Chapter 3
The Modification of Attributes, Affordances, Abilities, and Distance for Learning Framework and Its Applications to Interactions with Mathematics Virtual Manipulatives

Stephen I. Tucker

Abstract While extensive research has examined the outcomes of interacting with virtual manipulatives, less research has focused on constructs and relationships among constructs involved in user-tool interactions. This chapter presents the Modification of Attributes, Affordances, Abilities, and Distance (MAAAD) for Learning framework, which conceptualizes the relationships among these constructs to describe user-tool interactions, including those involving virtual manipulatives. The framework is primarily grounded in theories of representation and embodied cognition, as user-tool interactions in mathematics involve internalizing and externalizing representations through physically embodied mathematical practices. In the framework, attributes, affordance-ability relationships, and distance are interrelated, and modification of one construct contributes to modification of the other constructs. Each attribute can contribute to many affordance-ability relationships and to distance. Attribute modification can change the approach or degree of affordance access and alter the degree of distance present, which can, in turn, lead to attribute modification. This chapter illustrates the constructs and relationships among constructs that form the framework in the context of user-tool interactions in mathematics. The chapter then applies the framework to examples of children's interactions with mathematics virtual manipulative touchscreen tablet apps. The MAAAD for Learning framework has implications and applications relevant to theory, development, implementation, and research concerning technology tools, including virtual manipulatives.

Keywords Virtual manipulatives · Mathematics · Tools · Learning

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A menagerie of digital tools exist for learning mathematics and other content, including virtual manipulatives (Moyer et al. 2002), learning objects (Kay and Knaack 2007), mathematical cognitive tools (Sedig 2004), visual mathematical representations (Sedig and Liang 2006), and many others. Definitions of these tools often overlap, but a virtual manipulative is "an interactive, Web-based visual representation of a dynamic object that presents opportunities for constructing mathematical knowledge" (Moyer et al. 2002, p. 373). One interacts with content presented by a virtual manipulative via the interface (e.g., iPad touchscreen platform) that presents the virtual manipulative as part of a digital environment (e.g., app). Thus, characteristics contributing to interactions with a virtual manipulative, the digital environment, and the interface are interrelated.

Continued development and implementation has seen virtual manipulatives become important tools for learning. Recent research suggests that virtual manipulatives need not be web-based (Tucker et al. 2014) and can be used to construct knowledge in other content areas (e.g., Zacharia and de Jong 2014; Zacharia et al. 2008). Although a substantial body of research has indicated that virtual manipulatives can be effective tools for learning (e.g., Moyer-Packenham et al. 2015; Olympiou and Zacharia 2012; Satsangi and Bouck 2014), less research has examined why these tools are effective (e.g., Durmuş and Karakırık 2006; Moyer-Packenham and Westenskow 2013). In particular, little research has identified constructs and relationships among constructs that contribute to interactions with virtual manipulatives. Thus, the purpose of this chapter is threefold: (a) to introduce the Modification of Attributes, Affordances, Abilities, and Distance (MAAAD) for Learning Framework, which models constructs and relationships involved in user-tool interactions, (b) to apply the framework to describe interactions with virtual manipulatives, and c) to discuss potential implications and further applications of the framework.

3.1 Theoretical Grounding

The MAAAD for Learning framework is grounded in theories of representation and embodied cognition set in the context of interaction with technology tools, including virtual manipulatives. These tools offer varying levels of embodiment and fidelity, which also influence user-tool interactions.

3.1.1 Representing Mathematics

Learning mathematics involves interactions among and development of internal and external representations. Internal representations are individuals' mental configurations of mathematics and cannot be directly observed (Goldin and Kaput 1996). External representations are observable, physically embodied configurations of mathematics (i.e., pictures, words, equations, digital environments) which one can access with sufficient understanding of the representations. Interplay among representations includes internalizing external representations (e.g., interpreting graphs, symbols, and pictures) and externalizing internal representations (e.g., writing, speaking, manipulating concrete objects). Importantly, interactions with appropriate combinations of multiple external representations can enhance learning (Ainsworth 2006). Understanding of representations and connections among multiple representations is representational fluency (Zbiek et al. 2007) which influences interactions among and development of internal and external representations. Representational fluency can both facilitate and result from mathematical learning (Heinze et al. 2009; Nathan and Kim 2007) and thus both contributes to, and results from, learning mathematics.

3.1.2 Embodied Cognition: Physical Interactions with Representations

Mathematical practices that include physically interacting with external representations involve embodied cognition, as cognitive processes are part of bodily interactions with the environment. From an embodied cognition lens, human cognition is rooted in sensorimotor processing (Wilson 2002), which integrates perception of the environment with actions upon the environment. Thus, human cognition is based in action and perception, and is grounded in the physical environment (Alibali and Nathan 2012). Nemirovsky et al. (2013) suggested that "the intertwining of perceptual and motor aspects of tool use [is] *perceptuomotor integration,*" allowing one to perceive and interact with representations in such a way that integrates action and thought (p. 373, emphasis in original).

Applied to mathematics, mathematical thinking is equivalent to physical engagement in mathematical practices, and mathematical learning involves changes in these physically embodied practices (cf. Lakoff and Núñez 2000). Therefore, perceptuomotor integration is the way in which one uses bodily activity to facilitate interplay between internal and external representations and develop representational fluency. Thus, bodily engagement (external) in mathematical practices, such as interactions with mathematics virtual manipulatives, can provide evidence of (internal) representations of mathematics, and changes in these practices provide evidence of learning.

3.1.3 Technology for Interactions with Representations

Technology tools offer varying degrees of embodiment. Bodily engagement includes representational gestures, which bodily actions use in interplay among internal and external representations (Hostetter and Alibali 2008; Segal 2011). Gestures can help children retain and apply knowledge within similar contexts (Cook et al. 2008) when developmentally appropriate (Ginsburg et al. 2013; Shuler 2009) and mapped to the content (Segal 2011; Segal et al. 2014). Many technology tools feature multi-touch interfaces (e.g., iPads), which can support a wide variety of input gestures (e.g., Hamon et al. 2013) for user-tool interactions.

Although relatively few apps effectively incorporate multi-touch capabilities (Byers and Hadley 2013), apps that do use multi-touch capabilities may uniquely influence children's mathematical understandings and strategy development (Baccaglini-Frank and Maracci 2015). Multi-touch technology can thus afford users greater embodiment, relatively direct control over the manipulation of representations, and a wider range of mathematically meaningful gestures than mouse-based interaction, when tasks and the tools are appropriately designed.

3.1.4 Faithfulness of Technology Tools for Interacting with Representations

Researchers have theorized ways to design educational tools that facilitate mathematics learning (e.g., Ginsburg et al. 2013; Pelton and Francis Pelton 2011). Many guidelines originate with Dick (2008), who recommended that technology tool designers insure high levels of cognitive, pedagogical, and mathematical fidelity. Cognitive fidelity is the degree of alignment of the mathematical representations of the tool with the cognitive processes of the student. Pedagogical fidelity is the degree of alignment of the tool with design principles. Mathematical fidelity is the degree of mathematical appropriateness of the representations of the content.

