Chapter 6
Fingu—A Game to Support Children's Development of Arithmetic Competence: Theory, Design and Empirical Research

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Abstract This chapter aims at describing research on Fingu, a virtual manipulative housed in a game environment, which is designed to support young children's learning and development of number concepts and flexible arithmetic competence. More specifically Fingu targets the understanding and mastering of the basic numbers 1–10 as part-whole relations, which according to the literature on early mathematics learning is critical for this development. In the chapter, we provide an overview of the theoretical grounding of the design, development and research of Fingu as well as the theoretical and practical design rationale and principles. We point out the potential of Fingu as a research platform and present examples of some of the empirical research conducted to demonstrate the learning potential of Fingu. Methodologically, the research adopts a design-based research approach. This approach combines theory-driven design of learning environments with empirical research in educational settings, in a series of iterations. In a first series of iterations, a computer game—the Number Practice Game—was designed and researched, based on phenomenographic theory and empirical studies. In a second series of iterations, Fingu was designed and researched, based on ecological psychology in a

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socio-cultural framing. The design trajectory of NPG/Fingu thus involves both theoretical development and (re)design and development of specific educational technologies.

6.1 Introduction

This chapter describes research on Fingu, a virtual manipulative housed in a game environment, which is designed to support young children's learning and development of number concepts and flexible arithmetic competence. More specifically, Fingu targets the understanding and mastering of the basic numbers 1–10 as part-whole relations. Our primary aim is to give an overview of the theoretical grounding of the design, development and research of Fingu as well as the theoretical and practical design rationale and principles. The potential of Fingu as a research platform is pointed out. We also present examples of some of the empirical research conducted, with the aim of demonstrating the learning potential of Fingu.

6.2 The Fingu Game

In the Fingu game, the player is exposed to two moving sets of objects on the touch screen (in the case shown in Fig. 6.1, four apples on the left and three apples on the right), and is supposed to tell how many objects there are in total (seven) by touching the screen with the corresponding number of fingers. Fingers can be placed anywhere on the screen and there are no restrictions concerning which fingers are used, but they must be pressed down roughly at the same time, as a representation of the total number.

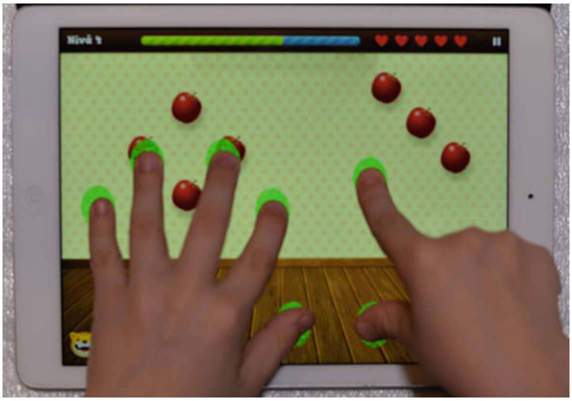

Fig. 6.1 A screen view of a child playing Fingu

The objective is to progress through different levels of the game, making as few wrong answers as possible. Each level comprises a set of tasks and the levels are progressively more demanding. In order to advance to the next level, the player is not allowed to give more than a certain number of wrong answers (represented by hearts in the upper right of the screen in Fig. 6.1).

6.3 Young Children's Learning of Mathematics

One of the key goals for mathematics education around the world is to ensure all children's proficiency with numbers. This means to have a flexible and adaptive understanding and knowledge of arithmetic that can be used with confidence and fluency in a variety of situations occurring in everyday life and at work (Kilpatrick et al. 2001). Learning to become proficient with numbers is a process that starts early on and then becomes a part of the growing child for several years (Baroody et al. 2013; Sarama and Clements 2009). One of the foundations in this process is to develop flexible arithmetic competence, that includes mastering number concepts for the basic numbers 1–10, and to this end, the use of reasoning strategies, based on using known sums and relations between them to deduce unknown sums (Baroody et al. 2013). For many children, this is a straightforward process, but some children seem to have trouble learning and using such reasoning strategies, as observed by Gray and Tall (1994), who found that children aged 7–12 with below average mathematical performance dominantly relied on counting strategies while performing basic sums. Neuman (1987, 2013) also found that counting one by one, as a dominant strategy for managing single-digit sums, is common among children with mathematical difficulties.

There are different views on the role of counting for the development of conceptual knowledge about numbers. A dominant idea is that basic sums are learned in a process, where children begin by using counting one by one to determine the sums of single-digit additions. They then are supposed to memorize these facts. But in order to develop fluency with basic sums (and not just basic facts), which includes the ability of using them both appropriately and in adaptable ways, children "typically progress through three overlapping phases in the meaningful learning of a particular basic sum or family of sums: (a) Phase 1 (counting strategies), (b) Phase 2 (conscious or deliberate reasoning strategies), and (c) Phase 3 (fluent retrieval)" (Baroody et al. 2013, p. 536). Here, the counting strategies of Phase 1 can be seen as the first steps in determining and exploring different sums, which in Phase 2 can turn into knowledge of the part-whole relations that constitute the sums. Continuing in Phase 2 to explore these relations further, may then lay the ground for a network of relations, which in Phase 3 will constitute itself in a more or less fluent performance. So Phase 2 will act as a mediator between the less efficient counting strategies in Phase 1 and the more efficient retrieval strategies in Phase 3. Note that these phases must be understood as overlapping, and that there is no absolute succession from one phase to another.

This network of relations within the first ten natural numbers is built upon a network of quantitative part-whole relations, which, in the terminology of Resnick (1989), is grounded in the principle of additive composition—that numbers are composed of other numbers, and any number can be decomposed into parts—combined with more qualitative knowledge of part-whole relations. Thus the ability to make use of part-whole number relations is foundational for developing the kind of number networks that are necessary for becoming fluent with mental computations (Baroody et al. 2009; Sarama and Clements 2009). Later, in the development of arithmetic fluency, the ability to use these part-whole relations in a flexible and adaptive way is also fundamental in understanding fractions (McMullen et al. 2014, 2015).

