 |\xi_m + i|\xi'| + i|^{2æ} |\sigma_+^{-2}(\xi)| |\xi_m|^{2j} (1 + |\xi|)^{2s} d\xi,$$

and passing to repeated integral, we first calculate

$$\int\limits_{-\infty}^{+\infty} |\xi_m + i|\xi'| + i|^{2æ} |\xi_m|^{2j} (1 + |\xi|)^{2s} d\xi_m,$$

which exists only if $æ + j + s < -1/2$. Hence we obtain after integration that $C_j \in H^{æ+j+s+1/2}(\mathbb{R}^{m-1})$.

Thus we have proved the following

Theorem 4.3. *If $s + æ < -1/2$, then the Eq. (3.1) has many solutions in the space $H^s(\mathbb{R}_+^m)$, and the formula for a general solution in Fourier image*

$$\tilde{u}_+(\xi', \xi_m) = (\xi_m + i|\xi'| + i)^{æ}\sigma_+^{-1}(\xi) Q_n(\xi) \tilde{g}_+(\xi', \xi_m) + (\xi_m + i|\xi'| + i)^{æ}\sigma_+^{-1}(\xi', \xi_m) \sum_{k=0}^{æ-1} c_k(\xi')\xi_m^k$$

holds, where $c_k \in H^{s_k}(\mathbb{R}^{m-1})$, $s_k = -æ + k + 1/2$, $k = 0, \cdots, æ - 1$ are arbitrary functions.

Corollary 4.4. *If under assumptions of the Theorem 4.3 $v \equiv 0$, then a general solution of the equation*

$$(\mathcal{D}u)(x) = 0, \quad x \in \mathbb{R}_+^m \tag{4.4}$$

has the form

$$\tilde{u}_+(\xi) = (\xi_m + i|\xi'| + i)^{æ}\sigma_+^{-1}(\xi', \xi_m) \sum_{k=1}^{n} \tilde{c}_k(\xi')\xi_m^{k-1}. \tag{4.5}$$

4.3 Boundary Conditions

For a brevity we consider a homogeneous equation using the Corollary 4.4. We need some additional conditions to uniquely determine the functions $\tilde{c}_k, k = 1, \cdots, n$. It is an interesting fact that we cannot use the same conditions for positive and negative æ. Moreover the boundary operators in a certain sense are determined by the formula for a general solution. We consider below very simple boundary operators. Usually such operators are traces of some pseudo-differential operators on the hyperplane $x_m = 0$. But it is possible not for all cases.

4.3.1 Positive Case

Let us assume we know the values of $\tilde{u}_+$ in n distinct hyperplanes from $\mathbb{R}^m$ of type $\xi_m = p_j$. We denote $\tilde{u}_+(\xi', p_j) \equiv \tilde{r}_j(\xi')$ and obtain from the formula (4.5) the following system of linear algebraic equations

$$\sum_{k=1}^{n} \tilde{c}_k(\xi')p_j^{k-1} = \tilde{r}_j(\xi')(p_j + i|\xi'| + i)^{æ}\sigma_+^{-1}(\xi', p_j), \qquad j = 1, \cdots, n.$$

Obviously the system is uniquely solvable because its matrix has the Vandermonde determinant. To formulate a corresponding boundary value problem, we need some preliminaries.

We take the following boundary conditions

$$\int_{-\infty}^{+\infty} u_+(x', x_m)e^{-ip_jx_m}dx_m = r_j(x'), \quad j = 1, \cdots, n. \tag{4.6}$$

It will mean $\tilde{u}_+(\xi', p_j) = \tilde{r}_j(\xi')$. If $u_+ \in H^s(\mathbb{R}^m_+)$ then $r_j \in H^{s-1/2}(\mathbb{R}^m_+)$ [1]. So we have the following

Theorem 4.5. *Let $r_j \in H^{s-1/2}(\mathbb{R}^{m-1}), j = 1, \cdots, n$. Then the boundary value problem* (4.4)*,* (4.6) *has a unique solution in the space $H^s(\mathbb{R}^m_+)$.*

Note. One can consider a linear combination of the conditions (4.6) and require nonvanishing the associated determinant.

4.3.2 Negative Case

This case admits integration for the right-hand side of the formula (4.5); thus, we take boundary conditions in the standard form

$$(A_ju_+)(x)_{|_{x_m=0}} = r_j(x'), \quad j = 1, \cdots, n, \tag{4.7}$$

where A_j are pseudo-differential operators with symbols $A_j(\xi', \xi_m)$ satisfying the condition

$$|A_j(\xi', \xi_m) \sim (1 + |\xi'| + |\xi_m|)^{\gamma_j}.$$

Let us denote

$$a_{jk}(\xi') = \int\limits_{-\infty}^{+\infty} A_j(\xi', \xi_m)(\xi_m + i|\xi'| + i)^{æ}\sigma_+^{-1}(\xi', \xi_m)\xi_m^{k-1}d\xi_m.$$

Theorem 4.6. *Let* $\gamma_j + æ + k < -1, r_j \in H^{s_j}(\mathbb{R}^{m-1}), s_j = s - \gamma_j - 1/2, \ \forall j, k = 1, \cdots, n$, *and the*

$$\inf_{\xi' \in \mathbb{R}^{m-1}} |\det(a_{jk}(\xi'))_{j,k=1}^n| > 0.$$

Then the boundary value problem (4.4),(4.7) *has a unique solution in the space* $H^s(\mathbb{R}^m_+)$.

5 Conclusion

There are a lot of possibilities to state distinct problems for the Eq. (3.1) adding some additional conditions. Also it seems to be interesting to transfer this approach and results to a discrete case, i.e., for spaces of a discrete variable. This will be discussed elsewhere.

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Discrete Dirac-Kähler and Hestenes Equations

Volodymyr Sushch

Abstract A discrete analogue of the Dirac equation in the Hestenes form is constructed by introduction of the Clifford product on the space of discrete forms. We discuss the relation between the discrete Dirac-Kähler equation and the discrete Hestenes equation.

Keywords Dirac-Kähler equation • Hestenes equation • Clifford product • Discrete models • Difference equations

Mathematics Subject Classification (2000): 81Q05, 39A12, 81R05

1 Introduction

The purpose of this paper is to discuss the relation between the discrete Dirac-Kähler equation which was constructed in [9, 10] and a discrete analogue of the Hestenes equation. We show that the geometric discretization scheme as developed in [10] can be used to find a new discrete formulation of the Dirac equation for a free electron in the Hestenes form.

We first briefly review some definitions and basic notation on the Dirac-Kähler equation [6, 8]. Let $M = \mathbb{R}^{1,3}$ be Minkowski space with metric signature $(+,-,-,-)$. Denote by $\Lambda^r(M)$ the vector space of smooth differential r-forms, $r = 0, 1, 2, 3, 4$. We consider $\Lambda^r(M)$ over $\mathbb{C}$. Let $d : \Lambda^r(M) \rightarrow \Lambda^{r+1}(M)$ be the exterior differential and let $\delta : \Lambda^r(M) \rightarrow \Lambda^{r-1}(M)$ be the formal adjoint of d with respect to the natural inner product in $\Lambda^r(M)$ (codifferential). We have $\delta = *d*$, where $*$ is the Hodge star operator $* : \Lambda^r(M) \rightarrow \Lambda^{4-r}(M)$ with respect to the Lorentz metric. Denote by $\Lambda(M)$ the set of all differential forms on M. We have

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$$\Lambda(M) = \Lambda^0(M) \oplus \Lambda^1(M) \oplus \Lambda^2(M) \oplus \Lambda^3(M) \oplus \Lambda^4(M) = \Lambda^{ev}(M) \oplus \Lambda^{od}(M),$$

where $\Lambda^{ev}(M) = \Lambda^0(M) \oplus \Lambda^2(M) \oplus \Lambda^4(M)$ and $\Lambda^{od}(M) = \Lambda^1(M) \oplus \Lambda^3(M)$.