Tools and tasks, such as those that involve virtual manipulatives, vary in fidelity (Moyer-Packenham et al. 2008). Some researchers consider the greatest challenge in designing digital tools for learning mathematics to be insuring cognitive fidelity by allowing effective externalization of a child's mathematical thinking (Olive 2013). For many concepts, digital tools can offer "idealized" representations that are more mathematically faithful than concrete representations (de Kirby 2013), allowing users embodied interactions with visual models of concepts that formerly were only accessible in mental models (Carpenter 2013). Discussions of pedagogical fidelity often include pedagogical approaches of digital tools (e.g., instructive, manipulable, and constructive: Highfield and Goodwin 2013; self-leveling, collaborative, and sandbox: Zanchi et al. 2013). Each type of fidelity influences the design of the tool and the users' perception of and interactions with the tool, thereby influencing the internalization and externalization of representations via perceptuomotor integration.

3.1.5 Summary of the Theoretical Framework

Embodied cognition and representation, in the form of perceptuomotor integration and representational fluency, influence the transformation of internal representations, while gestures assist the externalization and internalization of representations. Technology affords embodiment in human-computer interaction, and cognitive, pedagogical, and mathematical fidelity influence how users interact with technology tools. These theories provide theoretical grounding for the MAAAD for Learning framework.

3.2 Building Toward the Conceptual Framework

The MAAAD for Learning framework integrates attributes, affordances-ability relationships, and distance to model user-tool interactions. Thus, the conceptual framework emerges from a synthesis of empirical and theoretical research involving these constructs and relationships among these constructs (Tucker 2015).

3.2.1 Roles of the Constructs

Attributes, affordance-ability relationships, and distance each play roles in children's interactions with technology tools, including virtual manipulatives.

Attributes. Attributes are characteristics of people or things (Attribute [Def. 5] 2014). Relevant attributes of tools (e.g., virtual manipulatives) and users are involved in user-tool interactions. Using an embodied cognition perspective of learning mathematics, attributes contribute to physical engagement in mathematical practices (i.e., mathematical thinking) and changes in the physically embodied practices (i.e., mathematical learning). Users and tools both have attributes related to content (e.g., mathematics), technology (i.e., physical interactions with the tool), and other aspects of user-tool interactions (e.g., user: personal characteristics; tool: structural characteristics).

In a study examining children's interactions with mathematics virtual manipulative iPad apps, Tucker (2015) categorized app attributes and user attributes involved in physical interactions with mathematical representations. Both apps and users had mathematical attributes (i.e., content attributes), which were characteristics involved in representing mathematical content. Apps and users had subcategories of mathematical attributes related to content (e.g., decimals) and representation (e.g., number line). Users also had a subcategory of mathematical attributes related to flexibility (e.g., transfer from shaded rectangles to shaded circles). Other literature also implies evidence of content attributes of technology tools (e.g., fraction models: Rick 2012) and content attributes of users (e.g., understandings of fraction models: Moyer-Packenham et al. 2014b).

Research has identified empirical evidence of technological attributes pertaining to physical interactions between user and tool in user-app interactions (Tucker 2015). For apps, technological attribute subcategories included input range (i.e., scope of gestures accepted by the [tool] for a given function) and input complexity (i.e., intricacy of the required gestures). For users, technological attribute subcategories included motor skills (i.e., facility with which a user performed the relevant physical actions) and input familiarity (i.e., how conversant a user was in a given input). Other literature also implies the presence of technological attributes of technology tools (e.g., touch input types: Lao et al. 2009) and technological attributes of users (e.g., motor skills: Dejonckheere et al. 2014). Users and tools each have an additional, unique category of attributes: personal and structural, respectively (Tucker 2015). Personal attributes are characteristics of one's personality that influence how one interacts with a tool, including affect, persistence, and goals (e.g., goal of accuracy or speed). Structural attributes are non-content presentation features, including feedback, context, and scaffolding (e.g., hint scaffold reveals worked example). Other literature also implies evidence of personal attributes of users (e.g., affect: Goldin et al. 2011) and structural attributes of technology tools (e.g., scaffolding: Belland and Drake 2013).

Attributes are not static, and attribute alignment influences attribute modification (Tucker 2015). Alignment of content (e.g., mathematical) or technological attributes varies. Finding a missing addend by adding on objects is developmentally appropriate for many 4–5 year-old children (i.e., relatively aligned mathematical attributes), but is likely to be developmentally inappropriate for 2–3 year-old children (i.e., misaligned mathematical attributes) (Sarama and Clements 2009). When interacting with the mathematics virtual manipulative iPad app Motion Math: Zoom, some children efficiently performed a pinching input gesture (i.e., aligned technological attributes), while other children struggled to do so (i.e., misaligned technological attributes) (Tucker et al. 2016a). Users and tools influence attribute alignment through attribute modification, which can be proactive or reactive (Tucker 2015). Reactive modification occurs when tools modify tool attributes and in response, users apply and modify user attributes.

When attributes align and the user successfully completes the task, the tool may in turn modify tool attributes. The user responds by applying and modifying user attributes, continuing the cycle. Proactive modification occurs when tools modify tool attributes, users apply and modify user attributes, and users modify tool attributes. For example, some users repeatedly attempted the same level of mathematics virtual manipulative iPad apps despite consistently poor performance (i.e., reactively modified attributes), whereas other users returned to a previous level with related content to build back toward the more challenging content (i.e., proactively modified attributes). From an embodied cognition perspective of learning mathematics, attribute modification can contribute to and result from changes in physically embodied mathematical practices (i.e., learning).

Affordance-ability relationships. Research suggests that affordance-ability relationships play a complex but key role in how children learn while interacting with technology (e.g., Tucker 2015; Tucker et al. 2016b). Greeno (1994), drawing on Gibson (e.g., 1986), posited that an affordance related attributes of an object in the environment (e.g., tool) to an interactive activity undertaken by an agent (e.g., user). The agent applied an ability based on its attributes as part of this interactive activity. Thus, an interactive activity links an affordance of a tool with the ability of a user. Each affordance exists only in relation to an ability, and vice versa (Greeno 1994), and the two are coupled in a continuous system (Chemero 2003).

Some authors discuss the idea of constraints, which one can consider part of what the app affords. However, widely varying conceptions of affordances led Burlamaqui and Dong (2014) to state that “the only uncontroversial claim about affordances is that they are about action possibilities relative to the agent” (p. 13). From an embodied cognition perspective of learning mathematics, affordance-ability relationships are interactive links between user and tool as part of physically embodied mathematical practices.

Authors have applied the concept of affordances to technology (e.g., Gaver 1991; Sedig and Liang 2006), such as virtual manipulatives for various content areas (e.g., fractions: Moyer-Packenham et al. 2014a) and as part of multiple technology tools (e.g., mouse-controlled computer applets: Moyer-Packenham et al. 2013; touch-screen tablet apps: Tucker et al. 2016b). Although meta-analyses of affordances of virtual manipulatives show that instruction using virtual manipulatives has positive effects on learning (Moyer-Packenham and Westenskow 2013, 2016), accession of the same affordance can vary greatly by user ability and by context (Moyer-Packenham et al. 2016; Tucker and Moyer-Packenham 2014).

Furthermore, applying the same approach to affordance access may still result in different outcomes (e.g., Tucker et al. 2016b) For example, DragonBox Algebra 12+ affords efficient precision by guiding completion of the additive equality and additive identity properties (Tucker 2015). Some children used the same approach, efficiently combining these properties for the first part of a task when possible without combining these properties for later stages of the same task. However, outcomes varied, as some of these children correctly completed both properties, while others only completed the additive equality property.