One of the basic arithmetic competences that support the development of good number sense is the ability to quickly and reliably determine the numerosity of a collection of objects. This can be done by counting all the objects, but can also be speeded up either by different forms of skip counting or by using some kind of subitizing. Humans are born with an innate ability to *subitize*, that is "the direct and rapid perceptual apprehension of the numerosity of a group" (Sarama and Clements 2009, p. 29). Kaufman et al. (1949) invented the term subitizing to distinguish it from estimating, which is also a rapid but only approximate way of quantifying. They emphasized that subitizing goes together with a high degree of accuracy and confidence. Thus subitizing appears to be a distinct way of quantifying, different from counting and estimating. From the neuroscientific literature the conclusion is that "true" subitizing only occurs in adults for the numerosities 1, 2, and 3 (Dehaene 2011). For some young children, the number 3 may, according to Sarama and Clements (2009), be questionable. This fact does however not exclude the possibility of rapid recognition of higher numerosities, but this is limited to identifying certain familiar configurations such as those used on dice or dominoes.

Noting that "subitizing develops considerably as children grow and combines with other mental processes", Sarama and Clements (2009, p. 44) distinguish between perceptual and conceptual subitizing. They define perceptual subitizing as "recognizing a number without consciously using other mental or mathematical processes and then naming it" (p. 44). This ability, limited to the quantities 1, 2, and 3, includes the decomposing of 2 items as 1 and 1, and 3 items as 2 and 1 without having to count. This ability then provides a foundation for conceptually experiencing somewhat larger numbers as composed of smaller parts (e.g., 4 as 2 + 2 or 3 + 1, 5 as 2 + 3, or 2 + 2+1, and 6 as 3 + 3 or 2 + 2 + 2). This process of expanding perceptual subitizing into immediate recognition of combinations of smaller units they call conceptual subitizing (Sarama and Clements 2009). Conceptual subitizing thus presupposes a network of part-whole relations for small numbers, and the richer it is the more effectively, flexibly and adaptively, it can be used for performing different computations more rapidly.

Today there is a growing awareness that the concept of embodiment is an important dimension of learning mathematics. As stated by Edwards and Robutti (2014), "although mathematics may be socially constructed, this construction is not arbitrary or unconstrained but rather is rooted in and shaped by the body" (p. 2). The body then becomes "an important resource in the construction and communication of meaning". Therefore, in addition to the traditional view of learning modalities such as visual and auditory, an expanded view will include motor modalities such as gesture and touch. A growing neuroscientific literature connecting fingers to numerical cognition (Fischer et al. 2012) is an example of this embodiment. From a mathematics education perspective, it is well-known that fingers play a central role in learning arithmetic. Children can use them in two ways, either by making a finger pattern displaying a certain number, for instance showing the index, middle, ring and pinkie fingers on one hand to denote the number four, or as a tool, while performing a calculation, keeping track of their counting, giving attention to one finger at a time. Historically, fingers as a means of representing numbers have also played important roles in the early development of computational methods found in commerce and administration (Ifrah 2000). So, from an embodied perspective, using fingers to form different number representations is a way to enrich the learning of basic numbers.

6.4 Theoretical Framework

Methodologically, the research on Fingu adopts a design-based research (DBR) approach (Brown 1992; Cobb et al. 2003). This approach combines theory-driven design of learning environments with empirical research in educational settings, in a series of iterations. In a first series of iterations a computer game—the Number Practice Game—was designed and researched, based on phenomenographic theory and empirical studies (Lindström et al. 2011; Marton and Booth 1997; Neuman 1987). In a second series of iterations Fingu was designed and researched, based on ecological psychology (Gibson 1986, 2000) in a socio-cultural framing. The design trajectory of NPG/Fingu thus involves both theoretical development and (re)design and development of specific educational technologies.

The design-based research methodology is grounded in cultural-historical activity theory (CHAT). There are different versions of CHAT. The work by Engeström (Engeström and Sannino 2010) emphasizes structural and systemic aspects, while the foundational work of Vygotsky and Leontiev (cf. Hodkinson et al. 2008) focuses more on human agency. As a foundation for the design of environments and tools for learning, recent developments of CHAT come closer to the originators (cf. Kaptelinin and Nardi 2006).

In CHAT, human activity is the core unit of analysis. Activities should be understood as socially and historically situated. For one thing, this means that they are multi-layered. For example, the activity of playing a mathematics computer game is inevitably a part of a larger activity system or practice, of institutional schooling or day-care/preschool or play or family upbringing. At the same time, an activity is realized by the actions performed by the participating individuals. Actions in turn are composed of operations, which in the knowledgeable individual (expert) are unconscious.

An activity unfolds over time, realized by human actions and operation. Humans change (i.e., learn) by participating in the activity. Learning can then be understood as a by-product of participation, appropriating whatever patterns of actions and operations are (deemed) functional given the activity. Human abilities, knowledge and skill, in general, have this dynamic and relational character. They are not to be regarded as something static "in the head" or, for that matter, "in the body", but only realized when acted out "in situ".

An activity is understood as historically situated on different levels. In our case, not only the individuals have a specific history with respect to playing computer games, mathematics learning activities, money counting, etc. This is clearly demonstrated in our empirical studies of Fingu in pre-school and school. The game playing activity became a part of a larger process of teaching and learning arithmetic. Despite our attempts to offer the game as a mathematics activity not belonging to the regular practice, the Fingu activities were organized and fitted into the everyday classroom scheme. Thus, how children frame the activity (Goffman 1974), for example as gaming or as mathematics learning or both, is to be understood in a historical perspective, both on the collective sociogenetic level and on an individual ontogenetic level.

Thus, a specific activity, such as playing a mathematics game, does not necessarily belong only to one larger activity system. Sometimes it belongs to several and even larger activity systems. As pointed out, playing a mathematics game can be both a gaming activity and/or a school mathematics activity. The activity then becomes a boundary activity and the game can be considered a boundary object (cf. Akkerman and Bakker 2011). This boundary character of game playing is important to consider when evaluating the general argument that game playing offers a learning potential that breaks away from institutional schooling (Gee 2003).