Let $\Omega \in \Lambda(M)$ be an inhomogeneous differential form and then $\Omega = \sum_{r=0}^{4} \overset{r}{\omega}$, where $\overset{r}{\omega} \in \Lambda^r(M)$. Denote by Ω^{ev} and by Ω^{od} the even and odd parts of Ω, i.e. $\Omega^{ev} = \overset{0}{\omega} + \overset{2}{\omega} + \overset{4}{\omega}$ and $\Omega^{od} = \overset{1}{\omega} + \overset{3}{\omega}$. The Dirac-Kähler equation is given by

$$i(d + \delta)\Omega = m\Omega, \tag{1}$$

where i is the usual complex unit ($i^2 = -1$) and m is a mass parameter. It is easy to show that Eq. (1) is equivalent to the set of equations

$$i(d + \delta)\Omega^{od} = m\Omega^{ev}, \qquad i(d + \delta)\Omega^{ev} = m\Omega^{od}.$$

The operator $d + \delta$ is the analogue of the gradient operator in Minkowski space-time $\nabla = \sum_{\mu=0}^{3} \gamma_\mu \partial^\mu$, $\mu = 0, 1, 2, 3$, where γ_μ is the Dirac gamma matrix. Think of $\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$ as a vector basis in space-time. Then the gamma matrices γ_μ can be considered as generators of the Clifford algebra of space-time $C\ell(1, 3)$ [2]. The complex Clifford algebra $C\ell(1, 3)$ is a complex 16-dimensional vector space. It is known that an inhomogeneous form Ω can be represented as element of $C\ell(1, 3)$ over the complex field $\mathbb{C}$. Then the Dirac-Kähler equation can be written as an algebraic equation in $C\ell(1, 3)$ over $\mathbb{C}$

$$i\nabla\Omega = m\Omega, \quad \Omega \in C\ell(1, 3). \tag{2}$$

Equation (2) is equivalent to the four Dirac equations (traditional column-spinor equations) for a free electron. Let $C\ell^{ev}(1, 3)$ be the even subalgebra of the real algebra $C\ell(1, 3)$. The equation

$$-\nabla\Omega\gamma_1\gamma_2 = m\Omega\gamma_0, \quad \Omega \in C\ell^{ev}(1, 3) \tag{3}$$

is called the Hestenes form of the Dirac equation [4, 5]. The Hestenes equation is equivalent to the Dirac equation [5, 7]. Suppose that for exterior forms (elements of $\Lambda(M)$) the Clifford multiplication is defined. In this case the basis covectors $e^\mu = dx^\mu$ and $\mu = 0, 1, 2, 3$ of space-time are considered as generators of the Clifford algebra. The resulting algebra $\Lambda(M)$ with two multiplications is called the Grassmann-Clifford bialgebra [7]. Thus, Eq. (3) can be rewritten in terms of inhomogeneous forms as

$$-(d + \delta)\Omega e^1 e^2 = m\Omega e^0, \tag{4}$$

where $\Omega \in \Lambda^{ev}(M)$ is a real-valued form.

In this paper we construct a discrete analogue of the Hestenes Eq. (4) by introduction of the Clifford product on the space of discrete forms. In much the same way as in the continuum case [1], it is shown that a solution of the discrete Dirac-Kähler equation gives rise to four independent solutions of the discrete Hestenes equation. Note that the discrete model is expressed clearly in terms of difference equations.

2 Discrete Dirac-Kähler Equation

We use a discretization scheme based on the language of differential forms and the double complex construction which is described in our preceding paper [10]. Due to space limitations, this paper does not include the relevant material from [10]. We refer the reader to [9, 10] for full mathematical details of the approach. This approach was originated by Dezin [3]. Let $K(4) = K \otimes K \otimes K \otimes K$ be a cochain complex with complex coefficients, where K is the one-dimensional complex generated by zero- and one-dimensional basis elements x^κ and e^κ, $\kappa \in \mathbb{Z}$, respectively. Then an arbitrary r-dimensional basis element of $K(4)$ can be written as $s^k_{(r)} = s^{k_0} \otimes s^{k_1} \otimes s^{k_2} \otimes s^{k_3}$, where s^{k_μ} is either x^{k_μ} or e^{k_μ}, $k = (k_0, k_1, k_2, k_3)$ and $k_\mu \in \mathbb{Z}$. The dimension r of a basis element $s^k_{(r)}$ is given by the number of factors e^{k_μ} that appear in it. For example, the one-dimensional basis elements of $K(4)$ can be written as

$$e^k_0 = e^{k_0} \otimes x^{k_1} \otimes x^{k_2} \otimes x^{k_3}, \qquad e^k_1 = x^{k_0} \otimes e^{k_1} \otimes x^{k_2} \otimes x^{k_3},$$
$$e^k_2 = x^{k_0} \otimes x^{k_1} \otimes e^{k_2} \otimes x^{k_3}, \qquad e^k_3 = x^{k_0} \otimes x^{k_1} \otimes x^{k_2} \otimes e^{k_3},$$

where the subscript $\mu = 0, 1, 2, 3$ indicates a place of e^{k_μ} in e^k_μ. The complex $K(4)$ is a discrete analogue of $\Lambda(M)$. The cochains we will call forms, emphasising their relationship with the corresponding continuum objects, namely differential forms. Denote by $K^r(4)$ the set of all r-forms. Then we have

$$K(4) = K^0(4) \oplus K^1(4) \oplus K^2(4) \oplus K^3(4) \oplus K^4(4) = K^{ev}(4) \oplus K^{od}(4),$$

where $K^{ev}(4) = K^0(4) \oplus K^2(4) \oplus K^4(4)$ and $K^{od}(4) = K^1(4) \oplus K^3(4)$. Any r-form $\overset{r}{\omega} \in K^r(4)$ can be expressed as

$$\overset{0}{\omega} = \sum_k \overset{0}{\omega}_k x^k, \qquad \overset{4}{\omega} = \sum_k \overset{4}{\omega}_k e^k, \tag{5}$$

where $x^k = x^{k_0} \otimes x^{k_1} \otimes x^{k_2} \otimes x^{k_3}$ and $e^k = e^{k_0} \otimes e^{k_1} \otimes e^{k_2} \otimes e^{k_3}$, and

$$\overset{1}{\omega} = \sum_k \sum_{\mu=0}^{3} \omega^\mu_k e^k_\mu, \qquad \overset{2}{\omega} = \sum_k \sum_{\mu<\nu} \omega^{\mu\nu}_k e^k_{\mu\nu}, \qquad \overset{3}{\omega} = \sum_k \sum_{\iota<\mu<\nu} \omega^{\iota\mu\nu}_k e^k_{\iota\mu\nu}, \tag{6}$$

where e^k_μ, $e^k_{\mu\nu}$ and $e^k_{\iota\mu\nu}$ are one-, two- and three-dimensional basic elements of $K(4)$. The components $\overset{0}{\omega}_k$, $\overset{4}{\omega}_k$, ω^μ_k, $\omega^{\mu\nu}_k$ and $\omega^{\iota\mu\nu}_k$ are complex numbers. A discrete inhomogeneous form $\Omega \in K(4)$ is defined to be

$$\Omega = \overset{0}{\omega} + \overset{1}{\omega} + \overset{2}{\omega} + \overset{3}{\omega} + \overset{4}{\omega}. \tag{7}$$