Research also indicates that affordance-ability relationships are not static and can interact with one another. For example, children can access simultaneous linking of pictorial representations, symbolic representations, and actions in different ways, and some children use different approaches as their relevant ability changes (Tucker et al. 2016b). From an embodied cognition perspective of learning mathematics, changes in physically embodied mathematical practices (i.e., learning) can both contribute to and result from changes in affordance-ability relationships.

Furthermore, multiple affordance-ability relationships can influence one another, such as efficient precision, creative variation, and focused constraint (Tucker 2015). For example, while interacting with a mathematics virtual manipulative iPad app, a child attempted efficient, mathematically correct input that the app disallowed, constraining focus on another mathematical concept. During a more advanced level, the app permitted the mathematical input it had previously disallowed, emphasizing efficiency. The child then creatively applied this input by combining it with other mathematical input. Thus, affordance-ability relationships play a role in user-tool interactions.

Distance. Sedig and Liang (2006) define distance as the "degree of difficulty in understanding how to act upon [something] and interpret its responses" (p. 184). From an embodied cognition perspective of learning mathematics, distance characterizes the degree of difficulty in engaging in mathematical practices through user-tool interactions. Cognitive, pedagogical, and mathematical fidelity (Dick 2008) may contribute to distance, as they influence tool design and user perception of the tool. Distance can be reduced if one designs the tool to fit the learner's understandings or if the learner determines how to use the tool, and maintaining appropriate distance through purposeful, stepwise distance modification by a tool can facilitate learning (Sedig et al. 2001). Maintaining appropriate distance relates to Vygotsky's (1978) Zone of Proximal Development (ZPD), applied to technology as when tasks remain developmentally appropriate while users progressively master instructional objectives (Murray and Arroyo 2002). Progressive mastery implies that users also change to maintain an appropriate degree of distance. Thus, both users and technology tools change during interactions to maintain distance, which facilitates the learning process.

There are multiple types of distance, including mathematical (i.e., content) and technological (c.f., Sedig and Liang 2006). *Mathematical distance* is "the degree of difficulty of the mathematical aspects of interactions between the user and the tool (e.g., a mathematics virtual manipulative iPad app)" (Tucker 2015, p. 82). A high degree of mathematical distance is evident when one struggles to complete the mathematical aspects of a task, whereas a low degree of mathematical distance is evident when one has less difficulty completing the mathematical aspects of a task. For example, when navigating a number line displaying intervals of one tenth to find the range in which 0.05 is located (i.e., 0.0–0.1), choosing 0.5–0.6 shows evidence of a higher degree of distance than choosing the range of 0.0–0.1. *Technological distance* is "the degree of difficulty of the technological aspects of interactions between the user and the tool" (Tucker 2015, p. 83). A high degree of technological distance is evident when one struggles to complete the technological aspects of a task, whereas a low degree of technological distance is evident when one has less difficulty completing the technological aspects of a task. For example, some children had difficulty controlling mouse input when interacting with computer-based virtual manipulatives (i.e., higher degree of mathematical distance), whereas other children found these interactions less difficult (i.e., lower degree of mathematical distance) (Highfield and Mulligan 2007).

Distance is not static and distance types can influence each other (Tucker 2015). Research implies that distance differs by context. Distance can decrease, as when children initially struggled to accurately complete a task involving the splitting model of fractions, yet successfully completed the task after additional experience with the content (i.e., decreasing mathematical distance) (Martin et al. 2013).

Distance can also increase, as when children initially appropriately used input gestures but later chose inappropriate gestures that hindered task completion (i.e., increasing technological distance) (Tucker 2015). Using an embodied cognition perspective of learning mathematics, changes in distance can both lead to and result from changes in physically embodied mathematical practices (i.e., learning).

Research also implies that these distance types can interact. Difficulty performing required input such as controlled mouse movements (technological distance) while interacting with virtual Pattern Blocks can lead to unintended mathematical outcomes such as unintentionally rotating shapes instead of sliding them (mathematical distance) (Highfield and Mulligan 2007). A high degree of technological distance in the form of difficulty using appropriate input can also lead to a user focusing attention on performing the gestures (i.e., decreasing technological distance) rather than attending to the mathematical content (i.e., high degree of mathematical distance) (Rick 2012). Thus, distance plays a role in user-tool interactions.

3.2.2 Relationships Among the Constructs

Relationships among attributes, affordance-ability relationships, and distance also play roles in children's interactions with technology tools, including virtual manipulatives.

Attributes and affordance-ability relationships. In this context, (a) tools have attributes that combine to provide affordances, (b) users have attributes that combine to create abilities, and (c) an affordance-ability relationship exists between user and tool (see Fig. 3.1).

Modification of attributes can lead to modification of affordance-ability relationships and vice versa (Tucker 2015). Modifying user attributes can lead to modification of ability as part of affordance-ability relationships. For example, research indicates that children may be less likely to access audio feedback as part of efficient precision as they become more proficient at completing tasks while

Fig. 3.1 Relationship between attributes and affordance-ability relationships set within user-tool interactions (adapted from Tucker 2015, p. 19)

interacting with mathematics iPad apps (Bartoschek et al. 2013; Paek 2012). Modification to affordance-ability relationships can lead to modification of app attributes, such as when children access efficient precision by placing the appropriate number of fingers on the screen to indicate an answer while interacting with the mathematics virtual manipulative app Fingu, contributing to advancement to a level featuring different content (Barendregt et al. 2012).

Modifying app attributes can modify an affordance as part of the affordance-ability relationship, which can lead to modification of user attributes (Tucker 2015). For example, during interactions with the mathematics virtual manipulative iPad app DragonBox Algebra 12+, the app initially prohibits children from combining the additive inverse property and the additive equality property to efficiently move a variable from one side of an equation to the other, focusing their attention on separately applying these properties. In a more advanced level, the app removed this constraint, permitting the "drag across" move. After this, some children creatively and efficiently combined the properties when possible, providing evidence of a change in user mathematical attributes and abilities as part of the corresponding affordance-ability relationships. Each attribute can contribute to multiple affordance-ability relationships, such as coordination (user technological: motor skills) contributing to accession of both planning (efficient precision) and navigation restrictions (focused constraint). Improving the coordination attribute could lead to a different approach to planning and fewer encounters with navigation restrictions. Thus, there are relationships between attributes and affordance-ability relationships during user-app interactions.

Attributes and distance. Attributes also relate to distance, and attribute modification can lead to modification of distance, while modification of distance can contribute to modification of attributes (Tucker 2015). Distance is the degree of alignment (i.e., difference) between clusters of relevant user attributes and app attributes (see Fig. 3.2).

Evidence of relationships between attributes and distance is present in research literature. Specifically, distance can be conceived of as the degree of alignment between clusters of relevant attributes of the tool and user (Tucker 2015). For example, during the *Pop the Bubble* activity in the *MathemAntics* app suite, children compare schools of fish by counting each set and popping the bubble that

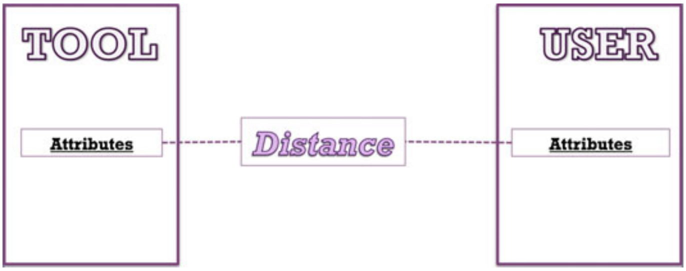


Fig. 3.2 Relationship between attributes and distance set within user-tool interactions

contains more fish (Ginsburg et al. 2013). The app represents quantity and comparison using sets of objects (fish grouped in bubbles) and the user must have sufficient knowledge of this representation of quantity, as well as counting and quantity comparison skills to decide which set has more objects in it. The degree of alignment of these attributes is the mathematical distance. The app requires the user to click on the bubble to indicate a response, so the user must have sufficient familiarity with this input method and sufficient motor skills to perform this gesture. The degree of alignment of these attributes is the technological distance.