A basic tenet in CHAT is that teaching and learning are intrinsically related, that is, two aspects of an unfolding activity. Teaching should then be understood in a generic sense. It might be an active involvement by a teacher, but a game playing activity involving a single player and a mathematics learning game also involves a teaching or instructional component. Vygotsky had a specific term for this learning/teaching process, *obuchenie* (Cole 2009). Learning/teaching is thus a two-sided process and understanding learning is thus a matter of understanding a teaching/learning activity in relational terms, as a relation between the individual and the social and material environment.

CHAT, or more generally socio-cultural theory, is often used as a foundation for designing learning environments with collaborative activities. Whereas not incorrect, an epistemology that premises human learning and development as culturally and socially situated does not necessarily imply this kind of educational or instructional model. Any activity, collaborative or not, is socially and culturally situated. Reading a book or playing a computer game in private is also a socially and culturally situated activity.

In summary, CHAT provides a framework for both design and analysis. Designing a mathematics computer game is a design for certain teaching/learning activities to take place, just like designing a set of tasks in a mathematics workbook. However, the actual use of the game—the teaching/learning activity—is also highly dependent upon the overall system of activities it is part of. This means that the context of use is always an issue to be considered. This is also an analytic consideration in doing research on game use. How do the game playing activities relate to other mathematical activities in which children are involved? How do children perceive the game playing activities?

Given this general framework, the theoretical underpinnings for designing and studying Fingu as a virtual manipulative are given by Gibsonian ecological psychology (Gibson, 1986), and in particular the theory of perceptual learning (cf. Gibson and Pick 2000). Acknowledging the contributions of both James and Eleanor Gibson, Gibsonian theory is intrinsically relational and non-dualistic. As pointed out above, this is something that is shared with cultural-historical activity theory. It is also non-representational; it rejects the idea that cognition and learning are about constructing inner representations of the world "outside" the individual.

Ecological psychology is grounded in a "realist" ontology, which like CHAT, acknowledge that human activity is grounded in the material world. Perception, for example, is the selection and "picking up" of (invariant) information in the course of acting in a concrete physical environment. It should be noted that, when it comes to social and intellectual activities, for example, teaching/learning, these are constituted in interactions unfolding over time.

The Gibsons (Gibson and Gibson 1955; Gibson and Pick 2000) argue against "enrichment theories", where perception or "sensory reception is enriched and supplemented by the addition of something" (Gibson and Pick 2000, p. 7). Instead "perception begins as unrefined, vague impressions and is progressively *differentiated* into more specific percepts" (p. 7). In development, through perceptual learning, the individual becomes more and more apt to learn the specific affordances of the environment.

The concept of affordance, which takes on a number of different definitions in contemporary social and behavioral sciences, is pivotal to Gibsonian theory and was developed to account for the relational nature between the individual and the environment. It "refers to the 'fit' between an individual's capabilities and the scaffolds/support and opportunities that makes a certain activity possible" (Gibson and Pick 2000, p. 15). An affordance can be thought of as an offering for meaning in a given situation, or put in more general terms, an offering for action.

That differentiation is fundamental to learning and development has a number of consequences. In the present context, it means that the development of flexible and adaptive competence in dealing with part-whole-relations in the range from 1 to 10 is a matter of successive refinement of the understanding of numbers, from a more undifferentiated whole (for example the number 7) to grasping a differentiated network of relations between parts that can make up the whole (i.e., 6|1; 5|2; 4|3; 3|3|1 etc.). The whole can take on different forms, for example number words of a more symbolic nature; sets or constellations of concrete objects; visually presented patterns (of objects); or even procedures (of which the counting sequence is an example).

Gibsonian theory also emphasizes that perception is not static but dynamic; thus, building on our actions in the environment, in which individuals engage in activities that are extended in time (e.g., moving around). The construct "perception-action cycles" captures this dynamic, emphasizing that perception is not a prerequisite for action. Rather, action is foundational for perception and perception is foundational for action, making up a perceptual system. Thus, perceptual learning is

> the means of discovering distinctive features and invariant properties of things and events ..... Learning to distinguish faces from one another or to distinguish letters of the alphabet are such cases. Discovering a repeated theme in a symphony and the variations on it is another. Discovering distinctiveness and invariance is another kind of meaning, also a product of perceptual learning. (Gibson 2000, p. 295)

The idea of discovering and retaining information as invariant features of the environment is central in James Gibson's seminal work (Gibson 1986). That perceiving invariance(s) (which presumes variation) is fundamental to perception, action and learning, is similar to what variation theory proposes (Marton and Pang 2006). This is particularly important since pre-cursors of the Fingu game were designed from variations of theoretical ideas (cf. Lindström et al. 2011).

An important aspect of perceptual (and cognitive) systems is that they are typically not uni-modal, but multi-modal, building on the use of several sensory modalities (Neisser 1976). This is important in the present context, where children are exploring a game environment that explicitly draws on both visual and kinesthetic modalities. Embodiment, then, is in this view building on multi-modal agency.

Going further, perceptual learning builds on two complementary processes: exploratory activity and performatory activity. Gibson (2000) states that exploratory activity

> is itself an event, a perception–action sequence that has consequences. It brings about new information of two kinds: information about changes in the world that the action produces and information about what the active perceiver is doing. (p. 296)

This kind of learning tends toward flexibility and is geared to maintain an adaptive relation with the environment. But learning is also geared towards economy and efficiency, and this results in a tendency for specificity, resulting in actions that from the beginning of an encounter can be more varied and exploratory. This then develops into more specifically limited actions that effectively fulfill the goals of a task. This is what the theory of perceptual learning conceptualizes as performatory activity: "Activity that starts as exploratory can become performatory as an affordance is discovered. This shift is marked by making contact with the environment and ensuing control of it" (Gibson 2000, p. 297). This idea resonates with CHAT in that operations that build up actions (for example "seeing" or counting number patterns) can become unconscious in the course of learning.

Building on the theory of perceptual learning, Kellman and Garrigan (2009) formulated design principles for interventions aimed at developing expertise with some key areas in mathematics learning. These principles include many and varied short tasks, where the child has the opportunity to develop rapid selection of task-relevant information, and the pick-up of higher-order relations and invariances in different modalities such as visual, auditory and kinesthetic. The latter principle corresponds with the core idea in variation theory and phenomenography (Marton and Booth 1997; Marton 2015).