Let $d^c : K^r(4) \to K^{r+1}(4)$ be a discrete analogue of the exterior derivative d and let $\delta^c : K^r(4) \to K^{r-1}(4)$ be a discrete analogue of the codifferential δ. For definitions of these operators and other discrete operations (the $\cup$-multiplication, the discrete Hodge star and so on), we refer the reader to [10]. In this paper we give only the difference expression for d^c and δ^c. Let the difference operator Δ_μ be defined by

$$\Delta_\mu \omega^{(r)}_k = \omega^{(r)}_{\tau_\mu k} - \omega^{(r)}_k, \tag{8}$$

where $\omega^{(r)}_k \in \mathbb{C}$ is a component of $\overset{r}{\omega} \in K^r(4)$ and τ_μ is the shift operator which acts as $\tau_\mu k = (k_0, \dots k_\mu + 1, \dots k_3)$, $\mu = 0, 1, 2, 3$. For forms (5) and (6), we have

$$d^c \overset{0}{\omega} = \sum_k \sum_{\mu=0}^{3} \left(\Delta_\mu \overset{0}{\omega}_k \right) e^k_\mu, \qquad d^c \overset{1}{\omega} = \sum_k \sum_{\mu<\nu} \left(\Delta_\mu \omega^\nu_k - \Delta_\nu \omega^\mu_k \right) e^k_{\mu\nu}, \tag{9}$$

$$\begin{aligned} d^c \overset{2}{\omega} = \sum_k \big[&\left(\Delta_0 \omega^{12}_k - \Delta_1 \omega^{02}_k + \Delta_2 \omega^{01}_k \right) e^k_{012} + \left(\Delta_0 \omega^{13}_k - \Delta_1 \omega^{03}_k + \Delta_3 \omega^{01}_k \right) e^k_{013} \\ &+ \left(\Delta_0 \omega^{23}_k - \Delta_2 \omega^{03}_k + \Delta_3 \omega^{02}_k \right) e^k_{023} + \left(\Delta_1 \omega^{23}_k - \Delta_2 \omega^{13}_k + \Delta_3 \omega^{12}_k \right) e^k_{123} \big], \end{aligned} \tag{10}$$

$$d^c \overset{3}{\omega} = \sum_k \left(\Delta_0 \omega^{123}_k - \Delta_1 \omega^{023}_k + \Delta_2 \omega^{013}_k - \Delta_3 \omega^{012}_k \right) e^k, \qquad d^c \overset{4}{\omega} = 0, \tag{11}$$

$$\delta^c \overset{0}{\omega} = 0, \qquad \delta^c \overset{1}{\omega} = \sum_k \left(\Delta_0 \omega^0_k - \Delta_1 \omega^1_k - \Delta_2 \omega^2_k - \Delta_3 \omega^3_k \right) x^k, \tag{12}$$

$$\begin{aligned} \delta^c \overset{2}{\omega} = \sum_k \big[&\left(\Delta_1 \omega^{01}_k + \Delta_2 \omega^{02}_k + \Delta_3 \omega^{03}_k \right) e^k_0 + \left(\Delta_0 \omega^{01}_k + \Delta_2 \omega^{12}_k + \Delta_3 \omega^{13}_k \right) e^k_1 \\ &+ \left(\Delta_0 \omega^{02}_k - \Delta_1 \omega^{12}_k + \Delta_3 \omega^{23}_k \right) e^k_2 + \left(\Delta_0 \omega^{03}_k - \Delta_1 \omega^{13}_k - \Delta_2 \omega^{23}_k \right) e^k_3 \big], \end{aligned} \tag{13}$$

$$\begin{aligned} \delta^c \overset{3}{\omega} = \sum_k \big[&\left(-\Delta_2 \omega^{012}_k - \Delta_3 \omega^{013}_k \right) e^k_{01} + \left(\Delta_1 \omega^{012}_k - \Delta_3 \omega^{023}_k \right) e^k_{02} \\ &+ \left(\Delta_1 \omega^{013}_k + \Delta_2 \omega^{023}_k \right) e^k_{03} + \left(\Delta_0 \omega^{012}_k - \Delta_3 \omega^{123}_k \right) e^k_{12} \\ &+ \left(\Delta_0 \omega^{013}_k + \Delta_2 \omega^{123}_k \right) e^k_{13} + \left(\Delta_0 \omega^{023}_k - \Delta_1 \omega^{123}_k \right) e^k_{23} \big], \end{aligned} \tag{14}$$

$$\delta^c \overset{4}{\omega} = \sum_k \left[\left(\Delta_3 \overset{4}{\omega}_k \right) e^k_{012} - \left(\Delta_2 \overset{4}{\omega}_k \right) e^k_{013} + \left(\Delta_1 \overset{4}{\omega}_k \right) e^k_{023} + \left(\Delta_0 \overset{4}{\omega}_k \right) e^k_{123} \right]. \quad (15)$$

Let $\Omega \in K(4)$ be given by (7). A discrete analogue of the Dirac-Kähler Eq. (1) can be defined as

$$i(d^c + \delta^c)\Omega = m\Omega. \quad (16)$$

We can write this equation more explicitly by separating its homogeneous components as

$$i\delta^c \overset{1}{\omega} = m\overset{0}{\omega}, \quad i(d^c \overset{1}{\omega} + \delta^c \overset{3}{\omega}) = m\overset{2}{\omega}, \quad id^c \overset{3}{\omega} = m\overset{4}{\omega},$$
$$i(d^c \overset{0}{\omega} + \delta^c \overset{2}{\omega}) = m\overset{1}{\omega}, \qquad i(d^c \overset{2}{\omega} + \delta^c \overset{4}{\omega}) = m\overset{3}{\omega}. \quad (17)$$

Substituting (9)–(15) into (17), one obtains the set of 16 difference equations [10].

3 Clifford Multiplication in $K(4)$ and Discrete Hestenes Equation

Let us define the Clifford multiplication in $K(4)$ by the following rules:

1. $x^k e^k_\mu = e^k_\mu x^k = e^k_\mu, \quad \mu = 0, 1, 2, 3;$
2. $e^k_\mu e^k_\nu + e^k_\nu e^k_\mu = 2g_{\mu\nu}x^k$, where $g_{\mu\nu} = diag(1, -1, -1, -1)$ is the metric tensor;
3. $e^k_{\mu_1} \cdots e^k_{\mu_s} = e^k_{\mu_1 \cdots \mu_s}$ for $0 \le \mu_1 < \cdots < \mu_s \le 3$.