Patterns related to attributes and distance are also evident in user-app interactions. For example, when children advanced to a new level while interacting with an app, mathematical distance often increased (Tucker 2015). Children applied and modified user mathematical attributes in an attempt to decrease mathematical distance. Some children decreased mathematical distance by proactively modifying app attributes to select different mathematical content. This provided an environment in which they could strengthen relevant user attributes, leading to decreased distance when returning to levels with content that was initially too difficult. Users can also modify user technological attributes to align with requirements for interacting with apps, such as by using gestures that the tool can recognize after initial attempts are unsuccessful (Ladel and Kortenkamp 2012).

Modification of structural attributes and personal attributes can also influence distance. For example, upon first encountering the needle timer (structural) during interactions with Motion Math: Zoom, children struggled to accurately complete mathematical tasks (increased mathematical distance) or efficiently perform appropriate gestures (increased technological distance) (Tucker 2015). One child proactively chose when to activate and deactivate the needle timer while as a way to increase or decrease the degree of difficulty (i.e., goal led to changing distance by modifying app attributes). Thus, there are relationships between attributes and distance during user-app interactions.

Distance and affordance-ability relationships. Distance also relates to affordance-ability relationships, as accession of affordances can influence distance, and distance can influence accession of affordances (Tucker 2015) (see Fig. 3.3).

Fig. 3.3 Relationship between distance and affordance-ability relationships set within user-tool interactions

Research also provides evidence of relationships between distance and affordance-ability relationships, as seen in Fig. 3.3. Affordance-ability relationships can influence distance, such as when a child stated that each group of levels in a mathematics virtual manipulative iPad app “starts off easy and then gets harder and it tells you what [math] to do at first and then you do that on your own on the next one” (Tucker 2015, p. 116). The group of levels began with a level that focused attention on one mathematical property, before modifying constraints by providing a level that required the user to apply the newest property with others to complete the task, often resulting in increased mathematical distance. Distance also influences affordance-ability relationships, such as when high achieving students (i.e., implied lower degree of distance) ignored pictorial models as part of simultaneous linking, whereas lower achieving students (i.e., implied higher degree of distance) relied on the linked pictorial models (Moyer-Packenham and Suh 2012).

Interactions between mathematical distance and technological distance can also influence accession of motivation. For example, a child described interactions with a mathematics virtual manipulative iPad app as, “easy, but it wouldn’t give me enough time to do stuff because it was super-hard to get to areas you wanted to go to” (Tucker 2015, p. 116). This implied the perception that the mathematical distance was not worth overcoming due to the degree of technological distance, which led to a high degree of access to negative motivation and the decision to stop interacting with the app.

However, other research indicated that children who struggled to overcome technological difficulties that interfered with mathematical accuracy were still motivated to persist with mathematical activities (Paek and Hoffman 2014). Thus, there are relationships between distance and affordance-ability relationships during user-app interactions. Research provided evidence of attributes, affordance-ability relationships, distance, and relationships among these constructs in user-tool interactions. Integration of these constructs and relationships among these constructs led to the development of a conceptual framework to model the roles they play during user-tool interactions.

3.3 The Modification of Attributes, Affordances, Abilities, and Distance for Learning Framework

Syntheses of theoretical and empirical research on user-tool interactions, such as those involving virtual manipulatives, provides evidence of the interconnected relationships among attributes, affordance-ability relationships, and distance that form the Modification of Attributes, Affordances, Abilities, and Distance (MAAAD) for Learning framework (see Fig. 3.4).

As seen in Fig. 3.4, the MAAAD for Learning framework for user-tool interactions begins with attributes (Tucker 2015). The difference between clusters of relevant tool and user attributes forms distance. Modification of attributes may

Fig. 3.4 Modification of Attributes, Affordances, Abilities, and Distance (MAAAD) for Learning framework (Tucker 2015, p. 117)

bring attributes into alignment (e.g., the user masters the content the tool presents) leading to decreased distance, or it may misalign attributes (e.g., the tool presents relatively challenging content after a user successfully completes tasks) leading to increased distance. Abilities, stemming from clusters of related user attributes, relate to specific affordances, which are based on related clusters of app attributes. A variety of approaches or degrees of affordance access may emerge from variations in user attributes. A particular attribute can contribute to a multitude of affordance-ability relationships and to distance. Distance influences affordance-ability relationships; for example, a high degree of distance due to misaligned attributes may induce different affordance access than when a low degree of distance is present because attributes are aligned. Figure 3.5 presents a version of the MAAAD for Learning framework applied to learning mathematics through user-app interactions.

As illustrated in Fig. 3.5, the MAAAD for Learning framework can be applied to user interactions with mathematics virtual manipulative apps, and includes distance types and attribute categories and subcategories identified in prior research (Tucker 2015). In this application of the framework, mathematical distance is the difference between clusters of app mathematical attributes and the corresponding clusters of user mathematical attributes, while technological distance is the difference between clusters of app technological attributes and the corresponding clusters of user technological attributes. Clusters of user attributes (mathematical, technological, and personal) form abilities to access app affordances, which stem from clusters of app attributes (mathematical, technological, and structural). Each attribute can contribute to multiple affordance-ability relationships (e.g., user: mathematical: content: comparison contributes to accessing efficient precision of range contents and focused constraint of navigation restrictions); therefore, an affordance-ability relationship can influence another affordance-ability relationship if they share contributing attributes. Distance types (mathematical and technological) can interact, as well as influence affordance-ability relationships, which contribute to variations in accession of affordances.

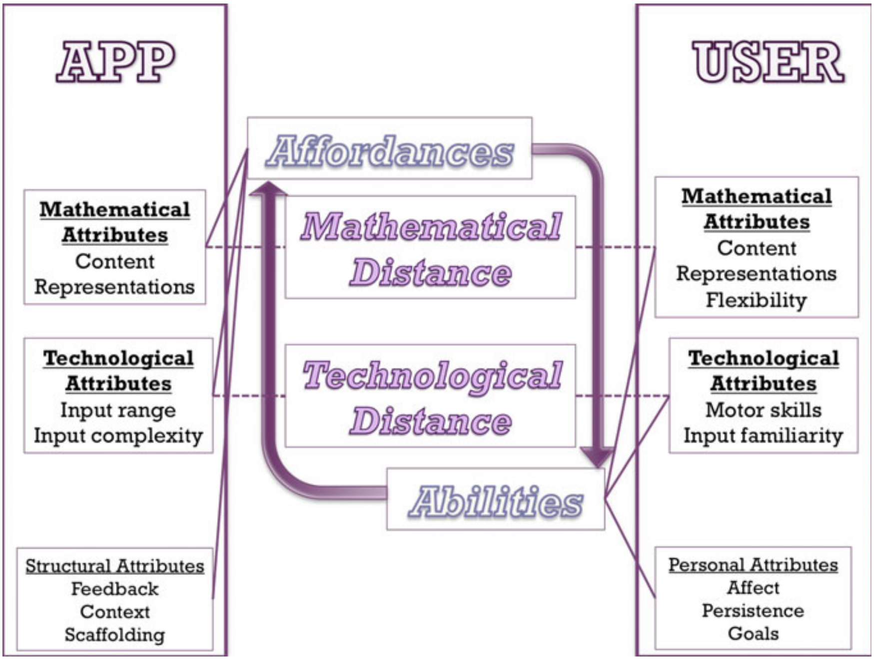


Fig. 3.5 Expanded version of the Modification of Attributes, Affordances, Abilities, and Distance for Learning framework applied to learning mathematics through user-app interactions (Tucker 2015, p. 119)

3.4 Applying the MAAAD for Learning Framework

Researchers have applied the MAAAD for Learning framework to describe user-tool interactions. The following examples are drawn from a study in which Tucker (2015) examined fifth-grade children's interactions with two mathematics virtual manipulative iPad apps during one-to-one semi-structured interviews.