6.5 Design Principles for Fingu

In this section, we describe the design principles of Fingu and how these principles are realized/materialized in the game. As previously discussed, these design principles are grounded in empirical research on mathematics learning and instruction and in more general theories of learning and instruction (Neuman 1987; Lindström et al. 2011).

Design of a computer game for learning is not a straightforward derivation from theory. The design is influenced further by contemporary and historical mathematics education practice and by more general game design. Furthermore, Fingu is a second-generation implementation of design ideas for a new technological platform (i.e., tablets). This platform offers possibilities to implement new design elements (e.g., using finger patterns and thus multiple fingers to manipulate the game) that were not available in earlier generations of the technology (such as laptops).

Since the design of Fingu is encapsulated in a design-based research process, the design principles and design elements outlined in this section can, to some extent, also be regarded as results.

6.5.1 Overall Design

When designing a mathematics game for children, there are a number of alternatives for framing the mathematics tasks. A common practice is to embody the mathematics in a cover story or activity, for example using a route metaphor. When advancing from START to END, the player meets a number of obstacles that have to be handled with specific tools (for example collecting a number of keys to open a door). This type of design was rejected for Fingu. Recent research shows children might be focusing on completing the gaming elements rather than engaging in the desired content (cf. Linderoth 2012). Since the goal was to design a learning environment that maximized children's attention to part-whole relations, we chose a design that resembles a microworld (cf. Papert 1980). In this case, it is a world of all possible part-whole relations in the number range from 1 to 10.

As argued above, mastering part-whole relations in the number range from 1 to 10 is pivotal to the development of flexible and adaptive arithmetic competence. From an arithmetic point of view being able to decompose every number in the range into two parts in all the possible ways (2 = 1|1; 3 = 2|1; 4 = 3|1 = 2|2; 5 = 4|1 = 3|2; …) and, conversely, to construct larger numbers by combining these parts into new numbers, is an important developmental milestone.

In accordance with the theory of perceptual learning and variation theory, children are presented with a set of tasks covering all the possible part-whole relations in the number range from 1 to 10. Each task targets a specific part-whole relation (for example 5 = 3|2). Playing Fingu makes it possible to discern invariances (for example that 5 is 5 regardless of if the parts are 2|3 or 4|1 and regardless of modality) and extract information about higher order relations (a network of part-whole relations).

Fingu is designed as a game in order to encourage extensive experience with many and varied tasks, as the theory of perceptual learning prescribes (Kellman and Garrigan 2009). A basic game design element is a progression of mastery from an introductory level comprised of tasks with small numbers (1–5 and canonical visual patterns) to an end level with tasks with large numbers (6–10) and non-canonical visual patterns. Mastery is represented by the number of lives preserved while going through the tasks on a level. With mastery of one level, it is then possible to advance to the next higher level. Another game design element is immediate and simple feedback on correctness. This feedback is presented both auditory and visually, to be clearly recognized by the child before taking on the next task. Fingu is also packaged as a computer game, in terms of graphical layout and in terms of vocabulary. For example, the child has to pick an icon for player and name the player.

As pointed out above, a problem in developing arithmetic competence might be that children develop non-productive or even counter-productive counting procedures that are not used in a flexible and adaptive way. Fingu is designed to afford conceptual learning (with a focus on part-whole relations) and to minimize or even prevent counting. The main design element is to put time-constraints on the individual tasks. Even if the task design affords both counting and perception of part-whole relations using subitizing, the latter is a more efficient approach to solve the task. In the next section we elaborate more on how time-constraints are used in the design of tasks.

A distinctive feature is that the design draws on the embodied nature of arithmetic. As discussed above, there are two distinct aspects of this. One is making subitizing (both perceptual and conceptual) a basic design element in the development of the understanding of part-whole relations. The other is making children use fingers as tools in dealing with the tasks. More specifically, Fingu affords the use of finger patterns. Essentially, it is the latter characteristic that makes Fingu a virtual manipulative. Fingu is thus fundamentally a multi-modal learning environment. First, it allows children to find out invariances across modalities, as pointed out by variation theory and perceptual learning theory. Second, it affords transformations across modalities, for example making a visually presented part-whole relation (for example $6 = 4|2$) into another relation ($6 = 5|1$) using the fingers and preserving the whole.

Of note, Fingu was not designed as a symbolic activity (i.e., invoking the use of number symbols). There is however not any principled reason for this. On the contrary, from a CHAT perspective, language is a critically important tool in human activity. Thus, the game, as currently envisioned, does not capitalize on children's communication through number vocabulary. Furthermore, Fingu is used in educational contexts where language and number symbols play a decisive role. In our empirical research the educational use of the game was embedded in language. However, the present version of Fingu aims primarily at developing non-symbolic aspects of arithmetic competence, partly for the reason of avoiding the bias on using simple counting procedures and rote learning that might come with language.

From a methodological point of view, however, the fact that Fingu is a non-symbolic game, meaning that children learn part-whole relations not explicitly coupled to number symbols, might pose a problem. It might be difficult to capture the embodied form of knowledge and understanding that Fingu affords with the common practice of testing children's understanding of arithmetic by using interviews, which heavily rely on language.

6.5.2 Design of Tasks

The basic structure of a single task in Fingu resembles an IRE-sequence (Initiation—Response—Evaluation, Mehan 1979), with a problem presentation in visual mode (I); an answer given by the fingers (R); and feedback about the correctness of the answer (E). However, it is critically important to appreciate that the task activity comprises all phases in the sequence. The affordances for learning are tied to the whole activity sequence.

6.5.2.1 Visually Presented Collections of Objects

In the problem presentation phase, collections or sets of objects (e.g., pieces of fruit) are presented visually. Either one collection is presented, which is assumed perceivable as an undifferentiated whole (e.g., 5), or two collections of objects are presented that together make up a differentiated whole (e.g., 5 = 3|2). Fingu then essentially affords building up differentiated wholes, drawing on the ability to subitize the parts, and develop a conceptual subitizing of the whole.