Note that the multiplication is defined for the basis elements of $K(4)$ with the same multi-index $k = (k_0, k_1, k_2, k_3)$ supposing the product to be zero in all other cases. The operation is linearly extended to arbitrary discrete forms. For example, for any $\overset{1}{\omega}, \overset{1}{\varphi} \in K^1(4)$, we have

$$\overset{1}{\omega}\overset{1}{\varphi} = \left(\sum_k \sum_{\mu=0}^{3} \omega^\mu_k e^k_\mu \right) \left(\sum_k \sum_{\mu=0}^{3} \varphi^\mu_k e^k_\mu \right) = \sum_k \left(\omega^0_k \varphi^0_k - \omega^1_k \varphi^1_k - \omega^2_k \varphi^2_k - \omega^3_k \varphi^3_k \right) x^k$$
$$+ \sum_k \left[\left(\omega^0_k \varphi^1_k - \omega^1_k \varphi^0_k \right) e^k_{01} + \left(\omega^0_k \varphi^2_k - \omega^2_k \varphi^0_k \right) e^k_{02} + \left(\omega^0_k \varphi^3_k - \omega^3_k \varphi^0_k \right) e^k_{03} \right.$$
$$\left. + \left(\omega^1_k \varphi^2_k - \omega^2_k \varphi^1_k \right) e^k_{12} + \left(\omega^1_k \varphi^3_k - \omega^3_k \varphi^1_k \right) e^k_{13} + \left(\omega^2_k \varphi^3_k - \omega^3_k \varphi^2_k \right) e^k_{23} \right].$$

Proposition 1. *For any inhomogeneous form $\Omega \in K(4)$, we have*

$$(d^c + \delta^c)\Omega = \sum_{\mu=0}^{3} e_\mu \Delta_\mu \Omega, \quad (18)$$

where

$$e_\mu = \sum_k e_\mu^k, \qquad \mu = 0, 1, 2, 3, \tag{19}$$

and Δ_μ is the difference operator which acts on each component of Ω by the rule (8).

Proof. We prove the claim only for the case of even forms. Similar calculations apply to the case of odd forms. Let $\Omega^{ev} = \overset{0}{\omega} + \overset{2}{\omega} + \overset{4}{\omega}$ be the even part of Ω. We have

$$\sum_{\mu=0}^{3} e_\mu \Delta_\mu \overset{0}{\omega} = \sum_k \left(\Delta_0 \overset{0}{\omega}_k e_0^k + \Delta_1 \overset{0}{\omega}_k e_1^k + \Delta_2 \overset{0}{\omega}_k e_2^k + \Delta_3 \overset{0}{\omega}_k e_3^k \right),$$

$$\begin{aligned}
\sum_{\mu=0}^{3} e_\mu \Delta_\mu \overset{2}{\omega} = \sum_k \big[& \left(\Delta_1 \omega_k^{01} + \Delta_2 \omega_k^{02} + \Delta_3 \omega_k^{03} \right) e_0^k \\
& + \left(\Delta_0 \omega_k^{01} + \Delta_2 \omega_k^{12} + \Delta_3 \omega_k^{13} \right) e_1^k + \left(\Delta_0 \omega_k^{02} - \Delta_1 \omega_k^{12} + \Delta_3 \omega_k^{23} \right) e_2^k \\
& + \left(\Delta_0 \omega_k^{03} - \Delta_1 \omega_k^{13} - \Delta_2 \omega_k^{23} \right) e_3^k \big] \\
& + \sum_k \big[\left(\Delta_0 \omega_k^{12} - \Delta_1 \omega_k^{02} + \Delta_2 \omega_k^{01} \right) e_{012}^k \\
& + \left(\Delta_0 \omega_k^{13} - \Delta_1 \omega_k^{03} + \Delta_3 \omega_k^{01} \right) e_{013}^k \\
& + \left(\Delta_0 \omega_k^{23} - \Delta_2 \omega_k^{03} + \Delta_3 \omega_k^{02} \right) e_{023}^k \\
& + \left(\Delta_1 \omega_k^{23} - \Delta_2 \omega_k^{13} + \Delta_3 \omega_k^{12} \right) e_{123}^k \big],
\end{aligned}$$

$$\sum_{\mu=0}^{3} e_\mu \Delta_\mu \overset{4}{\omega} = \sum_k \left(\Delta_0 \overset{4}{\omega}_k e_{123}^k + \Delta_1 \overset{4}{\omega}_k e_{023}^k - \Delta_2 \overset{4}{\omega}_k e_{013}^k + \Delta_3 \overset{4}{\omega}_k e_{012}^k \right).$$

Summing both sides of the above and using (9)–(15), we obtain

$$\begin{aligned}
\sum_{\mu=0}^{3} e_\mu \Delta_\mu \Omega^{ev} &= \sum_{\mu=0}^{3} e_\mu \Delta_\mu (\overset{0}{\omega} + \overset{2}{\omega} + \overset{4}{\omega}) \\
&= d^c(\overset{0}{\omega} + \overset{2}{\omega}) + \delta^c(\overset{2}{\omega} + \overset{4}{\omega}) = (d^c + \delta^c)\Omega^{ev}.
\end{aligned}$$

Thus the discrete Dirac-Kähler equation can be rewritten in the form

$$i \sum_{i=0}^{3} e_\mu \Delta_\mu \Omega = m\Omega.$$

Let $\Omega \in K^{ev}(4)$ be a real-valued even inhomogeneous form. A discrete analogue of the Hestenes Eq. (4) is defined by

$$-(d^c + \delta^c)\Omega e_1 e_2 = m\Omega e_0, \tag{20}$$

where e_0, e_1 and e_2 are given by (19). This equation can be expressed in terms of difference equations. Substituting (9)–(15) into (20) and by the rules (1)–(3), we obtain

$$\Delta_0\omega_k^{12} - \Delta_1\omega_k^{02} + \Delta_2\omega_k^{01} + \Delta_3\overset{4}{\omega}_k = m\overset{0}{\omega}_k,$$
$$\Delta_2\overset{0}{\omega}_k + \Delta_0\omega_k^{02} - \Delta_1\omega_k^{12} + \Delta_3\omega_k^{23} = m\omega_k^{01},$$
$$-\Delta_1\overset{0}{\omega}_k - \Delta_0\omega_k^{01} - \Delta_2\omega_k^{12} - \Delta_3\omega_k^{13} = m\omega_k^{02},$$
$$-\Delta_1\omega_k^{23} + \Delta_2\omega_k^{13} - \Delta_3\omega_k^{12} - \Delta_0\overset{4}{\omega}_k = m\omega_k^{03},$$
$$-\Delta_0\overset{0}{\omega}_k - \Delta_1\omega_k^{01} - \Delta_2\omega_k^{02} - \Delta_3\omega_k^{03} = m\omega_k^{12},$$
$$-\Delta_0\omega_k^{23} + \Delta_2\omega_k^{03} - \Delta_3\omega_k^{02} - \Delta_1\overset{4}{\omega}_k = m\omega_k^{13},$$
$$\Delta_0\omega_k^{13} - \Delta_1\omega_k^{03} + \Delta_3\omega_k^{01} - \Delta_2\overset{4}{\omega}_k = m\omega_k^{23},$$
$$\Delta_3\overset{0}{\omega}_k + \Delta_0\omega_k^{03} - \Delta_1\omega_k^{13} - \Delta_2\omega_k^{23} = m\overset{4}{\omega}_k.$$

Let us introduce the following constant forms:

$$P_{\pm 0} = \frac{1}{2}(x \pm e_0), \qquad P_{\pm 12} = \frac{1}{2}(x \pm ie_1e_2), \tag{21}$$

where $x = \sum_k x^k$ is the unit 0-form, and e_μ is given by (19). Note that x plays a role of the unit element in $K(4)$. It is easy to check that

$$(P_{\pm 0})^2 = P_{\pm 0}P_{\pm 0} = P_{\pm 0}, \qquad (P_{\pm 12})^2 = P_{\pm 12}P_{\pm 12} = P_{\pm 12}.$$

Hence, the forms $P_{\pm 0}$ and $P_{\pm 12}$ are projectors.