Applying the framework to interactions with Motion Math: Zoom. Children's interactions with the mathematics virtual manipulative tablet app Motion Math: Zoom provides evidence of the MAAAD for Learning framework. Motion Math: Zoom includes content related to number comparisons, magnitude, and estimation on an idealized number line that is navigable and scalable. This representation is more mathematically faithful to the infinite number line than a static physical representation. The user employs single-touch and multi-touch input to change scales and navigate the number line and place target numbers (see Fig. 3.6). Figures 3.7, 3.8, 3.9 and 3.10 illustrate the MAAAD for Learning framework as applied to an excerpt from a sequence of a child's interactions with Motion Math: Zoom.

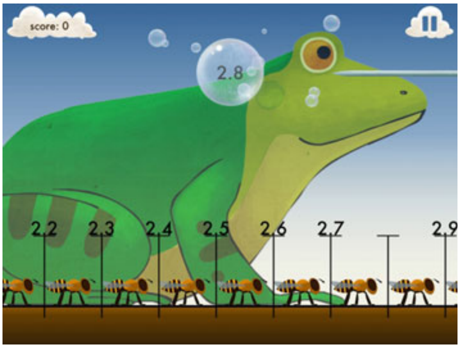


Fig. 3.6 Screenshot of Motion Math: Zoom (Tucker 2015, p. 33)

As Figs. 3.7, 3.8, 3.9 and 3.10 demonstrate, relationships within the MAAAD for Learning framework can be found throughout the child's interactions with Motion Math: Zoom. Throughout these interactions, technological attributes remained aligned and there was a low degree of technological distance. For much of the time, the child consistently attempted planning when possible. Initial

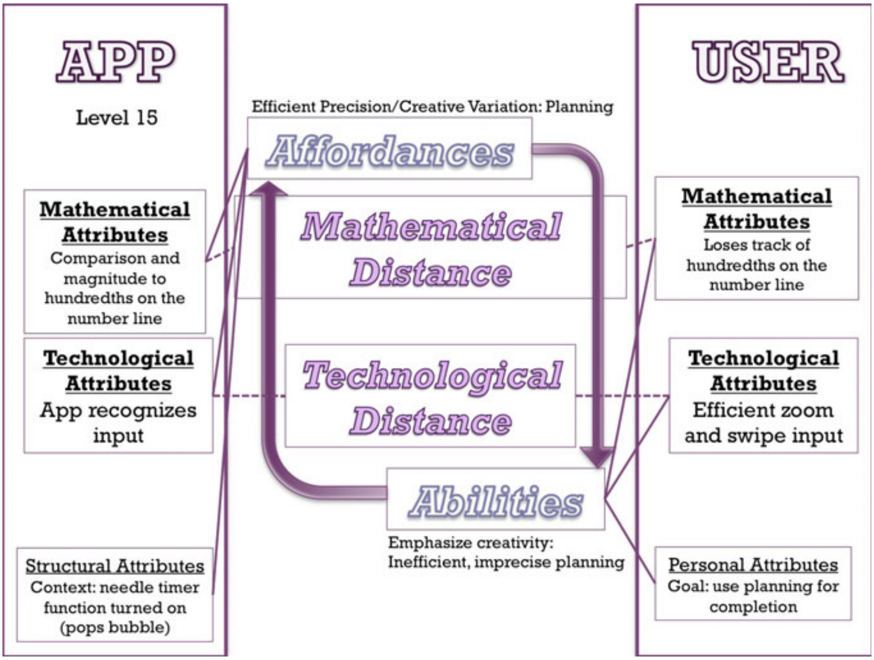


Fig. 3.7 The child experimented with planning as part of completing the level but there was a high degree of mathematical distance as the child lost track of hundredths on the idealized number line. The needle popped the bubble as time for task completion expired. The child made efficient input gestures that the app recognized, so there was a low degree of technological distance. The child restarted the level (adapted from Tucker 2015, pp. 125–127)

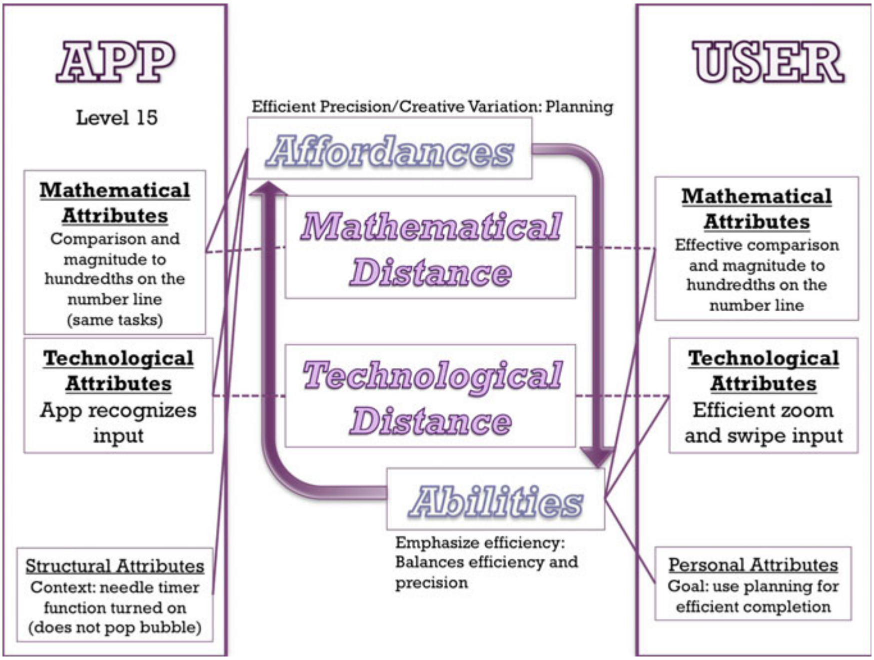


Fig. 3.8 The child modified user mathematical attributes, effectively navigating hundredths on the number line, thus decreasing mathematical distance. The child changed approach to accessing planning, emphasizing efficiency and precision instead of creatively experimenting. Technological distance remained minimal (adapted from Tucker 2015, pp. 125–127)

experimentation with a creative approach to planning contributed to failure to complete the level in a timely manner (Fig. 3.7). The child then modified the approach to planning to focus on efficiency, which in tandem with modification of mathematical attributes led to decreased mathematical distance and successful completion of the level (Fig. 3.8).