The spatial arrangements of these collections of objects are regular and symmetric and often have the same configuration as a dice pattern, although there are many less familiar configurations (see Table 6.1). We call the familiar dice configurations canonical and the rest of the configurations non-canonical, since only the dice patterns (which is supported by our empirical results), are familiar to

Table 6.1 Visual number configurations used in Fingu

N	1	2	3	4	5	6	7	8	9	10
Variant a										
Variant b										

Swedish 5–7-year-old children. All configurations were chosen to be possible to subitize either perceptually or conceptually.

As can be seen in Table 6.1, using our definition strictly, there are 6 canonical configurations and 11 non-canonical. There are two configurations for each of the numbers 3–9 (allowing for invariance across visual patterns), but only one configuration for the numbers 1, 2 and 10. We use the nomenclature of 3a and 3b etc., to refer to the different representations of a given number.

In total, there are 60 tasks with different combinations of configurations and sums ranging from 1 to 10. As an example, 5a + 5b stands for the task where the canonical die-5 configuration (5a) is combined with the non-canonical configuration (5b).

A progression, or trajectory of learning, is built into the game design with seven levels of difficulty involving increasing sums and more unfamiliar visual patterns of objects. On all levels tasks are presented in random order and each task is presented twice.

As mentioned previously, time-constraints are important to the game design. The visual part-whole patterns are therefore displayed for a limited period of time. This design can be seen as a modernized version of Kühnel's (1916) flash card activities. In order to afford perception of part-whole relations, rather than counting the whole or the parts, the second presentation of a task is of shorter duration than the first presentation, given that the child has arrived at the correct number. This affords counting in examining the numerosity of a part, something that typically is done when new and complex patterns are met. However, the shorter duration of the second presentation should encourage more use of subitizing.

6.5.2.2 Answering with Coordinated Finger Patterns

A key part of the design is that children are forced to use a coordinated finger pattern to complete each task. The child/player cannot sequentially touch the screen with one finger at a time, but has to touch it with all the fingers that constitute the patterned response *at the same time*. A limited touch input latency (default 0.25 s, adjustable) is used to assure this. Thus, the player is stimulated to focus on the parts of the presented problem and the total sum instead of resorting to counting one by one in presenting the total sum.

This limited touch input latency can make it inconvenient in the initiation phase due to the risk of the game interpreting a multi-touch response as a single finger response. The task, which on the surface level may appear a simple skill-focused activity, in this way becomes more of a problem solving activity focusing on part-whole relations.

This is amplified by the player's freedom of choosing which fingers and which partition to use in the completion of the task. Learning to manage the fingers to express sums is in this way (as structured finger patterns) an essential part of what *Fingu* affords, coupled with the time constraint to reduce the likelihood that the fingers are used as tools for counting.

6.6 Fingu as a Research Platform

Fingu is designed for use in different contexts and for different purposes. The default version is a mathematical computer game to be used by children as is, preferably introduced by a more knowledgeable person (teacher/parent/sibling/peer), but not necessarily so. This is also the basic or default mode of usage.

Fingu is also designed as a research platform, essentially allowing the researcher to tailor the game for different research purposes. Part of this functionality is made available in the Settings menu of the game where game parameters can be changed. Examples of game parameters are exposure time for individual tasks (ExposureTime), time allowed to give an answer (AnswerTime), number of errors allowed on each level (Lives) and how long the player has to hold down a stable number of fingers before the answer is registered (TouchInputLatency). The default values of these parameters are based on earlier research of the forerunner of Fingu, NPG (Lindström et al. 2011) and pilot studies of Fingu (Barendregt et al. 2012).

Another level of flexibility is the ability to change the game by re-designing the content, structure and sequence of tasks in the game. This can be done by defining new individual tasks and collections of tasks for the different levels. Even the number of levels can be altered. In this way, it is possible to change the task trajectory of the game, and potentially different learning trajectories. It is, for example, possible to make different versions of the game for different age groups or to adapt the game to children's individual needs. In our empirical research we have developed re-designs in order to make the game less challenging for weak children. These re-designs are implemented in xml-code and have been locally entered into the game as customized game behavior files.

In addition, Fingu provides tools for logging and real-time playback of children's game playing behavior including answering latency and finger placement for further analysis through a visualization function built into the program. Log-files can be saved in the system (as xml-files) and can be exported for further analyses. However, logging is optional, with no logging as the default.

The functionalities that make Fingu a flexible research platform are also accessible for a teacher or a parent. The settings are easy to alter and experiment with. However, re-designing the game by developing new xml-files is more demanding. It presupposes knowledge about the architecture of the game and basic XML-coding.

Furthermore, in order to analyze progress in the game, there is a simple statistics function available that presents the success rates summed for each quantity in the range from 1 to 10. There are records for the current session and the accumulated results over all sessions. It is also possible for a researcher, teacher or parent to use the replay function as an audit trail in a debriefing session with a child.

6.7 Software

Fingu was developed in collaboration with a company developing game software and built on an open source platform. The research group, in discussion with the software company, made the design and the company did the technical implementation. Revisions to the design and development of new versions were driven by empirical research (Barendregt et al. 2012).

Fingu is available for free on the App Store. One reason for this open access was to have a stable method for distribution of the program in our naturalistic research settings. Another reason was to make Fingu available for teachers, parents, children and other researchers, as an output of our research. The app comes in different languages (presently Swedish and English).

6.8 Empirical Research

In this section, we will give examples of the empirical research we have conducted based on a larger study involving children aged 5–7 years. First, we describe the design of the study and empirical data generated. The first example is a quantitative analysis of game playing. The second example is an analysis of effects of playing on children's arithmetic abilities based on group data. The third example is an analysis of different ways of playing the game, based on group data. The fourth example is an analysis of individual development in playing the game, based on a single case.

6.8.1 Study Design and Empirical Data

In order to investigate if and how Fingu is a productive learning tool, we carried out a larger study in pre-schools and in schools, educational settings with high ecological validity, where children were given opportunities to play Fingu extensively. The study was designed with pre-, post-, and delayed tests. We gathered data from 112 children (approximately equally as many children in each age group; see Table 6.2), and with equal numbers of girls and boys. Before playing Fingu children were tested with a set of arithmetic tests. Then they played the game for 8 weeks as a part of their ordinary practice. Immediately after the playing period, children completed post-tests and 8 weeks later they completed delayed tests. The same set of test instruments were used as pre-, post-, and delayed tests. Children were not allowed to play Fingu between post- and delayed testing.