Proposition 2. *The projectors $P_{\pm 0}$ and $P_{\pm 12}$ have the following properties:*

$$P_{\pm 0}P_{\pm 12} = P_{\pm 12}P_{\pm 0}, \quad e_0P_{\pm 0} = P_{\pm 0}e_0, \quad e_1e_2P_{\pm 12} = P_{\pm 12}e_1e_2, \tag{22}$$
$$P_{\pm 0} = \pm P_{\pm 0}e_0, \qquad P_{\pm 12} = \pm iP_{\pm 12}e_1e_2. \tag{23}$$

Proof. The proof is a computation.

Let

$$P_{++} = P_{+0}P_{+12}, \quad P_{+-} = P_{+0}P_{-12}, \quad P_{-+} = P_{-0}P_{+12}, \quad P_{--} = P_{-0}P_{-12}. \tag{24}$$

It is obvious that (24) are projectors again.

Proposition 3. *Any inhomogeneous form $\Omega \in K(4)$ decomposes into four parts*

$$\Omega = \Omega P_{++} + \Omega P_{-+} + \Omega P_{+-} + \Omega P_{--}. \tag{25}$$

Proof. By (21) Ω can be represented as

$$\Omega = \Omega P_{+0} + \Omega P_{-0} \quad \text{or} \quad \Omega = \Omega P_{+12} + \Omega P_{-12}.$$

This yields

$$\Omega = (\Omega P_{+0} + \Omega P_{-0})P_{+12} + (\Omega P_{+0} + \Omega P_{-0})P_{-12}.$$

Hence, by (24) we obtain (25).

Recall that the Hestenes equation is defined on real-valued even forms. First suppose that the discrete Hestenes Eq. (20) acts in $K(4)$, i.e. acts in the same space as the discrete Dirac-Kähler equation.

Proposition 4. *Let $\Omega \in K(4)$ be a solution of the discrete Dirac-Kähler equation and then ΩP_{++} and ΩP_{--} satisfy Eq. (20), while ΩP_{-+} and ΩP_{+-} satisfy the same equation but the sign of the right-hand side changed to its opposite.*

Proof. It suffices to prove the claim for one of the projectors (24), say for P_{++}. The other cases are similar. Multiplying Eq. (16) from the right by the projector P_{++}, we obtain

$$i(d^c + \delta^c)\Omega P_{++} = m\Omega P_{++}. \tag{26}$$

Since P_{++} is constant, using (22) and (23), we have

$$\begin{aligned} i(d^c + \delta^c)\Omega P_{++} &= i(d^c + \delta^c)\Omega P_{+0}P_{+12} = i^2(d^c + \delta^c)\Omega P_{+0}P_{+12}e_1e_2 \\ &= -(d^c + \delta^c)(\Omega P_{++})e_1e_2, \end{aligned}$$

$$\Omega P_{++} = \Omega P_{+0}P_{+12} = \Omega P_{+12}P_{+0}e_0 = \Omega P_{++}e_0.$$

Substituting this into (26) yields

$$-(d^c + \delta^c)(\Omega P_{++})e_1e_2 = m(\Omega P_{++})e_0.$$

Let $\overline{\Omega}$ be the complex conjugate of Ω. Consider the real-valued forms Ω_+ and Ω_- given by

$$\Omega_\pm = \pm\frac{1}{2}(\Omega + \overline{\Omega})e_0 \pm \frac{i}{2}(\Omega - \overline{\Omega})e_1e_2. \tag{27}$$

By (22) and (23), it is easy to check that

$$\Omega P_{++} = \Omega_+ P_{++}, \qquad \Omega P_{--} = \Omega_- P_{--}.$$

Hence, if Ω is a solution of the discrete Dirac-Kähler equation, then $\Omega_+ P_{++}$ and $\Omega_- P_{--}$ are solutions of Eq. (20). The forms $\Omega_+ P_{++}$ and $\Omega_- P_{--}$ are complex-valued again. However, if $\Omega_+ P_{++}$ and $\Omega_- P_{--}$ are solutions of Eq. (20) then the real and image parts of these complex-valued forms are also solutions of Eq. (20). This is obvious since the discrete Hestenes equation is real and linear. The real and image parts of $\Omega_+ P_{++}$ are

$$\mathrm{Re}(\Omega_+ P_{++}) = \frac{1}{4}(\Omega_+ + \Omega_+ e_0), \qquad \mathrm{Im}(\Omega_+ P_{++}) = \frac{1}{4}(\Omega_+ e_1e_2 + \Omega_+ e_0e_1e_2).$$

Set

$$\Omega_1 = \Omega_+, \qquad \Omega_2 = \Omega_+ e_0, \qquad \Omega_3 = \Omega_+ e_1e_2, \qquad \Omega_4 = \Omega_+ e_0e_1e_2.$$

Now we take the even part of these forms. A direct computation gives

$$\begin{aligned}
\Omega_1^{ev} &= \tfrac{1}{2}(\Omega^{od} + \overline{\Omega}^{od})e_0 + \tfrac{i}{2}(\Omega^{ev} - \overline{\Omega}^{ev})e_1e_2,\\
\Omega_2^{ev} &= \tfrac{1}{2}(\Omega^{ev} + \overline{\Omega}^{ev}) + \tfrac{i}{2}(\Omega^{od} - \overline{\Omega}^{od})e_0e_1e_2,\\
\Omega_3^{ev} &= \tfrac{1}{2}(\Omega^{od} + \overline{\Omega}^{od})e_0e_1e_2 - \tfrac{i}{2}(\Omega^{ev} - \overline{\Omega}^{ev}),\\
\Omega_4^{ev} &= \tfrac{1}{2}(\Omega^{ev} + \overline{\Omega}^{ev})e_1e_2 - \tfrac{i}{2}(\Omega^{od} - \overline{\Omega}^{od})e_0,
\end{aligned} \tag{28}$$

where Ω^{ev} and Ω^{od} are the even and odd parts of $\Omega = \Omega^{ev} + \Omega^{od}$.

Thus, we have proved the following

Proposition 5. *Let $\Omega \in K(4)$ be a solution of the discrete Dirac-Kähler equation. Then $\Omega_j^{ev} \in K^{ev}(4)$ and $j = 1, 2, 3, 4$ in the form (28) are four independent solutions of the discrete Hestenes Eq. (20).*

It should be noted that taking $\Omega_- P_{--}$ instead of $\Omega_+ P_{++}$, we also obtain four independent solutions of Eq. (20) in the same form (28).

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A Remark on the Fractional Step Theta Scheme for the Nonstationary Stokes Equations

Werner Varnhorn and Florian Zanger

Abstract The second-order convergence of the fractional step theta scheme for the linearized nonstationary Stokes equations with error bounds growing exponentially in time is known (Müller-Urbaniak, Dissertation Universität Heidelberg, 1994). As pointed out in Zanger (Proc Appl Math Mech 12:587–588, 2012), this convergence can even be shown with error bounds independent of time. The proof requires a suitable choice of test functions depending on quite sophisticated coefficients, the existence of which is proved in the present paper. This problem does not appear in the case of simpler methods like the first-order Euler or the second-order Crank-Nicolson scheme: since these schemes do not depend on additional parameters, here the functions $2e^k$ and $e^k + e^{k-1}$, respectively, can be used as test functions (Varnhorn, Math Meth Appl Sci 15(1):39–55, 1992), e^k denoting the discretization error at the k-th time level. In the case of the fractional step theta scheme, however, the question, whether suitable coefficients do exist, proves to be so difficult that the solution of the resulting inequalities constitutes a problem of its own.