On the following level, which presented similar but slightly more advanced tasks, the degree of mathematical distance was so great that the child did not successfully complete even the first task and thus did not have the opportunity to plan (Fig. 3.9). During this attempt, the child struggled to access the affordance of efficient precision in the form of consistent range contents (i.e., 0–10 contains 0, 1, 2–10). In the final attempt in the excerpt, the child proactively changed app attributes by choosing to attempt a different level with related but easier content, decreasing mathematical distance (Fig. 3.10). The child could again access the planning affordance, but discontinued planning after realizing it was not more efficient to do so in that context.

Applying the framework to interactions with DragonBox Algebra 12+. Children's interactions with the mathematics virtual manipulative tablet app DragonBox Algebra 12+ also provide evidence of the MAAAD for Learning

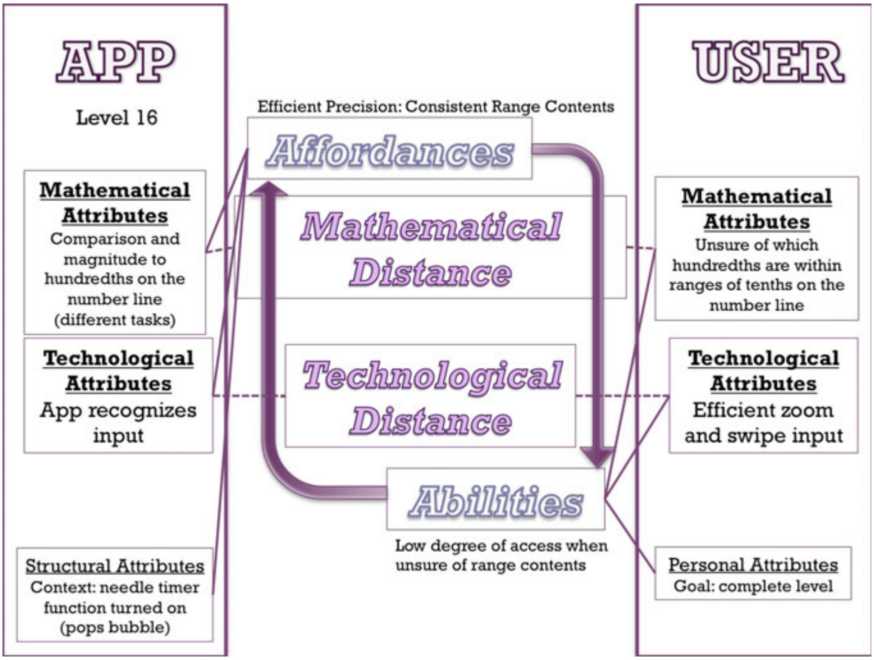


Fig. 3.9 The app presented different tasks based on similar content, showing a change in mathematical attributes. The child did not effectively transfer understanding of ranges to the new tasks, and mathematical distance increased. The child struggled to access efficient precision in the form of consistent range contents. Technological distance remained minimal (adapted from Tucker 2015, pp. 125–127)

framework. DragonBox Algebra 12+ includes content related to solving expressions and equations, operations, negative and positive values, additive and multiplicative thinking, and fractions, presented using pictorial and symbolic tiles (see Fig. 3.11). The user employs single-touch input to drag or tap tiles to complete each expression or equation. Figures 3.12, 3.13, 3.14 and 3.15 illustrate the MAAAD for Learning framework as applied to an excerpt from a sequence of a child's interactions with DragonBox Algebra 12+.

As Figs. 3.12, 3.13, 3.14 and 3.15 demonstrate, relationships within the MAAAD for Learning framework can be found throughout children's interactions with DragonBox Algebra 12+. While accessing a high degree of negative motivation during struggles to complete level 2:13, the child proactively used the solution scaffold to model the steps to complete the level (Fig. 3.12). However, the child rushed to replicate the solution, blurring gestures and increasing technological distance (Fig. 3.13). During this failure to replicate the solution, the child showed signs of frustration and a high degree of access to negative motivation.

During the next attempt, the child's goal changed to accurate completion (Fig. 3.14). The use of relatively precise gestures decreased technological distance,

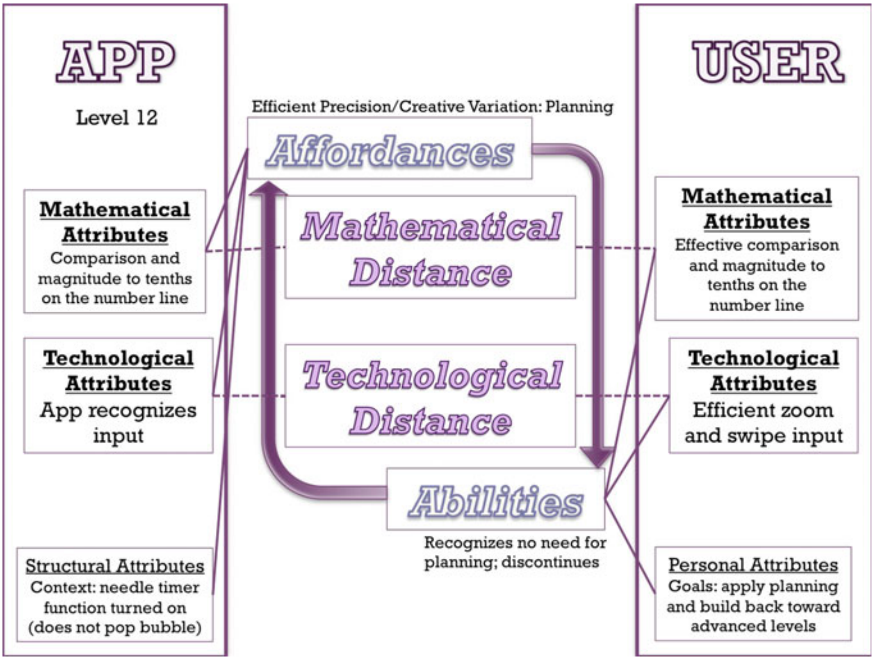
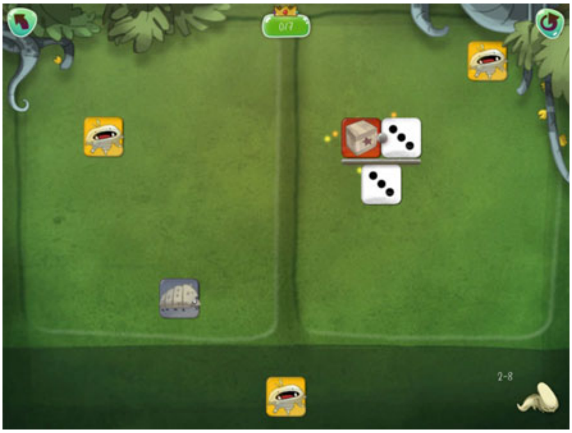


Fig. 3.10 The child proactively modified app attributes, reducing the level and changing to less advanced mathematical content, decreasing mathematical distance. Technological distance remained minimal. During this attempt, the child stopped planning after recognizing it did not contribute to efficiency in this level (adapted from Tucker 2015, pp. 125–127)

and the child recognized the need to apply the reverse order of operations. After restarting the level, the child correctly applied the reverse order of operations, demonstrating changes in mathematical attributes contributing to a decrease in

Fig. 3.11 Screenshot of DragonBox Algebra 12+ (Tucker 2015, p. 35)