The tests were administered through individual interviews, and focused on general or more specific mathematical abilities, to measure changes in the arithmetic knowledge of the children. We used: The Test of Early Mathematics Ability, version 3 (TEMA-3) (Ginsburg and Baroody 2003); A test of part-whole knowledge, PWK, using tasks focusing on part-whole relations (some with finger patterns, some with other patterns), most of them inspired from the Early Numeracy Research Project, ENRP, in Australia (ENRP 2015); A problem solving test, PS, consisting of arithmetic problems of change or combine type with sums less than or equal to 10, and similar to the problems used by Neuman (1987); and a pattern recognition test, PR, where single configurations from the Fingu design were exposed in random order for half a second each, and children responded verbally (Holgersson et al. 2016).

During the study, the Fingu log function was used to gather log data, including data on tasks, answering times, and childrens' responses (including how many fingers were registered and their coordinates). To complement these data, we also video-recorded the children three times when they played the game: first when introduced to the game, secondly after a few weeks, and thirdly towards the end of the intervention period. For the study of a child's success in playing the game and how it develops, analyses of answering times and correctness of different trials are important, and in our third and fourth examples of analysis, we use individual and task specific median answering times (MATs) of the correctly answered trials, and mean proportions of correct answers (PCAs) to reach our conclusions. In our fourth example of analysis, we also complement this information with regression analyses (linear and binary for answering times and correctness respectively), and a study of the finger patterns used in responding to different tasks.

6.8.2 Example 1—Playing Fingu

Here we present overall data on Fingu playing time and children's success. Although the teachers were instructed to give the children opportunities to play at least three times a week, the variation in children's playing was substantial (see Table 6.2). The 5-year-olds have a median number of trials equal to around 1100 and an IQR equal to about 1200 trials, which is much bigger than the IQRs for the 6- and 7-year-olds who have medians equal to about 900 trials with IQRs equal to around 600 and 950 respectively.

Table 6.2 Number of trials made during the intervention period, disaggregated by age

Age	N	Median	Q_1	Q_3	IQR	Min	Max
5	35	1114	683	1896	1213	186	4572
6	38	905	630	1214	584	279	3189
7	39	916	654	1599	945	320	2445
All	112	953	654	1620	966	186	4572

Note Q_1 Lower quartile, Q_3 Upper quartile, and *IQR* Interquartile range

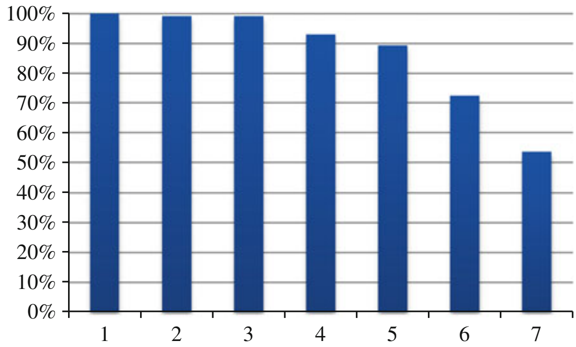

Fig. 6.2 The percentage of children playing on different levels

There is also a large variation in how many of the children played on different levels, where level 5 and 6 were the hardest to pass (see Fig. 6.2). Only 54 % (60 out of 112) of the children played on all the levels.

6.8.3 Example 2—Learning by Playing Fingu

To study the effects on children's arithmetic abilities, we performed paired-samples t-tests on the mean results on the separate arithmetic tests. The results (see Table 6.3) were that between the pre- and post-tests, there is a small effect on the TEMA-3, small to moderate effects on the PWK and the PS tests, and a large effect on the PR test.

Between the post- and the delayed post-tests (see Table 6.4) the only significant effects were found on the TEMA-3 and the PWK test. Thus, playing Fingu did have an immediate and delayed effect on children's mathematics knowledge. The effects we observed varied with age group and were largest for the 7-year-olds while the effects for the 5- and 6-year-olds were similar and comparable to the overall effects.

Table 6.3 Results from a paired-samples T-test of the mean values of the different pre- and post-tests complemented by effect sizes

Test	N	Pre-test	Post-test	Difference	Effect size	t	*p*-value
Tema3	81	28.15	31.40	3.25	0.34	7.54	< 0.001
PWK	82	15.88	18.65	2.77	0.42	6.33	< 0.001
PS	74	4.08	5.28	1.20	0.46	5.17	< 0.001
PR	75	11.64	14.44	2.80	0.79	8.54	< 0.001

Table 6.4 Results from a paired-samples T-test of the mean values of the different post- and delayed-post-tests complemented by effect sizes

Test	N	Post-test	Delayed test	Difference	Effect size	t	*p*-value
Tema3	82	31.43	33.16	1.73	0.18	4.64	< 0.001
PWK	80	18.65	19.68	1.03	0.16	2.38	0.020
PS	81	5.12	5.09	−0.04	−0.01	−0.22	0.829
PR	78	14.46	13.97	−0.49	−0.15	−1.76	0.082

6.8.4 Example 3—Two Different Ways of Playing Fingu

Our third example focuses on how two different groups of children, identified by their tendency to take shorter or longer amounts of time to answer the different tasks, differ in their ability to make use of the structural affordances that become available when playing the game.

A core idea in the design of Fingu was that children would develop subitizing strategies rather than relying on only counting strategies. Signs that such strategies were used should be evident as shorter MATs and/or greater PCAs. When we analyzed the answering times for different tasks in the game, there were some tasks (e.g., 2 + 0, 5a + 0, and 5a + 5a), where almost 100 % of the answering times were shorter than 3.0 s. In other tasks (e.g. 7a + 1 and 7a + 2), the majority of the answering times were longer than 4.0 s. However, there were a few tasks (e.g., 9a + 1 and 10 + 0), where the distribution of answering times displayed a clear bimodal form, with one bump between 1 and 3 s, and the other between 4 and 6 s (see Fig. 6.3). An interpretation of these bumps in the data is that the first bump is the result of strategies using retrieval or subitizing, whereas the second bump is the result of counting all the objects using a one-by-one method. Of course the answering time is the result of the joint time it takes to determine the sum and to form the finger response pattern.