Keywords Stokes equations • Fractional step theta • Finite differences • Semi-discretization • Convergence

Mathematics Subject Classification (2000): 35Q35, 35J25, 76D07, 76M20

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1 Introduction

Let $T > 0$ be given and $G \subseteq \mathbb{R}^n$ be a bounded domain with a sufficiently smooth boundary ∂G. In $(0,T) \times G$, we consider a viscous incompressible fluid flow and assume that it can be described by the nonstationary linearized Stokes equations

$$\begin{aligned} u_t - \nu \Delta u + \nabla p &= f, \\ \nabla \cdot u &= 0, \\ u_{|\partial G} = 0\,, \quad u_{|t=0} &= u^0. \end{aligned} \tag{1}$$

These equations represent a system of linear partial differential equations concerning $n+1$ unknown functions: the vector $u = (u_1(t,x), \cdots, u_n(t,x))$ denotes the velocity field and the scalar $p = p(t,x)$ the kinematic pressure function of the fluid at time $t \in (O,T)$ at position $x \in G$ in the linearized approach. The constant $\nu > 0$ is the kinematic viscosity, and the external force density f together with the initial velocity u^0 are given data. In (1), u_t means the partial derivative with respect to the time t, Δ is the Laplacian in $\mathbb{R}^n$, and $\nabla = (\partial_1, \cdots, \partial_n)$ is the gradient, where ∂_j denotes the partial derivative with respect to x_j $(j = 1, \cdots, n)$. The term $\nabla \cdot u = \partial_1 u_1 + \cdots + \partial_n u_n$ denotes the divergence of u, which vanishes due to the incompressibility of the fluid. Finally, the no-slip boundary condition $u_{|\partial G} = 0$ expresses that the fluid adheres to the boundary ∂G. In hydrodynamics, the cases $n = 2$ (planar flow) and $n = 3$ (spatial flow) are considered mostly.

2 The Fractional Step Theta Scheme for the Stokes Equations

Throughout the paper, let $G \subset \mathbb{R}^n$ with $n \geqslant 2$ be a bounded domain with boundary ∂G of class C^6, and let $T > 0$ be given. To introduce our notation, by $C^\infty_{0,\sigma}(G)$, we denote the space of all solenoidal (i. e., $\nabla \cdot \varphi = 0$) vector fields $\varphi \in C_0^\infty(G)^n$, by $H^m(G)$ the Sobolev space $W^{m,2}(G)^n$, and by $L^2_\sigma(G)$ and $V(G)$ the closures of $C^\infty_{0,\sigma}(G)$ in $L^2(G)^n$ and $H^1(G)$, respectively. If

$$P \;:\; L^2(G)^n \longrightarrow L^2_\sigma(G)$$

denotes the Helmholtz projection (see [4]) such that

$$L^2(G)^n \;=\; L^2_\sigma(G) \;\oplus\; \{v \in L^2(G)^n \mid v = \nabla p \;\text{ for some }\; p \in W^{1,2}(G)\},$$

then the following result is well known [6, 7]:

Let $\nu > 0$ be some given viscosity coefficient, let $u^0 \in H^4(G) \cap V(G)$ be some given initial velocity, and let $f \in C([0,T]; H^2(G) \cap L^2_\sigma(G))$ with $f_t \in C([0,T];$

$L^2_\sigma(G)) \cap L^2(0,T;V(G))$ and $f_{tt} \in L^2(0,T,V'(G))$ be some given external force density. Let, moreover, the nonlocal compatibility condition (compare [2, 5])

$$f(0) + \nu P\Delta u^0 \in V(G)$$

be satisfied.[1] Then the linear nonstationary Stokes equations

$$u_t - \nu P\Delta u = f \quad \text{in } (0,T)\times G, \qquad u(0) = u^0 \quad \text{in } G$$

have a unique strongly H^4-continuous solution uniformly in the closed interval $[0,T]$, i. e., it holds $u \in C([0,T];H^4(G)\cap V(G))$ with $u_t \in C([0,T];H^2(G)\cap L^2_\sigma(G))$, $u_{tt} \in C([0,T];L^2_\sigma(G)) \cap L^2(0,T;V(G))$, and $u_{ttt} \in L^2(0,T;V'(G))$.

To approximate the above solution u on a discrete time grid, the so-called fractional step theta scheme can be used. It works as follows: first, for every $N \in \mathbb{N}$, define a time step size $h := T/N$. Then, setting

$$\vartheta := 1 - \frac{1}{2}\sqrt{2} = 0,29289\ldots,$$

this leads to a finite fractional step time grid on $[0,T]$ of the form

$$\{t_i = ih\ ;\ i = 0, \vartheta, 1-\vartheta, 1, 1+\vartheta, 2-\vartheta, 2, 2+\vartheta, \ldots, N-\vartheta, N\}$$

containing $3N+1$ non-equidistant grid points t_i. The corresponding functions $u^i := u(t_i) := u(t_i,\cdot)$ at time $t_i := ih$ of the exact solution u are now approximated by the $3N+1$ vector fields

$$v^0,\ v^\vartheta,\ v^{1-\vartheta},\ v^1,\ v^{1+\vartheta},\ v^{2-\vartheta},\ v^2,\ \ldots\ ,\ v^{N-\vartheta},\ v^N,$$

satisfying $v^0 = u^0$ and the fractional step theta scheme

$$\begin{aligned}
\frac{1}{\vartheta h}\big(v^{n+\vartheta} - v^n\big) - \nu P\Delta\big(\alpha v^{n+\vartheta} + (1-\alpha)\, v^n\big) &= f^n,\\
\frac{1}{(1-2\vartheta)\,h}\big(v^{n+1-\vartheta} - v^{n+\vartheta}\big) - \nu P\Delta\left(\left(1-\alpha\right) v^{n+1-\vartheta} + \alpha v^{n+\vartheta}\right) &= f^{n+1-\vartheta}, \quad (2)\\
\frac{1}{\vartheta h}\big(v^{n+1} - v^{n+1-\vartheta}\big) - \nu P\Delta\big(\alpha v^{n+1} + (1-\alpha)\, v^{n+1-\vartheta}\big) &= f^{n+1-\vartheta},
\end{aligned}$$

where $n = 0,\ldots,N-1$ and α is a fixed parameter satisfying $\frac{1}{2} < \alpha < 1$.

Following the arguments in [7, 8], it can be shown that the finite sequence v^i, $i = 0, \vartheta, 1-\vartheta, \ldots, N$ is uniquely determined by the above fractional step theta scheme, and it holds $v^i \in H^4(G) \cap V(G)$ for all $i = 0, \vartheta, 1-\vartheta, \ldots, N$ using Cattabriga's estimate [1].

[1]For reasons on why nontrivial data u^0 and f that satisfy the compatibility condition actually exist, see [9] and [11, Remark 64].