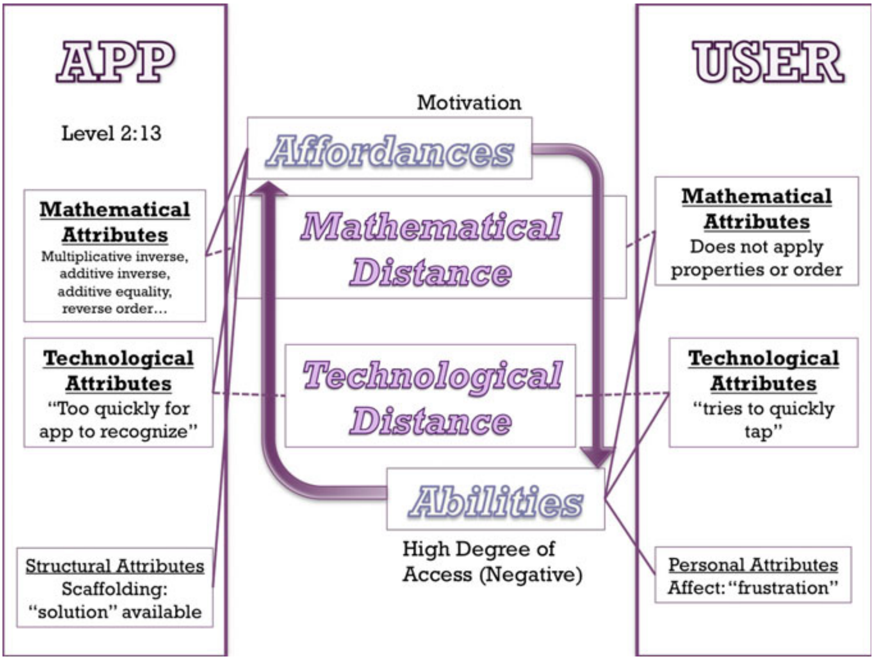


Fig. 3.12 The child attempted to decrease mathematical distance due to unaligned mathematical attributes through proactive modification of app structural attributes (solution scaffolding). The child showed a high degree of access to negative motivation (adapted from Tucker 2015, pp. 121–123)

mathematical distance. The child also honed input gestures, contributing to both a decrease in technological distance and a change in ability to access the affordance of simultaneously linking mathematical representations with actions. These examples show that the MAAAD for Learning framework models relationships among attributes, affordance-ability relationships, and distance in user-tool interactions, such as children's interactions with mathematics virtual manipulative iPad apps.

3.5 Implications and Applications

The MAAAD for Learning framework has implications and applications relevant to theory, development, implementation, and research concerning interactions with technology tools, including virtual manipulatives. These implications and applications build on the descriptive power of the framework (e.g., analyzing user-tool interactions), which has not been applied for prescriptive purposes (e.g., hypothesizing specific user-tool interactions).

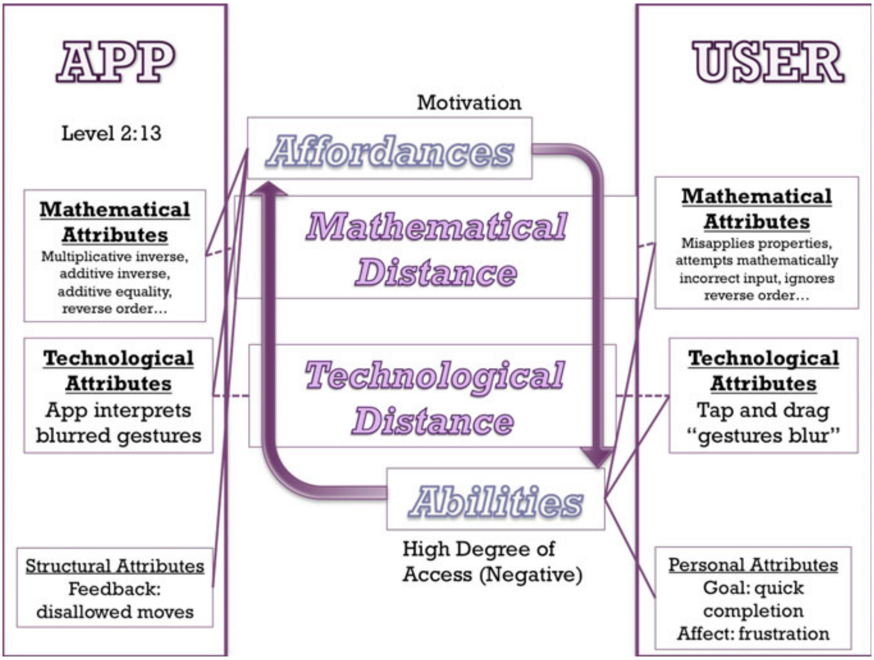


Fig. 3.13 The child failed to correctly replicate solution while attempting to quickly complete the level. A high degree of mathematical distance remained and technological distance increased as the child struggled to make the app recognize some input gestures. The child continued to have a high degree of access to negative motivation and reset the level (adapted from Tucker 2015, pp. 121–123)

3.5.1 Theory

One can view the MAAAD for Learning framework through a variety of theoretical lenses, but embodied cognition and representation provided a context for the developmental phases (Tucker 2015). The framework models specific constructs and relationships among constructs that contribute to bodily engagement in mathematical practices (i.e., physically embodied interactions with representations) that constitute mathematical thinking and learning. However, researchers could consider MAAAD for Learning using other theoretical lenses and conceptual frameworks. Potential approaches include: (a) different theories of affordances (see Burlamaqui and Dong 2014), (b) multimedia learning (e.g., Interactive Multimedia Model for Cognitive Learning: Daghestani 2013), (c) complex cognitive activities (e.g., EDIFICE-AP: Sedig and Parsons 2013), and (d) activity theory (e.g., Artifact-centric activity theory: Ladel and Kortenkamp 2013). These and other theoretical discussions could continue development of the framework, such as by locating the teacher, interviewer, or peer. Along with embodied cognition and representation,

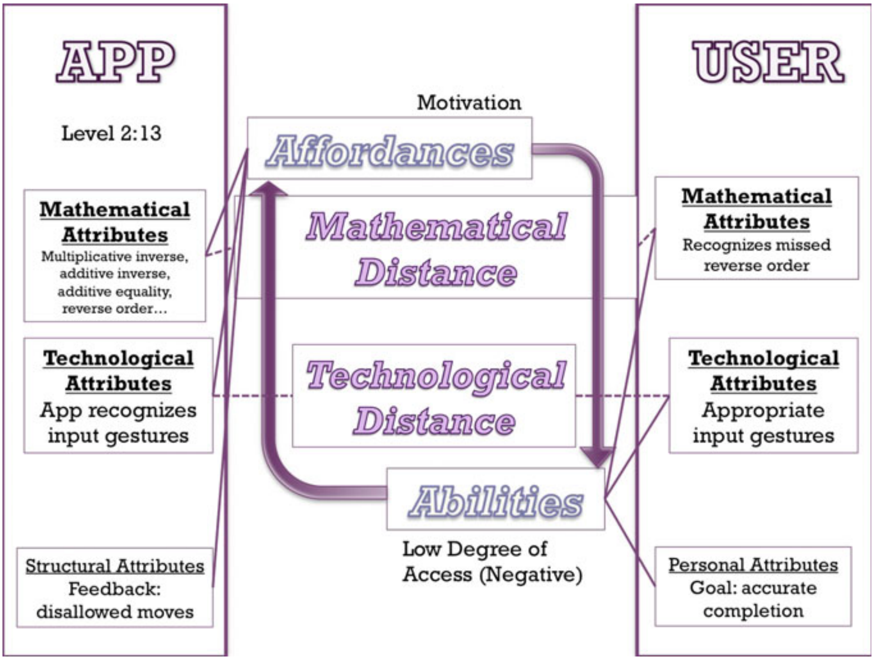


Fig. 3.14 The child attempted to increase accuracy, but failed to correctly replicate solution. However, the child noticed the missed use of the reverse order of operations for solving. The degree of mathematical distance remained high but technological distance decreased as the child produced recognizable input gestures. The child reduced the degree of access to negative motivation and reset the level (adapted from Tucker 2015, pp. 121–123)

various theories could inform further development and application of the framework.