Since the tasks 10 + 0 and 9a + 1 only appear on level 7, we restricted our analysis of this question to those 60 children, that is 54 % of all children, (see Fig. 6.2), who played all the levels of Fingu. Using the individual MATs from only three special tasks, 10 + 0, 9a + 1 and 5a + 5b, we identified three different groups (with roughly equally numbers of subjects in each): F (Fast) individuals (with MATs shorter than 3.0 s on all three tasks); S (Slow) individuals (with MATs longer than 4.0 s on all three tasks); and M (Medium) individuals (individuals with mixed answering time patterns).

To compare the performance of these three groups, we performed independent samples t-tests between the F and S groups' mean MATs on each of the 60 tasks. What we found is that the F and S groups had significantly different mean MATs on 52 of the 60 tasks (with $p < 0.001$ for 26 of the tasks and p-values between 0.001 and 0.05 for the remaining 26 tasks). The tasks that, divided the F and S groups most significantly were tasks where the configuration 5b was one of the elements.

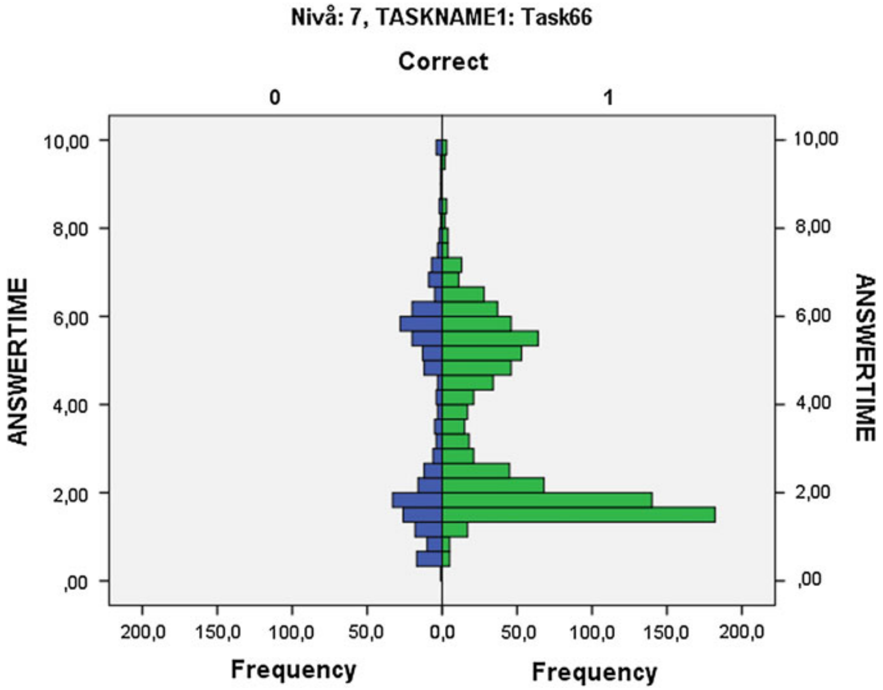

Fig. 6.3 Distribution of answer time for correct and non-correct answers from 60 players that answered the task 10 + 0

The ability to perceive configuration 5b as the quantity five, in the same way as 5a is perceived by almost all of the players, thus seems to be a crucial difference between the F and the S groups.

Other tasks that showed strong significant differences between the F and S groups included tasks with one large (>5) single element, or tasks of the N + 1 type, where N was larger than 3. These tasks represented at least two different types of increases in complexity. The first was to learn the affordance to subitize a non-canonical configuration as 6b, 7b, or 9b. The other type of complexity was to be able to quickly add +1, an ability that presupposes the ability to conceptually subitize the larger part.

In summary, we identified two types of strategies that children developed as they played Fingu. The first was to use counting to find out how many fingers to use and then respond accordingly. The other was to pick up some of the affordances that the game offers to use some kind of subitizing (either perceptual or conceptual), to directly recognize either (a) the single configuration of a task, (b) the two configurations separately, or (c) the totality of these tasks, all resulting in shorter answering times. These strategies were used by all of the different groups, but in very different proportions. There were tasks such as 1 + 0, 5a + 0 and 5a + 5a, which both the F and the S groups solved by subitizing and there were tasks such as

6b + 3a and 7b + 2, which the majority in both groups solved by enumeration (although the F group was faster on these tasks). The biggest observed difference between the groups was in how they managed to utilize the affordances of Fingu. As a sign of an individual's ability to take advantage of these affordances, his or her relation to configuration 5b seems to be indicative.

6.8.5 Example 4—An Individual Developmental Trajectory

Our last example is an analysis of individual development while playing the game based on a single case.

Adam is a five-year old boy attending pre-school, who made large improvements between pre- and post-testing on all four tests, with delayed test results about the same as the post-test results. Among the participants he completed the most trials, with a total of 4572 (IRE-sequences). These trials fell into two periods. In the first 5 weeks of the intervention, he completed 2140 trials during 195 attempts in levels 1–6. In the last three weeks of the intervention, he completed 2432 trials during 141 attempts in levels 6 and 7 while also replaying other levels (see Table 6.5).

In Period 1, Adam was busy trying to advance in the game. What is striking in Table 6.5 is the display of endurance and persistence even with low proportions of correct answers. He completed Levels 1 and 2 quickly but, on Levels 3–5, he made a number of attempts before he succeeded. On Level 6, he made 17 unsuccessful attempts. In Period 2, Adam started by successfully replaying all the levels that he had completed before, and then he continued with his attempts to succeed on Levels 6 and 7. This took another 12 attempts on Level 6 and 28 attempts on Level 7. After that, there was a long period where he repeatedly (mostly successfully) replayed Levels 1–5. Levels 6 and 7 remained difficult with only 2 successes on Level 6 and 4 successes on Level 7. Altogether, what emerged was Adam's perseverance in the game and his willingness to replay it, becoming a more confident Fingu player.