As the vector fields v^i are considered as approximations of the corresponding values of the exact solution $u^i = u(t_i)$ at time t_i, we call the differences

$$e^i := v^i - u^i \in H^4(G) \cap V(G), \qquad i = 0, \vartheta, 1-\vartheta, \ldots, N$$

the discretization errors. As outlined in the following section and in [10], these errors can be shown to satisfy the second-order bound

$$\|e^n\|^2 + \kappa h \sum_{k=0}^{n} \|\nabla e^k\|^2$$

$$\leqslant h^4 \int_0^{t_n} \left(\eta_{-1}\|u_{ttt}\|_{-1}^2 + \eta_1\|\nabla u_{tt}\|^2 + \varphi_{-1}\|F_{tt}\|_{-1}^2 + \varphi_1\|\nabla F_t\|^2\right) dt \tag{3}$$

with numbers $\kappa, \eta_{-1}, \eta_1, \varphi_{-1}, \varphi_1 > 0$ depending on α and ν.[2] Due to the regularity of the exact solution u and the assumptions on the external force f as stated above, all norms appearing on the right-hand side of this estimate are finite.

3 Structure of the Convergence Proof

In this section we give a rough outline of the convergence proof with an emphasis on the coefficients the existence of which we intend to prove. The proof is based on ideas from [3] and follows the common strategy of plugging the errors into the scheme, thereby obtaining the expressions

$$\begin{aligned} E^n &:= e^{n+\vartheta} - e^n - \vartheta h\nu P\Delta\big(\alpha e^{n+\vartheta} + (1-\alpha)\, e^n\big), \\ E^{n+\vartheta} &:= e^{n+1-\vartheta} - e^{n+\vartheta} - (1-2\vartheta)\, h\nu P\Delta\big((1-\alpha)\, e^{n+1-\vartheta} + \alpha e^{n+\vartheta}\big), \\ E^{n+1-\vartheta} &:= e^{n+1} - e^{n+1-\vartheta} - \vartheta h\nu P\Delta\big(\alpha e^{n+1} + (1-\alpha)\, e^{n+1-\vartheta}\big), \end{aligned}$$

which are then multiplied by suitable test functions. Depending on whether the parameter α satisfies

$$\alpha \leqslant \frac{1}{4\vartheta} \tag{4}$$

or not, different test functions are used. We focus on the case (4) since this is where the coefficients we are interested in come up. In this case E^n, $E^{n+\vartheta}$, and $E^{n+1-\vartheta}$ are multiplied (scalar in $L^2(G)^n$) by

[2] In [10] the numbers $\kappa, \eta_{-1}, \eta_1, \varphi_{-1}$, and φ_1 are mistakenly stated to depend on nothing but α.

$$e^{n+\vartheta} + e^n - \vartheta \left(4 + \frac{1}{2\left(\alpha - \frac{1}{2}\right)}\right) h\nu P\Delta \left(\alpha e^{n+\vartheta} + (1-\alpha)e^n\right),$$

$$e^{n+1-\vartheta} + e^{n+\vartheta} - \frac{\delta}{2\left(\alpha - \frac{1}{2}\right)} h\nu P\Delta \left((1-\alpha)e^{n+1-\vartheta} + \alpha e^{n+\vartheta}\right),$$

$$e^{n+1} + e^{n+1-\vartheta} - \frac{\vartheta}{2\left(\alpha - \frac{1}{2}\right)} h\nu P\Delta \left(\alpha e^{n+1} + (1-\alpha)e^{n+1-\vartheta}\right),$$

respectively, with δ to be chosen suitably. The mixed terms in the expansions of the resulting scalar products $\langle\cdot|\cdot\rangle$ need to be estimated in a way that enables their absorption by available terms. Two of these estimations are performed with Hölder's and Young's inequalities with yet undetermined coefficients

$$C_1, C_2 > 0, \tag{5}$$

namely,

$$|\langle\nabla e^n|\nabla e^{n+\vartheta}\rangle| \leqslant \frac{C_1}{2}\|\nabla e^n\|^2 + \frac{1}{2C_1}\|\nabla e^{n+\vartheta}\|^2,$$

$$|\langle\nabla e^{n+\vartheta}|\nabla e^{n+1-\vartheta}\rangle| \leqslant \frac{C_2}{2}\|\nabla e^{n+\vartheta}\|^2 + \frac{1}{2C_2}\|\nabla e^{n+1-\vartheta}\|^2.$$

All terms are then arranged in preparation of telescopic canceling and the following observations can be made:

The $\|e^{n+1}\|$-terms can be handled provided

$$\frac{\alpha^2}{\alpha - \frac{1}{2}} - \left(3 + \frac{1}{2\left(\alpha - \frac{1}{2}\right)}\right)(1-\alpha) - 4\left(\alpha - \frac{1}{2}\right)C_1 > 0,$$

which is equivalent to

$$C_1 < 1. \tag{6}$$

The $\|e^{n+\vartheta}\|$-terms can be handled provided

$$\vartheta\alpha\,(5\alpha - 2) + \left(\alpha - \frac{1}{2}\right)\left((1-2\vartheta)\left(\alpha - \frac{1}{2}C_2\right) - 4\vartheta\left(\alpha - \frac{1}{2}\right)\frac{1}{C_1}\right)$$
$$-\left(\alpha + \left(\alpha - \frac{1}{2}\right)C_2\right)\delta > 0 \tag{7}$$

and the $\|e^{n+1-\vartheta}\|$-terms can be handled provided

$$\left((1-\alpha)-\left(\alpha-\frac{1}{2}\right)\frac{1}{C_2}\right)\delta+(1-2\vartheta)(1-\alpha)\left(\alpha-\frac{1}{2}\right)$$
$$-\vartheta(1-\alpha)^2-(1-2\vartheta)\left(\alpha-\frac{1}{2}\right)\frac{1}{2C_2}>0. \tag{8}$$

The condition

$$\delta>0 \tag{9}$$

comes in because it permits to drop the term

$$(1-2\vartheta)\,\delta\left(\alpha-\frac{1}{2}\right)^{-1}h\nu\|P\Delta\left((1-\alpha)\,e^{n+1-\vartheta}+\alpha e^{n+\vartheta}\right)\|^2.$$

The satisfiability of the conditions (5)–(9) is crucial for telescopic canceling, which in combination with estimates on norms of the quantities E^i stated in [10, Lemma 3.2] finally yields the desired error bound (3).

4 Main Result

As pointed out in the previous section, the convergence proof rests upon the following result concerning the existence of suitable coefficients for a test function and for Young's inequalities.