3.5.2 *Design*

The MAAAD for Learning framework has implications and applications for those who design virtual manipulatives and other technology tools. During research and development, designers could use MAAAD for Learning to examine and organize the attributes that contribute to affordance-ability relationships involved in the user-tool interactions, as well as the myriad of potential manifestations of these relationships. Within the framework, designers could also consider purposeful modification of the constructs, including when and how the tool could modify attributes that in turn modify distance and affordance-ability relationships, as well as possible outcomes of these modifications. Additionally, by clarifying for users which tool attributes are modifiable, designers could encourage proactive modification. The

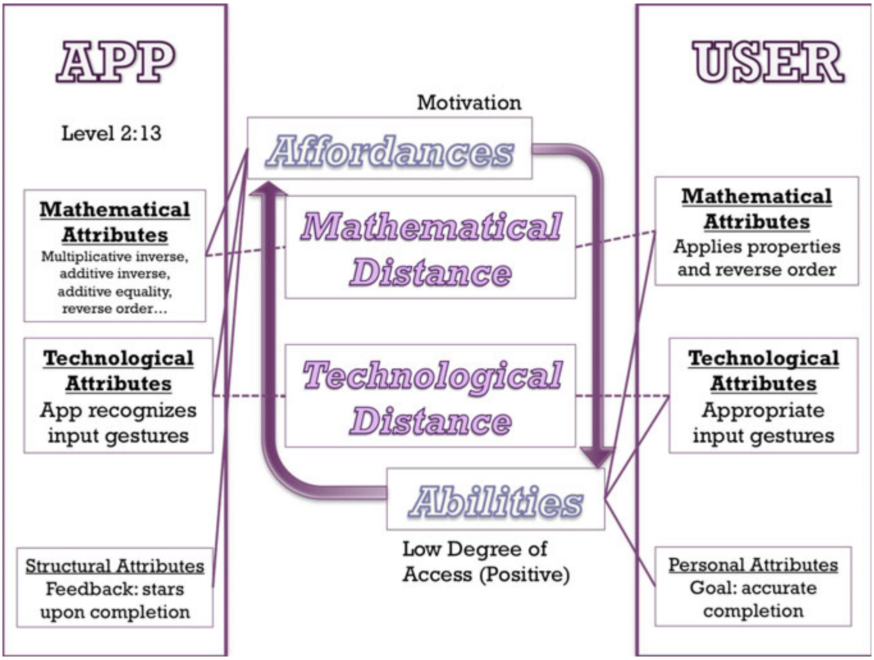


Fig. 3.15 The child slowed interactions and accurately completed the level, having changed mathematical attributes and reduced mathematical distance by correctly applying the properties in the correct (reverse) order. The child also reduced technological distance, completing the level without struggling to perform recognizable input gestures. The child showed a low degree of access to positive motivation (adapted from Tucker 2015, pp. 121–123)

framework may be of use to technology research and development groups when examined in relation to human-computer interaction research in the technology design field, including decision making, information visualization, and adaptive systems (e.g., Jacko 2012).

3.5.3 Implementation

The MAAAD for Learning framework also has implications and applications for implementers of virtual manipulatives and other technology tools and for those who train others to implement these tools. Teacher educators could consider the framework in relation to literature about teachers' use of technology tools, such as teacher beliefs about technology integration (e.g., Ertmer 2005) and Technological Pedagogical Content Knowledge (Mishra and Koehler 2006). Teacher educators could also develop a practitioner presentation of the framework that would permit teachers to use MAAAD for Learning to evaluate the appropriateness of a given tool for a particular child, including the alignment of user attributes and tool attributes. This may help teachers decide when to provide targeted external scaffolding to encourage appropriate proactive attribute modification, such as by helping users recognize opportunities to modify tool attributes. Although mathematical thinking and learning can occur throughout these interactions, children may not be aware they are engaged in mathematical practices. Thus, teachers could also use the framework to examine user-tool interactions for evidence of mathematical thinking and learning as part of informal assessment, supporting facilitation of intentional discussions of these mathematical interactions that could aid recognition of the mathematical thinking and learning.

3.5.4 Research

The MAAAD for Learning framework has potential implications and applications for those who research learning and technology tools, including virtual manipulatives. Fine-grained applications of the framework, such as using it to analyze user-tool interactions during specific mathematical learning trajectories (e.g., Sarama and Clements 2009) could aid research into the potential influences of multi-touch technology on the ways that children learn mathematics (e.g., Baccaglini-Frank and Maracci 2015). Researchers could also investigate manifestations of specific attributes (e.g., representing Base 10) or attribute categories (e.g., personal, structural). Lateral applications of MAAAD for Learning include applying it to other user-tool interactions. These studies could involve different subject matter (e.g., science) to develop subject-specific variants (e.g., science attributes and scientific distance). Additional investigations could apply the framework to interactions with other technology tools (e.g., video games) in a variety of settings (e.g., classroom) involving various users (e.g., diverse learners). These applications would inform research on user-tool interactions in multiple contexts, such as using virtual manipulatives to teach children with learning disabilities in mathematics in general education classes (e.g., Satsangi and Bouck 2014).

Broader applications of the framework are also possible. Researchers could investigate MAAAD for Learning in relation to specific outcomes, such as achievement on learning assessments, particularly when conducting longitudinal examinations of user-tool interactions. This could build on research that indicates use of virtual manipulative touchscreen apps can positively influence performance on mathematical tasks (e.g., Riconscente 2013; Zhang et al. 2015). Extensions of this research could identify long-term patterns in user-tool interactions that correlate with learning outcomes.

3.6 Conclusion

The MAAAD for Learning framework models relationships among attributes, affordance-ability relationships, and distance in the context of user-tool interactions, and primarily emerges from studies focusing on interactions with mathematics virtual manipulatives. The framework can be a useful tool for developers, educators, and researchers whose work involves technology tools. Developers of technology tools can use the framework to model relationships among constructs that play a role in user-tool interactions and the resulting experiences. Educators who implement technology tools to support learning can use the framework to evaluate learning that occurs during children's classroom-based interactions with technology tools. Researchers can apply the framework to investigate constructs contributing to children's learning during interactions with technology, in addition to potential outcomes of these interactions.

Importantly, consistent use of the MAAAD for Learning framework across these applications could provide a common language for modeling and discussing user-tool interactions. These applications may also lead to further development of the framework, such as through clarification of constructs, relationships, and emergent themes, or creation of versions for different content areas. To aid both aims, it may be beneficial to clarify potential differences between a tool and an object embedded within the tool (e.g., attributes of the touchscreen device, the app, and each mathematics virtual manipulative), or if the entire tool should be considered as one when interacting with the embedded object (i.e., attributes of a mathematics virtual manipulative touchscreen app). Future research involving connections to learning outcomes, diverse populations, various contexts, different content areas, and additional technology tools will advance the literature concerning user-tool interactions and contribute to the development and application of the MAAAD for Learning framework.

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