What kind of mathematics has Adam learned? Looking at the PCAs and MATs of the different tasks on different levels reveals several patterns. Since the MATs for correct answers were only greater than 4.0 s for 5 of the tasks and less than 3.0 s for 53 of the tasks, we concluded that Adam had a clear tendency of answering quickly and avoiding counting methods. Regression analyses showed that Adam significantly improved the PCA on 30 of the 60 tasks, and became significantly faster in giving correct answers on 25 of the tasks.

Table 6.5 Adam's number of attempts on different levels and success rates

	Level	1	2	3	4	5	6	7	Sum
Period 1	Attempts	5	7	32	46	88	17	0	195
	Successful (%)	0.60	0.29	0.03	0.02	0.01	0.00		0.04
Period 2	Attempts	21	9	10	9	8	34	50	141
	Successful (%)	0.95	0.78	0.80	0.78	0.88	0.06	0.08	0.39

Using the playback system (see Fig. 6.4 for an illustration), we analyzed the finger patterns that Adam used. During the first period, there were often flashes of dots flickering by and many of the patterns that emerged were interpreted as missing one finger. The first two videos of Adam's play, which showed that he was very impulsive and kinesthetically imprecise, confirmed this observation. An illustrative example is Adam's trajectory for the task 7b + 0. In the first period, he usually answered with the pattern 3 + 5 (3 fingers on the left hand +5 fingers on the right) together with occasional correct answers with the pattern 3 + 4. The most reasonable interpretation of this observation is that he used the pattern 3 + 5 all the time, but occasionally the pinky of his right hand was placed on the touch-insensitive frame of the iPad, which resulted in a correct answer. In this example, the feedback must have been confusing to him, most often being negative, but occasionally positive, for what may have appeared to Adam as the same response. However, as he continued playing, he resolved this dilemma during period 2 by changing his response to 2 + 5.

Another observation made from Adam's response patterns is his tendency to rely on subitizing in solving the tasks. From the beginning, Adam quickly responded with the pattern 0 + 5 on the task 5a + 0 and it did not take long before he used the same pattern on the task 5b + 0. On the tasks 6a + 0 and 6b + 0, however, he quickly developed the response pattern 3 + 3. In the configuration 6a, it was easy for Adam to recognize a 3 + 3 pattern, but in the configuration 6b, this was not as easy. However, it is possible that Adam recognized the 3b triangular configuration on the top of a linear 3-dot pattern. To explain why the task 8b + 0 was harder for Adam to learn than the task 8a + 0, there is the possibility he saw the configuration 8a as composed of the configurations 5b and the same linear 3-dot pattern.

The 3 + 3 response pattern for 6a + 0 and 6b + 0 from the later part of Period 1 into the whole of Period 2 becomes the dominating response pattern for all tasks with 6 as the total sum, with the exception of the task 5a + 1, where the response pattern 1 + 5 persists. In the first part of Period 1 he uses the semi-decimal response pattern 1 + 5 also for tasks 4a + 2, 4b + 2, and 3a + 3a. Our interpretation of these observations is that in the beginning of his playing Adam uses the response pattern 1 + 5 to represent the number 6, while he later establishes the response pattern 3 + 3 as a form of mapping, and as his favorite representation of the number 6. The exception is the task 5a + 1 where he uses the pattern 1 + 5, because it is a direct mapping of the task.

6.8.6 Additional Empirical Observations

In summing up, our analyses have shown that individuals exhibited large variations, not only in the amount of time they played, but also in the strategies they developed to manage the game. Our experience was that on tasks where there was a configuration that children did not immediately recognize, they either explicitly or tacitly counted in order to determine the number of objects that were presented on the

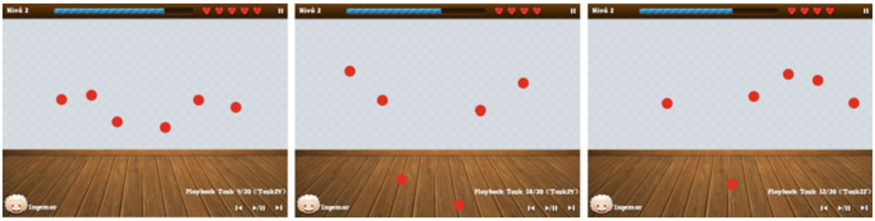

Fig. 6.4 Adam's different ways of answering using six fingers

screen. However, they were much less inclined to count their fingers to determine the number of fingers to put down in responding to the task. Instead, they seemed to develop a kind of personal canonical finger pattern for each of the ten basic numbers. For total quantities of 1–5 these patterns were most often formed by one hand with 1–5 adjacent fingers. For total quantities of 6–10, these patterns could be semi-decimal (i.e., consisting of all the fingers on one hand complemented by fingers on the other hand), or they could be a symmetrical pattern (e.g., 4 fingers on each hand representing 8, or 3 fingers on each hand representing 6). Another way children determined the number of fingers to use, as part of their response was to map each set of the presented objects separately. When this strategy was used, most of the children did not count the number of objects in either of the sets. Instead, they seemed to subitize these numbers. In this way, their strategy was more efficient than counting. In Fig. 6.4, both mapping patterns and a semi-decimal pattern is illustrated. All these analyses of different finger patterns have made use of the built-in replay function of Fingu. As the figure shows, due to the pattern of the red spots, it is very often possible to be almost certain if one or two hands have been used, or even which fingers are used. This interpretation becomes much stronger when compared to the information from the corresponding videos.

6.9 Concluding Remarks

In this chapter, we have described the Fingu game as a virtual manipulative, outlined the design principles, and discussed the underlying theoretical rationale. We have also illustrated some of the affordances of Fingu and the potential effects of playing the game. Our conclusion is that Fingu is a game that offers valuable experience for teaching and learning early numeracy, whether in school or in home settings.

As pointed out above, design, development, use and research of Fingu are part of a research program adopting a DBR-approach. Fingu has gone through several iterations of revision of a number of its design elements, including layout changes and enhancements of the game packaging. We also aim to develop other versions of Fingu, with designs for other affordances than the present version.

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