Theorem 1. *Let $\vartheta=1-\frac{1}{2}\sqrt{2}$. Suppose the previously fixed parameter $\frac{1}{2}<\alpha<1$ used in the fractional step theta scheme (2) satisfies the condition (4). Then there are coefficients $\delta, C_1, C_2>0$ such that the inequalities (5)–(9) hold.*

Proof. Due to (4), the number

$$L:=-2(1-\vartheta)^2\underbrace{\left(\alpha+\frac{1}{2}+\frac{1}{4\vartheta}-\frac{1-3\vartheta}{1-\vartheta}\right)}_{>0}\underbrace{\left(\alpha-\frac{1}{2}\right)}_{>0}\underbrace{\left(\alpha-\frac{1}{4\vartheta}\right)}_{\leqslant 0}\geqslant 0$$

is nonnegative. If we express $L=l_0+l_1\alpha+l_2\alpha^2+l_3\alpha^3$ as a polynomial in α and compare it to the number

$$H:=2\vartheta\alpha(5\alpha-2)(1-\vartheta)(1-\alpha)+2\alpha(1-\vartheta)(1-\alpha)(1-2\vartheta)\left(\alpha-\frac{1}{2}\right)$$
$$-8\vartheta\left(\alpha-\frac{1}{2}\right)(1-\vartheta)(1-\alpha)\left(\alpha-\frac{1}{2}\right)$$
$$-\left((1-\vartheta)\alpha+\vartheta-\sqrt{\vartheta}\right)(1-2\vartheta)\left(\alpha-\frac{1}{2}\right),$$

also expressed as a polynomial $H = h_0 + h_1\alpha + h_2\alpha^2 + h_3\alpha^3 \in \mathbb{R}[\alpha]$, we observe that

$$\begin{aligned} l_0 &= -2\,(1-\vartheta)^2 \left(\frac{1}{2} + \frac{1}{4\vartheta} - \frac{1-3\vartheta}{1-\vartheta}\right)\frac{1}{8\vartheta} \\ &< -\frac{1}{2}\,(1-2\vartheta)\left(\sqrt{\vartheta}-\vartheta\right) - 2\,(1-\vartheta)\,\vartheta \;=\; h_0, \end{aligned}$$

$$\begin{aligned} l_1 &= -2\,(1-\vartheta)^2 \left(-\frac{1}{16\vartheta^2} - \frac{1}{8\vartheta} - \frac{1}{4} + \frac{1-3\vartheta}{1-\vartheta}\left(\frac{1}{4\vartheta} + \frac{1}{2}\right)\right) \\ &< (1-2\vartheta)\left(\sqrt{\vartheta}-\vartheta\right) + (1-\vartheta)\left(7\vartheta - \frac{1}{2}\right) \;=\; h_1, \end{aligned}$$

$$l_2 \;=\; 2\,(1-\vartheta)\,(1-3\vartheta) = h_2, \qquad l_3 \;=\; -2\,(1-\vartheta)^2 \;=\; h_3.$$

Therefore, H is positive. The division of the inequality $H > 0$ by the positive number $(1-\vartheta)\,(1-\alpha)\,(1-2\vartheta)\left(\alpha - \frac{1}{2}\right)$ yields

$$\frac{(1-\vartheta)\,\alpha + \vartheta - \sqrt{\vartheta}}{(1-\vartheta)\,(1-\alpha)} \;<\; \frac{2\vartheta\alpha\,(5\alpha-2)}{(1-2\vartheta)\left(\alpha-\frac{1}{2}\right)} \;+\; 2\alpha \;-\; 8\vartheta\,\frac{\alpha-\frac{1}{2}}{1-2\vartheta}. \tag{10}$$

The positivity of ϑ implies

$$\frac{(1-\vartheta)\,\alpha + \vartheta - \sqrt{\vartheta}}{(1-\vartheta)\,(1-\alpha)} \;<\; \frac{(1-\vartheta)\,\alpha + \vartheta + \sqrt{\vartheta}}{(1-\vartheta)\,(1-\alpha)}. \tag{11}$$

In view of (10) and (11), there is a number C_2 that satisfies the three conditions

$$C_2 \;>\; \frac{(1-\vartheta)\,\alpha + \vartheta - \sqrt{\vartheta}}{(1-\vartheta)\,(1-\alpha)}, \tag{12}$$

$$C_2 \;<\; \frac{2\vartheta\alpha\,(5\alpha-2)}{(1-2\vartheta)\left(\alpha-\frac{1}{2}\right)} + 2\alpha - 8\vartheta\,\frac{\alpha-\frac{1}{2}}{1-2\vartheta}, \tag{13}$$

$$C_2 \;<\; \frac{(1-\vartheta)\,\alpha + \vartheta + \sqrt{\vartheta}}{(1-\vartheta)\,(1-\alpha)}. \tag{14}$$

From (12) we deduce

$$C_2 > \frac{(1-\vartheta)\,\alpha + \vartheta - \sqrt{\vartheta}}{(1-\vartheta)\,(1-\alpha)} = \frac{(1-\vartheta)\left(\alpha-\frac{1}{2}\right) + \frac{1}{2}\left(1-\sqrt{\vartheta}\right)^2}{(1-\vartheta)\,(1-\alpha)} \geqslant \frac{\alpha-\frac{1}{2}}{1-\alpha}, \tag{15}$$

which also implies $C_2 > 0$. In order to see that the inequality

$$\frac{N_2}{D_2} := \frac{\vartheta\alpha\,(5\alpha-2) + \left(\alpha-\frac{1}{2}\right)\left((1-2\vartheta)\left(\alpha-\frac{1}{2}C_2\right) - 4\vartheta\left(\alpha-\frac{1}{2}\right)\right)}{\alpha+\left(\alpha-\frac{1}{2}\right)C_2} > 0 \tag{16}$$

holds, observe that the denominator D_2 is positive since $C_2 > 0$ and that the numerator N_2 vanishes if the constant C_2 is replaced by the right-hand side of (13).

The inequality

$$\frac{N_1}{D_1} := \frac{(1-2\vartheta)\left(\alpha-\frac{1}{2}\right)\frac{1}{2C_2} + \vartheta\,(1-\alpha)^2 - (1-2\vartheta)\,(1-\alpha)\left(\alpha-\frac{1}{2}\right)}{(1-\alpha)-\left(\alpha-\frac{1}{2}\right)\frac{1}{C_2}} < \frac{N_2}{D_2} \tag{17}$$

can be shown as follows. Due to (15) the denominator D_1 is positive. Therefore (17) is equivalent to the positivity of the term $N_2D_1 - N_1D_2$. The latter can be written as a product

$$N_2D_1 - N_1D_2 = \frac{(1-\vartheta)\,(1-\alpha)^2\left(\alpha-\frac{1}{2}\right)}{C_2}\cdot\left(C_2 - \frac{(1-\vartheta)\,\alpha+\vartheta-\sqrt{\vartheta}}{(1-\vartheta)\,(1-\alpha)}\right)\cdot \left(\frac{(1-\vartheta)\,\alpha+\vartheta+\sqrt{\vartheta}}{(1-\vartheta)\,(1-\alpha)} - C_2\right)$$

of three factors all of which are positive: the first one because $C_2 > 0$, the second one because of (12), and the third one because of (14). This shows (17). The inequalities (16) and (17) imply the existence of a number δ that satisfies the three conditions

$$\delta > 0, \qquad \delta > \frac{N_1}{D_1}, \qquad \delta < \frac{N_2}{D_2}, \tag{18}$$

which we refer to as (18_1), (18_2), and (18_3). Condition (18_1) is nothing but (9), condition (18_2) yields (8), and condition (18_3) yields

$$\vartheta\alpha\,(5\alpha-2) + \left(\alpha-\frac{1}{2}\right)\left((1-2\vartheta)\left(\alpha-\frac{1}{2}C_2\right) - 4\vartheta\left(\alpha-\frac{1}{2}\right)\right) - \left(\alpha+\left(\alpha-\frac{1}{2}\right)C_2\right)\delta > 0.$$

This inequality shows that the choice $C_1 = 1$—impossible because it contradicts (6)—would satisfy (7). Hence for continuity reasons, there is some constant $0 < C_1 < 1$ such that (7) holds. This completes the proof. □

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